Math 6337 - Homework 5 - Caitlin Beecham

Question 1
Let E be a m'ble subset of $\mathbb{R}^n$ with $|E| < \infty$ and $f : E \to [0, \infty)$ be m'ble. Let

$$\omega(\alpha) = |\{f > \alpha\}|, \ 0 \leq \alpha < \infty$$

be its distribution function. Let $0 < p < \infty$. Prove that

$$\int_E f^p < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} 2^{kp} \omega\left(2^k\right) < \infty.$$

Hint: You may use Lemma 5.38.

Now, we have a slight problem with trying to bound $\int_E f^p$ be our desired quantity but that problem can be solved by instead bounding above as

$$\int_E f^p \leq 2^p \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) \tag{1}$$

because certainly $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$ if and only if $2^p \sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$ which means that we still get $\int_E f^p$. So, let's show Equation 1 or equivalently that

$$\int_E f^p \leq \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \omega(2^k). \tag{2}$$

We break our domain into the following regions $E_k := \{2^k < f \leq 2^{k+1}\}$.

Now, we note that since $f \geq 0$ and since $g(x) = x^p$ is monotonically increasing for all $x \geq 0$ we have for all $k \in \mathbb{Z}$

$$\int_{E_k} f^p \leq \int_{E_k} (2^{k+1})^p.$$

which implies that

$$\begin{aligned}
\int_E f^p &= \int_{\cup_{k \in} E_k} f^p && (3)\\
&= \sum_{k \in \mathbb{Z}} \int_{E_k} f^p && (4)\\
&\leq \int_E \chi_{E_k}(x)(2^{k+1})^p\\
\int_E f^p &\leq \int_E \sum_{k \in \mathbb{Z}} \chi_{E_k}(x)(2^{k+1})^p && (5)
\end{aligned}$$

Now, we let $G_N(x) = \sum_{k=-N}^{N} \chi_{E_k}(x)(2^{(k+1)})^p$ and note that $G_N(x) \nearrow$ as $N \nearrow$.

Then, by Equation 5 and noting that $G(x) := \sum_{k\in\mathbb{Z}} \chi_{E_k}(x)(2^{k+1})^p = \lim_{N\to\infty} G_N(x)$, the fact that $G_N \nearrow G$ tells us by the Monotone Convergence Theorem that

$$\int_E \lim_{N\to\infty} \left(\sum_{k=-N}^{N} \chi_{E_k}(x)(2^{k+1})^p \right) = \lim_{N\to\infty} \int_E \left(\sum_{k=-N}^{N} \chi_{E_k}(x)(2^{k+1})^p \right). \tag{6}$$

Then, Lemma 5.4 tells us that

$$\int_E \left(\sum_{k=-N}^{N} \chi_{E_k}(x)(2^{k+1})^p \right) = \sum_{k=-N}^{N} (2^{k+1})^p |E_k| \tag{7}$$

which along with Equation 6 means that

$$\int_E \sum_{k=-\infty}^{\infty} \chi_{E_k}(x)(2^{k+1})^p =$$

$$\int_E \lim_{N\to\infty} \left(\sum_{k=-N}^{N} \chi_{E_k}(x)(2^{k+1})^p \right) = \lim_{N\to\infty} \left(\sum_{k=-N}^{N} (2^{k+1})^p |E_k| \right) \tag{8}$$

$$\int_E \sum_{k=-\infty}^{\infty} \chi_{E_k}(x)(2^{k+1})^p = \sum_{k=-\infty}^{\infty} (2^{k+1})^p |E_k| \tag{9}$$

and recall that Equation 5 says that

$$\int_E f^p \leq \int_E \sum_{k\in\mathbb{Z}} \chi_{E_k}(x)(2^{k+1})^p,$$

meaning that all together we have

$$\int_E f^p \leq \sum_{k=-\infty}^{\infty} (2^{k+1})^p |E_k|. \tag{10}$$

Finally, since $E_k \subseteq \{f > 2^k\}$ and $\omega(2^k) = |\{f > 2^k\}|$ we have by monotonicity that $|E_k| \leq \omega(2^k)$ for all $k \in$ meaning that

$$\sum_{k=-\infty}^{\infty} (2^{k+1})^p |E_k| \leq \sum_{k=-\infty}^{\infty} (2^{k+1})^p \omega(2^k). \tag{11}$$

Now, we note that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty}(2^{k+1})^p\omega(2^k) &= \lim_{N\to\infty}\sum_{k=-N}^{N}(2^{k+1})^p\omega(2^k) \\
&= \lim_{N\to\infty}\int_E \sum_{k=-N}^{N}(2^{k+1})^p\chi_{\{f>2^k\}} \\
&= \int_E \lim_{N\to\infty}\sum_{k=-N}^{N}(2^{k+1})^p\chi_{\{f>2^k\}} && (12) \\
&\geq \int_E \lim_{N\to\infty}\left(\sum_{k=-N}^{N}(2^{k+1})^p\chi_{\{2^{k+1}\geq f>2^k\}}\right) && (13) \\
&= \int_E \lim_{N\to\infty}\left(\sum_{k=-N}^{N}(2^{k+1})^p\chi_{E_k}\right) && (14) \\
&\geq \int_E f^p. && (15)
\end{aligned}
$$

where Equation 12 holds by the Monotone Convergence Theorem, Equation 13 holds since $\chi_{\{f>2^k\}} \geq \chi_{\{2^{k+1}\geq f>2^k\}}$, Equation 14 holds by definition of E_k, and Equation 15 holds by Equation 5 which is what we set out to prove, namely that

$$\int_E f^p \leq \sum_{k=-\infty}^{\infty}(2^{k+1})^p\omega(2^k),$$

which proves implication in one direction, namely that $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$ implies that $\int_E f^p < \infty$.

Now, we would like to prove that

$$\int_E f^p < \infty$$

implies that

$$\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$$

.

Note that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) &= \sum_{k=-\infty}^{\infty} 2^{kp}\big(\sum_{i=k}^{\infty}|E_i|\big)\\
&= \sum_{i=-\infty}^{\infty}\big(|E_i|\sum_{k=-\infty}^{i}2^{kp}\big)\\
&= \sum_{i=-\infty}^{-1}\big(|E_i|\sum_{k=-\infty}^{i}2^{kp}\big) + \sum_{i=0}^{\infty}\big(|E_i|\sum_{k=-\infty}^{i}2^{kp}\big)\\
&= \sum_{i=-\infty}^{-1}\big(|E_i|\sum_{k=-\infty}^{i}2^{kp}\big) + \sum_{i=0}^{\infty}|E_{-i}|\big(\sum_{k=-\infty}^{-1}2^{kp} + \sum_{k=0}^{i}2^{kp}\big)\\
&= \sum_{i=1}^{\infty}\big(|E_{-i}|\sum_{k=i}^{\infty}2^{-kp}\big) + \sum_{i=0}^{\infty}|E_i|\big(\sum_{k=1}^{\infty}2^{-kp} + \sum_{k=0}^{i}2^{kp}\big)\\
&= \sum_{i=1}^{\infty}\big(|E_{-i}|\sum_{k=i}^{\infty}(\frac{1}{2^k})^p\big) + \sum_{i=0}^{\infty}|E_i|\Big(\sum_{k=1}^{\infty}(\frac{1}{2^k})^p + \sum_{k=0}^{i}2^{kp}\Big)\\
&= \sum_{i=1}^{\infty}\big(|E_{-i}|\Big((\frac{1}{2^i})^p\sum_{k=0}^{\infty}(\frac{1}{2^k})^p)\Big)\\
&+ \sum_{i=0}^{\infty}|E_i|\Big(\Big((\frac{1}{2^i})^p(\sum_{k=0}^{\infty}(\frac{1}{2^k})^p\Big) + \sum_{k=0}^{i}2^{kp}\Big)\\
&= \sum_{i=1}^{\infty}\big(|E_{-i}|\Big((\frac{1}{2^i})^p 2^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\Big)\\
&+ \sum_{i=0}^{\infty}|E_i|\Big(\Big((\frac{1}{2^i})^p 2^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\Big) + \sum_{k=0}^{i}2^{kp})\Big)\\
\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) &= \sum_{i=1}^{\infty}\big(|E_{-i}|\Big((\frac{1}{2^{i-1}})^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p)\Big)\\
&+ \sum_{i=0}^{\infty}|E_i|\Big(\Big((\frac{1}{2^{i-1}})^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\Big) + \sum_{k=0}^{i}2^{kp}\Big), \qquad (16)
\end{aligned}
$$

and for $p \geq 1$ we have $\sum_{k=1}^{\infty}(\frac{1}{2^k})^p \leq \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$ so that

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|\Big((\frac{1}{2^{i-1}})^p \sum_{k=1}^{\infty}(\frac{1}{2^k})^p)\Big) + \sum_{i=0}^{\infty} |E_i|\Big(\Big((\frac{1}{2^{i-1}})^p (\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\Big) + \sum_{k=0}^{i} 2^{kp}\Big)
$$

$$
\le \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp}\Big)\Big) \tag{17}
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} \frac{2^{ip}}{2^{ip}} 2^{kp}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} \sum_{k=0}^{i} \frac{2^{kp}}{2^{ip}}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} \sum_{k=0}^{i} 2^{(k-i)p}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} \sum_{l=0}^{i} 2^{-kp}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} 2^p \sum_{l=1}^{i} 2^{-kp}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} 2^p \sum_{l=1}^{i} 2^{-kp}\Big)\Big)
$$

$$
\le \sum_{i=1}^{\infty} \big(|E_i|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{ip} 2^p\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}|(\frac{1}{2^{i-1}})^p\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}| 2^{-(i-1)p}\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)\Big)
$$

$$
= \sum_{i=1}^{\infty} \big(|E_{-i}| 2^{(-i+1)p}\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)\Big)
$$

$$
= \sum_{i=-\infty}^{-1} \big(|E_i| 2^{(i+1)p}\big) + \sum_{i=0}^{\infty} \Big(|E_i|\Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)\Big)
$$

$$
= \sum_{i=-\infty}^{-1} \int_{E_i} 2^{(i+1)p} + \sum_{i=0}^{\infty} \int_{E_i} \Big((\frac{1}{2^{i-1}})^p + 2^{(i+1)p}\Big)
$$

$$
\le \sum_{i=-\infty}^{-1} \int_{E_i} 2^{(i+1)p} + \sum_{i=0}^{\infty} \int_{E_i} \Big(2^{(i+1)p} + 2^{(i+1)p}\Big)
$$

$$
= \sum_{i=-\infty}^{-1} \int_{E_i} 2^{(i+1)p} + \sum_{i=0}^{\infty} \int_{E_i} \Big(2^{((i+1)p)+1}\Big)
$$

$$
= \sum_{i=-\infty}^{-1} 2^p \int_{E_i} 2^{ip} + \sum_{i=0}^{\infty} 2^{p+1} \int_{E_i} \Big(2^{ip}\Big)
$$

and quickly recalling that we are assuming throughout this calculation that $p \geq 1$ we continue on to get

$$
\begin{aligned}
&\leq \sum_{i=-\infty}^{-1} 2^{p+1} \int_{E_i} 2^{ip} + \sum_{i=0}^{\infty} 2^{p+1} \int_{E_i} \left(2^{ip}\right) \\
&= 2^{p+1} \left(\sum_{i=-\infty}^{-1} \int_{E_i} 2^{ip} + \sum_{i=0}^{\infty} \int_{E_i} \left(2^{ip}\right) \right) \\
&= 2^{p+1} \left(\sum_{i=-\infty}^{\infty} \int_{E_i} 2^{ip} \right) \\
&\leq 2^{p+1} \left(\sum_{i=-\infty}^{\infty} \int_{E_i} f^p \right) \\
&= 2^{p+1} \left(\int_{\cup_{i=-\infty}^{\infty} E_i} f^p \right) \\
&= 2^{p+1} \left(\int_{E} f^p \right)
\end{aligned}
$$

So, finally, throughout that whole calculation, what we concluded is that for $p \geq 1$ we have that

$$
\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) \leq 2^{p+1} \left(\int_E f^p \right), \tag{18}
$$

which means that if $\int_E f^p < \infty$ then of course $2^{p+1}\left(\int_E f^p \right) < \infty$ and in turn $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$, which concludes the proof of the reverse implication if $p \geq 1$.

Now, what if $p < 1$. In such a case, we cannot use $\frac{1}{2}^p \leq \frac{1}{2}$ which was crucial in Equation 17.

However, we have a similar result. Pick $n \in \mathbb{N}$. We observe that

$$\begin{aligned}
\sum_{k=0}^{\infty} (\frac{1}{2^k})^{\frac{1}{n}} &= \sum_{a=0}^{\infty} \sum_{b=0}^{n-1} (\frac{1}{2^{an+b}})^{\frac{1}{n}} \\
&\leq \sum_{a=0}^{\infty} \sum_{b=0}^{n-1} (\frac{1}{2^{an}})^{\frac{1}{n}} \\
&= \sum_{a=0}^{\infty} \sum_{b=0}^{n-1} \frac{1}{2^a} \\
&= \sum_{a=0}^{\infty} \frac{n}{2^a} \\
&= n \sum_{a=0}^{\infty} \frac{1}{2^a} \\
&= 2n \\
\sum_{k=0}^{\infty} (\frac{1}{2^k})^{\frac{1}{n}} &\leq 2n. \qquad (19)
\end{aligned}$$

Now, we adjust this inequality slightly to note

$$\begin{aligned}
\sum_{k=1}^{\infty} (\frac{1}{2^k})^{\frac{1}{n}} &= \left(\sum_{k=0}^{\infty} (\frac{1}{2^k})^{\frac{1}{n}} \right) - 1 \\
&\leq 2n - 1 \\
\sum_{k=1}^{\infty} (\frac{1}{2^k})^{\frac{1}{n}} &\leq 2n - 1. \qquad (20)
\end{aligned}$$

So, find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < p \leq \frac{1}{n_0 - 1}$. Then, we have that $(\frac{1}{2^k})^{\frac{1}{n_0}} \geq (\frac{1}{2^k})^p$ for all $k \geq 1$. So, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} (\frac{1}{2^k})^p &\leq \sum_{k=1}^{\infty} (\frac{1}{2^k})^{\frac{1}{n_0}} \\
&\leq 2n_0 - 1. \qquad (21)
\end{aligned}$$

and

$$\sum_{k=0}^{\infty} (\frac{1}{2^k})^p \leq 2n_0. \qquad (22)$$

So, now we return to the result given in Equation 16 which did not depend on the value of p. Namely, we concluded that

$$\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) = \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p \sum_{k=1}^{\infty}(\frac{1}{2^k})^p)\bigg)$$
$$+\sum_{i=0}^{\infty}|E_i|\bigg(\bigg((\frac{1}{2^{i-1}})^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\bigg)+\sum_{k=0}^{i}2^{kp}\bigg)$$

which along with Equation 21 gives

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) &= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p \sum_{k=1}^{\infty}(\frac{1}{2^k})^p)\bigg) \\
&+\sum_{i=0}^{\infty}|E_i|\bigg(\bigg((\frac{1}{2^{i-1}})^p\sum_{k=1}^{\infty}(\frac{1}{2^k})^p\bigg)+\sum_{k=0}^{i}2^{kp}\bigg) \\
&\leq \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg) \\
&+\sum_{i=0}^{\infty}|E_i|\bigg(\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg)+\sum_{k=0}^{i}2^{kp}\bigg) \\
&= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg) \\
&+\sum_{i=0}^{\infty}|E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg)+\sum_{i=0}^{\infty}|E_i|\bigg(\sum_{k=0}^{i}2^{kp}\bigg) \\
&= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg) \\
&+\sum_{i=0}^{\infty}|E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0-1\bigg)\bigg)+\sum_{i=0}^{\infty}|E_i|\bigg(\sum_{k=0}^{i}2^{kp}\bigg). \qquad (23)
\end{aligned}$$

Now, let $K := max\bigg(1, \big(2n_0-1\big)\bigg)$.

Then, we continue on from Equation 23 to get

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0 - 1\bigg)\bigg) \\
&+ \sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(2n_0 - 1\bigg)\bigg) + \sum_{i=0}^{\infty} |E_i|\bigg(\sum_{k=0}^{i} 2^{kp}\bigg) \\
&= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(K\bigg)\bigg) \\
&+ \sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(K\bigg)\bigg) + \sum_{i=0}^{\infty} |E_i|\bigg(\sum_{k=0}^{i} 2^{kp}\bigg)K \\
&= \sum_{i=1}^{\infty} \big(|E_{-i}|\bigg((\frac{1}{2^{i-1}})^p\bigg(K\bigg)\bigg) \\
&+ \sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(K\bigg)\bigg) + \sum_{i=0}^{\infty} |E_i|\bigg(\sum_{k=0}^{i} 2^{kp}\bigg)K \\
&= \sum_{i=-\infty}^{-1} \big(|E_i|\bigg((\frac{1}{2^{-i-1}})^p\bigg(K\bigg)\bigg) \\
&+ \sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p\bigg(K\bigg)\bigg) + \sum_{i=0}^{\infty} |E_i|\bigg(\sum_{k=0}^{i} 2^{kp}\bigg)K \\
&= \sum_{i=-\infty}^{-1} \big(|E_i|\bigg((\frac{1}{2^{-i-1}})^p\bigg(K\bigg)\bigg) \\
&+ \sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp}\bigg)K \\
&= K \sum_{i=-\infty}^{-1} \big(|E_i|\bigg((\frac{1}{2^{-i-1}})^p\bigg) \\
&+ K\sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp}\bigg) \\
&= K \sum_{i=-\infty}^{-1} \big(|E_i|(2^{i+1})^p\bigg) \\
&+ K\sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp}\bigg) \\
\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) &\leq K \sum_{i=-\infty}^{-1} \big(|E_i|(2^{i+1})^p\bigg) \\
&+ K\sum_{i=0}^{\infty} |E_i|\bigg((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp}\bigg), \qquad (24)
\end{aligned}
$$

and finally, note that

$$
\begin{aligned}
\sum_{k=0}^{i} 2^{kp} &= 2^{ip} \sum_{k=0}^{i} \frac{1}{2^{kp}} \\
&= 2^{ip} \sum_{k=0}^{i} (\frac{1}{2^k})^p,
\end{aligned}
\tag{25}
$$

which along with Equation 22 gives

$$
\begin{aligned}
\sum_{k=0}^{i} 2^{kp} &= 2^{ip} \sum_{k=0}^{i} (\frac{1}{2^k})^p, \\
&\leq 2^{ip} \sum_{k=0}^{\infty} (\frac{1}{2^k})^p \\
&\leq 2^{ip}(2n_0) \\
\sum_{k=0}^{i} 2^{kp} &\leq 2^{ip}(2n_0).
\end{aligned}
\tag{26}
$$

(Please recall that n_0 depends solely on p which we are viewing as a fixed constant; namely it is the number $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < p \leq \frac{1}{n_0 - 1}$).

Now, plugging the result of Equation 26 into Equation 24 gives

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) &\leq K \sum_{i=-\infty}^{-1} \Big(|E_i|(2^{i+1})^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \left((\frac{1}{2^{i-1}})^p + \sum_{k=0}^{i} 2^{kp} \right) \\
&\leq K \sum_{i=-\infty}^{-1} \Big(|E_i|(2^{i+1})^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \left((\frac{1}{2^{i-1}})^p + (2^{ip}(2n_0)) \right) \\
&= K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \left((\frac{1}{2^{i-1}})^p + (2^{ip}(2n_0)) \right) \\
&\leq K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big((1 + (2^{ip}(2n_0)) \Big) \\
&\leq K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big((1 + (2^{ip}(2n_0)) \Big) \\
&\leq K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big(((2^{ip}(2n_0)) + (2^{ip}(2n_0)) \Big) \\
&= K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p\Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big((4n_0) 2^{ip} \Big)
\end{aligned}
$$

Now, let $R := max\big(2^p, 4n_0\big)$ and continue on to get

$$
\begin{aligned}
&= K \sum_{i=-\infty}^{-1} \Big(|E_i| 2^p (2^i)^p \Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big((4n_0) 2^{ip} \Big) \\
&\leq K \sum_{i=-\infty}^{-1} \Big(|E_i| R (2^i)^p \Big) \\
&+ K \sum_{i=0}^{\infty} |E_i| \Big(R 2^{ip} \Big) \\
&= KR \sum_{i=-\infty}^{-1} \Big(|E_i| (2^{ip}) \Big) \\
&+ KR \sum_{i=0}^{\infty} |E_i| \Big(2^{ip} \Big) \\
&= KR \sum_{i=-\infty}^{\infty} \Big(|E_i| (2^{ip}) \Big) \\
&= KR \sum_{i=-\infty}^{\infty} \int_{E_i} (2^{ip}) \\
&\leq KR \sum_{i=-\infty}^{\infty} \int_{E_i} f^p \\
&= KR \int_{\cup_{i=-\infty}^{\infty} E_i} f^p \\
&= KR \int_E f^p
\end{aligned}
$$

$$\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) \leq KR \int_E f^p. \tag{27}$$

Since, K, R are constants depending only on p and n_0, really that means that K, R depend only on p since n_0 depends only on p. So, indeed if for fixed p we have that

$$\int_E f^p < \infty,$$

then certainly

$$KR \int_E f^p < \infty,$$

and thus

$$\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty,$$

which is what we set out to prove, namely that $\int_E f^p$ finite implies that $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k)$ is finite.

Now we have proven both direction of the implication and we are done.

Question 2

(a) Let $E \subset \mathbb{R}^2$ be measurable and such that for a.e. $x \in \mathbb{R}$, $\{y : (x, y) \in E\}$ has $\mathbb{R}^1-$measure 0. Show that E has measure 0 and for a.e. $y \in \mathbb{R}$, $\{x : (x, y) \in E\}$ has $\mathbb{R}^1-$measure 0.

First I show that E has measure 0.

Plan:

We first assume $|E| < \infty$, then apply Fubini's Theorem (Lemma 6.6). We showed that $\int_E 1 = |E|$ for any lesbegue measurable set E.

Then, Fubini tells us that

$$\int \int_E 1dxdy = \int \int_E \chi_E dxdy = \int_{\mathbb{R}} \big(\int_{\mathbb{R}} \chi_E(x, y)dy \big)dx.$$

We then use Theorem 5.10 (b) as described below.

Namely, let $F := \{x \in E : |\{y : (x, y) \in E\}| \neq 0\}$ and let $E' = E \setminus F$.

Then, we note that $\chi_E(x, y) = \chi_{E'}(x, y)$ except for those $x \in F$.

Note that for $x_0 \in E'$ we have that $\int_{\mathbb{R}} \chi_E(x, y)dy\Big|_{x_0} = \int_{\mathbb{R}} \chi_{E'}(x, y)dy\Big|_{x_0}$.

If $x_0 \in F$ we have

$$\int_{\mathbb{R}} \chi_E(x, y)dy\Big|_{x_0} \neq \int_{\mathbb{R}} \chi_{E'}(x, y)dy\Big|_{x_0},$$

possibly.

If we let

$$f(x) = \int_{\mathbb{R}} \chi_E(x, y)dy\Big|_{x_0}$$

and

$$g(x) = \int_{\mathbb{R}} \chi_{E'}(x, y)dy\Big|_{x_0}$$

then Theorem 5.10 tells us that since $f(x) = g(x)$ a.e. we have that

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} g(x)dx$$

or using the old notation that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x, y)dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_{E'}(x, y)dy \right) dx.$$

Then, recall that for $x \in E'$ we have that

$$\int_{\mathbb{R}} \chi_{E'}(x, y)dy = 0.$$

Thus,

$$\int \int_E 1dxdy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_{E'}(x, y)dy \right) dx = \int_{\mathbb{R}} \Big(0 \Big) dx = 0.$$

Now, we note that if $|E| = \infty$ we can now consider $B_k := \{x \in \mathbb{R}^2 : k \leq |x| < k+1\}$ and define $E_k = E \cap B_k$ for all $k \in \mathbb{N}$. Then, by monotonicity we have that $|E_k| < \infty$ for each $k \in \mathbb{N}$, meaning that the result stands for each E_k. Finally note that

$$|E| = \sum_{k=0}^{\infty} \int \int_{E_k} \chi_{E_k}(x,y) dxdy = 0.$$

Now, it still remains to show that for a.e. $y \in \mathbb{R}$ that $|\{x : (x,y) \in E\}| \neq 0$ has $\mathbb{R}^1$ measure 0.

Otherwise, assume for contradiction that the set $G := \{y : |\{x : (x,y) \in E\}| \neq 0\}$ has nonzero measure $|G| > 0$.

Once again we reduce to the case in which $|E|$ finite in order to apply Fubini's theorem. Then, we examine the integral

$$\int \int \chi_E(x,y) dydx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y) dy \right) dx \geq \int_G \int_{\mathbb{R}} \chi_E(x,y) dydx = |\{(x,y) \in E : y \in G\}|.$$

Now, $\int_{\mathbb{R}} \chi_E(x,y) dy > 0$ for all $x \in G$ means that $\int_G \int_{\mathbb{R}} \chi_E(x,y) dydx > 0$ (since $|G| = 0$ by assumption), so that $\int \int \chi_E(x,y) dydx > 0$, which is a contradiction.

Now, for the case in which $|E| = \infty$ we once again write $E = \bigsqcup_{k=0}^{\infty} E_k$ and note that

$$\int \int \chi_E(x,y) dydx = \sum_{k=0}^{\infty} \int \int \chi_{E_k}(x,y) dydx \geq \sum_{k=0}^{\infty} \int_G \int_{\mathbb{R}} \chi_E(x,y) dydx > 0,$$

which again gives the result.

(b) Let f be a non-negative m'ble function on $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}$, $f(x,y)$ is finite for a.e. y. Show that for a.e. $y \in \mathbb{R}$, $f(x,y)$ is finite for a.e. x.

Let $E = \{(x,y) \in \mathbb{R}^2 : f(x,y) = \infty\}$. Note that E is m'able since $S = \{(x,y) : f(x,y) \neq \infty\} = f^{-1}((-1,\infty))$ is m'able since f is m'able. Then $E = \overline{S}$ is measurable as well. Then, the condition "for a.e. $x \in \mathbb{R}$, $f(x,y)$ is finite for a.e." implies that for a.e. $x \in \mathbb{R}$, $\{y : (x,y) \in E\}$ has $\mathbb{R}^1-$measure 0. Then, the results from part (a) give that for a.e. $y \in \mathbb{R}$, $\{x : (x,y) \in E\}$ has $\mathbb{R}^1-$measure 0 or that for a.e. y, $|\{x : f(x,y) = \infty\}| = 0$, or equivalently that for a.e. y, $f(x,y)$ is finite for a.e. x.

Question 3

Let $g : \mathbb{R}^m \to \mathbb{R}$ be m'ble and $h : \mathbb{R}^n \to \mathbb{R}$ be m'ble.

First, we define some notation.
Let $G_1 := \{(x, y) \in E : g(x) \geq 0, h(y) \geq 0\}$
and $G_2 := \{(x, y) \in E : g(x) < 0, h(y) < 0\}$.
Then, let $G = G_1 \sqcup G_2$.

Furthermore, let $L_1 := \{x \in E : g(x) \geq 0, h(y) < 0\}$
and let $L_2 := \{x \in E : g(x) < 0, h(y) \geq 0\}$.
Finally, let $L := L_1 \sqcup L_2$.

(a) Prove that $f = gh : \mathbb{R}^{m+n} \to \mathbb{R}$ is m'ble. You may assume that if E is a m'ble set in $\mathbb{R}^m$ and F is a m'ble set in $\mathbb{R}^n$, then $E \times F$ is m'ble in $\mathbb{R}^{m+n}$.

Break $\mathbb{R}^{n+m}$ into G_1, G_2, L_1, L_2 as defined in above the statement of Question 3, Part (a).

Now, for G_1, G_2 we use a sequence of functions approximating $g(x)$.

Namely, for $k \geq 1$ and $g(x) \leq 0$:
Let
$g_k(x) = \frac{-r+1}{2^k}$ if $\left(\frac{-r+1}{2^k} \geq g(x) > \frac{-r}{2^k} \text{ and } g(x) \geq -k\right)$,
and $g_k(x) = -k$ otherwise.

Likewise, for $g(x) \geq 0$:
Let
$g_k(x) = \frac{r-1}{2^k}$ if $\left(\frac{r-1}{2^k} \leq g(x) < \frac{r}{2^k} \text{ and } g(x) \leq k\right)$,
and $g_k(x) = k$ otherwise.

Note in passing that $|g_k(x)| \leq |g(x)|$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ and that for fixed $x \in \mathbb{R}^n$ we have that $|g_k(x)| \nearrow$ as $k \nearrow$.

We note that $g_k(x) \nearrow g(x)$ wherever $g(x) \geq 0$ (so on G_1, L_1).
So, note that on G_1 since $h(y) \geq 0$, we have that $g_k(x)h(y) \nearrow g(x)h(y) = f((x, y))$ on G_1.
Also, since $h(y) < 0$ on L_1, we have that $g_k(x)h(y) \searrow g(x)h(y) = f((x, y))$ on L_1.
Furthermore we have that $g_k(x) \searrow g(x)$ on G_2, L_2 which means that
$g_k(x)h(y) \nearrow g(x)h(y) = f((x, y))$ on G_2

and $g_k(x)h(y) \searrow g(x)h(y) = f((x,y))$ on L_2.

Define $F^k_{G_1}((x,y)) = \chi_{G_1}((x,y))g_k(x)h(y)$,
$F^k_{G_2}((x,y)) = \chi_{G_2}((x,y))g_k(x)h(y)$,
$F^k_{L_1}((x,y)) = \chi_{L_1}((x,y))g_k(x)h(y)$,
and $F^k_{L_2}((x,y)) = \chi_{L_2}((x,y))g_k(x)h(y)$.

Now, we expand $f = gh$ in terms of the sets G_1, G_2, L_1, L_2 enumerated above. We note

$$\begin{aligned}
f((x,y)) &= f((x,y))\Big(\chi_{G_1}((x,y)) + \chi_{G_2}((x,y)) + \chi_{L_1}((x,y)) + \chi_{L_2}((x,y))\Big) \\
&= \chi_{G_1}((x,y))f((x,y)) + \chi_{G_2}((x,y))f((x,y)) \\
&+ \chi_{L_1}((x,y))f((x,y)) + \chi_{L_2}((x,y))f((x,y)) \\
&= \chi_{G_1}((x,y))\big(g(x)h(y)\big) + \chi_{G_2}((x,y))\big(g(x)h(y)\big) \\
&+ \chi_{L_1}((x,y))\big(g(x)h(y)\big) + \chi_{L_2}((x,y))\big(g(x)h(y)\big).
\end{aligned}$$

Now, recall that on G_1, G_2 we have that $g_k(x)h(y) \nearrow g(x)h(y) = f((x,y))$. Also, we have that on L_1, L_2 we have that $g_k(x)h(y) \searrow g(x)h(y) = f((x,y))$.

Note that

$$\begin{aligned}
f((x,y)) &= \chi_{G_1}((x,y))\big(g(x)h(y)\big) + \chi_{G_2}((x,y))\big(g(x)h(y)\big) \\
&+ \chi_{L_1}((x,y))\big(g(x)h(y)\big) + \chi_{L_2}((x,y))\big(g(x)h(y)\big) \\
&= \chi_{G_1}((x,y))\big(\lim_{k\to\infty} g_k(x)h(y)\big) + \chi_{G_2}((x,y))\big(\lim_{k\to\infty} g_k(x)h(y)\big) \\
&+ \chi_{L_1}((x,y))\big(\lim_{k\to\infty} g_k(x)h(y)\big) + \chi_{L_2}((x,y))\big(\lim_{k\to\infty} g_k(x)h(y)\big) \\
&= \lim_{k\to\infty} \chi_{G_1}((x,y))\big(g_k(x)h(y)\big) + \lim_{k\to\infty} \chi_{G_2}((x,y))\big(g_k(x)h(y)\big) \\
&+ \lim_{k\to\infty} \chi_{L_1}((x,y))\big(g_k(x)h(y)\big) + \lim_{k\to\infty} \chi_{L_2}((x,y))\big(g_k(x)h(y)\big) \\
&= \lim_{k\to\infty} F^k_{G_1}((x,y)) + \lim_{k\to\infty} F^k_{G_2}((x,y)) \\
&+ \lim_{k\to\infty} F^k_{L_1}((x,y)) + \lim_{k\to\infty} F^k_{L_2}((x,y)) \qquad (28)
\end{aligned}$$

Provided we can show that $F^k_{G_1}((x,y)), F^k_{G_2}((x,y)), F^k_{L_1}((x,y))$, and $F^k_{L_2}((x,y))$ are all measurable functions, then we can use the fact that a monotone limit of measurable functions is measurable. Then, indeed since the sum of measurable functions is measurable, we will have that $f((x,y))$ is measurable.

So, how do we show that each of $F^k_{G_1}((x,y)), F^k_{G_2}((x,y)), F^k_{L_1}((x,y))$, and $F^k_{L_2}((x,y))$ is measurable?

Let us first show that for each of the sets G_1, G_2, L_1, L_2 in the domain we have that $g_k(x)h(y)$ is measurable for all $k \in \mathbb{N}$.

We want to show that for all $\alpha \in \mathbb{R}$ we have that $\{g_k(x)h(y) > \alpha\}$ is measurable. Note first that $g_k(x)$ is a simple function and in particular we can split up the domain E into $E = \left(\bigsqcup_{r\in I} \{ \frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\} \right) \bigsqcup \left(\{|g| > k\} \right)$

$$g_k(x) = \left(\sum_{r\in I} \chi_{\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\}} \frac{r-1}{2^k} \right) + k\chi_{\{|g|>k\}} \tag{29}$$

(where I just indicates some indexing set, and I am taking the union over $r \in I$ to be over the set where $\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}$ and $|g| \leq k$, which means that I will be a finite indexing set).

Then, note that $\{g_k(x)h(y) > \alpha\} = \{h(y) > \frac{\alpha}{g_k(x)}\}$. So, we aim to determine systematically what the set $\{h(y) > \frac{\alpha}{g_k(x)}\}$ is. By Equation 29, we note that for (x, y) with $g(x) \geq 0$ we have that $h(y) > \frac{\alpha}{g_k(x)} = \alpha(g_k(x))^{-1}$ if and only if

$$\begin{aligned} h(y) &> \alpha \left(\left(\sum_{r\in I} \chi_{\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\}} \frac{r-1}{2^k} \right) + k\chi_{\{|g|>k\}} \right)^{-1} \\ &= \alpha \left(\left(\sum_{r\in I} \chi_{\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\}} \frac{2^k}{r-1} \right) + \frac{1}{k}\chi_{\{|g|>k\}} \right). \end{aligned}$$

So,

$$\begin{aligned} \left\{ h(y) > \frac{\alpha}{g_k(x)} \right\} = &\left(\bigcup_{r\in I} \left(\left\{ \frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k \right\} \cap \left\{ h(y) > \frac{\alpha(2^k)}{r-1} \right\} \right) \right) \\ &\bigcup \left(\left\{ |g| > k \right\} \cap \left\{ h(y) > \frac{\alpha}{k} \right\} \right). \end{aligned}$$

Now, since for all $r \in I$ we have that each of

$$\left\{ \frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k \right\} = \left\{ \frac{r-1}{2^k} \leq |g| < \frac{r}{2^k} \right\} \cap \left\{ |g| \leq k \right\},$$

$$\left\{ h(y) > \frac{\alpha(2^k)}{r-1} \right\},$$

$$\left\{ |g| > k \right\},$$

and

$$\left\{ h(y) > \frac{\alpha}{k} \right\},$$

are measurable, we know that indeed $\{h(y) > \frac{\alpha}{g_k(x)}\} \cap \{g(x) \geq 0\}$ is measurable, which means that $F^k_{G_1}((x, y))$ and $F^k_{L_1}((x, y))$ are measurable functions.

Likewise, we note that for (x, y) with $g(x) < 0$ we have that $g_k(x)h(y) > \alpha$ if and only if $h(y) < \frac{\alpha}{g_k(x)} = \alpha(g_k(x))^{-1}$ if and only if

$$
\begin{aligned}
h(y) &< \alpha\left(\left(\sum_{r\in I}\chi_{\{\frac{r-1}{2^k}\leq|g|<\frac{r}{2^k},|g|\leq k\}}\frac{-(r-1)}{2^k}\right)+(-k)\chi_{\{|g|>k\}}\right)^{-1} \\
&= \alpha\left(\left(\sum_{r\in I}\chi_{\{\frac{r-1}{2^k}\leq|g|<\frac{r}{2^k},|g|\leq k\}}\frac{2^k}{-(r-1)}\right)+\frac{1}{-k}\chi_{\{|g|>k\}}\right).
\end{aligned}
$$

Then, we observe that

$$
\left\{h(y) < \frac{\alpha}{g_k(x)}\right\} = \left(\bigcup_{r\in I}\left(\left\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\right\} \cap \left\{h(y) < \frac{\alpha 2^k}{-(r-1)}\right\}\right)\right) \tag{30}
$$

$$
\bigcup\left(\left\{|g| > k\right\} \cap \left\{h(y) < \frac{\alpha}{-k}\right\}\right). \tag{31}
$$

Now, since for all $r \in I$ we have that each of

$$
\left\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}, |g| \leq k\right\} = \left\{\frac{r-1}{2^k} \leq |g| < \frac{r}{2^k}\right\} \cap \left\{|g| \leq k\right\},
$$

$$
\left\{h(y) < \frac{\alpha(2^k)}{-(r-1)}\right\},
$$

$$
\left\{|g| > k\right\},
$$

and

$$
\left\{h(y) < \frac{\alpha}{-k}\right\},
$$

are measurable, we know that indeed $\{h(y) > \frac{\alpha}{g_k(x)}\} \cap \{g(x) < 0\}$ is measurable, which means that $F^k_{G_2}((x, y))$ and $F^k_{L_2}((x, y))$ are measurable functions.

Recall that now, having shown that $F^k_{G_1}((x, y)), F^k_{G_2}((x, y)), F^k_{L_1}((x, y))$, and $F^k_{L_2}((x, y))$ are measurable function, now we can use the fact that these monotonically increase or decrease to $f(x)h(y)$ on their respective supports (I believe support means input for which the function is non-zero).

Precisely, $F^k_{G_i}((x, y)) \nearrow \chi_{G_i}((x, y))f((x, y))$ for $i \in [2]$ means that $\chi_{G_i}((x, y))f((x, y))$ is measurable for $i \in [2]$.
Also, $F^k_{L_i}((x, y)) \searrow \chi_{L_i}((x, y))f((x, y))$ for $i \in [2]$ means that $\chi_{L_i}((x, y))f((x, y))$ is measurable for $i \in [2]$.
Thus, by Equation 28 we have shown that $f((x, y))$ is measurable.

(**b**) Assume that both g, h are integrable. Show that $f = gh$ defined by $f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x})\, h(\mathbf{y})$is integrable over $\mathbb{R}^{m+n}$ and that

$$
\int_{\mathbb{R}^{m+n}} gh = \left(\int_{\mathbb{R}^m} g\right)\left(\int_{\mathbb{R}^n} h\right).
$$

Hint: first use Tonelli's Theorem, then Fubini's Theorem.

By part (a) we know that f is measurable which means that $\{f \geq 0\}$ and $\{f \leq 0\}$ are both measurable. Thus, the integrals $\int_{\{f \geq 0\}} f^+$ and $\int_{\{f \geq 0\}} f^-$ exist if and only if f^+ and f^- are measurable which indeed they are.

Then, we note that

$$\begin{aligned}
\int_{\{\mathbb{R}^{n+m}\}} f &= \int_{\{\mathbb{R}^{n+m}\}} (f^+ - f^-) \\
&= \int_{\{\mathbb{R}^{n+m}\}} (f^+) + \int_{\{\mathbb{R}^{n+m}\}} (-f^-) \\
&= \int_{\{f \geq 0\}} (f^+) + \int_{\{f \leq 0\}} (-f^-) \\
&= \int_{\{f \geq 0\}} (f^+) - \int_{\{f \leq 0\}} (f^-)
\end{aligned}$$

Then, since $f \geq 0$ on $\{f \geq 0\}$ and $f^- \geq 0$ on $\{f \leq 0\}$ we can apply Tonelli's theorem.

Now, we have that

$$
\begin{aligned}
\int_{\{f\geq 0\}}(f^{+})-\int_{\{f\leq 0\}}(f^{-}) &= \int_{\{f\geq 0\}}(f^{+})-\int_{\{f\leq 0\}}(f^{-})\\
&= \int_{\{f\geq 0\}}(f)-\int_{\{f\leq 0\}}(-f)\\
&= \int_{G}(f)-\int_{L}(-f)\\
&= \int_{G}(gh)-\int_{L}(-gh)\\
&= \int_{\{h\geq 0\}}\int_{\{g\geq 0\}} g(x)h(y)dxdy+\int_{\{h<0\}}\int_{\{g<0\}} g(x)h(y)dxdy\\
&- \int_{\{h\leq 0\}}\int_{\{g\geq 0\}} -g(x)h(y)dxdy-\int_{\{h\geq 0\}}\int_{\{g<0\}} -g(x)h(y)dxdy\\
&= \int_{\{h\geq 0\}}\int_{\{g\geq 0\}} g(x)h(y)dxdy-\int_{\{h\geq 0\}}\int_{\{g<0\}} -g(x)h(y)dxdy\\
&+ \int_{\{h<0\}}\int_{\{g<0\}} g(x)h(y)dxdy-\int_{\{h\leq 0\}}\int_{\{g\geq 0\}} -g(x)h(y)dxdy\\
&= \int_{\{h\geq 0\}}\int_{\{g\geq 0\}} g(x)h(y)dxdy-\int_{\{h\geq 0\}}\int_{\{g<0\}} -g(x)h(y)dxdy\\
&+ \int_{\{h<0\}}\int_{\{g<0\}} g(x)h(y)dxdy-\int_{\{h<0\}}\int_{\{g\geq 0\}} -g(x)h(y)dxdy\\
&= \int_{\{h\geq 0\}}\left(\int_{\{g\geq 0\}} g(x)h(y)dx-\int_{\{g<0\}} -g(x)h(y)dx\right)dy\\
&+ \int_{\{h<0\}}\left(\int_{\{g<0\}} g(x)h(y)dx-\int_{\{g\geq 0\}} -g(x)h(y)dx\right)dy\\
&= \int_{\{h\geq 0\}}\left(\left(h(y)\right)\left(\int_{\{g\geq 0\}} g(x)dx-\int_{\{g<0\}} -g(x)dx\right)\right)dy\\
&+ \int_{\{h<0\}}\left(\left(h(y)\right)\left(\int_{\{g<0\}} g(x)dx-\int_{\{g\geq 0\}} -g(x)dx\right)\right)dy\\
&= \int_{\{h\geq 0\}}\left(\left(h(y)\right)\left(\int_{\{g\geq 0\}} g(x)dx+\int_{\{g<0\}} g(x)dx\right)\right)dy\\
&+ \int_{\{h<0\}}\left(\left(h(y)\right)\left(\int_{\{g<0\}} g(x)dx+\int_{\{g\geq 0\}} g(x)dx\right)\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\{h\geq 0\}} \left(\left(h(y) \right) \left(\int_{\mathbb{R}^m} g(x)dx \right) \right) dy \\
&+ \int_{\{h<0\}} \left(\left(h(y) \right) \left(\int_{\mathbb{R}^m} g(x)dx \right) \right) dy \\
&= \left(\int_{\mathbb{R}^m} g(x)dx \right) \int_{\{h\geq 0\}} \left(\left(h(y) \right) \right) dy \\
&+ \left(\int_{\mathbb{R}^m} g(x)dx \right) \int_{\{h<0\}} \left(\left(h(y) \right) \right) dy \\
&= \left(\int_{\mathbb{R}^m} g(x)dx \right) \left(\int_{\{h\geq 0\}} \left(h(y) \right) dy \right. \\
&\left. + \int_{\{h<0\}} \left(h(y) \right) dy \right) \\
&= \left(\int_{\mathbb{R}^m} g(x)dx \right) \left(\int_{\mathbb{R}^n} \left(h(y) \right) dy \right).
\end{aligned}
$$

Question 4

Let ϕ be a bounded measurable function on $\mathbb{R}^n$ such that $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 1$ and

$$\int_{\mathbb{R}^n} \phi = 1.$$

For $\varepsilon > 0$, let

$$\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-n} \phi(\mathbf{x}/\varepsilon), x \in \mathbb{R}^n.$$

Let $f \in L(\mathbb{R}^n)$ and define

$$f * \phi_\varepsilon(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) \phi_\varepsilon(\mathbf{y}) \, d\mathbf{y}.$$

Show that for $\mathbf{x}$ in the Lebesgue set of f, we have

$$\lim_{\varepsilon \to 0} f * \phi_\varepsilon(\mathbf{x}) = f(\mathbf{x}).$$

(You may assume the substitution rule for integrals of functions of one variable applied repeatedly).

Note

$$\begin{aligned}
f * \phi_\epsilon(x) &= \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y)dy \\
&= \int_{\mathbb{R}^n} (f(x-y) - f(x) + f(x))\phi_\epsilon(y)dy \\
&= \int_{\mathbb{R}^n} \Big((f(x-y) - f(x))\phi_\epsilon(y) + (f(x)\phi_\epsilon(y)) \Big) dy \\
f * \phi_\epsilon(x) &= \int_{\mathbb{R}^n} \Big((f(x-y) - f(x))\phi_\epsilon(y) \Big) dy + \int_{\mathbb{R}^n} \Big((f(x)\phi_\epsilon(y)) \Big) dy. \qquad (32)
\end{aligned}$$

Now, we note that we can pull out the $f(x)$ in the right-most term to get

$$\int_{\mathbb{R}^n} (f(x)\phi_\epsilon(y)dy) = f(x) \int_{\mathbb{R}^n} \phi_\epsilon(y)dy.$$

Now, letting $y = \epsilon w$ we see that $\phi_\epsilon(y) = \epsilon^{-n}\phi(y/\epsilon) = \epsilon^{-n}\phi(w)$. Also, note that (I believe dy denotes the infinitesimal domain for volume) $dy = \epsilon^n dw$. So,

$$\begin{aligned}
\int_{\mathbb{R}^n} (f(x)\phi_\epsilon(y)dy) &= f(x) \int_{\mathbb{R}^n} \phi_\epsilon(y)dy \\
&= f(x) \int_{\mathbb{R}^n} \epsilon^{-n}\phi(y/\epsilon)dy \\
&= f(x) \int_{\mathbb{R}^n} \epsilon^{-n}\phi(w)\epsilon^n dw \\
&= f(x) \int_{\mathbb{R}^n} \phi(w)dw \\
&= f(x) \qquad (33)
\end{aligned}$$

Now, we deal with the first term on the RHS, namely $\int_{\mathbb{R}^n}\Bigg((f(x-y) - f(x))\phi_\epsilon(y)\Bigg)dy$. Note

$$\begin{aligned}\int_{\mathbb{R}^n}\Bigg((f(x-y)-f(x))\phi_\epsilon(y)\Bigg)dy &= \int_{\mathbb{R}^n}\Bigg((f(x-y)-f(x))\epsilon^{-n}\phi(y/\epsilon)\Bigg)dy \\ &= \int_{\mathbb{R}^n}\Bigg((f(x-y)-f(x))\frac{1}{\epsilon^n}\phi(y/\epsilon)\Bigg)dy \\ &= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\Bigg((f(x-y)-f(x))\phi(y/\epsilon)\Bigg)dy \quad (34)\end{aligned}$$

Let $C_x(\epsilon) = [x_1-\epsilon, x_1+\epsilon]\times\cdots\times[x_n-\epsilon, x_n+\epsilon]$ and then note that $B(x,\epsilon) = \{y \in \mathbb{R}^n : |y| < \epsilon\} \subseteq C_x(\epsilon)$ and note that $|C_x(\epsilon)| = 2^n\epsilon^n$.

So, since $\phi(x)$ is bounded, we know there exists $M \in \mathbb{R}^+$ such that $\phi(x) \leq M$ for all $x \in \mathbb{R}^n$. Now,

$$\begin{aligned}\frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\Bigg((f(x-y)-f(x))\phi(y/\epsilon)\Bigg)dy &\leq \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\Bigg|\Bigg((f(x-y)-f(x))\phi(y/\epsilon)\Bigg)\Bigg|dy \\ &= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n}\Bigg|(f(x-y)-f(x))\Bigg|\Bigg|\phi(y/\epsilon)\Bigg|dy \\ &= \frac{1}{\epsilon^n}\int_{\{|\phi(y/\epsilon)|\geq 0\}}\Bigg|(f(x-y)-f(x))\Bigg|\Bigg|\phi(y/\epsilon)\Bigg|dy \\ &\leq \frac{1}{\epsilon^n}\int_{B(0,\epsilon)}\Bigg|(f(x-y)-f(x))\Bigg|\Bigg|\phi(y/\epsilon)\Bigg|dy \\ &\leq \frac{1}{\epsilon^n}\int_{C_0(\epsilon)}\Bigg|(f(x-y)-f(x))\Bigg|\Bigg|\phi(y/\epsilon)\Bigg|dy \\ &\leq \frac{1}{\epsilon^n}\int_{C_0(\epsilon)}\Bigg|(f(x-y)-f(x))\Bigg|M\,dy \\ &= M\frac{1}{\epsilon^n}\int_{C_0(\epsilon)}\Bigg|(f(x-y)-f(x))\Bigg|dy \end{aligned} \quad (35)$$

Now, let $w = x-y$. So, then $y \in C_0(\epsilon) = [-\epsilon,\epsilon]^n$ if and only if $x-y \in C_x(\epsilon)$. So, finally, noting that if $w = x-y$, then $dy = -dw$ we get

$$
\begin{aligned}
M\frac{1}{\epsilon^n}\int_{C_0(\epsilon)}\bigg|(f(x-y)-f(x))\bigg|dy &= M\frac{1}{\epsilon^n}\int_{C_x(\epsilon)}-\bigg|(f(w)-f(x))\bigg|dw \\
&= -M\frac{1}{\epsilon^n}\int_{C_x(\epsilon)}\bigg|(f(w)-f(x))\bigg|dw \\
&= -M\frac{2^n}{2^n\epsilon^n}\int_{C_x(\epsilon)}\bigg|(f(w)-f(x))\bigg|dw \\
&= -M\frac{2^n}{|C_x(\epsilon)|}\int_{C_x(\epsilon)}\bigg|(f(w)-f(x))\bigg|dw
\end{aligned} \tag{36}
$$

Finally, we note that by the definition of a Lesbegue point we have that

$$
\begin{aligned}
\lim_{\epsilon\to 0}\left(-M\frac{2^n}{|C_x(\epsilon)|}\int_{C_x(\epsilon)}\bigg|(f(w)-f(x))\bigg|dw\right) &= -2^nM\lim_{\epsilon\to 0}\left(\frac{1}{|C_x(\epsilon)|}\int_{C_x(\epsilon)}\bigg|(f(w)-f(x))\bigg|dw\right) \\
&= -2^nM(0) \\
&= 0
\end{aligned} \tag{37}
$$

So, to summarize, if one wishes to calculate

$$\lim_{\epsilon\to 0} f*\phi_\epsilon(x).$$

One notes that by Equation 32 we have

$$\lim_{\epsilon\to 0} f*\phi_\epsilon(x) = \lim_{\epsilon\to 0}\left(\int_{\mathbb{R}^n}\bigg((f(x-y)-f(x))\phi_\epsilon(y)\bigg)dy + \int_{\mathbb{R}^n}\bigg((f(x)\phi_\epsilon(y))\bigg)dy\right),$$

and then, by Equations 33, 34, 35, 36, and 37 we indeed get that

$$
\begin{aligned}
\lim_{\epsilon\to 0} f*\phi_\epsilon(x) &= \lim_{\epsilon\to 0}\left(\int_{\mathbb{R}^n}\bigg((f(x-y)-f(x))\phi_\epsilon(y)\bigg)dy + \int_{\mathbb{R}^n}\bigg((f(x)\phi_\epsilon(y))\bigg)dy\right) \\
&= 0 + f(x) \\
&= f(x),
\end{aligned}
$$

which is what we set out to prove.

Question 6

Show that

$$\lim_{A\to\infty}\lim_{B\to\infty}\int_0^A\left[\int_0^B e^{-xy}\sin x\ dy\right]\ dx = \lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Note

$$\begin{aligned}
\lim_{A\to\infty}\lim_{B\to\infty}\int_0^A\left[\int_0^B e^{-xy}\sin x\ dy\right]\ dx &= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A\left[sin(x)\int_0^B e^{-xy}\ dy\right]\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A sin(x)\left(\frac{1}{-x}e^{-xy}\ \right)\Bigg|_{y=0}^{B}\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \frac{sin(x)}{-x}\left(e^{-xy}\ \right)\Bigg|_{y=0}^{B}\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \frac{sin(x)}{-x}\left(e^{-xB}-e^{0}\right)\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \frac{sin(x)}{-x}\left(e^{-xB}-1\right)\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \left(\frac{sin(x)}{-xe^{Bx}}+\frac{sin(x)}{x}\right)\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \left(\frac{sin(x)}{-xe^{Bx}}+\frac{sin(x)}{x}\right)\ dx \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A \frac{sin(x)}{-xe^{Bx}}dx+\int_0^A\frac{sin(x)}{x}\ \ dx\right) \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A \frac{sin(x)}{-xe^{Bx}}dx+\int_0^A\frac{sin(x)}{x}\ \ dx\right) \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A \frac{sin(x)}{-xe^{Bx}}dx\right)+\lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A\frac{sin(x)}{x}\ \ dx\right) \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A \frac{sin(x)}{-xe^{Bx}}dx\right)+\lim_{A\to\infty}\left(\int_0^A\frac{sin(x)}{x}\ \ dx\right) \\
&= \lim_{A\to\infty}\lim_{B\to\infty}\left(\int_0^A \frac{sin(x)}{-xe^{Bx}}dx\right)+\lim_{A\to\infty}\left(\int_0^A\frac{sin(x)}{x}\ \ dx\right)
\end{aligned}$$

Then, note that

$$\left|\frac{sin(x)}{-xe^{Bx}}\right|\leq\phi(x)=\left|\frac{sin(x)}{x}\right|.$$

Then, since $\phi(x)$ is integrable, we can apply Lesbegue's Dominated Convergence Theorem to get

$$\begin{aligned}
\lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \frac{sin(x)}{-xe^{Bx}}dx &= \lim_{A\to\infty}\int_0^A \lim_{B\to\infty}\frac{sin(x)}{-xe^{Bx}}dx \\
&= \lim_{A\to\infty}\int_0^A 0dx \\
&= \lim_{A\to\infty}(0) \\
&= 0.
\end{aligned}$$

Hence,

$$\lim_{A\to\infty}\lim_{B\to\infty}\int_0^A \left[\int_0^B e^{-xy}\sin x\ dy\right]\ dx = \lim_{A\to\infty}\left(\int_0^A \frac{sin(x)}{x}\ \ dx\right).$$

Also, we have that

$$\lim_{A\to\infty}\frac{sin(x)}{x}dx = \frac{\pi}{2}.$$

Does this mean that $\int_0^\infty \frac{\sin x}{x}dx$ exists as a Lebesgue integral? Hint: use Fubini's theorem.

No. We might like to apply Fubini's theorem to the function

$$f(x,y) = e^{-xy}sin(x)$$

on the measurable set $E = [0,A]\times[0,B]$.
Successful application of the theorem would ensure that

$$\int_0^B e^{-xy}\sin x\ dy = \frac{sin(x)}{-e^{-Bx}} + \frac{sin(x)}{x}$$

exists and is integrable, meaning

$$\int_0^A \left|\frac{sin(x)}{-e^{-Bx}} + \frac{sin(x)}{x}\right|dx < \infty.$$

We expand

$$\begin{aligned}
\int_0^A \left|\frac{sin(x)}{-e^{-Bx}} + \frac{sin(x)}{x}\right| &= \int_0^A \left|sin(x)\right|\left|\frac{1}{-e^{-Bx}} + \frac{1}{x}\right| dx \\
&= \int_0^A \left|sin(x)\right|\left|\frac{x - e^{-Bx}}{-xe^{-Bx}}\right| dx \\
&= \int_0^A \left|sin(x)\right|\frac{|x - e^{-Bx}|}{|-xe^{-Bx}|} dx \\
&= \int_0^A \left|sin(x)\right|\frac{|x - e^{-Bx}|}{|x||e^{-Bx}|} dx \\
&= \int_0^A \frac{|sin(x)|}{|x|}\frac{|x - e^{-Bx}|}{|e^{-Bx}|} dx \\
&= \int_0^A \left|\frac{sin(x)}{x}\right|\frac{|x - e^{-Bx}|}{|e^{-Bx}|} dx \\
&= \int_0^A \left|\frac{sin(x)}{x}\right|\frac{|e^{-Bx} - x|}{|e^{-Bx}|} dx \\
&\leq \int_0^A \left|\frac{sin(x)}{x}\right| 1 dx \\
&\leq \int_0^A \left|\frac{sin(x)}{x}\right| dx
\end{aligned}$$

In order to apply Fubini we would need that $f(x, y)$ is integrable, which I do not think it is. However, the string of equations above shows that we have an even bigger issue which is that even if Fubini applies still the nice consequence that $\int_0^A \left|\frac{sin(x)}{-e^{-Bx}} + \frac{sin(x)}{x}\right| < \infty$ does NOT imply that $\int_0^A \left|\frac{sin(x)}{x}\right| dx < \infty$ (which is equivalent to the statement that $\frac{sin(x)}{x}$ is integrable. Still it could happen theoretically that $\frac{sin(x)}{x}$ is not integrable but the integral indeed exists.
However, that cannot happen. Recall that we showed last homework that $\int_0^A \frac{sin(x)}{x} dx = \int_0^A ((\frac{sin(x)}{x})^+ - (\frac{sin(x)}{x})^-) dx$ is finite as an improper riemann integral. However, that and the fact that as shown last homework $\frac{sin(x)}{x}$ is not integrable or that $\int_0^\infty |\frac{sin(x)}{x}| dx = \infty$. Note that $\infty = \int_0^\infty |\frac{sin(x)}{x}| dx = \int_0^\infty (\frac{sin(x)}{x})^+ + (\frac{sin(x)}{x})^- dx$ which means that at least one of $\int_0^\infty (\frac{sin(x)}{x})^+ dx$ or $\int_0^\infty (\frac{sin(x)}{x})^- dx$ is infinite. But then the Improper Riemann integral being finite means that both $\int_0^\infty (\frac{sin(x)}{x})^+ dx$ and $\int_0^\infty (\frac{sin(x)}{x})^- dx$ are infinite since the improper riemann integral is $\int_0^\infty (\frac{sin(x)}{x})^+ dx - \int_0^\infty (\frac{sin(x)}{x})^- dx$ is finite (where indeed we are allowing the two terms there to be infinity, which may be an abuse of notation).