Math 6441 - Homework 2

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4. Show a path connected space X is simply connected if and only if every map $\phi : S^1 \to X$ extends to a map $\phi' : D^2 \to X$.
First if X is simple connected, that means that for any fixed $x_0 \in X$ (we recall from class that for $x_0, x_0' \in X$ such that x_0, x_0' are connected by a path, one has that $\pi_1(X, x_0) \simeq \pi_1(X, x_0')$; since X is path connected, we indeed have that the fundamental group is independent of the base point since any two points are connected by a path) we have that $\pi_1(X, x_0) = \{e\}$ is the trivial group. In particular, this means that any loop is null-homotopic.
Precisely, if we take $x_0 := \phi((1, 0))$ then X simply connected implies that there exists a homotopy
$$h : S_1 \times [0, 1] \to X$$
such that $h(x, 0) = \phi(x)$ and $h(x, 1) = x_0$ for all $x \in S^1$.
Now, we extend to a map $\phi' : D^2 \to X$ by defining using polar coordinates $\phi'((r, \theta)) = h((1, \theta), 1 - r)$.
Now, I convert to rectangular coordinates and then prove continuity.
we have that $\phi'((x, y)) = h((\frac{1}{\sqrt{x^2+y^2}}x, \frac{1}{\sqrt{x^2+y^2}}y), 1 - (\sqrt{x^2 + y^2}))$.
Now, I prove continuity. In particular, it holds because we have that ϕ' is a composition of continuous functions.
Namely,
$$\phi' = h \circ \Big((x, y) \mapsto^f ((\frac{1}{\sqrt{x^2 + y^2}}x, \frac{1}{\sqrt{x^2 + y^2}}y), 1 - (\sqrt{x^2 + y^2})\Big).$$
Why is $f : D^2 \to S^1 \times [0, 1]$ continuous? We must show pre-images of open sets are open.
To make this easier we recall that there exists a continuous bijection (with a continuous inverse) (so homeomorphism) $g : S^1 \times [0, 1] \to [0, 1) \times [0, 1]$ defined by $g((x, y), s) = ((\frac{1}{2\pi}arctan(y/x), s)$ if $x \neq 0$ and $g((x, y), s) = ((\frac{1}{2\pi}arcsin(y), s)$ if $x = 0$. (Note that I am defining $arctan(s)$ to have principal branch (i.e. range $[0, 2\pi)$)). Thus, one has that $f : D^2 \to S^1 \times [0, 1]$ is continuous if and only if
$$g \circ f : D^2 \to S^1 \times [0, 1] \to [0, 1) \times [0, 1]$$
is continuous.
We note that
$$g(f(x, y)) = g((\frac{1}{\sqrt{x^2 + y^2}}x, \frac{1}{\sqrt{x^2 + y^2}}y), 1 - (\sqrt{x^2 + y^2})) = (arctan(\frac{y}{x}), 1 - (\sqrt{x^2 + y^2}))$$
if $x \neq 0$ and
$$g(f(x, y)) = g((\frac{1}{\sqrt{x^2 + y^2}}x, \frac{1}{\sqrt{x^2 + y^2}}y), 1 - (\sqrt{x^2 + y^2})) = (arcsin(y), 1 - (\sqrt{x^2 + y^2}))$$
if $x = 0$.
To simplify later notation, we define functions g_1, g_2 by
$$g_1((x, y)) = arctan(y/x)$$
if $x \neq 0$ and
$$g_1((x, y)) = arcsin(y)$$
if $x = 0$, then also
$$g_2((x, y)) = 1 - \sqrt{x^2 + y^2}.$$
That means that
$$g \circ f((x, y)) = (g_1(x, y), g_2(x, y)).$$
Now, to check whether $g \circ f$ is continuous we must check that $(g \circ f)^{-1}(U)$ is open in D^2 for all $U \subseteq [0, 1) \times [0, 1]$ open.
By definition of the product topology U open means that U has the form $U = \cup_{i \in I}(U_1^i \times U_2^i)$ where $U_1^i \in \mathcal{T}([0, 1))$ is

open in S^1 and $U_2^i \in \mathcal{T}([0,1])$ is open in $[0,1]$ for each $i \in I$ (where I is just some indexing set).
Furthermore, we have bases for $[0,1)$ and $[0,1]$ meaning we can write open sets U_1^i, U_2^i as unions of basis elements in the respective spaces. (I am assuming we are using the subspace topology on $\mathbb{R}$ for both $[0,1)$ and $[0,1]$.
So, we have that

$$U_1^i = \cup_{j \in J}\Big([0,1) \cap (a_j^i, b_j^i)\Big)$$

where $a_j^i, b_j^i \in [-0.001, 1]$ and $a_j^i < b_j^i$. Also, we have that

$$U_2^i = \cup_{k \in K}\Big([0,1] \cap (c_k^i, d_k^i)\Big)$$

where $c_k^i, d_k^i \in [-0.001, 1.001]$ and $c_k^i < d_k^i$.
We denote $(a_j^i, b_j^i)' := [0,1) \cap (a_j^i, b_j^i)$.
Likewise, we denote $(c_k^i, d_k^i)'' := [0,1] \cap (c_k^i, d_k^i)$ so that we may simply write

$$U_1^i = \cup_{j \in J}(a_j^i, b_j^i)'$$

and

$$U_2^i = \cup_{k \in K}(c_k^i, d_k^i)''.$$

Now, we may rewrite

$$U_1^i \times U_2^i = \cup_{(j,k) \in J \times K}\big((a_j^i, b_j^i)' \times (c_k^i, d_k^i)''\big)$$

Thus, we have

$$\begin{aligned}
(g \circ f)^{-1}(U) = (g \circ f)^{-1}(\cup_{i \in I}(U_1^i \times U_2^i)) &= \cup_{i \in I}(g \circ f)^{-1}(U_1^i \times U_2^i)) \\
&= \cup_{i \in I}(g \circ f)^{-1}\Big(\cup_{(j,k) \in J \times K} \big((a_j^i, b_j^i)' \times (c_k^i, d_k^i)''\big)\Big) \\
&= \cup_{i \in I} \cup_{j \in J} (g \circ f)^{-1}\Big(\cup_{k \in K} \big((a_j^i, b_j^i)' \times (c_k^i, d_k^i)''\big)\Big) \\
&= \cup_{i \in I} \cup_{j \in J} (g \circ f)^{-1}\Big(\cup_{k \in K} \big((a_j^i, b_j^i)' \times (c_k^i, d_k^i)''\big)\Big) \\
&= \cup_{i \in I} \cup_{j \in J} \cup_{k \in K}(g \circ f)^{-1}\Big(\big((a_j^i, b_j^i)' \times (c_k^i, d_k^i)''\big)\Big) \\
&= \cup_{i \in I} \cup_{j \in J} \cup_{k \in K}(g \circ f)^{-1}\Big(\big([0,1) \cap (a_j^i, b_j^i)\big) \times \big([0,1] \cap (c_k^i, d_k^i)\big)\Big) \\
&= \cup_{i \in I} \cup_{j \in J} \cup_{k \in K}\left(g_1^{-1}\Big([0,1) \cap (a_j^i, b_j^i)\Big) \cap g_2^{-1}\Big([0,1] \cap (c_k^i, d_k^i)\Big)\right)
\end{aligned}$$

Now, since g_1, g_2 are each continuous we know that $V_j^i := g_1^{-1}\Big([0,1) \cap (a_j^i, b_j^i)\Big) \subseteq D^2$ is open and that $W_k^i := g_2^{-1}\Big([0,1] \cap (c_k^i, d_k^i)\Big) \subseteq D^2$ is open. Thus, we can continue on to say

$$= \cup_{i \in I} \cup_{j \in J} \cup_{k \in K}\left(V_j^i \cap W_k^i\right)$$

Finally, since the intersection of two open sets is open we have that

$$G_{(j,k)}^i := V_j^i \cap W_k^i$$

is open, thus to summarize we have

$$(g \circ f)^{-1}(U) = \cup_{i \in I} \cup_{j \in J} \cup_{k \in K}\left(G^{i_{(j,k)}}\right)$$

which is a union of open sets and is thus open. So, $g \circ f$ is continuous which as previously argued means that f is continuous.
Thus, finally as mentioned before

$$\phi' = h \circ f$$

is continuous, meaning we have successfully extended ϕ to ϕ' using the homotopy h.

5. Recall from class if $p \in S^1$ is a chosen point, then $\pi_1(X, x_0) = [S^1, X]_0$ (that is homotopy classes of base point preserving maps $(S^1, p) \to (X, x_0)$). There is a natural map

$$\psi : \pi_1(X, x_0) \to [S^1, X],$$

which recall $[S^1, X]$ is the set of homotopy classes of maps (with no condition on the base point). Show that if X is path connected then ψ is onto.
Well, if X is path connected then one has that for every class $[f] \in [S^1, X]$ that there exists a representative $f_0 \in [f]$ (recall that $f_0 : S^1 \to X$) such that $f_0(p) = x_0$.
Why? Because X is path connected that means that there exists a path h from x_0 to $f(p)$. Also, note that clearly f defines a closed path (using the obvious parameterization) $q_f : [0,1] \to X$ with $q_f(0) = q_f(1) = f(p)$.
The punchline is that we can then use h to define a closed path $q_{f_0} : [0,1] \to X$ such that $q_{f_0}(0) = x_0$ and $q_{f_0}(1) = x_0$.
We simply let

$$q_{f_0} := h * q_f * \overline{h}.$$

Finally, we define f_0 by

$$f_0((x,y)) = q_{f_0}(\tau^{-1}((x,y)))$$

where τ is the cannonical choice of map $\tau : [0,1) \to S^1$ given by $\tau(\theta) = (cos(2\pi\theta), sin(2\pi\theta))$.
Then, indeed since $[f_0] \in \pi_1(X, x_0)$ and $\psi([f_0]) = [f]$, and since $[f] \in [S^1, X]$ was chosen arbitrarily, we have indeed shown that ψ is onto since we have found $[f_0] \in \pi_1(X, x_0)$ that maps to it.
Also, show that $\psi([\gamma]) = \psi([\eta])$ if and only if $[\gamma]$ and $[\eta]$ are conjugate in $\pi_1(X, x_0)$.
I show first the direction that $[\gamma], [\eta]$ conjugate in $\pi_1(X, x_0)$ implies that $\psi([\gamma]) = \psi([\eta])$. Note that $\psi([\gamma]) = \psi([\eta])$ amounts to the statement that $\gamma \sim \eta$, which is what we aim to show.
By definition if $[\gamma]$ and $[\eta]$ are conjugate in $\pi_1(X, x_0)$, that means that there exists $[\alpha] \in \pi_1(X, x_0)$ such that

$$[\gamma] = [\alpha]^{-1}[\eta][\alpha].$$

So, what we want to show is that $\eta \sim [\alpha]^{-1}[\eta][\alpha]$ where α is some fixed loop based at x_0.
Well, certainly in $\pi_1(X, x_0)$ we have that

$$[\eta] = [\eta]e_{\pi_1(X,x_0)} = [\eta][\alpha][\alpha]^{-1} = [\eta][\alpha][\overline{\alpha}] = [\eta * \alpha * \overline{\alpha}].$$

Now, we demonstrate a homotopy between $\eta * \alpha * \overline{\alpha} =: q : [0,1] \to X$ and $\overline{\alpha} * \eta * \alpha = q' : [0,1] \to X$.
In particular, we define

$$h : [0,1] \times [0,1] \to X$$

by

$$h(s,t) = q((s + \frac{2}{3}t) \mod 1)$$

so that $h(s,0) = q(s)$ and $h(s,1) = q'(s) = q((s + \frac{2}{3}) \mod 1)$. So, putting that all together we have that

$$\eta \sim \eta * \alpha * \overline{\alpha} \sim \overline{\alpha} * \eta * \alpha$$

which implies that

$$\psi([\eta]) = \psi([\overline{\alpha} * \eta * \alpha]) = \psi([\overline{\alpha}][\eta][\alpha]) = \psi([\alpha]^{-1}[\eta][\alpha]) = \psi(\gamma),$$

which proves one direction of the claim.

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