Midterm II

April 22-25, 2014

Math 465/501 - Introduction to Differential Geometry   Name

You may use any references or sources you wish, but you must work independently.

Problem 1. (25 points) Let V be a 3-dimensional vector space, and let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis, with dual basis $\mathcal{B}^* = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$.

Let $\eta = 2\mathbf{e}^1 - \mathbf{e}^3$ and let $\gamma = \frac{1}{2}\mathbf{e}^2 + \mathbf{e}^3$. Set $T = \eta \wedge \gamma$ and $S = \eta \odot \gamma$.

a) (5 points) Express S and T as tensors (that is, without the symbols "$\wedge$" or "$\odot$").

b) (5 points) Determine the values of the following symbols: T_{33}, T_{12}, S_{33}, and S_{12}.

c) (5 points) Suppose V has a metric $g = g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$ where

$$g_{ij} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find $Tr(S)$. (*Hint: The matrix g is easy to invert by hand.*)

d) (10 points) Compute the norm-square of T; that is, compute $|T|^2$.

Problem 2. (15 points)

Recall the following operations in classical 3-dimensional vector field theory:
Given vector fields $X = (a, b, c)^T$ and $Y = (\alpha, \beta, \gamma)^T$ on $\mathbb{R}^3$, we have the operations

- The inner product: $\langle X, Y \rangle \; = \; a\alpha + b\beta + c\gamma$.
- The cross product: $X \times Y = (b\gamma - c\beta,\, c\alpha - a\gamma,\, a\beta - b\alpha)^T$
- The curl: $\nabla \times X = \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z},\, \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x},\, \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right)^T$
- The divergence: $\nabla \cdot X \; = \; \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$

In the formalism of exterior forms, these operations can be expressed with combinations of the Hodge star $*$ and the exterior derivative d as follows: If $\eta = X_\flat$ and $\gamma = Y_\flat$ are 1-forms on $\mathbb{R}^3$, then

i) The inner product of η with γ: $\langle \eta, \gamma \rangle \; = \; *(\eta \wedge *\gamma)$.

ii) The cross product of η with γ: $*(\eta \wedge \gamma)$

iii) The curl of η: $curl(\eta) \; = \; *d\eta$

iv) The divergence of η: $div(\eta) \; = \; *d(*\eta)$

(You are not asked to prove these; you may accept them as facts.) One advantage of our new formalism is that it becomes easy to determine vector calculus identities: for instance the classical identity $div(curl(X)) = 0$ is simply

$$div(curl(\eta)) \; = \; *d**d\eta \; = \; *dd\eta = 0$$

where we used that $** : \bigwedge^k \to \bigwedge^k$ equals $(-1)^{k(n-k)}$ and $dd : \bigwedge^k \to \bigwedge^{k+2}$ is always zero.

Use the formalism of exterior forms to prove the classic vector calculus identity:

$$\nabla \cdot (X \times Y) = \langle Y, \nabla \times X \rangle - \langle X, \nabla \times Y \rangle \, .$$

(*Hint: You will have to use the identity* $** = (-1)^{k(n-k)}$ <u>*twice*</u>*: once at the beginning of the problem, and then later in order to use the identity (i) in conjunction with (iii).*)

Problem 3. (35 points) We have often used stereographic projection to create coordinate charts on the sphere, but there are other ways to construct charts. Let

$$\mathbb{S}^2 \;=\; \left\{\, (x^1, x^2, x^2)^2 \in \mathbb{R}^2 \;\middle|\; (x^1)^2 + (x^2)^2 + (x^3)^2 \;=\; 1 \,\right\}$$

be the unit sphere. Let U and V be the following hemispheres:

$$\begin{aligned} U \;&=\; \{(x^1, x^2, x^3)^T \in \mathbb{S}^2 \mid x^3 > 0\}, \quad \text{(this is the upper hemisphere)} \\ V \;&=\; \{(x^1, x^2, x^3)^T \in \mathbb{S}^2 \mid x^2 > 0\} \end{aligned}$$

We will use the *projection charts*

$$\begin{aligned} \varphi_U : U \to \mathbb{R}^2, \qquad \varphi_U(x^1, x^2, x^3) \;&=\; \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\ \varphi_V : V \to \mathbb{R}^2, \qquad \varphi_V(x^1, x^2, x^3) \;&=\; \begin{pmatrix} x^1 \\ x^3 \end{pmatrix} \end{aligned}$$

Give the U-chart coordinates u^1, u^2 and give the V-chart coordinates v^1, v^2.

a) (5 points) As best you can, sketch the two domains U and V on the sphere $\mathbb{S}^2 \in \mathbb{R}^3$. Indicate the location (or locations) where they overlap. Do the sets U, V with the maps φ_U, φ_V constitute an atlas for $\mathbb{S}^2$?

b) (5 points) Show that

$$\begin{aligned} \varphi_U^{-1}(u^1,\, u^2) \;&=\; \begin{pmatrix} u^1 \\ u^2 \\ (1 - (u^1)^2 - (u^2)^2)^{\frac{1}{2}} \end{pmatrix} \\ \varphi_V^{-1}(v^1,\, v^2) \;&=\; \begin{pmatrix} v^1 \\ (1 - (v^1)^2 - (v^2)^2)^{\frac{1}{2}} \\ v^2 \end{pmatrix} \end{aligned}$$

c) (5 points) Compute the U-to-V transition φ_{VU}. What is v^1 as a function of u^1 and u^2? What is v^2 as a function of u^1 and u^2?

d) (5 points) Consider the function $h : \mathbb{S}^2 \to \mathbb{R}$ given by $h(x^1, x^2, x^3) = x^3$ (this sometimes called the height function). For points in the U-chart, express h as a function of u^1, u^2. For points in the V-chart, express h as a function of v^1, v^2.

e) (5 points) Compute dh in the U-chart and in the V-chart.

f) (10 points) Show by computation that your two expressions for dh, though they look very different, are identical up to change of coordinates.

Problem 4. (25 points) Consider the two paths

$$a : [-0.5,\, 0.5] \longrightarrow \mathbb{R}^2, \qquad a(t) \;=\; \begin{pmatrix} t \\ \frac{1}{2}t \end{pmatrix}$$

$$b : [-0.5,\, 0.5] \longrightarrow \mathbb{R}^2, \qquad b(t) \;=\; \begin{pmatrix} t \\ \frac{1}{2}\sin(t) \end{pmatrix}$$

These paths obviously coincide at $t = 0$. Assume $\mathbb{R}^2$ has coordinates u^1 and u^2.

a) (10 points) Do the paths a and b determine the same vector at the point $(0, 0) \in \mathbb{R}^2$? To check this, you must use the definitions—in other words if f is an arbitrary differentiable function, you must show that the differential operators associated to $a(t)$ and $b(t)$ at $t = 0$ have the same action on f.

b) (5 points) Express the vector $\mathbf{v} = \frac{da}{dt}\big|_{t=0} \in T_{(0,0)^T}\mathbb{R}^2$ in terms of the coordinate fields $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$.

c) (10 points) Consider again the coordinate chart $\varphi_U : U \to \mathbb{R}^2$ from Problem (3). Let $\psi_U = \varphi_U^{-1}$ be the associated parametrization. The vector $\mathbf{v}$ pushes forward along ψ_U to a vector on the sphere. Answer the following two questions: At what point on $\mathbb{S}^2$ is the vector $\psi_{U*}(\mathbf{v})$ based? What is the expression of $\psi_{U*}(\mathbf{v})$ in terms of the standard coordinate fields $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ of $\mathbb{R}^3$?