MATH 6338 (Prof. Heil) HOMEWORK #1 Due date: January 27, 2020

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together. Hand in your solutions in class on the due date.

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1. (Part of Problem 7.1.24 in the text). Assume that $1 \le p,\, q,\, r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Given $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$ and $y = (y_k)_{k\in\mathbb{N}} \in \ell^1$, prove that $xy = (x_k y_k)_{k\in\mathbb{N}}$ belongs to ℓ^r, and

$$||xy||_r \le ||x||_p\, ||y||_q.$$

Hint: Consider $|x|^r$ and $|y|^r$.

Let

$$u = \frac{x}{||x||_p}$$
$$w = \frac{y}{||y||_q}.$$

so that

$$||u||_p = 1$$
$$||w||_q = 1.$$

Note

$$||u^r||_1 = ||u||_r^r$$

and

$$||w^r||_1 = ||w||_r^r.$$

We want to show that

$$\left(\frac{||xy||_r}{||x||_p||y||_q}\right)^r = (||uw||_r)^r = ||u^r w^r||_1 \le 1.$$

Now, note that

$$\frac{1}{(p/r)} + \frac{1}{(q/r)} = \frac{r}{p} + \frac{r}{q}$$
$$= 1.$$

Thus, (p/r) and (q/r) are dual which means we can use Equation 7.3.

Well, note that

$$\begin{aligned}
||u^r w^r||_1 &= \sum_{k\in\mathbb{N}} |uw|^r \\
&= \sum_{k\in\mathbb{N}} |u^r||w^r| \\
&\leq \sum_{k\in\mathbb{N}} (\frac{|u^r|^{p/r}}{(p/r)} + \frac{|w^r|^{q/r}}{(q/r)}) \text{ (Equation 7.3)} \\
&= \sum_{k\in\mathbb{N}} (\frac{|u|^p}{(p/r)} + \frac{|w|^q}{(q/r)}) \\
&= \frac{1}{(p/r)} \sum_{k\in\mathbb{N}} (|u|^p) + \frac{1}{(q/r)} \sum_{k\in\mathbb{N}} (|w|^q) \\
&= \frac{||u||_p^p}{(p/r)} + \frac{||w||_q^q}{(q/r)} \\
&= \frac{1}{(p/r)} + \frac{1}{(q/r)} \\
&= 1.
\end{aligned}$$

So, indeed we have shown

$$(\frac{||xy||_r}{||x||_p||y||_q})^r = (||uw||_r)^r = ||u^r w^r||_1 \leq 1.$$

which means that

$$\frac{||xy||_r}{||x||_p||y||_q} \leq 1$$

and in turn

$$||xy||_r \leq ||x||_p||y||_q,$$

which concludes the proof.

2. Problem 7.1.25. Choose $1 \leq p \leq \infty$, and let $D = \{x \in \ell^p : ||x||_p \leq 1\}$ be the "closed unit disk" in ℓ^p. Observe that D is a bounded subset of ℓ^p, since it is contained in an open ball of finite radius. Prove the following statements.

(a) D is a closed subset of ℓ^p, i.e., if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in D such that $x_n \to x$ in ℓ^p-norm, then $x \in D$. Assume not. Say there exists a convergent sequence $x_n \to x$ with $\{x_n\}_{n\in\mathbb{N}} \subseteq D$ but $x \notin D$. Then, $||x||_p > 1$. By the triangle inequality one has that

$$||x||_p \leq ||x_n||_p + ||x - x_n||_p$$

for all $n \in \mathbb{N}$. However, $||x - x_n||_p \to 0$ as $n \to \infty$. Precisely, for all $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that for all $m \geq N$ one has that $||x - x_m||_p < \epsilon$.
Now, $1 < ||x||_p$ implies that $||x||_p = 1 + \epsilon$ for some $\epsilon > 0$. Then, as above we have that one has that $||x_{N(\frac{\epsilon}{2})} - x||_p < \frac{\epsilon}{2}$.
So, all together that says

$$1 + \epsilon = ||x||_p \leq ||x_n||_p + ||x - x_n||_p \leq 1 + \frac{\epsilon}{2}$$

or that

$$1 + \epsilon \leq 1 + \frac{\epsilon}{2}$$

meaning

$$\epsilon \leq \frac{\epsilon}{2}$$

or

$$1 \leq \frac{1}{2},$$

a contradiction.

(b) The sequence of standard basis vectors $\{\delta_n\}_{n\in\mathbb{N}}$ contains no convergent subsequences.
In a metric space, every convergent sequence is Cauchy (Lemma 1.1.3). So, it suffices to prove that the sequence of standard basis vectors has no Cauchy subsequence.
In particular, we prove that the difference $||\delta_n - \delta_{n'}||_p = 1$ for all $n \neq n' \in \mathbb{N}$. We compute

$$||\delta_n - \delta_{n'}||_p^p = \sum_{k\in\mathbb{N}} |\delta_n(k) - \delta_{n'}(k)|^p = |\delta_n(n) - \delta_{n'}(n)|^p + |\delta_n(n') - \delta_{n'}(n')|^p = 1^p + 1^p = 2.$$

So, $||\delta_n - \delta_{n'}||_p = 2^{1/p}$ for all $n \neq n' \in \mathbb{N}$. Thus, the sequence of standard basis vectors has no Cauchy subsequence.

(c) D is not a compact subset of ℓ^p.
By Theorem 1.1.10, since $D \subseteq \ell^p$ and ℓ^p is a metric space, one has that D is compact if and only if D is sequentially compact. However, part (b) demonstrates exactly that D is not sequentially compact.

3. Problem 7.2.16, part (a) only.

(a) Let E be a measurable subset of $\mathbb{R}^d$, and fix $0 < p < q \leq \infty$. If $0 < |E| < \infty$, then $L^q(E) \subsetneq L^p(E)$ and

$$||f||_p \leq |E|^{\frac{1}{p}-\frac{1}{q}} \, ||f||_q, \quad f \in L^p(E).$$

Hint: To show that the inclusion is proper, let A_k be disjoint subsets of E such that $|A_k| = 2^{-k} |E|$ (these exist by Problem 2.3.20) and consider $f = \sum c_k \chi_{A_k}$.

We first show inclusion.
Take $f \in L^q(E)$. This means that

$$\left(\int_E |f|^q\right)^{1/q} < \infty.$$

and in turn

$$\left(\int_E |f|^q\right) < \infty.$$

Now, we wish to show that

$$\left(\int_E |f|^p\right)^{1/p} < \infty.$$

or better yet show instead

$$\left(\int_E |f|^p\right) < \infty.$$

We decompose our integral into two parts.
In particular, we note

$$\begin{aligned}
\int_E |f|^p &= \int_{|f|<1} |f|^p + \int_{|f|\geq 1} |f|^p \text{ (since one can split up integrals of non-negative functions} \\
&\qquad \text{over their domains, if domain is finite)} \\
&\leq \int_{\{|f|<1\}} |f|^p + \int_{\{|f|\geq 1\}} |f|^p \\
&\leq \int_{\{|f|<1\}} 1 + \int_{\{|f|\geq 1\}} |f|^p \\
&\leq \int_{\{|f|<1\}} 1 + \int_{\{|f|\geq 1\}} |f|^q \text{ (since } a^p \leq a^q \text{ for } p < q, a \geq 1) \\
&\leq \int_{\{|f|<1\}} 1 + \int_E |f|^q \\
&= |\{x \in E : |f(x)| < 1\}| + ||f||_q^q \\
&\leq |E| + ||f||_q^q \\
&< \infty.
\end{aligned}$$

So, indeed $f \in L^p(E)$.

Now, to show

$$||f||_p \leq |E|^{\frac{1}{p}-\frac{1}{q}}\,||f||_q,$$

We first see

$$||f^p||_1 = ||f||_p^p < \infty. \tag{1}$$

We then note that

$$\begin{gathered}(1/q) + ((1/p) - (1/q)) = (1/p)\\ (1/q) + ((q-p)/(pq)) = (1/p)\\ (p/q) + ((q-p)/q) = 1\\ \frac{1}{(q/p)} + \frac{1}{q/(q-p)} = 1,\end{gathered}$$

which means by Equation 7.18 that

$$\begin{aligned}||1 * f^p||_1 &\leq ||1||_{q/(q-p)}||f^p||_{q/p}\\ &= (\int_E 1)^{(q-p)/q}(\int_E |f^p|^{q/p})^{p/q}\\ &= |E|^{(q-p)/q}(\int_E |f^p|^{q/p})^{p/q}.\end{aligned}$$

(Quick note that we may apply Equation 7.18 since both

$$||1||_{q/(q-p)} = |E|^{(q-p)/q}$$

and

$$||f^p||_{q/p} = (||f||_q)^p$$

are finite).

Thus,

$$\begin{aligned}||1 * f^p||_1^{1/p} &\leq |E|^{(q-p)/(pq)}(\int_E |f^p|^{q/p})^{1/q} = |E|^{(1/p)-(1/q)}(\int_E |f|^q)^{1/q}\\ &= |E|^{(1/p)-(1/q)}||f||_q\end{aligned}$$

So, using Equation 1 we see that

$$||f||_p \leq |E|^{(1/p)-(1/q)}||f||_q.$$

Now we show that the inclusion is proper.

Let $c_k = 2^{(k/\sqrt{pq})}$. Then, $f = 2^{(k/\sqrt{pq})}\chi_{A_k}$.

Now, note that $c_k^p = (2^{(k/\sqrt{pq})})^p = 2^{(kp/\sqrt{pq})} = 2^{(k\sqrt{p}\sqrt{p})/\sqrt{pq})} = 2^{((k\sqrt{p})/\sqrt{q})} = 2^{(k\sqrt{p/q})}$. So, we have

$$||f||_p^p = \int_E f^p = \sum_{k\in\mathbb{N}_{\geq 1}} \frac{|E|}{2^k} 2^{(k\sqrt{p/q})} = |E| \sum_{k\in\mathbb{N}_{\geq 1}} (2^{(\sqrt{p/q}-1)})^k.$$

Now, the ratio test says that $\sum_{k\in\mathbb{N}_{\geq 1}} 2^{((k\sqrt{p/q})-1)}$ is finite if the following ratio is less than 1 and infinite if it is greater than one. We compute:

$$\frac{2^{(\sqrt{p/q}-1)})^{k+1}}{(2^{(\sqrt{p/q}-1)})^k} = 2^{(\sqrt{p/q}-1)} = \frac{1}{2}2^{\sqrt{p/q}} < \frac{1}{2}2^1 = 1.$$

So, the above sum which is $||f||_p^p < \infty$, meaning that $f \in L^p(E)$.

However, note that $c_k^q = (2^{(k/\sqrt{pq})})^q = 2^{(kq/\sqrt{pq})} = 2^{(k\sqrt{q}\sqrt{q})/\sqrt{pq})} = 2^{((k\sqrt{q})/\sqrt{p})} = 2^{(k\sqrt{q/p})}$

$$||f||_q^q = \int_E f^q = \sum_{k\in\mathbb{N}_{\geq 1}} \frac{|E|}{2^k} 2^{(k\sqrt{q/p})} = |E| \sum_{k\in\mathbb{N}_{\geq 1}} \frac{(2^{(k\sqrt{q/p})}}{2^k} = |E| \sum_{k\in\mathbb{N}_{\geq 1}} (2^{(\sqrt{q/p}-1)})^k.$$

Now, the ratio test says that $\sum_{k\in\mathbb{N}_{\geq 1}} (2^{(\sqrt{q/p}-1)})^k$ is finite if the following ratio is less than one and infinite if it is greater than 1. We compute

$$\frac{(2^{(\sqrt{q/p}-1)})^{k+1}}{(2^{(\sqrt{q/p}-1)})^k} = 2^{(\sqrt{q/p}-1)} = \frac{1}{2}2^{\sqrt{q/p}} > \frac{1}{2}2^{\sqrt{p/p}} = 1.$$

Thus, $||f||_q^q = \infty$, meaning $f \notin L^q(E)$.

So, $f \in L^p(E) \setminus L^q(E)$ which (along with the earlier proof) proves $L^q(E) \subsetneq L^p(E)$.

4. Problem 7.2.17. Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable sets, let $f(x,y)$ be a measurable function on $E \times F$, and fix $1 \leq p < \infty$. Prove *Minkowski's Integral Inequality*:

$$\left(\int_E \left(\int_F |f(x,y)|\, dy \right)^p dx \right)^{1/p} \leq \int_F \left(\int_E |f(x,y)|^p\, dx \right)^{1/p} dy. \tag{2}$$

Proof:
Outline:
Approximate $F := |f|$ by a sequence of functions F_k with $F_k \nearrow F$ as $k \to \infty$. We do this by looking at an expanding cube over which we simultaneously divide into finer and finer subcubes. We then define F_k to be a simple function which is the infimum of F over each of these tiny subcubes and zero outside of our big cube.
Then, since we can know write integrals as finite sums, we can use the normal Minkowski inequality (Equation 7.19).
Then, the Monotone Convergence Theorem allows us to successfully translate the truth of the statement for our approximate simple functions F_k to a statement about the function F.
First, some definitions:

- I define the "k-th floor function" which is $\lfloor \cdot \rfloor_k : \mathbb{R} \to \mathbb{Z}$ by $\lfloor a \rfloor_k = max\{r \in \mathbb{Z} : a \geq \frac{1}{2^k} r\}$.
- For $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ I define the k-th floor function of the vector (x,y) to be $\lfloor (x,y) \rfloor_k := (\lfloor x_1 \rfloor_k, \lfloor x_2 \rfloor_k, \ldots, \lfloor x_m \rfloor_k, \lfloor y_1 \rfloor_k, \ldots, \lfloor y_n \rfloor_k)$.
- For any $r_1, \ldots, r_m, s_1, \ldots, s_n \in \mathbb{Z}$ I define the interval "in the $(r,s) = (r_1, \ldots, r_m, s_1, \ldots, s_n)$ position at the kth time step" to be $I^k_{(r,s)} = [\frac{r_1}{2^k}, \frac{r_1+1}{2_k}) \times \cdots \times [\frac{r_m}{2^k}, \frac{r_m+1}{2_k}) \times [\frac{s_1}{2^k}, \frac{s_1+1}{2_k}) \times \cdots \times [\frac{s_n}{2^k}, \frac{s_n+1}{2_k})$.
- Now, given some $(x,y) \in \mathbb{R}^{m+n}$ I can associate a vector $(r_k, s_k) \in \mathbb{Z}^{m+n}$ which is $(r_k, s_k) = (r_k(x), s_k(y)) = (\lfloor x_1 \rfloor_k, \ldots, \lfloor x_m \rfloor_k, \lfloor y_1 \rfloor_k, \ldots, \lfloor y_n \rfloor_k)$.
- Now for any $k \in \mathbb{N}$ I define

$$F_k = \begin{cases} 0, & \text{if } |x_i| \geq k \text{ or } |y_j| \geq k \text{ for some } i \in [m] \text{ or } j \in [n] \\ \inf_{(u,v) \in I^k_{(r_k(x), s_k(y))}} F(u,v), & \text{otherwise.} \end{cases}$$

Note, that we denote $\inf_{(u,v) \in I^k_{(r_k(x), s_k(y))}} F(u,v) =: A^k_{(r,s)}$ for $(r,s) \in \{-(k-1)2^k, (k-1)2^k\}^{n+m}$. Otherwise, we define $A^k_{(r,s)} = 0$. So, $A^k_{(r,s)}$ is just the value that the function F_k takes on the corresponding cube in $\mathbb{R}^{n+m}$. For fixed k, only finitely many of them are non-zero.

Now, clearly $F_k \nearrow F$ as $k \nearrow \infty$.

Now, as stated in the outline, we wish to show the statement for F_k where $k \in \mathbb{N}$ is fixed. So, we want to show that

$$\left(\int_E \left(\int_F F_k dy \right)^p dx \right)^{1/p} \leq \int_F \left(\int_E F_k^p dx \right)^{1/p} dy.$$

Well, we note that for fixed k we have that $F_k(x,y)$ is a simple function. In particular, we see that $F_k(x,y)=\sum_{(r,s)\in\{-(k-1)2^k,\dots,(k-1)2^k\}^{m+n}}(A^k_{(r,s)}\chi_{I^k_r}(x)\chi_{I^k_s}(y))$. Indeed we can write

$$\int_F F_k dy = \Big(\sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n} \frac{1}{2^k}A^k_{(r_k(x),s)}\Big)\chi_{I^k_{r_k}}(x)$$

where the sum runs only over s since the above is STILL a function of $x\in\mathbb{R}^m$.
Now, we recognize the LHS of our desired claim to be

$$||\int_F F_k dy||_p$$

which now becomes by our above deconstruction

$$||\int_F F_k dy||_p = ||\sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n} \frac{1}{2^k}A^k_{(r_k(x),s)}||_p.$$

(Note that the above is indeed a function of x still since the $r_k(x)$ in the subscript on the RHS takes in a value x).
By the usual Minkowski inequality (Equation 7.19) we know that $||\sum_{i\in I} f_i||_p \le \sum_{i\in I}||f_i||_p$, where I is any finite indexing set. So, we have

$$\begin{aligned}||\sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n}\Big(\frac{1}{2^k}A^k_{(r_k(x),s)}\Big)||_p &\le \sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n}||\frac{1}{2^k}A^k_{(r_k(x),s)}||_p.\\ &= \sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n}\frac{1}{2^k}||A^k_{(r_k(x),s)}||_p.\end{aligned}$$

Now, recall the definition of $||h(x)||_p = (\int_E |h(x)|^p dx)^{1/p}$. So, we have

$$\begin{aligned}&= \sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n}\frac{1}{2^k}||A^k_{(r_k(x),s)}||_p\\ &= \sum_{s\in\{-(k-1)2^k,\dots,(k-1)2^k\}^n}\frac{1}{2^k}\left(\int_E\left(A^k_{(r_k(x),s)}\right)^p dx\right)^{1/p}\\ &= \int_F\left(\int_E\left(A^k_{(r_k(x),s_k(y))}\right)^p dx\right)^{1/p} dy\end{aligned}$$

So, all together we have

$$\left(\int_E\left(\int_F F_k dy\right)^p dx\right)^{1/p} \le \int_F\left(\int_E F_k^p dx\right)^{1/p} dy.$$

So, since taking a limit preserves this inequality we have

$$\lim_{k\to\infty}\left(\left(\int_E\left(\int_F F_k dy\right)^p dx\right)^{1/p}\right) \le \lim_{k\to\infty}\left(\int_F\left(\int_E F_k^p dx\right)^{1/p} dy\right).$$

Now, we can apply the Monotone Convergence Theorem. Namely, since $F^k \nearrow F$, we have that

$$\lim_{k\to\infty}\left(\left(\int_E\left(\int_F F_k dy\right)^p dx\right)^{1/p}\right) = \left(\lim_{k\to\infty}\left(\int_E\left(\int_F F_k dy\right)^p dx\right)\right)^{1/p}$$

since by definition the limit on both sides is finite. We continue to see

$$= \left(\int_E \lim_{k\to\infty}\left(\left(\int_F F_k dy\right)^p\right) dx\right)^{1/p}$$

since $0 \le F_k \nearrow F$ implies that $\int_F F_k dy \nearrow \int_F F dy$ implies that $(\int_F F_k dy)^p \nearrow (\int_F F dy)^p$ which by the MCT gives the above. Then, just using basic limit rules which apply to finite limits we get

$$= \left(\int_E\left(\lim_{k\to\infty}\left(\int_F F_k dy\right)\right)^p dx\right)^{1/p}.$$

Then, we apply the MCT one more time to get

$$= \left(\int_E\left(\int_F (\lim_{k\to\infty} F_k) dy\right)^p dx\right)^{1/p}$$
$$= \left(\int_E\left(\int_F F dy\right)^p dx\right)^{1/p}.$$

So, to summarize, the above said that

$$\lim_{k\to\infty}\left(\left(\int_E\left(\int_F F_k dy\right)^p dx\right)^{1/p}\right) = \left(\int_E\left(\int_F F dy\right)^p dx\right)^{1/p}.$$

Now, we do the same for the RHS of our limiting equation in k (using the MCT over and over and the fact that if $\lim_{k\to\infty} L_k < \infty$ then we have that $(\lim_{k\to\infty} L_k)^p = \lim_{k\to\infty} L_k^p$ for any $p > 0$). We note that

$$\begin{aligned}
\lim_{k\to\infty}\left(\int_F\left(\int_E F_k^p dx\right)^{1/p} dy\right) &= \int_F \lim_{k\to\infty}\left(\left(\int_E F_k^p dx\right)^{1/p}\right) dy \text{ (by the MCT)}\\
&= \int_F \left(\lim_{k\to\infty}\left(\int_E F_k^p dx\right)\right)^{1/p} dy \text{ (by the limit rule above)}\\
&= \int_F \left(\int_E \left(\lim_{k\to\infty} F_k^p\right) dx\right)^{1/p} dy \text{ (by the MCT)}\\
&= \int_F \left(\int_E \left(\lim_{k\to\infty} F_k\right)^p dx\right)^{1/p} dy \text{ (by the limit rule above)}\\
&= \int_F \left(\int_E F^p dx\right)^{1/p} dy \text{ (since } F_k \nearrow F\text{)}.
\end{aligned}$$

So, indeed we have that

$$\begin{aligned}
\left(\int_E\left(\int_F F dy\right)^p dx\right)^{1/p} &= \lim_{k\to\infty}\left(\left(\int_E\left(\int_F F_k dy\right)^p dx\right)^{1/p}\right)\\
&\le \lim_{k\to\infty}\left(\int_F\left(\int_E F_k^p dx\right)^{1/p} dy\right) = \int_F\left(\int_E F^p dx\right)^{1/p} dy,
\end{aligned}$$

which is exactly what we set out to prove.