Math 6337 Real Analysis - HW 2

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1. Cantor sets of positive measure:
   The usual Cantor set is obtained by removing "open middle thirds", so that at the nth stage we obtain C_n by removing 2^{n-1} intervals of length $\left(\frac{1}{3}\right)^n$ from C_{n-1}. Now let $\delta \in (0,1)$. Assume that instead at the nth stage, we obtain C_n^δ from C_{n-1}^δ by removing 2^{n-1} intervals of length $\delta\left(\frac{1}{3}\right)^n$, one each from the middle of each interval in C_{n-1}^δ. Let

   $$C^\delta = \bigcap_{n=1}^{\infty} C_n^\delta.$$

   Show that

   $$\left|C^\delta\right| = 1 - \delta.$$

   Thus this is a Cantor type set of positive measure.

   Plan:

   (a) Let $V_i = \bigcap_{n=1}^{i} C_n^\delta$.
   (b) We show that $V_i = C_i^\delta$ for all $i \in \mathbb{N}_{\geq 1}$.
   (c) Note that $V_i \searrow V^\delta$.
   (d) Show that V_i measurable for all $i \in \mathbb{N}$.
   (e) Show that $|V_i| < \infty$ for some $i \in \mathbb{N}$.
   (f) Then we know that $lim_{i\to\infty}|V_i| = |C^\delta|$.
   (g) So, show that $lim_{i\to\infty}|V_i| = 1 - \delta$.

   First, we show 1b. Well, in particular note that $C_n^\delta = C\delta_{n-1} \setminus (\bigcup_{i=1}^{2^{n-1}} I_n^i)$, where $\{I_n^i : i \in [2^{n-1}]\}$ are the set of open intervals we are removing at this stage. Thus, $C_n^\delta \subseteq C_{n-1}^\delta$ for all $n \in \mathbb{N}_{\geq 1}$. Thus, $V_i = \bigcap_{n=1}^{i} C_n^\delta = C_i^\delta$ for all $i \in \mathbb{N}_{\geq 1}$.

   We show 1d inductively. Namely, note that $C_0^\delta = [0,1]$ which is closed and thus measurable, which finishes the base case. Then, for the inductive step, for all $n \in \mathbb{N}_{\geq 1}$ one has that $C_n^\delta = C\delta_{n-1} \setminus (\bigcup_{i=1}^{2^{n-1}} I_n^i)$, where $\{I_n^i : i \in [2^{n-1}]\}$ are the set of open intervals we are removing at this stage. Now, I_n^i is an open set for all $i \in [2^{n-1}]$ which means that I_n^i is measurable. Then $S_n := \bigcup_{i=1}^{2^{n-1}} I_n^i$ is a countable union of measurable sets and is thus measurable. By our inductive hypothesis C_{n-1}^{δ} is measurable. Now, since the set difference of any two sets is measurable we know that

$$C_n^{\delta} = C_{n-1}^{\delta} \setminus (\bigcup_{i=1}^{2^{n-1}} I_n^i)$$

is measurable.

Now, we show 1e. Note that $|C_0| = |C_0|_e = inf\{\sigma(S) : S \text{ covers } C_0\}$ (where S is a countable union of intervals). In particular, $C_0 = [0, 1]$ which is itself an interval. We proved consistency of exterior measure, namely that $|I| = v(I)$ for all intervals $I \in \mathbb{R}$. Here $|C_0| = v(C_0) = 1 < \infty$.

Finally, we show 1g. Recall that $V_i = C_i^{\delta}$ for all $i \in \mathbb{N}_{\geq 1}$. Namely, we aim to show that each C_n^{δ} is the disjoint union of some finite number of intervals of certain lengths to be calculated.

First, $C_0^{\delta} = [0, 1]$ and is one interval of length (or volume) 1.

NOTE from now on I have a change of notation! The I_i^j's are now intervals BELONGING to the cantor set C_i^{δ} (not what we are taking out). This is made precise below. I just realized I used the same notation twice for different things so I thought I should make a note.

Then, note that $C_1^{\delta} = I_1^1 \sqcup I_1^2$ where $v(I_1^i) = \frac{1-\delta\frac{1}{3}}{2}$ for $i \in [2]$.

Next, note that $C_2^{\delta} = I_2^1 \sqcup I_2^2 \sqcup I_2^3 \sqcup I_2^4$ where $v(I_2^1) = v(I_2^2) = \frac{v(I_1^1)-\delta(\frac{1}{3})^2}{2}$ and similarly $v(I_2^3) = v(I_2^4) = \frac{v(I_1^2)-\delta(\frac{1}{3})^2}{2}$.

In general, for $n \geq 1$, one has that

$$C_n^{\delta} = \bigsqcup_{k=1}^{2^n} I_n^k$$

where $v(I_n^k) = \frac{v(I_{n-1}^1)-\delta(\frac{1}{3})^n}{2}$.

We denote $a_i = v(I_i^1)$ for $i \in \mathbb{N}_{\geq 1}$ (and note that $v(I_i^k) = v(I_i^j)$ for all $k, j \in [2^i]$). Now, we aim to show that $2^i a_i \to 1 - \delta$ as $i \to \infty$.

Note that after some computation one sees that

$$a_i = \frac{1}{2^i}\big(1 - \delta(\sum_{j=0}^{i-1} 2^{i-j-1}\frac{1}{3}^{i-j})\big)$$

and thus,

$$2^i a_i = 1 - \delta(\sum_{j=0}^{i-1} 2^{i-j-1}\frac{1}{3}^{i-j}).$$

Now, we wish to show that $Q_i := (\sum_{j=0}^{i-1} 2^{i-j-1}\frac{1}{3}^{i-j}) \to 1$ as $i \to \infty$.

Note that by reindexing to get $k = i - j$ (which means $j = i - k$) one has

$$\begin{aligned} Q_i &= \frac{1}{2}\sum_{j=0}^{i-1} 2^{i-j}\frac{1}{3}^{i-j} \\ &= \frac{1}{2}\sum_{k=1}^{i} 2^{k-1}\frac{1}{3}^{k} \\ &= \frac{1}{2}\sum_{k=1}^{i} \frac{2}{3}^{k}. \end{aligned}$$

Then, recall that $\sum_{k=1}^{i} \frac{2}{3}^{k}$ is convergent by the geometric series test and one has that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2}{3}^{k} &= \frac{\frac{2}{3}}{(1-\frac{2}{3})} \\ &= \frac{\frac{2}{3}}{(\frac{1}{3})} \\ &= 2 \end{aligned}$$

where above I am using the fact that $\sum_{k=1}^{\infty} \frac{2}{3}^{k} =: x \in \mathbb{R}$ to note that $\frac{2}{3}x = x - \frac{2}{3}$ which implies that $x = \frac{\frac{2}{3}}{(1-\frac{2}{3})}$.

Thus, $lim_{i\to\infty} Q_i = \frac{1}{2} lim_{i\to\infty}(\sum_{k=1}^{i} \frac{2}{3}^{k}) = \frac{1}{2} * 2 = 1$.

Finally, note that

$$lim_{i\to\infty}|C_i^\delta| = lim_{i\to\infty}(2^i V(I_i^1))$$

since $C_i^\delta = \bigsqcup_{j\in[2^i]} I_i^j$ and we showed that the measure of disjoint measurable sets is the sum of their measure, which means that $|C_i^\delta| = \sum_{j\in[2^i]} |I_i^j| = \sum_{j\in[2^i]} v(I_i^j) = \sum_{j\in[2^i]} v(I_i^1) = 2^i v(I_i^1) = 2^i a_i$.

Continuing on one has

$$\begin{aligned} lim_{i\to\infty}|C_i^\delta| &= lim_{i\to\infty}(2^i a_i) \\ &= lim_{i\to\infty}(1 - \delta Q_i) \\ &= 1 - \delta(lim_{i\to\infty}(Q_i)) \\ &= 1 - \delta * 1 \\ &= 1 - \delta, \end{aligned}$$

and we are done.

Question 2 Measures of lim sup's and lim inf's of sets
Let $\{E_k\}_{k\geq 1}$ be a sequence of sets in $\mathbb{R}^n$. Define

$$\liminf_{k\to\infty} E_k = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k;$$
$$\limsup_{k\to\infty} E_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k.$$

(a) Show that $\liminf_{k\to\infty} E_k$ is the set of all points that belong to E_k for all large enough k.

We now want to show that $liminf(E_k)$ consists is the set $R := \{x \in R^d : |\{r \in \mathbb{N} : x \notin E_r\}| < \infty\}$. Why is that equivalent to the above statement? Because if $x \in R := \{x \in R^d : |\{r \in \mathbb{N} : x \notin E_r\}| < \infty\}$, then one can take $N = max\{r \in \mathbb{N} : x \notin E_r\}$ and then one notes that $x \in E_m$ for all $m > N$. Conversely, if there exists $N \in \mathbb{N}$ such that $x \in E_m$ for all $m > N$, then certainly $\{r \in \mathbb{N} : x \notin E_r\} \subseteq [N]$ which means that $|\{r \in \mathbb{N} : x \notin E_r\}| < \infty$.

First we show that $R \subseteq liminf(E_k) := \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}(E_k))$. Take $x \in R$. Consider the set $R(x) := \{k \in \mathbb{N} : x \notin E_k\}$. Let $M := max(k \in R(x))$. Now, note that $x \in E_m$ for all $m \geq M+1$. Thus, $x \in \bigcap_{k=M+1}^{\infty}(E_k))$. Thus, $x \in \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}(E_k))$. Finally, we show that $liminf(E_k) \subseteq R$. Take $x \in liminf(E_k)$. We want to show that $x \in R$. Well, $x \in liminf(E_k)$ implies that there exists $n \in \mathbb{N}$ such that $x \in \bigcap_{k=n}^{\infty} E_k$. Then, note that that means that $x \in E_k$ for all $k \geq n$. That implies that $R(x) \subseteq [n-1]$. Thus, $|R(x)| \leq |[n-1]| = n-1 < \infty$. Then, finally recall $R(x) = \{k \in \mathbb{N} : x \notin E_k\}$. Thus, if we denote $S(x) := \{k \in \mathbb{N} : x \in E_k\}$, then one has that $R(x) \sqcup S(x) = \mathbb{N}$ and also that $|R(X)| < \infty$, which was the desired result.

(b) Show that $\limsup_{k\to\infty} E_k$ is the set of all points that belong to E_k for infinitely many k and hence

$$\liminf_{k\to\infty} E_k \subset \limsup_{k\to\infty} E_k.$$

First we want to show that $limsup(E_k)$ consists is the set $S := \{x \in R^d : |\{k : x \in E_k\}| = \infty\}$. First we want to show that $S \subseteq limsup(E_k) := \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty}(E_k))$. Take $x \in S$. We know that for all $N \in \mathbb{N}$ one has that there exists $m \geq N$ such that $x \in E_m$. Otherwise, if not, then one would have that $\{k : x \in E_k\} \subseteq [N-1]$ implying that $|\{k : x \in E_k\}| \leq N-1 < \infty$. So, taking $N = n$, one knows there exists $m \in \{n, n+1, n+2, \dots\}$ such that $x \in E_m$. Thus, for all $n \in \mathbb{N}$ we have that $x \in \bigcup_{k=n}^{\infty} E_k$. Thus, $x \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k = limsup(E_k)$. Now, we wish to show that $limsup(E_k) \subseteq S$. Otherwise, assume for contradiction that there exists $x \in limsup(E_k) \setminus S$. Then, $x \in \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty}(E_k))$. However, $x \notin S$ means that $|\{k \in \mathbb{N} : x \in E_k\}| =: R < \infty$. Now, this means that there exists $N \in \mathbb{N}$ such that $x \notin E_m$ for all $m \geq N$. Why? Take $k_0 = max(k \in \mathbb{N} : x \in E_k)$. Then, take $N := k_0 + 1$. One has that for all $m \geq N$ that $x \notin E_m$. Thus, $x \notin \bigcup_{k=N}^{\infty} E_k$. Thus, $x \notin \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$.

Finally, note that indeed $x \in liminf_{k\to\infty} E_k$ if and only if $x \in E_m$ for all $m \geq N$ for some $N \in \mathbb{N}$, which means that $\{k \in \mathbb{N} : x \in E_k\} \supseteq \{N, N+1, N+2, \dots\}$ which means that $|\{k \in \mathbb{N} : x \in E_k\}| = \infty$ which is the characterization we just proved meaning that $x \in limsup_{k\to\infty} E_k$.

(c) Show that

$$\left|\liminf_{n\to\infty} E_n\right|_e \leq \liminf_{n\to\infty} |E_n|_e \,.$$

Define $D_n := \bigcap_{k=n}^{\infty} E_k$. Then, note that $\liminf_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} D_n$. Also, note that $D_i \subseteq D_j$ for $i \leq j$ with $i, j \in \mathbb{N}$. So, by definition $D_m \nearrow liminf_{k\to\infty} E_k$.

Now, by Theorem 3.27, we know that

$$|liminf_{k\to\infty} E_k|_e = lim_{k\to\infty} |D_k|_e.$$

Thus, we wish to show now that

$$lim_{n\to\infty} |D_n|_e \leq \liminf_{n\to\infty} |E_n|_e \,.$$

Also, embedded in Theorem 3.27, we have that $lim_{n\to\infty}|D_n|_e$ exists (though may not be finite), so in particular, that means $lim_{n\to\infty}|D_n|_e = \liminf_{n\to\infty} |D_n|_e = \limsup_{n\to\infty} |D_n|_e$.

Thus,

$$lim_{n\to\infty} |D_n|_e = \liminf_{n\to\infty} |D_n|_e,$$

and recall by definition of D_n that $D_n = \bigcap_{k=n}^{\infty} E_k$ which means $D_n \subseteq E_n$ for all $n \in \mathbb{N}$, meaning by monotonicity of exterior measure that

$$lim_{n\to\infty} |D_n|_e = \liminf_{n\to\infty} |D_n|_e \qquad\qquad \leq \liminf_{n\to\infty} |E_n|_e,$$

which concludes the proof.

(d) Suppose that the $\{E_k\}$ are measurable, and for some n, $\left|\bigcup_{k=n}^{\infty} E_k\right| < \infty$. Then

$$\left|\limsup_{n\to\infty} E_n\right| \geq \limsup_{n\to\infty} |E_n| \,.$$

Let $F_n := \bigcup_{k=n}^{\infty} E_k$. Now, we have that

$$F_n \searrow \limsup_{k\to\infty} E_k.$$

Thus, Theorem 3.27 says that

$$|\limsup_{k\to\infty} E_k|_e = \lim_{n\to\infty} |F_n|_e.$$

Once again, existence of limit implies that $\lim_{n\to\infty} |F_n|_e = \limsup_{n\to\infty} |F_n|_e$, so that

$$\begin{aligned} |\limsup_{k\to\infty} E_k|_e &= \lim_{n\to\infty} |F_n|_e \\ &= \limsup_{n\to\infty} |F_n|_e. \end{aligned}$$

Now, we wish to show that $\limsup_{n\to\infty} |F_n|_e \geq \limsup_{n\to\infty} |E_n|_e$. Note that for all $n \in \mathbb{N}$, one has $F_n \supseteq E_k$ for all $k \geq n$. In particular, $F_n \supseteq E_n$. By monotonicity that implies that $|F_n|_e \geq |E_n|e$ for all $n \in \mathbb{N}$. Thus,

$$\limsup_{n\to\infty} |F_n|_e \geq \limsup_{n\to\infty} |E_n|_e$$

and we are done.

(e) Show that if

$$\sum_{k=1}^{\infty} |E_k|_e < \infty,$$

then

$$\left|\limsup_{k\to\infty} E_k\right|_e = 0.$$

Denote $\sum_{k=1}^{\infty} |E_k|_e =: B \in \mathbb{R}$.

Recall by definition

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

We want to show that for all $\epsilon > 0$, that $\left|\limsup_{k\to\infty} E_k\right|_e < \epsilon$.

Note $\limsup_{n\to\infty} E_n \subseteq \bigcup_{k=m}^{\infty} E_k$ for all $m \in \mathbb{N}$.

Claim β: $|E_m|_e \to 0$ as $m \to \infty$.

Proof:

Denote $B_m := \sum_{k=1}^{m} |E_k|_e$. Now, the fact that $\lim_{m\to\infty} \left(\sum_{k=1}^{m} |E_k|_e\right)$ converges (to B) means that for all $\epsilon > 0$, there exists $M(\epsilon) \in \mathbb{N}$ such that $|B - B_m| < \epsilon$ for all $m \geq M(\epsilon)$.

it is a Cauchy sequence. Thus, by definition, for all $\epsilon > 0$ there there exists $M(\epsilon) \in \mathbb{N}$ such that $|B_r - B_m| < \epsilon$ for all $m, r \geq M(\epsilon)$. Now, without loss of generality $r \geq m$, meaning that

$$|B_r - B_m| = B_r - B_m = \sum_{k=m+1}^{r} |E_k|_e < \epsilon.$$

for all such r, m.

In particular, choose arbitrary $\epsilon > 0$ and we are guaranteed $M(\epsilon) \in \mathbb{N}$ such that $|B_r - B_m| < \epsilon$ for all $m, r \geq M(\epsilon)$. So, pick $r = m + 1$ and we get

$$|B_r - B_m| = B_{m+1} - B_m = |E_{m+1}|_e < \epsilon.$$

for all $m \geq M(\epsilon)$.

Thus, we have shown that $|E_{m+1}|_e \to 0$ as $m \to \infty$ or equivalently $|E_m|_e \to 0$ as $m \to \infty$, concluding proof of Claim β.

Claim α (proved in a couple paragraphs): Now I want to show that for any $\epsilon > 0$ for sufficiently large $M(\epsilon) \in \mathbb{N}$ one has that $T(M(\epsilon)) := \sum_{j=M(\epsilon)}^{\infty} |E_k|_e < \epsilon$.

Then, subadditivity says $|\bigcup_{j=M(\epsilon)}^{\infty} E_j|_e < \epsilon \leq \sum_{j=M(\epsilon)}^{\infty} |E_j|_e < \epsilon$ meaning intuitively that $|\bigcup_{j=r}^{\infty} E_j|_e \to 0$ as $r \to \infty$.

Let $U := \limsup_{n\to\infty} E_n$. Now, for all $x \in U$ for any $M \in \mathbb{N}$ there exists $r = r(x) \geq M$ such that $x \in E_r$. Thus, for any $M \in \mathbb{N}$, one has that $U = \limsup_{n\to\infty} E_n \subseteq \bigcup_{k=M}^{\infty} E_k$.

By monotonicity one has that

$$|\limsup_{n\to\infty} E_n|_e \leq |\bigcup_{k=M}^{\infty} E_k|_e.$$

Then, by subadditivity one has that

$$\begin{aligned} |\limsup_{n\to\infty} E_n|_e &\leq |\bigcup_{k=M}^{\infty} E_k|_e \\ &\leq \sum_{k=M}^{\infty} |E_k|_e. \end{aligned}$$

Upon proving Claim α, I will have shown that $0 \leq |\limsup_{n\to\infty} E_n|_e \leq \sum_{k=M}^{\infty} |E_k|_e =: T(M) \to 0$ as $n \to \infty$, which will then imply that for all $\epsilon > 0$

$$0 \leq |\limsup_{n\to\infty} E_n|_e \leq \sum_{k=M}^{\infty} |E_k|_e < \epsilon$$

for sufficiently large $M \in \mathbb{N}$, which then shows that

$$0 \leq |\limsup_{n\to\infty} E_n|_e \leq 0$$

or equivalently

$$|\limsup_{n\to\infty} E_n|_e = 0,$$

which was the goal.

So let's prove Claim α.
Proof: Fix $\epsilon > 0$. I have already shown (Claim β) that $|E_k|_e \to 0$ as $k \to \infty$.

Now, note that for any $M \in \mathbb{N}$ one can define

$$c(M) := \sum_{i=1}^{M} |E_i|_e$$

(so $c(M)$'s are our partial sums)

and

$$d(M) := \sum_{i=M+1}^{\infty} |E_i|_e$$

meaning that $c(M) + d(M) = B$.

If $\epsilon \geq B$ then we're done.

Otherwise, recall that $c(j) := \sum_{i=1}^{j} |E_i|_e \to B$ as $j \to \infty$ means for all $\gamma > 0$ there exists $Q(\gamma) \in \mathbb{N}$ such that

$$|B - c(r)| = |B - \sum_{i=1}^{r} |E_i|_e| = B - \sum_{i=1}^{r} |E_i|_e < \gamma \tag{1}$$

for all $r \geq Q(\gamma)$.

Now, to prove Claim α we want $M(\epsilon) \in \mathbb{N}$ such that $d(M(\epsilon)-1) = B - C(M(\epsilon)-1) = |B - C(M(\epsilon) - 1)| < \epsilon$.

We are guaranteed (by Equation 1) such $W \in \mathbb{N}$ with $|B - c(W)| = |B - \sum_{i=1}^{W} |E_i|_e| < \epsilon$. Namely take $W = Q(\epsilon)$.

Then, let $M(\epsilon) = Q(\epsilon)+1$ and by Equation 1 we get that $|B - c(r)| = |B - \sum_{i=1}^{r} |E_i|_e| = B - \sum_{i=1}^{r} |E_i|_e < \epsilon$ for all $r \geq Q(\epsilon)$ including all $r \geq M(\epsilon) = Q(\epsilon)+1$ and that concludes the proof of Claim α and this problem.

Question 3 Inner Measure
For sets E in $\mathbb{R}^n$, define its inner measure by

$$|E|_i = \sup\{|F| : F \text{ is closed and } F \subset E\}.$$

(i) Show that

$$|E|_i \leq |E|_e.$$

First, one has that for all $F \subseteq E$ with F closed and $F \subseteq E$, $|F|_e \leq |E|_e$ by monotonicity, and note that since F closed means F measurable we can simply write $|F| = |F|_e$.

Then, taking sups over such F gives

$$|E|_i = \sup\{|F| : F \text{ is closed and } F \subset E\} \leq |E|_e$$

(ii) Let $|E|_e < \infty$. Show that E is measurable iff

$$|E|_i = |E|_e.$$

(Hint: use Lemma 3.22)

By Lemma 3.22, E is measurable if and only if for all $\epsilon > 0$ one has that there exists a closed set $F \subseteq E$ such that $|E \setminus F|_e < \epsilon$.

We aim to show two things: first that E measurable implies $|E|_i = |E|_e$, then that $|E|_i = |E|_e$ implies E measurable.

So, assume E measurable. That means that for all $\epsilon_1 > 0$ there exists an open set $G \supseteq E$ such that $|G \setminus E|_e < \epsilon$. Also, by Theorem 3.22 it means that for all $\epsilon_2 > 0$ there exists a closed set $F \subseteq E$ such that $|E \setminus F|_e < \epsilon_2$.

Now, note by Theorem 3.6 that $|E|_e = \inf_{G \supseteq E : G \text{ open}} |G|_e$.

Also, by Caratheodory's theorem, E measurable implies that $|G|_e = |G \cap E|_e + |G \setminus E|_e = |E|_e + |G \setminus E|_e$ or we can write more usefully

$$|G|_e = |E|_e + |G \setminus E|_e. \tag{2}$$

Also, since E measurable once again by Caratheodory's theorem we have that $|E|_e = |E \cap F|_e + |E \setminus F|_e = |F|_e + |E \setminus F|_e$ or written more neatly

$$|F|_e = |E|_e - |E \setminus F|_e. \tag{3}$$

In particular, note that

$$\begin{aligned} |F|_e &= |E|_e - |E \setminus F|_e \\ &\leq |E|_e \\ &\leq |E|_e + |G \setminus E|_e = |G|_e, \end{aligned}$$

for all closed $F \subseteq E$ and all open $G \supseteq E$. Now, taking sups over all such closed $F \subseteq E$ one obtains

$$|E|_i = sup\{|F|_e\} \leq= |G|_e.$$

Then, taking infs over all open sets $G \supseteq E$ one obtains

$$|E|_i \leq inf\{|G|_e\} = |E|_e$$

and we are done.

Now, for the converse, we wish to show that if $|E|_i = |E|_e$ then E is measurable.

Well, $|E|_i = |E|_e$ implies by definition that

$$\begin{aligned} |E|_i &= sup\{|F| : F \text{ closed and } F \subseteq E\} && (4)\\ &= |E|_e && (5)\\ &= inf\{|G| : G \text{ open and } G \supseteq E\}. && (6) \end{aligned}$$

Now, the fact that $|E|_i = sup\{|F| : F \text{ closed and } F \subseteq E\} = inf\{|G| : G \text{ open and } G \supseteq E\} = |E|_e$ means that if both sides are finite then for all $\epsilon > 0$ there exist $F, G \subseteq \mathbb{R}^n$ with $F \subseteq E \subseteq G$ and F closed, G open such that $|G| - |F| < \epsilon$. Why?

Otherwise, if there existed $\epsilon_0 > 0$ such that $|G| - |F| \geq \epsilon_0$ for all such G, F, then equivalently $|G| \geq |F| + \epsilon_0$ for all such F, G. Then, taking sups over appropriate F of both sides gives

$$\sup\{|F|\} + \epsilon_0 \leq |G|$$

and then taking infs over appropriate G of both sides gives

$$\sup\{|F|\} + \epsilon_0 \leq inf\{|G|\}$$

meaning that

$$0 < \epsilon_0 \leq inf\{|G|\} - \sup\{|F|\} = |E|_e - |E|_i,$$

a contradiction.

As we will show in part (iii), the requirement that $|E|_e, |E|_i$ finite (which is equivalent to $|E|_e$ finite) does matter.

Now, recall that by Caratheodory's theorem one has that since F measurable

$$|G|_e = |G \cap F|_e + |G \setminus F|_e = |F|_e + |G \setminus F|_e,$$

which means that

$$|G|_e - |F|_e = |G \setminus F|_e.$$

Furthermore, note by monotonicity that

$$|G \setminus E|_e \leq |G \setminus F|_e$$

Then, as stated before $|G|_e - |F|_e < \epsilon$ and putting all the above together, we get that

$$|G \setminus E|_e \leq |G \setminus F|_e = |G|_e - |F|_e < \epsilon$$

and we are done, as the existence of such G for all $\epsilon > 0$ is the definition of measurability.

(iii) Give an example in 1 dimension, to show that if $|E|_e = \infty$, then the last conclusion can fail, i.e. we can have $|E|_i = |E|_e$, but E is not measurable.

First start with $S = \mathbb{R} \setminus \mathbb{Q}$. Next, use the Vitali set we saw in class (or a Vitali set). Namely, for $r, s \in \mathbb{R}$ define $r \sim s$ if and only if $r - s \in \mathbb{Q}$. Now, look at the equivalence classes of this equivalence relation (which partition $\mathbb{R}$). By Zermelo's axiom pick exactly one element a from each equivalence class which we then of course denote $E_a = a + \mathbb{Q}$ and call the union of these elements E.

Now, note that there is exactly one equivalence class which contains a rational. Namely, $\mathbb{Q}$ itself. Every other equivalence class consists entirely of irrationals. So, note that $E' := E \setminus \mathbb{Q} = E \setminus \{q\}$ where q is the chosen representative of the equivalence class $E_q = \mathbb{Q}$.

Now, as shown in class E does NOT have exterior measure 0. If it did, it would be measurable. So, likewise $|E'|_e = |E \setminus \{q\}|_e \neq 0$. Why?

Subadditivity says $|E|_e \leq |\{q\}|_e + |E \setminus \{q\}|_e = 0 + |E \setminus \{q\}|$. If one had that $|E \setminus \{q\}|_e = 0$, then that would imply $|E|_e \leq 0$ meaning $|E|_e = 0$, a contradiction.

Now, let $T = (\mathbb{R} \setminus \mathbb{Q}) \setminus E'$.

Once again subadditivity says that $\infty = |\mathbb{R} \setminus \mathbb{Q}|_e \leq |T|_e + |E'|_e$ meaning that $|T|_e = \infty$ or $|E'|_e = \infty$.

In fact, if E is the Vitali set with a nice set of representatives chosen one will have that $|T|_e = \infty$. Namely, note that if $\{C_e : e \in E\}$ is our set of equivalence classes of $\mathbb{R}$ where E is once again our arbitrarily chosen set of representatitives, then we also know that for all $e \in E$ there exists $a_e \in C_e$ such that $a_e \in [-1, 1]$, namely, if $e > 1$ let $a_e = e - \lfloor e \rfloor$. Otherwise if $e < -1$ let $a_e = -((-e) - (\lfloor (-e) \rfloor))$. Thus, there exists a Vitali set E such that $E \subseteq [-1, 1]$. Let $E' = E \setminus \{q\}$ where as before q is the unique rational in E.

Now, $E' \in [-1, 1]$ implies $|E'|_e \leq 2$. As before let $T = (\mathbb{R} \setminus \mathbb{Q}) \setminus E'$. We have that $\infty = |\mathbb{R} \setminus \mathbb{Q}|_e \leq |T|_e + |E'|_e \leq |T|_e + 2$ meaning that $|T|_e = \infty$ for this specific Vitali set.

Now, exercise 15 of the book says that for any measurable set $S \subseteq \mathbb{R}$ and any set $T \subseteq S$ one has that

$$|S| = |T|_i + |S \setminus T|_e.$$

Namely, here let $S = \mathbb{R} \setminus \mathbb{Q}$ and let T be the T above.

So,

$$\infty = |S| = |T|_i + |S \setminus T|_e = |T|_i + |E'|_e \leq |T|_i + 2$$

which implies that

$$|T|_i = \infty.$$

Thus, $|T|_i = |T|_e = \infty$, pending my proof of exercise 15 in the book.

Furthermore, I claim that T is not measurable.

Why? Well, $\mathbb{Q}$ m'able implies $\mathbb{R} \setminus \mathbb{Q}$ m'able. Then, if one had that T was m'able, that would imply $(R \setminus \mathbb{Q}) \setminus T = E'$ is m'able. However, then $E = E' \cup \{q\}$ would be measurable since $|\{q\}|_e = 0$, which we showed in class is not.

Thus, $|T|_i = |T|_e = \infty$ and T not measurable.

Question 4
Show that for m'ble E_1 and E_2,

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|.$$

Note that $E_1 \cup E_2 = (E_1 \cap E_2) \sqcup (E_1 \setminus E_2) \sqcup (E_2 \setminus E_1)$.
By Theorem 3.30 one has that

$$|E_2|_e = |E_2 \cap E_1|_e + |E_2 \setminus E_1|_e$$

and also

$$|E_1|_e = |E_1 \cap E_2|_e + |E_1 \setminus E_2|_e.$$

Thus,

$$|E_1|_e + |E_2|_e = |E_1 \cap E_2|_e + |E_1 \setminus E_2|_e + |E_1 \cap E_2|_e + |E_2 \setminus E_1|_e = |E_1 \cup E_2|_e + |E_1 \cap E_2|_e,$$

and we are done.

Question 5
Suppose that $|E|_e$ is finite. Then E is measurable iff given $\varepsilon > 0$, we can write

$$E = (S \cup N_1) \setminus N_2,$$

where S is a finite union of non-overlapping intervals, and $|N_1|_e < \varepsilon, |N_2|_e < \varepsilon$.
Remark
You can assume Theorem 1.11, namely that any open set can be written as a countable union of non-overlapping intervals.

Question 6 α-dimensional Hausdorff outer measure

Let $\alpha > 0$. We can think of Lebesgue measure on $\mathbb{R}^n$ as $n-$dimensional. In this exercise, you shall develop a small part of the theory of $\alpha-$dimensional Hausdorff outer measure. A ball B in $\mathbb{R}^n$ center $\mathbf{x}$, of radius r has the form

$$B = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}.$$

Its diameter is $diam\,(B) = 2r$. For $E \subset \mathbb{R}^n$, define its $\alpha-$dimensional Hausdorff outer measure

$$\Lambda_\alpha(E) = \inf\left\{\sum_{k=1}^{\infty} (\mathrm{diam}\,(B_k))^\alpha : E \subset \bigcup_{k=1}^{\infty} B_k\right\},$$

where the inf is taken over all sequences of balls $\{B_k\}$ in $\mathbb{R}^n$ covering E.

(a) Prove that Λ_α is subadditive: if

$$E = \bigcup_{k=1}^{\infty} E_k,$$

then

$$\Lambda_\alpha(E) \leq \sum_{k=1}^{\infty} \Lambda_\alpha(E_k).$$

Plan:

(a) For each $k \in \mathbb{N}$, take an arbitrary set of balls $\mathcal{B}_k$ covering E_k.

(b) Then, note that $\mathcal{B} := \cup_{k\in\mathbb{N}}\mathcal{B}_k$ covers E.

(c) (Small point: note that a countable union of countable sets is countable. Thus, we can enumerate $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$).

(d) Now $\Lambda_\alpha(E) \leq \sum_{i\in\mathbb{N}} diam(B_i)^\alpha$ since $\mathcal{B}$ is one a member of the set we are taking inf's over.

(e) Succesively take inf's for each $j \in \mathbb{N}$ over each $\mathcal{B}_j$ covering E_j.

(f) Indeed in the equation

$$\begin{aligned}\Lambda_\alpha(E) &\leq \sum_{i\in\mathbb{N}} diam(B_i)^\alpha \\ &= \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} diam(\mathcal{B}_j^i)^\alpha\end{aligned}$$

(where $\mathcal{B}_j^i$ denotes the ith element of the set $\mathcal{B}_j$ covering E_j), the left hand side is a fixed number which does NOT depend on any of the specific sets $\mathcal{B}_j$. Thus, we have that (dropping the middle term for clarity)

$$\Lambda_\alpha(E) \leq \ldots (inf_{\mathcal{B}_j \text{ covering } E_j}(\ldots(inf_{\mathcal{B}_2 \text{ covering } E_2}(inf_{\mathcal{B}_1 \text{ covering } E_1}(\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} diam(\mathcal{B}_j^i)^\alpha)))))$$

$$= \sum_{j\in\mathbb{N}} inf_{\mathcal{B}_j \text{ covering } E_j} \sum_{i\in\mathbb{N}} diam(\mathcal{B}_j^i)^\alpha$$

$$= \sum_{j\in\mathbb{N}} \Lambda_\alpha(E_k)$$

(b) Let $E \subset \mathbb{R}^n$. Prove that for $\alpha > n$,

$$\Lambda_\alpha(E) = 0.$$

(Hint: first prove this for bounded sets, and then for unbounded sets.)

(c) Prove that if $n = 1$ and C is the usual usual Cantor set in $[0, 1]$, then

$$\Lambda_\alpha(C) = 0 \text{ for } \alpha > \frac{\log 2}{\log 3}.$$

(One can show that C has "Hausdorff dimension" $\frac{\log 2}{\log 3}$

Question 7

Let E be a set in $\mathbb{R}^n$ with $0 < |E|_e < \infty$. Let $0 < \theta < 1$. Show that there is a set $E_\theta \subset E$ with

$$|E_\theta|_e = \theta \, |E_e| \, .$$

Hint: for $r > 0$, let $Q(r)$ denote the open cube centered on 0, with sides of length r parallel to the coordinate axes. Note that $Q(r)$ increases as r does. Use the $Q(r)$ in constructing your set.