# Math 6321 - Homework 1

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1. For $z, w \in \mathbb{C}$ show that
   (a) $\left|\frac{z-w}{1-\overline{z}w}\right| = 1$, if $|z| = 1$ and $w \neq z$
   (b) $\left|\frac{z-w}{1-\overline{z}w}\right| < 1$, if $|z| < 1$ and $|w| < 1$.

   (a) Note we want to show that $|z-w| = |1-\overline{z}w|$. Well. Note that $|1-\overline{z}w| = |\frac{z}{z}(1-\overline{z}w)| = |\frac{1}{z}(z-(z\overline{z})w)| = |\frac{1}{z}(z-|z|^2w)| = |\frac{1}{z}(z-w)| = |\frac{1}{z}||z-w| = \frac{1}{|z|}|z-w| = |z-w|$.

   (b) Let $\theta = Arg(z)$. Now, we note that

$$1|z-w| = |e^{-i\theta}||z-w| = |ze^{-i\theta} - we^{-i\theta}| = ||z| - we^{-i\theta}|.$$

Now, we would like to show that

$$||z| - we^{-i\theta}|^2 \leq |1 - |z|we^{-i\theta}|.$$

Note of course that $\overline{z} = |z|e^{-i\theta}$.
Now, for simplicity, define $\alpha = Arg(we^{-i\theta})$. Now, our claim becomes

$$||z| - |w|e^{i\alpha}|^2 \leq |1 - |z||w|e^{i\alpha}|^2.$$

We perform a simple computation. We compute the LHS as

$$\begin{aligned}
||z| - |w|e^{i\alpha}|^2 &= (|z| - |w|cos(\alpha))^2 + (|w|sin(\alpha))^2 \\
&= |z|^2 + |w|^2cos^2(\alpha) - 2|z||w|cos(\alpha) + |w|^2sin^2(\alpha) \\
&= |z|^2 + |w|^2(cos^2(\alpha) + sin^2(\alpha)) - 2|z||w|cos(\alpha) \\
&= |z|^2 + |w|^2 - 2|z||w|cos(\alpha).
\end{aligned}$$

Also, we compute the RHS as

$$\begin{aligned}
|1 - |z||w|e^{i\alpha}|^2 &= (1 - |z||w|cos(\alpha))^2 + (|z||w|sin(\alpha))^2 \\
&= 1 + |z|^2|w|^2cos^2(\alpha) - 2|z||w|cos(\alpha) + |z|^2|w|^2sin^2(\alpha) \\
&= 1 + |z|^2|w|^2(cos^2(\alpha) + sin^2(\alpha)) - 2|z||w|cos(\alpha) \\
&= 1 + |z|^2|w|^2 - 2|z||w|cos(\alpha).
\end{aligned}$$

Thus, we want to show that

$$|z|^2 + |w|^2 - 2|z||w|cos(\alpha) \leq 1 + |z|^2|w|^2 - 2|z||w|cos(\alpha)$$

which is true if and only if

$$|z|^2 + |w|^2 \leq 1 + |z|^2|w|^2.$$

which is true if and only if

$$|z|^2 - |z|^2|w|^2 \leq 1 - |w|^2$$

but $1 - |w|^2 > 0$ (this is where we use $|w| < 1$!) we can (factor the LHS as $|z|^2(1-|w|^2)$) then divide both sides by $(1-|w|^2)$ to find that the above is true if and only if

$$|z|^2 \leq 1$$

which is true if and only if

$$|z| \leq 1,$$

which was given in the problem, so we are done.

2. Prove that if $a_1, a_2, a_3$ for the vertices of an equilateral triangle, then $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3$.
Certainly if I have $a_1 = r, a_2 = re^{i\frac{2\pi}{3}}, a_3 = re^{i\frac{4\pi}{3}}$, then one has

$$a_1a_2 + a_2a_3 + a_3a_1 = r^2e^{i\frac{2\pi}{3}} + r^2e^{i\frac{6\pi}{3}} + r^2e^{i\frac{4\pi}{3}} = r^2e^{i\frac{2\pi}{3}} + r^2 + r^2e^{i\frac{4\pi}{3}}.$$

Now, we also have that

$$a_1^2 + a_2^2 + a_3^2 = r^2 + r^2e^{i4\pi/3} + r^2e^{i8\pi/3} = r^2 + r^2e^{i4\pi/3} + r^2e^{i2\pi/3}.$$

Thus, we have $a_1a_2 + a_2a_3 + a_3a_1 = a_1^2 + a_2^2 + a_3^2$ for our triangle centered at 0 with $a_1 \in \mathbb{R}$.
Now, say we have another equilateral triangle not centered at 0 and perhaps rotated in some non-standard way. Well, certainly we can bring it to the above form via a shift then a rotation. We compute the centroid of our trangle which is $c = (a_1 + a_2 + a_3)/3$.

$$\begin{aligned} a_1' &= a_1 - c \\ a_2' &= a_2 - c \\ a_3' &= a_3 - c. \end{aligned}$$

Now, the triangle $a_1', a_2', a_3'$ is centered at 0. Now, we wish to rotate it. We choose our principal branch (of angles) to be $[0, 2\pi)$. Now, let $\theta = min_{i\in[3]}Arg(a_i')$.

$$\begin{aligned} A_1 &= a_1'e^{-i\theta} \\ A_2 &= a_2'e^{-i\theta} \\ A_3 &= a_3'e^{-i\theta}. \end{aligned}$$

Now, by the first argument we know that since this is an equilateral triangle centered at 0 rotated so one vertex is on the real axis we have that

$$\begin{aligned} A_1A_2 + A_2A_3 + A_3A_1 &= A_1^2 + A_2^2 + A_3^3 \\ (a_1'e^{-i\theta})(a_2'e^{-i\theta}) + (a_2'e^{-i\theta})(a_3'e^{-i\theta}) + (a_3'e^{-i\theta})(a_1'e^{-i\theta}) &= (a_1'e^{-i\theta})^2 + (a_2'e^{-i\theta})^2 + (a_3'e^{-i\theta})^2 \\ (e^{-i\theta})^2(a_1'a_2' + a_2'a_3' + a_3'a_1') &= (e^{-i\theta})^2(a_1'^2 + a_2'^2 + a_3'^2) \\ a_1'a_2' + a_2'a_3' + a_3'a_1' &= a_1'^2 + a_2'^2 + a_3'^2 \\ (a_1 - c)(a_2 - c) + (a_2 - c)(a_3 - c) + (a_3 - c)(a_1 - c) &= (a_1 - c)^2 + (a_2 - c)^2 + (a_3 - c)^2 \\ a_1a_2 - a_1c - a_2c + c^2 + a_3a_2 - a_3c - a_2c + c^2 + a_3a_1 - a_3c - a_1c + c^2 &= a_1^2 - 2a_1c + c^2 + a_2^2 - 2a_2c + c^2 + a_3^2 - 2a_3c + c^2 \\ a_1a_2 + a_3a_2 + a_3a_1 - 2a_1c - 2a_2c - 2a_3c + 3c^2 &= a_1^2 - 2a_1c + a_2^2 - 2a_2c + a_3^2 - 2a_3c + 3c^2 \\ a_1a_2 + a_3a_2 + a_3a_1 &= a_1^2 + a_2^2 + a_3^2. \end{aligned}$$

So, we have shown that whenever one has an equilateral triangle with vertices $a_1, a_2, a_3$, the above equation will be satisfied.
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3. Write in a simple closed form the sums $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta)$ and $\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta)$.

Note that $cos(j\theta) = Re(e^{ij\theta})$ for all $j \in [0 : n]$.
So,

$$\sum_{j=0}^{n} Re(e^{ij\theta}) = Re(\sum_{j=0}^{n} e^{ij\theta}) = Re(\sum_{j=0}^{n}(e^{i\theta})^j) = Re(\frac{(e^{i\theta})^{(n+1)} - 1}{e^{i\theta} - 1}).$$

Likewise, we see that since $sin(j\theta) = Im(e^{ij\theta})$, we have that

$$0 + sin(\theta) + \cdots + sin(n\theta) = \sum_{j=0}^{n} Im(e^{ij\theta}) = Im(\sum_{j=0}^{n} e^{ij\theta}) = Im(\sum_{j=0}^{n}(e^{i\theta})^j) = Im(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}).$$

4. Find necessary and sufficient conditions for the complex numbers $a, b, c$ so that the equation $az + b\bar{z}b + c = 0$ represents a line.

First, note that the complex equation for a line is $\ell = \{tz' + z'' : t \in \mathbb{R}\}$. This can also be expressed in terms of $Re(z), Im(z)$. So, equivalently $\ell = \{z \in \mathbb{C} : Im(z) = m(Re(z) - a) + b\}$ where $m = \frac{Im(z')}{Re(z')}$ and where $a = Re(z'')$ and $b = Im(z'')$. So, now we write

$$\begin{aligned}
(a_1 + a_2 i)(x + iy) + (b_1 + b_2 i)(x - yi)) + (c_1 + c_2 i) &= 0 \\
(a_1 x - a_2 y) + (a_2 x + a_1 y)i + (b_1 x + b_2 y) + (b_2 x - b_1 y)i + (c_1 + c_2 i) &= 0 \\
(a_1 x - a_2 y + b_1 x + b_2 y + c_1) + (a_2 x + a_1 y + b_2 x - b_1 y + c_2)i &= 0 \\
(a_1 x - a_2 y + b_1 x + b_2 y + c_1) &= 0 \\
(a_2 x + a_1 y + b_2 x - b_1 y + c_2) &= 0 \\
(a_1 + b_1)x + (-a_2 + b_2)y + c_1 &= 0 \\
(a_2 + b_2)x + (a_1 - b_1)y + c_2 &= 0 \\
y &= \frac{(-a_1 - b_1)}{(-a_2 + b_2)}x - \frac{c_1}{(-a_2 + b_2)} \\
y &= \frac{(-a_2 - b_2)}{(a_1 - b_1)}x - \frac{c_2}{(a_1 - b_1)},
\end{aligned}$$

and the solution set to the last two equations forms a line if and only if each of the two solution sets to the last two equations above coincide. That means we need that

$$\begin{aligned}
\frac{(-a_1 - b_1)}{(-a_2 + b_2)} &= \frac{(-a_2 - b_2)}{(a_1 - b_1)} \\
\frac{c_1}{(-a_2 + b_2)} &= \frac{c_2}{(a_1 - b_1)}
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
(a_1 + b_1)(a_1 - b_1) &= (a_2 + b_2)(-a_2 + b_2) \\
c_1(a_1 - b_1) &= c_2(-a_2 + b_2)
\end{aligned}$$

and finally that means the given equation forms the equation for a line if and only if

$$\begin{aligned}
a_1^2 + a_2^2 - b_1^2 - b_2^2 &= 0 \\
c_1(a_1 - b_1) + c_2(a_2 - b_2) &= 0.
\end{aligned}$$

or better put if and only if

$$\begin{aligned}
|a| &= |b| \\
Re(c)(Re(a) - Re(b)) &= -Im(c)(Im(a) - Im(b)).
\end{aligned}$$

5. Show that $z$ and $z'$ correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z'} = -1$.

We write $z = r_1 e^{i\theta_1}, z' = r_2 e^{i\theta_2}$. Now, for the associated $Z, Z'$ we write them in the form $Z = (1, \theta_1', \phi_1')$ and $Z' = (1, \theta_2', \phi_2')$.
First, I claim that $\theta_1' = \theta_1$ and $\theta_2' = \theta_2$.
Now, two points on the sphere are diametrically opposed if the line segment connecting them passes through the origin.
If they $(Z, Z')$ are diametrically opposed, consider the line segment between them denoted $\ell(Z, Z') = (x(t), y(t), z(t))$ with $(x(0), y(0), z(0)) = Z$ and $(x(1), y(1), z(1)) = Z'$.
Now, consider the projection of this line into the XY plane denoted $\ell_p(Z, Z') = (x(t), y(t), 0)$.
Now, I claim that if we extend this line segment into a line that $z, z'$ belong to this line. Of course, since $(0,0,0) \in \ell(Z, Z')$ we also have that $(0,0,0) \in \ell_p(Z, Z')$ meaning that if we extend to a line denoted $\ell^*(Z, Z') = (x(t), y(t), 0) = (x(t), kx(t), 0)$ for some $k \in \mathbb{R}$. Now, we have that if $z = x_1 + iy_1$ and $z' = x_2 + iy_2$ that $(x_1, y_1)$ and $(x_2, y_2)$ are co-linear. We may write them in the form $(x_1, y_1, 0) = (r_1 cos(\theta_1), r_1 sin(\theta_1), 0)$ and $(x_2, y_2, 0) = (r_2 cos(\theta_2), r_2 sin(\theta_2), 0)$.
Now, since $(x_1, y_1)$ and $(x_2, y_2)$ are co-linear on a line through the origin, we know that the angle between them is $\pi$.
WLOG, say $\theta_1 \geq \theta_2$. Thus, $\theta_1 - \theta_2 = \pi$ and note that $\theta_1 - \theta_1 2 = Arg(\frac{z}{z'}) = Arg(\frac{z\overline{z'}}{z'\overline{z'}} = Arg(\frac{z\overline{z'}}{|z'|^2} = Arg(z\overline{z'}) = \pi$.
So, $zz' = -c$ for some $c \in \mathbb{R} \geq 0$.
Now we must examine the radii.
To reduce our attention to just the radii, we think of $S^2$ as a parameterized set of circles (which are points if their radii are zero) $\{\{(R(w)cos(\theta), R(w)sin(\theta)wz) : \theta \in [0, 2\pi)\} : w \in [-1, 1]\}$ where $R(w)$ is such that the circle $C_w = \{(R(w)cos(\theta), R(w)sin(\theta), w) : \theta \in [0, 2\pi)\} = S^2 \cap \{(u, v, w) : u, v \in \mathbb{R}\}$ is the intersection of the sphere with the plane $x_3 = w$. So the whole point is that we want to examine the relation between $R(w)$ and $r(w)$ where $r(w)$ is such that $ster.proj((r(w)cos(\theta), r(w)sin(\theta), 0) = (R(w)cos(\theta), R(w)sin(\theta), w$. Since we established earlier that any circle of the form $C_w$ will be mapped to a circle denoted $c_w$ in the X,Y plane centered at the origin I will use that fact without proof. Now, since $R(w) = dist((R(w)cos(\theta), R(w)sin(\theta), w), \{(0,0,q) : q \in \mathbb{R}\})$ is constant for all points on $X_\theta \in C_w$, we may choose any we like and likewise the corresponding quantity $c_w = |x_\theta|$ is constant no matter which point $X_\theta$ we chose. So, we choose $X_\theta$ corresponding to $\theta = 0$ meaning of the form $X_\theta = (R(w), 0, w)$. Then, if we wish to find $x_\theta = (a, b, 0)(== (a_\theta, b_\theta, 0))$ we note that means

$$w = \frac{a^2 - 1}{a^2 + 1}$$

and

$$0 = \frac{2b}{a^2 + 1}$$

and

$$R(w) = \frac{2a}{a^2 + 1}.$$

Thus, we use the first equation to solve for $a$ in terms of w which is given. Then, one notes that indeed $b = 0$. So, $r(w) = |a|$ which we have in terms of w per above. Now, I want to show that if I have a pair $w, -w$ that $r(w)r(-w) = 1$.
Once again I simplify things by looking at points of the sphere of the form $(R(w), 0, w)$ and $(R(-w), 0, -w)$. Well, for this pair we have that

$$w = \frac{(r(w))^2 - 1}{(r(w))^2 + 1}$$

and

$$-w = \frac{(r(-w))^2 - 1}{(r(-w))^2 + 1}.$$

Thus,

$$\frac{(r(w))^2 - 1}{(r(w))^2 + 1} = \frac{-((r(-w))^2 - 1)}{(r(-w))^2 + 1}$$

meaning that

$$((r(w))^2 - 1)((r(-w))^2 + 1) = (-(r(-w))^2 + 1)((r(w))^2 + 1)$$

or

$$(r(w))^2(r(-w))^2 + (r(w))^2 - r(-w))^2 - 1 = -(r(w))^2(r(-w))^2 - (r(-w))^2 + (r(w))^2 + 1.$$

Then, cancelling some terms we get

$$(r(w))^2(r(-w))^2 - 1 = -(r(w))^2(r(-w))^2 + 1.$$

which gives

$$2(r(w))^2(r(-w))^2 = 2,$$

or

$$(r(w)r(-w))^2 = 1,$$

or

$$|r(w)r(-w)| = 1,$$

and since $r(w), r(-w) \in \mathbb{R}_{\geq 0}$ we have that $r(w)r(-w) = 1$, which suffices to show that $z\overline{z'} = -1$ since if $Z = (a, b, c), Z' = (a', b', c')$ are diametrically opposed, that means that $c' = -c$ which by the above means that $r(c)r(c') = 1$ and indeed $z = r(c')e^{i\theta_1}, z' = r(c')e^{i\theta_2}$ whic means that $\overline{z'} = r(c')e^{i(-\theta_2)}$ so that $z\overline{z'} = r(c)r(c')e^{i(\theta_1-\theta_2)} = e^{i\pi} = -1$. FOR THE REVERSE DIRECTION. Say $z\overline{z'} = -1$. I want to show that $Z, Z'$ are diametrically opposed. Now once again we denote $z = r_1 e^{i\theta_1}, z' = r_2 e^{i\theta_2}$. So clearly $\theta_1 - \theta_2 = \pi$ (assuming WLOG $\theta_1 \geq \theta_2$ and recalling that multiplication is commutative in $\mathbb{C}$ so we really can just relabel them z as $z'$ if needed). Now, I claim that we can restrict our attention to cases where $\theta_1 = \pi$ and $\theta_2 = 0$. If that is not the case, we rotate our entire space (including the sphere so that that is the case). Rotation preserves whether points are diametrically opposed, so this is okay. Now, we have $z = r_1 e^{i\pi} = (-r_1, 0, 0)$ and $z' = r_2 e^{0i} = (r_2, 0, 0)$ and we know that $r_1 r_2 = 1$. Now, we note that when we project $z, z'$ onto the sphere we get $Z = (\frac{-2r_1}{r_1^2+1}, 0, \frac{r_1^2-1}{r_1^2+1})$ and $Z' = (\frac{2r_2}{r_2^2+1}, 0, \frac{r_2^2-1}{r_2^2+1})$. By an earlier argument, we know that the associated angles are $\pi$ apart. Now, we want to show that (meaning that our $x_3$ coordinates are opposites)

$$\frac{r_2^2 - 1}{r_2^2 + 1} = \frac{-(r_1^2 - 1)}{r_1^2 + 1}$$

if and only if

$$(r_2^2 - 1)(r_1^2 + 1) = (-r_1^2 + 1))(r_2^2 + 1)$$

if and only if

$$(r_1^2 r_2^2 - r_1^2 + r_2^2 - 1 = -r_1^2 r_2^2 - r_1^2 + r_2^2 + 1$$

if and only if

$$r_1^2 r_2^2 - 1 = -r_1^2 r_2^2 + 1$$

if and only if

$$(r_1 r_2)^2 = 1$$

if and only if $r_1 r_2 = \pm 1$. Now, since $r_1, r_2 \in \mathbb{R}_{\geq 0}$ we have that means that $r_1 r_2 = 1$. So, indeed since $r_1 r_2 = 1$ by following the reverse chain of implications going up above we get that our $x_3$ coordinates of $Z, Z'$ are opposites (additive inverses) of each other, which along with our angle argument given in the proof of the first implication gives that $Z, Z'$ are diametrically opposed.

6. (a) If $Q$ is a polynomial with distinct roots $\alpha_1, \alpha_2, \dots, \alpha_n$ and $P$ is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

We use the method of partial fractions.
In general we have that $Q(z) = \prod_{i\in[n]}(z-\alpha_i)$. (If the leading coefficient of $Q(z)$ is not 1 but rather some $c \neq 1$, we redefine $P(z)$ to be $\frac{1}{c}P(z)$.
Now, by the method of partial fractions we may write

$$\frac{P(z)}{\prod_{i\in[n]}(z-\alpha_i)} = \sum_{i\in[n]} \frac{r_i}{(z-\alpha_i)},$$

where $r_i \in \mathbb{C}$ for $i \in [n]$.
Now, we must determine the $r_i$. Well, for fixed $i$, we multiply both sides of the above equation by $(z-\alpha_i)$ which gives

$$\frac{P(z)(z-\alpha_i)}{\prod_{j\in[n]}(z-\alpha_j)} = r_i + \sum_{j\in[n]:j\neq i} \frac{r_j(z-\alpha_i)}{(z-\alpha_j)}.$$

Then, we simplify the left by canceling the common factor in the numerator and denominator of $(z-\alpha_i)$ to get

$$\frac{P(z)}{\prod_{j\in[n]:j\neq i}(z-\alpha_j)} = r_i + \sum_{j\in[n]:j\neq i} \frac{r_j(z-\alpha_i)}{(z-\alpha_j)}.$$

Now, we evaluate the above equation at $z = \alpha_i$ which zeroes out all but one term on the right to get

$$\frac{P(\alpha_i)}{\prod_{j\in[n]:j\neq i}(\alpha_i-\alpha_j)} = r_i.$$

Now, I claim that $\prod_{j\in[n]:j\neq i}(\alpha_i-\alpha_j) = Q'(\alpha_i)$. Why? Because of the product rule. We have that

$$(\prod_{j\in[n]}(z-\alpha_j))' = (z-\alpha_i)'(\prod_{j\in[n]:j\neq i}(z-\alpha_j)) + (z-\alpha_i)(\prod_{j\in[n]:j\neq i}(z-\alpha_j))'.$$

Now, we evaluate the above at $z = \alpha_i$ which means that the rightmost term on the RHS is now 0.
So, we have that

$$(\prod_{j\in[n]}(z-\alpha_j))'\Big|_{z=\alpha_i} = (z-\alpha_i)'(\prod_{j\in[n]:j\neq i}(z-\alpha_j)),$$

and since $(z-\alpha_i)' = 1$ we get the result. Namely, since $Q'(z)\Big|_{z=\alpha_i} = (\prod_{j\in[n]}(z-\alpha_j))'\Big|_{z=\alpha_i} = (\prod_{j\in[n]:j\neq i}(\alpha_i - \alpha_j))$ we do in fact have that $r_i = \frac{P(\alpha_i)}{Q'(\alpha_i)}$, we indeed get that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)},$$

as desired.

(b) Say there exist 2 distinct polynomials $P_1(z), P_2(z)$ such that $P_i(\alpha_k) = c_k$ for all $i \in [2]$ and all $k \in [n]$. Then, since $P_1(a_k) = P_2(a_k)$, by the above result we have that $\frac{P_1(z)}{Q(z)} = \frac{P_2(z)}{Q(z)}$ meaning that $P_1(z) = P_2(z)$ proving uniqueness.