Final exam, Math 6321

Please, upload the exam on Canvas by Monday, April 27 at 1:30pm

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Note:

- Please, provide **detailed** solutions to the problems below. You can use and quote all theorems and facts proved or stated in class, in homework assignments, in the notes, or in the textbook.
- The final exam must be your own work. In particular, collaboration with other people or the use of other sources is not allowed.
- Please, write clearly and concisely.
- I encourage you to submit your solutions typed.

Notation:
I denote

$$[n] := \{1, 2, \ldots, n\}$$

for all $n \in \mathbb{N}_{\geq 1}$.
Also, I denote

$$[a : b] := \{x \in \mathbb{Z} : a \leq x \leq b\}$$

for all $a, b \in \mathbb{Z}$ with $a \leq b$.

Problem 1. (a) Determine the set Ω of complex numbers z for which the series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges.
(b) Is the sum $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ analytic on Ω?

- The above series converges for $z \in \mathbb{C}$ such that $|z| < 1$ or $|z| > 1$.
- In particular, we show for $|z| < 1$ that $S(z) = \sum_{n \geq 1} \frac{z^n}{1+z^{2n}}$ converges to some function $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$.
- Then, since $\frac{z^n}{1+z^{2n}} = \frac{\frac{1}{z}^n}{1+\frac{1}{z}^{2n}}$ we have that $S(z) = S(\frac{1}{z})$. That means that for $|z| > 1$ we have $S(z)$ converges to $F(z) := f(\frac{1}{z})$ (where $F : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}$) is defined only outside the unit disk) (since $S(z) = S(\frac{1}{z}) = f(\frac{1}{z})$ for all $|z| > 1$).
- We then show that the partial sums $S_N(z) = \sum_{n=1}^{N} \frac{z^n}{1+z^{2n}}$ converge uniformly to $f(z)$ on balls $B_\delta(0) = \{z \in \mathbb{C} : |z| \leq \delta\}$ with radius $0 < \delta < 1$, which as stated in the lecture notes implies that $S_N(z)$ converges uniformly to f on all compact subsets of the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ which means by the theorem on the bottom of page 177 of Ahlfors that $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ is analytic.

- Then, that also implies that $F : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}$ is analytic since

$$F(z) = f(\frac{1}{z})$$

is the composition of two analytic functions namely $g(z) = \frac{1}{z}$ which is analytic on its domain $\{z \in \mathbb{C} : |z| > 1\}$ and f which is analytic on its domain $\{z \in \mathbb{C} : |z| < 1\} \supseteq range(g)$.

So, we now flesh out the above steps.

- Let's first show that the series converges for $|z| < 1$. Namely, note that for $|z| < 1$ we have that $\frac{z^n}{1+z^{2n}} = z^n \sum_{m=0}^{\infty}(-1)^m z^{2nm}$ and thus

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}} = \sum_{n=1}^{\infty}(z^n \sum_{m=0}^{\infty}(-1)^m z^{2nm}) = \sum_{n=1}^{\infty}(\sum_{m=0}^{\infty}(-1)^m z^{n(2m+1)}).$$

- We now show that the above series converges absolutely which then implies that reordering the summands in the above sum does not change the value, which will be useful when showing uniform convergence. Namely, we note that for $|z| < 1$ we have,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} |(-1)^m z^{n(2m+1)}| = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(|z|^{(2m+1)})^n &= \sum_{m=0}^{\infty} \frac{|z|^{(2m+1)}}{1-|z|^{(2m+1)}} \\ &\leq \frac{1}{1-|z|}\sum_{m=0}^{\infty} |z|^{(2m+1)} \leq (\frac{1}{1-|z|})^2\end{aligned}$$

thus proving absolute convergence.

- So, knowing that we can now say that

$$\sum_{n=1}^{\infty}(\sum_{m=0}^{\infty}(-1)^m z^{n(2m+1)}) = \sum_{m=0}^{\infty}(-1)^m(\sum_{n=1}^{\infty} z^{n(2m+1)})$$

and furthermore

$$S_N(z) = \sum_{n=1}^{N}(\sum_{m=0}^{\infty}(-1)^m z^{n(2m+1)}) = \sum_{m=0}^{\infty}(-1)^m(\sum_{n=1}^{N} z^{n(2m+1)})$$

for all $N \in \mathbb{N}$. This may not seem inherently useful, but we will use it to show uniform convergence. Namely, our function $f : \{z \in \mathbb{C} : |z| < 1\}$ is defined by

$$f(z) = \sum_{m=0}^{\infty}(-1)^m(\sum_{n=1}^{\infty} z^{n(2m+1)}) = \sum_{m=0}^{\infty}(-1)^m \frac{z^{(2m+1)}}{1-z^{(2m+1)}}.$$

- To show that $S_N(z)$ converges uniformly to $f(z)$, we fix $0 < \delta < 1$ and $\epsilon > 0$, then we show that there exists $N(\delta, \epsilon) \in \mathbb{N}$ such that

$$|S_N(z) - f(z)| < \epsilon$$

for all $N \geq N(\delta,\epsilon)$ and all $|z| < \delta$. Namely, for fixed δ, ϵ as above, we note that

$$\begin{aligned}
|S_N(z) - f(z)| &= \big|\big(\sum_{m=0}^{\infty}(-1)^m\big(\sum_{n=1}^{N} z^{n(2m+1)}\big)\big) - \big(\sum_{m=0}^{\infty}(-1)^m \frac{z^{(2m+1)}}{1-z^{(2m+1)}}\big)\big| \\
&= \big|\sum_{m=0}^{\infty}(-1)^m\big(\big(\sum_{n=1}^{N} z^{n(2m+1)}\big) - \frac{z^{(2m+1)}}{1-z^{(2m+1)}}\big)\big| \\
&= \big|\frac{1}{1-z^{(2m+1)}}\big|\big|\sum_{m=0}^{\infty}(-1)^m\Big(\Big((1-z^{(2m+1)})\sum_{n=1}^{N} z^{n(2m+1)}\Big) - z^{(2m+1)}\Big)\big| \\
&= \big|\frac{1}{1-z^{(2m+1)}}\big|\big|\sum_{m=0}^{\infty}(-1)^m\Big(\Big(z^{(2m+1)} - (z^{(2m+1)})^N\Big) - z^{(2m+1)}\Big)\big| \\
&= \big|\frac{1}{1-z^{(2m+1)}}\big|\big|\sum_{m=0}^{\infty}(-1)^{m+1}(z^{(2m+1)})^N\big| \\
&\leq \big|\frac{1}{1-z^{(2m+1)}}\big|\sum_{m=0}^{\infty}(|z|^{(2m+1)})^N.
\end{aligned}$$

Now, writing the above more judiciously we see that for $|z| < 1$

$$\begin{aligned}
|S_N(z) - f(z)| &\leq \big|\frac{1}{1-z^{(2m+1)}}\big|\sum_{m=0}^{\infty}(|z|^N)^{(2m+1)} \leq \frac{1}{1-|z|^{(2m+1)}}|z|^N\sum_{m=0}^{\infty}(|z|^N)^m \\
&\leq \frac{1}{1-|z|^{(2m+1)}}|z|^N\sum_{m=0}^{\infty}(|z|)^m \text{ (since } |z|^N \leq |z|) \\
&= \frac{1}{1-|z|^{(2m+1)}}\frac{|z|^N}{1-|z|} \leq \big(\frac{1}{1-|z|}\big)^2|z|^N.
\end{aligned}$$

So, in order to have $|S_N(z) - f(z)| \leq \epsilon$ for $|z| < \delta$ we note

$$|S_N(z) - f(z)| \leq \big(\frac{1}{1-|z|}\big)^2|z|^N < \big(\frac{1}{1-\delta}\big)^2\delta^N.$$

Then, setting

$$\big(\frac{1}{1-\delta}\big)^2\delta^N \leq \epsilon$$

and solving for $N = N(\delta,\epsilon)$ we see

$$\delta^N \leq \epsilon\big(1-\delta\big)^2.$$

Now, since $0 < \delta < 1$ one has that $\delta^N \to 0$ as $N \to \infty$ meaning that there exists $N(\delta,\epsilon)$ such that for all $N \geq N(\delta,\epsilon)$ one has that

$$\delta^N \leq \epsilon(1-\delta)^2,$$

which concludes our proof that $S_N(z) \to f(z)$ on compact balls $B_\delta(0)$ with radius $0 < \delta < 1$.

- That actually concludes our proof that $S(z) \to H(z)$ where $H : \mathbb{C} \setminus \{z : |z| = 1\} \to \mathbb{C}$ is defined by

$$H(z) = \begin{cases} f(z) \text{ if } |z| < 1 \\ F(z) \text{ if } |z| > 1 \end{cases}$$

and is analytic on its domain.

Problem 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, which is analytic in $\mathbb{C} \setminus \mathbb{R}$. Show that f is analytic in $\mathbb{C}$ (i.e. f is analytic also on $\mathbb{R}$).

- Let $H := \{z \in \mathbb{C} : im(z) \geq 0\}$ and $L := \{z \in \mathbb{C} : im(z) \leq 0\}$. Let $B = \{z \in \mathbb{C} : |z| > 1\}$ and let $D = B \cup \partial B$.
- By the Riemann Mapping Theorem and specifically the corollary on page 233 there exist functions $F : H \to D$ and $G : L \to D$ such that F, G are analytic and one to one on H, L, which by what we learned means technically that F is actually an analytic function on a region $U \supseteq H$ and likewise G is actually an analytic function on a region $V \supseteq L$.
- Furthermore, by the Local Mapping Theorem, we each point $z \in H$ (resp. $z \in L$) has a neighborhood $N(z) := B_\delta(z) \cap U$ (resp. $N(z) := B_\delta(z) \cap V$) such that $F|_{N(z)}$ is one to one. Thus, if we let $U' = H \cup (U \cap (\cup_{z \in \partial H} N(z)))$ and $V' = L \cup (V \cap (\cup_{z \in \partial L} N(z)))$. Then, $F : U' \to U'' \supseteq D$ and $F : V' \to V'' \supseteq D$ are one-to-one (where $U'' = im(F|_{U'})$ and likewise for V').
- So, that means that the functions $F^{-1} : U'' \to U'$ and $G^{-1} : V'' \to V'$ are each one to one and analytic.
- So, consider the functions $fF^{-1} : U'' \to \mathbb{C}$ and $fG^{-1} : V'' \to \mathbb{C}$. Note that each is analytic on B and thus has a power series expansion $p(z), q(z)$ such that $p(z) = fF^{-1}(z)$ for $z \in B$ and $q(z) = fG^{-1}(z)$ for $z \in B$.
- Furthermore, if we define $Y := \{z \in \mathbb{C} : p(z) \neq \infty\}$ and $W := \{z \in \mathbb{C} : q(z) \neq \infty\}$. Then $P = p|_Y$ and $Q = q|_W$ are continuous on Y, W respectively. Also, note that $Y \supseteq D$ since for $z \in \partial D$ we have $p(z) = \lim_{z' \to z : z' \in B} p(z') = \lim_{z' \to z : z' \in B} fF^{-1}(z') = fF^{-1}(z) \neq \infty$ (since $fF^{-1}(z)$ is continuous), which also implies that $F^{-1}(Y) \supseteq H$. By the same reasoning we have that $W \supseteq D$ and $G^{-1}(W) \supseteq L$.
- Also, $PF : U' \cap F^{-1}(Y) \to \mathbb{C}$ is analytic and $QG : V' \cap G^{-1}(W)$ is analytic. Furthermore, $PF(z) = fF^{-1}F(z) = f(z)$ for $z \in U' \cap F^{-1}(Y) \supseteq H$ and $QG(z) = fG^{-1}G(z) = f(z)$ for $z \in V' \cap G^{-1}(W) \supseteq L$.
- Additionally, $PF(z) = fF^{-1}F(z) = f(z) = fG^{-1}G(z) = QG(z)$ for all $z \in U' \cap V' \cap F^{-1}(Y) \cap G^{-1}(W)$. So, define the function
$$K(z) = \begin{cases} PF(z) \text{ if } z \in U' \cap F^{-1}(Y) \\ QG(z) \text{ if } z \in V' \cap G^{-1}(W) \end{cases}$$
which is analytic on all of $\mathbb{C}$ and as stated before satisfies $K(z) = f(z)$ for $z \in U' \cap F^{-1}(Y) \supseteq H$ and $K(z) = f(z)$ for $z \in V' \cap G^{-1}(W) \supseteq L$ meaning that $K|_{\mathbb{C} \setminus \mathbb{R}} = f|_{\mathbb{C} \setminus \mathbb{R}}$. Then, K, f continuous that means that $K(z) = f(z)$ for all $z \in \mathbb{C}$ and thus f is analytic on all of $\mathbb{C}$.

Problem 3. Suppose that f is analytic in some region containing the closed unit disc $\overline{D} = \{z : |z| \leq 1\}$ and $|f(z)| = 1$ when $|z| = 1$. Prove that f is a rational function.

- The key idea is to use Theorem 24 on page 172 of Ahlfors.
- In order to do so, we modify $f : D \to \mathbb{C}$ to obtain a function $g : H \to H$, where $H = \{z \in \mathbb{C} : im(z) \geq 0\}$, such that $g(\mathbb{R}) \subseteq \mathbb{R}$ on which we will use Theorem 24 to obtain an analytic extension $\hat{g} : \mathbb{C} \to \mathbb{C}$ with $g|_H = g$ and $\hat{g}(z) = \overline{\hat{g}(\overline{z})}$.
- More specifically the function g will be constructed as $g = T \circ f \circ T^{-1}$ where $T : \mathbb{C} \to \mathbb{C}$ is a Mobius transformation taking $\{z \in \mathbb{C} : |z| = 1\}$ to $\mathbb{R}$ and D to H.

- We define T so that $-i \mapsto 0$, $1 \mapsto 1$,$i \mapsto \infty$.
- So, in particular,
$$T(z) = \frac{1-i}{1+i}\frac{z+i}{z-i} = \frac{-iz+1}{z-i}$$
and also
$$T^{-1}(z) = \frac{-iz-1}{-z-i}.$$
since
$$\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} -i & -1 \\ -1 & -i \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$
implies that
$$T(T^{-1}(z)) = \frac{-2z}{-2} = z.$$
Likewise one has that
$$\begin{pmatrix} -i & -1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$
implies that
$$T^{-1}(T(z)) = \frac{-2z}{-2} = z.$$
- Note that $T^{-1}(\mathbb{R}) \subseteq \{z \in \mathbb{C} : |z| = 1\}$ since
$$|T^{-1}(x)| = \frac{|-ix-1|}{|-x-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$
Also, by theorems covered in class since "circles" map to "circles" we know that $T^{-1}(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\}$ and $T(\{z \in \mathbb{C} : |z| = 1\}) = \mathbb{R}$.
- Note that since T^{-1} has only one pole at $z = -i \notin H$ and T has only one pole at $z = i$ and for $g = TfT^{-1}$ one has that $im(g)$ is harmonic on $H^+ := \{z \in \mathbb{C} : im(z) > 0\}$ (note that the above is the strictly upper half plane) and zero on $\mathbb{R}$. It is also the imaginary part of the analytic function $g|_{H^+} = Tg|_{D'}T^{-1}|_{H^+}$ where $D' = D \setminus \partial D$.
- Thus, all the hypotheses of Theorem 24 on page 172 are satisfied meaning that there exists an analytic extension $\hat{g} : \mathbb{C} \to \mathbb{C}$ such that $\hat{g}(z) = \overline{\hat{g}(\overline{z})}$.

- Because of the argument given on pages 80 and 81 of Ahlfors, we know that $T^{-1}(\overline{z}) = \frac{1}{\overline{T^{-1}(z)}}$. Likewise that means that $T(\frac{1}{\overline{z}}) = \overline{T(z)}$.

- Then, by defining $\hat{f} : \mathbb{C} \to \mathbb{C}$ by $\hat{f}(z) = T^{-1}\hat{g}T(z)$ we see that firstly since $\hat{g}|_H = g = TfT^{-1}$ that means that $\hat{f}|_D = T^{-1}TfT^{-1}T = f$.

- Also, since $\hat{g}(z) = \overline{\hat{g}(\overline{z})}$ that means that
$$\hat{f}(\frac{1}{\overline{z}}) = T^{-1}\hat{g}T(\frac{1}{\overline{z}}) = T^{-1}\hat{g}(\overline{T(z)}) = T^{-1}(\overline{\hat{g}T(z)}) = \frac{1}{\overline{T^{-1}\hat{g}T(z)}} = \frac{1}{\overline{\hat{f}(z)}}.$$
In other words, that means that we have constructed a function $f : \mathbb{C} \to \mathbb{C}$ such that $\frac{1}{\overline{\hat{f}(\frac{1}{\overline{z}})}} = \hat{f}(z)$.

- Note that $\hat{f}(z)$ has no zeros outside of D. Namely, because if so, then $\frac{1}{\bar{z}} \in D$ and

$$\hat{f}(z) = \frac{1}{\overline{\hat{f}(\frac{1}{\bar{z}})}} = 0$$

which implies that $\overline{\hat{f}(\frac{1}{\bar{z}})} = \infty$ meaning that $\hat{f}$ has a pole at $\frac{1}{\bar{z}} \in D$, a contradiction.

- Now, in order to locate the poles of $\hat{f}$ we note that there exist finitely many zeros of $\hat{f}|_D = f$ in D since D compact implies by the Bolzano-Weierstrass theorem that if there were infinitely many then there would exist a convergent subsequence of them meaning that the zeros would have an accumulation point and thus not be isolated, a contradiction. Furthermore, f analytic in D means there are no poles of $\hat{f}_D = f$ in D.

- However, we now note that since $\hat{f}(z) = \frac{1}{\overline{\hat{f}(\frac{1}{\bar{z}})}}$ if $\{a_1, \dots, a_n\} \subseteq D$ are the zeros of f in D then $\hat{f}(\frac{1}{\bar{a}_i}) = \frac{1}{\overline{\hat{f}(a_i)}} = \infty$, which means by the definition of a pole given on page 127 of Ahlfors that $\{\frac{1}{\bar{a}_i} : i \in [n]\}$ are (all of) the (finite) poles of $\hat{f}$.

- Finally, we examine the behavior of $\hat{f}(z)$ as $z \to \infty$. So, note that $\hat{f}(z)$ has a zero of order $k \geq 0$ (if $k = 0$, $\hat{f}(0) \neq 0$). In particular, that means by definition that $\frac{\hat{f}(z)}{z^k}$ is analytic in a neighborhood of $z = 0$ and $\frac{\hat{f}(z)}{z^k}|_{z=0} \notin \{0, \infty\}$. Thus, note that

$$\frac{\hat{f}(z)}{z^k} = \frac{1}{z^k \overline{\hat{f}(\frac{1}{\bar{z}})}}$$

and

$$\overline{(\frac{z^k}{\hat{f}(z)})} = \bar{z}^k \hat{f}(\frac{1}{\bar{z}}) = \frac{\hat{f}(\frac{1}{\bar{z}})}{\frac{1}{(\bar{z})^k}}.$$

are analytic and nonzero in a neighborhood of $z = 0$. So, note that

$$\lim_{z \to 0} \overline{(\frac{z^k}{\hat{f}(z)})} = \lim_{z \to 0} \frac{\hat{f}(\frac{1}{\bar{z}})}{\frac{1}{(\bar{z})^k}} = \lim_{w \to \infty} \frac{\hat{f}(w)}{w^k} \notin \{0, \infty\}$$

implies by the definition given on page 129 of Ahlfors that $\hat{f}$ has a nonessential singularity at $z = \infty$.

- Now, we note that as proven on a past homework, if a function $F : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and has a non-essential singularity at $z = \infty$, then F is a polynomial. So, note that since $\hat{f}$ has poles at $\{\frac{1}{\bar{a}_i} : i \in [n]\}$ means that there exist $\{h_i \in \mathbb{N}_{\geq 1} : i \in [n]\}$ such that $\hat{f}(z) \prod_{i=1}^n (z - \frac{1}{\bar{a}_i})^{h_i}$ is analytic in all of $\mathbb{C}$. Thus, by the above theorem the function $F(z) = \hat{f}(z) \prod_{i=1}^n (z - \frac{1}{\bar{a}_i})^{h_i}$ is a polynomial, which means that

$$\hat{f}(z) = \frac{F(z)}{\prod_{i=1}^n (z - \frac{1}{\bar{a}_i})^{h_i}}$$

is a rational function.

Problem 4. Let g be a meromorphic function in $\mathbb{C}$. Show that the following conditions are equivalent:
(a) All poles of g are of order 1, and the residues at all poles are integers;
(b) There exists a meromorphic function f in $\mathbb{C}$ such that $f'/f = g$.

(a) We prove that (a) implies (b) in the following steps.

1. First we show that on the set $\mathbb{C} \setminus \{a_1, \ldots, a_k\}$

$$g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)} = H'(z)$$

is the derivative of an analytic function

$$H : \mathbb{C} \setminus \{a_1, \ldots, a_k\} \to \mathbb{C}.$$

2. Then, we show that H has only removable singularities.

3. Then, using an analytic extension $\hat{H} : \mathbb{C} \to \mathbb{C}$ we let

$$f(z) = exp(\hat{H}(z)) \prod_{i=1}^{k} (z - a_i)^{R_i}$$

and we show that

$$g(z) = \frac{f'(z)}{f(z)}.$$

We now fill in the above steps.

1. First we show that on the set $\mathbb{C} \setminus \{a_1, \ldots, a_k\}$

$$g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)} = H'(z)$$

is the derivative of an analytic function

$$H : \mathbb{C} \setminus \{a_1, \ldots, a_k\} \to \mathbb{C}.$$

   (a) We do so by showing that

   $$\int_{\gamma} g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)} dz = 0$$

   for any closed curve $\gamma \subseteq \mathbb{C} \setminus \{a_1, \ldots a_k\}$.

   (b) Then, that implies that the path integral

   $$\int_{\eta} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)}) dz$$

   depends only on its endpoints for any curve $\eta \subseteq \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ and thus as noted on page 107 of Ahlfors since $g(z) - \sum_{i=1}^{k} \frac{Res(g,a_i)}{(z-a_i)}$ is continuous and the above path integral depends only on its endpoints we see that

   $$g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)} = H'(z)$$

   is the derivative of an analytic function $H : \mathbb{C} \setminus \{a_1, \ldots, a_k\} \to \mathbb{C}$.

   (c) Now that we've laid out the plan let's jump into the calculations.

- Namely, if $\gamma \subseteq \mathbb{C} \setminus \{a_1, \ldots, a_k\}$ is a closed curve, decompose γ into $\gamma = \gamma_1 + \cdots + \gamma_s$ where the γ_i is a simple closed curve (meaning an injective mapping of $\gamma_i : (0,1) \to \mathbb{C}$ with $\gamma_i(0) = \gamma_i(1)$).
- So, it suffices to show that

$$\int_{\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz = 0$$

for all simple closed curves γ_i since

$$\int_{\gamma} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz = \sum_{i=1}^{s} \int_{\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz.$$

- So, we show the above by noting that we may decompose γ_i into

$$\gamma_i = \mu_1 + \cdots + \mu_{n_i}$$

where each each pole α_l such that $n(\gamma_i, \alpha_l) \neq 0$ is enclosed by one of the curves μ_j and so that each μ_j contains one or zero of the poles α_l such that $n(\gamma_i, \alpha_l) \neq 0$.
We carry out this construction below. (Each of these μ's will actually be constructed as ∂S_j^l for some regions S_j^l to be defined below).
- Namely, once we know the poles α_i we compose a grid of lines such that each α_i belongs to exactly one rectangle in the grid and each rectangle contains at most one α_i.
More precisely, say our poles have real and imaginary parts

$$\alpha_i = A_i + iB_i.$$

Then, consider the ordered lists of real and imaginary parts.

$$A_1' < A_2' < \cdots < A_v'$$

where $v = |\{A_i : i \in [k]\}|$ and $A_l' \in \{A_i : i \in [k]\}$ for all $l \in [v]$ and

$$B_1' < B_2' < \cdots < B_u'$$

where $u = |\{B_i : i \in [k]\}|$ and $B_l' \in \{B_i : i \in [k]\}$ for all $l \in [u]$. Then, let

$$A_i'' = \frac{A_i' + A_{i+1}'}{2}$$

for $i \in [v-1]$ and

$$B_i'' = \frac{B_i' + B_{i+1}'}{2}$$

for $i \in [u-1]$.
Now, one has the set of lines

$$Re(z) = A_i''$$

for $i \in [v-1]$ and

$$Im(z) = B_i''$$

for $i \in [u-1]$.

So, the above lines naturally divide the plane $\mathbb{C}$ into rectangles and "unbouded rectangle" which will be defined below.
Also, define

$$A_0'' = B_0'' = -\infty$$

and

$$A_v'' = B_u'' = \infty.$$

Now, for $i \in \{0, \ldots, v-1\}$ and $j \in \{0, \ldots, u-1\}$ we have regions (some bounded others not)

$$R_j^i = \{z \in \mathbb{C} : A_i'' < Re(z) < A_{i+1}'', B_j'' < Im(z) < B_{j+1}''\}.$$

So, note that

$$\mathbb{C} = \cup_{i=0}^{v-1} \cup_{j=0}^{u-1} closure(R_j^i).$$

- Note, for any simple closed curve γ_i the Jordan Curve Theorem implies that γ_i divides $\mathbb{C}$ into a bounded region G and another unbounded region. We are only interested in G.
  So, let
  $$S_j^i = R_j^i \cap G.$$
  Note that for certain $i \in [0 : v-1], j \in [0 : u-1]$ it may happen that $S_j^i = \varnothing$.
- Now, note that to show that
  $$\int_{\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz = 0$$
  we may assume without loss of generality that γ_i is traversed counterclockwise since otherwise take $\gamma_i' = -\gamma_i$ which is traversed counterclockwise and then if $\int_{-\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g,a_i)}{(z-a_i)})dz = 0$ then $\int_{\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g,a_i)}{(z-a_i)})dz = 0$ since $\int_{-\gamma} F(z)dz = -\int_{\gamma} F(z)dz$ for any function F.
- So, for our counterclockwise simple closed curve γ_i we note that
  $$\gamma_i = \sum_{i=0}^{v-1} \sum_{j=0}^{u-1} \partial S_j^i$$
  where $\partial\varnothing = \varnothing$.
- Furthermore, note that for each pole α_j with $n(\gamma_i, \alpha_j) \neq 0$ (which means by our assumptions that γ_i simple closed curve with counterclockwise orientation that $n(\gamma_i, \alpha_j) = 1$) there is exactly one $(i, k) \in [0 : v-1] \times [0 : u-1]$ such that $n(\partial S_k^i, \alpha_j) \neq 0$ and in fact $n(\partial S_k^i, \alpha_j) = 1$. So, each pole enclosed by γ_i is contained in exactly one of the subregions S_j^i. Also, each S_j^i contains at most one pole by construction of our grid.
- So,
  $$\int_{\gamma_i} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz = \sum_{l=0}^{v-1} \sum_{j=0}^{u-1} \int_{\partial S_j^l} (g(z) - \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)})dz$$
- Now, for each $(l, j) \in [0 : v-1] \times [0 : u-1]$ we examine the quantity $\int_{\partial S_j^l} (g(z) - \sum_{i=1}^{k} \frac{Res(g,a_i)}{(z-a_i)})dz$.

Namely, note that there is at most one $p \in [k]$ such that $n(\partial S_j^l, \alpha_p) \neq 0$.
If there is no such p, then $(g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})$ is analytic on $closure(S_j^l)$ implying that

$$\int_{\partial S_j^l} (g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})dz = 0.$$

If there does exist such a $p \in [k]$ then note that

$$\int_{\partial S_j^l} (g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})dz = \int_{\partial S_j^l} (g(z) - \frac{Res(g,a_p)}{(z-a_p)})dz - \int_{\partial S_j^l} \sum_{i\in[k], i\neq p} \frac{Res(g,a_i)}{(z-a_i)})dz.$$

Now, note that by definition of the residue as given in Ahlfors (given on page 149) one has that

$$\int_{\partial S_j^l} (g(z) - \frac{Res(g,a_p)}{(z-a_p)})dz = 0.$$

Furthermore, since $\frac{Res(g,a_i)}{(z-a_i)})$ is analytic on $closure(S_j^l)$ we have that $\int_{\partial S_j^l} \sum_{i\in[k], i\neq p} \frac{Res(g,a_i)}{(z-a_i)})dz = 0$.

- So, above we showed that for each $(l,j) \in [0:v-1] \times [0:u-1]$ one has that

$$\int_{\partial S_j^l} (g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})dz = 0.$$

Thus,

$$\int_{\gamma_i} (g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})dz = \sum_{l=0}^{v-1}\sum_{j=0}^{u-1} \int_{\partial S_j^l} (g(z) - \sum_{i=1}^k \frac{Res(g,a_i)}{(z-a_i)})dz = 0,$$

as we wished to prove.

2. Then, we show that H has only removable singularities.

   (a) Otherwise, if the singularity a_j was non-removable for some $j \in [k]$, then it would have an associated Laurent series expansion

$$H(z) = \sum_{n=-\infty}^{\infty} c_n (z-a_j)^n$$

   with $c_n \neq 0$ for at least one $n < 0$ valid for $0 < |z - a_j| < \min_{i\in[k], i\neq j}(|a_j - a_i|)$ since any function $H(z)$ analytic on the annulus $A = 0 < |z-a_j| < \min_{i\in[k], i\neq j}(|a_j - a_i|)$ has a convergent Laurent series on that annulus centered at a_j.

   (b) Then, likewise $H'(z)$ would have a non-removable singularity at $z - a_i$ since differentiating the above series term by term gives

$$H'(z) = \sum_{n=-\infty}^{-1} nc_n(z-a_j)^{n-1} + \sum_{n=1}^{\infty} nc_n(z-a_j)^{n-1} = \sum_{m=-\infty}^{-2} (m+1)c_{m+1}(z-a_j)^m + \sum_{m=0}^{\infty}(m+1)c_{m+1}(z-a_j)^m$$

   which is again valid for $0 < |z-a_j| < \min_{i\in[k], i\neq j}(|a_j - a_i|)$.

(c) Now, we note that since $g(z)$ is meromorphic with poles of order 1 we have that

$$g(z)\prod_{i=1}^{k}(z-a_i) = H'(z)\prod_{i=1}^{k}(z-a_i) + (\sum_{i=1}^{k}\frac{Res(g,a_i))}{(z-a_i)})(\prod_{i=1}^{k}(z-a_i)) =: H'(z)\prod_{i=1}^{k}(z-a_i) + R(z)$$

is analytic (where we define $R(z)$ which is also analytic as above) and now, plugging in our Laurent expansion which is valid on the annulus A we see

$$g(z)\prod_{i=1}^{k}(z-a_i) = (\sum_{m=-\infty}^{-2}(m+1)c_{m+1}(z-a_j)^m + \sum_{m=0}^{\infty}(m+1)c_{m+1}(z-a_j)^m)\prod_{i=1}^{k}(z-a_i) + R(z)$$

is analytic for $z \in A$. Now, since $g(z)\prod_{i=1}^{k}(z-a_i)$ and $R(z)$ are analytic on A, that means their difference

$$(\sum_{m=-\infty}^{-2}(m+1)c_{m+1}(z-a_j)^m + \sum_{m=0}^{\infty}(m+1)c_{m+1}(z-a_j)^m)\prod_{i=1}^{k}(z-a_i)$$

is also analytic. That then implies that $c_{m+1} = 0$ for all $m \leq -2$.

(d) Thus, our original Laurent Expansion actually has the form

$$H(z) = \sum_{n=0}^{\infty} c_n(z-a_j)^n$$

valid for $0 < |z-a_j| < \min_{i\in[k],i\neq j}(|a_j - a_i|)$.

(e) Thus, a_j was actually a removable singularity, a contradiction.

3. Then, using an analytic extension $\hat{H} : \mathbb{C} \to \mathbb{C}$ we let

$$f(z) = exp(\hat{H}(z))\prod_{i=1}^{k}(z-a_i)^{R_i}$$

and we show that

$$g(z) = \frac{f'(z)}{f(z)}.$$

(a) Namely, let

$$\hat{H}(z) = \begin{cases} H(z) \text{ if } z \notin \{a_1,\ldots,a_n\} \\ \sum_{n=0}^{\infty} c_n(a_j)z^n \text{ if } z = a_j \text{ for some } j \in [k] \end{cases}$$

where $\sum_{n=0}^{\infty} c_n(a_j)z^n$ is the Laurent series for H valid on the annulus

$$0 < |z-a_j| < \min_{i\in[k],i\neq j} |a_i - a_j|.$$

(b) Now, we prove the construction of f such that $g = f'/f$ is correct in steps. Namely, we first note that

$$\sum_{i=1}^{k}\frac{-Res(g,a_i)}{(z-a_i)} = \frac{h'}{h}$$

for some meromorphic function h.

(c) Then, of course

$$\hat{H}' = \frac{\frac{d}{dz}exp(\hat{H}(z))}{exp(\hat{H}(z))} =: \frac{r'}{r}$$

where $r(z) = exp(\hat{H}(z))$ which is the composition of two entire functions and thus entire.

(d) Finally, we note that the difference of two quotients of the above form has the same form, namely that

$$\frac{r'}{r} - \frac{h'}{h} = \frac{f'}{f}$$

for some meromorphic function f.

(e) Finally that will imply that

$$g = \frac{f'}{f}$$

since

$$g(z) = \hat{H}'(z) + \sum_{i=1}^{k} \frac{Res(g, a_i)}{(z - a_i)} = \frac{r'}{r} - \frac{h'}{h} = \frac{f'}{f}.$$

(f) Now, that we've heard our plan, we now jump back into those calculations and I claim that

$$\sum_{i=1}^{k} \frac{-Res(g, a_i)}{(z - a_i)} = \frac{\frac{d}{dz}(\prod_{i=1}^{k}(z - a_i)^{-Res(g,a_i)})}{\prod_{i=1}^{k}(z - a_i)^{-Res(g,a_i)}} =: \frac{h'}{h}.$$

Why? Because

$$\frac{d}{dz}log(\prod_{i=1}^{k}(z - a_i)^{-R_i}) = \frac{\frac{d}{dz}(\prod_{i=1}^{k}(z - a_i)^{-R_i})}{\prod_{i=1}^{k}(z - a_i)^{-R_i}}$$

and indeed

$$\frac{d}{dz}log(\prod_{i=1}^{k}(z - a_i)^{-R_i}) = \frac{d}{dz}(\sum_{i=1}^{k} -R_i log(z - a_i)) = \sum_{i=1}^{k} \frac{-R_i}{z - a_i}$$

which means that

$$\frac{d}{dz}(\prod_{i=1}^{k}(z - a_i)^{-R_i}) = (\sum_{i=1}^{k} \frac{-R_i}{z - a_i})(\prod_{i=1}^{k}(z - a_i)^{-R_i}).$$

and thus, dividing both sides by $\prod_{i=1}^{k}(z - a_i)^{-R_i}$ gives

$$\frac{\frac{d}{dz}(\prod_{i=1}^{k}(z - a_i)^{-R_i})}{\prod_{i=1}^{k}(z - a_i)^{-R_i}} = \sum_{i=1}^{k} \frac{-R_i}{z - a_i}.$$

So, indeed if $h(z) = \prod_{i=1}^{k}(z - a_i)^{-R_i}$, then

$$\sum_{i=1}^{k} \frac{-Res(g, a_i)}{(z - a_i)} = \frac{h'}{h}.$$

(g) Now, we show that whenever h, r are meromorphic functions then

$$\frac{r'}{r} - \frac{h'}{h} = \frac{f'}{f}$$

for some f also a meromorphic function.
Namely, let

$$f = \frac{r}{h}.$$

Then,

$$f' = \frac{r'h - h'r}{h^2}$$

meaning that

$$\frac{f'}{f} = \frac{r'h - h'r}{h^2}\frac{h}{r} = \frac{r'h - h'r}{rh} = \frac{r'}{r} - \frac{h'}{h}.$$

(h) So, finally putting all that together we see that

$$f(z) = \frac{r}{h} = \frac{exp(\hat{H}(z))}{\prod_{i=1}^{k}(z-a_i)^{R_i}}$$

satisfies

$$\frac{f'}{f} = \frac{r'}{r} - \frac{h'}{h} = \frac{H'exp(H)}{exp(H)} - \sum_{i=1}^{k}\frac{-Res(g,a_i)}{(z-a_i)} = g$$

as claimed.

Now, I show that (b) implies (a).

- Note that f meromorphic implies that

$$f(z) = F(z)\prod_{i=1}^{n}(z-a_i)^{k_i}$$

for some entire function $F(z)$ and negative integers $k_i \in \mathbb{Z}_{\leq}0$. In this chosen representation we also stipulate that $(z-a_i) \nmid F(z)$ for all $i \in [n]$. (Otherwise, we would factor that out and increment the appropriate power in the product in the right by 1).

- Now, notice that poles of $\frac{f'}{f}$ are zeros or poles of f' or f.

- Since F is entire we know that it has no poles and likewise F' has no poles.

- Now, we note that by iteration of the usual product rule we have

$$f'(z) = F'(z)(\prod_{i=1}^{n}(z-a_i)^{k_i}) + F(z)\sum_{i=1}^{n}k_i\frac{\prod_{j=1}^{n}(z-a_j)^{k_j}}{(z-a_i)}$$

and thus

$$\frac{f'(z)}{f(z)} = \frac{F'(z)(\prod_{i=1}^{n}(z-a_i)^{k_i})}{F(z)\prod_{i=1}^{n}(z-a_i)^{k_i}} + \frac{F(z)\sum_{i=1}^{n}k_i\frac{\prod_{j=1}^{n}(z-a_j)^{k_j}}{(z-a_i)}}{F(z)\prod_{i=1}^{n}(z-a_i)^{k_i}} = \frac{F'(z)}{F(z)} + \sum_{i=1}^{n}\frac{k_i}{(z-a_i)}.$$

- Now, certainly it is possible that the function $\frac{F'(z)}{F(z)}$ has poles, which since F and F' are analytic would have to be zeros of $F(z)$. However, zeros of F are isolated and as mentioned before a_i is not a zero of F for all $i \in [n]$ which means that $\frac{F'}{F}$ is analytic in a neighborhood of a_i and so is $\sum_{j\in[n]:j\neq i} \frac{k_j}{(z-a_j)}$ and thus
$$\frac{f'}{f} - \frac{k_i}{(z-a_i)} = \frac{F'}{F} + \sum_{j\in[n]:j\neq i} \frac{k_j}{(z-a_j)}$$
is analytic in a neighborhood of a_i which means by definition that $k_i = Res(\frac{f'}{f}, a_i)$. Also, since
$$(z-a_i)\frac{f'}{f} = k_i + (z-a_i)\frac{F'}{F} + (z-a_i)\sum_{j\in[n]:j\neq i} \frac{k_j}{(z-a_j)}$$
is analytic in a neighborhood of a_i we see that a_i is a pole of order 1 of $\frac{f'}{f}$.
- Now, as mentioned before we need to examine the zeros of $F(z)$. Since $F(z)$ may have infinitely many zeros, we look locally to see the nature of these zeros, which as learned in class are isolated. Of course, there also are finitely many poles $\{a_i\}_{i\in[n]}$ of f, which means that for each zero α of F there exists a $\delta > 0$ such that the neighborhood $B_\delta(\alpha)$ contains no zeros of F and no poles a_i for any $i \in [n]$. We note that locally near α we have that $F(z) = (z-\alpha)^k G(z)$ for $z \in B_\delta(\alpha)$ where $G(z)$ is an analytic function such that $G(\alpha) \neq 0$ and $k \geq 1$. Now, for $z \in B_\delta(\alpha)$ we have $F' = k(z-\alpha)^{k-1}G + (z-\alpha)^k G'$ meaning that
$$\frac{F'}{F} = \frac{k}{(z-\alpha)} + \frac{G'}{G}$$
and thus
$$\frac{f'}{f} = \frac{k}{(z-\alpha)} + \frac{G'}{G} + \sum_{i=1}^{n} \frac{k_i}{(z-a_i)}$$
for $z \in B_\delta(\alpha)$ meaning that
$$\frac{f'}{f} - \frac{k}{(z-\alpha)} = \frac{G'}{G} + \sum_{i=1}^{n} \frac{k_i}{(z-a_i)}$$
is analytic for $z \in B_\delta(\alpha)$ which means by definition that
$$Res(\frac{f'}{f}, \alpha) = k.$$
Once again we see that α is a pole of order 1 of $\frac{f'}{f}$ since
$$(z-\alpha)\frac{f'}{f} = k + (z-\alpha)\frac{G'}{G} + (z-\alpha)\sum_{i=1}^{n} \frac{k_i}{(z-a_i)}$$
is analytic for $z \in B_\delta(\alpha)$ which means by definition that α is a pole of order 1.

Problem 5. Let $\mathcal{H}(\Omega)$ denote the space of analytic functions on a region Ω with the metric ρ constructed in class. Show that for a set $\mathcal{F} \subset \mathcal{H}(\Omega)$ the following conditions are equivalent:
(a) $\mathcal{F}$ is normal;
(b) For every $\epsilon > 0$ there exists a number $c > 0$ such that $\{cf : f \in \mathcal{F}\} \subset B(0, \epsilon)$
(where $B(0, \epsilon)$ denotes the open ball in $(\mathcal{H}(\Omega), \rho)$ with center at 0 and radius ϵ).

We first show that (a) implies (b). Namely, if $\mathcal{F}$ is normal then, as stated in Theorem 15 on page 224 of Ahlfors, that means that the functions in $\mathcal{F}$ are uniformly bounded on every compact subset of Ω.
More, precisely that means that for all compact $E \subseteq \Omega$ we know that there exists $M(E) > 0$ such that

$$\sup_{f \in \mathcal{F}} \sup_{z \in E} (f(z)) \leq M(E).$$

Then we turn to our definition of $\rho(f, 0)$ given on page 220. In particular, first note that as defined on page 220

$$\delta(a, b) = \frac{d(a, b)}{1 + d(a, b)}$$

is a monotone increasing function in $d(a, b)$ and is also bounded above by 1, which means that a uniform bound on the function d gives a uniform bound on the function δ. More, precisely, to prove monotonicity we take the derivative of the function $g(x) = \frac{x}{1+x}$, noting that

$$g'(x) = \frac{1}{1+x} + \frac{-x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0.$$

Furthermore $g(x)$ is bounded above by 1. So, that means the fact that $d(f(z), 0) \leq M(E)$ for all $f \in \mathcal{F}$ and all $z \in E$ implies that there exists $N(E) \in (0, 1)$ such that

$$\delta(f(z), 0) \leq N(E)$$

for all $z \in E$ and all $f \in \mathcal{F}$.
Now, returning to the definition of ρ we first note that since each set E_k is compact we may apply the above argument to obtain real numbers $N(E_k)$ with the above property.
Then, for fixed ϵ we construct the desired $c > 0$ as follows. Choose $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \epsilon/2$. Then, note that since $\delta_k(f, 0) \leq 1$ for all $f \in \mathcal{F}$ we have that

$$\sup_{f \in \mathcal{F}} \rho(f, 0) = \sum_{k \geq 1} \sup_{f \in \mathcal{F}} \delta_k(f, 0) 2^{-k} \leq \sum_{k=1}^{k_0} \sup_{f \in \mathcal{F}} \delta_k(f, 0) 2^{-k} + \sum_{k \geq k_0+1} 2^{-k} = \sum_{k=1}^{k_0} \sup_{f \in \mathcal{F}} \delta_k(f, 0) 2^{-k} + 2^{-k_0}$$

$$< \sum_{k=1}^{k_0} \sup_{f \in \mathcal{F}} \delta_k(f, 0) 2^{-k} + \frac{\epsilon}{2}.$$

Now, using our above knowledge of the relation between d and δ we now note that $g(x) = \frac{x}{1+x} \to 0$ as $x \to 0^+$ which means that for all $\alpha > 0$ there exists $\beta > 0$ such that for all $0 < x < \beta$ one has that $g(x) < \alpha$.

So, now note that by choice of E_k as an increasing set of subsets we have that $\delta_k(f, 0)$ is an increasing function of k for all f which means

$$d(f(z), 0) \leq M(E_k) \leq M(E_{k_0})$$

for all $k \in [k_0]$.
So, let $\alpha = \frac{\epsilon}{2k_0}$. That means there exists $\beta > 0$ as above. Now, we know that we may set c such that

$$cM_{k_0} < \beta.$$

Say we choose

$$c = 0.99 \frac{\beta}{M_{k_0}}.$$

That then implies that

$$\delta_k(cf,0) \leq \delta_{k_0}(cf,0) < \alpha$$

for all $f \in \mathcal{F}$. Then, we have that

$$\sup_{f\in\mathcal{F}} \rho(f,0) < \sum_{k=1}^{k_0} \sup_{f\in\mathcal{F}} \delta_k(f,0)2^{-k} + \frac{\epsilon}{2} < \sum_{k=1}^{k_0} \alpha 2^{-k} + \frac{\epsilon}{2}.$$

Now, since $\alpha = \frac{\epsilon}{2k_0}$ that means that

$$\sum_{k=1}^{k_0} \alpha 2^{-k} = \frac{\epsilon}{2k_0} \sum_{k=1}^{k_0} 2^{-k} \leq \frac{\epsilon}{2k_0} < \frac{\epsilon}{2},$$

and thus,

$$\rho(f,0) < \sum_{k=1}^{k_0} \sup_{f\in\mathcal{F}} \delta_k(f,0)2^{-k} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as desired.

Now, we show that (b) implies (a) by contrapositive. Namely, assume that $\mathcal{F}$ is not normal. That means that there exists some compact set $E \subseteq \Omega$ such that

$$\sup_{f\in\mathcal{F}} \sup_{z\in E} |f(z)| = \infty.$$

Then, that implies that for all $c > 0$ that

$$\sup_{f\in\mathcal{F}} \sup_{z\in E} |cf(z)| = \infty.$$

Note that by construction of E_k, $E \subseteq \Omega$ compact implies that there exists $k_1 \in \mathbb{N}$ such that $E \subseteq E_k$ for all $k \geq k_1$ and of course then that implies that

$$\sup_{f\in\mathcal{F}} \sup_{z\in E_k} |cf(z)| = \infty$$

for all $k \geq k_1$ and all $c > 0$. That implies that for all $k \geq k_1$ we have that

$$\sup_{f\in\mathcal{F}} \delta_k(f,0) = \sup_{f\in\mathcal{F}} \sup_{z\in E_k} \frac{d(cf,0)}{1+d(cf,0)} = \lim_{x\to\infty} \frac{x}{1+x} = 1.$$

So, for all $c > 0$ and all $k \geq k_1$ we have that $\delta_k(f,0) = 1$ meaning that

$$\sup_{f\in\mathcal{F}} \rho(cf,0) = \sum_{k\geq 1} \sup_{f\in\mathcal{F}} \delta_k(cf,0)2^{-k} = \sum_{k=1}^{k_1-1} \sup_{f\in\mathcal{F}} \delta_k(cf,0)2^{-k} + \sum_{k\geq k_1} 2^{-k} \geq 2^{-k_1+1},$$

and thus for any $\epsilon < 2^{-k_1+1}$ there does not exist $c > 0$ such that $\rho(cf,0) < \epsilon$ for all $f \in \mathcal{F}$.