Princeton Complex Analysis Qualifying Exam Questions

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1 Basic Complex Analysis

Question 1.1. *Use the del operator to reformulate the Cauchy–Riemann equations. State the generalized Cauchy–Riemann equations.*

The Cauchy-Riemann equations give a necessary condition for a function, $f : \mathbb{C} \to \mathbb{C}$, to be differentiable at a point.

So, to answer the question, for a given function $f : \mathbb{C} \to \mathbb{C}$, the Cauchy-Riemann equations state that $\nabla(f)(z) \cdot (i, -1) = 0$ if and only if f is differentiable at z. Just in case more precision of notation is desired, I will note that if we write $z = x + iy$ then $f : \mathbb{C} \to \mathbb{C}$ implicitly defines a function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ defined by $\hat{f}(x, y) := f(x + iy)$ and then the notation $\nabla(f)(z)$ is somewhat incorrect or ambiguous and really denotes $\nabla(f)(z) = \nabla(\hat{f})(Re(z), Im(z))$.

It is important to note that the satisfaction of the Cauchy-Riemann equations is also sufficient for f to be differentiable if the relevant partial derivatives, $\Re(f)_x, \Re(f)_y, \Im(f)_x, \Im(f)_y$, are continuous. It is to be understood that in that statement all conditions apply to a specific point z_0, e.g. the CR equations are satisfied at z_0, f is differentiable at z_0, and the relevant partial derivatives are continuous functions of x, y at the specific point $z_0 = x_0 + iy_0$.

I will note in passing that differentiability of f at a point, z, is related to the notion of f being holomorphic at that point but is a weaker condition since a function f is holomorphic at a point, z, only if there is a neighborhood, U, of z such that f is differentiable at all points $u \in U$. Put simply and intuitively, differentiability is a condition that depends only on the given point whereas the condition of being holomorphic depends on a neighborhood around the point. There are functions f which are, for instance, differentiable at every point on the imaginary axis since the satisfy the Cauchy-Riemann equations there and have continuous partial derivatives $\Re(f)_x, \Re(f)_y, \Im(f)_x, \Im(f)_y$, but who fail to be holomorphic at any point on the imaginary axis since f is not differentiable at points to the left and right of the imaginary axis and any neighborhood of a point on the imaginary axis contains some of these points. In summary, the Cauchy-Riemann equations are concerned with differentiability, and as such they give information about f only at a specific point.

Question 1.2. *What's the radius of convergence of the Taylor series of* $1/(z^2 + 1)$ *at* 100*?*

It is $\sqrt{10001}$. In particular, using the partial fraction decomposition we see that $\frac{1}{z^2+1} = \frac{\frac{i}{2}}{z+i} + \frac{\frac{-i}{2}}{z-i}$. Using simple algebra we can rewrite that as $\frac{1}{z^2+1} = \frac{\frac{i}{2}}{i+100+(z-100)} + \frac{\frac{-i}{2}}{-i+100+(z-100)}$

whose power series $\frac{i}{2}\frac{1}{i+100}\sum_{n\in\mathbb{N}}\left(\frac{-(z-100)}{i+100}\right)^n+\frac{-i}{2}\frac{1}{-i+100}\sum_{n\in\mathbb{N}}\left(\frac{-(z-100)}{-i+100}\right)^n$ converges to it when $\left|\frac{z-100}{i+100}\right|<1$ and $\left|\frac{z-100}{-i+100}\right|<1$ or equivalently when $|z-100|<\sqrt{10001}$.

Question 1.3. *What's the orientation of curves if you do a line integral on the boundary of an annulus?*

The orientation is so that if you are traversing the curve, then the region is always on your left. So, the outer circle is oriented counterclockwise and the inner circle is oriented clockwise.

Question 1.4. *How do you define the residue of a function at a pole? What's it good for? What other consequences are there?*

If the function f has a pole of order n at a. Then, $Res(f,a)=\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}((z-a)^n f(z))\right)\Big|_{z=a}$.

Question 1.5. *How would you integrate*

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^4)^2}?$$

Concepts, not calculations.

We note that the desired integral is $I=\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(1+x^4)^2}$. Now, we recognize this as part of the integral of $\frac{1}{(1+x^4)^2}$ over the counterclockwise region bounding the semi-circle $S_R:=D(0,R)\cap\{z\in\mathbb{C}: Im(z)\geq 0\}$. We then show that the integral of $\frac{1}{(1+x^4)^2}$ along the upper arc of the semi-circle tends to zero as $R\to\infty$. So, applying the Residue Theorem and letting $R\to\infty$ we recognize the stated integral has value $I=2\pi i(Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}}+Res(\frac{1}{(1+z^4)^2})|_{z=e^{i3\pi/4}})$. We flesh out these details now. Here $R\geq 2$.

- First, note that $\partial S_R=C_{1,R}+C_{2,R}$ where $C_{1,R}:[0,1]\to\mathbb{C}$ is the curve defined by $C_{1,R}(t)=-R+2\pi Rt$ (the particular parameterization of the curve is does not matter, but we must choose one so we choose the above) and $C_{1,R}:[0,1]\to\mathbb{C}$ is the curve defined by $C_{2,R}(t)=Re^{i\pi t}$.

- We directly compute an upper bound on the absolute value of the integral along $C_{2,R}$. In particular, note that $\int_{C_{2,R}}\frac{1}{(1+z^4)^2}dz=\int_0^1\frac{Ri\pi e^{i\pi t}}{(1+(Re^{i\pi t})^4)^2}dt$ and thus $|\int_{C_{2,R}}\frac{1}{(1+z^4)^2}dz|\leq\int_0^1|\frac{Ri\pi e^{i\pi t}}{(1+(Re^{i\pi t})^4)^2}|dt\leq\int_0^1\frac{R\pi}{(R^4-1)^2}dt=\frac{R\pi}{(R^4-1)^2}$.

- So, we see that $\lim_{R\to\infty}|\int_{C_{2,R}}\frac{dz}{(1+z^4)^2}|\leq\lim_{R\to\infty}\frac{R\pi}{(R^4-1)^2}=0$ and thus $\lim_{R\to\infty}\int_{C_{2,R}}\frac{dz}{(1+z^4)^2}=0$.

- Now, by the Residue Theorem $\int_{C_{1,R}+C_{2,R}}\frac{dz}{(1+z^4)^2}=2\pi i(Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}}+Res(\frac{1}{(1+z^4)^2})|_{z=e^{i3\pi/4}})$ for all $R\geq 2$. So, we also have that $\lim_{R\to\infty}(\int_{C_{1,R}+C_{2,R}}\frac{dz}{(1+z^4)^2})=I=2\pi i(Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}}+Res(\frac{1}{(1+z^4)^2})|_{z=e^{i3\pi/4}})$.

- As a final step we compute the value of $2\pi i(Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}}+Res(\frac{1}{(1+z^4)^2})|_{z=e^{i3\pi/4}})$. Namely, note that $\frac{1}{(1+z^4)^2}=(\frac{1}{((z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4}))^2}$ and thus $Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}}=\frac{d}{dz}(\frac{1}{((z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4}))^2})|_{z=e^{i\pi/4}}$. Now, note that $\frac{1}{((z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4}))^2}=$

$\frac{1}{((z-(\frac{1-i}{\sqrt{2}}))(z-(\frac{-1-i}{\sqrt{2}}))(z-(\frac{1-i}{\sqrt{2}})))^2} = \frac{1}{((z^2+\sqrt{2}iz+1)(z-(\frac{1-i}{\sqrt{2}})))^2} = \frac{1}{(z^3+(\sqrt{2}i+\frac{-1+i}{\sqrt{2}})z^2+(-i)z-(\frac{1-i}{\sqrt{2}}))^2} = \frac{1}{(z^3+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)z^2+(-i)z-(\frac{1-i}{\sqrt{2}}))^2}$. So, $\frac{d}{dz}(\frac{1}{((z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4}))^2}) = \frac{d}{dz}(\frac{1}{(z^3+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)z^2+(-i)z-(\frac{1-i}{\sqrt{2}}))^2}) = \frac{-2}{(z^3+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)z^2+(-i)z-(\frac{1-i}{\sqrt{2}}))^3}(3z^2+(\sqrt{2}+2\sqrt{2}i)z-i)$ which tells us that $Res(\frac{1}{(1+z^4)^2})|_{z=e^{i\pi/4}} = \left(\frac{-2}{(z^3+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)z^2+(-i)z-(\frac{1-i}{\sqrt{2}}))^3}(3z^2+(\sqrt{2}+2\sqrt{2}i)z-i)\right)\Big|_{z=e^{i\pi/4}=\frac{1+i}{\sqrt{2}}} = \frac{-2}{(e^{i3\pi/4}+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)e^{i\pi/2}+(-i)(\frac{1+i}{\sqrt{2}})-(\frac{1-i}{\sqrt{2}}))^3}$ $(\sqrt{2}+2\sqrt{2}i)\frac{1+i}{\sqrt{2}} - i) = \frac{-2}{(\frac{-1+i}{\sqrt{2}}+(\frac{-\sqrt{2}}{2}+\sqrt{2}i)i+(-i)(\frac{1+i}{\sqrt{2}})-(\frac{1-i}{\sqrt{2}}))^3}(3i + (\sqrt{2}+2\sqrt{2}i)\frac{1+i}{\sqrt{2}} - i) = \frac{-2}{(-\frac{3\sqrt{2}}{2})^3}(-1+5i) = \frac{2^{5/2}}{3^3}(-1+5i) = -\frac{2^{5/2}}{27} + 5\frac{2^{5/2}}{27}i$. The computation of the second residue would be similar, but since the prompt says concepts, not computations, I assume I should stop here now that the process for answering the question is completely illustrated.

Question 1.6. *If f is meromorphic, what is the meaning of the contour integral of f'/f?*

It allows us to count the number of zeros contained within the contour minus the number of poles contained within the contour, both counted with multiplicity also accounting for their index with respect to the curve. Namely, for any function f which is meromorphic on an open set $\Omega \subseteq \mathbb{C}$ and any closed contour $C \subseteq \Omega$ which is contractible to a point within Ω (that condition could fail for the contour $C = \{z \in \mathbb{C} : |z| = 2\}$ within the annulus $\Omega = \{z \in \mathbb{C} : 1 < |z| < 3\}$) one has that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{\{a\in\Omega: f(a)=0\}} n(a, C)\text{mult}(a, f) - \sum_{\{b\in\Omega: f(b)=\infty\}} n(b, C)\text{ord}(b, f)$ where $\text{mult}(a, f) \in \mathbb{N}$ is defined for a a zero of f as the unique $n \in \mathbb{N}$ such that $\frac{f(z)}{(z-a)^n} \neq 0$ and $\frac{f(z)}{(z-a)^n} \neq \infty$. Similarly $\text{ord}(b, f)$ is defined for a pole b of f as the unique $n \in \mathbb{N}$ such that $f(z)(z-a)^n \neq 0$ and $f(z)(z-a)^n \neq \infty$.

Question 1.7. *Under what conditions is the pointwise limit of holomorphic functions holomorphic?*

If the sequence $\{f_n\}$ of holomorphic functions "converges compactly" to f then f is holomorphic. Note: "Converges compactly" means that $\{f_n\}$ converges uniformly to f on any compact subset of the domain.

Question 1.8. *What do you know about several complex variables?*

Osgood's Theorem says that any function $f : \mathbb{C}^n \to \mathbb{C}$ that is a continuous function from $\mathbb{C}^n \to \mathbb{C}$ and is holomorphic in each individual complex variable is holomorphic. There is another, more difficult to prove theorem that removes the requirement that f be a continuous function of all n variables collectively (but of course still uses the fact that f is continuous in each individual variable, which is true as a result of f being holomorphic in each individual variable).

Definition: A function, $f : \mathbb{C}^n \to \mathbb{C}$, of several variables is "holomorphic" if, for each point $(z_1^0, z_2^0, \ldots, z_n^0)$ in the domain, $f(z_1, z_2, \ldots, z_n)$ has a multi-variable power series that converges to f in a neighborhood of $(z_1^0, z_2^0, \ldots, z_n^0)$.

2 Entire Functions

Question 2.1. *What is your favorite proof of Liouville's theorem?*

- We use Cauchy's Integral Formula.
- Note that for all $z_0 \in \mathbb{C}$ and all $R > 0$ we have that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D(z_0,R)} \frac{f(z)}{z - z_0} dz$$

  and thus

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial D(z_0,R)} \frac{f(z)}{(z - z_0)^{k+1}} dz.$$

  Note that since f is bounded there exists $M \in \mathbb{R}_{\geq 0}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Now, since every entire function is analytic, we have that $f(z) = \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} z^k$.

- Now, we see that

$$|f^{(k)}(z_0)| \leq \frac{k!}{2\pi i} \int_{\partial D(z_0,R)} |\frac{f(z)}{(z - z_0)^{k+1}}||dz| \leq \frac{k!}{2\pi} \frac{M}{R^{k+1}} \int_{\partial D(z_0,R)} |dz| = \frac{Mk!}{R^k}.$$

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- The inequality $|f^{(k)}(z_0)| \leq \frac{Mk!}{R^k}$ holds for all $R > 0$ and all $z_0 \in \mathbb{C}$. So, letting R tend to infinity we see that $|f^{(k)}(0)| = 0$ and thus

$$f(z) = f(0) + \sum_{k \geq 1} 0 z^k = f(0).$$

Question 2.2. *If f is entire and $\Re(f)$ is bounded, does f have to be constant?*

Yes. Note that either $\Re(f)$ is constant or not. What we actually show is that $\Re(f)$ bounded implies that $\Re(f)$ is constant which implies by the Cauchy-Riemann equations that $\Im(f)$ is constant and thus so is f. (In particular, $0 = \Re(f)_x(z) = \Im(f)_y(z)$ and $0 = \Re(f)_y(z) = -\Im(f)_x(z)$ for all $z \in \mathbb{C}$ implies $\Im(f)$ is constant since the directional derivative, $u \cdot (\Im(f)_x(z), \Im(f)_y(z))$ in the direction of any unit vector u, at any point z is zero). Our proof that $\Re(f)$ is constant is actually a proof that any bounded harmonic function is constant.

- Recall that for any harmonic function $G : \mathbb{R}^2 \to \mathbb{R}$ and any $z_0 \in \mathbb{R}^2$ one has that

$$G(z_0) = \frac{1}{Vol(B(z_0, R))} \int_{B(z_0,R)} G(z) dA$$

  where $\partial(B(z_0, R)) = \{z \in \mathbb{C} : |z - z_0| = R\}$ is the boundary of the disc $B(z_0, R)$ oriented counterclockwise.

- We choose two arbitrary $z_1, z_2 \in \mathbb{C}$ and show that $G(z_1) = G(z_2)$. Namely, we first consider radii R_1, R_2 such that $B(z_1, R_1) \subseteq B(z_2, R_2)$ which allows us to show that $G(z_1) \leq G(z_2)$ and then pick different radii S_1, S_2 such that $B(z_2, S_2) \subseteq B(z_1, S_1)$ which allows us to show that $G(z_2) \leq G(z_1)$.

- To flesh out those details, let $R_1 = R$ where $R > 0$ is arbitrary (we will soon let it tend to infinity). Let $R_2 = R + |z_1 - z_2|$. Since $|G| \leq M$ is bounded, that implies that $G \geq -M$ and thus $G + M$ is a non-negative, harmonic function (adding a constant to a harmonic function produces a harmonic function).

- Then, by monotonicity of a non-negative integral we see that $\int_{B(z_1,R_1)} GdA \leq \int_{B(z_2,R_2)} GdA$. Note that the left side is exactly $Vol(B(z_1, R_1))G(z_1)$ and the right hand side is exactly $Vol(B(z_2, R_2))G(z_2)$. So, the result we obtain is that $G(z_1) \leq \frac{Vol(B(z_2,R_2))}{Vol(B(z_1,R_1))} G(z_2) = (\frac{R_2}{R_1})^2 G(z_2)$. Note that $(\frac{R_2}{R_1})^2 \to 1$ as $R \to \infty$ and thus $G(z_1) \leq G(z_2)$.

- Then, following the above process with radii $S_2 = R$ and $S_1 = R + |z_1 - z_2|$ we obtain the reverse inequality and thus $G(z_2) \leq G(z_1)$, which means $G(z_1) = G(z_2)$.

- So, we have shown any bounded harmonic function, such as $\Re(f)$ for instance, is constant, which by our argument given at the beginning implies f is constant.

Question 2.3. *How does the complex function* $\sin z$ *differ from the real function* $\sin x$*? On the real line,* $\sin^2 x + \cos^2 x = 1$*, but in the complex plane, is* $\sin^2 z + \cos^2 z$ *even bounded? Are the given definitions for* $\sin z$ *and* $\cos z$ *the right ones? Why?*

They differ in that $\sin(z)$ is not bounded. However, they are similar in that both are periodic. Namely, $\sin(x)$ is 2π periodic and $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix} - e^{y-ix}}{2i} = \frac{e^{-y}e^{ix} - \frac{e^y}{e^{ix}}}{2i}$ is 2π periodic in $x = Re(z)$. The behavior could be summarized by saying that $Re(z)$ controls the argument of $\sin(z)$ and $Im(z)$ controls the magnitude of $\sin(z)$.

Now, we note that $\sin^2(z) + \cos^2(z) = (\frac{e^{iz} - e^{-iz}}{2i})^2 + (\frac{e^{iz} + e^{-iz}}{2})^2 = \frac{-1}{4}(e^{2iz} - 2 + e^{-2iz}) + \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) = 1$. They are the right ones in the sense that both are 2π periodic in $\Re(z)$ and satisfy $\sin^2(z) + \cos^2(z) = 1$. Also, perhaps most importantly, the $\sin(z)$ and $\cos(z)$ functions have the same exact Taylor series as $\sin(x)$ and $\cos(x)$ on the real line respectively.

3 Singularities

Question 3.1. *On an annulus, if a function's Laurent series has infinitely many terms in the principal part, can you say it has an essential singularity?*

No. We can show so by constructing an example of a function whose Laurent series converges on an annulus, which we do by adding together two functions, one whose Laurent series converges outside some disc and another whose Laurent series converges inside some smaller disc. In particular, let $f_1(z) = \frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}$ whose Laurent series $\sum_{n\in\mathbb{N}} \frac{1}{z^n}$ converges to f_1 when $|z| > 1$. Now, consider $f_2(z = \frac{1}{2-z} = \frac{\frac{1}{2}}{1-\frac{z}{2}}$ whose Laurent series $\sum_{n\in\mathbb{N}} \frac{1}{2}(\frac{z}{2})^n$ converges to f_2

on the disc $|z| < 2$. Thus, if we let $f(z) = f_1(z) + f_2(z) = \frac{z}{z-1} + \frac{1}{2-z} = \frac{-z^2+3z-1}{(z-1)(2-z)}$, we see that the Laurent series $\sum_{n\in\mathbb{N}} \frac{1}{z^n} + \sum_{n\in\mathbb{N}} \frac{1}{2}(\frac{z}{2})^n$ converges to f on the annulus $1 < |z| < 2$. Indeed, the given Laurent series has infinitely many terms, but the function f has no singularities on the annulus at all.

Question 3.2. *Talk about the singularity Logz has at* 0*.*

It has a branch singularity. It is not isolated. To elaborate more on that, technically $log(z)$ is not a function since it is multi-valued due to the argument multi-function arg being multi-valued. However, we may pick a single-valued branch for the function Arg (which means in precise mathematics that we stipulate $a \leq Arg(z) < b$ for some chosen $a, b \in \mathbb{R}$ with $b - a = 2\pi$) allows us to define a single-valued function $Log(z) = ln|z| + iarg(z)$ on $\mathbb{C} \setminus \{0\}$. While well-defined, it will be discontinuous at all z on the branch cut which is a ray from the origin, namely along the ray $\{z \in \mathbb{C} : arg(z) = a\}$. Finally, even after mitigating the issue of the argument not being single valued, it is even more unclear what $arg(0)$ should be. One could assign it any value between a and b, but this choice is not canonical. One important consequence of that fact is that the singularity at 0 is not removable, i.e. there is no $z_0 \in \mathbb{C}$ such that assigning $Log(0) := z_0$ results in $Log(z)$ being holomorphic in a neighborhood of the origin.