MATH 6338 (Prof. Heil) HOMEWORK 6 Due: April 17, 2020

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Suppose that X is a normed space and X^* is separable.

(a) Prove that there is a countable set $\{\lambda_n\}_{n\in\mathbb{N}}$ in X^* that is dense in the closed unit sphere

$$D^* = \{\mu \in X^* : \|\mu\| = 1\}.$$

Since X^* is separable, it has a countable dense subset. Call it $\{\alpha_n\}_{n\in\mathbb{N}}$. Now, let $\{\lambda_n\}_{n\in\mathbb{N}} = \{\frac{\alpha_n}{||\alpha_n||}\}_{n\in\mathbb{N}}$. I claim that $\{\lambda_n\}_{n\in\mathbb{N}}$ is dense in D^* and show so using the triangle inequality and continuity of the norm. Namely, for each $\mu \in X^*$ and all $\epsilon > 0$ I would like to find λ_{n_ϵ} such that $||\mu - \lambda_{n_\epsilon}|| < \epsilon$. Well, consider the original set $\{\alpha_n\}_{n\in\mathbb{N}}$ and note that there existed a sequence $\{\alpha_{n_i} : i \in \mathbb{N}\}$ with $\alpha_{n_i} \to \mu$ in norm, or specifically that $||\alpha_{n_i} - \mu|| \to 0$ as $i \to \infty$. So, there exists $m(\epsilon/2) \in \mathbb{N}$ such that $||\alpha_m - \mu|| < \frac{\epsilon}{2}$ for all $m \geq m(\epsilon/2)$. Also, by continuity of the norm, we know that for this sequence, since $||\mu|| = 1$ we have that $||\alpha_{n_i}|| \to 1$ as $i \to \infty$. Then, since $||\alpha_{n_i}|| \to 1$ one has that $||\alpha_{n_i} - \frac{\alpha_{n_i}}{||\alpha_{n_i}||}|| = (1 - \frac{1}{||\alpha_{n_i}||})||\alpha_{n_i}|| \to 0$ as $i \to \infty$. So, there exists $k(\epsilon/2) \in \mathbb{N}$ such that $||\alpha_k - \frac{\alpha_k}{||\alpha_k||}|| < \epsilon/2$ for all $k \geq k(\epsilon/2)$. Now, let $N(\epsilon) = max(m(\epsilon/2), k(\epsilon/2))$. Now, for all $n \geq N(\epsilon)$ we have that $||\frac{\alpha_n}{||\alpha_n||} - \mu|| \leq ||\frac{\alpha_n}{||\alpha_n||} - \alpha_n|| + ||\alpha_n - \mu|| < \epsilon$. So, since by definition $\lambda_n = \frac{\alpha_n}{||\alpha_n||}$ the set is dense in D^*.

(b) For each $n \in \mathbb{N}$, find a unit vector $x_n \in X$ such that $|\lambda_n(x_n)| \geq 1/2$. Well, since $||\lambda_n|| = 1$ we know that $1 = \sup_{||x||=1} |\lambda_n(x)|$ which means that for all $\epsilon > 0$ there exists x^n_ϵ with $||x^n_\epsilon|| = 1$ such that $|\lambda_n(x^n_\epsilon)| > 1 - \epsilon$. Thus, set $\epsilon = 1/2$ and the result follows by choosing $x^n_{1/2}$. Let $M = \overline{span}\{x_n\}_{n\in\mathbb{N}}$. This is a closed subspace of X, and by Problem 7.4.6 we know that M is separable.

(c) Suppose that $M \neq X$, and use the Hahn–Banach Theorem to derive a contradiction. Conclude that $X = M$, and therefore X is separable.

Note that then by Corollary 19.1.5 to the Hahn Banch Theorem we know that since there exists $x_0 \in X \setminus M$ that then $x_0' = x_0/||x_0||$ is a unit vector in X and thus there exists a functional $\mu \in X^*$ such that $|\mu(x_0')| = 1$, $\mu|_M = 0$, and $||\mu|| = 1/d$. Now, let $\alpha = d\mu$ which means that $\alpha(x_0') = d$ $\alpha|_M = 0$ and $||\alpha|| = 1$. Now, note that for $\epsilon = 1/4$ there exists $\lambda_k \in \{\lambda_n\}_{n\in\mathbb{N}}$ such that $||\lambda_k - \alpha|| < 1/4$. However, note that

$$||(\lambda_k - \alpha)x_k = ||\lambda_k(x_k) - 0|| \geq 1/2$$

which implies that $||\lambda_k - \alpha|| \geq 1/2$, a contradiction.

2. Exercise 19.2.5. Let X and Y be Banach spaces, and fix a bounded linear operator $A \in \mathcal{B}(X, Y)$.

(a) Choose $\mu \in Y^*$, and define a functional $A^*\mu \colon X \to \mathbf{F}$ by

$$(A^*\mu)(x) \;=\; \mu(Ax), \qquad \text{for } x \in X.$$

Show that $A^*\mu$ is linear and bounded, and therefore $A^*\mu \in X^*$.

Note that $A^*\mu(x+y) = \mu(A(x+y)) = \mu(Ax+Ay) = \mu(Ax)+\mu(Ax) = A^*\mu(x)+A^*\mu(y)$ and also $A^*\mu(kx) = \mu(kAx) = k\mu(Ax) = kA^*\mu(x)$ for all $k \in F$ thus proving linearity. Furthermore note that

$$||A^*\mu|| = \sup_{||x||=1} \mu(Ax) \le \sup_{||x||=1} ||\mu|| ||Ax|| \le \sup_{||x||=1} ||\mu|| ||A|| = ||\mu|| ||A||$$

thus proving boundedness.

(b) Show that the mapping $A^* \colon \mu \mapsto A^*\mu$ is a bounded linear mapping of Y^* into X^*, and the operator norm of this mapping is $\|A^*\| = \|A\|$. I show that it is bounded by showing that the operator norm satisfies the above equality. Namely, we calculate

$$||A^*|| = \sup_{\mu \in Y^*:||\mu||=1} ||A^*\mu|| = \sup_{\mu \in Y^*:||\mu||=1} \Big(\sup_{x \in X:||x||=1} |A^*\mu(x)|\Big)$$
$$= \sup_{\mu \in Y^*:||\mu||=1} \Big(\sup_{x \in X:||x||=1} |\mu(Ax)|\Big) = \sup_{x \in X:||x||=1} \Big(\sup_{\mu \in Y^*:||\mu||=1} |\mu(Ax)|\Big) = \sup_{x \in X:||x||=1} ||Ax|| = ||A||.$$

where the second to last equality holds by Corollary 19.1.4 to the Hahn-Banach Theorem.

(c) Prove that A^* is the unique operator mapping Y^* into X^* that satisfies

$$\mu(Ax) \;=\; (A^*\mu)(x), \qquad \text{for all } x \in X \text{ and } \mu \in Y^*. \qquad \diamondsuit$$

This is true almost entirely by definition. Assume that there are two operators A^*, B^* which satisfy

$$\mu(Ax) = (A^*(\mu))(x) = (B(\mu))(x)$$

Then, note that $B(\mu) = A^*(\mu) = \mu \circ A$ for all $\mu \in Y^*$ and thus $B = A^*$.

3. Exercise 19.2.12. Let X and Y be Banach spaces, and let $T \colon X \to Y$ be a topological isomorphism. Prove that the adjoint map $T^* \colon Y^* \to X^*$ is a topological isomorphism, and

$$(T^{-1})^* \;=\; (T^*)^{-1}.$$

I first prove that $(T^{-1})^* = (T^*)^{-1}$ and use that to prove that T^* is a topological isomorphism. In particular I show that $((T^{-1})^* \circ T^*)(\mu) = \mu$ and $(T^* \circ (T^{-1})^*)(\alpha) = \alpha$ for all $\mu \in Y^*$ and all $\alpha \in X^*$. Indeed note that $((T^{-1})^*(T^*(\mu))(y) = (T^*(\mu))(T^{-1}(y)) = \mu(T(T^{-1}(y))) = \mu(y)$. Also, we have that $(T^*((T^{-1})^*(\mu))(y) = ((T^{-1})^*(\mu))(T(y)) = \mu(T^{-1}T(y)) = \mu(y)$.

Now, we show that if $T : X \to Y$ is a topological isomorphism for space some Banach spaces X, Y, then one has that $T^* : Y^* \to X^*$ is a topological isomorphism as well thus proving that T^* is continuous and that $(T^*)^{-1} = (T^{-1})^*$ is continuous by applying the claim to the topological isomorphism $T^{-1} : Y \to X$.

Proof: We must show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $||\alpha - \beta||_{Y^*} < \delta$ implies that $||T^*\alpha - T^*\beta||_{X^*} < \epsilon$. Indeed, note that

$$||T^*\alpha - T^*\beta||_{X^*} = \sup_{||x||=1} |(T^*\alpha - T^*\beta)(x)| = \sup_{||x||=1} |(T^*(\alpha - \beta))(x)|$$

$$= \sup_{||x||=1} |((\alpha - \beta)(T(x)))| \leq \sup_{||x||=1} ||\alpha - \beta||_{Y^*}|T(x)| \leq ||\alpha - \beta||_{Y^*}||T||.$$

So, let $\delta = \epsilon/||T||$ and then indeed the claim follows.

4. Problem 19.2.14. Let M be a closed subspace of a Banach space X. Let $\epsilon \colon M \to X$ be the embedding map defined by $\epsilon(x) = x$ for $x \in M$. Prove that $\epsilon^* \colon X^* \to M^*$ is the restriction map defined by $\epsilon^*\mu = \mu|_M$ for $\mu \in X^*$.

Well, note that $\mu|M(m) = \mu(\epsilon(m))$ for all $m \in M$ and all $\mu \in X^*$ and thus by problem 2(c) we know that $\epsilon^*(\mu) = \mu|M$.