MATH 6338 (Prof. Heil) FINAL EXAM Due: 5:30 p.m., April 30, 2020

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only. Upload your solutions to Canvas. This exam is worth a total of 50 points.

1. Problem 20.1.17.

(a) Prove that if $X$ is an infinite-dimensional Banach space, then every Hamel basis for $X$ is uncountable.

Otherwise, if it has a countable (Hamel) basis $\{e_n\}_{n\in\mathbb{N}}$ that means that

$$X = \cup_{N=0}^{\infty} span(\{x_0, x_1, \ldots, x_N\}).$$

Then, by the Baire Category Theorem we know that at least one of the sets

$$E_{N'} := span(\{x_0, x_1, \ldots, x_{N'}\})$$

is not nowhere dense. However, that is a contradiction since each space

$$E_N := span(\{x_0, x_1, \ldots, x_{N'}\})$$

is a finite-dimensional subspace which is thus proper and closed and is thus nowhere dense as noted on page 242 of the textbook.

(b) Let $X$ be an infinite-dimensional Banach space. Suppose that $M$ is an infinite-dimensional subspace of $X$ that has a countable Hamel basis. Prove that $M$ is a meager subset of $X$, and $M$ is not closed.

Note that by part (a) the fact that $M$ has a countable Hamel basis implies that $M$ is not complete. The fact that it is not complete implies that it is not closed since incompleteness means that $\overline{M} \supsetneq M$ and since $\overline{M} = \cap\{Y \subseteq X : Y \subseteq X \text{ closed and } Y \supseteq M\}$ is the smallest closed set containing $M$ that also means that $M$ is not closed.

Furthermore, $M$ is meager since if $\{h_n\}_{n\in\mathbb{N}}$ is our Hamel basis for $M$ then

$$M = \cup_{N\in\mathbb{N}} span(\{h_0, h_1, \ldots, h_N\})$$

and since $span(\{h_0, h_1, \ldots, h_N\})$ is a proper (of course $span(\{h_0, h_1, \ldots, h_N\}) \subseteq M \subsetneq X$) since if $M = X$ then $M$ would not have a countable Hamel basis), finite-dimensional subspace for each $N \in \mathbb{N}$ we have that $span(\{h_0, h_1, \ldots, h_N\})$ is nowhere dense for all $N \in \mathbb{N}$, meaning by definition that $M$ is meager.

(c) Prove that $C_c(\mathbb{R})$ is a meager, dense subspace of $C_0(\mathbb{R})$.

Indeed we see that $C_c(\mathbb{R})$ is meager since

$$C_c(\mathbb{R}) = \cup_{N\in\mathbb{N}} C_N(\mathbb{R})$$

where $C_N(\mathbb{R}) := \{f \in C_c(\mathbb{R}) : f(x) = 0 \forall x \in \mathbb{R} \text{ such that } |x| > N\}$. Note that $C_N(\mathbb{R})$ is nowhere dense in $C_0(\mathbb{R})$ for each $N \in \mathbb{N}$ since $\overline{C_N(\mathbb{R})}$ has empty interior. Otherwise, there would exist an open ball contained in $\overline{C_N(\mathbb{R})}$ meaning that for some $f \in \overline{C_N(\mathbb{R})}$ there exists $r > 0$ such that $g \in \overline{C_N(\mathbb{R})}$ for all $g \in C_0(\mathbb{R})$ with $||g - f|| < r$. However, that fails. Namely, take

$$g(x) = \chi_{||x|| \le N} f(x) + max(0, -r|x - (N+1)| + r) + max(0, -r|x - (-N-1)| + r) \in C_0(\mathbb{R})$$

and note that

$$(g-f)(x) = max(0, -r|x-(N+1)|+r) + max(0, -r|x-(-N-1)|+r)$$

and also that

$$||g-f|| = \sup_{x\in\mathbb{R}}(max(0, -r|x-(N+1)|+r) + max(0, -r|x-(-N-1)|+r)) = r$$

yet clearly $g \notin \overline{C_N(\mathbb{R})}$ since $g$ is not a limit points of functions $f_n \in C_N(\mathbb{R})$ because for any sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq C_N(\mathbb{R})$ one has that

$$(\lim_{n\to\infty} f_n)(N+1) = \lim_{n\to\infty}(f_n(N+1)) = \lim_{n\to\infty} 0 = 0 < r = g(N+1).$$

Thus, $C_c(\mathbb{R})$ is a meager subspace.

However, it is a dense subspace since for any $h \in C_0(\mathbb{R})$ we may find a sequence of functions $g_n \in C_c(\mathbb{R})$ so that

$$\lim_{n\to\infty} ||g_n - h||_u = 0.$$

Namely, for $n \in \mathbb{N}$ let

$$g_n(x) = \chi_{|x|\le n}h(x) + \chi_{|x|\ge n+1}0 + \chi_{-n-1<x<-n}(h(-n))(x+n+1) + \chi_{n<x<n+1}(-h(n))(x-n-1)$$

and note that since $h \in C_0(\mathbb{R})$ meaning that $\lim_{|x|\to\infty}|h(x)| = 0$ we have that

$$\begin{aligned}
\lim_{n\to\infty} ||g_n - h||_u &= \lim_{n\to\infty} ||(\chi_{|x|\le n}h(x) + \chi_{|x|\ge n+1}0 + \chi_{-n-1<x<-n}(h(-n))(x+n+1) \\
&+ \chi_{n<x<n+1}(-h(n))(x-n-1)) - h(x)|| \\
&= \lim_{n\to\infty}\sup_{x\in\mathbb{R}} |\chi_{|x|\le n}0 + \chi_{|x|\ge n+1}(-h(x)) + \chi_{-n-1<x<-n}((h(-n))(x+n+1) - h(x)) \\
&+ \chi_{n<x<n+1}((-h(n))(x-n-1) - h(x))| \\
&= \lim_{n\to\infty}(max\Big(\sup_{x\in\mathbb{R}:||x||\ge n+1}|(-h(x))|, \sup_{-n-1<x<-n}|(h(-n))(x+n+1) - h(x)|, \\
&\sup_{n<x<n+1}|(-h(n))(x-n-1) - h(x)|\Big)) \\
&= max\Big(\lim_{n\to\infty}\big(\sup_{x\in\mathbb{R}:||x||\ge n+1}|(-h(x))|\big), \lim_{n\to\infty}\big(\sup_{-n-1<x<-n}|(h(-n))(x+n+1) - h(x)|\big), \\
&\lim_{n\to\infty}\big(\sup_{n<x<n+1}|(-h(n))(x-n-1) - h(x)|\big)\Big)
\end{aligned}$$

but since $\lim_{|x|\to\infty}|h(x)| = 0$ that means that for all $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that $|h(x)|$ for all $x \in \mathbb{R}$ with $|x| \geq M(\epsilon)$. Thus, continuing on we see

$$\leq max\Big(\epsilon, \lim_{n\to\infty}\big(\sup_{-n-1<x<-n}|(h(-n))(x+n+1)| + |h(x)|\big),$$

$$\lim_{n\to\infty}\big(\sup_{n<x<n+1}|(-h(n))(x-n-1)| + |h(x)|\big)\Big)$$

$$\leq max\Big(\epsilon, \lim_{n\to\infty}\big(\sup_{-n-1<x<-n}|(h(-n))| + |h(x)|\big), \lim_{n\to\infty}\big(\sup_{n<x<n+1}|(-h(n))| + |h(x)|\big)\Big)$$

(since $0 \leq |x+n+1| \leq 1$ for $x \in [-n-1,-n]$ and $0 \leq |x-n-1| \leq 1$ for $x \in [n, n+1]$)

$$\leq max\Big(\epsilon, 2\epsilon, 2\epsilon\Big)$$

$$\leq 2\epsilon,$$

but now since $\epsilon > 0$ was arbitrary we see that

$$\lim_{n\to\infty}||g_n - h||_u = 0,$$

proving that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

2. Let $X$ and $Y$ be Banach spaces. Prove that $A \in \mathcal{B}(X, Y)$ is surjective if and only if range$(A)$ is not a meager subset of $Y$.

Hint: Lemma to the Open Mapping Theorem.

First, note that if $A \in \mathcal{B}(X, Y)$ is surjective, then $range(A) = Y$ then by the Baire Category Theorem since $Y$ is a complete metric space we know that $Y = range(A)$ is not a meager subset of $Y$.

Now, assume that $range(A)$ is not a meager subset of $Y$. That means in particular that $range(A)$ is not nowhere dense meaning that $Y \setminus (\overline{range(A)})$ is not dense in $Y$ meaning that there exists a ball $B_s^Y(y_0) \subseteq Y \setminus (Y \setminus (\overline{range(A)})) = \overline{range(A)}$.

Now, one has that

$$\sup_{z\in B_s^Y(y_0)}||z|| \leq \sup_{z\in B_s^Y(y_0)}(||y_0|| + ||z - y_0||) \leq ||y_0|| + s.$$

Furthermore, the inverse mapping tells us that $||A^{-1}|| < \infty$ which means that for all $z \in B_s^Y(y_0) \cap range(A)$ we have

$$||A^{-1}(z)|| \leq ||A^{-1}||||z|| \leq ||A^{-1}||(||y_0|| + s) =: R$$

meaning that

$$B_s^Y(y_0) \cap range(A) \subseteq A(B_R^X(0))$$

and thus

$$B_s^Y(y_0) \subseteq \overline{B_s^Y(y_0) \cap range(A)} \subseteq \overline{A(B_R^X(0))}.$$

Now, from $X$ construct an auxilliary map $B : X \to X$ defined by $B(x) = Rx$.

Now, let $A' = A \circ B$ which means that

$$A'x = A(B(x)) = A(Rx)$$

Now, note that

$$A'(B_1^X(0)) = A(B(B_1^X(0))) = A(B_R^X(0)),$$

which implies that $A'$ satisfies the hypotheses of Lemma 20.3.3 since

$$\overline{A'(B_1^X(0))} = \overline{A(B_R^X(0))} \supseteq B_s^Y(y_0)$$

which then implies that $A'(B_1^X(0))$ contains an open ball $B_r^Y(0)$ for some radius $r > 0$.

But now since

$$B_r^Y(0) \subseteq A'(B_1^X(0)) = A(B_R^X(0))$$

We simply note that

$$range(A) = \cup_{n=1}^{\infty} A(B_{nR}^X(0))$$

and likewise

$$Y = \cup_{n=1}^{\infty} B_{nr}^Y(0)$$

Now, I claim that $Y \subseteq range(A)$ which can be shown by noting that

Observation A: For any $y \in Y$ we have that $y \in B_{n'r}^Y(0)$ for some $n' \in \mathbb{N}$.

Now, I claim that

$$B_{n'r}^Y(0) \subseteq A(B_{mR}^X(0)) \text{ for some } m \in \mathbb{N}. \tag{0.1}$$

Indeed the above holds if and only if

$$A^{-1}(B_{n'r}^Y(0)) \subseteq A^{-1}(A(B_{mR}^X(0))) = B_{mR}^X(0), \tag{0.2}$$

so we show that.

Namely, for any $y \in B_{n'r}^Y(0)$ we see that $||A^{-1}y|| \leq ||A^{-1}||||y|| \leq n'r||A^{-1}||$ , meaning that

$$A^{-1}(B_{n'r}^Y(0)) \subseteq B_{nr'||A^{-1}||}^X(0)$$

So, if one takes $m \in \mathbb{N}$ such that

$$mR \geq n'r||A^{-1}||$$

so say

$$m = \lceil \frac{n'r||A^{-1}||}{R} \rceil$$

then we have that

$$B_{nr'||A^{-1}||}^X(0) \subseteq B_{mR}^X(0)$$

which shows Equation 0.2 and thus the claim Equation 0.1 which along with Observation A shows that $Y \subseteq range(A)$.

3. Problem 20.2.10, parts (a)–(d).

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space $X$. Fix $1 \leq p \leq \infty$, and suppose that for every $\mu \in X^*$ we have that

$$T(\mu) \;=\; \big(\mu(x_n)\big)_{n\in\mathbb{N}} \in \ell^{p'}.$$

We call $T$ the *analysis operator* associated with the sequence $\{x_n\}_{n\in\mathbb{N}}$. Prove the following statements.

(a) $T\colon X^* \to \ell^{p'}$ is bounded and linear.
To show linearity we simply note that

$$\begin{aligned} T(a\mu+b\nu) = ((a\mu+b\nu)(x_n))_{n\in\mathbb{N}} = (a(\mu(x_n))+b(\nu(x_n)))_{n\in\mathbb{N}} &= a(\mu(x_n))_{n\in\mathbb{N}} + b(\nu(x_n))_{n\in\mathbb{N}} \\ &= aT(\mu)+bT(\nu). \end{aligned}$$

To show boundedness we will apply the Uniform Boundedness Principle.
Note that as stated on page 106 of the textbook, since $X$ is a Banach space we know that $X^*$ is as well and of course $\ell^{p'}$ is a normed space. Now, note that we have a sequence of operators $A_i : X^* \to \ell^{p'}$ for $i \in \mathbb{N}$ defined by

$$A_i(\mu) = (\chi_{k\le i}\mu(x_k))_{k\in\mathbb{N}}.$$

Now, note that since $||(\chi_{k\le i}\mu(x_k))_{k\in\mathbb{N}}||_{\ell^{p'}} \nearrow ||(\mu(x_k))_{k\in\mathbb{N}}||_{\ell^{p'}}$ as $i\to\infty$, we have that

$$\sup_{i\in\mathbb{N}} ||A_i\mu||_{\ell^{p'}} = \lim_{i\to\infty} ||A_i\mu||_{\ell^{p'}} = ||\lim_{i\to\infty} A_i\mu||_{\ell^{p'}} = ||T\mu||_{\ell^{p'}} < \infty$$

So, the Uniform Boundedness Principle implies that

$$\begin{aligned} \sup_{i\in\mathbb{N}} ||A_i|| = \sup_{||\mu||=1} \sup_{i\in\mathbb{N}} ||A_i\mu||_{\ell^{p'}} = \sup_{||\mu||=1} \lim_{i\to\infty} ||A_i\mu||_{\ell^{p'}} &= \sup_{||\mu||=1} ||\lim_{i\to\infty} A_i\mu||_{\ell^{p'}} \\ &= \sup_{||\mu||=1} ||T\mu||_{\ell^{p'}} = ||T|| < \infty. \end{aligned}$$

(b) If $1 \le p < \infty$ then the series $\sum_{n=1}^{\infty} c_n x_n$ converges for each sequence $(c_n)_{n\in\mathbb{N}} \in \ell^p$, and the *synthesis operator* $Uc = \sum_{n=1}^{\infty} c_n x_n$ is a bounded map of $\ell^p$ into $X$.
Hint: Hahn–Banach (Corollary 19.1.4).
Denote

$$\sum_{n=1}^{N} c_n x_n =: S_N.$$

We show convergence of $\sum_{n\in\mathbb{N}} c_n x_n$ through the following steps:

- We note that for each $N \in \mathbb{N}$ we may obtain a sequence of operators

  $$(\pi(S_N))_{N\in\mathbb{N}} \subseteq B(X^*, F)$$

  via the map $\pi : X \to X^{**}$ given in the book.
- By definition of $\pi$ we note that

  $$\pi(S_N)(\mu) = \sum_{n=1}^{N} c_n\mu(x_n).$$

- Then, we know by Holder's Inequality that

  $$\sum_{n=1}^{\infty} |c_n\mu(x_n)| = ||(c_n\mu(x_n))_{n\in\mathbb{N}}||_{\ell^1} \le ||c_n||_{\ell^p} ||\mu(x_n))_{n\in\mathbb{N}}||_{\ell^{p'}} < \infty,$$

Now, since $F$ is complete we also know by Theorem 1.2.8 that

$$\sum_{n=1}^{\infty} c_n \mu(x_n)$$

converges.

- Thus,
$$\lim_{N\to\infty} \pi(S_N)(\mu) = \sum_{n=1}^{\infty} c_n \mu(x_n) \neq \infty$$
exists for all $\mu \in X^*$ which means we can define a function $S : X^* \to F$ by
$$S\mu = \sum_{n=1}^{\infty} c_n \mu(x_n).$$
- Then, the Banach-Steinhaus Theorem implies that
$$S \in B(X^*, F).$$
- Now, we further show that
$$\pi(S_N) \to S$$
in operator norm.
  - In particular, note that
$$||S - \pi(S_N)||_{op} = \sup_{||\mu||=1} |(S - \pi(S_N))(\mu)|.$$
  - So, for arbitrary $\mu \in X^*$ with $||\mu|| = 1$ we compute
$$\begin{aligned} |(S - \pi(S_N))(\mu)| &= |\sum_{n=1}^{\infty} c_n \mu(x_n) - \sum_{n=1}^{N} c_n \mu(x_n)| \\ &= |\sum_{n=N+1}^{\infty} c_n \mu(x_n)|. \end{aligned}$$
  - Then, we use Holder's Inequality which says that for $1 \leq p \leq \infty$ whenever $(a_n)_{n\in\mathbb{N}} \in \ell^p$ and $(b_n)_{n\in\mathbb{N}} \in \ell^{p'}$ then one has that $||(a_n b_n)_{n\in\mathbb{N}}||_1 \leq ||(a_n)_{n\in\mathbb{N}}||_{\ell^p} ||(b_n)_{n\in\mathbb{N}}||_{\ell^{p'}}$. In our particular case, we have the sequences
$$(a_n^N)_{n\in\mathbb{N}} = (\chi_{n\geq N} c_n)_{n\in\mathbb{N}} \in \ell^p$$
and
$$(b_n^N)_{n\in\mathbb{N}} = (\chi_{n\geq N} |\mu(x_n)|)_{n\in\mathbb{N}} \in \ell^{p'}$$
where the above are in $\ell^p$ and $\ell^{p'}$ respectively since
$$\begin{aligned} ||(\chi_{n\geq N} c_n)_{n\in\mathbb{N}}||_{\ell^p} &= (\sum_{n=1}^{\infty} |\chi_{n\geq N} c_n|^p)^{1/p} \\ &\leq (\sum_{n=1}^{\infty} |c_n|^p)^{1/p} \\ &= ||(c_n)_{n\in\mathbb{N}}||_{\ell^p} < \infty, \end{aligned}$$

and

$$
\begin{aligned}
||(\chi_{n\geq N}|\mu(x_n)|)_{n\in\mathbb{N}}||_{\ell^{p'}} &= (\sum_{n=1}^{\infty}(\chi_{n\geq N}|\mu(x_n)|)^{p'})^{1/p'}\\
&\leq (\sum_{n=1}^{\infty}|\mu(x_n)|^{p'})^{1/p'}\\
&= ||(\mu(x_n))_{n\in\mathbb{N}}||_{\ell^{p'}}\\
&= ||T(\mu)||_{\ell^{p'}} \text{ (by definition of } T)\\
&\leq ||T||||\mu|| \text{ (by definition of operator norm)}\\
&= ||T|| \text{ (since } ||\mu||=1)\\
&< \infty(\text{ by part A}).
\end{aligned}
$$

due to the main hypothesis of the problem statement. So, we have that

$$
\begin{aligned}
||a_n^N b_n^N||_1 = \sum_{n=N}^{\infty}|c_n||\mu(x_n)| &\leq ||(a_n^N)_{n\in\mathbb{N}}||_{\ell^p}||(b_n^N)_{n\in\mathbb{N}}||_{\ell^{p'}}\\
&= ||(\chi_{n\geq N}c_n)_{n\in\mathbb{N}}||_{\ell^p}||||(\chi_{n\geq N}|\mu(x_n)|)_{n\in\mathbb{N}}||_{\ell^{p'}}\\
&\leq ||(\chi_{n\geq N}c_n)_{n\in\mathbb{N}}||_{\ell^p}||||T||\\
&= (\sum_{n=N}^{\infty}|c_n|^p)^{1/p}||T||.
\end{aligned}
$$

All in all, we have concluded that

$$|\sum_{n=N}^{\infty}c_n\mu(x_n)| \leq (\sum_{n=N}^{\infty}|c_n|^p)^{1/p}||T||$$

for all $\mu \in X^*$ with $||\mu|| = 1$ which means that

$$\sup_{||\mu||=1}|\sum_{n=N}^{\infty}c_n\mu(x_n)| \leq (\sum_{n=N}^{\infty}|c_n|^p)^{1/p}||T||.$$

and since $||(c_n)_{n\in\mathbb{N}}||_{\ell^p} = (\sum_{n=0}^{\infty}|c_n|^p)^{1/p} < \infty$ we have that

$$||(c_n)_{n\in\mathbb{N}}||_{\ell^p}^p = \sum_{n=0}^{\infty}|c_n|^p < \infty$$

meaning that

$$\sum_{n=N}^{\infty}|c_n|^p \to 0$$

as $N \to \infty$ or more precisely, that for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n=M}^{\infty}|c_n|^p < \epsilon$$

for all $M \geq N(\epsilon)$.

- Finally, to show that

$$\sup_{||\mu||=1} |\sum_{n=N}^{\infty} c_n \mu(x_n)| \to 0$$

as $N \to \infty$, fix $\epsilon > 0$. Now, let $\epsilon' := (\frac{\epsilon}{||T||})^p$. Now, for $M \geq N(\epsilon')$ we have that

$$\sum_{n=M}^{\infty} |c_n|^p < \epsilon' = (\frac{\epsilon}{||T||})^p$$

meaning that

$$(\sum_{n=M}^{\infty} |c_n|^p)^{1/p} ||T|| < ((\frac{\epsilon}{||T||})^p)^{1/p} ||T|| = \epsilon$$

for all $M \geq N(\epsilon')$ thus proving the statement that $\sup_{||\mu||=1} |\sum_{n=N}^{\infty} c_n \mu(x_n)| \to 0 \to 0$ as $N \to \infty$.

- So, putting all that together we get that

$$||S - \pi(S_N)||_{op} = \sup_{||\mu||=1} |(S - \pi(S_N))(\mu)| = \sup_{||\mu||=1} |\sum_{n=N+1}^{\infty} c_n \mu(x_n)| \to 0$$

as $N \to \infty$.

- So, the above fact that $(\pi(S_N))_{N\in\mathbb{N}} \subseteq B(X^*, F)$ is convergent implies that it is Cauchy.
- Then, since $\pi : X \to B(X^*, F)$ is a linear isometry that implies that $(S_N)_{N\in\mathbb{N}} \subseteq X$ is Cauchy.
  - Proof of the claim that $(\pi(S_N))_{N\in\mathbb{N}}$ Cauchy implies that $(S_N)_{N\in\mathbb{N}}$ is also Cauchy:
  - Note that for all $n, m \in \mathbb{N}$ one has that

$$||\pi(S_n) - \pi(S_m)||_{op} = ||\pi(S_n - S_m)||_{op} = ||S_n - S_m||_X.$$

  - Now, $(\pi(S_N))_{N\in\mathbb{N}}$ Cauchy means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ one has that

$$||\pi(S_n) - \pi(S_m)||_{op} < \epsilon.$$

However, that also implies that for all $n, m \geq N$ one has that

$$||S_n - S_m||_X < \epsilon,$$

proving that $(S_N)_{N\in\mathbb{N}}$ is also Cauchy.

- Finally, note that since $X$ is complete the fact that $(S_N)_{N\in\mathbb{N}}$ is Cauchy means that $(S_N)_{N\in\mathbb{N}}$ is convergent, meaning that

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} c_n x_n \in X$$

exists.

Now, to show boundedness note that

$$\begin{aligned}
||Uc|| &= ||\sum_{n=1}^{\infty} c_n x_n|| \\
&= \sup_{||\mu||=1} |\mu(\sum_{n=1}^{\infty} c_n x_n)| \\
&= |\mu'(\sum_{n=1}^{\infty} c_n x_n)| \text{ (for some } \mu' \in X^* \text{ with } ||\mu'|| = 1 \text{ by 19.1.4)} \\
&= |\sum_{n=1}^{\infty} c_n \mu'(x_n)| \text{ (by linearity and continuity of } \mu') \\
&\leq \sum_{n=1}^{\infty} |c_n||\mu'(x_n)| \text{ (by the triangle inequality)} \\
&= ||(c_n \mu'(x_n))_{n\in\mathbb{N}}||_1 \\
&\leq ||(c_n)_{n\in\mathbb{N}}||_{\ell^p} ||(\mu'(x_n))_{n\in\mathbb{N}}||_{\ell^{p'}} \text{ (by Holder's Inequality)} \\
&= ||(c_n)_{n\in\mathbb{N}}||_{\ell^p} ||T(\mu')||_{\ell^{p'}} \\
&\leq ||(c_n)_{n\in\mathbb{N}}||_{\ell^p} ||T|| ||\mu'|| \\
&= ||(c_n)_{n\in\mathbb{N}}||_{\ell^p} ||T|| \text{ since} ||\mu'|| = 1).
\end{aligned}$$

So, indeed we see that $U$ is a bounded linear operator with $||U|| \leq ||T||$.

(c) If $p = \infty$ and $(c_n)_{n\in\mathbb{N}} \in \ell^\infty$, then the series $\sum_{n=1}^{\infty} c_n x_n$ converges *weakly*. That is, $\sum_{n=1}^{\infty} c_n \mu(x_n)$ converges for each $\mu \in X^*$. However, show by example that $\sum c_n x_n$ need not converge in the norm of $X$.

In particular, note that the problem hypothesis states that $T(\mu) = (\mu(x_n))_{n\in\mathbb{N}} \in \ell^1$. Then once again Holder's inequality implies that

$$|\sum_{n=1}^{\infty} c_n \mu(x_n)| \leq \sum_{n=1}^{\infty} |c_n \mu(x_n)| = ||(c_n \mu(x_n))_{n\in\mathbb{N}}||_1 \leq ||(c_n)_{n\in\mathbb{N}}||_\infty ||\mu(x_n)_{n\in\mathbb{N}}||_1 < \infty,$$

which means by Theorem 1.2.8 that

$$\sum_{n=1}^{\infty} c_n \mu(x_n)$$

converges since $F \in \{\mathbb{R}, \mathbb{C}\}$ is complete.

Likewise for the tails

$$||(\chi_{n\geq N} c_n \mu(x_n))_{n\in\mathbb{N}}||_1 \leq ||(\chi_{n\geq N} c_n)_{n\in\mathbb{N}}||_\infty ||\chi_{n\geq N} \mu(x_n)_{n\in\mathbb{N}}||_1$$

$$\leq ||(c_n)_{n\in\mathbb{N}}||_\infty ||\chi_{n\geq N} \mu(x_n)_{n\in\mathbb{N}}||_1 = ||(c_n)_{n\in\mathbb{N}}||_\infty (\sum_{n=N}^{\infty} |\mu(x_n)|).$$

Once again we show that the tails go to zero by noting that

$$|\sum_{n=N}^{\infty} c_n\mu(x_n)| \leq \sum_{n=N}^{\infty} |c_n||\mu(x_n)| = ||(\chi_{n\geq N}c_n\mu(x_n))_{n\in\mathbb{N}}||_1 \leq ||(c_n)_{n\in\mathbb{N}}||_\infty(\sum_{n=N}^{\infty} |\mu(x_n)|).$$

Now, since $\sum_{n=1}^{\infty} |\mu(x_n)| < \infty$ that implies that for all $\epsilon > 0$ there exists $M(\epsilon) \in \mathbb{N}$ such that $\sum_{n=K}^{\infty} |\mu(x_n)| < \epsilon$ for all $K \geq M(\epsilon)$.

Now, to show that $|\sum_{n=N}^{\infty} c_n\mu(x_n)| \to 0$ as $N \to \infty$, we fix $\epsilon > 0$ and now let $\epsilon_0 = \frac{\epsilon}{||(c_n)_{n\in\mathbb{N}}||_\infty}$ and now for all $K \geq M(\epsilon_0)$ we see that

$$|\sum_{n=K}^{\infty} c_n\mu(x_n)| \leq ||(c_n)_{n\in\mathbb{N}}||_\infty(\sum_{n=K}^{\infty} |\mu(x_n)|) < ||(c_n)_{n\in\mathbb{N}}||_\infty\epsilon_0 = ||(c_n)_{n\in\mathbb{N}}||_\infty\frac{\epsilon}{||(c_n)_{n\in\mathbb{N}}||_\infty} = \epsilon,$$

proving that $|\sum_{n=N}^{\infty} c_n\mu(x_n)| \to 0$ as $N \to \infty$.

Now, the above implies that $\sum_{n=1}^{N} c_n\mu(x_n)$ is Cauchy since if one fixes $\epsilon > 0$ and lets $\epsilon_1 = \frac{\epsilon}{2|(c_n)_{n\in\mathbb{N}}||_\infty}$, then for all $N, P \geq M(\epsilon_1)$ one has that

$$\begin{aligned}
||(\sum_{n=1}^{N} c_n\mu(x_n)) - (\sum_{n=1}^{P} c_n\mu(x_n))|| &= ||(\sum_{n=1}^{\infty} c_n\mu(x_n)) - (\sum_{n=1}^{\infty} c_n\mu(x_n)) + (\sum_{n=1}^{N} c_n\mu(x_n)) - (\sum_{n=1}^{P} c_n\mu(x_n))|| \\
&= ||((\sum_{n=1}^{\infty} c_n\mu(x_n)) - (\sum_{n=1}^{P} c_n\mu(x_n))) + (-(\sum_{n=1}^{\infty} c_n\mu(x_n)) + (\sum_{n=1}^{N} c_n\mu(x_n)))|| \\
&\leq ||(\sum_{n=1}^{\infty} c_n\mu(x_n)) - (\sum_{n=1}^{P} c_n\mu(x_n))|| + || - (\sum_{n=1}^{\infty} c_n\mu(x_n)) + (\sum_{n=1}^{N} c_n\mu(x_n))|| \\
&= ||(\sum_{n=1}^{\infty} c_n\mu(x_n)) - (\sum_{n=1}^{P} c_n\mu(x_n))|| + ||(\sum_{n=1}^{\infty} c_n\mu(x_n)) - (\sum_{n=1}^{N} c_n\mu(x_n))|| \\
&= ||\sum_{n=P+1}^{\infty} c_n\mu(x_n)|| + ||\sum_{n=N+1}^{\infty} c_n\mu(x_n)|| \\
&< ||(c_n)_{n\in\mathbb{N}}||_\infty(\sum_{n=P+1}^{\infty} |\mu(x_n)|) + ||(c_n)_{n\in\mathbb{N}}||_\infty(\sum_{n=N+1}^{\infty} |\mu(x_n)|) \\
&< ||(c_n)_{n\in\mathbb{N}}||_\infty\epsilon_1 + ||(c_n)_{n\in\mathbb{N}}||_\infty\epsilon_1 \\
&= \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

thus proving the sequence $\sum_{n=1}^{N} c_n\mu(x_n)$ is Cauchy and thus convergent since $\mathbb{R}, \mathbb{C}$ are both complete.

Now, we give a counterexample to show that $\sum c_nx_n$ need not converge in the norm of $X$. In particular, take $(c_n)_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}}$ the sequence of all ones. Now, for $n = 1$ let $x_n = e_n \in \ell^2$ and for $n \geq 2$ let $x_n = e_n - e_{n-1}$.

Note that all $\mu \in (\ell^2)^*$ have the form $\mu_y \in (\ell^2)^*$ for some $y \in \ell^2$ with $\mu_y(x) = \langle x, y\rangle$ for all $x \in \ell^2$. (Of course, as mentioned before $y = (y(n)) := (\mu(e_n))_{n\in\mathbb{N}}$).

So, to show that $\sum_{n=1}^{\infty} \mu(x_n)$ converges for all $\mu \in (\ell^2)^*$ note that $\mu = \mu_y$ with $y \in \ell^2$ defined as above and now we show that the norms of the tails $|\sum_{n=N}^{\infty} \mu(x_n)| \to 0$ as $N \to \infty$. In particular note that

$$|\sum_{n=N}^{\infty} \mu(x_n)| = |\sum_{n=N}^{\infty} \langle y, e_n - e_{n-1}\rangle| = |\sum_{n=N}^{\infty} \langle y, e_n\rangle - \langle y, e_{n-1}\rangle|$$

$$= |\sum_{n=N}^{\infty} \langle y, e_n\rangle - \langle y, e_{n-1}\rangle| = |\langle y, e_{N-1}\rangle| = |y(N-1)|.$$

Now, I claim that since $y \in \ell^2$ we know that

$$\sum_{n=M}^{\infty} |y(n)|^2 \to 0$$

as $M \to \infty$, which I also claim means that $|y(M)| \to 0$ as $M \to \infty$.

In particular, for fixed $\epsilon > 0$ we want to find $M \in \mathbb{N}$ such that $|y(m)| < \epsilon$ for all $m \geq M$. Well, note that for all $M \in \mathbb{N}$ we have that

$$|y(M)|^2 < \sum_{n=M}^{\infty} |y(n)|^2.$$

So, if we set $\epsilon' = \epsilon^2$ then since $\sum_{n=M}^{\infty} |y(n)|^2 \to 0$ we know there exists $N(\epsilon')$ such that

$$|\sum_{n=P}^{\infty} |y(n)|^2 < \epsilon'$$

for all $P \geq N(\epsilon')$ and thus that gives that our desired $M$ is $M = N(\epsilon')$ and we have that

$$|y(P)|^2 < \sum_{n=P}^{\infty} |y(n)|^2 < \epsilon' = \epsilon^2$$

for all $P \geq M$ and thus

$$|y(P)| < \epsilon$$

for all $P \geq M$ which means $|Y(N-1)| \to 0$ and thus $|\sum_{n=N}^{\infty} \mu(x_n)| \to 0$ as $N \to \infty$ meaning that $\sum_{n=1}^{\infty} \mu(x_n)$ converges (since that means that $\sum_{n=1}^{M} \mu(x_n)$ is Cauchy and thus convergent sine $F \in \{\mathbb{R}, \mathbb{C}\}$ is complete).

Now, however I claim that $\sum_{n=1}^{\infty} x_n$ does not converge in the norm of $\ell^2$. Namely, to have convergence one would need that (Condition A) for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $M \geq N$ one has that

$$|\sum_{n=M}^{\infty} x_n| < \epsilon.$$

However, we note that for all $P \in \mathbb{N}$

$$||\sum_{n=P}^{\infty} x_n||_{\ell^2} = ||\sum_{n=P}^{\infty} e_n - e_{n-1}||_{\ell^2} = ||e_{P-1}||_{\ell^2} = 1.$$

Thus, for $\epsilon < 1$ there does not exist the desired $N \in \mathbb{N}$ such that Condition A holds.

(d) If $1 \leq p < \infty$ then $U^*\colon X^* \to (\ell^p)^*$ and $T = U^*$ in the sense of identifying $\ell^{p'}$ with $(\ell^p)^*$. Specifically, for each $\mu \in X^*$ the linear functional on $\ell^p$ that is determined by the sequence $T\mu \in \ell^{p'}$ equals the linear functional $U^*\mu \in (\ell^p)^*$.

Note that $U^* : X^* \to (\ell^p)^*$ is the unique operator satisfying

$$(U^*\mu)(\{y_n\}_{n\in\mathbb{N}}) = \mu(U(\{y_n\}_{n\in\mathbb{N}}))$$

for all $\mu \in X^*$ and all $\{y_n\}_{n\in\mathbb{N}} \in \ell^p$.

Now, note that if one regards $T\mu = (\mu(x_n))_{n\in\mathbb{N}} \in \ell^{p'}$ as a functional on $\ell^p$ via the action

$$(T\mu)(\{y_n\}_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} y_n\mu(x_n)$$

we notice also by linearity and continuity of $\mu$ that

$$\mu(U(\{y_n\}_{n\in\mathbb{N}})) = \mu(\sum_{n=1}^{\infty} y_n x_n) = \sum_{n=1}^{\infty} y_n\mu(x_n).$$

Thus, we see that

$$(U^*\mu)(\{y_n\}_{n\in\mathbb{N}}) = (T\mu)(\{y_n\}_{n\in\mathbb{N}})$$

for all $\mu \in X^*$ and all $(\{y_n\}_{n\in\mathbb{N}}) \in \ell^p$ proving that $T = U^*$.

4. Exercise 20.5.7. Let $\{x_n\}_{n\in\mathbb{N}}$ be a minimal sequence in a Banach space $X$, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be its biorthogonal sequence in $X^*$.

Prove that the following four statements are equivalent.

Remark: You can assume without proof that a biorthogonal sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ exists in $X^*$. This sequence satisfies $\alpha_m(x_n) = \delta_{mn}$ for $m$, $n \in \mathbb{N}$.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for $X$.

(b) $S_N x \to x$ for each $x \in X$.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is complete, and $\sup \|S_N x\| < \infty$ for each $x \in X$.

(d) $\{x_n\}_{n\in\mathbb{N}}$ is complete, and $\sup \|S_N\| < \infty$.

Note that (a) implies (b). Since $\{x_n\}_{n\in\mathbb{N}}$ a Schauder basis means that for any $x \in X$ there exists a unique sequence of scalars $c_n$ such that

$$x = \sum_{n=1}^{\infty} c_n x_n$$

and the fact that $\{\alpha_n\}_{n\in\mathbb{N}}$ is biorthogonal with each $\alpha_n$ continuous we have that

$$\alpha_m(x) = \alpha_m(\sum_{n=1}^{\infty} c_n x_n) = \sum_{n=1}^{\infty} c_n\alpha_m(x_n) = c_m\alpha_m(x_m) = c_m \tag{0.3}$$

meaning that

$$x = \sum_{n=1}^{\infty} \alpha_n(x)x_n = \lim_{N\to\infty} S_N x$$

thus showing $(a) \implies (b)$.

We also have that $(b) \implies (a)$. In particular, (b) states that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n$$

for all $x \in X$. Furthermore, we note that if one has another expression

$$x = \sum_{n=1}^{\infty} c_n x_n$$

for some $c_n \in \mathbb{F}$ then as shown in Equation 0.3 we obtain $c_n = \alpha_n(x)$ which shows uniqueness and thus proves that (b) implies (a).

Now, we show that $(b) \implies (d)$. Fix $x \in X$.

We know that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n = \lim_{N \to \infty} S_N x,$$

which shows firstly that $\{x_n\}_{n \in \mathbb{N}}$ is complete. Furthermore, it so happens that $||S_N|| < \infty$ for each $N \in \mathbb{N}$ since

$$||S_N(x)|| = ||\sum_{n=1}^{N} \alpha_n(x) x_n|| \leq \sum_{n=1}^{N} |\alpha_n(x)|||x_n|| \leq \sum_{n=1}^{N} |||\alpha_n||||x|||||x_n||$$
$$\leq (N+1)||x|| \max_{n \in \{0,1,\ldots,N\}} ||x_n|| \max_{n \in \{0,1,\ldots,N\}} ||\alpha_n||$$

meaning that

$$||S_N|| \leq (N+1)||x|| \max_{n \in \{0,1,\ldots,N\}} ||x_n|| \max_{n \in \{0,1,\ldots,N\}} ||\alpha_n|| < \infty.$$

Now, the Banach-Steinhaus Theorem tells us that

$$||id|| \leq \sup_{N \in \mathbb{N}} ||S_N|| < \infty$$

which concludes our proof that $(b) \implies (d)$.

Now, we show that $(d) \implies (c)$ by noting that

$$\sup_{N \in \mathbb{N}} ||S_N x|| \leq \sup_{N \in \mathbb{N}} ||S_N||||x|| = ||x|| \sup_{N \in \mathbb{N}} ||S_N||.$$

Now, (d) states that $M := \sup_{N \in \mathbb{N}} ||S_N|| < \infty$ which then gives that

$$\sup_{N \in \mathbb{N}} ||S_N x|| \leq M||x|| < \infty$$

for all $x \in X$ which proves (c).

Likewise I show that $(c) \implies (d)$. Note that (c) says $\sup_{N \in \mathbb{N}} ||S_N x|| < \infty$. Then, we may apply the Uniform Boundedness Principle to obtain

$$\sup_{N \in \mathbb{N}} ||S_N|| < \infty.$$

Finally we show that $(d) \implies (b)$. Namely, for fixed $\epsilon > 0$ we would like to find $P(\epsilon) \in \mathbb{N}$ such that $||S_N x - x|| < \epsilon$ for all $N \geq P(\epsilon)$. So, denote $M := \sup_{N \in \mathbb{N}} ||S_N||$. Now, note that since $\{x_n\}_{n\in\mathbb{N}}$ is complete, for fixed $\epsilon' = \frac{\epsilon}{2max(M,1)}$ there exist constants $\{d_n(\epsilon')\}_{n\in\{0,1,\ldots,m(\epsilon')\}}$ such that the approximation $\hat{x} = \sum_{n=1}^{m(\epsilon')} d_n(\epsilon')x_n$ satisfies

$$||\hat{x} - x|| < \epsilon'.$$

Furthermore, note that

$$S_N\hat{x} = \hat{x}$$

since for $N \geq m(\epsilon')$ one has that

$$\begin{aligned}
S_N\hat{x} &= \sum_{n=1}^{N} \alpha_n(\hat{x})x_n \\
&= \sum_{n=1}^{N} \alpha_n(\sum_{k=0}^{m(\epsilon')} d_k(\epsilon')x_k)x_n \\
&= \sum_{n=1}^{N} \sum_{k=0}^{m(\epsilon')} d_k(\epsilon')\alpha_n(x_k)x_n \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') \sum_{n=1}^{N} \alpha_n(x_k)x_n \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon')\alpha_k(x_k)x_k \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon')x_k = \hat{x}.
\end{aligned}$$

Then, by the triangle inequality we have that

$$\begin{aligned}
||S_Nx - x|| &= ||S_Nx - S_N\hat{x} + S_N\hat{x} - \hat{x} + \hat{x} - x|| \\
&\leq ||S_Nx - S_N\hat{x}|| + ||S_N\hat{x} - \hat{x}|| + ||\hat{x} - x|| \\
&= ||S_N(x - \hat{x})|| + ||\hat{x} - x|| \\
&\leq M||x - \hat{x}|| + ||\hat{x} - x|| \\
&< M\epsilon' + \epsilon' = M\frac{\epsilon}{2max(M,1)} + \frac{\epsilon}{2max(M,1)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

thus proving that $S_Nx \to x$ as $N \to \infty$.

Of course, the above means that I have showed TFAE since there is a path of implication (using the implications I proved) from $i$ to $j$ for all $i, j \in \{a, b, c, d, e\}$.