Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 2.7
Show that each character of G which is zero for all $s \neq 1$ is an integral multiple of the character r_G of the regular representation.

Note that since $(\chi|1)$ is the number of times the associated representation contains the unit representation, it is an integer. Recall that

$$(\chi|1) = \frac{1}{g}\sum_{s\in G}\chi(s)$$

but $\chi(s) = 0$ for $s \neq 1$ and thus

$$(\chi|1) = \frac{1}{g}\chi(1)$$

which implies that g divides $\chi(1)$, thus proving that $\chi = mr_G$ for some $m \in \mathbb{Z}$.

Exercise 2.8
Let H be the vector space of linear mappings $h : W_i \to V$ such that $\rho_s h = h\rho_s$ for all $s \in G$. Each $h \in H_i$ maps W_i into V_i.

(a) Show that the dimension of H_i is equal to the number of times that W_i appears in V, i.e. to $\dim(V_i)/\dim(W_i)$. (Reduce to the case where $V = W_i$ and use Schur's Lemma).

- Fix any non-zero $h \in H_i$. Denote it h_1 and let $U_i^1 := im(h_1) \subseteq V_i$.
- Show that $U_i^1 \cong W_i$
  - To do so, we note that U_i^1 is a subspace of V_i, which I also claim is stable under the action of ρ_g for all g which along with the fact that U_i^1 is irreducible (Is it?) would imply that $U_i^1 \cong W_i$.
  - Indeed, U_i^1 is stable under the action of ρ_g since for all $u \in U_i^1$ one has by definition that $u = h_1(w)$ for some $w \in W_i$ and then

    $$\rho_g(u) = \rho_g(h_1(w)) = h_1(\rho_g(w)) = h(w')$$

    for some $w' \in W_i$ since W_i is stable under the action of G. and thus $\rho_g(u) \in U_i^1 = \mathrm{im}(h_1)$.
  - Also, U_i^1 is irreducible for the same reason. Otherwise, if there were some subspace $U_i^{1\prime} \subsetneq U_i^1$ which was stable under the action of G, then note that if $W_1' := h^{-1}(U_i^{1\prime}) := \{w \in W_i : h_1(w) \in U_i^{1\prime}\}$ denotes the set of pre-images under h_1, then $W_1' \subsetneq W_i$ and since W_i is irreducible one knows that W_1' is not stable under G. More precisely, there exists $g \in G, w' \in W_1'$ such that $u' := \rho_g(w_1') \notin W_1'$. Thus,

    $$\rho_g(h_1(w_1')) = h_1(\rho_g(w_1')) = h_1(u') \notin U_i^{1\prime}.$$

    since W_1' was by definition all pre-images $\{w \in W_i : h_1(w) \in U_i^{1\prime}\}$.
  - Also, up to isomorphism W_i is the only irreducible subspace of V_i.
- Then, use Schur's Lemma.
  - Namely, $W_i \cong U_i^1$ implies that there exists an isomorphism $\phi_i^1 : U_i^1 \to W_i$. So, define $\hat{h} : U_i^1 \to U_i^1$ by $\hat{h}(w) = h(\phi_i^1(w))$.
  - Now, part (2) of Schur's lemma says that since $\rho_g\hat{h} = \hat{h}\rho_g$ for all $g \in G$, $\hat{h}$ is a homothety.

- Now, we first show that $\dim(H_i) \geq n_i$ by an inductive argument. Namely, for $k < n_i$ suppose we have
$$\{h_1, \ldots, h_k\} \subseteq H_i \text{ a linearly independent set}$$
such that
$$U_i^j := \operatorname{im}(h_j) \cong W_i$$
and
$$Y_k := \operatorname{span}(\{U_i^j : j \in [k]\}) \cong \bigoplus_{j \in [k]} W_i.$$
Then, we show that there exists
$$h_{k+1} \in H_i$$
such that
  - $U_i^{k+1} := \operatorname{im}(h_{k+1}) \cong W_i$,
  - $Y_k \cap U_i^{k+1} = \{0\}$,
  - and $\{h_1, \ldots, h_k, h_{k+1}\} \subseteq H_i$ is a linearly independent set.
- The first two points of course imply that
$$Y_{k+1} := \operatorname{span}(\{U_i^j : j \in [k+1]\}) \cong \bigoplus_{j \in [k+1]} U_i^j \cong \bigoplus_{j \in [k+1]} W_i.$$
- All together, this will show that we can find a set of n_i linearly independent functions $\{h_1, \ldots, h_{n_i}\} \subseteq H_i$ which shows $\dim(H_i) \geq n_i$.
- After we carry out the above argument we will also show that $\dim(H_i) \leq n_i$.
- Now, we perform the above inductive proof by constructing the desired $h_{k+1} \in H_i$ from above as follows.
  - First, recall by Theorem 1 that there exists a complement Y'_{k+1} of Y_k in V_i which is stable under the action of G.
  - So, since $Y_k = \bigoplus_{j \in [k]} U_i^j \cong \bigoplus_{j \in [k]} W_i$ we have that
  $$Y'_{k+1} \cong \bigoplus_{j \in [k+1:n_i]} W_i$$
  since otherwise,
  $$V_i \cong \bigoplus_{j \in [n_i]} W_i$$
  and
  $$V_i \cong Y_k \oplus Y'_{k+1} \cong (\bigoplus_{j \in [k]} W_i) \oplus Y'_{k+1} \ncong (\bigoplus_{j \in [k]} W_i) \oplus (\bigoplus_{j \in [k+1:n_i]} W_i) \cong \bigoplus_{j \in [n_i]} W_i$$
  provide a contradiction. TODO: Prove definitively that if $W \ncong V$ then $U \oplus W \ncong U \oplus V$.
  - So, since $Y'_{k+1} \cong \bigoplus_{j \in [k+1:n_i]} W_i$ that implies that there exist subspaces $U_i^{k+1}, \ldots, U_i^{n_i} \subseteq Y'_{k+1}$ such that $U_i^j \cong W_i$ are isomorphic as representations for all $j \in [k+1 : n_i]$, $U_i^j \cap U_i^l = \{0\}$ for $j \neq l \in [k+1 : n_i]$ and so that $Y'_{k+1} = \operatorname{span}\{U_i^j : j \in [k+1 : n_i]\}$.
  - So, we choose $w' \in U_i^{k+1}$ and recall that since $U_i^k \cong W_i$ we have an isomorphism of representations
  $$\phi_i^k : U_i^k \to W_i$$
  and likewise we have an isomorphism of representations
  $$\phi_i^{k+1} : U_i^{k+1} \to W_i$$
  which means that for all $j \in [k+1 : n_i]$ we have an isomorphism of representations
  $$\phi_i^{k+1} \circ (\phi_i^k)^{-1} : U_i^k \to U_i^{k+1}$$

– Then, if one defines

$$h_{k+1} : W_i \to U_i^{k+1}$$

by

$$h_{k+1}(w) = \phi_i^{k+1}((\phi_i^k)^{-1}(h_k(w))) \text{ for all } w \in W_i,$$

then one has that

$$\text{im}(h_{k+1}) = U_i^{k+1} \text{ since } \text{im}(h_k) = U_i^k,$$

Also, we define

$$\widehat{h}_{k+1} : U_i^{k+1} \to U_i^{k+1}$$

by

$$\widehat{h}_{k+1}(w) = h_{k+1}(\phi_i^{k+1}(w))$$

then I claim that

$$\rho_g h_{k+1} = h_{k+1}\rho_g$$

since

$$\rho_g h_{k+1} = \rho_g \tau h_k$$

and

$$h_{k+1}\rho_g = \tau h_k \rho_g = \tau \rho_g h_k = \rho_g \tau h_k$$

since τ is an isomorphism of representations.

– Then, by Schur's Lemma,

$$\hat{h}_{k+1} : U_i^{k+1} \to U_i^{k+1}$$

is a scalar operator.

– Now, we show that

$$\{h_1, \ldots, h_{k+1}\} \subseteq H_i \text{ is a linearly independent set}$$

and that

$$Y_k \cap U_i^{j'} = \{0\}$$

as follows.

– First, assume for contradiction that $\{h_1, \ldots, h_{k+1}\}$ is dependent. Then there exist $c_1, \ldots, c_{k+1} \in F$, not all zero, such that

$$\sum_{j \in [k+1]} c_j h_j = 0$$

is the zero function, then for arbitrary $w \in W_i$ and in particular arbitrary $w \neq 0$ we have that

$$\sum_{j \in [k+1]} c_j h_j(w) = 0$$

meaning that if $c_{j'} \neq 0$ for $j' \in [k+1]$ then

$$h_{j'}(w) = \sum_{j \in [k+1]: j \neq j'} \frac{-c_j}{c_{j'}} h_j(w).$$

However, $h_{j'}(w) \in U_i^{j'}$ and $h_j(w) \in U_i^j$ for all $j \in [k+1] \setminus \{j'\}$.

– Furthermore, $U_i^j \cap U_i^{j'} = \{0\}$ for all $j \neq j'$ and $dim(Y_k \oplus U_i^{k+1}) = \sum_{j \in [k+1]} dim(U_i^j)$ which implies that

$$\left(\text{span}(\{U_i^j : j \in [k+1], j \neq j'\})\right) \cap U_i^{j'} = \{0\}$$

since

$$Y_k \oplus U_i^{k+1} \cong (\bigoplus_{j \in [k]} U_i^j) \oplus U_i^{k+1} \cong (\bigoplus_{j \in [k+1]: j \neq j'} U_i^j) \oplus U_i^{j'} \cong \text{span}(\{U_i^j : j \in [k+1], j \neq j'\}) \oplus U_i^{j'}$$

which by definition of direct sum gives the above.

– Now, we have a contradiction since

$$h_{j'}(w) \in \left(\text{span}(\{U_i^j : j \in [k+1], j \neq j'\})\right) \cap U_i^{j'}$$

implies that $h_{j'}(w) = 0$ but that implies that $w = 0$ since $h_{j'}$ is a bijection, a contradiction.

- So, we have shown that $\dim(H_i) \geq [n_i]$.
- Now, to show that $\dim(H_i) \leq [n_i]$, we show that one cannot have a set of n_i+1 linearly independent functions $\{h_1, \ldots, h_{n_i+1}\}$. The idea is to show that

$$\text{im}(h_{n_i+1}) \cap \left(\text{span}(\{U_i^j : j \in [n_i]\})\right) = \{0\}$$

where we denote $U_i^j := \text{im}(h_j)$, which will provide a contradiction if we can also show that

$$\text{im}(h_j) \cong W_i$$

for all $j \in [n_i + 1]$.

  – Once again, by the same logic as our earlier argument that $U_i^1 \cong W_i$, we have that $U_i^j \cong W_i$ for all $j \in [n_i+1]$ (since the earlier logic actually shows that $\text{im}(h) \cong W_i$ for all $h \in H_i$).
  – Now, if we assume for contradiction that $U_i^{n_i+1} \cap \left(\text{span}(\{U_i^j : j \in [n_i]\})\right) \supsetneq \{0\}$, note that the intersection of subspaces $U_i^{n_i+1} \cap \left(\text{span}(\{U_i^j : j \in [n_i]\}\right)$ is a subspace and then the preimage of a subspace is a subspace. Thus,

$$h_{n_i+1}^{-1}(U_i^{n_i+1} \cap \left(\text{span}(\{U_i^j : j \in [n_i]\})\right) \text{ is a subspace of } W_i.$$

I claim that $h_j^{-1}(U_i^j \cap U_i^k)$ is a proper (since $U_i^{n_i+1} \not\subseteq \left(\text{span}(\{U_i^j : j \in [n_i]\})\right)$ TODO: double check!), non-zero subspace that is stable under the action of G, a contradiction since W_i is irreducible, thus proving that $U_i^{n_i+1} \cap \left(\text{span}(\{U_i^j : j \in [n_i]\})\right) = \{0\}$, thus providing a contradiction since $V_i = \text{span}(\{U_i^j : j \in [n_i]\})$ and $U_i^{n_i+1} \subseteq V_i$ implies that $U_i^{n_i+1} = \{0\}$ which would imply that $h_{n_i+1} = 0$ is the zero function, a contradiction.

(b) Let G act on $H_i \otimes W_i$ through the tensor product of the trivial representation of G on H_i and the given representation on W_i. show that the map

$$F : H_i \otimes W_i \to V_i$$

defined by the formula

$$F(\sum h_\alpha \cdot w_\alpha) = \sum h_\alpha(w_\alpha)$$

is an isomorphism of $H_i \otimes W_i$ onto V_i. [Same method].

We must show that

$$\rho_g F = F \rho_g$$

for all $g \in G$. So, first note that

$$\rho_g(\sum c_\alpha h_\alpha \otimes w_\alpha) = \sum c_\alpha h_\alpha \otimes (\rho_g(w_\alpha)),$$

and thus

$$F(\rho_g(\sum c_\alpha h_\alpha \otimes w_\alpha)) = F(\sum c_\alpha h_\alpha \otimes (\rho_g(w_\alpha)))$$

but by definition of F we have

$$F(\sum c_\alpha h_\alpha \otimes (\rho_g(w_\alpha))) = \sum c_\alpha h_\alpha(\rho_g(w_\alpha))$$

but since $h_\alpha \in H_i$ we have that

$$h_\alpha \rho_g = \rho_g h_\alpha$$

and thus

$$\sum c_\alpha h_\alpha(\rho_g(w_\alpha)) = \sum c_\alpha \rho_g(h_\alpha(w_\alpha)) = \rho_g(\sum c_\alpha h_\alpha(w_\alpha)) = \rho_g(F(\sum c_\alpha h_\alpha \otimes w_\alpha))),$$

proving the claim. We now must show that F is a bijection. In particular, my work in part (a) shows that $V_i \cong \bigoplus_{j\in[n_i]} \operatorname{im}(h_j)$ where $\{h_j\}$ is any basis for H_i and also that $\operatorname{im}(h_j) \cong W_i$. So, we note that if $\{w^1, \dots, w^m\}$ is a basis for W_i, then

$$\{h_j \otimes w^k : j \in [n_i], k \in [m]\}$$

is a basis for $H_i \otimes W_i$. Now, to show that F is onto, we note that for any $v \in V_i$ we have since $V_i \cong \bigoplus_{j\in[n_i]} \operatorname{im}(h_j)$ that

$$\bigsqcup_{j\in[n_i]} \{u^j_{i,k} : k \in [m]\}$$

is a basis for V_i where $\{u^j_{i,k} : k \in [m]\}$ is a basis for $\operatorname{im}(h_j)$ for each $j \in [n_i]$. Thus, for all $v \in V_i$, there exist c^j_k for $j \in [n_i], k \in [m]$ such that

$$v = \sum_{j\in[n_i]} \sum_{k\in[m]} c^j_k u^j_{i,k}.$$

Then, $u^j_{i,k} \in \operatorname{im}(h_j)$ for all $j \in [n_i]$ and $k \in [m]$ implies that there exist $w^j_k \in W_i$ (not necessarily distinct) such that $h_j(w^j_k) = u^j_{i,k}$, and thus

$$v = \sum_{j\in[n_i]} \sum_{k\in[m]} c^j_k h_j(w^j_k) = F(\sum_{(j,k)\in[n_i]\times[m]} c^j_k h_j \otimes w^j_k).$$

Finally, F is injective, since otherwise if $F(\sum_{(j,k)\in[n_i]\times[m]} c^j_k h_j \otimes w_k) = 0$ for $\sum_{(j,k)\in[n_i]\times[m]} c^j_k h_j \otimes w_k \neq 0$, then that means precisely that

$$\sum_{(j,k)\in[n_i]\times[m]} c^j_k h_j(w_k) = 0$$

but

$$\sum_{(j,k)\in[n_i]\times[m]} c^j_k h_j(w_k) = \sum_{j\in[n_i]} (\sum_{k\in[m]} h_j(c^j_k w_k)).$$

Now, $c^{j'}_{k'} \neq 0$ for at least one (j', k') which implies that

$$h_{j'}(w_{k'}) = \sum_{(j,k)\in[n_i]\times[m]\setminus(j',k')} \frac{-c^j_k}{c^{j'}_{k'}} h_j(w_k).$$

However, $h_j(c^j_k w_k) \in U^j_i$ for all $j \in [n_i]$ and as shown in (a), $U^{j'}_i \cap \operatorname{span}(\{U^j_i : j \in [n_i] \setminus \{j'\}\}) = \{0\}$ which implies that $h_{j'}(w_{k'}) = 0$ and thus $w_{k'} = 0$ since h'_j is not the zero function, proving injectivity.

(c) Let $(h_1, \dots, h_k)$ be a basis of H_i and form the direct sum $W_i \oplus \dots \oplus W_i$ of k copies of W_i. The system $(h_1, \dots, h_k)$ defines in an obvious way a linear mapping h of $W_i \oplus \dots \oplus W_i$ into V_i; show that it is an isomorphism of representations and that each isomorphism is thus obtainable [apply (b), or argue directly]. In particular, to decompose V_i into a direct sum of representations isomorphic to W_i amounts to choosing a basis for H_i.

The obvious mapping is defined as follows. As I showed in part (a), for this basis $(h_1, \dots, h_{n_i})$ we have that

$$V_i \cong \bigoplus_{j\in[n_i]} U^j_i$$

and

$$U_i^j \cong W_i$$

where U_i^j denotes $U_i^j = \mathrm{im}(h_j)$. Thus, for $h \in H_i$ there of course exist $c_1, \ldots, c_{n_i} \in F$ so that $h = \sum_{j\in[n_i]} c_j h_j$ and then we define

$$h : \bigoplus_{j\in[n_i]} W_i \to V_i$$

by

$$h(\bigoplus_{j\in[n_i]} w_j) = \sum_{j\in[n_i]} c_j h_j(w_j).$$

I claim that $h : \bigoplus_{j\in[n_i]} W_i \to V_i$ as defined above is an isomorphism of representations for all $h \in H_i$. Namely, I must show that

$$h\rho_g = \rho_g h$$

for all $g \in G$ where we define action of G on V_i as usual and action of G on $\bigoplus_{j\in[n_i]} W_i$ by

$$\rho_g(\bigoplus_{j\in[n_i]} w_j) = \bigoplus_{j\in[n_i]} \rho_g(w_j).$$

So,

$$h\rho_g(\bigoplus_{j\in[n_i]} w_j) = h(\bigoplus_{j\in[n_i]} \rho_g(w_j)) = \sum_{j\in[n_i]} c_j h_j(\rho_g(w_j)) = \sum_{j\in[n_i]} c_j \rho_g(h_j(w_j)) \text{ (since } h \in H_i)$$

but then

$$\sum_{j\in[n_i]} c_j \rho_g(h_j(w_j)) = \rho_g(\sum_{j\in[n_i]} c_j h_j(w_j)) = \rho_g(h(\bigoplus_{j\in[n_i]} w_j)).$$