"A First Course in Modular Forms" by Schur and Diamond
Chapter 1.1 Exercises

3. (a) Show that the set $\mathcal{M}_k(SL_2(\mathbb{Z})$ of modular forms of weight k forms a vector space over $\mathbb{C}$.

Note that a function $f : \mathcal{H} \to \mathbb{C}$ is a modular form by definition if

- f is homolorphic on $\mathcal{H}$
- f is weakly modular of weight k, and
- f is holomorphic at ∞.

In order to show that $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is a vector space we must show

i. that addition is associative and commutative,
ii. the existence of an additive identity denoted 0 and an additive inverse for each element,
iii. that scalar multiplication is associative,
iv. that scalar multiplication distributes over vector addition and vice versa,
v. and that $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is closed under addition and scalar multiplication.

Indeed

i. Addition is associative and commutative since for $f, g : \mathcal{H} \to \mathbb{C}$ one has that $f + g$ is defined by $(f + g)(z) = f(z) + g(z)$ and thus the claim follows from the fact that addition in $\mathbb{C}$ is associative and commutative.
ii. Also, one has that the zero map $0 \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ is a modular form of weight k due to quick verification of the thRee necessary properties. Also, it is the additive identity due to the definition of function addition as above.
iii. FurthermoRe, scalar multiplication is associative once again due to definition of scalar multiplication of functions meaning that $a(bf)$ is defined by $(a(bf))(z) = a((bf)(z)) = ab(f(z)) = ((ab)(f))(z)$.
iv. Also scalar multiplication distributes over vector addition since $(a(f+g))(z) = a((f+g)(z)) = af(z)+ag(z) = (af + ag)(z)$.
v. Finally, $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is closed under addition and scalar multiplication.
   - Namely, if $f, g : \mathcal{H} \to \mathbb{C}$ are holomorphic on $\mathcal{H}$ then so is $(f + g) : \mathcal{H} \to \mathbb{C}$.
   - Likewise I claim that if $f, g : \mathcal{H} \to \mathbb{C}$ are weakly modular of weight k then so is $(f + g) : \mathcal{H} \to \mathbb{C}$. In particular, we know by definition that

   $$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ and } g(\gamma(\tau)) = (c\tau + d)^k g(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

   Then, of course

   $$(f+g)(\gamma(\tau)) = (c\tau+d)^k f(\tau)+(c\tau+d)^k g(\tau) = (c\tau+d)^k((f+g)(\tau)) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

   - Finally, I claim that the fact that f, g are holomorphic at ∞ implies that $f + g$ is also holomorphic at ∞. Namely, f, g holomorphic at ∞ means by definition that there exist $z_1, z_2 \in \mathbb{C}$ such that the functions

   $$\hat{f}(z) = \begin{cases} z_1 & \text{if } z = 0 \\ f\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ and } \hat{g}(z) = \begin{cases} z_2 & \text{if } z = 0 \\ g\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ are holomorphic on } D,$$

   where $D = \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the unit disk.

   Then, we see that

   $$\widehat{f+g}(z) = \begin{cases} z_1 + z_2 & \text{if } z = 0 \\ f\left(\frac{log(z)}{2\pi i}\right) + g\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ is holomorphic on } D,$$

   since it is the sum of two holomorphic functions, and thus $f + g$ is holomorphic at infinity, concluding the proof of closuRe under addition.
   - Likewise if $f : \mathcal{H} \to \mathbb{C}$ is holomorphic, then so is $af : \mathcal{H} \to \mathbb{C}$ for all $a \in \mathbb{C}$.
   - FurthermoRe, if

   $$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H},$$

then

$$(af)(\gamma(\tau)) = (c\tau+d)^k (af)(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

since $(af)(\gamma(\tau)) = a(f(\gamma(\tau))) = a(c\tau+d)^k f(\tau) = (c\tau+d)^k (af)(\tau)$.

- Finally, if $f : \mathcal{H} \to \mathbb{C}$ is holomorphic at infinity, that means by definition that there exists $z' \in \mathbb{C}$ such that the function

$$\hat{f}(z) = \begin{cases} z' & \text{if } z = 0 \\ f\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases}$$

is holomorphic on D. Thus, the function

$$\widehat{(af)}(z) = \begin{cases} az' & \text{if } z = 0 \\ (af)\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} = a\hat{f}(z)$$

is holomorphic on D, proving closuRe under scalar multiplication.

(b) If $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k and $g : \mathcal{H} \to \mathbb{C}$ is a modular form of weight ℓ, show that $(fg) : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $k + \ell$.

We need to prove that

i. $(fg) : \mathcal{H} \to \mathbb{C}$ is holomorphic,
ii. $(fg) : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight $k + \ell$,
iii. and $(fg) : \mathcal{H} \to \mathbb{C}$ is holomorphic at infinity.

Indeed,

i. we see that $(fg) : \mathcal{H} \to \mathbb{C}$ is holomorphic since the the product of holomorphic functions is holomorphic.
ii. Likewise, $(fg) : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight $k + \ell$ since

$$(fg)(\gamma(\tau)) = f(\gamma(\tau)g(\gamma(\tau) = ((c\tau+d)^k f(\tau))((c\tau+d)^\ell g(\tau)) = (c\tau+d)^{k+\ell} f(\tau)g(\tau) = (c\tau+d)^{k+\ell}(fg)(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and all $\tau \in \mathcal{H}$.

iii. Finally note that $(fg) : \mathcal{H} \to \mathbb{C}$ is holomorphic at infinity since $f, g : \mathcal{H} \to \mathbb{C}$ holomorphic at infinity implies existence of $z_1, z_2 \in \mathbb{C}$ such that

$$\hat{f}(z) = \begin{cases} z_1 & \text{if } z = 0 \\ f\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ and } \hat{g}(z) = \begin{cases} z_2 & \text{if } z = 0 \\ g\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ are holomorphic on } D,$$

which implies that the function

$$\hat{fg}(z) = \begin{cases} z_1 z_2 & \text{if } z = 0 \\ f\left(\frac{log(z)}{2\pi i}\right) g\left(\frac{log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} = (\hat{f}\hat{g})(z) \text{ is holomorphic on } D,$$

since it is product of holomorphic functions, and thus (fg) is holomorphic at infinity as claimed.

(c) Show that $\mathcal{S}_k(SL_2(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$ and that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.

- Since $\mathcal{S}_k(SL_2(\mathbb{Z})) \subseteq \mathcal{M}_k(SL_2(\mathbb{Z}))$ one need only show that $0 \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ (where 0 denotes the zero function $0 : \mathcal{H} \to \mathbb{C}$), and closuRe under addition and scalar multiplication to show that $\mathcal{S}_k(SL_2(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$.
  - Indeed, $0 \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ since the $0 \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ and constant term in the Fourier expansion for 0 is clearly 0.
  - Also, if $f, g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ then I claim that $(f + g) \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ since
    * $(f + g) \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ by our earlier arguments
    * and since the Fourier expansion of $(f + g)$ is the sum of those for each of f and g meaning that the constant term in the Fourier expansion of $(f + g)$ is zero.
  - Likewise, if $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ then $(af) \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ for all $a \in \mathbb{C}$ since
    * $(af) \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ by our earlier arguments
    * and since the 0th coefficients of the Fourier expansion of (af) is ac_0 where c_0 is the leading coefficient of the Fourier series for f which is zero since $f \in \mathcal{S}_k(SL_1(\mathbb{Z}))$

- Finally, I want to show that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$, meaning we need to show that
  i. $f+g \in \mathcal{S}(SL_2(\mathbb{Z}))$ for all $f, g \in \mathcal{S}(SL_2(\mathbb{Z}))$,
  ii. and $hf \in \mathcal{S}(SL_2(\mathbb{Z}))$ and $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ for all $f \in \mathcal{S}(SL_2(\mathbb{Z}))$ and all $h \in \mathcal{M}(SL_2(\mathbb{Z}))$.

  Indeed,
  i. if one has
  $$f = \bigoplus_{k\in\mathbb{Z}} f_k \in \mathcal{S}(SL_2(\mathbb{Z})) \quad \text{and} \quad g = \bigoplus_{k\in\mathbb{Z}} g_k \in \mathcal{S}(SL_2(\mathbb{Z}))$$
  then we note that
  $$f+g = \bigoplus_{k\in\mathbb{Z}} (f_k + g_k) \in \mathcal{S}(SL_2(\mathbb{Z}))$$
  since for each $k \in \mathbb{Z}$ we have as shown earlier that $f_k + g_k \in \mathcal{S}_k(SL_2(\mathbb{Z}))$.
  ii. FurthermoRe, given any function $h = \bigoplus_{k\in\mathbb{Z}} h_k \in \mathcal{M}(SL_2(\mathbb{Z}))$ and any $f = \bigoplus_{k\in\mathbb{Z}} f_k \in \mathcal{S}(SL_2(\mathbb{Z}))$ one has that $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ due to the following argument. Namely, we note that
  $$fh = \bigoplus_{n\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} f_{n-k} h_k)$$
  and now what Remains to show is that
  $$(\sum_{k\in\mathbb{Z}} f_{n-k} h_k) \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$
  for all $n \in \mathbb{Z}$. Well, we know that
  $$f_{n-k}(\tau) := \sum_{j=1}^{\infty} a_j q^j$$
  and
  $$h_k(\tau) = \sum_{r=0}^{\infty} b_r q^r.$$
  So, we have that
  $$(f_{n-k}h_k)(\tau) = \sum_{j=1}^{\infty} \sum_{r=0}^{\infty} a_j b_r q^{j+r} = \sum_{s=1}^{\infty} (\sum_{m=0}^{s-1} b_s a_{s-m}) q^s.$$
  Thus,
  $$f_{n-k}h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$
  for all $n, k \in \mathbb{Z}$ which gives
  $$\sum_{k\in\mathbb{Z}} f_{n-k}h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$
  for all $n \in \mathbb{Z}$ concluding the proof that $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ as desired. Finally, since $fh = hf$ for all $f, h : \mathcal{H} \to \mathbb{C}$ we see that $hf \in \mathcal{S}(SL_2(\mathbb{Z}))$ as well, proving that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.

4. Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 - \{(0,0)\}$.

   (a) Show that the series $S = \sum_{(c,d)\in L'} (\sup\{|c|, |d|\})^{-k}$ converges by considering the partial sums over expanding squares.

   We decompose the sum as

$$\begin{aligned} S_n &= \sum_{(c,d)\in[-n,n]^2\setminus\{(0,0)\}} (\sup\{|c|,|d|\})^{-k} \\ &= \sum_{c=0} \sum_{d=-n}^{n} |d|^{-k} + \sum_{c=-n}^{n} \sum_{d=0} |c|^{-k} + \sum_{\substack{(c,d)\in[-n,n]^2 \\ :c\neq 0\neq d, |d|>|c|}} |d|^{-k} + \sum_{\substack{(c,d)\in[-n,n]^2 \\ :c\neq 0\neq d, |c|>|d|}} |c|^{-k} + \sum_{\substack{(c,d)\in[-n,n]^2 \\ :c\neq 0\neq d, |c|=|d|}} |c|^{-k}. \end{aligned}$$

Thus,

$$\begin{aligned}
S_n &= 2\sum_{d=2}^{n}(d-1)(d^{-k}) + 8\sum_{d=1}^{n} d^{-k} = 2\sum_{d=2}^{n}(d-1)(d^{-k}) + 8 + 8\sum_{d=2}^{n} d^{-k} \\
&= 2\sum_{d=2}^{n} d(d^{-k}) - 2\sum_{d=2}^{n} d^{-k} + 8 + 8\sum_{d=2}^{n} d^{-k} \\
&= 2\sum_{d=2}^{n} d^{-k+1} + 8 + 6\sum_{d=2}^{n} d^{-k} \\
&= 8 + 2\sum_{d=2}^{n} d^{-k+1} + 6\sum_{d=2}^{n} d^{-k}.
\end{aligned}$$

We now want to show that as $n \to \infty$ we have that $\lim_{n\to\infty} S_n < \infty$. So, we note that

$$0 \leq S_n \leq 8 + 2\sum_{d=2}^{n} d^{-k+1} + 6\sum_{d=2}^{n} d^{-k}$$

Of course, that means

$$0 \leq S_n \leq 8 + 8\sum_{d=2}^{n} d^{-k+1} \leq 8 + 8\sum_{d=2}^{n} d^{-2}$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

we have that

$$0 \leq \lim_{n\to\infty} S_n \leq 8 + \frac{8\pi^2}{6} < \infty,$$

concluding the proof.

(b) Fix positive numbers A, B and let $\Omega = \{\tau \in \mathcal{H} : |Re(\tau)| \leq A, Im(\tau) \geq B\}$. Prove that there is a constant $C > 0$ such that $|\tau + \delta| > C \sup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbb{R}$.

There are two cases: either $\sup\{1, |\delta|\} = 1$ or $\sup\{1, |\delta|\} = |\delta|$.

If $\sup\{1, |\delta|\} = 1$ then simply let $C = B$ and note that $|\tau + \delta| > Im(\tau + \delta) \geq B = B\sup\{1, |\delta|\}$.

If $\sup\{1, |\delta|\} = |\delta|$ then we have two natural subcases, either $|\delta| > 2A$ or $1 \leq |\delta| \leq 2A$. If $|\delta| > 2A$ then $A < \frac{|\delta|}{2}$ meaning that

$$\begin{aligned}
|\tau + \delta| \geq |Re(\tau - \delta)| = |Re(\tau) - \delta| \geq ||Re(\tau)| - |\delta|| &= max(|Re(\tau)| - |\delta|, |\delta| - |Re(\tau)|) \geq max(|Re(\tau)| - |\delta|, |\delta| - A) \\
&\geq |\delta| - A > |\delta| - \frac{|\delta|}{2} = \frac{|\delta|}{2}
\end{aligned}$$

and thus

$$|\tau + \delta| > \frac{1}{2}\sup\{1, |\delta|\}.$$

Finally if $\sup\{1, |\delta|\} = |\delta|$ and $1 \leq |\delta| \leq 2A$ then one has $Im(\tau) > A$ or $B \leq Im(\tau) \leq A$.

If $B \leq Im(\tau) \leq A$ then note that $-A + \delta \leq Re(\tau + \delta) \leq A + |\delta|$ and $B \leq Im(\tau + \delta) \leq A$ proving that $B \leq |\tau + \delta| \leq 2A + |\delta|$. Then, since $1 \leq |\delta| \leq 2A$ one has that $F(\tau, \delta) = \frac{|\tau+\delta|}{|\delta|}$ is a continuous function $F : (\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A]) \to \mathbb{C}$ where $\Omega' = \Omega \cap \{z \in \mathbb{C} : Im(z) \leq A\}$ and since $(\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A])$ is a compact set, that implies that $M := min(\frac{|\tau+\delta|}{|\delta|})$ is attained by some $(\tau, \delta) \in (\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A])$ and thus

$$|\tau + \delta| \geq M|\delta| = M \sup\{1, |\delta|\}.$$

If $Im(\tau) > A$ then note that

$$|\tau + \delta| \geq |Im(\tau + \delta)| = Im(\tau) > A$$

and since $1 \leq |\delta| \leq 2A$ that implies that $A \geq \frac{|\delta|}{2}$ implying that

$$|\tau + \delta| > \frac{1}{2}|\delta| = \frac{1}{2}\sup\{1, |\delta|\}.$$

So, we let

$$C = min(B, \frac{1}{2}, M).$$

(c) Use parts (a) and (b) to prove that the series defining G_k converges absolutely and uniformly for $\tau \in \Omega$. Conclude that G_k is holomorphic on $\mathcal{H}$. Let $\mathbb{Z}^{2\,\prime} = \mathbb{Z}^2 \setminus \{(0,0\}$.

- We first note that $|c\tau + d| = |c||\tau + \frac{d}{c}|$.
- Then, by part (b), $|c\tau + d| \geq |c|C\sup\{1, |\frac{d}{c}|\} = C\sup\{|c|, |d|\}$ for all $\tau \in \Omega$.
- Then,

$$\sum_{(c,d)\in(\mathbb{Z}^2)'} \frac{1}{|c\tau+d|^k} \leq \sum_{(c,d)\in(\mathbb{Z}^2)'} \frac{1}{(C\sup\{|c|,|d|\})^k} = \frac{1}{C^k}\sum_{(c,d)\in(\mathbb{Z}^2)'} \frac{1}{\sup\{|c|,|d|\}^k},$$

which means by part (a) that

$$\sum_{(c,d)\in(\mathbb{Z}^2)'} \frac{1}{|c\tau+d|^k} < \infty \text{ for all } \tau \in \Omega,$$

proving that the series representing G_k converges absolutely for $z \in \Omega$.

Now, to show that the series representing G_k converges uniformly we note that the fact that the series representing G_k converges absolutely implies that the series converges unconditionally meaning that any reordering of the series converges. We have

$$G_k(\tau) = \sum_{(c,d)\in(\mathbb{Z}^2)'} \frac{1}{(c\tau+d)^k} = \sum_{n=1}^{\infty}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{(c\tau+d)^k}\Big).$$

We denote

$$T_N(\tau) = \sum_{n=1}^{N}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{(c\tau+d)^k}\Big).$$

I claim that for fixed $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|G_k(\tau) - T_M(\tau)|$ for all $M \geq N$. Namely, note that for arbitrary $m \in \mathbb{N}$ and arbitrary $\tau \in \Omega$ one has that

$$\begin{aligned}
|G_k(\tau) - T_m(\tau)| = \left|\sum_{n=m+1}^{\infty}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{(c\tau+d)^k}\Big)\right| &\leq \sum_{n=m+1}^{\infty}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{|c\tau+d|^k}\Big) \\
&\leq \sum_{n=m+1}^{\infty}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{(C\sup\{|c|,|d|\})^k}\Big) \\
&= \frac{1}{C^k}\sum_{n=m+1}^{\infty}\Big(\sum_{\substack{\{(c,d)\in(\mathbb{Z}^2)':\\ |c|+|d|=n\}}} \frac{1}{\sup\{|c|,|d|\}^k}\Big) \\
&= \frac{1}{C^k}\sum_{n=m+1}^{\infty}\Bigg(\frac{1}{\sup\{|-n|,|0|\}^k} + \frac{1}{\sup\{|n|,|0|\}^k} \\
&\quad + \sum_{c\in[-n+1,n-1]}\Big(\frac{1}{\sup\{|c|,n-|c|\}^k} + \frac{1}{\sup\{|c|,n-|c|\}^k}\Big)\Bigg) \\
&= \frac{1}{C^k}\sum_{n=m+1}^{\infty}\Bigg(\frac{2}{\sup\{|n|,|0|\}^k} + \sum_{c\in[-n+1,n-1]}\frac{2}{\sup\{|c|,n-|c|\}^k}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{2}{\sup\{|n|,|0|\}^k} + \frac{2}{\sup\{0,n\}^k} \right. \\
&\quad \left. + \sum_{c\in[1,n-1]} \frac{4}{\sup\{|c|, n-|c|\}^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{\sup\{|n|,|0|\}^k} + \sum_{c\in[1,n-1]} \frac{4}{\sup\{|c|, n-|c|\}^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{\sup\{|n|,|0|\}^k} + \sum_{c\in[1,\lfloor \frac{n}{2} \rfloor]} \frac{4}{(n-|c|)^k} \right. \\
&\quad \left. + \sum_{c\in[\lfloor \frac{n}{2} \rfloor+1,n-1]} \frac{4}{|c|^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{n^k} \right) + \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c\in[1,\lfloor \frac{n}{2} \rfloor]} \frac{4}{(n-|c|)^k} \right. \\
&\quad \left. + \sum_{c\in[\lfloor \frac{n}{2} \rfloor+1,n-1]} \frac{4}{|c|^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{n^k} \right) + \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c\in[n-\lfloor \frac{n}{2} \rfloor,n-1]} \frac{4}{c^k} \right. \\
&\quad \left. + \sum_{c\in[\lfloor \frac{n}{2} \rfloor+1,n-1]} \frac{4}{c^k} \right) \\
&= \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c\in[n-\lfloor \frac{n}{2} \rfloor,n-1]} \frac{1}{c^k} \right) \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^k} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{\lfloor \frac{n}{2} \rfloor + 1}{(\lfloor \frac{n}{2} \rfloor + 1)^k} = \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^{k-1}} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^2} \leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^2} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} = \frac{12}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} \leq \frac{12}{C^k} \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{2}{l^2} = \frac{24}{C^k} \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l^2}.
\end{aligned}
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Now, note that

$$\frac{1}{l^2} \leq \frac{1}{l(l-1)}$$

which implies that

$$\sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l^2} \leq \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l(l-1)} = \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l(l-1)} = \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \left(\frac{1}{l-1} - \frac{1}{l}\right) = \frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)}.$$

So, for fixed $\epsilon > 0$, let $N(\epsilon)$ be defined so that

$$\frac{24}{C^k} \frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)} < \epsilon$$

for all $m \geq N$ since that will imply that

$$|G_k(\tau) - T_m(\tau)| < \epsilon$$

for all $m \geq N$. In particular, let N be so that

$$\frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)} < \frac{\epsilon C^k}{24}$$

for all $m \geq N$ or equivalently

$$(\lfloor \frac{m+1}{2} \rfloor - 1) > \frac{24}{\epsilon C^k}$$

meaning

$$\lfloor \frac{m+1}{2} \rfloor > \frac{24}{\epsilon C^k} + 1$$

for all $m \geq N$. So, let N be so that

$$\frac{m+1}{2} - 1 > \frac{24}{\epsilon C^k} + 1$$

for all $m \geq N$ or more precisely so that

$$m + 1 > \frac{48}{\epsilon C^k} + 4$$

for all $m \geq N$ meaning we may let

$$N = \left\lfloor \frac{48}{\epsilon C^k} + 4 \right\rfloor,$$

proving uniform convergence.

(d) Show that for $\gamma \in SL_2(\mathbb{Z})$, right multiplication by γ defines a bijection from L' to L'.

Consider

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Since $det(\gamma) \neq 0$, that implies that $\gamma : \mathbb{C} \to \mathbb{C}$ is injective which implies that $\gamma|_{L'}$ is also injective. To show that $\gamma|_{L'} : L' \to L'$ is surjective we note that

$$\gamma^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus, arbitrary $(x, y)^T \in L'$ one can produce $(u, w)^T \in L'$ such that $\gamma((u, w)^T) = (x, y)^T$. Namely, let

$$(u, w)^T = \gamma^{-1}((x, y)^T)$$

and note that since $\gamma^{-1} \in SL_2(\mathbb{Z})$ we have that $(u, w) \in \mathbb{Z}^2$ and since $(x, y) \neq (0, 0)$ and γ^{-1} is injective we have that $(u, w) \neq (0, 0)$ meaning $(u, w)^T \in L'$ as desired.

(e) Use the calculation from (c) to show that G_k is bounded on Ω.

Indeed, we see that as noted in part (c)

$$\sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{|c\tau + d|^k} \leq \frac{1}{C^k} \sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{\sup\{|c|, |d|\}^k},$$

and by part (a)

$$\sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{\sup\{|c|, |d|\}^k} \leq 8 + \frac{8\pi^2}{6},$$

and thus

$$\left|G_k(\tau)\right| = \left| \sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{(c\tau + d)^k} \right| \leq \frac{1}{C^k}(8 + \frac{8\pi^2}{6})$$

for all $\tau \in \Omega$, proving boundedness.

(f) From the text and part (d), G_k is weakly modular so in particular $G_k(\tau + 1) = G_k(\tau)$. Show that therefore $G_k(\tau)$ is bounded as $Im(\tau) \to \infty$.

Indeed, fix $A = 1$, $B = 1$ and consider the corresponding region $\Omega(A, B)$ and constant $C = C(A, B)$. Note that for all $\tau \in \Omega$ one has that $|G_k(\tau)| \leq \frac{1}{C^k}(8 + \frac{8\pi^2}{6})$. In particular, for arbitrary $\tau \in \mathbb{C}$ with $Im(\tau) \geq 2$ one has that

$$\tau = Re(\tau) + iIm(\tau) = \lfloor Re(\tau) \rfloor + (Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau).$$

Since $|(Re(\tau) - \lfloor Re(\tau) \rfloor)| < 1$ and $Im(\tau) \geq 2 \geq B$ one has that $(Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau) \in \Omega$. Also, $\lfloor Re(\tau) \rfloor \in \mathbb{Z}$ implies that $G_k(\lfloor Re(\tau) \rfloor + s) = G_k(s)$ for all $s \in \mathbb{C}$. Thus,

$$G_k(\tau) = G_k((Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau))$$

and thus

$$|G_k(\tau)| \leq \frac{1}{C^k}(8 + \frac{8\pi^2}{6})$$

for all $\tau \in \mathbb{C}$ with $Im(\tau) \geq 2$. Thus,

$$0 \leq \lim_{Im(\tau)\to\infty} |G_k(\tau)| \leq \lim_{Im(\tau)\to\infty} \frac{1}{C^k}(8 + \frac{8\pi^2}{6}) = \frac{1}{C^k}(8 + \frac{8\pi^2}{6})$$

since C depends only on A, B and A, B do not depend on τ.