Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 3.4
Show that each irreducible representation of G is contained in a representation induced by an irreducible representation of H. Obtain from this another proof of the Corollary to Theorem 9.
We complete the above by way of the following steps:

- Note that any irreducible representation (ρ', V') is contained as an isomorphic copy $(\rho|_{V_0'}, V_0')$ of the regular representation (ρ, V) via the bijection $k : V' \to V_0'$. Specifically, that means that $V_0' \subseteq V$ and that $(\rho|_{V_0'})_g \circ k = k \circ \rho_g'$ for all $g \in G$. We note that either $V' = 0$ or $V' \neq 0$. If $V' = 0$ then $V_0' = 0$ and thus V_0' is contained in the representation induced by the zero representation $(\rho|_H, 0)$ of H, concluding the proof. So, from now on we assume $V' \neq 0$.

- Now, note that the restriction $(\rho|_{V_0'}|_H, V_0')$ of the representation $\rho_{V_0'} : G \to V_0'$ is a representation of H and, as such, contains a non-zero irreducible representation $V_{0,H}' \subseteq V_0'$ of H.

- Now, one can form the induced representation $V_{0,ind} := \sum_{r \in R} \rho_r V_{0,H}'$ by letting R be a set of representatives for the left cosets of H in G, and we note that the induced representation $(\rho_{0,ind}, V_{0,ind})$, with action defined by $(\rho_{0,ind})_g = \rho_g$, is a subset $V_{0,ind} \subseteq V$ of the regular representation.

- Now, we may apply the key lemma of this chapter which allows us to extend a linear function f defined on an irreducible representation of H to a linear function F defined on the induced representation of G, which respects the structure of the associated representations.

- Namely, let f be the natural inclusion map $f : V_{0,H}' \to V'$. The aforementioned lemma allows us to extend f to a linear map $F : V_{0,ind} \to V'$ such that $F \circ (\rho_{0,ind})_g = \rho_g' \circ F$ for all $g \in G$.

- Now, we note that since (ρ', V') is an irreducible representation of G we have that F is surjective or the zero map. To be a little more clear about the details, note that $F \circ (\rho_{0,ind})_g = \rho_g' \circ F$ implies that $\rho_g'(F(v)) \subseteq im(F)$ for all $v \in V_{0,ind}$, which put simply says that $\rho_g'(im(F)) \subseteq im(F)$ for all $g \in G$ or that $im(F)$ is stable under the action of ρ_g'. By irreducibility of V', we have that $im(F) = 0$ or $im(F) = V'$. Note that F is not the zero map since $F|_{V_{0,H}'} = id_{V_{0,H}'}$.

- Also, we note that $ker(F) \subseteq V_{0,ind}$ is stable under the action of $(\rho_{0,ind})_g$. In more detail, $F \circ (\rho_{0,ind})_g = \rho_g' \circ F$ for all $g \in G$ means that for $w \in ker(F)$ we have that $F((\rho_{0,ind})_g(w)) = 0$ for all $g \in G$ and thus $(\rho_{0,ind})_g(w)$ for all $g \in G$ and all $w \in ker(F)$ meaning that $ker(F)$ is stable under the action of $\rho_{0,ind}$. Since G is a finite group, we know that the orthogonal complement $ker(F)^\perp$ of $ker(F)$ inside of $V_{0,ind}$, which exists since $V_{0,ind}$ is a finite dimensional complex vector space, and thus a Hilbert space, is also stable under the action of $\rho_{0,ind}$.

- So, finally we have that the map $F|_{ker(F)^\perp} : ker(F)^\perp \to V'$ is an isomorphism of representations, which means precisely that $F|_{ker(F)^\perp} \circ (\rho_{0,ind})_g = \rho_g' \circ F|_{ker(F)^\perp}$ for all $g \in G$, or put more simply $(\rho_g', V') \subseteq (\rho_{0,ind}, V_{0,ind})$ is contained in the representation $(\rho_{0,ind}, V_{0,ind})$ induced by the irreducible representation $((\rho_{V_0'})|_H, V_{0,H}')$ of H.

Exercise 3.5
Let (W, θ) be a linear representation of H. ....

Exercise 3.6