Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 3.1
Show directly, using Schur's Lemma, that each irreducible representation of an abelian group, finite or not, has degree 1.
Consider an irreducible representation (ρ, V) of G. Now, G abelian implies that (for fixed $g \in G$) $\rho_g \rho_h = \rho_h \rho_g$ for all $h \in G$. Thus, we may apply Schur's lemma to conclude that $\rho_g = \lambda_g I$ is a scalar multiple of the identity. This applies for all $g \in G$ meaning that $\rho_g = \lambda_g I$ for all $g \in G$. However, this provides a contradiction unless $dim(V) = 1$ since otherwise $span(\{e_1\}) \subseteq V$ is a proper non-trivial subrepresentation.

Exercise 3.2
Let ρ be an irreducible representation of G of degree n and character χ; let C be the center of G (i.e. the set of $s \in G$ such that $st = ts$ for all $t \in G$), and let c be its order.

(a) Show that ρ_s is a homothety for each $s \in C$. [Use Schur's Lemma]. Deduce from this that $|\chi(s)| = n$ for all $s \in C$.

Well, for $s \in C$ we have that $\rho_s \rho_h = \rho_h \rho_s$ for all $h \in G$ which implies by Schur's lemma that $\rho_s = \lambda_s I$ is a scalar multiple of the identity for all $s \in C$. The proof of Proposition 1 part (ii) on page 11 shows that $|\lambda_s| = 1$ (here $||$ denotes complex modulus). Thus, $|\chi(s)| = |\lambda(s)|n = n$.

(b) Prove the inequality $n^2 \leq g/c$.
Note that $\sum_{s \in G} |\chi(s)|^2 = g$ and $\sum_{s \in C} |\chi(s)|^2 + \sum_{s \in G \setminus C} |\chi(s)|^2 = cn^2 + \sum_{s \in G \setminus C} |\chi(s)|^2$. So, $cn^2 \leq g$ which proves the claim.

(c) Show that, if ρ is faithful (i.e. $\rho_s \neq 1$ for $s \neq 1$), the group C is cyclic.
It suffices to show that $\{\lambda_s : s \in C\}$ is cyclic. Note that since C is a group we have that $\{\rho_s : s \in C\}$ is a group and thus $\{\lambda_s : s \in C\} \subseteq \{z \in \mathbb{C} : |z| = 1\}$ is also a group. I claim that any finite subgroup of $\{z \in \mathbb{C} : |z| = 1\}$ is cyclic. Of course, any finite group is finitely generated, so assume for contradiction that $\{\lambda_s : s \in C\}$ has a generating set of minimum size $\langle \lambda_{s_1}, \ldots, \lambda_{s_j} \rangle$ of at least two elements ($j \geq 2$). Then, note that $\lambda_{s_1} = \zeta_{ord(s_1)}^{r_1}$ for some r_1 coprime to $ord(s_1)$. Likewise $\lambda_{s_2} = \zeta_{ord(s_2)}^{r_2}$ for some r_2 coprime to s_2. Then, note that we have that $\lambda_{s_1} = \zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)}$ and $\lambda_{s_2} = \zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)}$. Now, by Bezout's identity there exist $a, b \in \mathbb{Z}$ such that $ar_1 ord(s_2) + br_2 ord(s_1) = gcd(r_1 ord(s_2), r_2 ord(s_1))$. Furthermore, for all $m \in \mathbb{Z}$ we have that $mgcd(r_1 ord(s_2), r_2 ord(s_1)) = mar_1 ord(s_2) + mbr_2 ord(s_1)$ meaning that $\zeta_{ord(s_1)ord(s_2)}^{mgcd(r_1 ord(s_2), r_2 ord(s_1))} = \zeta_{ord(s_1)ord(s_2)}^{mar_1 ord(s_2) + mbr_2 ord(s_1)} = (\zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)})^{ma} (\zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)})^{mb}$ or in other words that

$$\langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))} \rangle \subseteq \langle \zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)}, \zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)} \rangle = \langle \lambda_{s_1}, \lambda_{s_2} \rangle.$$

Furthermore

$$\lambda_{s_1} = (\zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))})^{\frac{r_1 ord(s_2)}{gcd(r_1 ord(s_2), r_2 ord(s_1))}}$$

and

$$\lambda_{s_2} = (\zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))})^{\frac{r_2 ord(s_1)}{gcd(r_1 ord(s_2), r_2 ord(s_1))}},$$

implying that

$$\lambda_1, \lambda_2 \in \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))}$$

which means

$$\langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))} \rangle = \langle \lambda_{s_1}, \lambda_{s_2} \rangle.$$

So, we have obtained a contradiction, namely that

$$\langle \lambda_{s_1}, \lambda_{s_2}, \lambda_{s_3}, \ldots, \lambda_{s_j} \rangle = \langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))}, \lambda_{s_3}, \ldots, \lambda_{s_j} \rangle,$$

which is impossible since we said that $\lambda_{s_1}, \lambda_{s_2}, \lambda_{s_3}, \ldots, \lambda_{s_j}$ was a generating set for $\{\lambda_s : s \in C\}$ of minimum size. Thus, C is cyclic.

Exercise 3.3