A First Course in Modular Forms
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1 Chapter 1 Notes and Exercises

1.1 Chapter 1.1

Exercise 1.1.4

Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 - \{(0,0)\}$.

- (a) Show that the series $S = \sum_{(c,d)\in L'} (sup\{|c|,|d|\})^{-k}$ converges by considering the partial sums over expanding squares.
  Indeed we decompose the sum as $S_n = \sum_{c=-n}^{n}\sum_{d=-n}^{n}(sup\{|c|,|d|\})^{-k} = \sum_{c=-n}^{n}\sum_{d=-n}^{n}(sup\{|c|,|d|\})^{-k} = \sum_{c=0}\sum_{d=-n}^{n}|d|^{-k} + \sum_{c=-n}^{n}\sum_{d=0}|c|^{-k} + \sum_{(c,d)\in[-n,n]^2:c\neq 0\neq d,|d|>|c|}|d|^{-k} + \sum_{(c,d)\in[-n,n]^2:c\neq 0\neq d,|c|>|d|}|c|^{-k} + \sum_{(c,d)\in[-n,n]^2:c\neq 0\neq d,|c|=|d|}|c|^{-k}$.
  Thus, $S_n = 2\sum_{d=2}^{n}(d-1)(d^{-k}) + 8\sum_{d=1}^{n}d^{-k} = 2\sum_{d=2}^{n}(d^{-k+1}) + \sum_{d=2}^{n}(-2(d^{-k})) + \sum_{d=1}^{n}8d^{-k} = 2\sum_{d=2}^{n}(d^{-k+1}) + 8 + \sum_{d=2}^{n}6d^{-k}$. We want to show that as $n\to\infty$ we have that $S_n \to S$ where $S<\infty$. Well, note that $0 \leq S_n \leq 2\sum_{d=2}^{n}(d^{-2}) + 8 + \sum_{d=2}^{n}6d^{-2} \leq 8 + \sum_{d=1}^{n}8d^{-2}$. Of course, that means $0 \leq S_n \leq 8 + 8\sum_{d=1}^{n}\frac{1}{d^2}$ and since $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ we have that $0 \leq \lim_{n\to\infty} S_n \leq 8 + \frac{8\pi^2}{6} < \infty$. So, indeed, this sum is finite.
- (b) Fix positive numbers A, B and let $\Omega = \{\tau \in \mathcal{H} : |Re(\tau)| \leq A, Im(\tau) \geq B\}$. Prove that there is a constant $C > 0$ such that $|\tau + \delta| > Csup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbb{R}$. (More to come).
- (c)

Exercise 1.2.2

Let $N \in \mathbb{N}_{\geq 0}$. Let $\gamma \in SL_2(\mathbb{Z}/N\mathbb{Z})$ be given. Lift to matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$$

- (a) Show that $gcd(c,d,N) = 1$. Show that $gcd(c',d') = 1$ for some $c' = c + sN$ and $d' = d + tN$ where $s,t \in \mathbb{Z}$.
  First, note $\gamma \in SL_2(\mathbb{Z}/N\mathbb{Z})$ means that

  $$\gamma = \begin{pmatrix} [a] & [b] \\ [c] & [d] \end{pmatrix}$$

  where $[a],[b],[c],[d] \in \mathbb{Z}/N\mathbb{Z}$ (meaning $a,b,c,d \in \mathbb{Z}$ are the corresponding representatives) and

  $$[a][d] - [b][c] = [1]$$

  or equivalently

  $$ad - bc = 1 + rN$$

  for some $r \in \mathbb{Z}$.
  So, indeed by definition, since we have $a, -b, -r \in \mathbb{Z}$ such that

  $$-bc + ad - rN = 1$$

  we have that $gcd(c,d,N) = 1$. (In particular, if c,d,N did have some common divisor $\alpha \in \mathbb{N}_{\geq 2}$ then we could factor $c = \alpha c', dd = \alpha d'$ and $N = \alpha N'$ which would then mean that for any numbers $B, A, R \in \mathbb{Z}$ we would have

  $$\alpha \mid Bc + Ad + RN.$$

  (More to come).