Galois Representations (Chapter 9 in A First Course in Modular Forms)

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Works Cited

Diamond, Fred, and Jerry Michael Shurman. A First Course in Modular Forms. Springer, 2016.

Note: I am simply making this presentation to force myself to read this text more carefully than I otherwise would. I am also starting from the last chapter as recommended by Professor Matthew Baker. From there I am going back to see where all previous material fits in and to read that material in more detail. (I am uploading exercises from previous chapters as part of that effort). I will update this periodically.

Basic Concepts

- All elliptic curves over $\mathbb{Q}$ arise from modular forms.
- We can think of elliptic curves and modular curves as Riemann surfaces.
- Elliptic curves form abelian groups. Modular curves do not.
- The set of weight 2 cusp forms associated to a modular curve form a vector space over $\mathbb{C}$. The dimension of this space is the genus of the curve.
- Then, once one has such a vector space, one notes that Hecke operators act (linearly) on this space.
- "Integral homology is a lattice in the dual space and is stable under the Hecke action".

- Equations over number fields define elliptic curves (usually over $\mathbb{Q}$).
- We can reduce equations modulo p where p is a prime to glean more insight.
- In particular, by reducing elliptic and modular curves modulo p we can obtain useful relations, namely that
  - $a_p(E) = \sigma_{p,*} + \sigma_p^*$ is an endomorphism of $Pic^0(\tilde{E})$, and that
  - $T_p = \sigma_{p,*} + \sigma_p^*$ is an endomorphism of $Pic^0(\tilde{X_0}(N))$ (which is famously known as the Eichler-Shimura Relation).
- Each of these relations hold for ALL BUT FINITELY MANY primes p.
- As one varies the primes p, the associated geometric objects change.

- One can lift the above relations from characteristic p to infinite characteristic.
- For any prime ℓ, one obtains a vector space $V_\ell(E)$ over the ℓ-adic field $\mathbb{Q}_\ell$ arising from the ℓ-power torsion group of the associated elliptic curve.
- Likewise in the case of modular curves, for any prime p one obtains a ℓ-adic vector space $V_\ell(X)$ arising from the $\ell-$power torsion groups of the Picard group.
- Then, one notes that the action of the ABSOLUTE GALOIS GROUP of $\mathbb{Q}$ on $V_\ell(E)$ and $V_\ell(X)$ respectively is a valid group action. (Note that the absoulte Galois group of $\mathbb{Q}$ is defined to be the group $G_\mathbb{Q} = \{\text{field automorphisms of the algebraic closure } \overline{Q}\}$.
- This group "subsumes" the Galois groups of all finite degree extensions. Also, for any maximal ideal $\mathfrak{p} \subseteq \overline{Z}$ lying over rational primes p, one notes that $G_\mathbb{Q}$ contains the ABSOLUTE FROBENIUS ELEMENTS $Frob_\mathfrak{p}$. (To double check: It seems that ℓ and p are allowed to be distinct primes).

- From the above, we obtain the following relations, which HOLD for a DENSE SUBSET OF ELEMENTS $Frob_{\mathfrak{p}}$ in $G_{\mathbb{Q}}$:
  - $Frob_{\mathfrak{p}}^2 - a_p(E)Frob_{\mathfrak{p}} + p = 0$ (which is an endomorphism of $V_\ell(E)$), and that
  - $Frob_{\mathfrak{p}}^2 - T_p Frob_{\mathfrak{p}} + p = 0$ (which is an endomorhpism of $V_\ell(X_0(N))$.
- Note that now, because of the above relations, each of the above relations "involves a single vector space as $Frob_{\mathfrak{p}}$ varies".
- Also, note that the second equation relates the Hecke action and the Galois action on the associated vector spaces for modular curves.

- Punchline: The vector spaces V_ℓ (for various primes ℓ) are GALOIS REPRESENTATIONS of the group $G_{\mathbb{Q}}$.
- IMPORTANT: The associated Galois representation to a modular curve can be decomposed into pieces associated to modular forms.
- In this context, the MODULARITY THEOREM says that **The Galois representation associated to any elliptic curve over** $\mathbb{Q}$ **arises from such a piece as described above**.
- Note that the above is part of a lager aim which is to show that Galois representations which show up in algebraic geometry arise as Galois representations associated to modular forms.

The End