

**Introduction to Representation Theory: Math 18.712**  
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**[1.20]** Let  $V$  be a nonzero finite dimensional representation of an algebra  $A$ . Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.

Either  $V$  is irreducible or it is not. If not, then it has some proper non-trivial subrepresentation  $V_1$  of dimension  $\dim(V_1) < \dim(V)$ . We continue in this fashion finding subrepresentations

$$V_i \subsetneq V_{i-1} \subsetneq V_1 \subsetneq V$$

until  $V_i$  is irreducible which we know is possible for the following reason. We note that any representation of dimension 1 is irreducible and indeed the dimension decreases by at least one at each step, or written precisely that  $\dim(V_i) \leq \dim(V_{i-1}) - 1$ , meaning that  $1 \leq \dim(V_i) \leq \dim(V) - i$ . So, indeed provided that  $\dim(V) > 0$  this process will make sense and terminate since  $1 \leq \dim(V) - i$  means that  $i \leq \dim(V) - 1 < \infty$ .

**[1.21]** Problem 1.21. Let  $A$  be an algebra over a field  $k$ . The center  $Z(A)$  of  $A$  is the set of all elements  $z \in A$  which commute with all elements of  $A$ . For example, if  $A$  is commutative then  $Z(A) = A$ .

- (a) Show that if  $V$  is an irreducible finite dimensional representation of  $A$  then any element  $z \in Z(A)$  acts in  $V$  by multiplication by some scalar  $\chi_V \in k$ . Show that  $\chi_V : Z(A) \rightarrow k$  is a homomorphism. It is called the central character of  $V$ .

This makes intuitive sense since the center of the  $GL_n(\mathbb{R})$  for instance is the set of scalar multiples of the identity. Now I provide a formal proof.

First, I note that we also need the field  $k$  to be algebraically closed otherwise taking  $A = \mathbb{C}$  (as an  $\mathbb{R}$ -algebra) we note that  $Z(A) = A$ . Then, one notes that  $V = A$  is a 2-dimensional representation over  $k = \mathbb{R}$  (not algebraically closed). Indeed, taking the regular representation and the element  $g = 1 + i = (1, 1) \in Z(A)$ , we note that  $g$  acts on an element  $v = (a, b) \in V$  by

$$(a, b) \mapsto^g (a - b, a + b),$$

and clearly taking  $(a, b) = (0, 1)$  we see that

$$(a - b, a + b) = (-1, 1) \neq \lambda(0, 1)$$

for any  $\lambda \in k = \mathbb{R}$ .

However, assuming that  $k$  is algebraically closed, we proceed to prove the statement. Indeed, we cite Corollary 1.17, noting that for any  $z \in Z(A)$  we have that  $\rho(z)$  is an intertwining operator within  $\rho(A)$  since for any  $a \in A$  to verify  $\rho(z)$  is an intertwining operator we need to show that

$$(\rho(z))(\rho(a)v) = \rho(a)(\rho(z))(v),$$

for all  $v \in V$  and all  $a \in A$ .

Indeed, we note that

$$(\rho(z))(\rho(a)v) = \rho(za)v = \rho(az)v = \rho(a)\rho(z)v.$$

Then Corollary 1.17 gives the result.

- (b) Show that if  $V$  is an indecomposable finite dimensional representation of  $A$  then for any  $z \in Z(A)$  the operator  $\rho(z)$  by which  $z$  acts in  $V$  has only one eigenvalue  $\chi_V(z)$ , equal to the scalar by which  $z$  acts on some irreducible subrepresentation of  $V$ . Thus  $\chi_V : Z(A) \rightarrow k$  is a homomorphism, which is again called the central character of  $V$ .

First, I show that if  $\rho$  has an eigenvalue then it is unique and then I will show at the end that it has an eigenvalue, which is clearly true by virtue of  $k$  being algebraically closed.

Suppose there exist  $\rho_1, \rho_2$  such that

$$\rho(z)v_1 = \lambda_1 v_1$$

and

$$\rho(z)v_2 = \lambda_2 v_2$$

for some  $\lambda_1, \lambda_2 \in k$  and  $v_1, v_2 \in V$  with  $v_1, v_2 \neq 0$ .

Now, for this fixed  $z \in A$  let  $W = \{v \in V : \rho(z)v = \lambda_1 v\}$ . This is a vector subspace.

I would like to show that  $W$  is a subrepresentation, namely that for all  $w \in W$  and all  $a \in A$  we have that  $\rho(a)w \in W$ .

Assume not. Assume that there exists  $w \in W$  such that  $\rho(a)w \notin W$  meaning that

$$\rho(z)(\rho(a)w) \neq \lambda_1(\rho(a)w).$$

Then note that  $a = z^{-1}az$  where  $z \in Z(A)$  is the same fixed  $z$  from above.

So,  $\rho(a) = \rho(z^{-1}az)$  meaning that

$$\rho(a)w = \rho(z^{-1}az)w.$$

Now multiplying on both sides by  $\rho(z)$  we get

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}az)w$$

which means that

$$\begin{aligned} \rho(z)\rho(a)w &= \rho(z)\rho(z^{-1}a)\rho(z)w = \rho(z)\rho(z^{-1}a)\lambda_1 w = \rho(zz^{-1})\rho(a)\lambda_1 w \\ &= \lambda_1 \rho(e)\rho(a)w = \lambda_1 \rho(a)w. \end{aligned}$$

We now have a contradiction since the above implies that

$$\rho(z)(\rho(a)w) = \lambda_1(\rho(a)w),$$

which means by definition that  $\rho(a)w \in W$ . So, indeed  $W$  is a subrepresentation.

Now, either  $W$  is irreducible or it is not. If not, then as shown in Exercise 1.20, we know that it contains some irreducible representation  $W'$ . Now, indeed we have shown that for any  $z \in V$  one has that  $\rho(z)$  has only one eigenvalue equal to the scalar by which  $z$  acts on  $W'$ .

Finally, we show that  $\rho(z)$  actually has an eigenvalue  $\lambda \in k$ . Since  $k$  is algebraically closed, indeed it does since all roots of the characteristic polynomial belong to  $k$ .

So, indeed we have shown the desired result where  $\lambda$  which exists as argued above is an eigenvalue and we take  $W_0 = \{w \in V : \rho(z)w = \lambda w\}$ . Either  $W_0$  is irreducible or it is not. If not we follow the same procedure above Equation we find  $W_i$  irreducible such that  $\rho(z)w = \lambda w$  for all  $w \in W_i$ .

(c) Does  $\rho(z)$  in (b) have to be a scalar operator?

No, it does not. For instance, take  $A = \mathbb{Z}/2\mathbb{Z}a + \mathbb{Z}/2\mathbb{Z}b$  as a  $\mathbb{Z}/2\mathbb{Z}$  algebra where  $a, b$  are indeterminants and we declare  $ab = ba$  and that  $\bar{0} = 0a + 0b$ . Then, define  $\rho : A \rightarrow GL(\mathbb{R}^2)$  by

$$\begin{aligned} \rho(0) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \rho(1a + 0b) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since  $A$  is commutative one has that  $Z(A) = A$ . So, let  $z = 1a + 1b$ . We see that

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not a scalar operator since

$$\rho(1a + 1b)(1, 1)^T = (0, 1)^T$$

but

$$(1, 1)^T \neq c(0, 1)^T$$

for any  $c \in \mathbb{Z}/2\mathbb{Z}$ .

I provide another slightly different example in which  $A$  is an algebra over an infinite field. Namely, take  $A = \mathbb{Q} + \mathbb{Q}x$  where  $x$  is an indeterminant and we stipulate that  $x^2 = 0$ . (So really this is just the ring  $\mathbb{Q}[x]/(x^2)$ ).

Then, let  $\rho : A \rightarrow GL(\mathbb{R}^2)$  be defined by

$$\rho(a + bx) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Indeed, we see that

$$\begin{aligned} \rho((a + bx)(c + dx)) &= \rho(ac + (ad + bc)x) \\ &= \begin{pmatrix} ac & ad + bc \\ 0 & ac \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \\ &= \rho(a + bx)\rho(c + dx), \end{aligned}$$

which shows that  $\rho$  is a homomorphism of algebras.

However,  $\rho(1 + x)$  is not a scalar operator since in fact, similar to before, we obtain

$$\rho(1 + x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and again

$$\rho(1 + x)(0, 1)^T = (1, 1)^T \neq \alpha(0, 1)^T$$

for any  $\alpha \in \mathbb{Q}$ .

Now, say we require  $k$  to be algebraically closed and ask whether one can still find a counterexample. Indeed we can. Take  $A = \mathbb{C}[x]/(x^n)$  as a  $\mathbb{C}$  algebra and let  $\rho : \mathbb{C}[x]/(x^n) \rightarrow \mathbb{C}^n$  be defined by

$$\rho(x) = J(2, n).$$

First, I verify that  $\rho$  is a homomorphism of algebras. Namely, I note

$$\begin{aligned} \rho((a + bi)(c + di)) &= \rho(ac - bd + (ad + bc)i) \\ &= \begin{pmatrix} (ac - bd) & (ad + bc) \\ -(ad + bc) & (ac - bd) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \end{aligned}$$

Once again  $Z(\mathbb{C}) = \mathbb{C}$ . Take  $z = i$ . Then,

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and note that

$$\rho(i)(1, 0)^T = (0, -1)^T \neq \alpha(1, 0)^T$$

for any  $\alpha \in \mathbb{C}$ .

We must also verify however that  $\rho, \mathbb{C}^2$  is an indecomposable representation.

It suffices to show that it is irreducible, which it is. Otherwise, there would exist some non-trivial proper subrepresentation  $W \subseteq V$ , which must be of dimension  $\dim(W) = 1$  since  $\dim(V) = 2$ . So, it must be of the form  $W = \text{span}(w)$  such that

$$\rho(a + bi)w \in W$$

for all  $a, b \in \mathbb{R}$ , meaning that

$$\rho(a + bi)w = (\alpha + \beta i)w$$

for some  $\alpha, \beta \in \mathbb{R}$ . Then, writing  $w = (w_1, w_2)^T$  for some  $w_1, w_2 \in \mathbb{C}$  gives

$$\begin{aligned} \rho(a + bi)(w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} (w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\ \begin{pmatrix} aw_1 + bw_2 \\ -bw_1 + aw_2 \end{pmatrix} &= \begin{pmatrix} (\alpha + \beta i)w_1 \\ (\alpha + \beta i)w_2 \end{pmatrix} \end{aligned}$$

which means that

$$aw_1 + bw_2 = \alpha w_1 + \beta i w_2$$

and

$$-bw_1 + aw_2 = \alpha w_2 + \beta i w_1.$$

The above statement must hold for all  $a, b \in \mathbb{R}$  so take  $(a, b) = (0, 1)$  meaning  $a + bi = i$ .

If  $(a, b) = (a, 0)$  then

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and indeed the eigenvalues of  $\rho(i)$  are the roots of  $x^2 + 1$  which are  $\lambda_1 = i$  and  $\lambda_2 = -i$  with corresponding eigenvectors  $v_1 = (-i, 1)^T$  and  $v_2 = (i, 1)^T$ , which we note form a basis for  $\mathbb{C}^2$ . If  $\rho$  is reducible, that would require some non-trivial stable subspace  $U \subseteq V$ , which would need to be one-dimensional. Then, the requirement that  $\dim(U) = 1$  implies that  $U = \text{Span}(u)$  for some  $u \in V$ .

Note that  $u \in \{v_1, v_2\}$ . Otherwise, if  $u = c_1 u_1 + c_2 u_2 = ((-c_1 + c_2)i, c_1 + c_2)^T$  for some  $c_1, c_2 \in \mathbb{C}$  with  $c_1, c_2 \neq 0$ , then

$$\begin{aligned} \rho(i)(u) &= \rho(i)((-c_1 + c_2)i, c_1 + c_2)^T \\ &= c_1(\rho(i)(u_1)) + c_2(\rho(i)(u_2)) \\ &= c_1 \lambda_1 u_1 + c_2 \lambda_2 u_2 \\ &= c_1(i(-i, 1)^T) + c_2(-i(i, 1)^T) \\ &= c_1(1, i)^T + c_2(1, -i)^T \\ &= (c_1 + c_2, (c_1 - c_2)i)^T. \end{aligned}$$

If  $\text{span}(U)$  is stable under the action of  $A$ , then that would imply that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} \in \text{Span} \left( \begin{pmatrix} (-c_1 + c_2)i \\ c_1 + c_2 \end{pmatrix} \right)$$

meaning that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} = \begin{pmatrix} d(-c_1 + c_2)i \\ d(c_1 + c_2) \end{pmatrix}$$

for some  $d \in \mathbb{C}$ .

Then, that implies that

$$c_1 + c_2 = di(-c_1 + c_2) \tag{1}$$

$$c_1 - c_2 = -di(c_1 + c_2). \tag{2}$$

Then, adding the above equations gives

$$2c_1 = -2c_1 di$$

implying that  $c_1 = 0$ , which cannot happen by assumption, or that  $1 = -di$  meaning that  $d = i$ .

Now, substituting back into our original pair of equations 1 and 2 gives

$$c_1 + c_2 = c_1 - c_2$$

$$c_1 - c_2 = c_1 + c_2$$

which then implies that  $c_2 = -c_2$  or that  $c_2 = 0$ , which contradicts our assumption that  $u = c_1 v_1 + c_2 v_2$  with  $c_1, c_2 \neq 0$ .

Thus,  $U = \text{Span}(v_1)$  or  $U = \text{Span}(v_2)$ . However, neither subspace is stable under the action of  $A$ . Namely, take  $a + bi = 2 + i$ . Then,

$$\rho(2 + i) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

so that

$$\begin{aligned} \rho(2 + i)v_1 &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} v_1 \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} (-i, 1)^T \\ &= \begin{pmatrix} 1 - 2i \\ 2 - i \end{pmatrix} \end{aligned}$$

Now, if we were to have  $\rho(2 + i)w_1 \in W_1$  we would need  $r + si \in \mathbb{C}$  such that

$$1 - 2i = -i(r + si)$$

and

$$2 - i = r + si$$

which means we would have  $r = 2$  and  $s = -1$  implying that

$$1 - 2i = -i(2 - i) = -2i + i^2 = -2i - 1,$$

which is a contradiction. So, indeed we see that this representation is in fact irreducible and consequently also indecomposable, which means this is a counterexample. So, even if we require the field  $k$  to be algebraically closed,  $\rho(z)$  is not necessarily a scalar operator.

[1.22]

Let  $A$  be an associative algebra, and  $V$  a representation of  $A$ . By  $End_A(V)$  one denotes the algebra of all homomorphisms of representations  $V \rightarrow V$ . Show that  $End_A(A) = A^{op}$ , the algebra  $A$  with opposite multiplication. I assume that  $\rho(a)(b) = ab$  for all  $a, b \in A$  is the regular representation.

We want to construct a bijection between  $\{\psi \in GL(A) : \psi(\rho(a)b) = \rho(a)\psi(b) \forall a, b \in A\}$  (Condition (\*)) and  $A$ . Let's try

$$\tau : \langle A, \cdot_{op} \rangle \rightarrow GL(A)$$

$$\tau : a \mapsto \rho(a)$$

for each  $a \in A$ . Now, is this a homomorphism of algebras? Yes, namely we have that  $\tau(c_1a + c_2b) = \rho(c_1a + c_2b) = c_1\rho(a) + c_2\rho(b)$ . Also, note that  $\rho(a)$  satisfies Condition (\*) since  $(\rho(\rho(a)b))(c) = \rho(ab)(c) = (ab) * c = a * (bc) = a * \rho(b)(c) = \rho(a)(\rho(b)(c))$ . Also, we have that for all  $c \in A$  that  $\tau(ab)(c) = \rho(ab)(c) = (ab) * (c) = \rho(a)(\rho(b)(c)) = \tau(a)(\tau(b)c)$ . Now since multiplication in the endomorphism ring of two endomorphisms  $f, g$  is defined by  $(f * g)(c) = g(f(c))$  we have that  $\tau(ab) = \tau(b) * \tau(a)$  proving the claim.

[1.23]

Prove the following “Infinite dimensional Schur’s lemma” (due to Dixmier): Let  $A$  be an algebra over  $\mathbb{C}$  and  $V$  be an irreducible representation of  $A$  with at most countable basis. Then any homomorphism of representations  $\phi : V \rightarrow V$  is a scalar operator.

First, I show that  $D$  is at most countable dimensional. To do so, I exhibit a countable spanning set of an even larger space namely  $End(V)$ .

Namely, if  $\{w_n\}_{n \in \mathbb{N}}$  is a countable basis for  $V$  (we may assume the basis is countably infinite since if it is finite there is nothing to show). Then consider the set  $S = \cup_{i \in \mathbb{N}} \cup_{j \in \mathbb{N}} f_j^i$  where  $f_j^i \in End(V)$  is the endomorphism defined by  $f_j^i : v_i \mapsto v_j$  and  $f_j^i : v_k \mapsto 0$  if  $k \neq i$ . Then, note that any endomorphism  $g : V \rightarrow V$  is defined by  $g(v_i)$  for each  $i \in \mathbb{N}$  and since  $v_i$  spans  $V$  one has that  $g(v_i) = \sum_{j \in \mathbb{N}} a_j^i v_j$  and thus  $g = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} a_j^i f_j^i$ . Thus since  $|S| = |\cup_{n \in \mathbb{N}} \mathbb{N}| = |\mathbb{N}|$  one has that  $dim(End(V)) \leq |\mathbb{N}|$  and since  $End_A(V) \subseteq End(V)$  the claim follows.

Then, assume that  $\phi$  is not a scalar and as the hint suggests I show that  $\mathbb{C}(\phi)$  is a transcendental extension of  $\mathbb{C}id \subseteq D$ . Otherwise it is algebraic meaning that there exists  $p(x) = \sum_{i=0}^n (a_i id)x^i$  such that  $p(\phi) = 0$ . However, since  $\mathbb{C}$  is algebraically closed we know that  $p(x)$  splits into linear factors  $p(x) = \prod_{i=0}^n (x - \lambda_i id)$  meaning that  $\phi = \lambda_i id$  for some  $i \in [0 : n]$ , a contradiction. Thus,  $\mathbb{C}(\phi)$  is a transcendental extension and is thus uncountably infinite dimensional as vector space over  $\mathbb{C}$ . However, that provides a contradiction as  $\mathbb{C}(\phi) \subseteq D$  which is at most countable dimensional. Thus,  $\phi$  is a scalar operator.

[1.24]

Let  $A = k[x_1, \dots, x_n]$  and  $I \neq A$  be any ideal in  $A$  containing all homogenous polynomials of degree  $\geq N$ . Show that  $A/I$  is an indecomposable representation of  $A$ .

Note that  $1 + I \in A/I$  is cyclic and in particular  $\rho(A)(1 + I) = A/I$  meaning that  $A/I$  is not decomposable since if it were of the form  $A/I = V \oplus W$  with  $V, W$  non-empty subrepresentations meaning that  $\rho(a)V \subseteq V$  and  $\rho(a)W \subseteq W$ . Now, without loss of generality one has that  $1 + I \in V$  and then however since  $W \neq \emptyset$  there exists  $f + I \in W$  and then  $\rho(f)(1 + I) = f + I \in W$ , a contradiction since  $1 + I \in V$ .

[1.25]

Let  $V \neq 0$  be a representation of  $A$ . We say that a vector  $v \in V$  is cyclic if it generates  $V$ , i.e.,  $Av = V$ . A representation admitting a cyclic vector is said to be cyclic. Show that

- (a)  $V$  is irreducible if and only if all nonzero vectors of  $V$  are cyclic.

If all nonzero  $v \in V$  are cyclic, then  $V$  is irreducible. otherwise, if there existed a subrepresentation  $W$  with  $0 \subsetneq W \subsetneq V$ , then taking any  $w \in W$  we have that  $\{\rho(a)w : a \in A\} = V \supsetneq W$  a contradiction to  $W$  a subrepresentation. Now, to show the converse we show that if some non-zero

vector  $v \in V$  is NOT cyclic then,  $V$  is reducible. In particular, if one has such  $v$  let  $W = Av$  and note that  $W$  is closed under the action of  $A$  and is thus a subrepresentation. Since  $Av \neq V$  and  $v \neq 0$  implying  $Av \neq 0$  we have that  $0 \neq W \subsetneq V$  is a subrepresentation.

- (b)  $V$  is cyclic if and only if it is isomorphic to  $A/I$ , where  $I$  is a left ideal in  $A$ .

Define the map  $\psi_v : A \rightarrow V$  by  $\psi_v(a) = \rho(a)(v)$ . Note that by the Ring Isomorphism Theorem we have that  $\text{Im}(\psi_v) \cong A/\ker(\psi_v)$ . Since  $\text{Im}(\psi_v) = V$  and since  $\ker(\psi_v)$  is an ideal by definition of a ring isomorphism we have that  $V \cong A/I$  where  $I = \ker(\psi_v)$  if  $V$  is cyclic. Now, if  $V \cong A/I$ , then

- (c) Give an example of an indecomposable representation which is not cyclic.

Note that one has an obvious isomorphism  $\phi : A^* \rightarrow \mathbb{R}^3$  given by  $\phi(f) = (f(1), f(x), f(y))^T$ . Then, by definition of  $\rho$  one can write

$$\rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that  $A^*$  is not cyclic since if one did have a generator  $f$  with  $\phi(f) = (f(1), f(x), f(y))^T$  then of course one would need  $(f(1), f(x), f(y))^T \neq (0, 0, 0)^T$ , and then one has that  $\rho(1)(\phi(f)) = (f(1), f(x), f(y))^T$ ,  $\rho(x)(\phi(f)) = (f(x), 0, 0)^T$ , and  $\rho(y)(\phi(f)) = (f(y), 0, 0)^T$ . Also, note that  $A = \{k_1 + k_2x + k_3y : k_i \in k \forall i \in [3]\}$ . Thus,

$$\{\rho(a) : a \in A\} = \left\{ \begin{pmatrix} k_1 & 0 & 0 \\ k_2 & k_1 & 0 \\ k_3 & 0 & k_1 \end{pmatrix} : k_i \in k \forall i \in [3] \right\}.$$

In order for  $A^*$  to be cyclic we would need  $f \in A^*$  such that  $\rho(A)f = A^*$ . Now, if one had such  $f$  then certainly  $f \neq 0$ . If  $f(1) = 0$  then note that  $(1, 0, 0)^T \notin \rho(A)f$  and thus  $f$  is not a cyclic vector. Now, if another  $v = (f(1), f(x), f(y))^T$  with  $f(0) \neq 0$  is a generator then note

$$\{\rho(a)f : a \in A\} = \left\{ \begin{pmatrix} k_1 & 0 & 0 \\ k_2 & k_1 & 0 \\ k_3 & 0 & k_1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(x) \\ f(y) \end{pmatrix} : k_i \in k \forall i \in [3] \right\} = \left\{ \begin{pmatrix} k_1 f(1) \\ k_2 f(1) + k_1 f(x) \\ k_3 f(1) + k_1 f(y) \end{pmatrix} : k_i \in k \forall i \in [3] \right\}.$$

If  $g \in \rho(A)f$  for arbitrary  $g \in A^*$  then note that  $k_1 = g(1)/f(1)$  meaning that  $k_2 = (g(x) - g(1)/f(1)f(x))/f(1)$  and  $k_3 = (g(y) - g(1)/f(1)f(y))/f(1)$  which I think would show that  $A^*$  is cyclic simply taking  $f \in A^*$  to be  $f : 1 \mapsto 1, f : x \mapsto 0, f : y \mapsto 0$  then for any  $g \in A^*$  we simply take  $k_1 = g(1), k_2 = g(x), k_3 = g(y)$ . I may need to come back and review this. However, for now I'll think of another example.

Let

$$M = \{M_{a,b,c} := \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R}\}$$

Note that  $M$  is an algebra since  $M$  is closed under matrix multiplication. Also, note that if one takes the representation  $V = \mathbb{R}^3$  with  $\rho(M)v = Mv$  then one notes that  $V$  is indecomposable since otherwise

it has the form  $V = U \oplus W$  with  $U$  irreducible. Now, without loss of generality  $\dim(U) = 1$ . One wishes to find all 1-dimensional subrepresentations of  $V$  which would be of the form  $U = \{k(x, y, z)^T : k \in \mathbb{R}\}$  for some fixed  $(x, y, z)^T \in \mathbb{R}^3$  which would imply that

$$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \right\}$$

for all  $a, b, c \in \mathbb{R}$  meaning

$$\left\{ \begin{pmatrix} ay + bz \\ cz \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_{a,b,c}x \\ \lambda_{a,b,c}y \\ \lambda_{a,b,c}z \end{pmatrix} \right\}$$

meaning  $\lambda_{a,b,c}z = 0$  which implies that  $\lambda_{a,b,c} = 0$  or  $z = 0$ . First I handle the case in which  $z \neq 0$  which implies that  $\lambda_{a,b,c} = 0$  which implies that  $cz = 0$  meaning  $c = 0$ , a contradiction since the above equation in fixed  $(x, y, z)$  but varied  $\lambda_{a,b,c}$  must hold for all  $a, b, c \in \mathbb{R}$ .

If  $z = 0$  we have that  $cz = \lambda_{a,b,c}y = 0$  and also as long as  $x \neq 0$  we have  $\lambda_{a,b,c} = \frac{ay}{x}$  and thus  $M(x, y, 0)^T = (\frac{ay}{x}x, \frac{ay}{x}y, 0)^T = (ay, cz, 0)^T = (ay, 0, 0)^T$  which implies that  $\frac{ay}{x}y = 0$  meaning  $y^2 = 0$  meaning  $y = 0$  (since  $\mathbb{R}$  has no zero divisors) since this holds for all  $a \in \mathbb{R}$  (provided  $z = 0$ ) which then means that  $(x, y, z)^T = (x, 0, 0)^T$  with  $x \neq 0$ . Indeed this is an irreducible subrepresentation. However,  $V$  does not decompose as  $U \oplus W$  since  $M_{1,1,1}(0, 1, 0)^T = (1, 0, 0)^T \in U$  but  $(0, 1, 0)^T \notin U$ .

Finally, note that  $V = \mathbb{R}^3$  is not cyclic since  $(0, 0, 1)^T \in V$  but  $(0, 0, 1)^T \neq M_{a,b,c}(x, y, z)^T$  for all  $a, b, c, x, y, z \in \mathbb{R}$ .

### [1.26]

Let  $A$  be the Weyl algebra, generated by two elements  $x, y$  with the relation

$$yx - xy - 1 = 0.$$

- (a) If  $\text{char}(k) = 0$ , what are the finite dimensional representations of  $A$ ? What are the two-sided ideals in  $A$ ?

If  $\text{char}(k) = 0$  then there are no finite-dimensional representations of  $A$  since  $yx - xy - 1 = 0$  implies that  $\chi(\rho(yx) - \rho(xy) - I) = \chi(\rho(yx)) - \chi(\rho(xy)) - \chi(I) = -\chi(I) = 0$  implies that  $\chi(I) = \dim(V) = 0$  meaning  $\dim(V) = 0$  meaning that  $V = 0$ .

For the second part, consider a non-zero ideal  $I \subseteq A$ , meaning there exists  $p(x, y) \in I$ .

Otherwise,  $p(x, y)$  is of course a sum of terms  $p(x, y) = \sum_{i=0}^N a_i \prod_{j=0}^{n_i} x^{r_j^i} y^{s_j^i} =: \sum_{i=0}^N t_i(x, y)$  where  $r_j^i, s_j^i \in \{0, 1\}$  and  $r_j^i \neq s_j^i$ . I claim that we may write  $p(x, y)$  in the form  $p(x, y) = \sum_{i=0}^M b_i y^{s_i} x^{r_i}$ . I prove so by induction on the quantity  $M := \max_{i \in [0:N]} (w(t_i(x, y)))$  where

$w(t_i(x, y)) := \sum_{j \in [1:n_i]: s_j^i=1} s_j^i$  and  $r_l^i=1$  for some  $l < j$   $|\{l < j : r_l^i=1\}|$  i.e. the sum over all  $y$ 's that appear not grouped to the left in the  $i$ th term or the number of  $x$ 's that appear before them. Note for clarity that  $M$  is a function of our expression of  $p(x, y)$ . Indeed  $p(x, y)$  is not changing throughout the proof below. So, in our base case where  $M = 0$  there is nothing to prove. So, we may assume that  $M \geq 1$ . Then, we note that  $p(x, y) = \sum_{i \in [0:N]: w(t_i(x, y))=0} t_i(x, y) + \sum_{i \in [0:N]: w(t_i(x, y)) \neq 0} t_i(x, y)$ . Now, note that for each  $i \in [0 : N]$  such that  $w(t_i(x, y)) \neq 0$  we know that  $t_i(x, y) = a_i y^{d_i} x^{e_i} x y s_i(x, y)$  where  $s_i(x, y) = \prod_{k \in [0:K_i]} x^{\alpha_k} y^{\beta_k}$  and  $d_i, e_i \in \mathbb{N}_0$ . Then, indeed  $p(x, y) = \sum_{i \in [0:N]: w(t_i(x, y))=0} t_i(x, y) + \sum_{i \in [0:N]: w(t_i(x, y)) \neq 0} a_i y^{d_i} x^{e_i} x y s_i(x, y)$ , but now note that  $a_i y^{d_i} x^{e_i} x y s_i(x, y) = a_i y^{d_i} x^{e_i} y x s_i(x, y) - a_i y^{d_i} x^{e_i} s_i(x, y)$  and that

$$w(a_i y^{d_i} x^{e_i} y x s_i(x, y)) = w(a_i y^{d_i} x^{e_i} x y s_i(x, y)) - 1 \text{ and } w(a_i y^{d_i} x^{e_i} s_i(x, y)) \leq w(a_i y^{d_i} x^{e_i} x y s_i(x, y)) - 1.$$

Thus, for our new expression we have

$$\begin{aligned} M \left( \sum_{i \in [0:N]: w(t_i(x, y))=0} t_i(x, y) + \sum_{i \in [0:N]: w(t_i(x, y)) \neq 0} (a_i y^{d_i} x^{e_i} y x s_i(x, y) - a_i y^{d_i} x^{e_i} s_i(x, y)) \right) \\ = M \left( \sum_{i \in [0:N]: w(t_i(x, y))=0} t_i(x, y) + \sum_{i \in [0:N]: w(t_i(x, y)) \neq 0} t_i(x, y) \right) - 1. \end{aligned}$$



So, indeed we have shown by induction that we may write  $p(x, y)$  in the form  $p(x, y) = \sum_{i=0}^M b_i y^{s_i} x^{r_i}$ . Now, letting  $p_0(x, y) = \sum_{i=0}^M b_i y^{s_i} x^{r_i}$  (Note here when we write  $p(x, y)$  we mean using the exact formal expression specified) iterate the following process.

Then, note that  $x p(x, y) \in I$  and  $-p(x, y)x \in I$ . Let  $P(x, y) = x p(x, y) - p(x, y)x$ . I claim firstly that we can write  $P(x, y) = \sum_{i=0}^{M'} B_i y^{s'_i} x^{r'_i} = \sum_{i=0}^{M'} S_i(x, y)$  and that  $\max\{s'_i : i \in [0 : M']\} < \max\{s_i : i \in [0 : M]\}$  (i.e. the max power of  $y$  appearing in any term goes down by at least 1) meaning that after finitely iterations of the process one obtains a polynomial solely in  $x$ .

I show this by looking term by term. Consider the term  $S_i(x, y) = b_i y^{s_i} x^{r_i}$ . Either  $s_i = 0$  in which case  $x S_i(x, y) - S_i(x, y)x = 0$ . Otherwise if  $s_i \neq 0$  then note that  $x b_i y^{s_i} x^{r_i} - b_i y^{s_i} x^{r_i} x = b_i x y^{s_i} x^{r_i} - b_i y^{s_i-1} y x x^{r_i} = b_i x y^{s_i} x^{r_i} - b_i y^{s_i-1} x y x^{r_i} - b_i y^{s_i-1} x^{r_i} x = b_i x y^{s_i} x^{r_i} - b_i y^{s_i-2} x y^2 x^{r_i} - b_i y^{s_i-1} x^{r_i} x - b_i y^{s_i-1} x^{r_i} = b_i x y^{s_i} x^{r_i} - b_i y^{s_i-3} x y^3 x^{r_i} - b_i y^{s_i-1} x^{r_i} x - b_i y^{s_i-1} x^{r_i} x = b_i x y^{s_i} x^{r_i} - b_i y^{s_i-1} x^{r_i} x - \sum_{l=1}^{s_i} b_i y^{s_i-1} x^{r_i} = b_i (x y - y x) y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i} = -b_i y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i}$ . So, indeed we see that

$$x p(x, y) - p(x, y)x = 0 + \sum_{i \in [0 : M'] : s_i \neq 0} (-b_i y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i})$$

concluding the proof that  $\max\{s'_i : i \in [0 : M']\} < \max\{s_i : i \in [0 : M]\}$  and also, since we assumed that  $p(x, y) \notin k[x]$  to start with that means that  $x p(x, y) - p(x, y)x \neq 0$ , concluding the proof.

Now, I claim that for arbitrary  $P(x) \in k[x]$  note that for sufficiently large  $K \in \mathbb{N}$  one has that  $y P(x) - P(x)y \in K$ . We consider  $y x^n - x^n y$  for  $n \in \mathbb{N}_0$ . If  $n = 0$  then  $y x^n - x^n y = 0$ . If  $n = 1$  then  $y x^n - x^n y = 1$ . Now, note that for any  $n \in \mathbb{N}_{\geq 2}$  one has  $y x^n - x^n y = y x^n - x^{n-1} x y = y x^n - x^{n-1} y x + x^{n-1} = y x^n - x^{n-2} y x x + 2x^{n-1} = y x^n - x y x^{n-1} + \sum_{i=0}^n x^{n-1} = y x^n - x y x^{n-1} + \sum_{i=0}^n x^{n-1} = (y x - x y) x^{n-1} + \sum_{i=2}^n x^{n-1} = n x^{n-1}$ . Thus, one has that  $y P(x) - P(x)y = \frac{d}{dx} P(x)$ . Now, clearly if  $P(x) = A_N x^N + Q(x)$  where  $\deg(Q) \leq N - 1$  then  $\frac{d^N}{dx^N} P(x) = N! A_N$ . Thus,  $\frac{1}{N! A_N} \frac{d^N}{dx^N} P(x) = 1$ , meaning that if one denote  $Q(P(X)) = y P(x) - P(x)y =: Q_1(P(x))$  and  $Q_n(P(x)) = Q(Q_{n-1}(P(x)))$  then since  $\frac{d^N}{dx^N} P(x) = Q_N(P(X)) \in I$  one has that  $1 = \frac{1}{N! A_N} \frac{d^N}{dx^N} P(x) \in I$  meaning that  $I = A$ . Thus the only non-zero two-sided ideal of  $A$  is  $I = A$ .

- (b) Suppose for the rest of the problem that  $\text{char } k = p$ . What is the center of  $A$ ?

Note that as above  $y x^p - x^p y = p x^{p-1} = 0$  which implies  $y x^p = x^p y$ . Also, clearly  $x x^p = x^p x$  which implies that  $x^p \in Z(A)$ . Now, note that there is something close to symmetry between  $x, y$  in the given relation. Namely, one has that  $(x, y) = (a, b)$  satisfy the relation  $ba - ab - 1 = 0$  and so does  $(y, -x) = (a, b)$ . So, as shown throughout part (a) one has for  $a, b$  satisfying the given relation that  $ba^p - a^p b = p a^{p-1} = 0$  meaning for  $(a, b) = (y, -x)$  one has that  $(-x) y^p - y^p (-x) = p y^{p-1} = 0$  meaning that  $x y^p = y^p x$  and of course  $y y^p = y^p y$  implying that  $y^p \in Z(A)$ .

- (c) Find all irreducible finite dimensional representations of  $A$ .

Note that since  $\rho(x^p)$  (which I simply denote by  $x^p$  when the use is clear from context) is an intertwining operator we know that it is a scalar operator and likewise for  $\rho(y^p)$ . Thus one has that  $\rho(A)v = \text{span}\{x^i y^j v : i, j \in [0 : p-1]\}$  and since  $y^j v \in \text{span}(v)$  one has that  $\rho(A) = \text{span}\{x^i v : i \in [0 : p-1]\}$ . Then, by part (a) of Question 1.25 we know that every non-zero  $v \in V$  is cyclic meaning that  $Av = V$  for the  $v$  meaning that  $V = \text{span}\{x^i v : i \in [0 : p-1]\}$ . Finally note that  $\{x^i v : i \in [0 : p-1]\}$  is a linear independent set since otherwise  $\dim(V) = \dim(\text{span}\{x^i v : i \in [0 : p-1]\}) < p$  which provides a contradiction since then  $\chi(I) < p$  meaning  $\chi(I) \notin \mathbb{Z}p$  unless  $\chi(I) = 0$  contradicting  $0 = \chi(xy) - \chi(yx) = \chi(I)$  since  $V \neq 0$ .

### [1.27]

Let  $q$  be a nonzero complex number, and  $A$  be the  $q$ -Weyl algebra over  $\mathbb{C}$  generated by  $x^{\pm 1}$  and  $y^{\pm 1}$  with defining relations  $x x^{-1} = x^{-1} x = 1$ ,  $y y^{-1} = y^{-1} y = 1$ , and  $xy = qyx$ .

- (a) What is the center of  $A$  for different  $q$ ? If  $q$  is not a root of unity, what are the two-sided ideals in  $A$ ?

Note that  $c \in Z(A)$  if and only if  $xc = cx$  and  $yc = cy$ . Clearly if  $q = 1$  then  $A$  is abelian meaning that  $Z(A) = A$ . Now say that  $q^n = 1$  and  $q^s \neq 1$  for  $0 \leq s < n$  for some  $n \in \mathbb{N}$ . We see that if  $c = \prod_{i=0}^r x^{s_i} y^{t_i}$  is a monomial in  $x, y$  with  $s_i, t_i \in \mathbb{N}_{\geq 0}$  for each  $i \in [0 : r]$  then note that  $c \in Z(A)$  implies that  $xc = cx$  implying that  $n \mid \sum_{i \in [0:r]} t_i$  and then  $yc = cy$  implies that  $n \mid \sum_{i \in [0:r]} s_i$ . Thus,  $Z(A) = \langle \prod_{i=0}^r x^{s_i} y^{t_i} : r \in \mathbb{N}, s_i, t_i \in \mathbb{N}_{\geq 0}, n \mid \sum_{i=0}^r t_i, n \mid \sum_{i=0}^r s_i \rangle$ . (Here  $\langle \rangle$  means algebra generation meaning finite linear combinations of these terms).

- (b) For which  $q$  does this algebra have finite dimensional representations?

Note that if there exists a finite dimensional representation then since  $xy = qyx$  one has that  $\det(xy) = q^{\dim(V)}(\det(yx)) = q^{\dim(V)}\det(xy)$  meaning that  $q^{\dim(V)} = 1$  or that  $q$  is a root of unity of order  $\text{ord}(q)$  such that  $\text{ord}(q) \mid \dim(V)$ , meaning it is necessary that  $q$  be a root of unity. Indeed we show in part (c) we show that the condition that  $q$  be a root of unity is also sufficient.

- (c) Find all finite dimensional irreducible representations of  $A$  for such  $q$ .

Say that  $q$  is an  $n$ th root of unity. Now, I claim that if  $v \in V$  is an eigenvector of  $x$  then  $\{v, yv, y^2v, y^3v, \dots, y^{n-1}v\}$  is a basis for  $V$ . Note that by Problem 1.25 (a) one has that  $v$  is cyclic meaning that  $\{x^i y^j v : i, j \in \mathbb{Z}\}$  is a spanning set, but note that  $x^i y^j v = q^{ij} y^j x^i v = q^{ij} \lambda^i y^j v$  (where  $xv = \lambda v$ ) meaning since  $\lambda \neq 0$  (by the fact that  $\rho(x)$  is invertible) that  $\{y^i v : i \in \mathbb{Z}\}$  is a spanning set but since any  $a \in Z(A)$  acts as a scalar we have that  $\{y^i v : i \in [0 : n-1]\}$  is a spanning set since if  $i \notin [0 : n-1]$  one has that if  $i = ni' + i'''$  (where  $i''' \in [0 : n-1]$  and  $i' \in \mathbb{Z}$ ) one has that  $y^i v = y^{ni' + i'''} v = y^{ni'} y^{i'''} v = \alpha y^{i'''} v$  since  $y^{ni'} \in Z(A)$  implies  $y^{ni'}$  acts as a scalar. Finally, I claim that  $\{v, yv, y^2v, y^3v, \dots, y^{n-1}v\}$  is a linearly independent set. Otherwise  $\dim(V) = \dim(\text{span}\{v, yv, y^2v, y^3v, \dots, y^{n-1}v\}) < \text{ord}(q)$  a contradiction to the observation in part (b) that  $\text{ord}(q) \mid \dim(V)$ .

### [1.33]

Show that the algebra  $P_Q$  is generated by  $p_i$  for  $i \in I$  and  $a_h$  for  $h \in E$  with the defining relations:

- $p_i^2 = p_i, p_i p_j = 0$  for  $i \neq j$ ,
- $a_h p_{h'} = a_h, a_h p_j = 0$  for  $j \neq h'$ ,
- $p_{h''} a_h = a_h, p_i a_h = 0$  for  $i \neq h''$ .

Note that an oriented path is a sequence of edges and vertices  $v_1 e_1 v_2 e_2 \dots e_{n-1} v_n$  such that  $e_{i-1} = \overrightarrow{v_{i-1} v_i}$ . Indeed, conditions (2) and (3) in the problem are guaranteeing exactly that and condition (1) is simply mandating one delete any vertex listed more than once in a row.

Optional: TODO later.

**Definition [1.37]** Let  $(V_i, x_h)$  and  $(W_i, y_h)$  be representations of the quiver  $Q$ . A homomorphism  $\phi : (V_i) \rightarrow (W_i)$  of quiver representations is a collection of maps  $\phi_i : V_i \rightarrow W_i$  such that  $y_h \circ \phi_{h'} = \phi_{h''} \circ x_h$  for all  $h \in E$ .

[1.38] Let  $A$  be a  $\mathbb{Z}_+$ -graded algebra, i.e.,  $A = \bigoplus_{n \geq 0} A[n]$ , and  $A[n] \cdot A[m] \subseteq A[n+m]$ . If  $A[n]$  is finite dimensional, it is useful to consider the Hilbert series  $h_A(t) = \sum \dim A[n] t^n$  (the generating function of dimensions of  $A[n]$ ). Often, this series converges to a rational function, and the answer is written in the form of such function. For example, if  $A = k[x]$  and  $\deg(x^n) = n$  then

$$h_A(t) = 1 + t + t^2 + \dots + t^n + \dots = \frac{1}{1-t}.$$

Find the Hilbert series of:

(a)  $A = k[x_1, \dots, x_m]$  (where the grading is by degree of polynomials);

Note that  $\dim(A[n]) = \frac{m^n}{n!}$  since  $m^n = |\{(j_1, j_2, \dots, j_m) : j_i \in \mathbb{N}_{\geq 0} \text{ and } \sum_{i=1}^m j_i = n\}|$  and thus  $h_A(t) = \sum_{n=0}^{\infty} \frac{m^n}{n!} t^n$ .

(b)  $A = k\langle x_1, \dots, x_m \rangle$  (the grading is by length of words);

Note that  $\dim(A[n]) = m^n$  since  $m^n = |\{x_{i_1} x_{i_2} \dots x_{i_n} \text{ such that } i_n \in [1 : m]\}|$  meaning that  $h_A(t) = \sum_{n=0}^{\infty} m^n t^n$ .

(c)  $A$  is the exterior (=Grassmann) algebra  $\wedge_k[x_1, \dots, x_m]$ , generated over some field  $k$  by  $x_1, \dots, x_m$  with the defining relations  $x_i x_j + x_j x_i = 0$  and  $x_i^2 = 0$  for all  $i, j$  (the grading is by degree).

Note that  $\dim(A[n]) = \binom{m}{n} = \frac{m!}{(m-n)!n!}$  meaning that  $h_A(t) = \sum_{n=0}^{\infty} \frac{m!}{(m-n)!n!} t^n = \sum_{n=0}^m \frac{m!}{(m-n)!n!} t^n$ .

(d)  $A$  is the path algebra  $P_Q$  of a quiver  $Q$  (the grading is defined by  $\deg(p_i) = 0, \deg(a_h) = 1$ ).

Note that if  $M_Q$  is the adjacency matrix of  $Q$  (note it may not be symmetric) then  $\dim(A[n]) = f(M_Q^n)$  where  $f(M_Q^n) = \sum_{i=0}^{|V|} \sum_{j=0}^{|V|} (M_Q^n)_j^i$ .