Introduction to Representation Theory: Math 18.712 – Caitlin Beecham

[1.20] Let V be a nonzero finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.
Now, either V is irreducible or it is not. If not, then it has some proper non-trivial subrepresentation V_1 of dimension $dim(V_1) < dim(V)$. We continue in this fashion finding subrepresentations

$$V_i \subsetneq V_{i-1} \subsetneq V_1 \subsetneq V \tag{1}$$

until W_i is irreducible which we know is possible for the following reason. We note that any representation of dimension 1 is irreducible and indeed the dimension decreases by at least one at each step (meaning $dim(W_i) \leq dim(W_{i-1}) - 1$ meaning that $1 \leq dim(W_i) \leq dim(W) - i$ so that indeed provided $dim(W) > 0$ this process will make sense and terminate since $1 \leq dim(W) - i$ means that $i \leq dim(W) - 1 < \infty$.

[1.21] Problem 1.21. Let A be an algebra over a field k. The center Z(A) of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative then $Z(A) = A$.

(a) Show that if V is an irreducible finite dimensional representation of A then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V \in k$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the central character of V.

This makes intuitive sense since the center of the $GL_n(\mathbb{R})$ for instance is the set of scalar multiples of the identity. Now for a formal proof.
However, I believe that we also need k to be algebraically closed otherwise taking $A = \mathbb{C}$ (as an $\mathbb{R}$-algebra) we note that $Z(A) = A$. Then we just take $V = A$ which is a 2-dimensional representation over $k = \mathbb{R}$ (not algebraically closed). Then taking the regular representation and say the element $g = 1 + i = (1, 1) \in Z(A)$ we note that g acts on an element $v = (a, b) \in V$ by $(a, b) \mapsto^g (a-b, a+b)$ and clearly taking $(a, b) = (0, 1)$ we see that $(a - b, a + b) = (-1, 1) \neq \lambda(0, 1)$ for any $\lambda \in k = \mathbb{R}$.
However, assuming k is algebraically closed, we proceed.
Well, we just cite Corollary 1.17, noting that for any $z \in Z(A)$ we have that $\rho(z)$ is an intertwining operator within $\rho(A)$ since for any $a \in A$ to verify $\rho(z)$ is an intertwining operator we need to show that

$$(\rho(z))(\rho(a)v)) = \rho(a)(\rho(z))(v),$$

for all $v \in V$ and all $a \in A$.
Indeed we note that

$$(\rho(z))(\rho(a)v)) = \rho(za)v = \rho(az)v = \rho(a)\rho(z)v.$$

Then, indeed Corollary 1.17 gives the result.

(b) Show that if V is an indecomposable finite dimensional representation of A then for any $z \in Z(A)$ the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

First I want to show that if it has an eigenvalue then it is unique and then I will show at the end that it has an eigenvalue.

Say there existed ρ_1, ρ_2 such that $\rho(z)v_1 = \lambda_1 v_1$ and $\rho(z)v_2 = \lambda_2 v_2$ for some $\lambda_1, \lambda_2 \in k$ and $v_1, v_2 \in V$ with $v_1, v_2 \neq 0$.
Now, for this fixed $z \in A$ let $W = \{v \in V : \rho(z)v = \lambda_1 v\}$. This is a vector subspace.

I would like to show that W is a subrepresentation, namely that for all $w \in W$ and all $a \in A$ we have that $\rho(a)w \in W$.
Assume not. Assume that there exists $w \in W$ such that $\rho(a)w \notin W$ meaning that $\rho(z)(\rho(a)w) \neq \lambda_1(\rho(a)w)$ Then note that $a = z^{-1}az$ where $z \in Z(A)$ is the same fixed z from above.
So, $\rho(a) = \rho(z^{-1}az)$ meaning that
$\rho(a)w = \rho(z^{-1}az)w$ and then multiplying on both sides by $\rho(z)$ we get $\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}az)w$ which means

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}a)\rho(z)w = \rho(z)\rho(z^{-1}a)\lambda_1 w = \rho(zz^{-1})\rho(a)\lambda_1 w$$

$$= \lambda_1 \rho(e)\rho(a)w = \lambda_1 \rho(a)w,$$

but then that provides a contradiction since it says

$$\rho(z)(\rho(a)w) = \lambda_1(\rho(a)w),$$

which means by definition that $\rho(a)w \in W$. So, indeed W is a subrepresentation. Now, either W is irreducible or it is not. If not, then it has some proper non-trivial subrepresentation W_1 of dimension $dim(W_1) < dim(W)$. We continue in this fashion finding subrepresentations

$$W_i \subsetneq W_{i-1} \subsetneq W_1 \subsetneq W \tag{2}$$

until W_i is irreducible. Now, any representation of dimension 1 is irreducible and indeed the dimension decreases by at least one at each step (meaning $dim(W_i) \leq dim(W_{i-1}) - 1$ meaning that $0 \leq dim(W_i) \leq dim(W) - i$ so that indeed provided $dim(W) > 0$ this process will make sense and terminate since $1 \leq dim(W) - i$ means that $i \leq dim(W) - 1 < \infty$.

Finally, we need to show that $\rho(z)$ actually has an eigenvalue λ. Since k is algebraically closed it does since it is any root of the characteristic polynomial. So, indeed we have shown the desired result where λ which exists as argued above is an eigenvalue and we take $W_0 = \{w \in V : \rho(z)v = \lambda z\}$. Either W_0 is irreducible or it is not. If not we follow the same procedure above Equation 2 we find W_i irreducible such that $\rho(z)w = \lambda w$ for all $w \in W_i$.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

No, it does not (well, I am not requiring that our field be algebraically closed). For instance I am using $k = Z/2Z$ and no finite field is algebraically closed. For instance take $A = Z/2Za + Z/2Zb$ as a $Z/2Z$ algebra where a, b are indeterminants and we declare $ab = ba$ and that $\overline{0} = 0a + 0b$. Then, define ρ by

$$\rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\rho(1a + 0b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now, since A is commutative we have that $Z(A) = A$. So, let $z = 1a + 1b$. We see that

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is NOT a scalar operator since $\rho(1a+1b)(1,1)^T = (0,1)^T$ but $(1,1)^T \neq c(0,1)^T$ for any $c \in Z/2Z$.

For another example, which is slightly different, still not algebraically closed but this time A is an algebra over an infinite field. Namely, take $A = +$ where x is an indeterminant and where we stipulate that $x^2 = 0$. (So really this is just the ring $[x]/(x^2)$).

Then, let ρ be defined by

$$\rho(a + bx) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Indeed we see that

$$\begin{aligned}
\rho((a + bx)(c + dx)) &= \rho(ac + (ad + bc)x) \\
&= \begin{pmatrix} ac & ad + bc \\ 0 & ac \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \\
&= \rho(a + bx)\rho(c + dx).
\end{aligned}$$

So indeed, ρ is a homomorphism of algebras.

However, $\rho(1 + x)$ is not a scalar operator since in fact we get the same matrix as before that

$$\rho(1 + x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\rho(1+x)(0,1)^T = (1,1)^T \neq \alpha(0,1)^T$ for any $\alpha \in$.

Now, if we require k to be algebraically closed on the other hand, then take say $A = \mathbb{C}[x]/(x^n)$ as a $\mathbb{C}$ algebra and let $\rho : \mathbb{C}[x]/(x^n) \to \mathbb{C}^n$ be defined by

$$\rho(x) = J(2, n).$$

Then, ρ, V is indecomposable, since otherwise

First, I verify that ρ is a homomorhism of algebras.
Namely, we verify

$$\begin{aligned}
\rho((a+bi)(c+di)) &= \rho(ac - bd + (ad+bc)i) \\
&= \begin{pmatrix} (ac-bd) & (ad+bc) \\ -(ad+bc) & (ac-bd) \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix}
\end{aligned}$$

Once again $Z(\mathbb{C}) = \mathbb{C}$. Take $z = i$. Then,

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and note that

$$\rho(i)(1,0)^T = (0,-1)^T \neq \alpha(1,0)^T$$

for any $\alpha \in \mathbb{C}$.
We must also verify however that $\rho, \mathbb{C}^2$ is an indecomposable representation.

Well, is it reducible? If so there must be some non-trivial proper subrepresentation, which must be of dimension 1 since $dim(V) = 2$. So, it must be of the form $W = span(w)$ such that $\rho(a+bi)w \in W$ for all $a, b \in \mathbb{R}$, meaning that $\rho(a+bi)w = (\alpha + \beta i)w$ for some $\alpha, \beta \in R$. Then, expanding that out where we write $w = (w_1, w_2)^T$ for some $w_1, w_2 \in \mathbb{C}$ gives

$$\begin{aligned}
\rho(a+bi)(w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\
\begin{pmatrix} a & b \\ -b & a \end{pmatrix} (w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\
\begin{pmatrix} aw_1 + bw_2 \\ -bw_1 + aw_2 \end{pmatrix} &= \begin{pmatrix} (\alpha + \beta i)w_1 \\ (\alpha + \beta i)w_2)^T \end{pmatrix}
\end{aligned}$$

which means that $aw_1 + bw_2 = \alpha w_1 + \beta i w_2$ and $-bw_1 + aw_2 = \alpha w_2 + \beta i w_2$.
This must hold for all $a, b \in \mathbb{R}$ so take $(a, b) = (0, 1)$ meaning $a + bi = i)$
If $(a, b) = (a, 0)$ then

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and indeed the eigenvalue of $\rho(i)$ are the roots of x^2+1 which are $\lambda_1 = i$ and $\lambda_2 = -i$ with corresponding eigenvectors $v_1 = (-i, 1)^T$ and $v_2 = (i, 1)^T$. Can we decompose $\mathbb{C}^2$ as a representation?

No, if we were able to we would need to set $V_1 = span(v_1)$ and $V_2 = span(v_2)$ and then we would want to show that V_1 is stable under the action of A. However, it is not. Namely, take $a + bi = 2 + i$. Then,

$$\rho(2+i) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

so that

$$\begin{aligned} \rho(2+i)v_1 &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} v_1 \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} (-i, 1)^T \\ &= \begin{pmatrix} 1-2i \\ 2-i \end{pmatrix} \end{aligned}$$

Now, if we were to have $\rho(2+i)w_1 \in W_1$ we would need $r + si \in \mathbb{C}$ such that

$$1 - 2i = -i(r + si)$$

and

$$2 - i = r + si$$

which means we would have $r = 2$ and $s = -1$ which then says that

$$1 - 2i = -i(2 - i) = -2i + i^2 = -2i - 1,$$

which is a contradiction. So, indeed we see that this representation is in fact indecomposable, which means this is a counterexample. So, we do NOT have that $\rho(z)$ is a scalar operator EVEN if we require k to be algebraically closed.