Linear Represention of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 2.5
Let ρ be a linear representation with character χ. Show that the number of times that ρ contains the unit representation is equal to $(\chi|1) = \frac{1}{g}\sum_{s\in G}\chi(s)$.

We note by Theorem 2 that every representation is a direct sum of irreducible representations. Thus, we have that

$$V = \oplus_{i=1}^{m} k_i W_i.$$

where $W_i \neq W_j$ for $i \neq j$ and W_i is irreducible and $k_i \in \mathbb{N}_{\geq 1}$ for all $i \in [m]$. Then, if χ is the character of V and χ_i is the character of W_i for each $i \in [m]$ Proposition 2 tells us that $\chi = \sum_{i\in[m]} k_i\chi_i$. Thus,

$$(\chi \mid 1) = \sum_{i\in[m]} k_i(\chi_i \mid 1) = \sum_{i\in[m]} k_i\chi_{W_i=1},$$

which is by definition the number of times V contains the unit representation.

Exercise 2.6
Let X be a finite set on which G acts. Let ρ be the corresponding permutation representation and let χ be its character.

(a) The set Gx of images under G of an element $x \in X$ is called an *orbit*. Let c be the number of distinct orbits. Show that c is equal to the number of times that ρ contains the unit representation 1; deduce from this that $(\chi \mid 1) = c$. In particular, if G is transitive (i.e., if $c = 1$), ρ can be decomposed into $1 \oplus \theta$ and θ does not contain the unit representation. If ψ is the character of θ, we have $\chi = 1+\psi$ and $(\psi \mid 1) = 0$.

We show that c is equal to the number of times that ρ contains the unit representation 1 in a few steps.

- Note that an elementary theorem in algebra tells us that the orbits $\{Gx : x \in X\}$ partition the set X we know that if $\{x_i : i \in [N]\}$ is a set of representatives for the orbits, then

$$X = \cup_{i\in[N]} Gx_i.$$

- Then, by definition of the permutation representation which has basis

$$B = \{e_{x_j} : j \in [|X|]\}$$

I claim that

$$V = \bigoplus_{i\in[N]} \operatorname{span}(\{\rho_g(e_{x_i}) : g \in G\}) =: \bigoplus_{i\in[N]} W_i.$$

(Note that the W_i are not irreducible but are stable under the action of G).

  - Namely, each W_i is stable under the action of G since

$$\rho_g(w_i) = \rho_g(\sum_{h\in G} c_h\rho_h(e_{x_i})) = \sum_{h\in G} c_h\rho_{gh}(e_{x_i}) = \sum_{k\in G} c_{g^{-1}k}\rho_k(e_{x_i}) \in W_i.$$

  - Also, we can now show that the W_i span V or more precisely that

$$V \subseteq \bigoplus_{i\in[N]} W_i$$

by noting that $V = \operatorname{span}(\{e_{x_j} : j \in [|X|]\})$ and for all $j \in [|X|]$ there exists a unique $i(j) \in [N]$ such that $x_j \in Gx_{i(j)}$ which implies that $e_{x_j} = g(j)x_{i(j)} \in W_i$ for some $g(j) \in G$. Then, since any $v \in V$ has the form $\sum_{j\in[|X|]} c_{x_j}e_{x_j}$ we know that

$$v = \sum_{j\in[|X|]} c_{x_j}e_{x_j} = \sum_{j\in[|X|]} c_{x_j}g(j)e_{x_{i(j)}} = \sum_{j\in[N]} e_{x_j} \sum_{\{k\in[|X|]:i(k)=j\}} c_{x_k}g(k)$$

Then the fact that the above is a finite linear combination of elements of W_i for $i \in [N]$ proves the claim.

– Finally we show that

$$W_i \cap W_j = \{0\}$$

for all $i, j \in [N]$ with $i \neq j$.
This follows from the fact that

$$\rho_g(e_{x_i}) = e_{gx_i} \in B$$

for all $g \in G$ and all $i \in [|X|]$. Namely, if we denote

$$Gx_i = \{x_1^i, x_2^i, \dots, x_{n_i}^i\}$$

then that implies that

$$\{\rho_g(e_{x_i}) : g \in G\} = \{e_{x_1^i}, e_{x_2^i}, \dots, e_{x_{n_i}^i}\} =: S_i$$

with

$$B = \bigsqcup_{i \in [N]} S_i.$$

– Then, one has that $W_i = \text{span}(S_i)$, and since $\{S_i : i \in [N]\}$ are pairwise disjoint subsets of basis vectors that means that $W_i \cap W_j = \{0\}$ for all $i \neq j$ since otherwise if there exists non-zero $w \in W_i \cap W_j$ for some $i \neq j$ then that means that $w = \sum_{k \in [n_i]} c_k e_{x_k^i} = \sum_{l \in [n_j]} b_l e_{x_l^j}$ where $\{c_k : k \in [n_i], c_k \neq 0\} \neq \emptyset$ and $\{b_l : l \in [n_j], b_l \neq 0\} \neq \emptyset$. Thus, one has that

$$0 = \sum_{k \in [n_i]} c_k e_{x_k^i} - \sum_{l \in [n_j]} b_l e_{x_l^j}$$

a contradiction since $\{e_{x_k^i} : k \in [n_i]\} \sqcup \{e_{x_l^j} : l \in [n_j]\}$ is a linearly independent set.

- Finally, I claim that each W_i contains the unit representation exactly once. Namely, note that for each $i \in [N]$ one has that the subspace $U_i = \text{span}(u_i)$ spanned by the vector

$$u_i := \sum_{g \in G} \rho_g(e_{x_i})$$

is invariant under G since

$$\rho_h(u_i) := \rho_h(\sum_{g \in G} \rho_g(e_{x_i})) = \sum_{g \in G} \rho_{hg}(e_{x_i}) = \sum_{k \in G} \rho_k(e_{x_i}) = u_i$$

for all $h \in G$ and furthermore for $u = cu_i \in \text{span}(u_i)$ for some $c \in F$ one has by linearity that $\rho_h(u) = c\rho_h(u_i) = cu_i \in \text{span}(u_i)$, meaning the subspace is stable and thus each W_i such that $i \in [N]$ contains the unit representation at least once.

- **TODO: Show that each orbit contains the unit representation at MOST once**.

(b) Let G act on the product $X \times X$ by means of the formula

$$s(x, y) = (sx, sy).$$

Show that the character of the corresponding permutation representation is equal to χ^2.

Let $\hat{\rho}$ denote the representation on the product $X \times X$ and $\hat{\chi}$ denote the corresponding character and recall from Exercise 2.2 that for any given permutation representation $\chi(s)$ is the number of elements $x \in X$ that are fixed by $s \in G$. Now, consider $X^s = \{x \in X : sx = x\}$. We then have that $X^s \times X^s = \{(x, y) \in X \times X : sx = x, sy = y\} = (X \times X)^s$. Of course then $\hat{\chi}(s) = |X^s \times X^s| = |X^s|^2 = (\chi(s))^2$, which concludes the proof.

(c) Suppose the G is transitive on X and that X has at least two elements. We say that G is doubly transitive if, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ such that $x' = sx$ and $y' = sy$. Prove the equivalence of the following properties:

(1) G is doubly transitive.

(2) The action of G on $X \times X$ has two orbits, the diagonal and its complement.

(3) $(\chi^2|1) = 2$.

(4) The representation defined in (a) is irreducible.

Clearly (1) implies (2), namely since considering the action of s on (x, y) we have exactly two cases, namely either $x \neq y$ or $x = y$.

If $x = y$ then $s(x, x) = (sx, sx)$ and in fact doubly transitive implies transitive since for any $u \in X$ with $u \neq x$ we may find s such that $sx = u$. Namely, we may set a dummy variable $w \neq x$ and then require s such that $s(x, w) = (u, sw)$ which exists since $sw \neq u = sx$ by basic facts of a group action (namely cancellation). So, indeed for any $x \in X$ and any $u \in X$ with $u \neq x$ we have $s \in G$ such that $s(x, x) = (u, u)$. Of course if $u = x$ we take $s = e$.

If $x \neq y$, then for any $x' \neq y'$ we can find $s \in G$ such that $s(x, y) = (x', y')$. Thus, the complement of the diagonal is indeed an orbit.

Now, clearly (2) implies (1) since the complement of the diagonal being an orbit is equivalent to (1). Thus (1) and (2) are equivalent.

Now, (3) says $(\chi^2|1) = 2$. By part (a) this means that the number of orbits of the action of G on $X \times X$ is 2. Of course, we know that the diagonal is one orbit since as argued above doubly transitive implies transitive. Then, the remains of $X \times X$ must be exactly the other orbit. Thus, (3) implies (2) and of course (2) implies (3), meaning they are equivalent.

Finally, (4) is equivalent to (3) since as noted in part (a) we have that $\chi = 1+\psi$ where ψ is the character of θ. Now, $\chi^2 = (1+\psi)^2 = 1+2\psi+\psi^2$ and since $2 = (\chi^2|1) = (1|1)+2(\psi|1)+(\psi^2|1) = 1+2(0)+(\psi^2|1)$ (where we are using $(\psi|1) = 0$ from part (a)) that gives $(\psi^2|1) = 1$.
Now, by definition $(\psi^2|1) = \frac{1}{g}\sum_{s\in G}\psi^2(s) = \frac{1}{g}\sum_{s\in G}\psi(s)\psi(s)^* = (\psi|\psi)$. Then, Theorem 5 says that $(\psi|\psi) = 1$ if and only if the representation (θ, V) is irreducible. So, indeed (3) implies (4) and also (4) implies (3) since Theorem 5 was an equivalence. Thus, (3) is equivalent to (4) and finally we have that (1), (2), (3), and (4) are all equivalent.