

Serre Exercises

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1 Exercise 2.1

Let χ and χ' be characters of two representations. Prove the formulas

$$(\chi + \chi')_{\sigma}^2 = \chi_{\sigma}^2 + \chi'_{\sigma}^2 + \chi\chi'$$

and

$$(\chi + \chi')_{\alpha}^2 = \chi_{\alpha}^2 + \chi'_{\alpha}^2 + \chi\chi'.$$

We use proposition 3. Namely, note

$$\begin{aligned} (\chi + \chi')_{\sigma}^2(s) &= \frac{1}{2}((\chi + \chi')(s))^2 + (\chi + \chi')(s^2)) \\ &= \frac{1}{2}(\chi^2(s) + \chi'^2(s) + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)) \\ &= \frac{1}{2}(\chi^2(s) + \chi(s^2)) + \frac{1}{2}(\chi'^2(s) + \chi'(s^2)) + \chi(s)\chi'(s) \\ &= (\chi_{\sigma}^2 + \chi'_{\sigma}^2 + \chi\chi')(s). \end{aligned}$$

The proof for the alternating square is analogous.

2 Exercise 2.2

Let X be a finite set on which G acts. Let ρ be the corresponding permutation representation. Let χ_X be the character of ρ . Let $s \in G$. Show that $\chi_X(s)$ is the number of elements of X fixed by s .

This is almost a tautology. However, I provide a formal proof. The fact that ρ is a permutation representation means that for every $s \in G$ we see that ρ_s is a permutation matrix. In particular, we order the elements of X as x_1, x_2, \dots, x_n . Then, say $sx_i =: x_{s,i}$. We associate e_i with x_i . Now, we define the matrix ρ_s by

$$\rho_s(e_i) = e_{s,i}$$

and we note that indeed since the map $\pi_s : x \mapsto sx$ is a permutation of X , that ρ_s is a permutation matrix. Now, we examine the diagonal of ρ_s .

We see that $(\rho_s)_i^i = (\rho_s(e_i))^i = 1$ if $\rho_s(e_i) = e_i$ and $= 0$ otherwise. So, $(\rho_s)_i^i = 1$ if $sx_i = x_i$ and $= 0$ otherwise. Thus,

$$\text{Tr}(\rho_s) = \sum_{i \in [|X|]} (\rho_s)_i^i = \sum_{i \in [|X|]: sx_i = x_i} 1 = |\{i \in [|X|] : sx_i = x_i\}|.$$

3 Exercise 2.3

Let $\rho : G \rightarrow GL(V)$ be a linear representation with character χ and let V' be the dual of V , i.e. the space of linear forms on V . For $x \in V$, $x' \in V'$ let $\langle x, x' \rangle$ denote the value of the linear form x' at x . Show that there exists a unique linear representation $\rho' : G \rightarrow GL(V')$ such that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle$$

for $s \in G, x \in V, x' \in V'$. This is called the contragradient (or dual) representation of ρ ; its character is χ^* .

Let ρ'_s be defined by $\rho'_s(x') = x' \circ \rho_s^{-1}$. Now,

$$\langle \rho_s(x), \rho'_s(x') \rangle = \langle \rho_s(x), x' \circ \rho_s^{-1} \rangle = (x' \circ \rho_s^{-1})(\rho_s(x)) = x'(x) = \langle x, x' \rangle.$$

4 Exercise 2.4

Let $\rho_1 : G \rightarrow GL(V_1)$ and $\rho_2 : G \rightarrow GL(V_2)$ be two representations with characters χ_1, χ_2 . Let $W = \text{Hom}(V_1, V_2)$ be the vector space of linear mappings $f : V_1 \rightarrow V_2$. For $s \in G$ and $f \in W$, let $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$; so $\rho_s f \in W$. Show that this defines a linear representation $\rho : G \rightarrow GL(W)$, and that its character is $\chi_1^* \cdot \chi_2$. This representation is isomorphic to $\rho'_1 \otimes \rho_2$, where ρ'_1 is the contragradient of ρ_1 .

We need to show that this map respects the group operation, namely that

$$\rho_{st}(f) = \rho_s(\rho_t(f)).$$

Well,

$$\begin{aligned} \rho_{st}(f) &= \rho_{2,st} \circ f \circ \rho_{1,st}^{-1} \\ &= \rho_{2,st} \circ f \circ \rho_{1,(st)}^{-1} \\ &= \rho_{2,s} \rho_{2,t} \circ f \circ \rho_{1,t^{-1}s^{-1}} \\ &= \rho_{2,s} \rho_{2,t} \circ f \circ \rho_{1,t^{-1}} \rho_{1,s^{-1}} \\ &= \rho_{2,s} \circ (\rho_{2,t} \circ f \circ \rho_{1,t^{-1}}) \circ \rho_{1,s^{-1}} \\ &= \rho_s(\rho_{2,t} \circ f \circ \rho_{1,t^{-1}}) \\ &= \rho_s(\rho_t(f)). \end{aligned}$$

We first show that $\rho \cong \rho'_1 \otimes \rho_2$.
We give an isomorphism.

Let $\tau : V_1^* \otimes V_2 \rightarrow \text{Hom}(V_1, V_2)$ be defined by $\tau(v'_1, v_2) = (v_1 \mapsto v'_1(v_1)v_2)$. Then, we note

$$\begin{aligned}\tau((\rho'_1 \otimes \rho_2)(v'_1, v_2)) &= \tau(\rho'_1(v'_1) \otimes \rho_2(v_2)) \\ &= (v_1 \mapsto \rho'_1(v'_1)\rho_2(v_2)) \\ &= (v_1 \mapsto v'_1(\rho_1^{-1}(v_1))\rho_2(v_2))\end{aligned}$$

and also note

$$\begin{aligned}\rho_s(\tau(v'_1, v_2)) &= \rho_{2,s} \circ (\tau(v'_1, v_2)) \circ \rho_{1,s}^{-1} \\ &= \rho_{2,s} \circ (w \mapsto v'_1(w)v_2) \circ \rho_{1,s}^{-1} \\ &= \rho_{2,s} \circ (v_1 \mapsto \rho_{1,s}^{-1}v_1 \mapsto v'_1(\rho_{1,s}^{-1}(v_1))v_2) \\ &= \rho_{2,s} \circ (v_1 \mapsto v'_1(\rho_{1,s}^{-1}(v_1))v_2) \\ &= (v_1 \mapsto v'_1(\rho_{1,s}^{-1}(v_1))\rho_{2,s}(v_2))\end{aligned}$$

So, indeed we have an isomorphism.

Now, to show that its character is $\chi_1^* \cdot \chi_2$, we simply cite the fact that $\chi(\rho'_1 \otimes \rho_2) = \chi(\rho'_1) \cdot \chi(\rho_2)$.

However, we also need to show that $\chi(\rho'_1) = \chi(\rho_1)^*$.

Well,

$$\chi(\rho'_1) = \text{Tr}(\rho'_1) = \sum_i \lambda_i(\rho'_1).$$

Now, what are the eigenvalues of ρ'_1 . Well, the eigenvalue, eigenvector pairs are (k, v'_1) where $k \in K$ and $v'_1 \in V'_1$ such that $\rho'_1(v'_1) = kv'_1$, and since $v'_1 : V_1 \rightarrow K$ is a map we are saying these maps differ by a constant on all inputs $v_1 \in V_1$, namely that $\rho'_1(v'_1)(v_1) = kv'_1(v_1)$ for all $v_1 \in V_1$.

Recall

$$\rho'_1(v'_1)(v_1) = v'_1(\rho_1^{-1}(v_1))$$

meaning that

$$v'_1(\rho_1^{-1}(v_1)) = kv'_1(v_1) = kv'_1(v_1)$$

for all $v_1 \in V_1$. So certainly if (\hat{k}, v_1) is an eigenpair of ρ_1 , then it suffices to take $v'_1 : V_1 \rightarrow K$ such that $v'_1(cv_1) = cv_1$ for all $c \in K$ and $v'_1(u_1) = 0$ for all $u_1 \perp v_1$.