A First Course in Modular FOrms
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1 Chapter 1 Notes and ExercisesExercises

1.1 Chapter 1.1

Exercise 1.1.3

- (a) Show that the set $\mathcal{M}_k(SL_2(\mathbb{Z})$ of modular forms of weight k forms a vector space over $\mathbb{C}$.
  Note that a function $f : \mathcal{H} \to \mathbb{C}$ is a modular form by definition if

  - f is homolorphic on $\mathcal{H}$,
  - f is weakly modular of weight k, and
  - f is holomorhpic at ∞.

  We need to show the following for all $a, b \in \mathbb{C}$ and all $f, g, h \in \mathcal{M}_k(SL_2(\mathbb{Z})$. We prove

  - existence of a zero element (additive identity),
    It is $0 := (z \mapsto 0, \forall z \in \mathcal{H})$.
  - $x + y \in \mathcal{M}_k(SL_2(\mathbb{Z})$
    First, f, g holomorphic implies $f + g$ holomorphic. Then, say f, g weakly modular of weight k. We want to show $f + g$ is too. Well, we know $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ and $g(\gamma(\tau)) = (c\tau + d)^k g(\tau)$. Then, $(f+g)(\gamma(\tau)) = (c\tau+d)^k f(\tau)+(c\tau+d)^k g(\tau) = (c\tau+d)^k((f+g)(\tau))$. Now, knowing that f, g are holomorphic at ∞ we would like to show that $f + g$ is too. Well, by definition that means that we have numbers z_1, z_2 such that the functions $\hat{f}(z) := f(\frac{log(z))}{(2\pi i)}$ and $\hat{g}(z) := g(\frac{log(z)}{2\pi i})$ for $z \in D'$ where we assign $\hat{f}(0) = z_1$ and $\hat{g}(0) = z_2$ are holomorphic on D (the unit disk, where note that $D' = D \setminus \{0\}$). Then, of course taking $f \hat{+} g(z) := f(\frac{log(z))}{(2\pi i)} + g(\frac{log(z))}{(2\pi i)}$ we set $f \hat{+} g(0) := z_1 + z_2$ and we see that $f \hat{+} g$ is also holomorphic on D since it is the (pointwise) sum of two holomorphic functions.
  - $ax \in \mathcal{M}_k(SL_2(\mathbb{Z})$
    Well, f holomorphic implies $af : z \to af(z)$ is as well. Then, $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ implies $af(\gamma(\tau)) = (c\tau + d)^k af(\tau)$. Finally, there exists $z_0 \in \mathbb{C}$ such that if we define $\hat{f}(z) := f(\frac{log(z))}{(2\pi i)}$ for $z \in D'$ and $\hat{f}(0) = z_0$ then $\hat{f}$ is holomorphic on D. Now, we set $\hat{af}(0) = az_0$ and see that $\hat{f}$ is holomorphic on D too, which is the definition of af being holomorphic at ∞.
  - $f + g = g + f$
    This holds just by commutativity of addition in $\mathbb{C}$.
  - $f + (g + h) = (f + g) + h$
    This holds by associativity of addition in $\mathbb{C}$.
  - $f + 0 = f$
    Taking the zero element to be the zero map, this is obvious.
  - $f + (-f) = 0$
    For given $f \in \mathcal{M}_k(SL_2(\mathbb{Z})$, we let $(-f)$ be defined as expected by $(-f)(z) = -f(z)$. Then, the claim follows.
  - $0f = 0$
    This is obvious since $0 * f(z) = 0 \in \mathbb{C}$ for all $z \in \mathcal{H}$. Thus, $0f$ is the zero function on $\mathcal{H}$.
  - $1f = f$
    This is clearly true since $1 \in \mathbb{C}$ is the multiplicative identity.
  - $a(bf) = (ab)f$
    This holds by associativity of multiplication in $\mathbb{C}$.

- $a(f+g) = af + ag$.
  This holds by distributativity of multiplication over addition in $\mathbb{C}$.
- $(a+b)f = af + bf$
  This holds by distributativity of multiplication over addititition ("from the RHS") in $\mathbb{C}$.

- (b) If f is a modular form of weight k and g is a modular form of weight ℓ, show that fg is a modular form of weight $k+\ell$.
  We need to show:
  - fg is holomorphic on $\mathcal{H}$, which holds since the class of holomorphic functions is closed under (pointwise) multiplication.
  - fg is weakly modular of weight $k+\ell$, which holds since $(fg)(\gamma(\tau)) = ((c\tau+d)^k f(\tau))((c\tau+d)^\ell g(\tau)) = (c\tau+d)^{k+\ell} f(\tau)g(\tau)$,
  - Once more, if we have z_1, z_2 which we use to set $\hat{f}(0) := z_2$ and $\hat{g}(0) := z_2$ where $\hat{f}, \hat{g}$ are defined on the rest of D as above. We set $\hat{fg}(0) := z_1 z_2$ and $\hat{fg}(z) = \hat{f}(z)\hat{g}(z)$ for $z \in D'$. Then, $\hat{fg}$ is holomorphic on D, and thus, fg is holomorphic at ∞.

- (c) Show that $\mathcal{S}_k(SL_2(\mathbb{Z})$ is a vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$ and that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.
  Since, the vector space opeartions are the same as those in the bigger space, we just need to check containment of the zero element, and closure of addition and scalar multiplication. Indeed we take the zero element to be the zero function again $0 : \mathcal{H} \to \mathbb{C}$. Now, since as noted on page 3, f, g holomorphic at ∞ is equivalent to saying that f, g have Fourier expansions equal to f, g respectively. So, $f(\tau) = \sum_{n=1}^{\infty} a_n (e^{2\pi i\tau})^n$ and $g(\tau) = \sum_{n=1}^{\infty} b_n (e^{2\pi i\tau})^n$. Then, $(f+g)(\tau) == \sum_{n=1}^{\infty} (a_n + b_n)(e^{2\pi i\tau})^n$. So indeed, $f + g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$. Now, given $a \in \mathbb{C}$, $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$, we note that since $f(\tau) = \sum_{n=1}^{\infty} a_n (e^{2\pi i\tau})^n$ indeed we have $af(\tau) = \sum_{n=1}^{\infty} a a_n (e^{2\pi i\tau})^n$ and we are done.
  Finally, I want to show that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$, meaning we need to show closure under addition from within, and absorbtion of multiplication from all of the bigger ring.
  So, first say $f = \oplus_{k\in\mathbb{Z}} f_k, g = \oplus_{k\in\mathbb{Z}} g_k \in \mathcal{S}(SL_2(\mathbb{Z})))$. Then we note that $f + g := \oplus_{k\in\mathbb{Z}}(f_k + g_k) \in \mathcal{S}(SL_2(Z))$ since for each $k \in \mathbb{Z}$ we have (as shown earlier) that $f_k + g_k \in \mathcal{S}_k(SL_2(\mathbb{Z}))$. Now, given any function $h = \oplus_{k\in\mathbb{Z}} h_k \in \mathcal{M}(SL_2(\mathbb{Z}))$ and any $f = \oplus_{k\in\mathbb{Z}} f_k \in \mathcal{S}(SL_2(\mathbb{Z})))$ we want to show that $fh \in \mathcal{S}(SL_2(\mathbb{Z})))$. Well, we write it in the following product structure.
  We note $fh = \oplus_{n\in\mathbb{Z}}(\sum_{k\in\mathbb{Z}} f_{n-k}h_k)$ and now what remains to show is that $(\sum_{k\in\mathbb{Z}} f_{n-k}h_k) \in \mathcal{S}_n(SL_2(\mathbb{Z}))$ for all $n \in \mathbb{Z}$. Well, we know that $f_{n-k}(z) := \sum_{j=1}^{\infty} a_j q^j$ and $h_k = \sum_{r=0}^{\infty} b_r q^r$. So, we have that $(f_{n-k}h_k)(z) = \sum_{j=1}^{\infty}\sum_{r=0}^{\infty} a_j b_r q^{j+r} = \sum_{s=1}^{\infty}(\sum_{m=0}^{s-1} b_s a_{s-m})q^s$. Thus, $f_{n-k}h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$ for all $n, k \in \mathbb{Z}$. Thus, $\sum_{k\in\mathbb{Z}} f_{n-k}h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$ for all $n \in \mathbb{Z}$. So, we have that $fh \in \mathcal{S}(SL_2(\mathbb{Z})))$ as desired.