Serre Exercises

Caitlin Beecham

1 Exercise 2.5

Let ρ be a linear representation with character χ. Show that the number of times that ρ contains the unit representation is equal to $(\chi|1) = \frac{1}{g}\sum_{s\in G}\chi(s)$.
Now, I note that by a "unit representation" I assume the author means a representation $\rho' : G \to GL(V)$ so that for some fixed suitable basis we have

$$\rho'_g = \begin{pmatrix} 1 & 0 \\ 0 & M_g \end{pmatrix}$$

for every $g \in G$ where M_g is a square matrix. So, indeed we just mean that there is some vector (which we have here designated to be the first basis vector) which is fixed by every $g \in G$ and so that some complement of the span of that vector is stable under G. Of course, by an earlier theorem in this book, indeed since $Span(e_1)$ is a subrepresentation then so is the orthogonal complement (indeed we then take any $n-1$ orthogonal vectors from this complement to be $e_2, e_3, \ldots, e_n$).
So, indeed we are really trying to calculate the number of pairwise orthogonal vectors v such that $\rho_g(v) = v$ for all $g \in G$, which just amounts to calculating the number of pairwise independent v with the above property, since for any such v_1, we see by an earlier theorem in this book that the orthogonal complement $Span(v_1)^\perp$ is a subrepresentation meaning stable under the action of G and then we search for $v_2 \in Span(v_1)^\perp$ and then $v_3 \in Span(v_1)^\perp \cap Span(v_2)^\perp$ and so on.
Now, indeed there is always such a vector. One can simply use some kind of averaging method to see that. Namely, take arbitrary fixed $v \in V$. Now, form the sum

$$\hat{v} = \sum_{g\in G}\rho_g(v)$$

and note that $\hat{v}$ is fixed by every element in $s \in G$. Namely, we note for any $s \in G$

$$\rho_s(\hat{v}) = \sum_{g\in G}\rho_s\rho_g(v) = \sum_{g\in G}\rho_{sg}(v) = \sum_{g\in G}\rho_g(v) = \hat{v}.$$

OK BIG NOTE: Something is WRONG here. By my interpretation, and my current understanding, this would mean that every representation decomposes as

$$\rho_g = \begin{pmatrix} I_{n-1}^{n-1} & 0 \\ 0 & c_g \end{pmatrix}$$

since by my above averaging argument in each of these blocks I am getting there is some fixed vector whose complement is also stable under G. So something is very off, but I want to work on something else right now. ANOTHER NOTE: Part (a) of the exercise below is a special case since we are looking at permutation representations, I also think Serre might be requiring that we use only permutation similarity transformations (meaning we need to keep our basis vectors fixed and can only reorder them), but I'm not sure. I have a lot to iron out.

2 Exercise 2.6

Let X be a finite set on which G acts. Let ρ be the corresponding permutation representation and let χ be its character.

2.1 (a)

Let c be the number of orbits of this group action. Show that c is equal to the number of times that ρ contains the unit representation 1; deduce from this that $c = (\chi|1)$. In particular, if G is transitive (i.e. if $c = 1$), ρ can be decomposed into $1 \oplus \theta$ and θ does not contain the unit representation. If ψ is the character of θ. We have that $\chi = 1 + \psi$ and $(\psi|1) = 0$.

First, we recall that we can decompose (ρ, V) into a direct sum of irreducible representations which we can in fact construct to consist of pairwise orthogonal spaces. We show that claim by induction.

Namely, either (ρ, V) is irreducible or their is some proper subspace $W \subsetneq V$ stable under ρ. Then, as noted as part of Theorem 1, if we take P to be the orthogonal projection matrix associated with W (so in particular we can take any orthonormal basis $w_1, \ldots, w_{n_1}$ of W then construct the matrix P as $P = WW^*$ where W has the vectors $w_1, \ldots, w_{n-1}$ as its columns). Then, we have that $W^\perp = range(I - P)$ and indeed $W^\perp$ is stable under ρ. Now, by induction, each of W and $W^\perp$ decomposes into a direct sum or pairwise orthogonal subrepresentations. So we have that W also decomposes as a direct sum of pairwise orthogonal subrepresentations by combining the two respective direct summands (I will spare the reader the details) and we have

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$

Now, I note that by a "unit representation" I assume the author means a representation $\rho' : G \to GL(V)$ so that for some fixed suitable basis we have

$$\rho'_g = \begin{pmatrix} 1 & 0 \\ 0 & M_g \end{pmatrix}$$

for every $g \in G$ where M_g is a square matrix. So, indeed we just mean that there is some vector (which we have here designated to be the first basis vector) which is fixed by every $g \in G$ and so that some complement of the span of that vector is stable under G. Of course, by an earlier theorem in this book, indeed since $Span(e_1)$ is a subrepresentation then so is the orthogonal complement (indeed we then take any $n - 1$ orthogonal vectors from this

complement to be $e_2, e_3, \dots, e_n$.
Now, indeed there is ALWAYS such a vector. One can simply use some kind of averaging method to see that. Namely, take arbitrary fixed $v \in V$. Now, form the sum

$$\hat{v} = \sum_{g \in G} \rho_g(v)$$

and note that $\hat{v}$ is fixed by every element in $s \in G$. Namely, we note for any $s \in G$

$$\rho_s(\hat{v}) = \sum_{g \in G} \rho_s \rho_g(v) = \sum_{g \in G} \rho_{sg}(v) = \sum_{g \in G} \rho_g(v) = \hat{v}.$$

So, indeed this is consistent with the fact that there is always at least one orbit for any group action.
Now, suppose one wants to look for other such vectors. One should however first restrict attention to the orthogonal complement of $Span(\hat{v})$.
First, we recall that we still have our original decomposition into pairwise orthogonal subspaces namely $V = V_1 \oplus \dots \oplus V_k$. Now, indeed since these span V, there is some $i_0 \in [k]$ such that $\hat{v} \in V_{i_0}$ and indeed since V_{i_0} is stable under G we have that $(Span(\hat{v}))^{\perp} \supseteq \bigoplus_{i \in [k]: i \neq i_0} V_i$. So, we choose the next subspace to restrict out attention to, which might as well just by from lowest to highest index. So, take $i_1 = min\{i \in [k] : i \neq i_0\}$. Now, take arbitrary $v_1 \in V_{i_1}$ and for the sum $\hat{v_1} = \sum_{g \in G} \rho_g v_1$. Clearly, $\hat{v_1} \in V_{i_1}$ is also still within the subspace V_{i_1} since it is stable under G. Now, once again we have that for all $s \in G$ that $\rho_s \hat{v_1} = \hat{v_1}$, meaning that v_1 "is a unit representation" (I am never sure what the proper terminology is, whether the representation is the map or the vector space). Now, we examine $Span(\hat{v})^{\perp} \cap Span(\hat{v_1})^{\perp}$ and iterate this process. Clearly, $Span(\hat{v})^{\perp} \cap Span(\hat{v_1})^{\perp} \supseteq \bigoplus_{i \in [k]: i \neq i_0, i \neq i_1} V_i$ and we see that we can find k pairwise mutually orthogonal unit subrepresentations which we label $\{\hat{v}, \hat{v_1}, \dots, \hat{v_{k-1}}\}$.
Furthermore, I need to argue that k which is the number of irreducible subrepresentations in our summand is in fact the number of orbits.
Well, clearly, intuitively, any orbit must be contained within one V_i or else the V_i are not stable under ρ, but we make this notion precise. Take $x_i \in X$ and corresponding basis vector e_i.
So, consider the set $\{j \in [k] : \exists g \in G \text{ s.t. } \rho_g e_i = e_j\}$. Now, as the V_i's span V, we know that $x_i \in V_{j_i}$ for some $j_i \in [k]$. Indeed, since V_{j_i} is stable under G we see that $\rho_g e_i (= e_{j(g)}) \in V_{j_i}$ for all $g \in G$. So, noting that the orbit of x_i in G is by definition the aforementioned set $\{j \in [k] : \exists g \in G \text{ s.t. } \rho_g e_i = e_j\}$ or the same but better described as $Orb(x_i) = \{x_{j(g)} : \rho_g e_i = e_{j(g)}\}$. Now, indeed we have seen that $Orb(x_i) \subseteq V_{j_i}$. So, that proves that either $c \geq k$ or there exists some (possibly more than one) $l \in [k]$ such that V_l "contains no orbit". Precisely, by that I mean that there is some V_l such that for all $x_i \in X$ we have that $Orb(x_i) \subseteq (V_l)^{\perp}$.
Now, we see that that cannot happen, but it is actually not as obvious as one might think. A nice first step would be to show that for all V_l there exists some $e_{j_l} \in V_l$ (a standard basis vector) which does turn out to be true but is not obvious without using the structure of ρ being a permutation representation. For instance it could a prior happen that V_l is one dimensional and is the span of a vector $\sum_{g \in G} \rho_g u$, which is not a standard basis vector where

$u \in V$ is some unspecified vector.

2.2 (b)

Let G act on the product $X \times X$ by means of the formula

$$s(x, y) = (sx, sy).$$

Show that the character of the corresponding permutation representation is equal to χ^2. Let $\hat{\rho}_s$ denote the representation on the product $X \times X$ and $\hat{\chi}$ denote the corresponding character.

Recall from Exercise 2.2 that for any given permutation representation χ_s is the number of elements $x \in X$ that are fixed by s.

Now, consider $X^s = \{x \in X : sx = s\}$. We then have that $X^s \times X^s = \{(x, y) \in X \times X : sx = x, sy = y\} = (X \times X)^s$. Of course then $\hat{\chi} = |X^s \times X^s| = |X^s|^2 = \chi_s^2$, which concludes the proof.

2.3 (c)

Suppose the G is transitive on X and that X has at least two elements. We say that G is doubly transitive if, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ such that $x' = sx$ and $y' = sy$. Prove the equivalence of the following properties:

1. G is doubly transitive.
2. The action of G on $X \times X$ has two orbits, the diagonal and its complement.
3. $(\chi^2|1) = 2$.
4. The representation defined in (a) is irreducible.

Clearly (1) implies (2), namely since considering the action of s on (x, y) we have exactly two cases, namely either $x \neq y$ or $x = y$.

If $x = y$ then $s(x, x) = (sx, sx)$ and in fact doubly transitive implies transitive since for any $u \in X$ with $u \neq x$ we may find s such that $sx = u$. Namely, we may set a dummy variable $w \neq x$ and then require s such that $s(x, w) = (u, sw)$ which exists since $sw \neq u = sx$ by basic facts of a group action (namely cancellation). So, indeed for any $x \in X$ and any $u \in X$ with $u \neq x$ we have $s \in G$ such that $s(x, x) = (u, u)$. Of course if $u = x$ we take $s = e$.

If $x \neq y$, then for any $x' \neq y'$ we can find $s \in G$ such that $s(x, y) = (x', y')$. Thus, the complement of the diagonal is indeed an orbit.

Now, clearly (2) implies (1) since the complement of the diagonal being an orbit is equivalent to (1). Thus (1) and (2) are equivalent.

Now, (3) says $(\chi^2|1) = 2$. By part (a) this means that the number of orbits of the action of G on $X \times X$ is 2. Of course, we know that the diagonal is one orbit since as argued above doubly transitive implies transitive. Then, the remains of $X \times X$ must be exactly the other orbit. Thus, (3) implies (2) and of course (2) implies (3), meaning they are equivalent.

Finally, (4) is equivalent to (3) since as noted in part (a) we have that $\chi = 1 + \psi$ where ψ is the character of θ. Now, $\chi^2 = (1+\psi)^2 = 1 + 2\psi + \psi^2$ and since $2 = (\chi^2|1) = (1|1) + 2(\psi|1) + (\psi^2|1) = 1 + 2(0) + (\psi^2|1)$ (where we are using $(\psi|1) = 0$ from part (a)) that gives $(\psi^2|1) = 1$.
Now, by defintion $(\psi^2|1) = \frac{1}{g}\sum_{s \in G} \psi^2(s) = \frac{1}{g}\sum_{s \in G} \psi(s)\psi(s)^* = (\psi|\psi)$. Then, Theorem 5 says that $(\psi|\psi) = 1$ if and only if the representation (θ, V) is irreducible. So, indeed (3) implies (4) and also (4) implies (3) since Theorem 5 was an equivalence. Thus, (3) is equivalent to (4) and finally we have that (1), (2), (3), and (4) are all equivalent.