Introduction to Representation Theory: Math 18.712
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[**1.20**] Let V be a nonzero finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.

Either V is irreducible or it is not. If not, then it has some proper non-trivial subrepresentation V_1 of dimension $dim(V_1) < dim(V)$. We continue in this fashion finding subrepresentations

$$V_i \subsetneq V_{i-1} \subsetneq V_1 \subsetneq V$$

until V_i is irreducible which we know is possible for the following reason. We note that any representation of dimension 1 is irreducible and indeed the dimension decreases by at least one at each step, or written precisely that $dim(V_i) \leq dim(V_{i-1}) - 1$, meaning that $1 \leq dim(V_i) \leq dim(V) - i$. So, indeed provided that $dim(V) > 0$ this process will make sense and terminate since $1 \leq dim(V) - i$ means that $i \leq dim(V) - 1 < \infty$.

[**1.21**] Problem 1.21. Let A be an algebra over a field k. The center Z(A) of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative then $Z(A) = A$.

(a) Show that if V is an irreducible finite dimensional representation of A then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V \in k$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the central character of V.

This makes intuitive sense since the center of the $GL_n(\mathbb{R})$ for instance is the set of scalar multiples of the identity. Now I provide a formal proof.

First, I note that we also need the field k to be algebraically closed otherwise taking $A = \mathbb{C}$ (as an $\mathbb{R}$-algebra) we note that $Z(A) = A$. Then, one notes that $V = A$ is a 2-dimensional representation over $k = \mathbb{R}$ (not algebraically closed). Indeed, taking the regular representation and the element $g = 1 + i = (1, 1) \in Z(A)$, we note that g acts on an element $v = (a, b) \in V$ by

$$(a, b) \mapsto^g (a - b, a + b),$$

and clearly taking $(a, b) = (0, 1)$ we see that

$$(a - b, a + b) = (-1, 1) \neq \lambda(0, 1)$$

for any $\lambda \in k = \mathbb{R}$.

However, assuming that k is algebraically closed, we proceed to prove the statement. Indeed, we cite Corollary 1.17, noting that for any $z \in Z(A)$ we have that $\rho(z)$ is an intertwining operator within $\rho(A)$ since for any $a \in A$ to verify $\rho(z)$ is an intertwining operator we need to show that

$$(\rho(z))(\rho(a)v)) = \rho(a)(\rho(z))(v),$$

for all $v \in V$ and all $a \in A$.
Indeed, we note that

$$(\rho(z))(\rho(a)v)) = \rho(za)v = \rho(az)v = \rho(a)\rho(z)v.$$

Then Corollary 1.17 gives the result.

(b) Show that if V is an indecomposable finite dimensional representation of A then for any $z \in Z(A)$ the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

First, I show that if ρ has an eigenvalue then it is unique and then I will show at the end that it has an eigenvalue, which is clearly true by virtue of k being algebraically closed.

Suppose there exist ρ_1, ρ_2 such that

$$\rho(z)v_1 = \lambda_1 v_1$$

and

$$\rho(z)v_2 = \lambda_2 v_2$$

for some $\lambda_1, \lambda_2 \in k$ and $v_1, v_2 \in V$ with $v_1, v_2 \neq 0$.
Now, for this fixed $z \in A$ let $W = \{v \in V : \rho(z)v = \lambda_1 v\}$. This is a vector subspace.

I would like to show that W is a subrepresentation, namely that for all $w \in W$ and all $a \in A$ we have that $\rho(a)w \in W$.

Assume not. Assume that there exists $w \in W$ such that $\rho(a)w \notin W$ meaning that

$$\rho(z)(\rho(a)w) \neq \lambda_1(\rho(a)w).$$

Then note that $a = z^{-1}az$ where $z \in Z(A)$ is the same fixed z from above.

So, $\rho(a) = \rho(z^{-1}az)$ meaning that

$$\rho(a)w = \rho(z^{-1}az)w.$$

Now multiplying on both sides by $\rho(z)$ we get

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}az)w$$

which means that

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}a)\rho(z)w = \rho(z)\rho(z^{-1}a)\lambda_1 w = \rho(zz^{-1})\rho(a)\lambda_1 w$$

$$= \lambda_1 \rho(e)\rho(a)w = \lambda_1 \rho(a)w.$$

We now have a contradiction since the above implies that

$$\rho(z)(\rho(a)w) = \lambda_1(\rho(a)w),$$

which means by definition that $\rho(a)w \in W$. So, indeed W is a subrepresentation.

Now, either W is irreducible or it is not. If not, then as shown in Exercise 1.20, we know that it contains some irreducible representation W'. Now, indeed we have shown that for any $z \in V$ one has that $\rho(z)$ has only one eigenvalue equal to the scalar by which z acts on W'.

Finally, we show that $\rho(z)$ actually has an eigenvalue $\lambda \in k$. Since k is algebraically closed, indeed it does since all roots of the characteristic polynomial belong to k.

So, indeed we have shown the desired result where λ which exists as argued above is an eigenvalue and we take $W_0 = \{w \in V : \rho(z)v = \lambda z\}$. Either W_0 is irreducible or it is not. If not we follow the same procedure above Equation we find W_i irreducible such that $\rho(z)w = \lambda w$ for all $w \in W_i$.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

No, it does not. For instance, take $A = \mathbb{Z}/2\mathbb{Z}a + \mathbb{Z}/2\mathbb{Z}b$ as a $\mathbb{Z}/2\mathbb{Z}$ algebra where a, b are indeterminants and we declare $ab = ba$ and that $\overline{0} = 0a + 0b$. Then, define $\rho : A \to GL(\mathbb{R}^2)$ by

$$\rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\rho(1a + 0b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since A is commutative one has that $Z(A) = A$. So, let $z = 1a + 1b$. We see that

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not a scalar operator since

$$\rho(1a + 1b)(1, 1)^T = (0, 1)^T$$

but

$$(1, 1)^T \neq c(0, 1)^T$$

for any $c \in \mathbb{Z}/2\mathbb{Z}$.

I provide another slightly different example in which A is an algebra over an infinite field. Namely, take $A = \mathbb{Q}+$ where x is an indeterminant and we stipulate that $x^2 = 0$. (So really this is just the ring $\mathbb{Q}[x]/(x^2)$).

Then, let $\rho : A \to GL(\mathbb{R}^2)$ be defined by

$$\rho(a + bx) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Indeed, we see that

$$\begin{aligned}
\rho((a+bx)(c+dx)) &= \rho(ac+(ad+bc)x) \\
&= \begin{pmatrix} ac & ad+bc \\ 0 & ac \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \\
&= \rho(a+bx)\rho(c+dx),
\end{aligned}$$

which shows that ρ is a homomorphism of algebras.

However, $\rho(1+x)$ is not a scalar operator since in fact, similar to before, we obtain

$$\rho(1+x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and again

$$\rho(1+x)(0,1)^T = (1,1)^T \neq \alpha(0,1)^T$$

for any $\alpha \in \mathbb{Q}$.

Now, say we require k to be algebraically closed and ask whether one can still find a counterexample. Indeed we can. Take $A = \mathbb{C}[x]/(x^n)$ as a $\mathbb{C}$ algebra and let $\rho : \mathbb{C}[x]/(x^n) \to \mathbb{C}^n$ be defined by

$$\rho(x) = J(2,n).$$

First, I verify that ρ is a homomorhism of algebras. Namely, I note

$$\begin{aligned}
\rho((a+bi)(c+di)) &= \rho(ac-bd+(ad+bc)i) \\
&= \begin{pmatrix} (ac-bd) & (ad+bc) \\ -(ad+bc) & (ac-bd) \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix}
\end{aligned}$$

Once again $Z(\mathbb{C}) = \mathbb{C}$. Take $z = i$. Then,

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and note that

$$\rho(i)(1,0)^T = (0,-1)^T \neq \alpha(1,0)^T$$

for any $\alpha \in \mathbb{C}$.
We must also verify however that $\rho, \mathbb{C}^2$ is an indecomposable representation.

It suffices to show that it is irreducible, which it is. Otherwise, there would exist some non-trivial proper subrepresentation $W \subseteq V$, which must be of dimension $dim(W) = 1$ since $dim(V) = 2$. So, it must be of the form $W = span(w)$ such that

$$\rho(a+bi)w \in W$$

for all $a, b \in \mathbb{R}$, meaning that

$$\rho(a+bi)w = (\alpha + \beta i)w$$

for some $\alpha, \beta \in \mathbb{R}$. Then, writing $w = (w_1, w_2)^T$ for some $w_1, w_2 \in \mathbb{C}$ gives

$$\begin{aligned}
\rho(a+bi)(w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\
\begin{pmatrix} a & b \\ -b & a \end{pmatrix}(w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\
\begin{pmatrix} aw_1 + bw_2 \\ -bw_1 + aw_2 \end{pmatrix} &= \begin{pmatrix} (\alpha + \beta i)w_1 \\ (\alpha + \beta i)w_2)^T \end{pmatrix}
\end{aligned}$$

which means that

$$aw_1 + bw_2 = \alpha w_1 + \beta i w_2$$

and

$$-bw_1 + aw_2 = \alpha w_2 + \beta i w_2.$$

The above statement must hold for all $a, b \in \mathbb{R}$ so take $(a, b) = (0, 1)$ meaning $a + bi = i$).
If $(a, b) = (a, 0)$ then

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and indeed the eigenvalues of $\rho(i)$ are the roots of $x^2 + 1$ which are $\lambda_1 = i$ and $\lambda_2 = -i$ with corresponding eigenvectors $v_1 = (-i, 1)^T$ and $v_2 = (i, 1)^T$, which we note form a basis for $\mathbb{C}^2$. If ρ is reducible, that would require some non-trivial stable subspace $U \subsetneq V$, which would need to be one-dimensional. Then, the requirement that $dim(U) = 1$ implies that $U = Span(u)$ for some $u \in V$.

Note that $u \in \{v_1, v_2\}$. Otherwise, if $u = c_1u_1 + c_2u_2 = ((-c_1+c_2)i, c_1+c_2)^T$ for some $c_1, c_2 \in \mathbb{C}$ with $c_1, c_2 \neq 0$, then

$$\begin{aligned}
\rho(i)(u) &= \rho(i)(((-c_1 + c_2)i, c_1 + c_2)^T) \\
&= c_1(\rho(i)(u_1)) + c_2(\rho(i)(u_2)) \\
&= c_1\lambda_1 u_1 + c_2\lambda_2 v_2 \\
&= c_1(i(-i, 1)^T) + c_2(-i(i, 1)^T) \\
&= c_1(1, i)^T + c_2(1, -i)^T \\
&= (c_1 + c_2, (c_1 - c_2)i)^T.
\end{aligned}$$

If $span(U)$ is stable under the action of A, then that would imply that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} \in Span\left(\begin{pmatrix} (-c_1 + c_2)i \\ c_1 + c_2 \end{pmatrix}\right)$$

meaning that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} = \begin{pmatrix} d(-c_1 + c_2)i \\ d(c_1 + c_2) \end{pmatrix}$$

for some $d \in \mathbb{C}$.
Then, that implies that

$$c_1 + c_2 = di(-c_1 + c_2) \tag{1}$$
$$c_1 - c_2 = -di(c_1 + c_2). \tag{2}$$

Then, adding the above equations gives

$$2c_1 = -2c_1 di$$

implying that $c_1 = 0$, which cannot happen by assumption, or that $1 = -di$ meaning that $d = i$.
Now, substituting back into our original pair of equations 1 and 2 gives

$$\begin{aligned} c_1 + c_2 &= c_1 - c_2 \\ c_1 - c_2 &= c_1 + c_2 \end{aligned}$$

which then implies that $c_2 = -c_2$ or that $c_2 = 0$, which contradicts our assumption that $u = c_1 v_1 + c_2 v_2$ with $c_1, c_2 \neq 0$.

Thus, $U = Span(v_1)$ or $U = Span(v_2)$. However, neither subspace is stable under the action of A. Namely, take $a + bi = 2 + i$. Then,

$$\rho(2+i) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

so that

$$\begin{aligned} \rho(2+i)v_1 &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} v_1 \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} (-i, 1)^T \\ &= \begin{pmatrix} 1 - 2i \\ 2 - i \end{pmatrix} \end{aligned}$$

Now, if we were to have $\rho(2+i)w_1 \in W_1$ we would need $r + si \in \mathbb{C}$ such that

$$1 - 2i = -i(r + si)$$

and

$$2 - i = r + si$$

which means we would have $r = 2$ and $s = -1$ implying that

$$1 - 2i = -i(2 - i) = -2i + i^2 = -2i - 1,$$

which is a contradiction. So, indeed we see that this representation is in fact irreducible and consequently also indecomposable, which means this is a counterexample. So, even if we require the field k to be algebraically closed, $\rho(z)$ is not necessarily a scalar operator.