

“A First Course in Modular Forms” by Schur and Diamond
Chapter 1.1 Exercises

3. (a) Show that the set $\mathcal{M}_k(SL_2(\mathbb{Z}))$ of modular forms of weight k forms a vector space over \mathbb{C} .

Note that a function $f : \mathcal{H} \rightarrow \mathbb{C}$ is a modular form by definition if

- f is holomorphic on \mathcal{H}
- f is weakly modular of weight k , and
- f is holomorphic at ∞ .

In order to show that $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is a vector space we must show

- i. that addition is associative and commutative,
- ii. the existence of an additive identity denoted 0 and an additive inverse for each element,
- iii. that scalar multiplication is associative,
- iv. that scalar multiplication distributes over vector addition and vice versa,
- v. and that $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is closed under addition and scalar multiplication.

Indeed

- i. Addition is associative and commutative since for $f, g : \mathcal{H} \rightarrow \mathbb{C}$ one has that $f + g$ is defined by $(f + g)(z) = f(z) + g(z)$ and thus the claim follows from the fact that addition in \mathbb{C} is associative and commutative.
- ii. Also, one has that the zero map $0 \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ is a modular form of weight k due to quick verification of the three necessary properties. Also, it is the additive identity due to the definition of function addition as above.
- iii. Furthermore, scalar multiplication is associative once again due to definition of scalar multiplication of functions meaning that $a(bf)$ is defined by $(a(bf))(z) = a((bf)(z)) = ab(f(z)) = ((ab)(f))(z)$.
- iv. Also scalar multiplication distributes over vector addition since $(a(f + g))(z) = a((f + g)(z)) = af(z) + ag(z) = (af + ag)(z)$.
- v. Finally, $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is closed under addition and scalar multiplication.
 - Namely, if $f, g : \mathcal{H} \rightarrow \mathbb{C}$ are holomorphic on \mathcal{H} then so is $(f + g) : \mathcal{H} \rightarrow \mathbb{C}$.
 - Likewise I claim that if $f, g : \mathcal{H} \rightarrow \mathbb{C}$ are weakly modular of weight k then so is $(f + g) : \mathcal{H} \rightarrow \mathbb{C}$. In particular, we know by definition that

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ and } g(\gamma(\tau)) = (c\tau + d)^k g(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

Then, of course

$$(f + g)(\gamma(\tau)) = (c\tau + d)^k f(\tau) + (c\tau + d)^k g(\tau) = (c\tau + d)^k ((f + g)(\tau)) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

- Finally, I claim that the fact that f, g are holomorphic at ∞ implies that $f + g$ is also holomorphic at ∞ . Namely, f, g holomorphic at ∞ means by definition that there exist $z_1, z_2 \in \mathbb{C}$ such that the functions

$$\hat{f}(z) = \begin{cases} z_1 & \text{if } z = 0 \\ f\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ and } \hat{g}(z) = \begin{cases} z_2 & \text{if } z = 0 \\ g\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ are holomorphic on } D,$$

where $D = \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the unit disk.

Then, we see that

$$\widehat{f + g}(z) = \begin{cases} z_1 + z_2 & \text{if } z = 0 \\ f\left(\frac{\log(z)}{2\pi i}\right) + g\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ is holomorphic on } D,$$

since it is the sum of two holomorphic functions, and thus $f + g$ is holomorphic at infinity, concluding the proof of closure under addition.

- Likewise if $f : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic, then so is $af : \mathcal{H} \rightarrow \mathbb{C}$ for all $a \in \mathbb{C}$.
- Furthermore, if

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H},$$

then

$$(af)(\gamma(\tau)) = (c\tau + d)^k (af)(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } \tau \in \mathcal{H}.$$

since $(af)(\gamma(\tau)) = a(f(\gamma(\tau))) = a(c\tau + d)^k f(\tau) = (c\tau + d)^k (af)(\tau)$.

- Finally, if $f : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic at infinity, that means by definition that there exists $z' \in \mathbb{C}$ such that the function

$$\hat{f}(z) = \begin{cases} z' & \text{if } z = 0 \\ f\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases}$$

is holomorphic on D . Thus, the function

$$\widehat{(af)}(z) = \begin{cases} az' & \text{if } z = 0 \\ (af)\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} = a\hat{f}(z)$$

is holomorphic on D , proving closure under scalar multiplication.

- (b) If $f : \mathcal{H} \rightarrow \mathbb{C}$ is a modular form of weight k and $g : \mathcal{H} \rightarrow \mathbb{C}$ is a modular form of weight ℓ , show that $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is a modular form of weight $k + \ell$.

We need to prove that

- $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic,
- $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is weakly modular of weight $k + \ell$,
- and $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic at infinity.

Indeed,

- we see that $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic since the product of holomorphic functions is holomorphic.
- Likewise, $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is weakly modular of weight $k + \ell$ since

$$(fg)(\gamma(\tau)) = f(\gamma(\tau))g(\gamma(\tau)) = ((c\tau + d)^k f(\tau))((c\tau + d)^\ell g(\tau)) = (c\tau + d)^{k+\ell} f(\tau)g(\tau) = (c\tau + d)^{k+\ell} (fg)(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and all $\tau \in \mathcal{H}$.

- Finally note that $(fg) : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic at infinity since $f, g : \mathcal{H} \rightarrow \mathbb{C}$ holomorphic at infinity implies existence of $z_1, z_2 \in \mathbb{C}$ such that

$$\hat{f}(z) = \begin{cases} z_1 & \text{if } z = 0 \\ f\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ and } \hat{g}(z) = \begin{cases} z_2 & \text{if } z = 0 \\ g\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} \text{ are holomorphic on } D,$$

which implies that the function

$$\hat{fg}(z) = \begin{cases} z_1 z_2 & \text{if } z = 0 \\ f\left(\frac{\log(z)}{2\pi i}\right) g\left(\frac{\log(z)}{2\pi i}\right) & \text{if } z \in D \setminus \{0\} \end{cases} = (\hat{f}\hat{g})(z) \text{ is holomorphic on } D,$$

since it is product of holomorphic functions, and thus (fg) is holomorphic at infinity as claimed.

- (c) Show that $\mathcal{S}_k(SL_2(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$ and that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.

- Since $\mathcal{S}_k(SL_2(\mathbb{Z})) \subseteq \mathcal{M}_k(SL_2(\mathbb{Z}))$ one need only show that $0 \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ (where 0 denotes the zero function $0 : \mathcal{H} \rightarrow \mathbb{C}$), and closure under addition and scalar multiplication to show that $\mathcal{S}_k(SL_2(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$.
 - Indeed, $0 \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ since the $0 \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ and constant term in the Fourier expansion for 0 is clearly 0 .
 - Also, if $f, g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ then I claim that $(f + g) \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ since
 - * $(f + g) \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ by our earlier arguments
 - * and since the Fourier expansion of $(f + g)$ is the sum of those for each of f and g meaning that the constant term in the Fourier expansion of $(f + g)$ is zero.
 - Likewise, if $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ then $(af) \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ for all $a \in \mathbb{C}$ since
 - * $(af) \in \mathcal{M}_k(SL_2(\mathbb{Z}))$ by our earlier arguments
 - * and since the 0th coefficients of the Fourier expansion of (af) is ac_0 where c_0 is the leading coefficient of the Fourier series for f which is zero since $f \in \mathcal{S}_k(SL_1(\mathbb{Z}))$

- Finally, I want to show that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$, meaning we need to show that
 - $f + g \in \mathcal{S}(SL_2(\mathbb{Z}))$ for all $f, g \in \mathcal{S}(SL_2(\mathbb{Z}))$,
 - and $hf \in \mathcal{S}(SL_2(\mathbb{Z}))$ and $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ for all $f \in \mathcal{S}(SL_2(\mathbb{Z}))$ and all $h \in \mathcal{M}(SL_2(\mathbb{Z}))$.

Indeed,

- if one has

$$f = \bigoplus_{k \in \mathbb{Z}} f_k \in \mathcal{S}(SL_2(\mathbb{Z})) \quad \text{and} \quad g = \bigoplus_{k \in \mathbb{Z}} g_k \in \mathcal{S}(SL_2(\mathbb{Z}))$$

then we note that

$$f + g = \bigoplus_{k \in \mathbb{Z}} (f_k + g_k) \in \mathcal{S}(SL_2(\mathbb{Z}))$$

since for each $k \in \mathbb{Z}$ we have as shown earlier that $f_k + g_k \in \mathcal{S}_k(SL_2(\mathbb{Z}))$.

- Furthermore, given any function $h = \bigoplus_{k \in \mathbb{Z}} h_k \in \mathcal{M}(SL_2(\mathbb{Z}))$ and any $f = \bigoplus_{k \in \mathbb{Z}} f_k \in \mathcal{S}(SL_2(\mathbb{Z}))$ one has that $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ due to the following argument. Namely, we note that

$$fh = \bigoplus_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} f_{n-k} h_k \right)$$

and now what Remains to show is that

$$\left(\sum_{k \in \mathbb{Z}} f_{n-k} h_k \right) \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$

for all $n \in \mathbb{Z}$. Well, we know that

$$f_{n-k}(\tau) := \sum_{j=1}^{\infty} a_j q^j$$

and

$$h_k(\tau) = \sum_{r=0}^{\infty} b_r q^r.$$

So, we have that

$$(f_{n-k} h_k)(\tau) = \sum_{j=1}^{\infty} \sum_{r=0}^{\infty} a_j b_r q^{j+r} = \sum_{s=1}^{\infty} \left(\sum_{m=0}^{s-1} b_s a_{s-m} \right) q^s.$$

Thus,

$$f_{n-k} h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$

for all $n, k \in \mathbb{Z}$ which gives

$$\sum_{k \in \mathbb{Z}} f_{n-k} h_k \in \mathcal{S}_n(SL_2(\mathbb{Z}))$$

for all $n \in \mathbb{Z}$ concluding the proof that $fh \in \mathcal{S}(SL_2(\mathbb{Z}))$ as desired. Finally, since $fh = hf$ for all $f, h : \mathcal{H} \rightarrow \mathbb{C}$ we see that $hf \in \mathcal{S}(SL_2(\mathbb{Z}))$ as well, proving that $\mathcal{S}(SL_2(\mathbb{Z}))$ is an ideal in $\mathcal{M}(SL_2(\mathbb{Z}))$.

4. Let $k \geq 3$ be an integer and let $L' = \mathbb{Z}^2 - \{(0,0)\}$.

- Show that the series $S = \sum_{(c,d) \in L'} (\sup\{|c|, |d|\})^{-k}$ converges by considering the partial sums over expanding squares.

We decompose the sum as

$$\begin{aligned} S_n &= \sum_{(c,d) \in [-n,n]^2 \setminus \{(0,0)\}} (\sup\{|c|, |d|\})^{-k} \\ &= \sum_{c=0}^n \sum_{d=-n}^n |d|^{-k} + \sum_{c=-n}^n \sum_{d=0}^n |c|^{-k} + \sum_{\substack{(c,d) \in [-n,n]^2 \\ : c \neq 0 \neq d, |d| > |c|}} |d|^{-k} + \sum_{\substack{(c,d) \in [-n,n]^2 \\ : c \neq 0 \neq d, |c| > |d|}} |c|^{-k} + \sum_{\substack{(c,d) \in [-n,n]^2 \\ : c \neq 0 \neq d, |c| = |d|}} |c|^{-k}. \end{aligned}$$

Thus,

$$\begin{aligned}
S_n &= 2 \sum_{d=2}^n (d-1)(d^{-k}) + 8 \sum_{d=1}^n d^{-k} = 2 \sum_{d=2}^n (d-1)(d^{-k}) + 8 + 8 \sum_{d=2}^n d^{-k} \\
&= 2 \sum_{d=2}^n d(d^{-k}) - 2 \sum_{d=2}^n d^{-k} + 8 + 8 \sum_{d=2}^n d^{-k} \\
&= 2 \sum_{d=2}^n d^{-k+1} + 8 + 6 \sum_{d=2}^n d^{-k} \\
&= 8 + 2 \sum_{d=2}^n d^{-k+1} + 6 \sum_{d=2}^n d^{-k}.
\end{aligned}$$

We now want to show that as $n \rightarrow \infty$ we have that $\lim_{n \rightarrow \infty} S_n < \infty$. So, we note that

$$0 \leq S_n \leq 8 + 2 \sum_{d=2}^n d^{-k+1} + 6 \sum_{d=2}^n d^{-k}$$

Of course, that means

$$0 \leq S_n \leq 8 + 8 \sum_{d=2}^n d^{-k+1} \leq 8 + 8 \sum_{d=2}^n d^{-2}$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

we have that

$$0 \leq \lim_{n \rightarrow \infty} S_n \leq 8 + \frac{8\pi^2}{6} < \infty,$$

concluding the proof.

- (b) Fix positive numbers A, B and let $\Omega = \{\tau \in \mathcal{H} : |Re(\tau)| \leq A, Im(\tau) \geq B\}$. Prove that there is a constant $C > 0$ such that $|\tau + \delta| > C \sup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbb{R}$.

There are two cases: either $\sup\{1, |\delta|\} = 1$ or $\sup\{1, |\delta|\} = |\delta|$.

If $\sup\{1, |\delta|\} = 1$ then simply let $C = B$ and note that $|\tau + \delta| > Im(\tau + \delta) \geq B = B \sup\{1, |\delta|\}$.

If $\sup\{1, |\delta|\} = |\delta|$ then we have two natural subcases, either $|\delta| > 2A$ or $1 \leq |\delta| \leq 2A$. If $|\delta| > 2A$ then $A < \frac{|\delta|}{2}$ meaning that

$$\begin{aligned}
|\tau + \delta| &\geq |Re(\tau + \delta)| = |Re(\tau) + \delta| \geq ||Re(\tau)| - |\delta|| = \max(|Re(\tau)| - |\delta|, |\delta| - |Re(\tau)|) \geq \max(|Re(\tau)| - |\delta|, |\delta| - A) \\
&\geq |\delta| - A > |\delta| - \frac{|\delta|}{2} = \frac{|\delta|}{2}
\end{aligned}$$

and thus

$$|\tau + \delta| > \frac{1}{2} \sup\{1, |\delta|\}.$$

Finally if $\sup\{1, |\delta|\} = |\delta|$ and $1 \leq |\delta| \leq 2A$ then one has $Im(\tau) > A$ or $B \leq Im(\tau) \leq A$.

If $B \leq Im(\tau) \leq A$ then note that $-A + \delta \leq Re(\tau + \delta) \leq A + |\delta|$ and $B \leq Im(\tau + \delta) \leq A$ proving that $B \leq |\tau + \delta| \leq 2A + |\delta|$. Then, since $1 \leq |\delta| \leq 2A$ one has that $F(\tau, \delta) = \frac{|\tau + \delta|}{|\delta|}$ is a continuous function $F : (\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A]) \rightarrow \mathbb{C}$ where $\Omega' = \Omega \cap \{z \in \mathbb{C} : Im(z) \leq A\}$ and since $(\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A])$ is a compact set, that implies that $M := \min(\frac{|\tau + \delta|}{|\delta|})$ is attained by some $(\tau, \delta) \in (\Omega' \times [-2A, -1]) \cup (\Omega' \times [1, 2A])$ and thus

$$|\tau + \delta| \geq M|\delta| = M \sup\{1, |\delta|\}.$$

If $Im(\tau) > A$ then note that

$$|\tau + \delta| \geq |Im(\tau + \delta)| = Im(\tau) > A$$

and since $1 \leq |\delta| \leq 2A$ that implies that $A \geq \frac{|\delta|}{2}$ implying that

$$|\tau + \delta| > \frac{1}{2} |\delta| = \frac{1}{2} \sup\{1, |\delta|\}.$$

So, we let

$$C = \min(B, \frac{1}{2}, M).$$

(c) Use parts (a) and (b) to prove that the series defining G_k converges absolutely and uniformly for $\tau \in \Omega$. Conclude that G_k is holomorphic on \mathcal{H} . Let $\mathbb{Z}^{2'} = \mathbb{Z}^2 \setminus \{(0, 0)\}$.

- We first note that $|c\tau + d| = |c||\tau + \frac{d}{c}|$.
- Then, by part (b), $|c\tau + d| \geq |c|C \sup\{1, |\frac{d}{c}|\} = C \sup\{|c|, |d|\}$ for all $\tau \in \Omega$.
- Then,

$$\sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{|c\tau + d|^k} \leq \sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{(C \sup\{|c|, |d|\})^k} = \frac{1}{C^k} \sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{\sup\{|c|, |d|\}^k},$$

which means by part (a) that

$$\sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{|c\tau + d|^k} < \infty \text{ for all } \tau \in \Omega,$$

proving that the series representing G_k converges absolutely for $z \in \Omega$.

Now, to show that the series representing G_k converges uniformly we note that the fact that the series representing G_k converges absolutely implies that the series converges unconditionally meaning that any reordering of the series converges. We have

$$G_k(\tau) = \sum_{(c,d) \in (\mathbb{Z}^2)'} \frac{1}{(c\tau + d)^k} = \sum_{n=1}^{\infty} \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{(c\tau + d)^k} \right).$$

We denote

$$T_N(\tau) = \sum_{n=1}^N \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{(c\tau + d)^k} \right).$$

I claim that for fixed $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|G_k(\tau) - T_M(\tau)|$ for all $M \geq N$. Namely, note that for arbitrary $m \in \mathbb{N}$ and arbitrary $\tau \in \Omega$ one has that

$$\begin{aligned} |G_k(\tau) - T_m(\tau)| &= \left| \sum_{n=m+1}^{\infty} \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{(c\tau + d)^k} \right) \right| \leq \sum_{n=m+1}^{\infty} \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{|c\tau + d|^k} \right) \\ &\leq \sum_{n=m+1}^{\infty} \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{(C \sup\{|c|, |d|\})^k} \right) \\ &= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{\substack{(c,d) \in (\mathbb{Z}^2)': \\ |c|+|d|=n}} \frac{1}{\sup\{|c|, |d|\}^k} \right) \\ &= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{\sup\{|-n|, |0|\}^k} + \frac{1}{\sup\{|n|, |0|\}^k} \right. \\ &\quad \left. + \sum_{c \in [-n+1, n-1]} \left(\frac{1}{\sup\{|c|, n-|c|\}^k} + \frac{1}{\sup\{|c|, n-|c|\}^k} \right) \right) \\ &= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{2}{\sup\{|n|, |0|\}^k} + \sum_{c \in [-n+1, n-1]} \frac{2}{\sup\{|c|, n-|c|\}^k} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{2}{\sup\{|n|, |0|\}^k} + \frac{2}{\sup\{0, n\}^k} \right. \\
&\quad \left. + \sum_{c \in [1, n-1]} \frac{4}{\sup\{|c|, n - |c|\}^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{\sup\{|n|, |0|\}^k} + \sum_{c \in [1, n-1]} \frac{4}{\sup\{|c|, n - |c|\}^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{\sup\{|n|, |0|\}^k} + \sum_{c \in [1, \lfloor \frac{n}{2} \rfloor]} \frac{4}{(n - |c|)^k} \right. \\
&\quad \left. + \sum_{c \in [\lfloor \frac{n}{2} \rfloor + 1, n-1]} \frac{4}{|c|^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{n^k} \right) + \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c \in [1, \lfloor \frac{n}{2} \rfloor]} \frac{4}{(n - |c|)^k} \right. \\
&\quad \left. + \sum_{c \in [\lfloor \frac{n}{2} \rfloor + 1, n-1]} \frac{4}{|c|^k} \right) \\
&= \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{4}{n^k} \right) + \frac{1}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c \in [n - \lfloor \frac{n}{2} \rfloor, n-1]} \frac{4}{c^k} \right. \\
&\quad \left. + \sum_{c \in [\lfloor \frac{n}{2} \rfloor + 1, n-1]} \frac{4}{c^k} \right) \\
&= \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \left(\sum_{c \in [n - \lfloor \frac{n}{2} \rfloor, n-1]} \frac{1}{c^k} \right) \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^k} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{\lfloor \frac{n}{2} \rfloor + 1}{(\lfloor \frac{n}{2} \rfloor + 1)^k} = \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^{k-1}} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^k} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{(\lfloor \frac{n}{2} \rfloor + 1)^2} \leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \left(\frac{1}{n^2} \right) + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} \\
&\leq \frac{4}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} + \frac{8}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} = \frac{12}{C^k} \sum_{n=m+1}^{\infty} \frac{1}{\lfloor \frac{n}{2} \rfloor^2} \leq \frac{12}{C^k} \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{2}{l^2} = \frac{24}{C^k} \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l^2}.
\end{aligned}$$

Now, note that

$$\frac{1}{l^2} \leq \frac{1}{l(l-1)}$$

which implies that

$$\sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l^2} \leq \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l(l-1)} = \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \frac{1}{l(l-1)} = \sum_{l=\lfloor \frac{m+1}{2} \rfloor}^{\infty} \left(\frac{1}{l-1} - \frac{1}{l} \right) = \frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)}.$$

So, for fixed $\epsilon > 0$, let $N(\epsilon)$ be defined so that

$$\frac{24}{C^k} \frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)} < \epsilon$$

for all $m \geq N$ since that will imply that

$$|G_k(\tau) - T_m(\tau)| < \epsilon$$

for all $m \geq N$. In particular, let N be so that

$$\frac{1}{(\lfloor \frac{m+1}{2} \rfloor - 1)} < \frac{\epsilon C^k}{24}$$

for all $m \geq N$ or equivalently

$$(\lfloor \frac{m+1}{2} \rfloor - 1) > \frac{24}{\epsilon C^k}$$

meaning

$$\lfloor \frac{m+1}{2} \rfloor > \frac{24}{\epsilon C^k} + 1$$

for all $m \geq N$. So, let N be so that

$$\frac{m+1}{2} - 1 > \frac{24}{\epsilon C^k} + 1$$

for all $m \geq N$ or more precisely so that

$$m+1 > \frac{48}{\epsilon C^k} + 4$$

for all $m \geq N$ meaning we may let

$$N = \left\lfloor \frac{48}{\epsilon C^k} + 4 \right\rfloor,$$

proving uniform convergence.

- (d) Show that for $\gamma \in SL_2(\mathbb{Z})$, right multiplication by γ defines a bijection from L' to L' .

Consider

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Since $\det(\gamma) \neq 0$, that implies that $\gamma : \mathbb{C} \rightarrow \mathbb{C}$ is injective which implies that $\gamma|_{L'}$ is also injective. To show that $\gamma|_{L'} : L' \rightarrow L'$ is surjective we note that

$$\gamma^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus, arbitrary $(x, y)^T \in L'$ one can produce $(u, w)^T \in L'$ such that $\gamma((u, w)^T) = (x, y)^T$. Namely, let

$$(u, w)^T = \gamma^{-1}((x, y)^T)$$

and note that since $\gamma^{-1} \in SL_2(\mathbb{Z})$ we have that $(u, w) \in \mathbb{Z}^2$ and since $(x, y) \neq (0, 0)$ and γ^{-1} is injective we have that $(u, w) \neq (0, 0)$ meaning $(u, w)^T \in L'$ as desired.

- (e) Use the calculation from (c) to show that G_k is bounded on Ω .

Indeed, we see that as noted in part (c)

$$\sum_{(c,d) \in (\mathbb{Z}^2)',} \frac{1}{|c\tau + d|^k} \leq \frac{1}{C^k} \sum_{(c,d) \in (\mathbb{Z}^2)',} \frac{1}{\sup\{|c|, |d|\}^k},$$

and by part (a)

$$\sum_{(c,d) \in (\mathbb{Z}^2)',} \frac{1}{\sup\{|c|, |d|\}^k} \leq 8 + \frac{8\pi^2}{6},$$

and thus

$$|G_k(\tau)| = \left| \sum_{(c,d) \in (\mathbb{Z}^2)',} \frac{1}{(c\tau + d)^k} \right| \leq \frac{1}{C^k} (8 + \frac{8\pi^2}{6})$$

for all $\tau \in \Omega$, proving boundedness.

- (f) From the text and part (d), G_k is weakly modular so in particular $G_k(\tau+1) = G_k(\tau)$. Show that therefore $G_k(\tau)$ is bounded as $Im(\tau) \rightarrow \infty$.

Indeed, fix $A = 1$, $B = 1$ and consider the corresponding region $\Omega(A, B)$ and constant $C = C(A, B)$. Note that for all $\tau \in \Omega$ one has that $|G_k(\tau)| \leq \frac{1}{C^k} (8 + \frac{8\pi^2}{6})$. In particular, for arbitrary $\tau \in \mathbb{C}$ with $Im(\tau) \geq 2$ one has that

$$\tau = Re(\tau) + iIm(\tau) = \lfloor Re(\tau) \rfloor + (Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau).$$

Since $|(Re(\tau) - \lfloor Re(\tau) \rfloor)| < 1$ and $Im(\tau) \geq 2 \geq B$ one has that $(Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau) \in \Omega$. Also, $\lfloor Re(\tau) \rfloor \in \mathbb{Z}$ implies that $G_k(\lfloor Re(\tau) \rfloor + s) = G_k(s)$ for all $s \in \mathbb{C}$. Thus,

$$G_k(\tau) = G_k((Re(\tau) - \lfloor Re(\tau) \rfloor) + iIm(\tau))$$

and thus

$$|G_k(\tau)| \leq \frac{1}{C^k} \left(8 + \frac{8\pi^2}{6}\right)$$

for all $\tau \in \mathbb{C}$ with $Im(\tau) \geq 2$. Thus,

$$0 \leq \lim_{Im(\tau) \rightarrow \infty} |G_k(\tau)| \leq \lim_{Im(\tau) \rightarrow \infty} \frac{1}{C^k} \left(8 + \frac{8\pi^2}{6}\right) = \frac{1}{C^k} \left(8 + \frac{8\pi^2}{6}\right)$$

since C depends only on A, B and A, B do not depend on τ .