Wyzant Traffic Flow Tutoring

Caitlin Beecham

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Problem 60.2 (a)

Define

$$F(X,t) := \int_0^X \rho(x,t)dx.$$

$$\frac{d}{dt}\big(F(b(t),t) - F(a,t)\big)$$
$$= F_X(b(t),t)\frac{db}{dt} + F_t(b(t),t) - F_X(a,t)a'(t) - F_t(a,t)$$

$$F_X(b(t),t) := \frac{d}{dX}\int_0^X \rho(x,t)dx\bigg|_{X=b(t)} = \rho(X,t)\big|_{X=b(t)}$$

Problem 60.2 (b)

Only the second term (the integral) changes. Then

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\rho u$$

tells us that

$$\int_a^{b(t)} \frac{\partial \rho}{\partial t} dx = \int_a^{b(t)} -\frac{\partial}{\partial x}(\rho u) dx = ??? \tag{1}$$

Problem 60.2 (c)

If the observer (at $x(t)$) is moving with traffic then how are u and x related? Then what does that mean for the quantity $u - x$?

Problem 60.3 (a)

What answer do you have here? It doesn't have to be precise. In fact it should be intuitive enough that it makes sense to you pragmatically.

Problem 60.3 (b)

$$\frac{d}{dt}\Big(F(b(t),t) - F(a(t),t)\Big) \tag{2}$$

$$= -\rho(a(t),t)a'(t) + \rho(b(t),t)b'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\rho(x,t)dx \tag{3}$$

$$=??? \tag{4}$$

Problem 60.3 (b) continued

Now you should carry out the rest using any conceptual information we have about the quantities that show up in the above expression.

Problem 60.3 (c)

I can explain and discuss as needed.

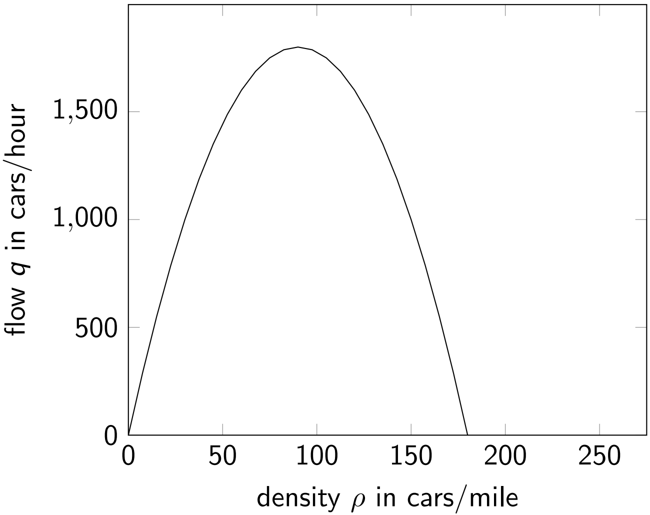
Problem 61.1

What did you get?

Problem 63

- How might you solve this?
- What answers did you get? Can you check whether they make sense?

Problem 63.4

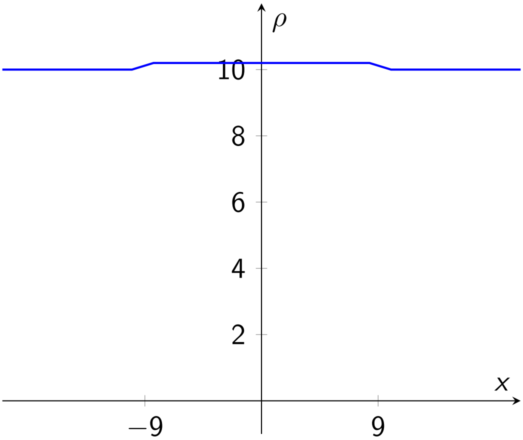


flow q in cars/hour
1,500
1,000
500
0
0
50
100
150
200
250
density ρ in cars/mile

Problem 67.2

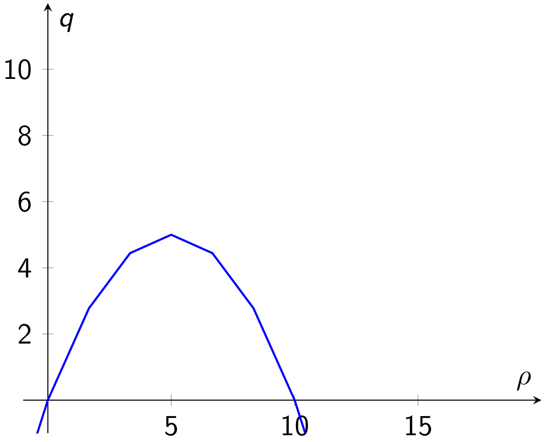
Here $a = 9$, $\rho_0 = 10$, $\epsilon = 0.2$ and

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \textit{otherwise} \end{cases}.$$



Problem 67.2 cont.

When plotting you have two cases: $c > 0$ and $c < 0$. What densities correspond to each case? What is c pictorially on the graph?



Problem 67.2 cont.

Then, for each case we know that

$$f(x-2c) = \begin{cases} 1 & \text{if } |x-2c| < a \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } -a+2c < x < a+2c \\ 0 & \text{otherwise} \end{cases}$$

From here you might be able to finish the problem and if not we can discuss further.

Problem 67.3

Solve using similar method as in Exercise 67.2.

Process to Follow for Chapter 73 Exercises

I recommend

- actually solving for $\rho(x, t)$,
- then sketching $\rho(x, 0), \rho(x, 2), \rho(x, 5)$ for various t values (which might be $0, 2, 5$ but you can pick).

So, basically solve to get $\rho(x, t)$ and then sketch, rather than guessing what the sketch would look like. So that covers the first part of the exercises in 73.

Then for the second part, (when the assignment has us come back to 1,5,6) having $\rho(x, t)$ as a function of x and t will allow us to find the position of a person moving in a car (that say starts at x_0) by solving an ODE.

How to find $\rho(x, t)$ in terms of x and t

So, here are the steps:

- Firstly we use the equation $u(\rho)$ we are given to write q solely in terms of ρ. Namely, we always have

$$q = \rho u$$

  which means that now $q = \rho u(\rho)$.

- Then, we can take the derivative of the above equation to get $\frac{dq}{d\rho}$.
- At this point we have the characteristic equations

$$x = \frac{dq}{d\rho} t + x_0. \tag{5}$$

  - Here we will either be given x_0 explicitly or we can find it somehow. It is the x intercept of the characteristic line that runs through (x, t)... As we will see, we will do this by looking that picture and filling in "fan-like" regions. You will get a feel for this. There isn't one cut and dry set of instructions to follow here.

- Then, we can solve for ρ in terms of x, t, which is what we were after.

Now you can very precisely sketch $\rho(x, T)$ at any specified time value T!

How to find $x(t)$ such that $x(t)$ is the path of a person moving in a car with traffic

The path of a person in a car (meaning one who is moving with traffic rather than standing on the side of the road observing or walking some weird path up and down the side of the road) is one so that

$$\frac{dx}{dt} = u.$$

How to find the path, $x(t)$, of a person moving in a car with traffic

- Now that we have that, the equation that says the observer is in a car moving with traffic is ...

$$\frac{dx(t)}{dt} = u(\rho(x,t)).$$

$$\frac{dx(t)}{dt} = u(\rho(x,t)).$$

- This is a differential equation which we solve to get $x(t)$. There will be some constant C involved. We then use an initial condition to find out what the C is and we finally have an equation $x(t)$ for the position of any time of a person in the car that started at position $-x_0$ (So, note that we will have a different equation for each x_0!)

Let's work through an example: red traffic light turns green

$$u(\rho) = u_{max}(1 - \frac{\rho}{\rho_{max}})$$

(That is what changes from scenario to scenario is $u(\rho)$. The rest of the process is the same). We will go through this together.

Exercise 73.1

We have that

$$q = \rho u = \rho u_{max}(1 - \frac{\rho}{\rho_{max}}) = u_{max}\rho - \frac{u_{max}}{\rho_{max}}\rho^2 \tag{6}$$

which gives

$$q'(\rho) = u_{max} - \frac{2u_{max}}{\rho_{max}}\rho. \tag{7}$$

That means that the equations of our characteristic lines are

$$x = (u_{max} - \frac{2u_{max}}{\rho_{max}}\rho)t + x_0 \tag{8}$$

which gives (for $t \neq 0$)

$$\rho(x,t) = \frac{\rho_{max}}{2u_{max}}(\frac{x_0 - x}{t} + u_{max})$$

We don't actually solve for x_0, but rather use the picture as the last line on page 337 suggests. For purposes of graphing I will assume that $u_{max} = 45$ and $\rho_{max} = 2$. As always, the slopes of the characteristic lines (**plotted where x is the horizontal axis and t is the vertical**) are

$$\frac{1}{q'(\rho)}.$$

So, as per equation (8) we have characteristic lines with slopes

$$\frac{1}{q'(\rho)} = \frac{1}{u_{max}}\Big(\frac{1}{1 - \frac{2\rho}{\rho_{max}}}\Big).$$

We note that our initial densities take one of 3 values as given in the problem statement. So, we have that

$$\frac{1}{q'(\rho)}\bigg|_{t=0} = \begin{cases} \frac{-1}{u_{max}} & \text{if } x < 0 \\ \infty & \text{if } 0 < x < a \\ \frac{1}{u_{max}} & \text{if } a < x \end{cases}$$

Exercise 73.1 cont.

NOTE: As in Figure 73.3, we have gap in our plot of characteristic lines which we fill in using a fan of lines just as on page 333 and indeed for the fan at $x = 0$ we set $q'(\rho) = \frac{x}{t}$ as on page 333. However, for the fan at $x = a$, we need to be a little more careful since the x intercept x_0 is now a not 1 meaning that the characteristic lines which fill in that fan are given by

$$x = q'(\rho)t + a. \tag{9}$$

Now, where $q'(\rho)$ is given by Equation (7).

Exercise 73.1 cont.

So, by actually writing that out we see that in the fan-like region at $x = a$ we have

$$x = (u_{max} - \frac{2u_{max}}{\rho_{max}}\rho)t + a \tag{10}$$

and thus in this region (Can you write out what equations in x and t define this region? Hint: see the line of text under Figure 73.3) solving for ρ gives

$$\rho = \left(\frac{\rho_{max}}{2u_{max}}\right)\left(u_{max} - \frac{x-a}{t}\right). \tag{11}$$

Similarly, in the fan-like region at $x = 0$ we can solve for ρ. So, eventually we have characteristic lines filling our x axis t axis plot, each of which having some density (it is possible that different characteristic lines have the same density). You can now just pick a few t values, draw horizontal lines across our plot of the characteristic lines and not the densities as each x value on that horizontal line and then plot those. That's it! (Not for each t value you will have a different x axis ρ axis plot. Please double check with your professor to see whether this was what was intended in the problem statement).

Exercises 73.5 and 73.6

Exercises 73.5 and 73.6 can be completed in the same manner as above. All that the statement in 73.6 is telling you is that you have initial density $\rho = \rho_{max}$ for $x < 0$ and $\rho = 0$ for $x \geq 0$.

Exercise 73.2

- At what time does maximum acceleration occur? (Just think about the physical scenario of you being in a car stopped at a traffic light. When do you accelerate the most?)
- How do we find acceleration given $\frac{dx}{dt}$? (Recall that upon seeing $\frac{dx}{dt}$ we are NOT to assume that the path $x(t)$ is moving with traffic because that is a special case when $\frac{dx}{dt} = u(x, t)$).
- Once we have that acceleration equation (in terms of time) and the time (in our first bullet point) at which the max acceleration occurs, just plug that time into the equation for acceleration and that tells you what the max acceleration of the car at $\frac{-1}{\rho_{max}}$ is.

Exercise 73.3

- At what time does the car start moving?
- Do we have an equation for velocity?
- If so, just plug that time into that equation for velocity.

Exercise 73.4

- I claim that something very strange happens if we consider the front of the car and end of the car separately. I claim the length between them is not L at all times. You should try this out! Try separately finding their position functions.
- If that is indeed the case then we should just find the position of the end of the car as L less than the position function of the front of the car. Also, read Section 74 to get a sense of what "for sufficiently large time" might mean.

Exercises 74.1 and 74.2

Solve them using the same method described in the slides regarding Section 73.

For Part 2 of 73.1,73.5,73.6

Just follow the instructions given in the slides titled "How to find the path, $x(t)$, of a person moving in a car with traffic". Basically, I am just telling you to use the hint given in the problem statement. We can go through any part of these slides in more detail in person which is what I intend.