From Definitions to Theorems via Proofs

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Overview

Motivation:	why am I telling you this stuff?
Logic:	various logics and their peculiarities
Definitions:	what makes a good definition
Assertions:	Propositions, Lemmata and Theorems
Proofs:	various methods for proving assertions
Coq:	an interactive proof assistant
Further Work:	projects

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Writing: is all about communicating your ideas

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This talk: how to do proofs involving logic

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Paraconsistent: some assertions may be true as well as false

Linear: every assumption must be used exactly once

Classical Logic: two fundamental principles

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i.e. at least one truth value

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Bivalent: every assertion is either true or else false

i.e. exactly one truth value

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Question: what about other ways to form natural numbers ?

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Question: what about other ways to form natural numbers ?
There are none! i.e. there are exactly two ways that n can be a natural number: either n is 0 or-else n is $S\ x$ for some already existing natural number x

Defining Odd and Even Natural Numbers

Notation: Assume that we have defined the usual arithmetical operations on natural numbers: e.g. equality ($=$), addition ($+$), subtraction ($-$), and multiplication ($*$) etc.

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Defn 5 (Odd): a natural number n is odd iff there exists a natural number k such that $n = 2 * k + 1$. e.g. $1, 3, 5, \cdots$

Notation: "iff" stands for "if and only if"

Exercise: can you think of an inductive definition for each?

Methods of Proof: direct, two versions

Claim: if the natural number n is even then $n + 1$ is odd

Proof: Assume n is even. Hence there is some natural number m such that $n = 2 * m$. Hence there is some natural number k such that $n + 1 = 2 * k + 1$. QED

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Proof: Assume n is even. By Definition 4 (LtoR), there is some natural number m such that $n = 2 * m$. Adding 1 to each side to obtain $n + 1 = 2 * m + 1$ tells us that there is some natural number $k = m$ such that $n + 1 = 2 * k + 1$. Definition 5 (RtoL) means $n + 1$ is odd. QED

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Question: what about the case where n is not even?

Classical logic: "if A then B" is "not-A or B"

Methods of Proof: by contradiction

Claim: if the natural number n is even then $n+1$ is odd

Proof: For a contradiction, assume n is even but that $n+1$ is not odd. Since n is even, Definition 4 tells us that there exists some natural number m such that $n = 2 * m$. Since $n+1$ is not odd, Definition 5 tells us that there is no natural number k such that $n+1 = 2 * k + 1$. Subtracting one from each side: there is no natural number k such that $n = 2 * k$. That is, for every natural number k, we must have $n \neq 2 * k$. Putting $k = m$ gives us that $n \neq 2 * m$. Contradiction, hence $n+1$ must be odd. QED

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Question: why can we ignore the case where $n + 1$ is odd?

Methods of Proof: by contraposition

Claim: if the natural number n is even then $n+1$ is odd

Lemma: For all natural numbers n, if the natural number $n+1$ is not odd then n is not even

Proof (direct): Suppose $n+1$ is not odd. Definition 5 tells us that there is no natural number k such that $n+1 = 2*k+1$. Subtracting one from each side gives that there is no natural number k such that $n = 2*k$. Renaming variables: there is no natural number m such that $n = 2*m$. Definition 4 then means that n is not even. QED

Corollary: By contraposition, for all natural numbers n, if the natural number n is even then $n+1$ is odd. QED

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Question: Why have I added the "For all natural numbers n"?

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Induction Hypothesis: Suppose the claim holds for all natural numbers up to some natural number $j > 0$. That is, for all natural numbers $0 \leq n \leq j$: if n is even then $n+1$ is odd.

Methods of Proof: by induction

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Induction Step: Consider the natural number $j+1$. Suppose $j+1$ is even. By definition of even, there is some natural number m such that $j+1 = 2*m$. Adding one to both sides gives that there is some natural number $k = m$ such that $j+1+1 = 2*m+1$. By definition of odd, $j+1+1$ is odd.

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Bogus Proofs!

Look again: The last proof is bogus!

IH: is never used!

Why?: it is just the direct proof in disguise!

Induction: not possible since the definitions of even and odd are not mutually inductive!

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Answer: use computer-based proof-assistants which are trusted

Coq: a proof assistant based upon intuitionistic logic

Syntax of Classical Logic:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \forall \vec{x}.P(\vec{x}) \mid \exists \vec{x}.P(\vec{x})$$

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Consequence: the following is not valid: $\neg\neg\varphi \to \varphi$

Connectives: all independent but $\neg\varphi := \varphi \to \bot$

ie $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$ not true!

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Coq: implements rules of a natural deduction proof calculus

User: interactively guides Coq to a proof from a collection of assumptions ie. assuming $\alpha_1 \cdots \alpha_n$ prove goal γ

Judgements and Proof States

Judgements: are terms of the form $t_1 : t_2$ where t_1 and t_2 are terms e.g. $n : nat$ $x : Set$ $t : Type$ $p : Prop$

Proof State: is always of the form below where the assumptions α_i are above the line and the goal(s) γ is (are) below the line:

```
1 subgoals, subgoal 1 (ID 9)
    alpha_1
    ...
    alpha_n
  ============================
    gamma
```

Interactive Proof: Reduce `gamma` to empty by invoking rules of Coq's Natural Deduction Calculus for manipulating judgements

Libraries and Definitions

Libaries: written by others can be imported into Coq as follows:

```
Require Import Arith.
Require Import Coq.omega.Omega.
```

Definitions: can be encode into Coq as follows:

```
Definition even (n: nat) := exists k, n = 2 * k.
Definition odd  (n: nat) := exists k, n = 2 * k + 1.
```

Lemma: can be encode into Coq as follows:

```
Lemma evenodd2 : forall n:nat, even n -> odd (n+1).
```

Lemma: can be encode into Coq as follows:

```
Lemma evenodd (n: nat) : even n -> odd (n+1).
```

Computation: for every `n`, `evenodd n` is a function that maps a proof of `even n` to a proof of `odd (n+1)`

Some proof rules of Coq: intro for forall

```
Lemma evenodd1 : forall (n: nat), even n -> odd (n+1).
Proof.

 1 subgoals, subgoal 1 (ID 9)

  ============================
   forall n : nat, even n -> odd (n + 1)
```

```
Lemma evenodd1 : forall (n: nat), even n -> odd (n+1).
Proof. intro n.

  1 subgoals, subgoal 1 (ID 10)

   n : nat
  ============================
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Some proof rules of Coq: intro for implication

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   Heven: even n
  ============================
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```

Some proof rules of Coq: applying a lemma

Suppose we have proved `Lemma ALem: forall x, A x → B x.`

```
Lemma BLem: B 5
Proof.
1 subgoals, subgoal 1 (ID 10)

 ============================
  B 5
```

```
Lemma BLem: B 5
Proof. apply ALem
1 subgoals, subgoal 1 (ID 11)

 ============================
  A 5
```

Finds an assignment `x := 5` which makes `B x` into `B 5` then reduces `B 5` to `A 5` using the same assignment to `A x`

Outline of our Proof:

Claim: if the natural number n is even then $n + 1$ is odd

Lemma 1: For all natural numbers n, even(n) $\vee$ odd(n).

Lemma 2: For all natural numbers n, odd(n) $\vee$ $\neg$ odd(n).

Lemma 3: For all natural numbers n, $\neg\neg$ odd(n) $\rightarrow$ odd(n).

Lemma 4: For all natural numbers n, even(n) $\rightarrow$ $\neg\neg$ odd(n+1)

Theorem: For all natural numbers n, even(n) $\rightarrow$ odd(n+1).

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Induction Hypothesis: Suppose that $even(n) \vee odd(n)$ holds.

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- If $odd(n)$ then there exists a natural number k such that $n = 2 * k + 1$. Adding 1 to both sides gives $n + 1 = 2 * k + 2 = 2 * (k + 1)$. Thus there exists a natural number $k + 1$ such that $n + 1 = 2 * (k + 1)$. By definition, we must have $even(n + 1)$.

Proof of Lemma 2

Lemma 2: For all natural numbers n, $\mathrm{odd}(n) \vee \neg\, \mathrm{odd}(n)$.

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Lemma 2: For all natural numbers n, odd(n) $\vee \neg$ odd(n).

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Lemma 2: For all natural numbers n, odd(n) $\vee \neg$ odd(n).

Proof: By Lemma 1, we know that for all natural numbers $even(n) \vee odd(n)$ holds, so proceed by cases:

- If $even(n)$ then there exists a natural number k such that $n = 2 * k$. For a contradiction, suppose $odd(n)$ also held. Thus there exists a natural number l such that $n = 2 * l + 1$. Thus there exist natural numbers n, k and l such that $2 * k = 2 * l + 1$: a contradiction, hence $\neg odd(n)$.

Proof of Lemma 2

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- If $odd(n)$ then we are done vacuously!

Proof of Lemma 3

Lemma 3: For all natural numbers n, ¬¬ odd(n) → odd(n).

Proof: Suppose $\neg\neg odd(n)$. By Lemma 2, for all natural numbers n, we know that $odd(n) \vee \neg odd(n)$, so proceed by cases:

Proof of Lemma 3

Lemma 3: For all natural numbers n, $\neg\neg$ odd(n) $\rightarrow$ odd(n).

Proof: Suppose $\neg\neg odd(n)$. By Lemma 2, for all natural numbers n, we know that $odd(n) \vee \neg odd(n)$, so proceed by cases:

- Suppose $\neg odd(n)$. This contradicts our assumption $\neg\neg odd(n)$, so is impossible.
- Suppose $odd(n)$. Then we are done vacuously.

Proof of Lemma 4

Lemma 4: For all natural numbers n, $even(n) \rightarrow \neg\neg odd(n+1)$.

Proof: Suppose $even(n)$. Thus there exists a natural number k such that $n = 2 * k$. To prove $\neg\neg odd(n+1)$, we have to prove that $odd(n+1) \rightarrow \perp \rightarrow \perp$. For a contradiction, suppose $\neg odd(n+1)$: ie $odd(n+1) \rightarrow \perp$. Then, we would obtain a contradiction if we could prove $odd(n+1)$. Adding one gives us a natural number $k+1$ such that $n+1 = 2 * k + 1$. That is, $odd(n+1)$ holds. Contradiction. Hence $\neg\neg odd(n+1)$ holds.

Proof of Lemma 5

Lemma 5: For all natural numbers n, $even(n) \to odd(n+1)$.

Proof: Suppose $even(n)$. By Lemma 4, we know that $\neg\neg odd(n+1)$ holds. Then by Lemma 3, we know that $odd(n+1)$ holds.