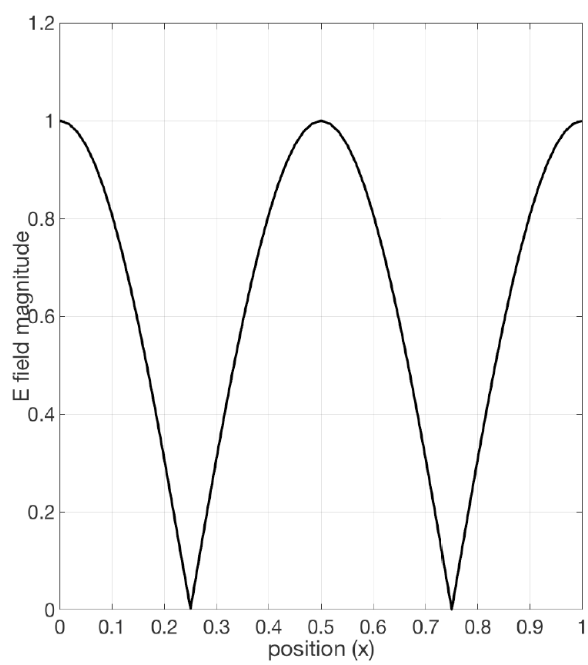
Problem Set #1 – Solution

1. The last row of the system should be modified so that

$$Z_{N,N-1} = \frac{-1}{\Delta}\int_{last\ cell}\left(\frac{dT_N}{dx}\frac{dB_{N-1}}{dx} - k_0^2\varepsilon_r T_N B_{N-1}\right)dx$$
$$= \frac{1}{\Delta^2} + \frac{k_0^2\varepsilon_{N-1}}{6}$$

$$Z_{N,N} = \frac{-1}{\Delta}\int_{last\ cell}\left(\frac{dT_N}{dx}\frac{dB_N}{dx} - k_0^2\varepsilon_r T_N B_N\right)dx$$
$$= \frac{-1}{\Delta^2} + \frac{k_0^2\varepsilon_{N-1}}{3}$$

The result produces a standing wave pattern, with nulls at 0.25 λ and 0.75 λ for a one wavelength domain:



Numerical results should converge to the above plot.

The zero Neumann boundary condition models a perfect magnetic conducting boundary, which produces a total reflection with a zero magnetic field at that location (and a maximum in the electric field at that location).

2. Applying extrapolation to the values in Table 5 produces:

For reflection coefficient:

4/8	0.41032767	60.588928
8/16	0.40395146	60.266784
16/32	0.40347251	60.245404
32/64	0.40336706	60.239908
exact	0.40338	60.245

For transmission coefficient

4/8	0.91449059	150.727737
8/16	0.91498399	150.286804
16/32	0.91511308	150.246938
32/64	0.91496756	150.240012
exact	0.91503	150.245

3. Suppose the two cells adjacent to node *m* have

$$\Delta_L = x_m - x_{m-1}$$

$$\Delta_R = x_{m+1} - x_m$$

Then for variable cells we obtain general expressions

$$\begin{aligned}
Z_{m,m-1} &= \int_{x_{m-1}}^{x_m} \frac{1}{\Delta_L}\left(\frac{-1}{\Delta_L}\right) dx - k_0^2 \int_{x_{m-1}}^{x_m} \varepsilon_r \left(\frac{x - x_{m-1}}{\Delta_L}\right)\left(\frac{x_m - x}{\Delta_L}\right) dx \\
&= -\frac{1}{\Delta_L} - \frac{k_0^2}{(\Delta_L)^2}\varepsilon_m^- \int_0^{\Delta_L} (x)(\Delta_L - x)\, dx \\
&= -\frac{1}{\Delta_L} - k_0^2 \varepsilon_m^- \frac{\Delta_L}{6}
\end{aligned}$$

$$\begin{aligned}
Z_{mm} &= \int_{x_{m-1}}^{x_m} \left(\frac{1}{\Delta_L}\right)^2 dx + \int_{x_m}^{x_{m+1}} \left(\frac{-1}{\Delta_R}\right)^2 dx \\
&\quad - \frac{k_0^2}{(\Delta_L)^2}\varepsilon_m^- \int_0^{\Delta_L} (x)^2\, dx - \frac{k_0^2}{(\Delta_R)^2}\varepsilon_m^+ \int_0^{\Delta_R} (\Delta_R - x)^2\, dx \\
&= \frac{1}{\Delta_L} + \frac{1}{\Delta_R} - k_0^2 \left(\varepsilon_m^- \frac{\Delta_L}{3} + \varepsilon_m^+ \frac{\Delta_R}{3}\right)
\end{aligned}$$

$$
\begin{aligned}
Z_{m,m+1} &= \int_{x_m}^{x_{m+1}} \left(\frac{-1}{\Delta_R}\right)\frac{1}{\Delta_R}\ dx - k_0^2 \varepsilon_m^+ \int_{x_m}^{x_{m+1}} \left(\frac{x_{m+1}-x}{\Delta_R}\right)\left(\frac{x-x_m}{\Delta_R}\right) dx \\
&= -\frac{1}{\Delta_R} - \frac{k_0^2}{(\Delta_R)^2}\varepsilon_m^+ \int_0^{\Delta_R} (\Delta_R - x)(x)\ dx \\
&= -\frac{1}{\Delta_R} - k_0^2 \varepsilon_m^+ \frac{\Delta_R}{6}
\end{aligned}
$$

The (1,1) entry of the system is different because of the absorbing boundary condition:

$$
\begin{aligned}
Z_{1,1} &= \left\{ \int_{first\ cell} \left(\frac{dT_1}{dx}\frac{dB_1}{dx} - k_0^2 \varepsilon_r T_1 B_1 \right) dx + jk_0 B_1\Big|_a \right\} \\
&= \frac{1}{\Delta_R} - k_0^2 \varepsilon_1 \frac{\Delta_R}{3} + jk_0
\end{aligned}
$$

where $\Delta_R = x_2 - x_1$. Similarly, the (*N*,*N*) entry is

$$
\begin{aligned}
Z_{N,N} &= \left\{ \int_{last\ cell} \left(\frac{dT_N}{dx}\frac{dB_N}{dx} - k_0^2 \varepsilon_r T_N B_N \right) dx + jk_0 B_N\Big|_b \right\} \\
&= \frac{1}{\Delta_L} - k_0^2 \varepsilon_N \frac{\Delta_L}{3} + jk_0
\end{aligned}
$$

where $\Delta_L = x_N - x_{N-1}$. For the RHS vector, the only nonzero entry is the first, given by

$$
\mathbf{RHS}_1 = 2\,jk_0 E_y^{inc}(a)
$$

Note that we cannot scale the equations through by a single factor Δ as we did in the uniform case (to make them look like the finite difference equations), since there is no longer a single factor Δ that arises from this arrangement.