Problem Set #6, Solution to Problem 2:

By combining the element matrix entries for 2 adjacent cells, the m-th FEM equation for the scalar Helmholtz equation (away from any boundaries) can be expressed

$$\left(\frac{1}{6\Delta}E_{m-2}-\frac{8}{6\Delta}E_{m-1}+\frac{14}{6\Delta}E_m-\frac{8}{6\Delta}E_{m+1}+\frac{1}{6\Delta}E_{m+2}\right)$$
$$-k^2\left(-\frac{\Delta}{15}E_{m-2}+\frac{2\Delta}{15}E_{m-1}+\frac{8\Delta}{15}E_m+\frac{2\Delta}{15}E_{m+1}-\frac{\Delta}{15}E_{m+2}\right)=0$$

From a single cell, the (m–1)st equation (mid-cell) is

$$\left(-\frac{8}{6\Delta}E_{m-2}+\frac{16}{6\Delta}E_{m-1}-\frac{8}{6\Delta}E_m\right)$$
$$-k^2\left(\frac{2\Delta}{15}E_{m-2}+\frac{16\Delta}{15}E_{m-1}+\frac{2\Delta}{15}E_m\right)=0$$

You may substitute the second equation into the first to eliminate E_{m-1} (and by symmetry E_{m+1}). Then, seek a solution to this equation in the form $E(x)=e^{-j\beta x}$, or equivalently

$$E_{m-2}=e^{+j\beta 2\Delta} \qquad E_m=1 \qquad E_{m+2}=e^{-j\beta 2\Delta}$$

and solve the system for β. You should get

$$\cos(2\beta\Delta)=\frac{15-26(k\Delta)^2+3(k\Delta)^4}{15+4(k\Delta)^2+(k\Delta)^4}$$

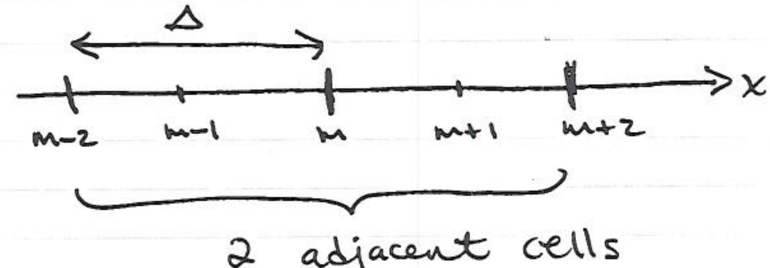
The following pages illustrate this derivation. Following those, there is a plot showing how $\cos(2\beta\Delta)$ compares with $\cos(2k\Delta)$. The agreement is good up to about $k\Delta=1.6$, or $\Delta\simeq 0.25\lambda$.

The result for β is approximately real-valued up to $k\Delta=\sqrt{15}\simeq 3.9$, but there is a region $\sqrt{10}/2\simeq 1.58<k\Delta<1.73=\sqrt{3}$ where $\cos(2\beta\Delta)$ is slightly less than –1 and β also becomes (slightly) complex. The final figure shows that the agreement between β and k from the expression derived above is only close in the range $k\Delta<1.6$, or $\Delta<0.25\lambda$.

Further analysis can be used to show that the error in β decreases as $O(\Delta^4)$.

P5.10 (a)

Consider 2 cells :



The basic eqn is :

$$\sum_i E_i \left\{ \int \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_i}{\partial x} - k^2 \int \phi_m \phi_i \right\} = 0$$

The m-th eqn (at cell junction in figure above) is :

$$\left(\frac{1}{6\Delta} E_{m-2} - \frac{8}{6\Delta} E_{m-1} + \frac{14}{6\Delta} E_m - \frac{8}{6\Delta} E_{m+1} + \frac{1}{6\Delta} E_{m+2} \right)$$

$$- k^2 \left(-\frac{\Delta}{15} E_{m-2} + \frac{2\Delta}{15} E_{m-1} + \frac{8\Delta}{15} E_m + \frac{2\Delta}{15} E_{m+1} - \frac{\Delta}{15} E_{m+2} \right) = 0$$

where the values are obtained directly from the matrices in P5.9.

The (m-1)th eqn (mid cell) is :

$$\left(-\frac{8}{6\Delta} E_{m-2} + \frac{16}{6\Delta} E_{m-1} - \frac{8}{6\Delta} E_m \right) - k^2 \left(\frac{2\Delta}{15} E_{m-2} + \frac{16\Delta}{15} E_{m-1} + \frac{2\Delta}{15} E_m \right) = 0$$

Multiplying the first by 30Δ produces :

$$\left[5 + 2(k\Delta)^2 \right] (E_{m-2} + E_{m+2}) + \left[-40 - 4(k\Delta)^2 \right] (E_{m-1} + E_{m+1})$$

$$+ \left[70 - 16(k\Delta)^2 \right] E_m = 0$$

— cont —

P5.10a, cont

and multiplying the second eqn by $\frac{-15\Delta}{2}$ produces

$$\left[10 + (k\Delta)^2\right](E_{m-2} + E_m) + \left[-20 + 8(k\Delta)^2\right] E_{m-1} = 0$$

Substituting the 2nd into the first to eliminate E_{m-1} (and E_{m+1}) leaves:

$$\left[(5+2(k\Delta)^2)(20-8(k\Delta)^2) - (40+4(k\Delta)^2)(10+(k\Delta)^2)\right](E_{m-2}+E_{m+2})$$

$$+\left[\{70-16(k\Delta)^2\}\{20-8(k\Delta)^2\} - 2\{40+4(k\Delta)^2\}\{10+(k\Delta)^2\}\right] E_m = 0$$

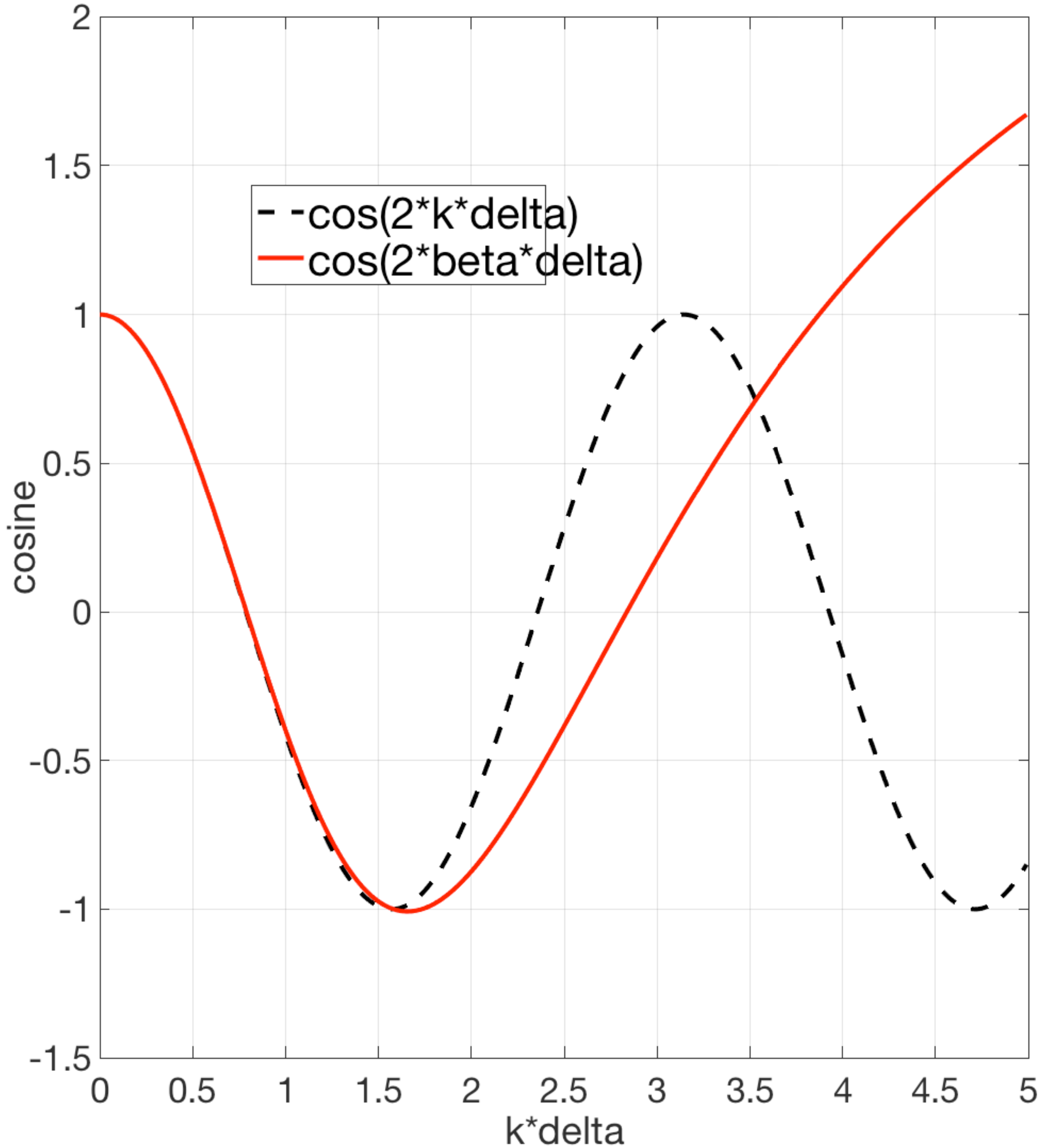
Now, assume $E(z) = e^{-j\beta z}$, so $E_{m-2} = e^{+j\beta 2\Delta}$,

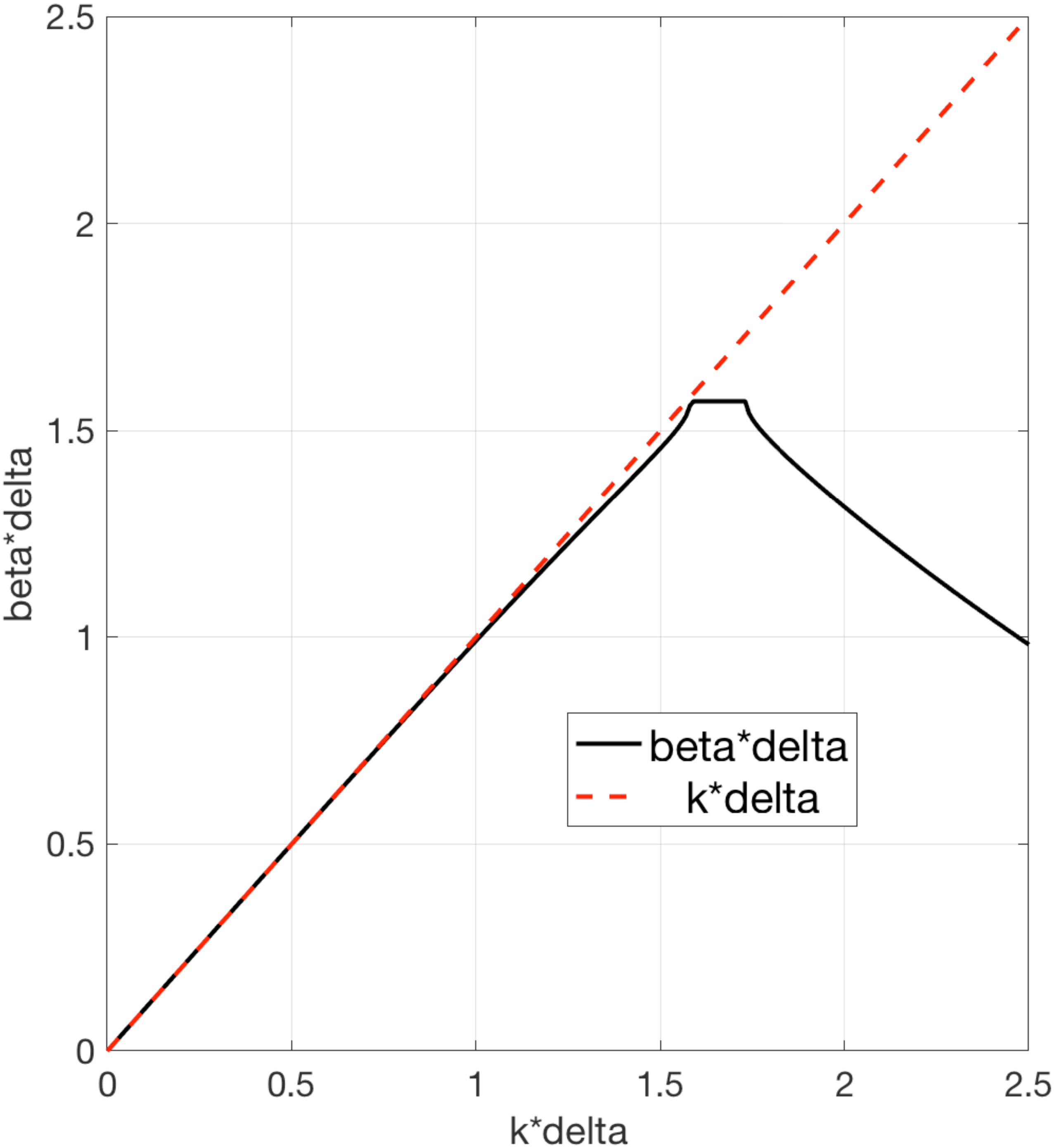
$E_m = 1$, $E_{m+2} = e^{-j\beta 2\Delta}$

Then, $E_{m-2} + E_{m+2} = 2\cos(2\beta\Delta)$ and

$$2\cos(2\beta\Delta) = \frac{2\{40+4(k\Delta)^2\}\{10+(k\Delta)^2\} - \{70-16(k\Delta)^2\}\{20-8(k\Delta)^2\}}{\{5+2(k\Delta)^2\}\{20-8(k\Delta)^2\} - \{40+4(k\Delta)^2\}\{10+(k\Delta)^2\}}$$

$$= \frac{15 - 26(k\Delta)^2 + 3(k\Delta)^4}{15 + 4(k\Delta)^2 + (k\Delta)^4}$$


cos(2*k*delta)
cos(2*beta*delta)
cosine
k*delta
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



2.5
2
1.5
1
0.5
0
beta*delta
0
0.5
1
1.5
2
2.5
k*delta
beta*delta
k*delta