Problem Set #5 : Solution

(1) Assume ϵ is constant for simplicity. Want to show that the FDTD update eqns preserve

$\oiint \bar{D} \cdot d\bar{s}$. In other words, want to show that

$$\frac{d}{dt} \oiint \bar{D} \cdot d\bar{s} = 0$$

Consider the stencil below:

$$\begin{array}{ccccc}
H_y^A & — & E_x^Q & — & H_y^B \\
| & & & & | \\
E_z^S & & & & E_z^T \\
| & & & & | \\
H_y^C & — & E_x^R & — & H_y^D \\
 & & \xleftrightarrow{\quad \Delta z \quad} & &
\end{array}
\quad \updownarrow \Delta x$$

$$\oiint \bar{D} \cdot d\bar{s} \simeq \epsilon E_x^Q \Delta z + \epsilon E_z^T \Delta x$$

$$- \epsilon E_z^S \Delta x - \epsilon E_x^R \Delta z$$

From the FDTD update eqns (10) and (11) in Note #8, we have

$$\frac{\Delta E_x^Q}{\Delta t} = \frac{-1}{\epsilon \Delta z} \left[H_y^B - H_y^A \right]$$

$$\frac{\Delta E_z^T}{\Delta t} = \frac{1}{\epsilon \Delta x}\left[H_y^B - H_y^D \right]$$

$$\frac{\Delta E_x^R}{\Delta t} = \frac{-1}{\epsilon \Delta z}\left[H_y^D - H_y^C \right]$$

$$\frac{\Delta E_z^S}{\Delta t} = \frac{1}{\epsilon \Delta x}\left[H_y^A - H_y^C \right]$$

Then

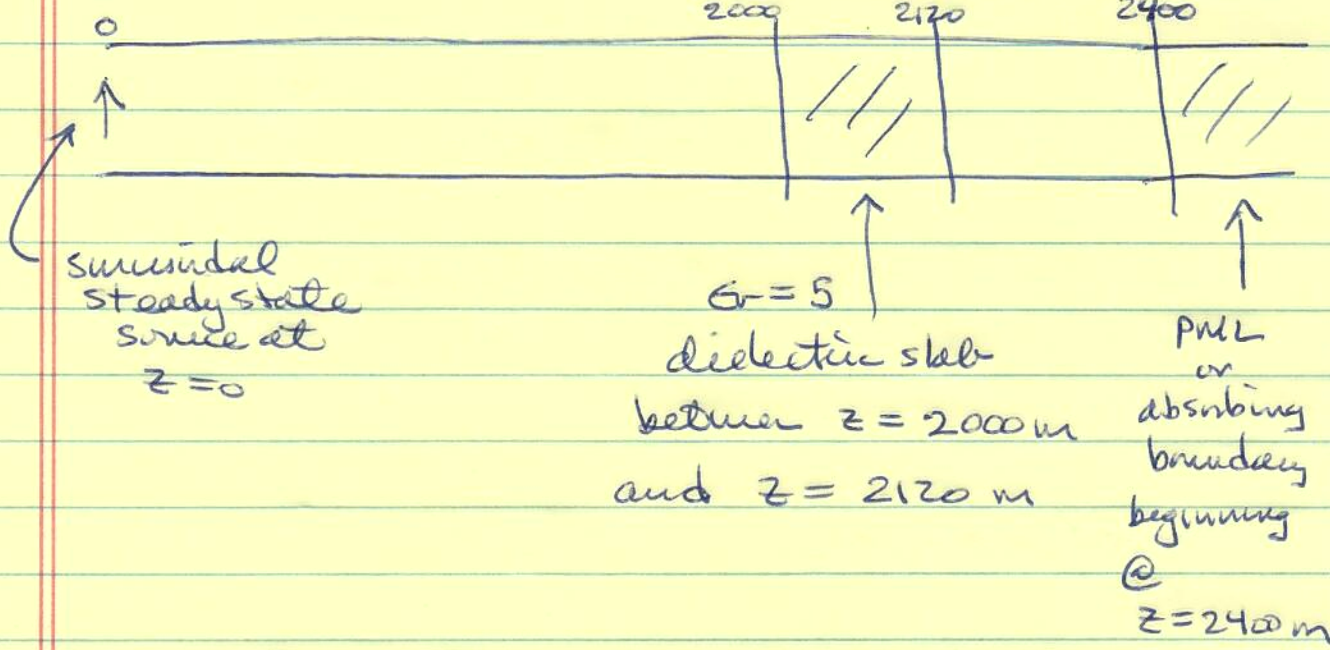
$$\frac{d}{dt}\left\{ \epsilon E_x^Q \Delta z + \epsilon E_z^T \Delta x - \epsilon E_z^S \Delta x - \epsilon E_x^R \Delta z \right\}$$

$$\cong \left(-H_y^B + H_y^A\right) + \left(H_y^B - H_y^D\right) + \left(-H_y^A + H_y^C\right)$$

$$+ \left(H_y^D - H_y^C\right) = 0$$

Thus, the FDTD update eqns preserve the divergence of the $\bar{D}$-field as time evolves.

(2) Possible Solution (there are many ways to do this)



→ choose $f = 1$ MHz, the slab thickness of 120 m is 0.4λ

→ Excite w/ sinusoidal steady state and wait until $t > 10^+$ μsec, then inspect the field in the region $2120 < z < 2400$ to determine the maximum values of the field ~ that gives $|\tau|$

→ To determine $|\Gamma|$, inspect the field $0 < z < 2000$ sometime around 13.33 μs (when the initial reflection has reached $z=0$ but not re-reflected)
~ capture E_x^{tot} at this time and subtract $E_x^{inc} = \sin(2\pi f [t - z/c])$ to get $E_x^{reflected}$.

~ Read maximum values of $E_x^{reflected}$ to get $|\Gamma|$

Numerical Results for 3 meshes:

cell size Δ	10 m	5 m	2.5 m
$\vert\tau\vert$	0.89	0.88	0.88
$\vert\Gamma\vert$	0.36	0.42	0.45

The exact results (Note #1) are
$$|\tau| = 0.876, \qquad |\Gamma| = 0.482$$

* Results for $|\Gamma|$ are less accurate because the total field has phase error that prevents proper cancellation with the analytical E^{inc}

Alternate approach for $|\Gamma|$: Determine the SWR in the region $0 < z < 2000$ and use that:
$$|\Gamma| = \frac{SWR - 1}{SWR + 1}$$