Problem Set #8

A second-order FEM solution on curved triangular cells in 2D is provided on Canvas in the "Sample Codes for Note 16" module. This code uses quadratic Lagrangian basis and testing functions to solve Laplace's equation using the curved-cell formulation described in Chapter 9, Section 9.4 of CEM Note #16.

Capacitance problem:

(1) Use the mesh generator (coaxmeshQ.m) and the FEM program (TriFEMQ.m) provided to generate numerical solutions for the capacitance per unit length of a coaxial cable represented by curved triangles. (The code to do this should be complete.) As in Problem Set #3, generate results for a coax with $b/a = 4$. Generate enough results to determine the convergence rate. Provide a table with numerical solutions for three or more meshes (for instance, use meshes similar to those in Table 1 of Note #3).

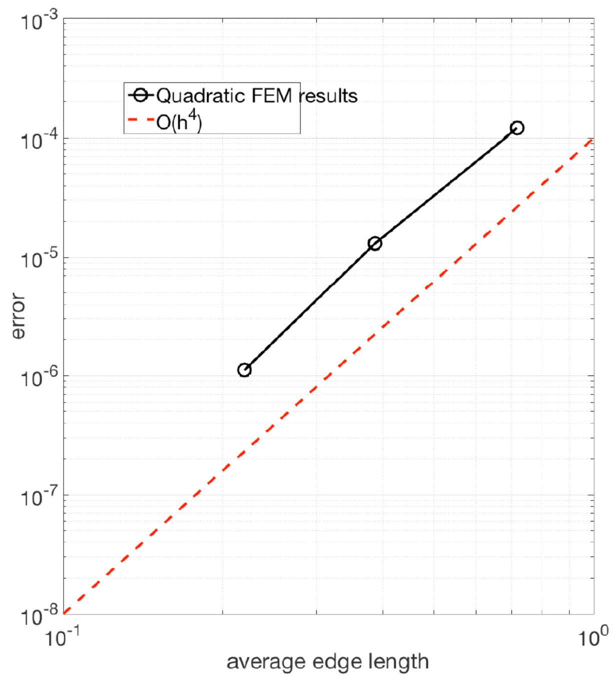
Some results are as follows:

Quadratic & Curved-cell FEM results for a coax with $b/a = 4$					
Layers in mesh	Nodes on inner layer	Unknowns	Avg. edge length	C/ε_0	error
5	12	396	0.719	4.532907	1.2×10^{-4}
9	24	1428	0.387	4.532419	1.3×10^{-5}
17	36	4416	0.219	4.532365	1.1×10^{-6}
exact				4.53236014	

These may be compared to the results from the approach used in Problem Set #3:

Linear & Straight-cell FEM results for a coax with $b/a = 4$					
Layers in mesh	Nodes on inner layer	Unknowns	Avg. edge length	C/ε_0	error
5	12	88	0.719	4.6291	0.09674
9	24	336	0.387	4.5602	0.02784
17	36	1120	0.219	4.5420	0.00964
20	48	1672	0.180	4.5386	0.00624
33	54	3776	0.123	4.5359	0.00354
exact				4.53236	

The convergence of the quadratic-basis capacitance results can be obtained by plotting the error versus the average cell edge lengths, to obtain:



The error tracks an $O(h^4)$ behavior.

Resonant cavity problem:

(2) Using TriFEMQ as an example, develop a new code that determines the TM modes of a cavity using the curved-cell quadratic representation. This should be a code similar to that you generated for Problem Set #4, but based on the curved triangular meshes produced by coaxmeshQ.m and the quadratic basis functions provided in TriFEMQ.m. Provide a listing of the new code.

The part of the code that is needed is the integrand for the T-matrix:

```
function [Tint] = Tint(u,v)

      global itest ibasis

      [~, ~, t] = basis(itest,u,v);
      [~, ~, b] = basis(ibasis,u,v);
      [rdet, ~] = rjacob(u,v);

%     return test times basis done in x-y coords (eq 9.87)

      Tint = t*b*rdet;

end
```

(3) Generate at least 3 different results for the 3 lowest TM resonant wavenumbers for a coaxial cavity with PEC walls with inner radius $a = 1.0$ and outer radius $b = 4.0$. Provide a table with your results.

Some results are as follows:

Quadratic & Curved-cell FEM results for a coax with $b/a = 4$		
5 layers, 16 nodes on inner 468 interior nodes avg. edge = 0.6583	10 layers, 30 nodes on inner 1826 interior nodes avg. edge = 0.3428	15 layers, 40 nodes on inner 4060 interior nodes avg. edge = 0.2323
1.02451	1.02580	1.02442
1.11218	1.11191	1.11188
1.11218	1.11410	1.11188
1.33053	1.33006	1.32998
1.33074	1.33121	1.32998
1.60775	1.60674	1.60664
1.60775	1.60720	1.60664

Observe that for the 15-layer mesh, the results for these wavenumbers agree with the exact solutions to 5 digits.

Results from Problem Set #4 are included below for comparison, these are far less accurate for a given mesh or for a given number of unknowns:

Linear & Straight-cell FEM results for a coax with $b/a = 4$		
5 layers, 16 nodes on inner 104 interior nodes avg. edge = 0.6583	10 layers, 30 nodes on inner 432 interior nodes avg. edge = 0.3428	15 layers, 40 nodes on inner 980 interior nodes avg. edge = 0.2323
1.04761	1.03037	1.02712
1.13901	1.11892	1.11507
1.13901	1.11892	1.11507
1.36459	1.33915	1.33413
1.37514	1.34185	1.33540
1.67064	1.62381	1.61448
1.67064	1.62382	1.61448