Problem Set #3: Solutions

1. Complete the FEM MATLAB code from Note #3, available on Canvas, to incorporate the capacitance calculation in Equation 42. Turn in a printout of the section of the MATLAB code that you added.

Solution: The MATLAB code should be modified as follows (the part that fills the matrix is the same but included here for clarity):

```
% loop through the cells, filling global matrix one cell at a time

  for icell=1:ncells

     n1=pcetond(icell,1);
     n2=pcetond(icell,2);
     n3=pcetond(icell,3);

%    compute 3 by 3 element matrix

     x(1)=xy(n1,1);  y(1)=xy(n1,2);
     x(2)=xy(n2,1);  y(2)=xy(n2,2);
     x(3)=xy(n3,1);  y(3)=xy(n3,2);

     b(1)=y(2)-y(3);
     b(2)=y(3)-y(1);
     b(3)=y(1)-y(2);

     c(1)=x(3)-x(2);
     c(2)=x(1)-x(3);
     c(3)=x(2)-x(1);

     Area = abs(b(3)*c(1) - b(1)*c(3))*0.5;

     for ii=1:3
        for jj=1:3
           elem(ii,jj)=(b(ii)*b(jj)+c(ii)*c(jj))*er(icell)/Area/4;
        end
     end

%    add contributions from cell 'icell' to global matrices

     for ii=1:3
        ig=pcetond(icell,ii); % 'ig' is the global node for 'ii'
        for jj=1:3
           jg=pcetond(icell,jj);  % 'jg' is the global node for 'jj'
           Wtilda(ig,jg) = Wtilda(ig,jg) + elem(ii,jj);
           if(ig <= nunks) % test function at interior node
               if(jg <= nunks) % basis function at interior node
                  W(ig,jg) = W(ig,jg) + elem(ii,jj);
               elseif(jg <= nunks+nouter) % basis function on outer bnd
                  V(ig) = V(ig) - elem(ii,jj);
               end
           end
```

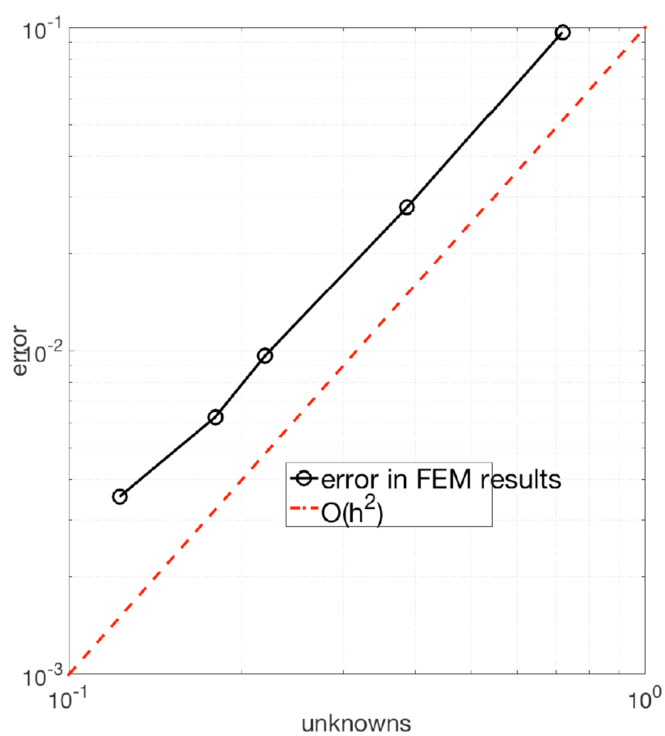
```
        end
     end
  end

%  solve the system of equations to find the potential function

  Pot = W\V;

%  add potential functions on the boundaries to the list

  Pot(nunks+nouter+ninner)=0;
  nstart=nunks+1;
  nend=nunks+nouter;
  for ii=nstart:nend
      Pot(ii)=1;
  end
  nstart=nend+1;
  nend=nunks+nouter+ninner;
  for ii=nstart:nend
      Pot(ii)=0;
  end

%  compute the capacitance

  Cap = Pot.'*Wtilda*Pot;
```

2. Use the code you developed in Problem 1 to generate some capacitance results for an airfilled coax with $b/a = 4$. Determine the approximate convergence rate by comparing to the exact result in Equation 44 of Note #3.

Solution: Some results are presented in the following table:

FEM results for a coax with $b/a = 4$					
Layers in mesh	Nodes on inner layer	Angle ratio	Avg. edge length	C/ε_0	error
5	12	2.1	0.719	4.6291	0.09674
9	24	2.3	0.387	4.5602	0.02784
17	36	2.4	0.219	4.5420	0.00964
20	48	2.4	0.180	4.5386	0.00624
33	54	2.6	0.123	4.5359	0.00354
exact				4.53236	

The following figure shows the error plotted versus the average edge length h. The plot shows close agreement with an $O(h^2)$ rate of decrease as the cell dimensions decrease.



3. *Problem 4.2 in Chapter 4 of PRM*: Find the condition number of the matrix

$$A = \begin{bmatrix} 1 & 10000 \\ 0 & 2 \end{bmatrix}$$

using the definition in Equation (4.11) of Chapter 4 (CEM Note #6). How does the condition number compare with the ratio of largest to smallest to eigenvalues?

Solution: Form the product

$$A^{\dagger}A = \begin{bmatrix} 1 & 10000 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 10000 & 2 \end{bmatrix} = \begin{bmatrix} 100{,}000{,}001 & 20000 \\ 20000 & 4 \end{bmatrix}$$

and find the eigenvalues of this matrix. One way is to set $\det(A^{\dagger}A - \lambda I) = 0$, which leads to the quadratic equation

$$(4\alpha - 4\times 10^{8}) - \lambda(\alpha+4) + \lambda^{2} = 0$$

where $\alpha = 100{,}000{,}001$. Using the quadratic formula, we obtain

$$\lambda = \frac{(\alpha+4) \pm \sqrt{(\alpha+4)^{2} - 16(\alpha - 10^{8})}}{2}$$

We make use of the approximation $\sqrt{1+\varepsilon} \simeq 1 + \varepsilon/2$ to simplify the calculation as follows:

$$\lambda = \frac{(\alpha+4)}{2}\left\{1 \pm \sqrt{1 - \frac{16}{(\alpha+4)^{2}}}\right\} \simeq \frac{(\alpha+4)}{2}\left\{1 \pm \left[1 - \frac{8}{(\alpha+4)^{2}}\right]\right\}$$

We obtain

$$\lambda_1 \simeq (\alpha+4) \simeq 10^{8}$$

$$\lambda_2 \simeq \frac{(\alpha+4)}{2}\frac{8}{(\alpha+4)^{2}} = \frac{4}{(\alpha+4)} \simeq 4\times 10^{-8}$$

Thus, the condition number is

$$\kappa = \sqrt{\lambda_1 / \lambda_2} \simeq 5\times 10^{7}$$

Compare this to the ratio of eigenvalues, which are 2 and 1. Thus the condition number of a non-symmetric matrix is not the ratio of largest to smallest eigenvalue!

4. *Problem 4.4 in Chapter 4 of PRM*: Derive (4.19)

Solution: Under the assumption that $\Delta b = 0$, equation (4.15) reduces to

$$(A + \Delta A)(x + \Delta x) = b$$

or

$$Ax + (\Delta A)(x) + A(\Delta x) + (\Delta A)(\Delta x) = b$$

Subtracting $Ax = b$ leaves

$$(\Delta A)(x) + A(\Delta x) + (\Delta A)(\Delta x) = 0$$

which we rearrange as

$$A(\Delta x) = -(\Delta A)(x + \Delta x)$$

and

$$\Delta x = -(A^{-1})(\Delta A)(x + \Delta x)$$

Using the Schwartz inequality,

$$||\Delta x|| \leq ||A^{-1}|| \; ||\Delta A|| \; ||x + \Delta x||$$

Therefore,

$$\frac{||\Delta x||}{||x + \Delta x||} \leq ||A^{-1}|| \; ||\Delta A|| = \frac{||A^{-1}|| \; ||A|| \; ||\Delta A||}{||A||} = \kappa(A) \frac{||\Delta A||}{||A||}$$

This is equation (4.19).