① forward kinematics

a) $g_1 = \left[\begin{array}{c|c} RotZ(\alpha_1) & \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$ $\quad g_2 = \left[\begin{array}{c|c} RotX(\alpha_2) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$ $\quad g_3 = \left[\begin{array}{c|c} RotX(\alpha_3) & \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$

$g_4 = \left[\begin{array}{c|c} RotX(\alpha_4) & \begin{bmatrix} 0 \\ l_2 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$ $\quad g_5 = \left[\begin{array}{c|c} RotX(\alpha_5) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$ $\quad g_6 = \left[\begin{array}{c|c} RotY(\alpha_6) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$ $\quad g_7 = \left[\begin{array}{c|c} \mathbb{1} & \begin{bmatrix} 0 \\ l_3 \\ 0 \end{bmatrix} \\ \hline 0 & 1 \end{array}\right]$

$$g_e = g_1 g_2 g_3 g_4 g_5 g_6 g_7$$

Pieper's Approach

$$g_e^* = g_w g_h \qquad g_w = g_1 g_2 g_3 \bar{g}_4 \qquad g_h = \tilde{g}_4 g_5 g_6 g_7$$

$$g_w^* = g_e^* (g_7)^{-1} \rightarrow \begin{aligned} x_w &= -\sin(\alpha_1)[l_1 \cos(\alpha_2) + l_2 \cos(\alpha_2 + \alpha_3)] \\ y_w &= \cos(\alpha_1)[l_1 \cos(\alpha_2) + l_2 \cos(\alpha_2 + \alpha_3)] \\ z_w &= l_0 + l_1 \sin(\alpha_2) + l_2 \sin(\alpha_2 + \alpha_3) \end{aligned}$$

$$\alpha_1 = \operatorname{atan2}(-x_w, y_w)$$

$$\alpha_3 = \operatorname{acos}\left(\frac{x_w^2 + y_w^2 + (z_w - l_0)^2 - l_1 - l_2}{2 l_1 l_2}\right)$$

$$r = x_w^2 + y_w^2, \quad w = \tan\left(\frac{\alpha_2}{2}\right) \xrightarrow{\text{roots of}} (r + l_1 + l_2\cos(\alpha_3))w^2 + 2 l_2 \sin(\alpha_3) w + (r - l_1 - l_2 \cos(\alpha_3)) = 0$$

$$\alpha_2 = 2\operatorname{atan}(w)$$

$(g_w)^{-1} g_e^* (g_7)^{-1} = \tilde{g}_4 g_5 g_6 \rightarrow$ will have $T=0$, so consider

$$R_h^* = R_4 R_5 R_6 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} =$$

$R_h = \text{RotX}(\alpha_4)\,\text{RotX}(\alpha_5)\,\text{RotY}(\alpha_6)$

$$= \begin{bmatrix} \cos(\alpha_5)\cos(\alpha_6) & -\sin(\alpha_5) & \cos(\alpha_5)\sin(\alpha_6) \\ \cos(\alpha_4)\sin(\alpha_5)\cos(\alpha_6) + \sin(\alpha_4)\sin(\alpha_6) & \cos(\alpha_4)\cos(\alpha_5) & \cos(\alpha_4)\sin(\alpha_5)\sin(\alpha_6) - \sin(\alpha_4)\cos(\alpha_6) \\ \sin(\alpha_4)\sin(\alpha_5)\cos(\alpha_6) - \cos(\alpha_4)\sin(\alpha_6) & \sin(\alpha_4)\cos(\alpha_5) & \sin(\alpha_4)\sin(\alpha_5)\sin(\alpha_6) + \cos(\alpha_4)\cos(\alpha_6) \end{bmatrix}$$

$\alpha_5 = \text{asin}(-R_{12})$

$$\vec{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$\alpha_4 = \text{atan2}(R_{32}, R_{22})$

$\alpha_6 = \text{atan2}(R_{13}, R_{11})$

b)

$\alpha_1 = \text{atan2}(-1.01, 1.7551) = -0.52218$

$$\alpha_3 = \text{acos}\left(\underbrace{\frac{1.01^2 + 1.7551^2 + (0.5947 - 0.5)^2 - 1^2 - 1^2}{2(1)(1)}}_{=1.0547}\right) =$$

$= 1.0547 \rightarrow$ use small angle approx. as this is outside range of acos

$\cos\theta = 1 - \frac{\theta^2}{2}$

$\cos\alpha_3 = 1.0547 = 1 - \frac{(\alpha_3)^2}{2}$

$\alpha_3 = \sqrt{(1 - 1.0547)2}$

calculations continued in MATLAB