ECE4560 - Solution #4

Problem 1. [10 pts] Let's keep going with the $SE(2)$ code.

(a) Using what is known about $SE(2)$ and its operations, complete the adjoint function.

(b) Verify that it works by showing that, given

$$g_1 = \left(\left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\}, R(\pi/3) \right) \quad \text{and} \quad g_2 = \left(\left\{ \begin{array}{c} -2 \\ 1 \end{array} \right\}, R(\pi/6) \right),$$

the code returns

$$\mathrm{Ad}_{g_1} g_2 = \left(\left\{ \begin{array}{c} -0.73 \\ -1.46 \end{array} \right\}, R(\pi/6) \right).$$

Solution 1. The code, copied from my files without the comments, is:

```
function g = adjoint(g1, g2)
g = g1*g2*inv(g1);
end
```

The corresponding Matlab test is

```
g1 = SE2([1;2], pi/3);
g2 = SE2([-2;1], pi/6);

g3 = adjoint(g1, g2)
```

which gives

```
g3 =

    0.8660   -0.5000   -0.7321
    0.5000    0.8660   -1.4641
         0         0    1.0000
```

and that's pretty much it for the homework.

Problem 2. [10 pts] In class we derived the Adjoint operation for groups. Let's work out an example to verify it. Consider the hammer problems from the previous homeworks. The hammer head center was considered to be the point of interest on the hammer for a few problems, while the base of the hammer defined its configuration for most of the problems. Given that these are two separate frames rigidly associated to the hammer, motion of the base frame and motion of the hammer head frame should be related by the Adjoint.

In previous homeworks, you've worked out some of the group configurations you'll be needing, but did not consider the hammer head as a frame. We know that its displacement relative to the base frame is $d = (1, 0)$, and from the previous homework pictures, the rotation appears to be $-\pi/2$.

(a) Given the hammer at frame $\mathcal{A}$ and frame $\mathcal{B}$ relative to the world frame $\mathcal{O}$, compute the equivalent configurations for the hammer head. Call these frames $\mathcal{A}'$ and $\mathcal{B}'$.

(b) Given the solutions to part (a), compute the motion of the hammer head from $\mathcal{A}'$ to $\mathcal{B}'$, e.g., $g_{\mathcal{B}'}^{\mathcal{A}'}$.

(c) The solution to the second part should be a motion consistent with the base moving from frame $\mathcal{A}$ to $\mathcal{B}$. If our interpretation of the Adjoint is correct, then using $g_{\mathcal{B}}^{\mathcal{A}}$ together with the proper Adjoint should give the same result. Verify that this is indeed true by actually working out the Adjoint based on the earlier answer to how the end-effector configuration changed from $\mathcal{A}$ to $\mathcal{B}$.

Solution 2. What the text is saying is that the hammer handle to hammer head configuration/transformation is $h = (1, 0, -\pi/2)$. From that and knowledge of the hammer handle configurations, we can solve for the hammer head information. I will do it in the vector plus rotation matrix form, but you could have done it differently (say homogeneous coordinates). To verify my work and get actual decimal values, I used the Matlab SE2 class we've been working on. Using the SE2 class makes everything easy, just a few lines of code and the homework is done! It took me much longer to work it out by hand to get the symbolic results. Basically three minutes versus twenty minutes. Effectively, if you are not already thinking naturally in terms of manipulating the group elements like you would addition or mulitplication as in algebra, then you are really gonna have problems. I highly suggest going back and reviewing everything until you feel that it is solid.

(a) First, let's work out the hammer head configurations given the hammer handle configurations,

$$\begin{aligned} g_{\mathcal{A}'}^{\mathcal{O}} = g_{\mathcal{A}}^{\mathcal{O}} h = \left(\left\{ \begin{array}{c} 5 \\ 12 \end{array} \right\}, R(\pi/3) \right) * \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right) &= \left(\left\{ \begin{array}{c} 5 + \frac{1}{2} \\ 12 + \frac{\sqrt{3}}{2} \end{array} \right\}, R(-\pi/6) \right) \\ &= \left(\left\{ \begin{array}{c} 5.5 \\ 12.866 \end{array} \right\}, R(-\pi/6) \right) \end{aligned}$$

and

$$g_{\mathcal{B}'}^{\mathcal{O}} = g_{\mathcal{B}}^{\mathcal{O}} h = \left(\left\{ \begin{array}{c} 2 \\ -1 \end{array} \right\}, R(\pi \right) * \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right) = \left(\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}, R(\pi/2) \right).$$

(b) Using this information, I can solve for the transformation from $\mathcal{A}'$ to $\mathcal{B}'$,

$$\begin{aligned} g_{\mathcal{B}'}^{\mathcal{A}'} &= \left(g_{\mathcal{A}'}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}'}^{\mathcal{O}} = \left(\left\{ \begin{array}{c} 5 + \frac{1}{2} \\ 12 + \frac{\sqrt{3}}{2} \end{array} \right\}, R(-\pi/6) \right)^{-1} * \left(\left\{ \begin{array}{c} 2 \\ 0 \end{array} \right\}, R(\pi/2) \right) \\ &= \left(\left\{ \begin{array}{c} 6 - 5\sqrt{3}/2 \\ -7/2 - 6\sqrt{3} \end{array} \right\}, R(\pi/6) \right) * \left(\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}, R(\pi/2) \right) = \left(\left\{ \begin{array}{c} 13/2 - 2\sqrt{3} \\ -3 - 13\sqrt{3}/2 \end{array} \right\}, R(2\pi/3) \right) \\ &= \left(\left\{ \begin{array}{c} 3.0359 \\ -14.2583 \end{array} \right\}, R(2\pi/3) \right) \end{aligned}$$

(c) The purpose here is to show that the answer for part (b) can be obtained as $g_{\mathcal{B}'}^{\mathcal{A}'} = \mathrm{Ad}_{h^{-1}} g_{\mathcal{B}}^{\mathcal{A}}$. For that we will need $g_{\mathcal{B}}^{\mathcal{A}}$. To see that the above really has to hold, we first work it out symbolically,

$$g_{\mathcal{B}'}^{\mathcal{A}'} = \left(g_{\mathcal{A}'}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}'}^{\mathcal{O}} = \left(g_{\mathcal{A}}^{\mathcal{O}} h\right)^{-1} g_{\mathcal{B}}^{\mathcal{O}} h = h^{-1} \left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}}^{\mathcal{O}} h = h^{-1} g_{\mathcal{B}}^{\mathcal{A}} h = \mathrm{Ad}_{h^{-1}} g_{\mathcal{B}}^{\mathcal{A}}.$$

Now to work it out numerically for verification purposes,

$$\begin{aligned} g_{\mathcal{B}}^{\mathcal{A}} &= \left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}}^{\mathcal{O}} = \left(\left\{ \begin{array}{c} 5 \\ 12 \end{array} \right\}, R(\pi/3) \right)^{-1} * \left(\left\{ \begin{array}{c} 2 \\ -1 \end{array} \right\}, R(\pi) \right) \\ &= \left(\left\{ \begin{array}{c} -5/2 - 6\sqrt{3} \\ -6 + 5\sqrt{3}/2 \end{array} \right\}, R(-\pi/3) \right) * \left(\left\{ \begin{array}{c} 2 \\ -1 \end{array} \right\}, R(\pi) \right) = \left(\left\{ \begin{array}{c} -3/2 - 13\sqrt{3}/2 \\ -13/2 + 3\sqrt{3}/2 \end{array} \right\}, R(-\pi/3) \right) \\ &= \left(\left\{ \begin{array}{c} -12.7583 \\ -3.9019 \end{array} \right\}, R(2\pi/3) \right) \end{aligned}$$

Lastly comes the adjoint operation,

$$
\begin{aligned}
\mathrm{Ad}_{h^{-1}} g_{\mathcal{B}}^{\mathcal{A}} &= \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right)^{-1} \left(\left\{ \begin{array}{c} -3/2 - 13\sqrt{3}/2 \\ -13/2 + 3\sqrt{3}/2 \end{array} \right\}, R(2\pi/3) \right) \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right) \\
&= \left(\left\{ \begin{array}{c} 0 \\ -1 \end{array} \right\}, R(\pi/2) \right) \left(\left\{ \begin{array}{c} -3/2 - 13\sqrt{/}23 \\ -13/2 + 3\sqrt{3}/2 \end{array} \right\}, R(2\pi/3) \right) \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right) \\
&= \left(\left\{ \begin{array}{c} 13/2 - 3\sqrt{3}/2 \\ -5/2 - 13\sqrt{3}/2 \end{array} \right\}, R(7\pi/6) \right) \left(\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}, R(-\pi/2) \right) \\
&= \left(\left\{ \begin{array}{c} 13/2 - 2\sqrt{3} \\ -3 - 13\sqrt{3} \end{array} \right\} /2, R(2\pi/3) \right) \\
&= \left(\left\{ \begin{array}{c} 3.0359 \\ -14.2583 \end{array} \right\}, R(2\pi/3) \right)
\end{aligned}
$$

which gives us the same as in part (b).

Problem 3. [15 pts] Depicted in Figure 1 are the before and after configurations for a manipulator. The task was to get a key fitted into the factory control board to activate the factory machinery. From the initial configuration (left side) to the final, key insertion configuration (right side), the manipulator end-effector frame configuration moved by

$$
g_{\mathcal{H},2}^{\mathcal{H},1} = \left[\begin{array}{c|c} R(-7\pi/12) & \left\{ \begin{array}{c} -3.151 \\ -3.906 \end{array} \right\} \\ \hline 0 & 1 \end{array} \right] = \left[\begin{array}{cc|c} -0.259 & 0.966 & -3.151 \\ -0.966 & -0.259 & -3.906 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right]
$$

to end up in the configuration

$$
g_{\mathcal{H},2}^{\mathcal{O}} = \left[\begin{array}{cc|c} -0.131 & -0.991 & 3.243 \\ 0.991 & -0.131 & 2.512 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right]
$$

The key frame relative to the end-effector frame is given by

$$
g_{\mathcal{K}}^{\mathcal{H}} = \left[\begin{array}{c|c} R(-\pi/4) & \left\{ \begin{array}{c} 3 \\ -1 \end{array} \right\} \\ \hline 0 & 1 \end{array} \right] = \left[\begin{array}{cc|c} 0.7071 & 0.7071 & 3 \\ -0.7071 & 0.7071 & -1 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right]
$$

a) In what configuration did the end-effector start out in relative to the base frame $\mathcal{O}$.

b) What is the final configuration of the key frame relative to the base frame $\mathcal{O}$?

c) What was the movement/displacement of the key frame from time 1 to time 2?

Solution 3. By now, the coordinate frames, adjoints, and all related operations should be pretty internalized. The solution will be worked out symbolically, then the true numerical answer simply given, as it follows from writing the equivalent Matlab code.

a) The question seeks the values in $g_{\mathcal{H},1}^{\mathcal{O}}$.

$$
\begin{aligned}
g_{\mathcal{H},1}^{\mathcal{O}} &= g_{\mathcal{H},2}^{\mathcal{O}} * g_{\mathcal{H},1}^{\mathcal{H},2} = g_{\mathcal{H},2}^{\mathcal{O}} * \left(g_{\mathcal{H},2}^{\mathcal{H},1} \right)^{-1} \\
&= \left[\begin{array}{cc|c} -0.131 & -0.991 & 3.243 \\ 0.991 & -0.131 & 2.512 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right] \left[\begin{array}{cc|c} -0.259 & 0.966 & -3.151 \\ -0.966 & -0.259 & -3.906 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right]^{-1} \\
&= \left[\begin{array}{cc|c} -0.924 & 0.383 & 1.827 \\ -0.383 & -0.924 & -2.303 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right].
\end{aligned}
$$

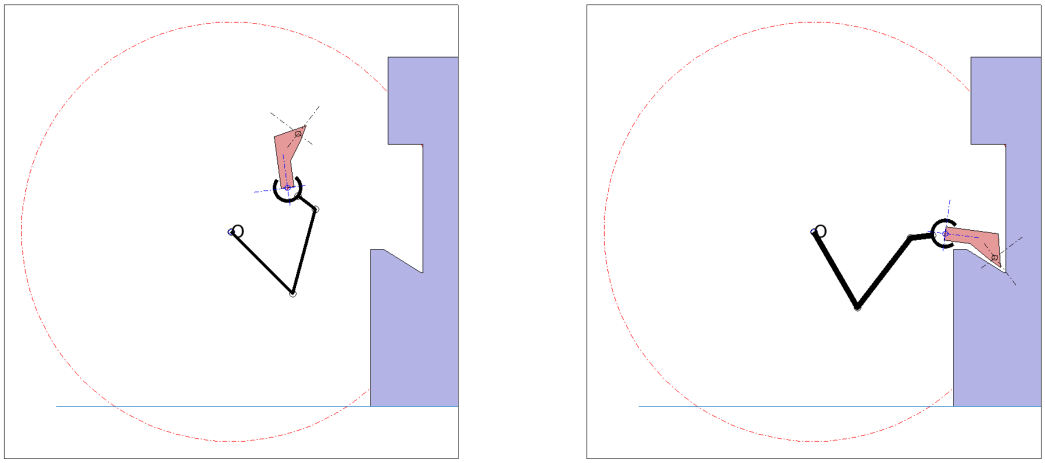

Figure 1: Manipulator attaching key to factory controller to turn it on.

b) The final configuration of the key frame relative to the base frame is

$$\begin{aligned}
g^{\mathcal{O}}_{\mathcal{K},2} &= g^{\mathcal{O}}_{\mathcal{H},2} * g^{\mathcal{H}}_{\mathcal{K}} \\
&= \left[\begin{array}{cc|c} -0.131 & -0.991 & 3.243 \\ 0.991 & -0.131 & 2.512 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right] \left[\begin{array}{cc|c} 0.707 & 0.707 & 3.000 \\ -0.707 & 0.707 & -1.000 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 0.609 & -0.793 & 3.843 \\ 0.793 & 0.609 & 5.616 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]
\end{aligned}$$

c) The displacement of the key frame from time 1 to time 2 is given by the adjoint operation,

$$\mathrm{Ad}_{\left(g^{\mathcal{H}}_{\mathcal{K}}\right)^{-1}} g^{\mathcal{H},1}_{\mathcal{H},2} = \left[\begin{array}{cc|c} -0.259 & 0.966 & -1.660 \\ -0.966 & -0.259 & -9.502 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]$$

Problem 4. [16 pts] In the space of 3×3 rotation matrices, $SO(3)$, there is no such thing as a unique way to represent a rotation matrix as a function of three variables (one for each axis of rotation). There is a way to map the rotation of a point about a single axis into a rotation matrix.

We will explore this in class, but for now, consider the three easy cases of rotating about one of the coordinate axes (the $\hat{x}$-axis, the $\hat{y}$-axis, or the $\hat{z}$-axis). Since the rotation is only about one axis, all cases should look like the planar case but with an extra dimension and re-ordered. In fact these single axis rotations do. They are:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \quad \& \quad R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can generate an arbitrary rotation matrix by taking combinations of these matrices. Peform the following computations:

(a) Compute $R_x(\pi/4)$.

(b) Compute $R_y(\pi/6)R_z(3\pi/4)$.

(c) Compute $R_x(\pi/3)R_z(-\pi/4)$.

(d) Show that $R_y(\theta_1)R_y(\theta_2) = R_y(\theta_1 + \theta_2)$.

Solution 4.

(a)

$$R_x(\pi/4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix}$$

(b)

$$\begin{aligned} R_y(\pi/6)R_z(\pi/2) &= \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0.8660 & 0.0000 & 0.5000 \\ 0.0000 & 1.0000 & 0.0000 \\ -0.5000 & 0.0000 & 0.8660 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 & 0.0000 \\ 0.7071 & -0.7071 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix} \\ &= \begin{bmatrix} -0.6124 & -0.6124 & 0.5000 \\ 0.7071 & -0.7071 & 0.0000 \\ 0.3536 & 0.3536 & 0.8660 \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} R_x(\pi/3)R_z(-\pi/4) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.866 \\ 0 & 0.866 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.3536 & 0.3536 & -0.8660 \\ -0.6124 & 0.6124 & 0.5000 \end{bmatrix} \end{aligned}$$

(d)

$$\begin{aligned} R_y(\theta_1)R_y(\theta_2) &= \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) \\ 0 & 1 & 0 \\ -\sin(\theta_1) & 0 & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)(\sin(\theta_2) & 0 & \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)(\sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)(\sin(\theta_2) & 0 & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)(\sin(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & 0 & \sin(\theta_1+\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_1+\theta_2) & 0 & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= R_y(\theta_1+\theta_2). \end{aligned}$$