ECE4560 - Solution #10

Problem 1. (40 pts) Consider the elbow manipulator depicted in Figure 1. The home position is the straight-out position. This manipulator has a reorientable wrist, hence it has a non-empty dextrous workspace (in the absence of joint limits). Supposing that the link lengths were $l_0 = \frac{1}{2}$, $l_1 = 1$, $l_2 = 1$, $l_3 = \frac{1}{2}$.

(a) What are the inverse kinematics?

(b) Solve the inverse kinematics corresponding to the following two configurations:

$$g_i = \left[\begin{array}{ccc|c} 0.9280 & 0.3536 & 0.1174 & 1.01 \\ -0.3245 & 0.6124 & 0.7209 & 1.7551 \\ 0.1830 & -0.7071 & 0.6830 & 0.5947 \\ \hline & 0 & & 1 \end{array}\right]$$

and

$$g_f = \left[\begin{array}{ccc|c} 0.25 & -0.6424 & -0.7244 & -1.071 \\ -0.4330 & 0.5950 & -0.6771 & 1.5966 \\ 0.8660 & 0.4830 & -0.1294 & 1.6075 \\ \hline & 0 & & 1 \end{array}\right].$$

Just as a reminder, recall that Pieper's approach can be used to break this problem into two separate problems, one for the wrist, then one for the hand. The solution then follows from what we've been discussing in class.

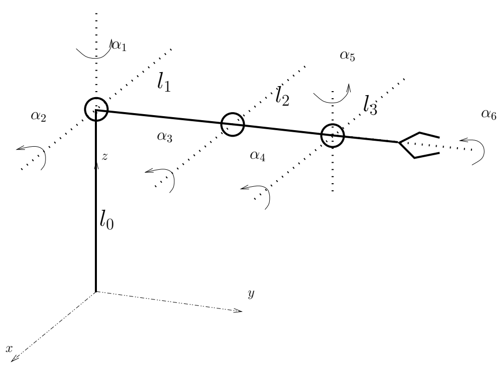


Figure 1: Elbow Manipulator.

Solution 1. The first thing to note is that the order of the rotations for the last three joints does not agree with the notes, so some adjustments need to be made with respect to the solutions from the notes. In what follows, we move forward with the solution.

(a) Consider that we have

$$g_e(\alpha) = g_1(\alpha_1)g_2(\alpha_2)g_3(\alpha_3)g_4(\alpha_4)g_5(\alpha_5)g_6(\alpha_6)g_7,$$

where each transform g_i goes from joint $i-1$ to joint i. The different g_i's are:

$$g_1(\alpha_1) = \left[\begin{array}{ccc|c} \cos(\alpha_1) & -\sin(\alpha_1) & 0 & 0 \\ \sin(\alpha_1) & \cos(\alpha_1) & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ \hline & 0 & & 1 \end{array}\right], \qquad g_2(\alpha_2) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_2) & -\sin(\alpha_2) & 0 \\ 0 & \sin(\alpha_2) & \cos(\alpha_2) & 0 \\ \hline & 0 & & 1 \end{array}\right],$$

$$g_3(\alpha_3) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_3) & -\sin(\alpha_3) & l_1 \\ 0 & \sin(\alpha_3) & \cos(\alpha_3) & 0 \\ \hline & 0 & & 1 \end{array}\right], \qquad g_4(\alpha_4) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_4) & -\sin(\alpha_4) & l_2 \\ 0 & \sin(\alpha_4) & \cos(\alpha_4) & 0 \\ \hline & 0 & & 1 \end{array}\right],$$

$$g_5(\alpha_5) = \left[\begin{array}{ccc|c} \cos(\alpha_5) & -\sin(\alpha_5) & 0 & 0 \\ \sin(\alpha_5) & \cos(\alpha_5) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline & 0 & & 1 \end{array}\right], \qquad g_6(\alpha_6) = \left[\begin{array}{ccc|c} \cos(\alpha_6) & 0 & \sin(\alpha_6) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\alpha_6) & 0 & \cos(\alpha_6) & 0 \\ \hline & 0 & & 1 \end{array}\right],$$

$$g_7 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ \hline & 0 & & 1 \end{array}\right],$$

We want to separate the "arm" part from the "hand" part. Thus, consider $g_e(\alpha)$ to be composed of a transformation that goes from the base to the wrist, $g_w(\alpha)$, and a transformation that goes from the wrist to the hand $g_h(\alpha)$, as per $g_e(\alpha) = g_w(\alpha)g_h(\alpha)$. To do this, we are going to have to break up the Lie group products into two parts. Of note, the transformation g_4 must be broken up into $g_4(\alpha) = \bar{g}_4\tilde{g}_4(\alpha)$, where $\bar{g}_4$ is the constant part and $\tilde{g}_4(\alpha)$ is the part that changes with the joint configuration. Under the desired decomposition,

$$g_w(\alpha) = g_1(\alpha_1)g_2(\alpha_2)g_3(\alpha_3)\bar{g}_4, \qquad \text{and} \qquad g_h(\alpha) = \tilde{g}_4(\alpha_4)g_5(\alpha_5)g_6(\alpha_6)g_7.$$

We need to find the joint configuration α leading to the desired end-effector configuration g_e^*,

$$g_e^* = g_w(\alpha)g_h(\alpha).$$

Now, recall that the last transformation of $g_h(\alpha)$ is g_7 which is constant, and the remaining transformations g_4, g_5, and g_6 do not have a translational component. A logical first step would be to move the constant transformation to the left hand side leading the desired wrist position,

$$g_w^* \equiv g_e^* g_7^{-1}.$$

The wrist positioning objective is to find α_1^*, α_2^*, and α_3^* such that

$$Position(g_w^*) = Position(g_w(\alpha^*)).$$

These inverse kinematics were discussed in class. Looking at the wrist forward kinematics provides clues as to how to proceed with the algebraic solution to the inverse kinematics problem,

$$\begin{aligned} x_w(\alpha) &= -\sin(\alpha_1)\left(l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3)\right) \\ y_w(\alpha) &= \cos(\alpha_1)\left(l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3)\right) \\ z_w(\alpha) &= l_0 + l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3). \end{aligned} \tag{1}$$

Following the strategy from the lecture notes, the solution for the first and third joint angles are

$$\begin{aligned} \alpha_1 &= \text{atan2}(-x_w, y_w) \\ \alpha_3 &= \cos^{-1}\left(\frac{x_w^2 + y_w^2 + (z_w - l_0)^2 - l_1^2 - l_2^2}{2l_1l_2}\right). \end{aligned}$$

For the second joint angle, let $w = \tan(\alpha_2/2)$, then seek that solution to

$$(r + l_1 + l_2\cos(\alpha_3))w^2 + (2l_2\sin(\alpha_3))w + (r - l_1 - l_2\cos(\alpha_3)) = 0,$$

where $r^2 = x_w^2 + y_w^2$. Once the solution to the quadratic equation is found, the correct solution of the two must be chosen and inverted to get, $\alpha_2 = 2\tan^{-1}(w)$. Here, because we solved for α_1 first, there really is a correct solution. If, as per lecture, α_1 were unsolved until the end, then either of the two solutions could be picked and would work. Of course, the solution α_1 would be arrived at differently from above as it would depend on the choices of α_2^* and α_3^*.

Once we have $g_w(\alpha)$, then we need to find the orientation part. Following the notes, we need to find the remaining α_4^*, α_5^*, and α_6^*, such that

$$g_e^* = g_w(\alpha^*) g_h(\alpha^*).$$

Since α_1^*, α_2^*, and α_3^* are known,

$$g_w^{-1}(\alpha^*) g_e^* = g_h(\alpha) = \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6) g_7.$$

Isolating the unknown components,

$$g_w^{-1}(\alpha^*) g_e^* g_7^{-1} = \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6).$$

At this point, both sides of the above equation should have zero translation. All that is left is to solve for the orientation. Therefore, we only need worry about the rotational portions of the above equation, which will involve a R_h^* from the left hand side and $R_h(\alpha) = R_4(\alpha_4) R_5(\alpha_5) R_6(\alpha_6)$ from the right hand side. The rotation matrix R_h^* something that will be computed based on given values and will consist of actual numerical values. For now, we will deal with it in its symbolic form,

$$R_h^* = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}.$$

The hand rotation matrix given the joint angles α_4, α_5, and α_6 is

$$\begin{aligned} R_h(\alpha) &= R_4(\alpha_4) R_5(\alpha_5) R_6(\alpha_6) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_4) & -\sin(\alpha_4) \\ 0 & \sin(\alpha_4) & \cos(\alpha_4) \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha_5) & -\sin(\alpha_5) & 0 \\ \sin(\alpha_5) & \cos(\alpha_5) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\alpha_6) & 0 & \sin(\alpha_6) \\ 0 & 1 & 0 \\ -\sin(\alpha_6) & 0 & \cos(\alpha_6) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha_5)\cos(\alpha_6) & -\sin(\alpha_5) & \cos(\alpha_5)\sin(\alpha_6) \\ \cos(\alpha_4)\sin(\alpha_5)\cos(\alpha_6) + \sin(\alpha_4)\sin(\alpha_6) & \cos(\alpha_4)\cos(\alpha_5) & \cos(\alpha_4)\sin(\alpha_5)\sin(\alpha_6) - \sin(\alpha_4)\cos(\alpha_6) \\ \sin(\alpha_4)\sin(\alpha_5)\cos(\alpha_6) - \cos(\alpha_4)\sin(\alpha_6) & \sin(\alpha_4)\cos(\alpha_5) & \sin(\alpha_4)\sin(\alpha_5)\sin(\alpha_6) + \cos(\alpha_4)\cos(\alpha_6) \end{bmatrix} \end{aligned}$$

which is actually the rotation matrix that the X-Z-Y Euler-type angle representation for a rotation matrix will give you. It is possible to solve for the three final joints angles. Recall that there were two cases that were going to have to be dealt with.

If $\cos(\alpha_5)$ does not vanish, then we can solve for all of the joint angles; this corresponds to the case where $R_{11} \neq 0$, $R_{13} \neq 0$, and $R_{12} \neq \mp 1$. In this case, we just have to look at the rotation matrix and figure out what trigonometric functions need to be inverted, using atan2 as much as possible.

The fifth joint angle can be obtained from the inverse sine function,

$$\alpha_5 = \arcsin(-R_{12}) = -\arcsin(R_{12}).$$

Since the solution to $\arcsin$ lies in the range $[-\pi/2, \pi/2]$ where $\cos$ achieves positive values, we can solve for α_4 using R_{22} and R_{32} directly,

$$\alpha_4 = \text{atan2}\left(R_{32}, R_{22}\right), \qquad \text{if } \cos(\alpha_5) \neq 0.$$

Likewise for α_6,

$$\alpha_6 = \text{atan2}\left(R_{13}, R_{11}\right), \qquad \text{if } \cos(\alpha_5) \neq 0.$$

If, instead, we do have $R_{11} = R_{13} = 0$ and $R_{12} = \mp 1$, then this corresponds to the case where $\cos(\alpha_5) = 0$ and $\sin(\alpha_5) = \pm 1$. The rotation matrix $R_h(\alpha^*)$ simplifies to

$$R_h(\alpha^*) = \begin{bmatrix} 0 & \mp 1 & 0 \\ \pm\cos(\alpha_6 \mp \alpha_4) & 0 & \pm\sin(\alpha_6 \mp \alpha_4) \\ -\sin(\alpha_6 \mp \alpha_4) & 0 & \cos(\alpha_6 \mp \alpha_4) \end{bmatrix}.$$

It is possible to solve only for the sum or difference of α_4 and α_6, meaning that one of the variables is arbitrary in this representation (if we were really doing trajectory design, then the choices would be so that α_4 and α_6 are close to the previous joint angles in the trajectory being designed, otherwise the manipulator would do funny things as it approached this singularity). Let $\alpha_4 = 0$, then

$$g_R(\alpha) = \left[\begin{array}{ccc|c} 0 & \mp 1 & 0 & 0 \\ \pm\cos(\alpha_6) & 0 & \pm\sin(\alpha_6) & 0 \\ -\sin(\alpha_6) & 0 & \cos(\alpha_6) & 0 \\ \hline & 0 & & 1 \end{array}\right].$$

Note that R_{31} and R_{33} are independent of the solution to α_5, so the solutions for the two extremal/singularity conditions are

$$\begin{cases} \alpha_4 = 0,\ \alpha_5 = \frac{\pi}{2},\ \text{and } \alpha_6 = \text{atan2}(-R_{31}, R_{33}) & \text{if } \sin(\alpha_5) = 1. \\ \alpha_4 = 0,\ \alpha_5 = -\frac{\pi}{2},\ \text{and } \alpha_6 = \text{atan2}(-R_{31}, R_{33}) & \text{if } \sin(\alpha_5) = -1. \end{cases}$$

It is also possible to choose $\alpha_6 = \text{atan2}(\pm R_{23}, \pm R_{21})$ for these cases. For these solutions, we are presuming that $\alpha = \text{atan2}\left(\sin(\alpha), \cos(\alpha)\right)$, which is how Matlab likes the atan2 arguments to be given.

(b) Using the above solution for the inverse kinematics, we get two possible solutions

$$\alpha_i = (-\frac{\pi}{6}, -\frac{\pi}{12}, \frac{\pi}{3}, -\frac{\pi}{2}, 0, -\frac{\pi}{12})^T \quad \text{or} \quad \alpha_i = (-\frac{\pi}{6}, \frac{\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, -\frac{\pi}{12})^T \tag{2}$$

and

$$\alpha_f = (\frac{\pi}{6}, 0, \frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{12}, -\frac{\pi}{2})^T, \quad \text{or} \quad \alpha_f = (\frac{\pi}{6}, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{12}, -\frac{\pi}{2})^T,$$

respectively, depending on the sign chosen for the inverse cosine solution. Performing the algebraic approach as per class would actually give four potential solutions. Two of them are a little trickier to get, while two of them will be correct when teting using the forward kinematics. Figure 2 visualizes the manipulator for the solutions given above.

The main source of error for many is typically in the use of the inverse tangent, or arctangent, function. Figuring out the correct argument order and sign assignation takes a while. Even then, one usually has to think about it. One can easily check the solution by plugging the angles returned into the forward kinematics.

Problem 2. (10 pts) Let's consider again the three-link rotational planar manipulator from the previous homeworks, c.f. Figure 3. The link lengths were specified to be $l_1 = 1$, $l_2 = \frac{1}{2}$, and $l_3 = \frac{1}{4}$. The task is to now generate trajectories for the manipulator.

If the initial joint angles are $\alpha(0) = (-\pi/6, \pi/4, -\pi/3)^T$ and the final joint angles are to be $\alpha(5) = (0, -\pi/12, \pi/4)^T$, design a splined trajectory in the joint space that will connect the two joint configurations. Note that the duration of the trajectory should be 5 seconds. In addition to turning in the polynomials, you need to turn in a plot of the manipulator performing the routine. Have it plot about 5-7 points along the trajectory.

Solution 2. Designing the spline is just a matter of solving for the polynomial coefficient associated to the polynomial for each joint angle. Applying the polynomial coefficient matrix solution for each of the joint angles gives the polynomials

$$\vec{\alpha}(t) = \left\{ \begin{array}{c} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \end{array} \right\} = \begin{bmatrix} -\frac{\pi}{6} & 0 & \frac{\pi}{50} & -\frac{\pi}{375} \\ \frac{\pi}{4} & 0 & -\frac{\pi}{25} & \frac{2\pi}{375} \\ -\frac{\pi}{3} & 0 & \frac{7\pi}{100} & -\frac{7\pi}{750} \end{bmatrix} \left\{ \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \end{array} \right\} = \left\{ \begin{array}{c} -0.0084t^3 + 0.0628t^2 - 0.5236 \\ 0.0168t^3 - 0.1257t^2 + 0.7854 \\ -0.0293t^3 + 0.2199t^2 - 1.0472 \end{array} \right\}$$

The associated figures can be found in Figure 4.

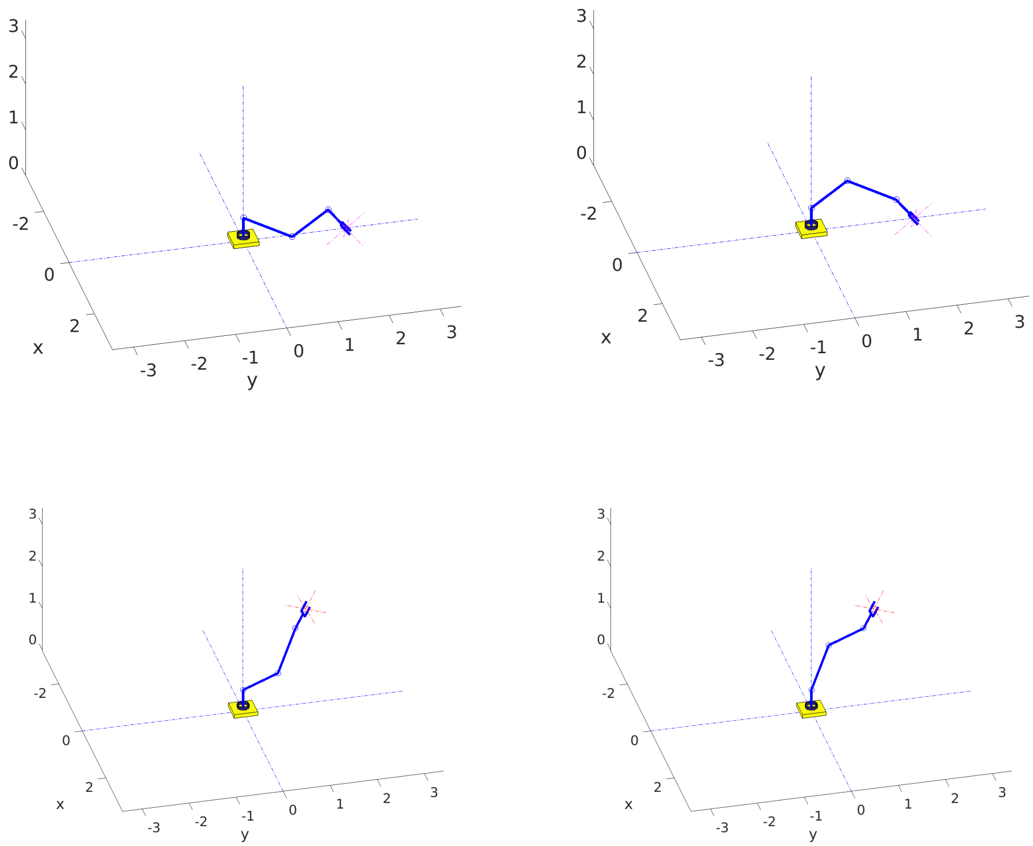


Figure 2: The top row has two solutions for the inverse kinematics of the start end-effector configuration. The bottom row has two solutions for the inverse kinematics of the final end-effector configuration.

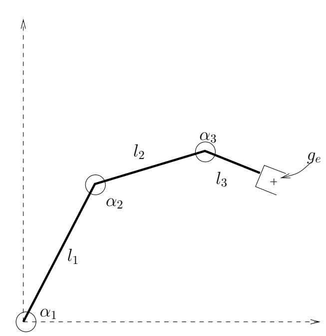
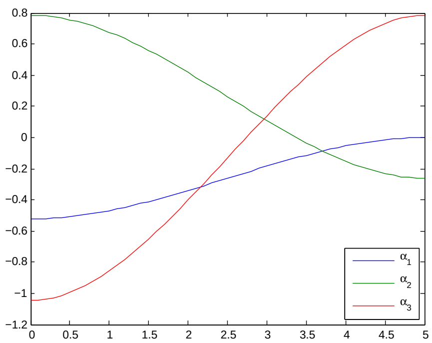
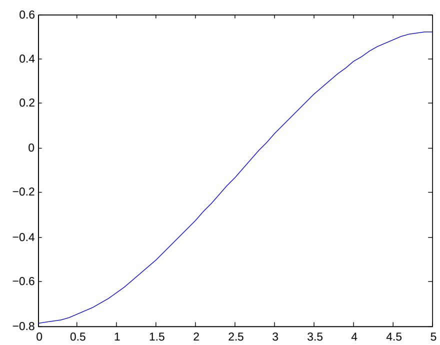


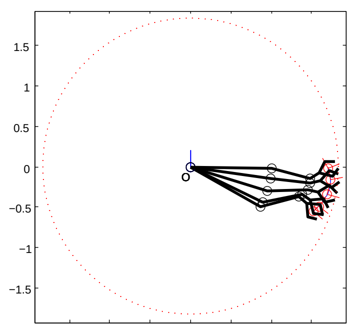
Figure 3: Planar 3R Manipulator.



(a) Joint Trajectory.



(b) Orientation vs. Time.



(c) Parametric plot w/Manipulator.

Figure 4: Polynomial trajectory in manipulator joint space.