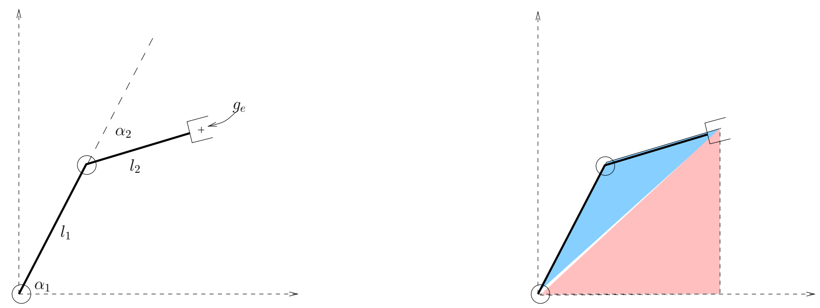
ECE4560 - Solution #7

Problem 1. (10 pts) The *inverse kinematics* of a manipulator are the equations that provide the manipulator joint angles given the end effector configuration. In this simple problem, let's consider that our only interest is in the position of the manipulator end-effector (orientation is not important) for the manipulator in Figure 1

(a) What are the forward kinematics of the planar manipulator? What is $d_e(\alpha) = (x_e(\alpha), y_e(\alpha))^T$.

(b) The inverse kinematics for this can be solved using the law of cosines and properties of right triangles, as hinted at in Figure 1(b). Suppose that $l_1 = 1$ and $l_2 = 0.5$, solve for the inverse kinematics given that $d_e = (1.3595, 0.2113)^T$. The inverse kinematics essentially asks: given the end-effector configuration of $(x_e, y_e)^T$, what are α_1 and α_2?

(c) When you solve the above problem, there will be cases where the computation cannot be performed. What cases of (x, y) cause the solution to not exist?



(a) Manipulator Geometry. (b) Geometry of Inverse Kinematics.

Figure 1: Simple planar manipulator with two revolute joints.

Solution 1.

(a) The forward kinematics are:

$$d_e(\alpha) = \left\{ \begin{array}{c} l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1 + \alpha_2) \\ l_1 \sin(\alpha_1) + l_2 \sin(\alpha_1 + \alpha_2) \end{array} \right\}$$

(b) As hinted, the solution to the inverse kinematics exploits properties of the manipulator geometry to illuminate the solution using the law of cosines. The law of cosines says that

$$\cos(\phi) = \frac{a^2 + b^2 - c^2}{2ab}.$$

Looking at the triangle from Figure 1(b), and the manipulator geometry from Figure 1(a), once can define a collection of angles which require both the law of cosines and the inverse tanget function to solve for. These angles and their meaning can be found in Figure 3. According to the figure, we see the following relationships,

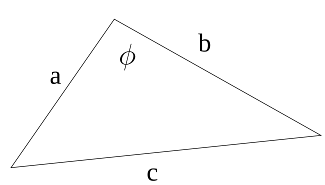


Figure 2: Depiction for law of cosines.

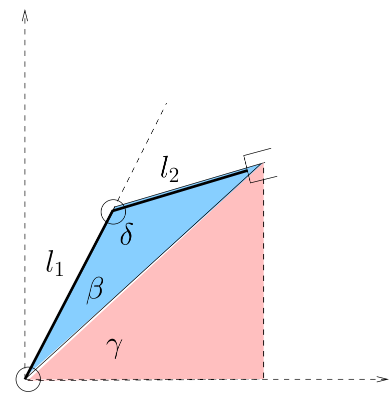


Figure 3: Depiction for resolving the inverse cos and inverse tangent relationships.

$$
\begin{aligned}
\alpha_2 &= -(\pi - \delta) = \text{acos}\left(\frac{l_1^2 + l_2^2 - r^2}{2l_1 l_2}\right) - \pi, \\
\alpha_1 &= \gamma + \beta, \\
\beta &= \text{acos}\left(\frac{l_1^2 + r^2 - l_2^2}{2l_1 r}\right), \text{ and} \\
\gamma &= \arctan2(y_e, x_e).
\end{aligned}
$$

Note that the solution to α_2 has a negative factor in it. That is because the angle that is computed is an absolute value of the angle. To figure out the sign, we have to compare against the zero reference line (the dotted line in Figure 3) and the right hand rule. In this case, the angle turns out to be negative. One must be careful since there are actually two solutions to the acos function, one positive and one negative (thus another solution could be $\alpha_2 = \pi - \delta$ and $\alpha_1 = \gamma - \beta$).

OK, given the above, we just need to compute the different variables. The computations lead to

$$
\begin{aligned}
r &= \left(1.3595^2 + 0.2113^2\right)^{\frac{1}{2}} = 1.3758 \\
\delta &= -\text{acos}(-0.6428) = 2.2689 \\
\gamma &= \arctan2(0.2113, 1.3595) = 0.1542 \\
\beta &= \text{acos}(0.9605) = 0.2821
\end{aligned}
\qquad \text{which effectively give} \qquad
\begin{aligned}
\alpha_1 &= \gamma + \beta = 0.4363 \\
\alpha_2 &= \delta - \pi = -0.8727
\end{aligned}
$$

Plugging the α_1 and α_2 variables into the forward kinematics from **(a)** gives the right answer, so we know the solution is correct.

Had the second solution been found, then the answer would have been $\alpha = (-0.1279; 0.8727)^T$. In degrees the two solutions are $\alpha = (25, -50)$ and $\alpha = (-7.3298, 50)^T$.

(c) For the inverse cosine function to exist, the value of the argument must be in the range $[-1, 1]$. Thus, some constraints on the solution are

$$
\begin{aligned}
-2l_1 l_2 &\leq l_1^2 - r^2 + l_2^2 \\
2l_1 l_2 &\geq l_1^2 + l_2^2 - r^2 \\
-2l_1 r &\leq l_1^2 + r^2 - l_2^2 \\
2l_1 r &\geq l_1^2 + r^2 - l_2^2
\end{aligned} \tag{1}
$$

The constraints essentially require that the point d_e^* not be too close in, nor too far out. Remembering the annulus of the reachable workspace, then manipulating the first two equations leads to,

$$
\begin{aligned}
r^2 &\leq (l_1 + l_2)^2 \\
r^2 &\geq (l_1 - l_2)^2
\end{aligned} \tag{2}
$$

which can be merged into

$$(l_1 - l_2)^2 \le r^2 \le (l_1 + l_2)^2 \tag{3}$$

and are the equations for an annulus. The other pair of equations simplify to

$$\begin{aligned} l_2^2 &\le (l_1 + r)^2 \\ l_2^2 &\ge (l_1 - r)^2 \end{aligned} \tag{4}$$

Since all quantities are positive, the first inequality is equivalent to

$$l_2 \le l_1 + r \tag{5}$$

which becomes

$$l_2 - l_1 \le r. \tag{6}$$

That inequality has already been established (it is the inner ring of the annulus). Taking into account for the sign of the right hand side, the second inequality leads to the condition

$$l_2 \ge l_1 - r \quad \text{and} \quad l_2 \ge r - l_1. \tag{7}$$

The first condition is related to the inner annulus and the second to the outer annulus since they are equivalent to

$$r \ge l_1 - l_2 \quad \text{and} \quad l_1 + l_2 \ge r. \tag{8}$$

In brief, we recover the condition that the desired end point be within the annulus $|l_1 - l_2| \le r \le l_1 + l_2$.

Problem 2. [10 pts] Using the $SE(2)$ class exponent and logarithm as example stubs, code in the exponent and logarithm for the $SE(3)$ class (they should be static member functions). Verify your solution by exponentiating the following twist

$$\xi = (2, 1, -3, \pi/10, -\pi/4, \pi/9)$$

for 2 seconds, and then taking the logarithm of the result. You should get the same thing back out. The group element you get should be

```
>> g = SE3.exp([2; 1; -3; pi/10; -pi/4; pi/9], 2)

ans =

   -0.1084   -0.7389   -0.6650    4.6977
   -0.0015    0.6691   -0.7432    4.5839
    0.9941   -0.0795   -0.0737   -0.8141
         0         0         0    1.0000

>> log(ans, 2)

ans =

    2.0000
    1.0000
   -3.0000
    0.3142
   -0.7854
    0.3491
```

You can also do this by hand as a test of your abilities to actually compute the exponent and logarithm.

Solution 2. The solution to this requires one to program the equations for the exponential into a Matlab fuction. Same for the logarithm. The checks should have been used to verify that the code was correct. Since the orientation part is done independent of the translation part, that would be the first one to work out and verify. Then the equation for the translation (or linear velocity) should be done and verified.

The main details are that the exponent should be a static member function and the logarithm should be a public member function. As a static member function, the exponent function definition would be:

```
function g = exp(xi, t)
```

whereas the logarithm function definition would be

```
function xi = log(g, t)
```

or

```
function xi = log(this, t)
```

depending on how one wants to label the group element argument.

Problem 3. (30 pts) From the piktul manipulator thread, work out Module 1, Adventure 2. At this point, it is just pencil and paper. The following week will be the manipulator implementation. It would probably be best to actually code the answer up in the inverse kinematics member function (which may be called `invkin` or something like that, within the `piktul` class).

Provide the joint angles associated to the following two $SE(2)$ configurations:

$$\left[\begin{array}{cc|c} 0.707 & 0.707 & 3.165 \\ -0.707 & 0.707 & -7.811 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 0.259 & -0.966 & 7.328 \\ 0.966 & 0.259 & 2.828 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right] \tag{9}$$

with the respective heights, 0.5 and 0.25. You can always test the output by applying your forward kinematics routine to the found joint angles. Or even to use the piktul display code to see if it really works out. You can plot the end-effector frame using the $SE(3)$ class with the proper up-conversion of the desired configurations, or the $SE(2)$ class but the height won't be proper.

Solution 3. This problem is essential an extension or application of the two-link inverse kinematics homework problem to the class manipulator. The two main differences are that the height is also an input, but that is resolved using the first joint α_1 modified according to the totip option. The last part is as hinted in the homework, which is the final joint angle α_4. The equation for α_4 given the other two rotational joint angles α_2 and α_3 was given in the problem statement.

Applying the inverse kinematics should have given the solutions:

$$\alpha_1 = [0.5, -60.8874, -15, 75.1020]^T \quad \text{and} \quad \alpha_2 = [0.25, 42.2085, -45, 4.1005]^T, \tag{10}$$

or the second equivalent solution for each one,

$$\alpha_1' = [0.5, -75, 15, 59.2146]^T \quad \text{and} \quad \alpha_2' = [0.25, 0, 45, -43.6910]^T, \tag{11}$$

or a mix. They don't include the gripper width and were computed using 4.5 and 4 inches as the two link lengths and going to the tip. If going to the mid-region of hand, then the first joint would be 0.5 more inches (1 and 0.75, respectively for the two solutions).

The equally important part was the member function definition,

```
function alpha = inversekin(this, gdes, z, totip, solfact)
```

where the first argument is the class instance, the second argument `gdes` is the desired end-effector $SE(2)$ configuration, the third argument is the desired end-effector height, the fourth argument specifies whether to go to the tip or to be centered in the gripper region (default if not given or is empty is `true`), and the fifth argument is the solution to return (the go left solution or the go right solution, if you will). this part might not have been done now, though it is probably best to have done so. In the homework that follows, it is essential to have this generic function written.

Problem 3. (30 pts) For the turtlebot groups, work out the first two enumerated items in the “Turtlebot: Sensing Part 1” module. Turn in the finite state machine associated to the bumper logic, as well as the code associated to the finite state machine. Naturally, you should also demo the final product.

Solution 3. Again, you should have received feedback directly from one of us, as well as submitted the answers to the ”Adventure” and ”Explore” questions. Hopefully the person reviewing the demo also asked questions regarding the implementation.

The important part is getting the finite state machine done properly, then having a state variable that cycled between the states based on triggers. After that working it into the code and properly entering the infinite loop would have been the next major issue to resolve. There are a few ways to do these things, so no solution will be provided.

Problem 3. (30 pts) For the biped track, work out Module Set 1, Part 2. The adventure explores the center of mass calculations for the robot. These will be important because the center of mass plays a big role is assessing the static stability (e.g. balance) of the robot. Ideally, the center of mass is always over the support region defined by the feet in contact with the ground. Submission details will be at the wiki page.

Solution 3. There is no solution here. You’ve got your group contact who will provide the feedback necessary to correct and advance. For sure, I would recommend getting in the habit of demoing a little early to course correct should there be minor details that were missed. The better state things are in at the official/final demo, the easier it will be to proceed.

Problem 3. (30 pts)**.** If you are working on a group project, then your group contact will review the submissions associated to the prior tasks and provide a new set of tasks. You should work out whatever baby steps this group contact assigns the group as opposed to the lab problem. It is your responsibility to turn in the results to the group contact. There should be sufficient output for them to assess progress.

Solution 3. For those in the group project, this step involved accomplishing what was requested, then turning in some kind of document or video as demonstration that the learning tasks were accomplished. The expectation is that the group response should be similar in details as the solutions to the homeworks are, which means that not only should they communicate accomplishments, but they should also convey the overall procedure involved.