ECE4560 - Solution #3

Problem 1. [20 pts] Figure 1 depicts a three-link rotational, planar manipulator. Let the link lengths be $l_1 = 1$, $l_2 = \frac{1}{2}$, and $l_3 = \frac{1}{4}$.

(a) Work out the 2D forward kinematics for the manipulator. This should be a function from the three joint angles to planar coordinates (the output is only the point coordinates in $\mathbb{E}(2)$ with orientation ignored). The full symbolic answer should be worked out.

(b) Given the joint angles $\alpha = (-\frac{\pi}{12}, \frac{\pi}{6}, -\frac{2\pi}{3})^T$, what is the end-effector position (x_e, y_e)?

(c) Work out the (manipulator) Jacobian for the function from part (a).

(d) Using the joint angles from part (b), and the joint velocities $\dot{\alpha} = (\frac{1}{6}, -\frac{1}{2}, \frac{1}{4})^T$, what is the end-effector velocity?

There should be a function called `planarR3_display` that can be used to visualize the manipulator. It is available through the class wiki page. You can even plot the vector in the figure using the `quiver` function in Matlab. Given a set of base points and an associated set of vector coordinates, the `quiver` function will plot the vectors at their associated base points. In this example case, the forward kinematics position would be the base point, and the end-effector velocity would be the vector.

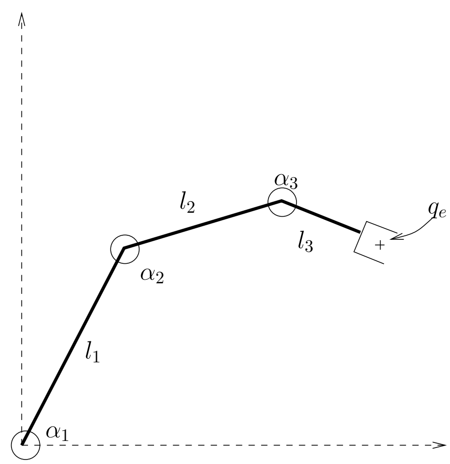

Figure 1: Planar 3R Manipulator.

Solution 1.

1. The forward kinematics for this case is performed as in class, by accumulating the rotations to generate the displacement vector from joint to joint,

$$\begin{aligned} q_e(\alpha) &= \left\{ \begin{array}{c} l_1 \cos\alpha_1 + l_2\cos(\alpha_1+\alpha_2) + l_3\cos(\alpha_1+\alpha_2+\alpha_3) \\ l_1 \sin\alpha_1 + l_2\sin(\alpha_1+\alpha_2) + l_3\sin(\alpha_1+\alpha_2+\alpha_3) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \cos\alpha_1 + \frac{1}{2}\cos(\alpha_1+\alpha_2) + \frac{1}{4}\cos(\alpha_1+\alpha_2+\alpha_3) \\ \sin\alpha_1 + \frac{1}{2}\sin(\alpha_1+\alpha_2) + \frac{1}{4}\sin(\alpha_1+\alpha_2+\alpha_3) \end{array} \right\} \end{aligned}$$

2. OK, so now it is just a matter of plugging in the joint angle and seeing what results

$$\begin{aligned} q_e(\alpha) &= \left\{ \begin{array}{c} \cos\left(-\frac{\pi}{12}\right) + \frac{1}{2}\cos\left(\frac{\pi}{12}\right) + \frac{1}{4}\cos\left(-\frac{7\pi}{12}\right) \\ \sin\left(\frac{\pi}{12}\right) + \frac{1}{2}\sin\left(\frac{-\pi}{12}\right) + \frac{1}{4}\sin\left(\frac{-7\pi}{12}\right) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right) + \frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right) - \frac{1}{16}\left(\sqrt{6}-\sqrt{2}\right) \\ \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right) - \frac{1}{8}\left(\sqrt{6}-\sqrt{2}\right) - \frac{1}{16}\left(\sqrt{6}+\sqrt{2}\right) \end{array} \right\} = \left\{ \begin{array}{c} 1.526 \\ 0.547 \end{array} \right\} \end{aligned}$$

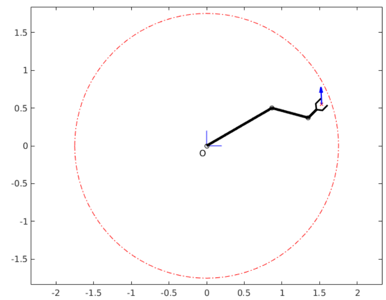

Figure 2: Depiction of the manipulator for the given configuration and also a plot of the end-effector motion vector with the given joint velocities.

3. The manipulator Jacobian is obtained by simply computing the Jacobian of the forward kinematics function,

$$J(\alpha) = \begin{bmatrix} -l_1 \sin\alpha_1 - l_2 \sin(\alpha_1+\alpha_2) - l_3 \sin(\alpha_1+\alpha_2+\alpha_3) & \cdots \\ l_1 \cos\alpha_1 + l_2 \cos(\alpha_1+\alpha_2) + l_3 \cos(\alpha_1+\alpha_2+\alpha_3) & \cdots \end{bmatrix}$$
$$\begin{matrix} -l_2 \sin(\alpha_1+\alpha_2) - l_3 \sin(\alpha_1+\alpha_2+\alpha_3) & -l_3 \sin(\alpha_1+\alpha_2+\alpha_3) \\ l_2 \cos(\alpha_1+\alpha_2) + l_3 \cos(\alpha_1+\alpha_2+\alpha_3) & l_3 \cos(\alpha_1+\alpha_2+\alpha_3) \end{matrix}\Bigg] .$$

4. The velocity of the end effector for the given joint angles and joint angular velocities is determined by

$$\dot{q}_e = J(\alpha)\cdot\dot{\alpha} = \begin{bmatrix} -0.371 & 0.129 & -0.177 \\ 1.349 & 0.483 & 0.177 \end{bmatrix} \cdot \begin{Bmatrix} 0.200 \\ 0.040 \\ -0.333 \end{Bmatrix} = \begin{Bmatrix} -0.010 \\ 0.230 \end{Bmatrix}$$

A depiction of the results can be found in Figure 2. The red plus is at the manipulator end-effector position, and the blue arrow is the direction of travel of the end-effector given the joint velocities.

Problem 2. [15 pts] Every planar transformation going from one frame to another is equivalent to a pure rotation about a unique point in the plane called the *pole*, see Figure 3. Basically, the pole is the point in the plane whose coordinates do not change when the point is rigidly transformed by g. Let q_p denote the location of the pole.

(a) First off, take the above statements and write down what they mean as an equation. In particular, what is the mathematical equation associated to the english of the second sentence? I neglected to include the super- and sub-scripts for the frames. Fill those in properly in your equation. (There are actually two equivalent interpretations, I just want one of them)

(b) If the planar transformation is given symbolically by $g = (\vec{d}, R)$, find the location of the pole symbolically as a function of $\vec{d}$ and R. In what frame did you compute the location of the pole?

(c) Suppose that the initial configuration of the object was $g^{\mathcal{O}}_{\mathcal{A}} = (7.0, 2.0, -3\pi/4)$ and the final configuration was $g^{\mathcal{O}}_{\mathcal{B}} = (0, 8.0, \pi/2)$. Where is the pole located? In what frame did you find the location of the pole?

Solution 2. To solve this, let's follow the description paragraph. First off, being a pole of the transformation g means that the location of the pole does not change under the transformation. Now, what does the transformation represent?

If it represents a rigid displacement as described in the problem, then being a pole means that the point does not move even though the entire body moves. This gives $q^{\mathcal{B}}_{p'} = q^{\mathcal{B}}_{p}$, where p corresponds to the original location and p' to the new location after being transformed. The same idea can be applied to the group element g if it represents a change of frame. In the change of frame case, then being a pole means that the coordinate representation of the point stays the same when changing coordinates from one frame to the next, $q^{\mathcal{B}}_{p} = q^{\mathcal{B}'}_{p}$. In both cases, the relationship takes a similar form. Let's work with the second form of the equation and work it out.

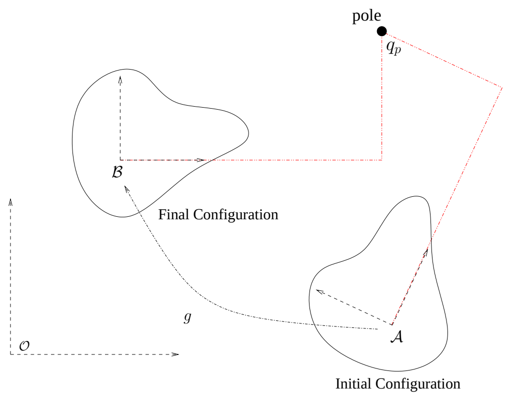

Figure 3: Pole of a planar transformation.

(a) From the English description of the pole to the mathematics we know that the relationship between the two coordinate descriptions should be $q_p^{\mathcal{B}} = g_{\mathcal{B}'}^{\mathcal{B}}\, q_p^{\mathcal{B}'}$, leading to the equation

$$q_p^{\mathcal{B}'} = g_{\mathcal{B}'}^{\mathcal{B}}\, q_p^{\mathcal{B}'}$$

for the pole. An equivalent description would have been to use $q_p^{\mathcal{B}}$ on both sides.

Had we used the rigid transformation pole, then the connection between the two coordinates would have been $q_{p'}^{\mathcal{A}} = g\, q_p^{\mathcal{A}}$, which has the same equality as above. Hence the pole computation would be the same.

Another way would be to note that since the point does not move, the following must hold:

$$g_{\mathcal{B}}^{\mathcal{O}} q^{\mathcal{B}} = g_{\mathcal{B}'}^{\mathcal{O}} q^{\mathcal{B}'}$$

which implies that

$$q^{\mathcal{B}} = g_{\mathcal{O}}^{\mathcal{B}} g_{\mathcal{B}'}^{\mathcal{O}} q^{\mathcal{B}'} = g_{\mathcal{B}'}^{\mathcal{B}} q^{\mathcal{B}'}$$

at which point we've arrived at the equation above. Following the same logic leads to the same conclusion.

(b) Since we know $g_{\mathcal{B}'}^{\mathcal{B}}$ but we don't know where q_p is located, we can use the above equation to actually solve for q_p. Using $g_{\mathcal{B}'}^{\mathcal{B}} = (d, R)$,

$$\begin{aligned} q_p^{\mathcal{B}'} &= (d, R) \cdot q_p^{\mathcal{B}'} \\ p^{\mathcal{B}'} &= d + R p^{\mathcal{B}'} \\ p^{\mathcal{B}'} &= (\mathbb{1} - R)^{-1} d, \end{aligned}$$

where $\mathbb{1}$ is the identity matrix. In homogeneous coordinates, this goes as follows,

$$\begin{aligned} \left\{ \begin{array}{c} q_p^{\mathcal{B}'} \\ 1 \end{array} \right\} &= \left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array} \right] \left\{ \begin{array}{c} q_p^{\mathcal{B}'} \\ 1 \end{array} \right\} \\ &= \left\{ \begin{array}{c} d + R q_p^{\mathcal{B}'} \\ 1 \end{array} \right\} \\ &= \left\{ \begin{array}{c} (\mathbb{1} - R)^{-1} d \\ 1 \end{array} \right\}. \end{aligned}$$

This gives us the pole in Frame $\mathcal{B}'$, but since $q_p^{\mathcal{B}} = q_p^{\mathcal{B}'}$, it also is the pole in Frame $\mathcal{B}$. Either response is acceptable. This is the perspective with regards to the coordinate frame description of g.

Had you done this with regards to the rigid transformation perspective, where the transformation maps the rigid body from aligning with $\mathcal{A}$ to aligning with frame $\mathcal{B}$, then the frame for both coordinate descriptions is $\mathcal{A}$, since it is the only frame used.

(c) Now, given the two configurations, we would like to figure out the pole associated to the transformation from the first configuration to the second. This means we need to compute what the transformation is,

$$g_{\mathcal{B}}^{\mathcal{A}} = g_{\mathcal{O}}^{\mathcal{A}} g_{\mathcal{B}}^{\mathcal{O}} = \left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}}^{\mathcal{O}}.$$

This one will be worked out in homogeneous form. Using the given values for the initial and final configurations,

$$\begin{aligned}
g_{\mathcal{B}}^{\mathcal{A}} &= \left[\begin{array}{c|c} R(-3\pi/4) & \left\{\begin{array}{c} 7 \\ 2 \end{array}\right\} \\ \hline 0 & 1 \end{array}\right]^{-1} \cdot \left[\begin{array}{c|c} R(\pi/2) & \left\{\begin{array}{c} 0 \\ 8 \end{array}\right\} \\ \hline 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} R(3\pi/4) & -R(3\pi/4)\left\{\begin{array}{c} 7 \\ 2 \end{array}\right\} \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} R(\pi/2) & \left\{\begin{array}{c} 0 \\ 8 \end{array}\right\} \\ \hline 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} R(5\pi/4) & R(3\pi/4)\left\{\begin{array}{c} -7+0 \\ -2+8 \end{array}\right\} \\ \hline 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{c|c} R(5\pi/4) & \begin{array}{c} \frac{\sqrt{2}}{2}(7-6) \\ \frac{\sqrt{2}}{2}(-7-6) \end{array} \\ \hline 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} -0.707 & 0.707 & 0.707 \\ -0.707 & -0.707 & -9.192 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]
\end{aligned}$$

The configuration $g_{\mathcal{B}}^{\mathcal{A}}$ goes from $\mathcal{A}$ to $\mathcal{B}$. The pole associated to this point in frame $\mathcal{B}$ is

$$q_p^{\mathcal{B}} = \left(\mathbb{1} - R(5\pi/4)\right)^{-1} \left\{\begin{array}{c} \sqrt{2}/2 \\ -13\sqrt{2}/2 \end{array}\right\} = \left\{\begin{array}{c} -1.5503 \\ -4.7426 \end{array}\right\}$$

The best way to code this up is using the $SE(2)$ class that you've been programming for the past few homeworks. You could have even programmed a function called pole that would have computed the pole given an $SE(2)$ element.

Problem 3. [10 pts] Consider the planar manipulator depicted in Figure 4. It has four rotational joint variables.

(a) What is the end-effector configuration of the following manipulator (symbolically), as a function of the angles $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$? (Recall vector form is (x, y, θ) form).

(b) How would you write it down as a kinematic chain, e.g., as a product of Lie group operations? (Use homogeneous coordinates.)

(c) What is the end-effector's configuration for $\alpha_1 = \pi$, $\alpha_2 = \pi/8$, $\alpha_3 = -\pi/4$, $\alpha_4 = \pi/8$, given that $l_1 = 1$, $l_2 = 0.75$, and $l_3 = 0.75$. (Acceptable in any representation/form)

Solution 3.

(a) The configuration of the end-effector in vector form is:

$$\left\{\begin{array}{c} l_1\cos(\alpha_1) + l_2\cos(\alpha_1+\alpha_2) + l_3\cos(\alpha_1+\alpha_2+\alpha_3) \\ l_1\sin(\alpha_1) + l_2\sin(\alpha_1+\alpha_2) + l_3\sin(\alpha_1+\alpha_2+\alpha_3) \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 \end{array}\right\}.$$

Of course anything that gives an equivalent solution is fine.

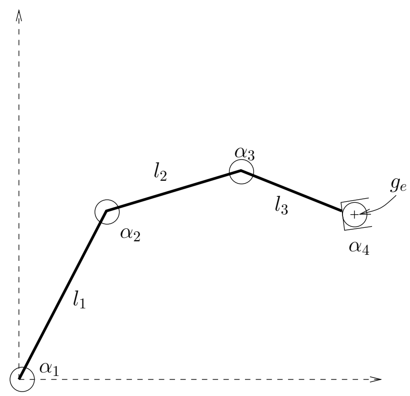

Figure 4: Problem 3 manipulator

(b) As a kinematic chain,

$$\left[\begin{array}{c|c} R(\alpha_1) & \vec{0} \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} R(\alpha_2) & \vec{d}_1 \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} R(\alpha_3) & \vec{d}_2 \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} R(\alpha_4) & \vec{d}_3 \\ \hline 0 & 1 \end{array}\right],$$

where

$$\vec{d}_1 = \left\{ \begin{array}{c} l_1 \\ 0 \end{array} \right\}, \quad \vec{d}_2 = \left\{ \begin{array}{c} l_2 \\ 0 \end{array} \right\}, \quad \text{and} \quad \vec{d}_3 = \left\{ \begin{array}{c} l_3 \\ 0 \end{array} \right\},$$

Working out the products results in,

$$g_e = \left[\begin{array}{c|c} R(\alpha_1+\alpha_2+\alpha_3+\alpha_4) & R(\alpha_1)\vec{d}_1 + R(\alpha_1+\alpha_2)\vec{d}_2 + R(\alpha_1+\alpha_2+\alpha_3)\vec{d}_3 \\ \hline 0 & 1 \end{array}\right],$$

which agrees with what we gave in part (a).

(c) Plugging the values into part (a), we get

$$g_e = \left\{ \begin{array}{c} 1\cos(\pi) + 0.75\cos(\pi+\pi/8) + 0.75\cos(\pi+\pi/8-\pi/4) \\ 1\sin(\pi) + 0.75\sin(\pi+\pi/8) + 0.75\sin(\pi+\pi/8-\pi/4) \\ \hline \pi+\pi/8-\pi/4+\pi/8 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cos(\pi) + 0.75\cos(9\pi/8) + 0.75\cos(7\pi/8) \\ \sin(\pi) + 0.75\sin(\pi/3) + 0.75\sin(\pi/12) \\ \hline \pi \end{array} \right\} = \left\{ \begin{array}{c} -2.3858 \\ 0.0000 \\ \hline -3.1416 \end{array} \right\}$$

As a homogeneous matrix, we let the Matlab code do the work for us, by typing

```
gm1 = SE2([0;0],  pi);
gm2 = SE2([1;0],  pi/8);
gm3 = SE2([0.75;0], -pi/4);
gm4 = SE2([0.75;0],  pi/8);

ge = gm1*gm2*gm3*gm4;
ge
```

and arrive at the homogeneous form of the solution a

$$g_e = \left[\begin{array}{c|c} R(\pi) & \begin{array}{c} -2.3858 \\ 0.0000 \end{array} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} -1.0000 & 0.0000 & -2.3858 \\ -0.0000 & -1.0000 & 0.0000 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]$$

Either answer is acceptable. To verify that the answer is correct, I used the following Matlab code:

```
figure(1);
  planarR4([ pi ; pi/8 ;  -pi/4 ; pi/8] , [1 ; 0.75 ; 0.75], 0.125);
  hold on;
  plot(ge,[],'b',[0.5]);
  hold off;
```

which resulted in Figure 5.

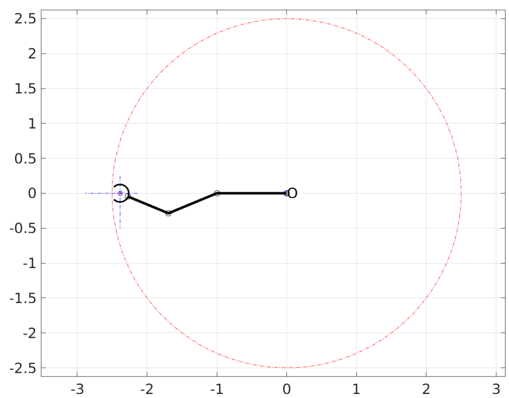

Figure 5: Verification of end-effector configuration.

Problem 4. [10 pts] Now is the time to start modifying the empty or incomplete functions in the SE2 Matlab class. Using what is known about $SE(2)$ and its operations, complete the functions (they should be inv, leftact, and mtimes). Note that this will define two product operations in Matlab, `*` and `.*` plus inversion. The two operations correspond to the `frame * frame` and `frame * point` operations (as well as the other operations of the matrix from class that have the same structure). Matlab nicely allows us to define these things so that the coded math follows our written math. How nice!

Verify that your coded functions work by showing that, given

$$g_1 = \left(\left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\}, R(\pi/3) \right) \quad \text{and} \quad g_2 = \left(\left\{ \begin{array}{c} -2 \\ 1 \end{array} \right\}, R(\pi/6) \right),$$

the code returns

$$g_1^{-1} = \left(\left\{ \begin{array}{c} -2.23 \\ -0.13 \end{array} \right\}, R(-\pi/3) \right), \quad \text{and} \quad g_1 * g_2 = \left(\left\{ \begin{array}{c} -0.87 \\ 0.77 \end{array} \right\}, R(\pi/2) \right).$$

In practice, this code can be used to verify or compute all of the work from the previous problems, and even to visualize the entire set of solutions. Using the code, also verify your work from Problem 3. If you used the code to solve Problem 3, then you cannot really verify without a second way to compute the answer. In that case, work it out by hand to verify that your code is correct with regards to the point transformations.

Representation: The code is configured to use the Real matrix homogeneous form, and is what has historically been used. However, the Complex matrix homogeneous form works just as well. Decide which of the two you want to use, then go ahead and make the code consistent with that formulation. For those choosing the complex form, the external use will still operate normally, it is just that internally complex numbers will be used. The whole point of classes is to abstract the underlying representation and keep that somewhat secret from the user perspective. The Complex version would be funner to implement, but the course will mostly cover the Real version with some discussion of the Complex equivalent after the Real derivations are done.

Solution 4. The code, copied from my files without the comments, is:

```
function invg = inverse(g)
invM = inv(g.M);

invg = SE2( invM(1:2,3) , invM(1:2, 1:2) );
end


function g = mtimes(g1, g2)

M = g1.M * g2.M;

g = SE2( M(1:2, 3), M(1:2, 1:2) );
end


function p2 = leftact(g, p)
p2 = g.M(1:2, :) * [p;1];
end
```

With the above functions defined, the following code

```
g1 = SE2([1;2], pi/3);
g2 = SE2([-2;1], pi/6);

ginv = inv(g1)
g3 = g1*g2
```

gives

```
ginv =

    0.5000    0.8660   -2.2321
   -0.8660    0.5000   -0.1340
         0         0    1.0000


g3 =

    0.0000   -1.0000   -0.8660
    1.0000    0.0000    0.7679
         0         0    1.0000
```

Now, there is the code associated to verifying the results of this homework. It will be done to show a neat trick that Matlab allows, which is the creation of inline functions. Each of the transformations for Problem 3 will be done that way.

```
>>   g1 = @(th1) SE2([0;0], th1);
>>   g2 = @(th2) SE2([1;0], th2);
>>   g3 = @(th3) SE2([0.75;0], th3);
>>   g4 = @(th4) SE2([0.75;0], th4);
```

```
>>   g = @(alpha)g1(alpha(1)) * g2(alpha(2)) * g3(alpha(3)) * g4(alpha(4));

>>   g([pi; pi/8; -pi/4; pi/8])

ans =

   -1.0000    0.0000   -2.3858
   -0.0000   -1.0000    0.0000
         0         0    1.0000
```

which outputs the same values as in the solutions, and that's pretty much it for the homework.