ECE4560 - Solution #2

Consider Figure 1 below. Suppose that the configurations associated with the two reference frames depicted were $g_{\mathcal{A}}^{\mathcal{O}} = (7, 4, 45°)$ and $g_{\mathcal{B}}^{\mathcal{O}} = (2, 7, 90°)$.

Problem 1. [10 pts] What are $g_{\mathcal{A}}^{\mathcal{O}}$ and $g_{\mathcal{B}}^{\mathcal{O}}$ in homogeneous representation?

Solution 1 In homogeneous representation, $g_{\mathcal{A}}^{\mathcal{O}}$ is

$$g_{\mathcal{A}}^{\mathcal{O}} = \left[\begin{array}{c|c} R(\pi/4) & \vec{d}_{\mathcal{O}A}^{\mathcal{O}} \\ \hline 0 & 1 \end{array}\right], = \left[\begin{array}{cc|c} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 7 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right], = \left[\begin{array}{cc|c} 0.7071 & -0.7071 & 7 \\ 0.7071 & 0.7071 & 4 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right],$$

and $g_{\mathcal{B}}^{\mathcal{O}}$ is

$$g_{\mathcal{B}}^{\mathcal{O}} = \left[\begin{array}{c|c} R(\pi/2) & \vec{d}_{\mathcal{O}B}^{\mathcal{O}} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0 & -1 & 2 \\ 1 & 0 & 7 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right].$$

In complex homogeneous form, they would be

$$g_{\mathcal{A}}^{\mathcal{O}} = \left[\begin{array}{c|c} 0.7071 + j0.7071 & 7 + j4 \\ \hline 0 & 1 \end{array}\right] \quad \text{and} \quad g_{\mathcal{B}}^{\mathcal{O}} = \left[\begin{array}{c|c} j & 2 + j7 \\ \hline 0 & 1 \end{array}\right]. \tag{1}$$

Note:*Though the next set of solutions could have more easily been done in homogeneous coordinates, they will be done in $(\vec{d}, R)$ notation. Mostly so you get used to the idea of a special product operation between different left/right pairs. It is important to understand that the meaning of the binary product really depends on what the two arguments are and what meaning is attached to them.*

Problem 2. [10 pts] Suppose the point q_1 in $\mathcal{O}$'s frame is given by $q_1^{\mathcal{O}} = (2, 2)^T$. Where is it in frame $\mathcal{A}$? Where is it in frame $\mathcal{B}$?

Solution 2 To figure out $q_1^{\mathcal{A}}$ and $q_1^{\mathcal{B}}$, just follow the blue arrows appropriately to get there (going in opposite direction to an arrow signifies inversion),

$$q_1^{\mathcal{A}} = g_{\mathcal{O}}^{\mathcal{A}} q_1^{\mathcal{O}} = \left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} q_1^{\mathcal{O}} \quad \text{and} \quad q_1^{\mathcal{B}} = g_{\mathcal{O}}^{\mathcal{B}} q_1^{\mathcal{O}} = \left(g_{\mathcal{B}}^{\mathcal{O}}\right)^{-1} q_1^{\mathcal{O}}.$$

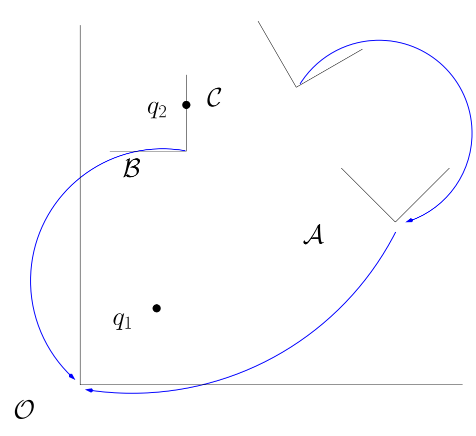


Figure 1: Coordinates frames and points.

Thus, the solution requires inverting the two given transformations,

$$\left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} = \left(\vec{d}_{\mathcal{O}\mathcal{A}}^{\mathcal{O}}, R(\pi/4)\right)^{-1} = \left(-R^{-1}(\pi/4)\vec{d}_{\mathcal{O}\mathcal{A}}^{\mathcal{O}}, R^{-1}(\pi/4)\right)$$
$$= \left(-\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\left\{ \begin{matrix} 7 \\ 4 \end{matrix} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\right) = \left(\left\{ \begin{matrix} -\frac{11\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{matrix} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\right)$$
$$= \left(\left\{ \begin{matrix} -7.7782 \\ 2.1213 \end{matrix} \right\}, \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}\right)$$

and

$$\left(g_{\mathcal{B}}^{\mathcal{O}}\right)^{-1} = \left(\vec{d}_{\mathcal{O}\mathcal{B}}^{\mathcal{O}}, R(\pi/2)\right)^{-1} = \left(-R^{-1}(\pi/2)\vec{d}_{\mathcal{O}\mathcal{B}}^{\mathcal{O}}, R^{-1}(\pi/2)\right)$$
$$= \left(-\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left\{ \begin{matrix} 2 \\ 7 \end{matrix} \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \left(\left\{ \begin{matrix} -7 \\ 2 \end{matrix} \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

Using the computed inverses and taking the appropriate product against the point $q_1^{\mathcal{O}}$ gives the answers,

$$q_1^{\mathcal{A}} = \left\{ \begin{matrix} -\frac{7\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{matrix} \right\} = \left\{ \begin{matrix} -4.9497 \\ 2.1213 \end{matrix} \right\} \qquad \text{and} \qquad q_1^{\mathcal{B}} = \left\{ \begin{matrix} -5 \\ 0 \end{matrix} \right\}.$$

Problem 3. [10 pts] Suppose the point q_2 in $\mathcal{B}$'s frame is given by $q_2^{\mathcal{B}} = (1, 0)^T$. Where is it in frame $\mathcal{O}$? Where is it in frame $\mathcal{A}$?

Solution 3. The solution to this problem is pretty much the same as for problem one except for the fact that the coordinate frames are different.

$$q_2^{\mathcal{O}} = g_{\mathcal{B}}^{\mathcal{O}} q_2^{\mathcal{B}} = \left(\left\{ \begin{matrix} 2 \\ 7 \end{matrix} \right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 8 \end{matrix} \right\}$$
$$q_2^{\mathcal{A}} = g_{\mathcal{B}}^{\mathcal{A}} q_2^{\mathcal{B}} = \left(\left\{ \begin{matrix} -\sqrt{2} \\ 4\sqrt{2} \end{matrix} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right)\left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} -\frac{\sqrt{2}}{2} \\ \frac{9\sqrt{2}}{2} \end{matrix} \right\} = \left\{ \begin{matrix} -0.7071 \\ 6.3640 \end{matrix} \right\}$$

where the transformation $g_{\mathcal{B}}^{\mathcal{A}}$ is taken from a later solution.

Problem 4. [10 pts] What is $g_{\mathcal{B}}^{\mathcal{A}}$ (frame $\mathcal{B}$'s configuration relative to frame $\mathcal{A}$)?

Solution 4. To obtain $g_{\mathcal{B}}^{\mathcal{A}}$ given what we do know, it's just a matter of following whatever path we can along the blue arrows. From the figure, we see that this can be done as follows:

$$g_{\mathcal{B}}^{\mathcal{A}} = g_{\mathcal{O}}^{\mathcal{A}} g_{\mathcal{B}}^{\mathcal{O}} = \left(g_{\mathcal{A}}^{\mathcal{O}}\right)^{-1} g_{\mathcal{B}}^{\mathcal{O}} = \left(\left\{ \begin{matrix} -\frac{11\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{matrix} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right)\left(\left\{ \begin{matrix} 2 \\ 7 \end{matrix} \right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \left(\left\{ \begin{matrix} -\sqrt{2} \\ 4\sqrt{2} \end{matrix} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right) = \left(\left\{ \begin{matrix} -1.4142 \\ 5.6569 \end{matrix} \right\} \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}\right)$$

Problem 5. [10 pts] Let coordinate frame $\mathcal{C}$ have the configuration $g_{\mathcal{C}}^{\mathcal{A}} = (1, 4, -15°)$ relative to frame $\mathcal{A}$. Then what are $g_{\mathcal{C}}^{\mathcal{O}}$ and $g_{\mathcal{C}}^{\mathcal{B}}$?

Solution 5. The solution to this is simply a matter of multiplication of the transformations,

$$
\begin{aligned}
g_{\mathcal{C}}^{\mathcal{O}} = g_{\mathcal{A}}^{\mathcal{O}} g_{\mathcal{C}}^{\mathcal{A}} &= \left(\left\{ \begin{array}{c} 7 \\ 4 \end{array} \right\}, \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \right) \left(\left\{ \begin{array}{c} 1 \\ 4 \end{array} \right\}, \begin{bmatrix} \cos(\pi/12) & \sin(\pi/12) \\ -\sin(\pi/12) & \cos(\pi/12) \end{bmatrix} \right) \\
&= \left(\left\{ \begin{array}{c} 7 - \frac{3\sqrt{2}}{2} \\ 4 + \frac{5\sqrt{2}}{2} \end{array} \right\}, \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \right) = \left(\left\{ \begin{array}{c} 4.8787 \\ 7.5355 \end{array} \right\}, \begin{bmatrix} 0.8660 & -0.5 \\ 0.5 & 0.8660 \end{bmatrix} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
g_{\mathcal{C}}^{\mathcal{B}} = g_{\mathcal{A}}^{\mathcal{B}} g_{\mathcal{C}}^{\mathcal{A}} = \left(g_{\mathcal{B}}^{\mathcal{A}}\right)^{-1} g_{\mathcal{C}}^{\mathcal{A}} &= \left(\left\{ \begin{array}{c} -\frac{11\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{array} \right\}, \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \right)^{-1} \left(\left\{ \begin{array}{c} 1 \\ 4 \end{array} \right\}, \begin{bmatrix} \cos(\pi/12) & \sin(\pi/12) \\ -\sin(\pi/12) & \cos(\pi/12) \end{bmatrix} \right) \\
&= \left(\left\{ \begin{array}{c} -3 \\ -5 \end{array} \right\}, \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \right) \left(\left\{ \begin{array}{c} 1 \\ 4 \end{array} \right\}, \begin{bmatrix} \cos(\pi/12) & \sin(\pi/12) \\ -\sin(\pi/12) & \cos(\pi/12) \end{bmatrix} \right) \\
&= \left(\left\{ \begin{array}{c} -3 + \frac{5\sqrt{2}}{2} \\ -5 + \frac{3\sqrt{2}}{2} \end{array} \right\}, \begin{bmatrix} \cos(4\pi/12) & \sin(4\pi/12) \\ -\sin(4\pi/12) & \cos(4\pi/12) \end{bmatrix} \right) \\
&= \left(\left\{ \begin{array}{c} -3 + \frac{5\sqrt{2}}{2} \\ -5 + \frac{3\sqrt{2}}{2} \end{array} \right\}, \begin{bmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \right) \\
&= \left(\left\{ \begin{array}{c} 0.5355 \\ -2.8787 \end{array} \right\}, \begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix} \right)
\end{aligned}
$$

A visualization of the solutions from the perspective of the different frames, $\mathcal{O}$, $\mathcal{A}$, $\mathcal{B}$, is depicted in the figure below.

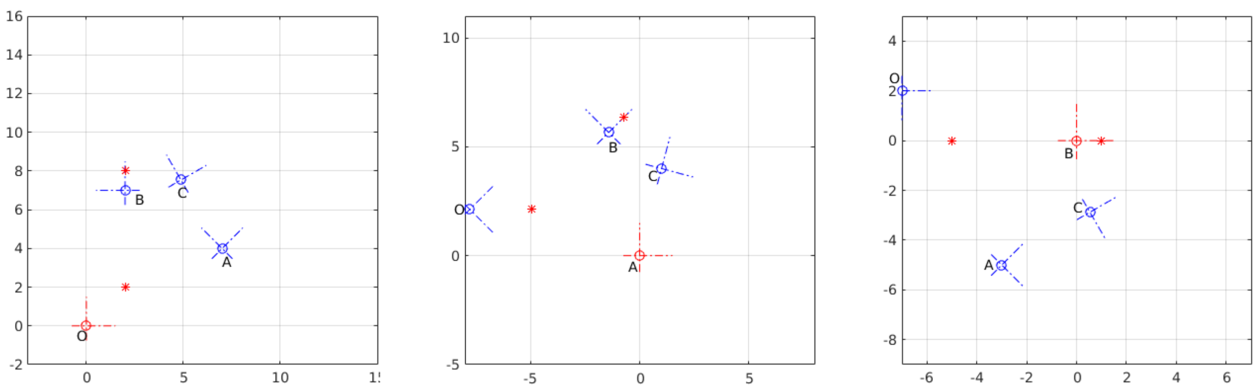


Figure 2: Depiction of the world from the perspective of the three main frames involved, $\mathcal{O}$, $\mathcal{A}$, and $\mathcal{B}$.

Problem 6. [20 pts] Figure 3 depicts a two-link rotational, planar manipulator. It is a standard example manipulator used to convey concepts simply and will figure in class. This problem examines the geometry associated with the manipulator's hand (or end-effector) position as a function of the free joint variables. In this case, these two variables are angular variables that rotate. The two joint angles, α_1 and α_2, are depicted in Figure 3. They are free to vary, wheras the link lengths l_1 and l_2 are fixed. This problem also explores the calculus of the hand as a function of the joint variables through the Jacobian.

Let the link lengths be $l_1 = 1$ and $l_2 = \frac{1}{2}$.

(a) Given the joint angles $\alpha = (\frac{\pi}{2}, -\frac{\pi}{3})^T$, what is the end-effector position (x_e, y_e)? Provide as a function $q_e : \mathbb{R}^2 \to \mathbb{R}^2$.

Regarding the notation: the q is interpreted to be the Euclidean position and the $\cdot_e$ subscript stands for end-effector.

(b) What is the Jacobian of this function? (i.e., Dq_e?)

(b) Using the Jacobian, the joint angles from part (a), and the joint velocities $\dot{\alpha} = (-\frac{1}{5}, \frac{1}{2})^T$, what is the end-effector velocity?

(c) Find a joint configuration α that causes the manipulator Jacobian to no longer be invertible. Describe this joint configuration if possible.

File for Problem 6. There should be a Matlab function called `planarR2_display` in the class wiki for visualizing the manipulator, along with some instructions.

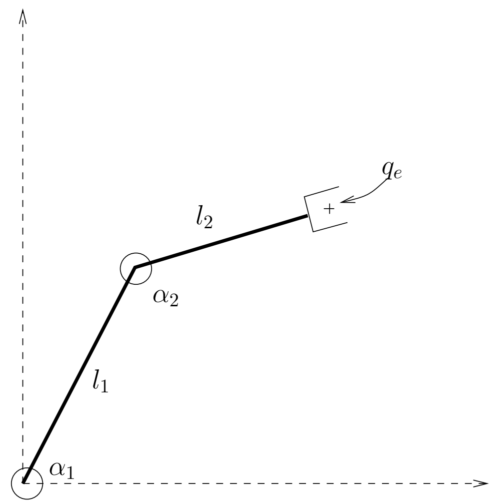


Figure 3: Planar 2R Manipulator.

Solution 6. Treating each link as a vector that aligns with the prior link's x-axis when its direction (angle) is zero, perform vector addition to get the position of the end-effector,

$$q_e(\alpha) = \left\{ \begin{array}{c} l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1 + \alpha_2) \\ l_1 \sin(\alpha_1) + l_2 \sin(\alpha_1 + \alpha_2) \end{array} \right\}$$

Computing the differential (or Jacobian) gives,

$$J(\alpha) = \begin{bmatrix} -l_1 \sin(\alpha_1) - l_2 \sin(\alpha_1 + \alpha_2) & -l_2 \sin(\alpha_1 + \alpha_2) \\ l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1 + \alpha_2) & +l_2 \cos(\alpha_1 + \alpha_2) \end{bmatrix}.$$

Note: *Some of you may have measured the angle in the classical way, which would have been via the complement. Admittedly, the figure is ambigous and should be revised. Those who did it the classical way should have obtained a solution involved $(\alpha_2 - \pi)$ of $(\alpha_2 - 180)$ rather than just α_2. This solution is acceptable for now, but in the future expect the zero angle to align with the previous $x - axis$ frame. Also, it is better to use radians since any functions I provide will also use radians and not degrees. The mismatch in units can lead to problems down the road. If your solution has a π shift, then it will differ slightly from that below.*

1. Using the given link lengths and joint angles, the end-effector position is

$$q_e(\alpha) = \left\{ \begin{array}{c} \cos(\frac{\pi}{2}) + \frac{1}{2}\cos(\frac{\pi}{6}) \\ \sin(\frac{\pi}{2}) + \frac{1}{2}\sin(\frac{\pi}{6}) \end{array} \right\} = \left\{ \begin{array}{c} 0.433 \\ 1.250 \end{array} \right\}$$

2. Using the Jacobian to get the end-effector velocity leads to

$$v_e = J(\alpha) \cdot \dot{\alpha} = \begin{bmatrix} -1.2500 & -0.2500 \\ 0.4330 & 0.4330 \end{bmatrix} \cdot \left\{ \begin{array}{c} -0.2 \\ 0.5 \end{array} \right\} = \left\{ \begin{array}{c} 0.125 \\ 0.130 \end{array} \right\}$$

3. We'll go over this in the future, but anything that drops the rank of the Jacobian or causes the determinant to vanish works. If $\alpha_2 = 0$, then it does not matter what α_1 is, because the determinant will vanish. This can be seen because the determinant of the Jacobian is:

$$\det(J(\alpha)) = l_1 l_2 \sin(\alpha_2).$$

The problem asked for *a* solution, so if you just found one such case where it happened, that's fine. For example $\alpha = (0,0)^T$ is a perfectly valid answer, as would any other version of the form $\alpha = (0, \alpha_2)$.

Problem 7. [20 pts] Consider again the two-link manipulator of Figure 3. At some point, it will be necessary to generate a trajectory for manipulators to follow. Let's start with a simple problem. Using the code stubs on t-square, integrate the differential equation

$$\dot{\alpha}_1 = \frac{1}{3}\cos(t)$$
$$\dot{\alpha}_2 = -\frac{1}{4}\sin(t)$$

given the initial joint angles $\alpha(0) = (-\frac{3\pi}{2}, \frac{\pi}{6})^T$.

(a) Plot the joint angles versus time, e.g., α_i versus t.

(b) Using the forward kinematics, plot the end-effector configuration versus time, e.g., x versus t and y versus t. You might want to write a function called `fkin2R` that computes this for you so that it can be easy to do (takes in the α and spits out the q_e). Even better would be to vectorize the function so that a matrix of multiple joint angles can be passed in, and a matrix of end-effector positions gets returned.

(c) Do a parametric plot of the end-effector position, e.g., $x(t)$ and $y(t)$ together.

You can use the `planar2R_display` function to visualize the results as an animation (the code loop is very similar to the animation code I sent for the Hilare robot in the last homework). Use the same link lengths as the earlier homework problem.

Solution 7.

1. The code to perform the integration is pretty much the same as that from Homework 1. Here is the differential equation function:

```
function adot = f(t, a)

adot = zeros(2,1);
adot(1) = (1/3)*cos(t);
adot(2) = -(1/4)*sin(t)];

end
```

Not too bad. Then the goal is to numerically integrate it in Matlab using `ode45` with the given initial conditions. I chose to integrate for about 8 seconds, to get a better sense for what was happening. Any longer and the thing start to look loopy. The plot is given in Figure 4(a).

2. Using the forward kinematics to map the joint angles to end-effector positions gives the positions over time. The plot is given in Figure 4(b).

3. Then using the planar manipulator plot function with the `hold` function in Matlab, as per

```
figure;
    planarR2_display(alpha(:,1), ll);
hold on;
planarR2_display(alpha(:,end), ll);
plot(qtraj(1,:), qtraj(2,:), 'b');
plot(qtraj(1,1), qtraj(2,1), 'rx','MarkerSize',15);
plot(qtraj(1,end), qtraj(2,end), 'r*','MarkerSize',15);
hold off;
```

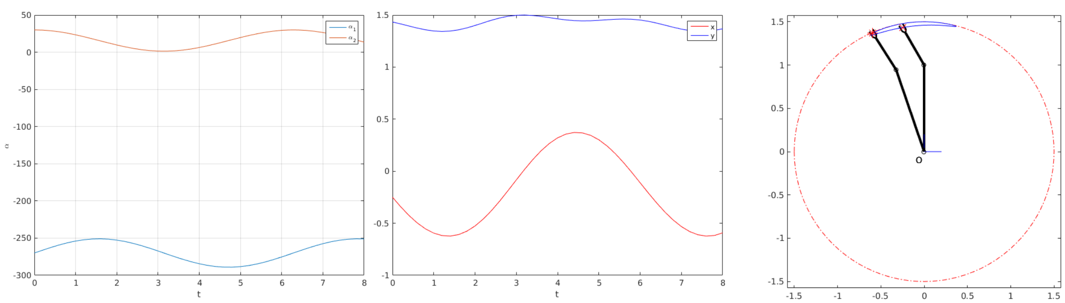
plots the trajectory parametrically and also the beginning and end configurations of the manipulator. The plot for this homework problem is given in Figure 4(c). To better see things, the beginning position is marked with a red x and the end position is marked with a red star.

Because the two functions in the differential equation of periodic, we see that the output of the numerical integration is also periodic. The shape of the end-effector position trajectory is some bent oval type shape.

To generate an animation, I used the following code:

```
figure(4); clf;

dt = diff(t);
dt = [dt; dt(end)];
for ii=1:size(alpha,2)
  planarR2_display(alpha(:,ii),ll);
  drawnow;
  pause(dt(ii));
end
```



(a) α vs t. (b) q_e vs t. (c) Parametric plot.

Figure 4: Plots for the two-link planar manipulator.

Problem 8. [10 pts] This problem relates to the earlier hammer questions (previous homework). Using the Matlab SE2 class provided on t-square, plot a visual representation of the frames $\mathcal{O}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and the three points $q_{\mathcal{A}}$, $q_{\mathcal{B}}$, and $q_{\mathcal{C}}$ all with respect to the observer frame $\mathcal{O}$. Also add the nudged hammer position to the figure (should have been Problem 7 in the previous homework), but label it differently than the other points (see `help plot`) to figure out how to change the point label.

If you have any problems with the code, please try to make sense of the Matlab error(s) and ask your fellow classmates prior to asking me. This is my weird way of encouraging dialog between students. Also, it is always best to see me in person for code questions, as e-mail is not the best media for resolving those kinds of issues. Trust me, I can't even help people over the phone, so e-mail is even worse.

The `SE2.m` file provides the class definition you need for the plotting. Use of the `SE2` class is very similar to classes in most other languages. We will explore this code and add to it over the course of the semester. An example is:

```
g = SE2();
figure(1);
g.plot();
```

I leave the rest up to you to figure out. There should be enough documentation in the code to do so. You'll have to make use of Matlabs `hold` function to plot all of the frames and points.

File for Problem 8. Should be found on class wiki at the "Matlab Class Stubs" link/page.

Solution 8. Now, assuming that the @SE2 directory is in your path so that Matlab knows it exists, the following code should result in Figure 5:

```
gOO = SE2([0 ;0], 0);
gOA = SE2([5 ;12], pi/3)
gOB = SE2([2; -1], pi)

gOC = SE2([5.7321 ; 9.2679] , pi/2);

qOA = [5.5 ; 12.866];
qOB = [1, -1];
qOC = [5.7321; 10.2679];
qOAm = [5.9848 ; 12.1737];

figure(1);
  clf;
  plot(gOO,'O','r');
  hold on;
    plot(gOA,'A');
    plot(gOB,'B');
    plot(gOC,'C','c');

    plot(qOA(1), qOA(2), 'b*', 'MarkerSize', 8);
    plot(qOB(1), qOB(2), 'b*', 'MarkerSize', 8);
    plot(qOC(1), qOC(2), 'b*', 'MarkerSize', 8);
    plot(qOAm(1), qOAm(2), 'm*', 'MarkerSize', 8);

  hold off;
  axis equal;
  axis([-5 10 -5 15]);
```

Note that the new hammer head point for the $\mathcal{A}$ frame is given by the magenta asterisk.

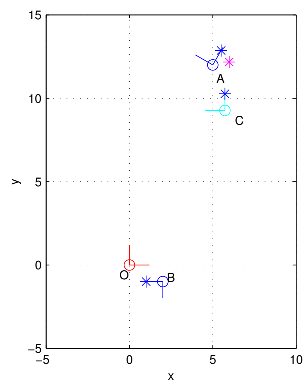


Figure 5: Visualization of solutions.