ECE4560 - Homework #1

Due: Aug. 21, 2017

The homework has two different problem categories. The first two problems are little primer or refresher problems, one on coordinates and one on Matlab. The second also serves to make sure that everyone has Matlab up and running. The rest is the beginning material of the course. You'll note that the way the frames are described differs a bit. Look at the notes to see what the different representations are (in short, there is vector form, translation + rotation form, and homogeneous form). They all describe the same thing, a coordinate frame or a displacement, depending on the problem.

I recommend doing as much of it as possible in Matlab, then using Matlab's publish option to generate the homework solutions. This should lead to a pdf file with both the code and the solution, for a clean homework submission.

Problem 1. [10 pts] The purpose of this problem is to cover some basic trigonometry. In this problem, we are examining displacements of an object. Recall that the rotation of a point $p = (p_1, p_2)^T$ through the rotation angle θ (in radians) is given by:

$$p'_1 = \cos(\theta)p_1 - \sin(\theta)p_2$$
$$p'_2 = \sin(\theta)p_1 + \cos(\theta)p_2$$

Consider Figure 1 below, where a hammer is depicted in two different configurations. The reference point and vectors of each hammer configuration is depicted in the figure. Suppose that in its unrotated state, and with the base of the hammer at the origin, the hammer head is located at $(6, 0)$. The figure depicts the hammer translated and also rotated relative to the origin at $\mathcal{O}$. What are the new hammer head points for the following rotation and translation information?

(a) The base is translated by $(10, 25)$ to $\mathcal{A}$, and the hammer undergoes a rotation of $\pi/6$ radians?

(b) The base is translated by $(21, 7)$ to $\mathcal{B}$, and the hammer undergoes a rotation of π radians?

Note that it makes a difference when you do the rotation and when you do the translation. One will give sensible results that agree with the figure. The other will give non-sensical results. What is the proper order?

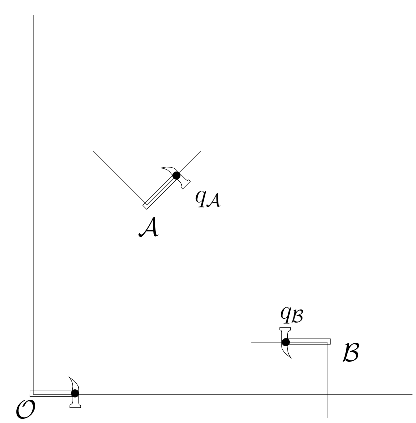


Figure 1: Rigid body displacements and point displacements.

Solution 1.

(a) First one should apply the rotation to the hammer head coordinates to get the new coordinates due to the rotation, then translate the by applying the translation to the coordinates.

$$q_A = \left\{ \begin{array}{c} 10 \\ 25 \end{array} \right\} + \left\{ \begin{array}{c} \cos(\pi/6)6 \\ \sin(\pi/6) \end{array} \right\} = \left\{ \begin{array}{c} 10 \\ 25 \end{array} \right\} + \left\{ \begin{array}{c} 3\sqrt{3} \\ 3 \end{array} \right\} = \left\{ \begin{array}{c} 15.2 \\ 28 \end{array} \right\}$$

(b) Applying the same idea to the other setup for the hammer gives

$$q_B = \begin{Bmatrix} 21 \\ 7 \end{Bmatrix} + \begin{Bmatrix} -6 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 7 \end{Bmatrix}$$

How to describe it is a little bit subject to interpretation, but the important thing to mention is that the rotation applies to the original hammer head coordinates $(6, 0)$. Here I described it as rotating the original coordinates then translating them.

Another possible solution would be to state that the translation must be applied in the desired coordinate frame, and the rotation must convert from the old frame to the desired frame. Thus for consistent application of the two coordinate systems, then original point must first be converted into the desired frame, then the translation applied (in the desired frame). Applying the translation first would result in addition of elements in two distinct coordinate systems (since the translation was given in the desired coordinate system).

Problem 2. [10 pts]

a. Numerically integrate the following differential equation using the code stub found on t-square:

$$\dot{x} = -2.5x + 0.75u, \qquad \text{where } x(0) = 1,$$

for $u(t) = \sin(t)$ and $u(t) = 8e^{-t}$. The current code works as though there were no input. Your job is to add the input part to the differential equation and numerically integrate. Hand in (i) a print-out of the modified code, and (ii) a plot of the system response versus time. Try to describe what happens with the response of the system to each of the two signals.

b. Integrate the following differential equation using the code stub found on t-square:

$$\begin{aligned} \dot{x} &= \cos(\theta)u(t), \\ \dot{y} &= \sin(\theta)u(t), \\ \dot{\theta} &= v(t). \end{aligned}$$

where $x(0) = 0, y(0) = 0, \theta(0) = 0$. Simulate for both **(i)** $u(t) = 0.2\sin(t)$ and $v(t) = 0.2$ and **(ii)** $u(t) = 0.5\cos(t)$ and $v(t) = 0.3\sin(t)$. Hand in (i) a print-out of the code, and (ii) a plot of the system response versus time. Try to describe what happens with the response of the system to each of the two signals.

To integrate this, you will have to vectorize the equations and define $x_1 = x$, $x_2 = y$, and $x_3 = \theta$ in Matlab. The stub sort of hints at how to do that.

File for Problem 2: odeProb.zip

Solution 2.

a. The code to do that is

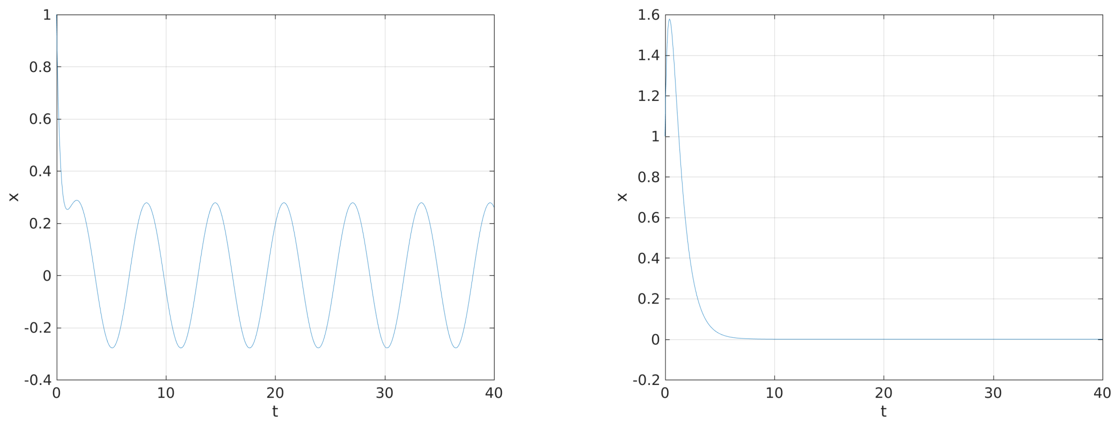
```
function xdot = f(t, x)

u = sin(t);
xdot = -2.5*x + 0.75*u;

end
```

Running it with the input above leads to Figure 2(a). Changing the input to be $8e^{-t}$ leads to Figure 2(b). In one case, the output stays oscillatory while, in the other case it dies down to zero. It looks like the output is the same as the input but with a lower amplitude and possibly also a phase shift.

b. Following the steps discussed, the code ends up being



(a) $u(t) = \sin(t)$ (b) $u(t) = 4e^{-t}$

Figure 2: Figures for Problem 2a.

```
function xdot = f(t, x)

u = 0.2*sin(t);
v = 0.2;

xdot(1) = cos(x(3))*u;
xdot(2) = sin(x(3))*u;
xdot(3) = v;

xdot = transpose(xdot);

end
```

For this set of inputs u and v, the motion is forwards and backwards with some rotation. Essentially the object is turning as it rocks forward and backwards. It does not stray far from the origin.

For the second set of inputs, the motion seems to be somewhat sideways (or uni-directional), as the object changes more along the y than the x. In both x and θ it seems to periodically return to the starting values, or near them. There is some diagonal drift.

Preface: The next set of problems are all related. Consider Figure **??** below, where a hammer is depicted in two different configurations. The reference frame of each hammer configuration specifies the position and orientation of the tip of the handle. Suppose that the configurations associated with the two reference frames depicted were $g^{\mathcal{O}}_{\mathcal{A}} = (5, 12, \pi/3)$ and $g^{\mathcal{O}}_{\mathcal{B}} = (2, -1, \pi)$, with $\mathcal{O}$ being the observer frame. We are interested in knowing where the location of the head is, supposing that the head is at $q = (1, 0)$ relative to the reference frame of the hammer.

Problem 3. [5 pts] What are $g^{\mathcal{O}}_{\mathcal{A}}$ and $g^{\mathcal{O}}_{\mathcal{B}}$ in (d, R) representation? What about in complex $(z_T, z_R)^T$ representation?

Solution 1. Conversion to $(\vec{d}, R)$ representation is a matter of bundling the position coordinates and generating a rotation matrix from the angle,

$$(x, y, \theta) \mapsto \left(\left\{ \begin{array}{c} x \\ y \end{array} \right\}, R(\theta) \right) = \left(\left\{ \begin{array}{c} x \\ y \end{array} \right\}, \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \right)$$

Consequently,

$$g^{\mathcal{O}}_{\mathcal{A}} = \left(\left\{ \begin{array}{c} 5 \\ 12 \end{array} \right\}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \right) \quad \text{and} \quad g^{\mathcal{O}}_{\mathcal{B}} = \left(\left\{ \begin{array}{c} 2 \\ -1 \end{array} \right\}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right).$$

Problem 4. [5 pts] Given the hammer is at configuration $\mathcal{A}$, where is the head located at? (e.g. what is $q^{\mathcal{O}}_{\mathcal{A}}$?)

Solution 4. Based on how points transform, we know that

$$q^{\mathcal{O}}_{\mathcal{A}} = \vec{d}^{\mathcal{O}}_{\mathcal{A}} + R^{\mathcal{O}}_{\mathcal{A}}\, q^{\mathcal{A}}_{\mathcal{A}}.$$

Since $q^{\mathcal{A}}_{\mathcal{A}}$ refers to the location of the point on the hammer at $\mathcal{A}$ in the reference frame of the hammer at $\mathcal{A}$, we know that $q^{\mathcal{A}}_{\mathcal{A}} = q$ coordinate-wise. Therefore,

$$q^{\mathcal{O}}_{\mathcal{A}} = \left\{\begin{array}{c} 5 \\ 12 \end{array}\right\} + \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}. = \left\{\begin{array}{c} 5 + \frac{1}{2} \\ 12 + \frac{\sqrt{3}}{2} \end{array}\right\} = \left\{\begin{array}{c} 5.5 \\ 12.866 \end{array}\right\}.$$

Problem 5. [5 pts] Given the hammer is at configuration $\mathcal{B}$, where is the head located at? (e.g. what is $q^{\mathcal{O}}_{\mathcal{B}}$?)

Solution 3. The same holds for this solution as for the prior one,

$$q^{\mathcal{O}}_{\mathcal{B}} = \vec{d}^{\mathcal{O}}_{\mathcal{B}} + R^{\mathcal{O}}_{\mathcal{B}}\, q^{\mathcal{B}}_{\mathcal{B}}. = \left\{\begin{array}{c} 2 \\ -1 \end{array}\right\} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 1 \\ -1 \end{array}\right\}.$$

Problem 6. [10 pts] Suppose that the hammer was moved from its configuration at $\mathcal{A}$ to yet another configuration, denoted by $\mathcal{C}$. Further, you are told that the new location, relative to frame $\mathcal{A}$, is $(-2, -2, \pi/6)$. What is $g^{\mathcal{O}}_{\mathcal{C}}$ and where is the head now located with respect to the observer frame (e.g., what is $q^{\mathcal{O}}_{\mathcal{C}}$)?

Solution 4. The first part of the problem is figuring out what $g^{\mathcal{O}}_{\mathcal{C}}$ should be. This requires computing $g^{\mathcal{O}}_{\mathcal{C}}$ through composition as per class. The solution to that is

$$\begin{aligned}
g^{\mathcal{O}}_{\mathcal{C}} &= g^{\mathcal{O}}_{\mathcal{A}} * g^{\mathcal{A}}_{\mathcal{C}} \\
&= (d^{\mathcal{O}}_{\mathcal{A}}, R^{\mathcal{O}}_{\mathcal{A}}) * (d^{\mathcal{A}}_{\mathcal{C}}, R^{\mathcal{A}}_{\mathcal{C}}) \\
&= (d^{\mathcal{O}}_{\mathcal{A}} + R^{\mathcal{O}}_{\mathcal{A}}\, d^{\mathcal{A}}_{\mathcal{C}}, R^{\mathcal{O}}_{\mathcal{A}}\, R^{\mathcal{A}}_{\mathcal{C}}) \\
&= \left(\left\{\begin{array}{c} 5 \\ 12 \end{array}\right\} + \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left\{\begin{array}{c} -2 \\ -2 \end{array}\right\}, \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \right) \qquad (1) \\
&= \left(\left\{\begin{array}{c} 4 + \sqrt{3} \\ 12 - \sqrt{3} \end{array}\right\}, R(\frac{\pi}{2}) \right) \\
&= \left(\left\{\begin{array}{c} 5.7321 \\ 9.2679 \end{array}\right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right).
\end{aligned}$$

Composition is achieved through the $*$ operation applied to the two configurations, $g^{\mathcal{O}}_{\mathcal{A}}$ and $g^{\mathcal{A}}_{\mathcal{C}}$. Once we have the configuration of the composition, we can get the coordinate location of the hammer head.

Given the transformation $g^{\mathcal{O}}_{\mathcal{C}}$, we can map points from frame $\mathcal{C}$ into frame $\mathcal{O}$.

$$q^{\mathcal{O}}_{\mathcal{C}} = d^{\mathcal{O}}_{\mathcal{C}} + R^{\mathcal{O}}_{\mathcal{C}}\, q^{\mathcal{C}}_{\mathcal{C}},$$

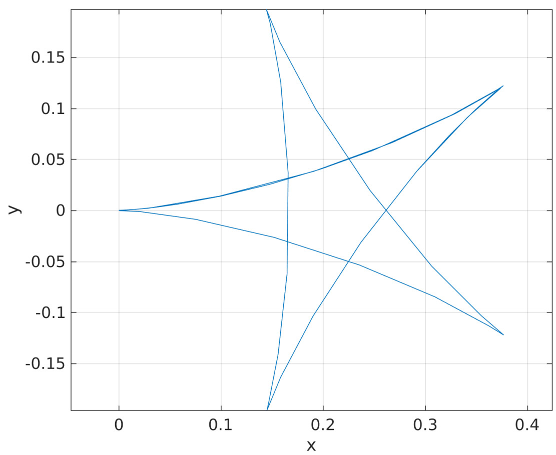
which, after substitution, gives

$$q^{\mathcal{O}}_{\mathcal{C}} = \left\{\begin{array}{c} 5.7321 \\ 9.2679 \end{array}\right\} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 5.7321 \\ 10.2679 \end{array}\right\}$$

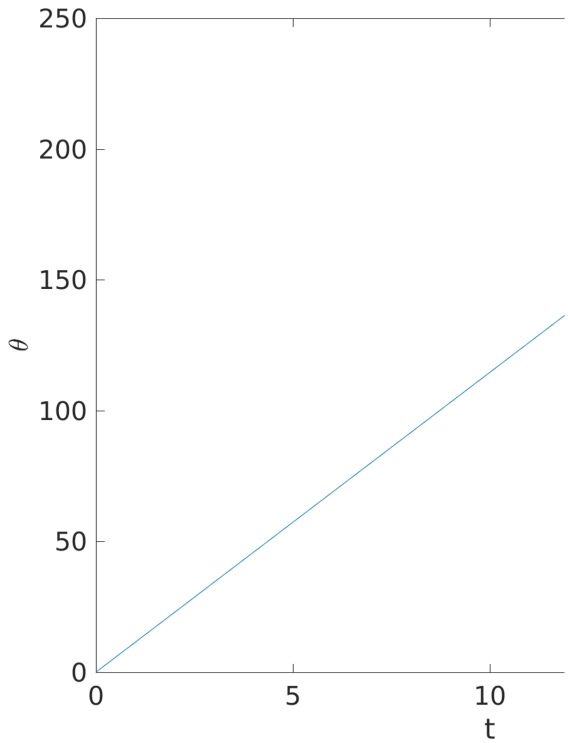
Problem 7. [10 pts] Suppose that someone accidentally nudges the hammer associated to frame $\mathcal{A}$. In particular, the head of the hammer rotates a bit in the frame. If the hammer head is now located in frame $\mathcal{A}$ at $(0.6428, -0.7660)$ and the base is at the origin as before, where is the hammer head located in frame $\mathcal{O}$?

Solution 7. The main piece of information is that $q_{\mathcal{A}}^{\mathcal{A}} = (0.6428, 0.7660)$. Repeating the computation from Problem 2 with the new information,

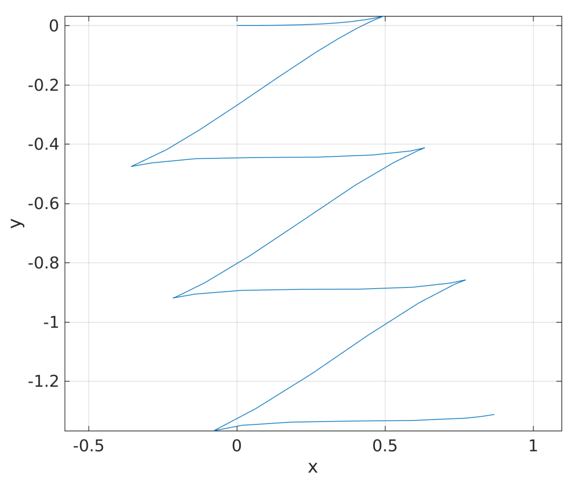
$$q_{\mathcal{A}}^{\mathcal{O}} = \left\{ \begin{array}{c} 5 \\ 12 \end{array} \right\} + \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left\{ \begin{array}{c} 0.6428 \\ -0.7660 \end{array} \right\} = \left\{ \begin{array}{c} 5 + 0.9848 \\ 12 + 0.1737 \end{array} \right\} = \left\{ \begin{array}{c} 5.9848 \\ 12.1737 \end{array} \right\}.$$



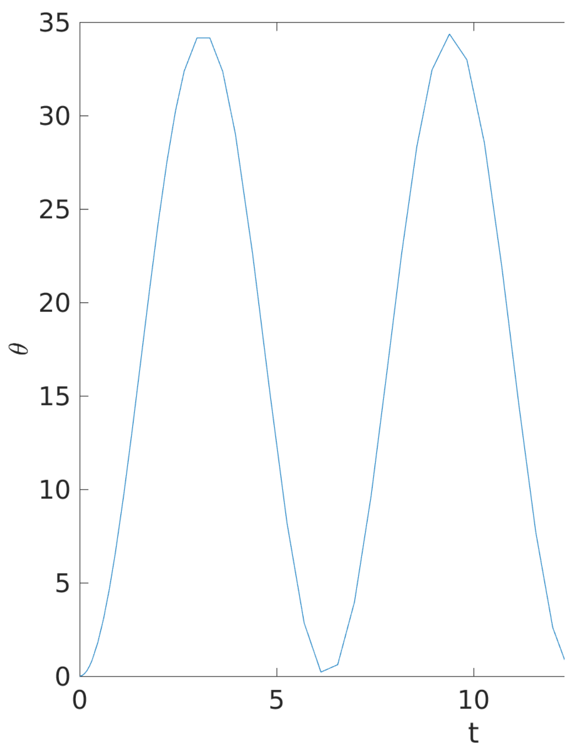
(a) First input ($y(t)$ vs. $x(t)$).



(b) First input (θ vs. t).

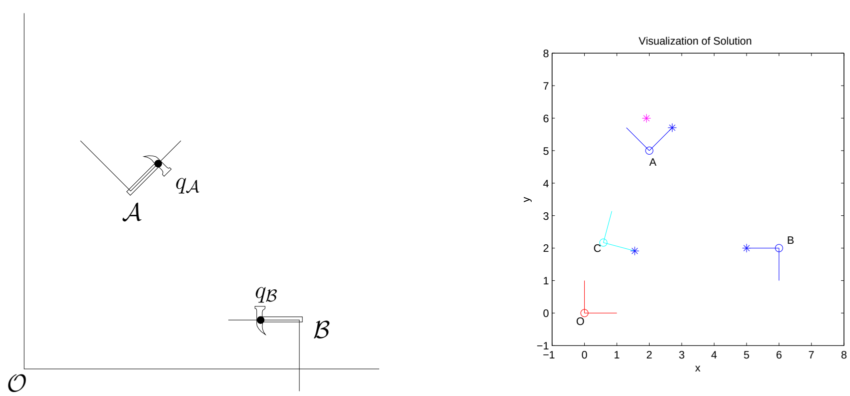


(c) Second input ($y(t)$ vs. $x(t)$).



(d) Second input (θ vs. t).

Figure 3: Figures for Problem 2b.



(a) Coordinates: Frames and Points.

(b) Sample Solution.