Exercise: Compute the Discrete Fourier Transform (DFT) of the sequence $\langle 1,2,3,4,5,6,7,8\rangle$ by using the FFT algorithm.

Solution: recall that the DFT of a sequence $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ is just the sequence of the values $\langle P(\omega_n^0), P(\omega_n), P(\omega_n^2), \ldots, P(\omega_n^{n-1})\rangle$ of the associated polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1}$.

In our case the polynomial is $P(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7$, and the roos of unity are $\omega_8^0, \omega_8^1, \omega_8^2, \omega_8^3, \ldots, \omega_8^7$.

Note that $\omega_8 = \mathrm{e}^{i\,\frac{2\pi}{8}} = \mathrm{e}^{i\,\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$. We now apply "divide and conquer"strategy, by splitting the polynomial into even and odd powers:

$$\begin{aligned}P(x) &= (1+3x^2+5x^4+7x^6) + (2x+4x^3+6x^5+8x^7)\\ &= (1+3x^2+5x^4+7x^6) + x(2+4x^2+6x^4+8x^6)\\ &= (1+3x^2+5(x^2)^2+7(x^2)^3) + x(2+4x^2+6(x^2)^2+8(x^2)^3)\end{aligned}$$

So we have to evaluate all $P(\omega_8^k)$, for $k = 0..7$, i.e.,

$$\begin{aligned}P(\omega_8^k) &= (1+3(\omega_8^k)^2+5((\omega_8^k)^2)^2+7((\omega_8^k)^2)^3) + \omega_8^k(2+4(\omega_8^k)^2+6((\omega_8^k)^2)^2+8((\omega_8^k)^2)^3)\\ &= (1+3\omega_4^k+5(\omega_4^k)^2+7(\omega_4^k)^3) + \omega_8^k(2+4\omega_4^k+6(\omega_4^k)^2+8(\omega_4^k)^3)\end{aligned}$$

because $(\omega_8^k)^2 = \omega_4^k$. Note that the two polynomials are of half the degree of the original polynomial, and that they are evaluated at roots of unity of order 4, and there are only 4 such, i.e., only half the number of roots of unity of order 8. Thus, we have reduced a problem of size 8 to two problems of size 4, plus 8 multiplications with numbers ω_8^k.

We further simplify the problem by noticing that for $k = 4,5,6,7$ we have $\omega_8^k = \omega_8^{4+m}$ for $m = 0,1,2,3$. We then get

$$\omega_4^k = \omega_4^{4+m} = \omega_4^4\omega_4^m = \omega_4^m,$$

and also

$$\omega_8^k = \omega_8^{4+m} = \omega_8^4\omega_8^m = \omega_2\omega_8^m = -\omega_8^m.$$

Thus we need to find

$$P(\omega_8^k) = (1 + 3\omega_4^k + 5(\omega_4^k)^2 + 7(\omega_4^k)^3) + \omega_8^k(2 + 4\omega_4^k + 6(\omega_4^k)^2 + 8(\omega_4^k)^3)$$

for $k = 0, 1, 2, 3$ and for $k = 4, 5, 6, 7$ we replace k with $4 + m$ for $m = 0, 1, 2, 3$ and obtain

$$\begin{aligned}
P(\omega_8^k) &= (1 + 3\omega_4^k + 5(\omega_4^k)^2 + 7(\omega_4^k)^3) + \omega_8^k(2 + 4\omega_4^k + 6(\omega_4^k)^2 + 8(\omega_4^k)^3) \\
&= (1 + 3\omega_4^{4+m} + 5(\omega_4^{4+m})^2 + 7(\omega_4^{4+m})^3) + \omega_8^{4+m}(2 + 4\omega_4^{4+m} + 6(\omega_4^{4+m})^2 + 8(\omega_4^{4+m})^3) \\
&= (1 + 3\omega_4^m + 5(\omega_4^m)^2 + 7(\omega_4^m)^3) - \omega_8^m(2 + 4\omega_4^m + 6(\omega_4^m)^2 + 8(\omega_4^m)^3)
\end{aligned}$$

Since now both k and m range only from 0 to 3, we can use the same variable and write

$$P(\omega_8^k) = (1 + 3\omega_4^k + 5(\omega_4^k)^2 + 7(\omega_4^k)^3) + \omega_8^k(2 + 4\omega_4^k + 6(\omega_4^k)^2 + 8(\omega_4^k)^3) \tag{1}$$
$$P(\omega_8^{4+k}) = (1 + 3\omega_4^k + 5(\omega_4^k)^2 + 7(\omega_4^k)^3) - \omega_8^k(2 + 4\omega_4^k + 6(\omega_4^k)^2 + 8(\omega_4^k)^3) \tag{2}$$

with k ranging only from 0 to 3.
We now evaluate the two polynomials, $P_0(y) = 1 + 3y + 5y^2 + 7y^3$ and $P_1(y) = 2 + 4y + 6y^2 + 8y^3$ again by splitting them into even and odd powers. We obtain

$$P_0(y) = (1 + 5y^2) + (3y + 7y^3) = (1 + 5y^2) + y(3 + 7y^2)$$

$$P_1(y) = (2 + 6y^2) + (4y + 8y^3 = (2 + 6y^2) + y(4 + 8y^2)$$

and we have to evaluate them at $y = \omega_4^k$ for $k = 0..3$. Just as above, we can let k range only from 0 to 1, and for $k = 3..4$ we let $k = 2 + m$ where $m = 0, 1$. Thus we get for $k = 0, 1$ and $m = 0, 1$,

$$\begin{aligned}
P_0(\omega_4^k) &= (1 + 5(\omega_4^k)^2) + \omega_2^k(3 + 7(\omega_4^k)^2) \\
&= (1 + 5\omega_2^k) + \omega_4^k(3 + 7\omega_2^k)) \\
P_0(\omega_4^{2+m}) &= (1 + 5(\omega_4^{2+m})^2) + \omega_4^{2+m}(3 + 7(\omega_4^{2+m})^2) \\
&= (1 + 5\omega_2^m) - \omega_4^m(3 + 7\omega_2^m)) \\
P_1(\omega_4^k) &= (2 + 6(\omega_4^k)^2) + \omega_2^k(4 + 8(\omega_4^k)^2) \\
&= (2 + 6\omega_2^k) + \omega_4^k(4 + 8\omega_2^k)) \\
P_1(\omega_4^{2+m}) &= (2 + 6(\omega_4^{2+m})^2) + \omega_4^{2+m}(4 + 8(\omega_4^{2+m})^2) \\
&= (2 + 6\omega_2^m) - \omega_4^m(4 + 8\omega_2^m))
\end{aligned}$$

Using the same variable $k = 0, 1$ we see that we have to evaluate:

$$P_0(\omega_4^k) = (1 + 5\omega_2^k) + \omega_4^k(3 + 7\omega_2^k)) \tag{3}$$
$$P_0(\omega_4^{2+k}) = (1 + 5\omega_2^k) - \omega_4^k(3 + 7\omega_2^k)) \tag{4}$$
$$P_1(\omega_4^k) = (2 + 6\omega_2^k) + \omega_4^k(4 + 8\omega_2^k)) \tag{5}$$
$$P_1(\omega_4^{2+k}) = (2 + 6\omega_2^k) - \omega_4^k(4 + 8\omega_2^k)) \tag{6}$$

Note that $\omega_4 = i$ and $\omega_2 = -1$; thus we get

$$P_0(\omega_4^0) = (1 + 5\omega_2^0) + \omega_4^0(3 + 7\omega_2^0)) = (1 + 5) + 1(3 + 7)) = 16 \tag{7}$$
$$P_0(\omega_4^1) = (1 + 5\omega_2^1) + \omega_4^1(3 + 7\omega_2^1)) = (1 - 5) + i(3 - 7)) = -4 - 4\,i \tag{8}$$
$$P_0(\omega_4^{2+0}) = P_0(\omega_4^2) = (1 + 5\omega_2^0) - \omega_4^0(3 + 7\omega_2^0)) = (1 + 5) - 1(3 + 7)) = -4 \tag{9}$$
$$P_0(\omega_4^{2+1}) = P_0(\omega_4^3) = (1 + 5\omega_2^1) - \omega_4^1(3 + 7\omega_2^1)) = (1 - 5) - i(3 - 7)) = -4 + 4i \tag{10}$$
$$P_1(\omega_4^0) = (2 + 6\omega_2^0) + \omega_4^0(4 + 8\omega_2^0)) = (2 + 6) + 1(4 + 8)) = 20 \tag{11}$$
$$P_1(\omega_4^1) = (2 + 6\omega_2^1) + \omega_4^1(4 + 8\omega_2^1)) = (2 - 6) + i(4 - 8)) = -4 - 4\,i \tag{12}$$
$$P_1(\omega_4^{2+0}) = P_1(\omega_4^2) = (2 + 6\omega_2^0) - \omega_4^0(4 + 8\omega_2^0)) = (2 + 6) - 1(4 + 8)) = -4 \tag{13}$$
$$P_1(\omega_4^{2+1}) = P_1(\omega_4^3) = (2 + 6\omega_2^1) - \omega_4^1(4 + 8\omega_2^1)) = (2 - 6) - i(4 - 8)) = -4 + 4\,i \tag{14}$$

Using the facts that:

$$\omega_8^0 = 1 \qquad \omega_8^1 = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2} \qquad \omega_8^2 = \omega_4 = i \qquad \omega_8^3 = -\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$$

we now get:

$$P(\omega_8^0) = P_0(\omega_4^0) + \omega_8^0 \cdot P_1(\omega_4^0) = P_0(\omega_4^0) + P_1(\omega_4^0) \tag{15}$$
$$P(\omega_8^4) = P(\omega_8^{4+0}) = P(\omega_4^0) - \omega_8^0 \cdot P_1(\omega_4^0) = P(\omega_4^0) - P_1(\omega_4^0) \tag{16}$$
$$P(\omega_8^1) = P_0(\omega_4^1) + \omega_8^1 \cdot P_1(\omega_4^1) = P_0(\omega_4^1) + \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) \cdot P_1(\omega_4^1) \tag{17}$$
$$P(\omega_8^5) = P(\omega_8^{4+1}) = P(\omega_4^1) - \omega_1 \cdot P_1(\omega_4^1) = P(\omega_4^1) - \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) P_1(\omega_4^1) \tag{18}$$
$$P(\omega_8^2) = P_0(\omega_4^2) + \omega_8^2 \cdot P_1(\omega_4^2) = P_0(\omega_4^2) + \omega_4 \cdot P_1(\omega_4^2) = P_0(\omega_4^2) + i \cdot P_1(\omega_4^2) \tag{19}$$
$$P(\omega_8^6) = P(\omega_8^{4+2}) = P(\omega_4^2) - \omega_8^2 \cdot P_1(\omega_4^1) = P(\omega_4^2) - \omega_4 \cdot P_1(\omega_4^1) = P(\omega_4^2) - i \cdot P_1(\omega_4^1) \tag{20}$$
$$P(\omega_8^3) = P_0(\omega_4^3) + \omega_8^3 \cdot P_1(\omega_4^3) = P_0(\omega_4^3) + \left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) P_1(\omega_4^3) \tag{21}$$
$$P(\omega_8^7) = P(\omega_8^{4+3}) = P_0(\omega_4^3) - \left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) P_1(\omega_4^3) \tag{22}$$

Finally we substitute 7-14 into 15- 22 and get

$$
\begin{aligned}
P(\omega_8^0) &= 36\\
P(\omega_8^4) &= -4 - 4\,i(1+\sqrt{2})\\
P(\omega_8^1) &= -4 - 4\,i\\
P(\omega_8^5) &= -4 + 4\,i(1-\sqrt{2})\\
P(\omega_8^2) &= -4\\
P(\omega_8^6) &= -4 - 4\,i(1-\sqrt{2})\\
P(\omega_8^3) &= -4 + 4\,i\\
P(\omega_8^7) &= -4 + 4\,i(1+\sqrt{2})
\end{aligned}
$$

Convince yourselves that this sequence of computations is what you get by "running" the FFT program:

function FFT(A)
    $n \leftarrow \text{length}[A]$
    if $n = 1$ **then return** A
    else
        $A^{[0]} \leftarrow (A_0, A_2, \dots A_{n-2})$;
        $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1})$;
        $y^{[0]} \leftarrow FFT(A^{[0]})$;
        $y^{[1]} \leftarrow FFT(A^{[1]})$;
        $\omega_n \leftarrow e^{i\frac{2\pi}{n}}$;
        $\omega \leftarrow 1$;
        for $k = 0$ to $k = \frac{n}{2} - 1$ **do**;
            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
            $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
            $\omega \leftarrow \omega \cdot \omega_n$;
        end for
        return y
    end if
end function