FFT Examples

Let's calculate the Discrete Fourier Transform (DFT) of the sequence $\vec{s} = \langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle$ using the Fast Fourier Transform (FFT).

(Remember, the FFT is just a fast algorithm for computing the DFT; unlike DFT it is not a transform itself!!)

The associated polynomial is

$$P(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7;$$

the $DFT(\vec{s})$ is just the sequence of values

$$DFT(\vec{s}) = \langle P(\omega_8^0), P(\omega_8^1), P(\omega_8^2), P(\omega_8^3), P(\omega_8^4), P(\omega_8^5), P(\omega_8^6), P(\omega_8^7) \rangle$$

Note that we have

$$\omega_8^0 = 1; \qquad \omega_8^1 = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}; \qquad \omega_8^2 = i; \qquad \omega_8^3 = -\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2};$$
$$\omega_8^4 = -1; \qquad \omega_8^5 = -\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}; \qquad \omega_8^6 = -\,i; \qquad \omega_8^7 = \frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2};$$

see Figure 1:

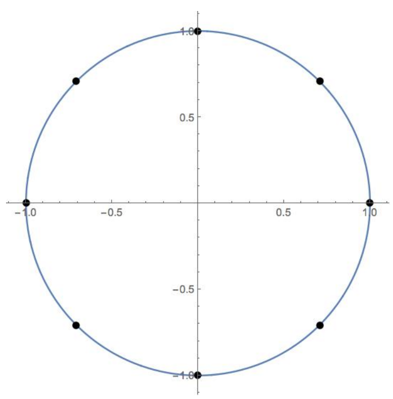

FIGURE 1.

Also,

$$\omega_4^0 = 1; \qquad \omega_4^1 = i; \qquad \omega_4^2 = -1; \qquad \omega_4^3 = -\,i$$

$$\omega_2^0 = 1; \qquad \omega_2^1 = -1.$$

We now proceed with the "Divide-And-Conquer" procedure: letting $y = x^2$ we get

$$\begin{aligned} P(x) &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 \\ &= (1 + 3x^2 + 5x^4 + 7x^6) + x(2 + 4x^2 + 6x^4 + 8x^6) \\ &= (1 + 3y + 5y^2 + 7y^3) + x(2 + 4y + 6y^2 + 8y^3). \end{aligned}$$

We now let

$$P_e(y) = 1 + 3y + 5y^2 + 7y^3; \qquad P_o(y) = 2 + 4y + 6y^2 + 8y^3;$$

Continuing with "Divide-And-Conquer" strategy we get, letting $z = y^2$

$$\begin{aligned} P_e(y) &= (1 + 5y^2) + y(3 + 7y^2); & P_o(y) &= 2 + 6y^2 + y(4 + 8y^2); \\ &= (1 + 5z) + y(3 + 7z); & &= 2 + 6z + y(4 + 8z). \end{aligned}$$

We now let:

$$\begin{aligned} P_{ee}(z) &= 1 + 5z; & P_{eo}(z) &= 3 + 7z; \\ P_{oe}(y) &= 2 + 6z; & P_{oo}(z) &= 4 + 8z. \end{aligned}$$

Note that, as powers of ω_n rotate once around the circle, the powers of the squares $\omega_n^2 = \omega_{\frac{n}{2}}$ rotate twice through only $\frac{n}{2}$ many distinct values; for example, for n=8, using the cancelation Lemma, i.e., $(\omega_n^k)^2 = \omega_{\frac{n}{2}}^k$ we get:

$$
\begin{array}{rcccccccc}
x = & \omega_8^0 & \omega_8^1 & \omega_8^2 & \omega_8^3 & \omega_8^4 & \omega_8^5 & \omega_8^6 & \omega_8^7 \\
x^2 = & (\omega_8^0)^2 & (\omega_8^1)^2 & (\omega_8^2)^2 & (\omega_8^3)^2 & (\omega_8^4)^2 & (\omega_8^5)^2 & (\omega_8^6)^2 & (\omega_8^7)^2 \\
 & \| & \| & \| & \| & \| & \| & \| & \| \\
 & \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & \omega_4^4 & \omega_4^5 & \omega_4^6 & \omega_4^7 \\
 & \| & \| & \| & \| & \| & \| & \| & \| \\
 & \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3
\end{array}
$$

Thus, if

$$P(x) = P^e(x^2) + x\, P^o(x^2) \tag{1}$$

and $P(x)$ is of degree $n-1$, then

$$P^e((\omega_n^k)^2) = P^e((\omega_n^{\frac{n}{2}+k})^2) = P^e(\omega_{\frac{n}{2}}^k); \tag{2}$$

$$P^o((\omega_n^k)^2) = P^o((\omega_n^{\frac{n}{2}+k})^2) = P^o(\omega_{\frac{n}{2}}^k); \tag{3}$$

$$\omega_n^{\frac{n}{2}+k} = -\omega_n^k. \tag{4}$$

Consequently, from (1)–(3) we have the following recursion for every n of the form $n = 2^m$:

if

- $\vec{s} = \langle s_0, s_1, s_2, s_3, \ldots, s_n \rangle$;
- $\vec{s^e} = \langle s_0, s_2, s_4, \ldots, s_n \rangle$ and $\vec{s^o} = \langle s_1, s_3, s_5, \ldots, s_{n-1} \rangle$
  (note: both of these sequences are of length n/2)

and if

- $DFT(\vec{s}) = \langle f_0, f_1, f_2, f_3, \ldots f_n \rangle$
- $DFT(\vec{s^e}) = \langle f_0^e, f_1^e, \ldots, f_{\frac{n}{2}}^e \rangle$ and $DFT(\vec{s^o}) = \langle f_0^o, f_1^o, \ldots, f_{\frac{n}{2}}^o \rangle$;

then for all $0 \leq k \leq \frac{n}{2} - 1$ we have

$$(5) \qquad f_k = f_k^e + \omega_n^k f_k^o; \qquad\qquad f_{\frac{n}{2}+k} = f_k^e + \omega_{\frac{n}{2}+k}^k f_k^o \quad = f_k^e - \omega_n^k f_k^o;$$

We now compute

$$\begin{aligned}
DFT(\langle 1,5\rangle) &= \langle P_{ee}(\omega_2^0), P_{ee}(\omega_2^1)\rangle = \langle P_{ee}(1), P_{ee}(-1)\rangle \\
&= \langle 1+5\cdot 1, \quad 1+5\cdot(-1)\rangle = \langle 6, \ -4\rangle \\
DFT(\langle 3,7\rangle) &= \langle P_{eo}(\omega_2^0), P_{eo}(\omega_2^1)\rangle = \langle P_{eo}(1), P_{eo}(-1)\rangle \\
&= \langle 3+7\cdot 1, \quad 3+7\cdot(-1)\rangle = \langle 10, \ -4\rangle \\
DFT(\langle 2,6\rangle) &= \langle P_{oe}(\omega_2^0), P_{oe}(\omega_2^1)\rangle = \langle P_{oe}(1), P_{oe}(-1)\rangle \\
&= \langle 2+6\cdot 1, \quad 2+6\cdot(-1)\rangle = \langle 8, \ -4\rangle \\
DFT(\langle 4,8\rangle) &= \langle P_{oo}(\omega_2^0), P_{oo}(\omega_2^1)\rangle = \langle P_{oo}(1), P_{oo}(-1)\rangle \\
&= \langle 4+8\cdot 1, \quad 4+8\cdot(-1)\rangle = \langle 12, \ -4\rangle
\end{aligned}$$

We now use equations (5) with $n = 4$ to start putting things together:

$$\begin{aligned}
DFT(\langle 1,3,5,7\rangle) &= \langle 6+\omega_4^0\cdot 10, \quad -4+\omega_4^1(-4), \quad 6-\omega_4^0\cdot 10, \quad -4-\omega_4^1(-4)\rangle \\
&= \langle 6+10, \quad -4+i(-4), \quad 6-10, \quad -4-i(-4)\rangle \\
&= \langle 16, \quad -4-4\,i, \quad -4, \quad -4+4\,i\rangle \\
DFT(\langle 2,4,6,8\rangle) &= \langle 8+\omega_4^0\cdot 12, \quad -4+\omega_4^1(-4), \quad 8-\omega_4^0\cdot 12, \quad -4-\omega_4^1(-4)\rangle \\
&= \langle 8+12, \quad -4+i(-4), \quad 8-12, \quad -4-i(-4)\rangle \\
&= \langle 20, \quad -4-4\,i, \quad -4, \quad -4+4\,i\rangle
\end{aligned}$$

Finally we obtain from (5) with $n = 8$,

$$\begin{aligned}
DFT(\langle 1,2,3,4,5,6,7,8\rangle) = \langle & 16+\omega_8^0\cdot 20, \quad -4-4\,i+\omega_8^1(-4-4\,i), \quad -4+\omega_8^2\cdot(-4), \quad -4+4\,i+\omega_8^3(-4+4\,i), \\
& 16-\omega_8^0\cdot 20, \quad -4-4\,i-\omega_8^1(-4-4\,i), \quad -4-\omega_8^2\cdot(-4), \quad -4+4\,i-\omega_8^3(-4+4\,i)\rangle
\end{aligned}$$

$$= \left\langle 16 + 1 \cdot 20, \quad -4 - 4\,i + \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)(-4 - 4\,i), \quad -4 + i \cdot (-4), \quad -4 + 4\,i + \left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)(-4 + 4\,i),\right.$$
$$\left. 16 - 1 \cdot 20, \quad -4 - 4\,i - \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)(-4 - 4\,i), \quad -4 - i \cdot (-4), \quad -4 + 4\,i - \left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)(-4 + 4\,i) \right\rangle$$

$$= \langle 36, \quad -4 - 4\,i(1 + \sqrt{2}), \quad -4 - 4\,i, \quad -4 + 4\,i(1 - \sqrt{2}), \quad -4, \quad -4 - 4\,i(1 - \sqrt{2}), \quad -4 + 4\,i, \quad -4 + 4\,i(1 + \sqrt{2})\rangle$$

Another example

Previous example might have resulted in confusingly symmetric calculations, so let us compute the DFT of the sequence $\langle 1, 8, 4, 3, 2, 5, 6, 7\rangle$. The corresponding polynomial is now

$$\begin{aligned} Q(x) &= 1 + 8x + 4x^2 + 3x^3 + 2x^4 + 5x^5 + 6x^6 + 7x^7 \\ &= 1 + 4x^2 + 2x^4 + 6x^6 + x(8 + 3x^2 + 5x^4 + 7x^6). \end{aligned}$$

Substituting $y = x^2$ we get

$$Q(x) = 1 + 4y + 2y^2 + 6y^3 + x(8 + 3y + 5y^2 + 7y^3).$$

We let

$$Q_e(y) = 1 + 4y + 2y^2 + 6y^3; \qquad Q_o(y) = 8 + 3y + 5y^2 + 7y^3;$$

We now have with substitution $z = y^2$

$$\begin{aligned} Q_e(y) &= 1 + 2y^2 + y(4 + 6y^2); & Q_o(y) &= 8 + 5y^2 + y(3 + 7y^2); \\ &= 1 + 2z + y(4 + 6z); & &= 8 + 5z + y(3 + 7z); \end{aligned}$$

We let

$$Q_{ee}(y) = 1 + 2z; \qquad Q_{eo}(y) = 4 + 6z;$$
$$Q_{oe}(y) = 8 + 5z; \qquad Q_{oo}(y) = 3 + 7z;$$

We now have:

$$DFT(\langle 1,2\rangle) = \langle Q_{ee}(\omega_2^0), Q_{ee}(\omega_2)\rangle = \langle Q_{ee}(1), Q_{ee}(-1)\rangle = \langle 3,-1\rangle;$$

$$DFT(\langle 4,6\rangle) = \langle Q_{eo}(\omega_2^0), Q_{eo}(\omega_2)\rangle = \langle Q_{eo}(1), Q_{eo}(-1)\rangle = \langle 10,-2\rangle;$$

$$DFT(\langle 8,5\rangle) = \langle Q_{oe}(\omega_2^0), Q_{oe}(\omega_2)\rangle = \langle Q_{ee}(1), Q_{ee}(-1)\rangle = \langle 13,3\rangle;$$

$$DFT(\langle 3,7\rangle) = \langle Q_{oo}(\omega_2^0), Q_{oo}(\omega_2)\rangle = \langle Q_{ee}(1), Q_{ee}(-1)\rangle = \langle 10,-4\rangle.$$

We now use equations (5) with $n = 4$ to start putting things together:

$$\begin{aligned}
DFT(\langle 1,4,2,6\rangle) &= \langle 3+\omega_4^0\cdot 10,\quad -1+\omega_4^1(-2),\quad 3-\omega_4^0\cdot 10,\quad -1-\omega_4^1(-2)\rangle \\
&= \langle 3+10,\quad -1+i(-2),\quad 3-10,\quad -1-i(-2)\rangle \\
&= \langle 13,\quad -1-2\,i,\quad -7,\quad -1+2\,i\rangle \\
DFT(\langle 8,3,5,7\rangle) &= \langle 8+\omega_4^0\cdot 12,\quad -4+\omega_4^1(-4),\quad 8-\omega_4^0\cdot 12,\quad -4-\omega_4^1(-4)\rangle \\
&= \langle 13+10,\quad 3+i(-4),\quad 13-10,\quad 3-i(-4)\rangle \\
&= \langle 23,\quad 3-4\,i,\quad 3,\quad 3+4\,i\rangle
\end{aligned}$$

Finally, from (5) with $n = 8$ we obtain

$$\begin{aligned}
DFT(\langle 1,8,4,3,2,5,6,7\rangle) = \langle &13+\omega_8^0\cdot 23,\quad -1-2\,i+\omega_8^1(3-4\,i),\quad -7+\omega_8^2\cdot 3,\quad -1+2\,i+\omega_8^3(3+4\,i), \\
&13-\omega_8^0\cdot 23,\quad -1-2\,i-\omega_8^1(3-4\,i),\quad -7-\omega_8^2\cdot 3,\quad -1+2\,i-\omega_8^3(3+4\,i)\rangle
\end{aligned}$$

$$\begin{aligned}
= \Bigg\langle &13+1\cdot 23,\quad -1-2\,i+\left(\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)(3-4\,i),\quad -7+i\cdot 3,\quad -1+2\,i+\left(-\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)(3+4\,i), \\
&13-1\cdot 23,\quad -1-2\,i-\left(\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)(3-4\,i),\quad -7-i\cdot 3,\quad -1+2\,i-\left(-\frac{\sqrt{2}}{2}+i\,\frac{\sqrt{2}}{2}\right)(3+4\,i)\Bigg\rangle
\end{aligned}$$

$$\begin{aligned}
= \Bigg\langle &36,\quad \frac{-2+7\sqrt{2}}{2}-i\,\frac{4+\sqrt{2}}{2},\quad -7+3\,i,\quad \frac{-2-7\sqrt{2}}{2}+i\,\frac{4-\sqrt{2}}{2}, \\
&-10,\quad \frac{-2-7\sqrt{2}}{2}-i\,\frac{4-\sqrt{2}}{2},\quad -7-3\,i,\quad \frac{-2+7\sqrt{2}}{2}+i\,\frac{4+\sqrt{2}}{2}\Bigg\rangle
\end{aligned}$$