Assignment 2

Due date: Wednesday, April 13, at Noon sharp - no extensions

Each question is worth 10 points; submit your solutions to problems the remaining problems are for your midterm practice

Submitting the second assignment:
Option 1: Login to a CSE server and run

`give cs3121 Assignment2 2_studentID.pdf`

or

`give cs3121 Assignment2 2_studentID.doc`

where "`2_studentID.pdf` " or "`2_studentID.doc`" is the file that contains your solutions **with your name and studentID number printed on top of the file**. For example, if your student ID is 323459 your would run:

`give cs3121 Assignment2 2_323459.pdf`

Option 2: If you do not have a CSE account or if you prefer it that way, you can also submit your assignment in either .pdf or .doc format via the web, https://cgi.cse.unsw.edu/~give/Student/give.php
Please **DO NOT** email your solutions. You should **type your solutions** because hand written notes tend not to be legible, and scanning them produces files of size which the system **CANNOT accept**.

1. Fibonacci numbers are defined by $F(0) = 0$, $F(1) = 1$ and $F(n) = F(n-1)+F(n-2)$ for all $n \geq 2$. Thus, the Fibonacci sequence looks as follows: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.
   (a) Show that $\begin{pmatrix} F(n+1) & F(n) \\ F(n) & F(n-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$.
   (b) Find $F(n)$ in $O(\log n)$ many steps.

2. (a) Multiply two complex numbers $a + i\,b$ and $c + i\,d$ using only three real numbers multiplications and any number of real number additions.
   (b) Find the square of a complex number $a + i\,b$ using only two real numbers multiplications and any number of real number additions.
   (c) Evaluate $(a + i\,b)^2(c + i\,d)^2$ using only 5 large number multiplications.

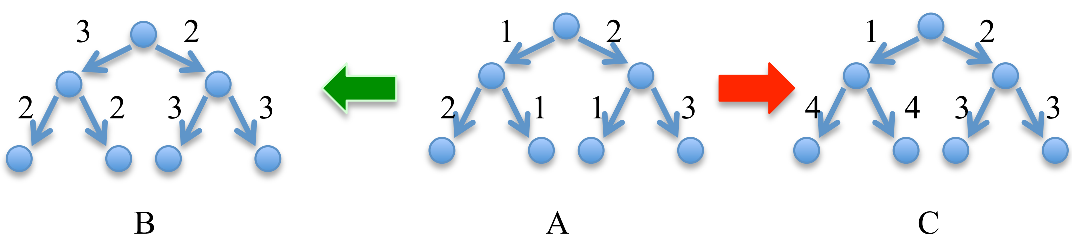
3. Multiply polynomials $P(x) = a_0 + a_{17}x^{17} + a_{19}x^{19} + a_{21}x^{21} + a_{23}x^{23}$ and $Q(x) = b_0 + b_{17}x^{17} + b_{19}x^{19} + b_{21}x^{21} + b_{23}x^{23}$ using
   (a) 16 multiplications of large numbers (involving the coefficients which can be arbitrarily large);
   (b) only 12 large number multiplications.

4. To apply the FFT to the sequence $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ we apply recursively FFT and obtain $FFT(a_0, a_2, a_4, a_6)$ and $FFT(a_1, a_3, a_5, a_7)$. Proceeding further with recursion, we obtain $FFT(a_0, a_4)$ and $FFT(a_2, a_6)$ as well as $FFT(a_1, a_5)$ and $FFT(a_3, a_7)$. Thus, from bottom up, $FFT(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is obtained using permutation $(a_0, a_4, a_2, a_6, a_1, a_5, a_3, a_7)$ as the leaves of the recursion tree of the

original input sequence. Given any input $(a_0, a_1, a_2, a_{2^n-1})$ describe the permutation of the leaves of the recursion tree.

Hint: write indices in binary and see what the relationship is of the bits of the i^{th} element of the original sequence and the i^{th} element of the resulting permutation of elements as they appear on the leaves on the recursion tree.

5. (Timing Problem in VLSI chips) Consider a complete binary tree with $n = 2^k$ leaves. Each edge has an associate positive number that we call the length of this edge (see figure below). The distance from the root to a leaf is the sum of the lengths of all edges from the root to this leaf. The root sends a clock signal and the signal propagates along the edges and reaches the leaf in time proportional to the distance from the root to this leaf. Design an algorithm which increases the lengths of some of the edges in the tree in a way that ensures that the signal reaches all the leafs at the same time while the total sum of the lengths of all edges is minimal. (For example, on the picture below if the tree A is transformed into trees B and C all leaves of B and C are on the distance 5 from the root and thus receive the clock signal in the same time, but the sum of lengths of edges in C is 17 while sum of lengths in B is only 15.)



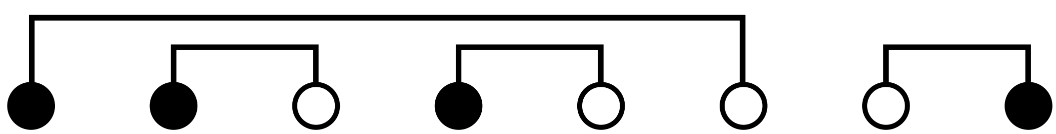
6. You are running a small manufacturing shop with plenty of workers but with a single milling machine. You have to produce n items; item i requires m_i machining time first and then p_i polishing time by hand. The machine can mill only one object at a time, but your workers can be polishing in parallel as many objects as you wish. You have to determine the order in which the objects should be machined so that the whole production is finished as quickly as possible. Prove that your solution is optimal.

7. You are given a connected graph with weighted edges. Find a spanning tree such that the largest weight of all of its edges is as small as possible.

8. Along the long, straight road from Loololong to Goolagong houses are scattered quite sparsely, sometimes with long gaps between two consecutive houses. Telstra must provide mobile phone service to people who live alongside the road, and the range of Telstras cell base station is 5km. Design an algorithm for placing the minimal number of base stations alongside the road, that is sufficient to cover all houses.

9. Alice wants to throw a party and is deciding whom to call. She has n people to choose from, and she has made up a list of which pairs of these people know each other. She wants to pick as many people as possible, subject to two constraints: at the party, each person should have at least five other people whom they know and at least five other people whom they do not know. Give an efficient algorithm that takes as input the list of n people and the list of all pairs who know each other and outputs a subset of these n people which satisfies the constraints and which has the largest number of invitees. Argue that your algorithm indeed produces a subset with the largest possible number of invitees.

10. Assume that you are given n white and n black dots, lying on a line, equally spaced. The dots appear in any order of black and white, see the example picture below. We need an algorithm which connects each black dot with a (different) white dot, so that the total length of wires used to form such connected pairs is minimal. The length of wire used to connect two dots is equal to their distance along the line.



    (a) Someone has proposed the following algorithm: start by connecting the closest pair of a black and a white dot. Repeat. Give an example where such an algorithm fails to produce an optimal solution.

    (b) Design an algorithm which produces an optimal solution.

11. Assume you are given n tasks each of which takes the same, unit amount of time to complete. Each task has an integer deadline and penalty associated with it which you pay if you do not complete the task in time. Design an algorithm that schedules the tasks so that the total penalty you have to pay is as small as possible.

12. You have to write a very long paper. You compiled a sequence of books in the order you will need them, some of them multiple times. Such a sequence might look something like this:

$$B_1, B_2, B_1, B_3, B_4, B_5, B_2, B_6, B_4, B_1, B_7, \ldots$$

Unfortunately, the library lets you keep at most 10 books at home at any moment, so every now and then you have to make a trip to the library to exchange books. On each trip you can exchange any number of books (of course, between 1 and all of 10 books you can keep at home). Design an algorithm which decides which books to exchange on each library trip so that the total number of trips which you will have to make to the library is as small as possible.

13. There is a line of 111 stalls, some of which need to be covered with boards. You can use up to 11 boards, each of which may cover any number of consecutive stalls. Cover all the necessary stalls, while covering as few total stalls as possible.

14. Given two sequences of letters A and B, find if B is a subsequence of A in the sense that one can delete some letters from A and obtain the sequence B.

Additional problems for extended classes only (COMP3821/9801)

15. Assume that you are given an $n \times n$ table; each square of the table contains a distinct number. A square is a local minimum if the number it contains is smaller than the numbers contained in all neighbouring squares which share an edge with that square. Thus, each of the four corner squares has only two neighbours sharing an edge; $4(n-2)$ non corner squares along the edges of the table have three neighbouring squares and all internal squares have four neighbouring squares. A square is a local minimum if the number it contains is smaller than the numbers contained in all of its neighbouring squares. You can only make queries which given numbers m and k, $1 \leq m, k \leq n$, return the value in the square (m, k).

    (a) Show by an example that if you search for a local minimum by picking a square and then moving to a neighbouring square with a smaller value (if there is such) might take $\Theta(n^2)$ many queries.

    (b) Design an algorithm which finds a local minimum and makes only $O(n)$ many queries.

    Hint: a surface $z = f(x, y)$ where (x, y) belongs to a bounded closed subset S of the plane attains its minimal height z either along the curve which is the edge of S or strictly in the interior of S.

16. Assume you are given two data bases A and B; A contains n numbers, B contains $n + 1$ numbers, all numbers are distinct. You can query each database only in the following way: you specify a number k and the database (A or B); the query returns the value of the k^{th} smallest element in that database. Design an algorithm which produces the median of the set of all elements which belong to either A or B using at most $O(\log n)$ queries.