Algorithms: COMP3121/3821/9101/9801

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LECTURE 3: FAST LARGE INTEGER MULTIPLICATION

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

```
          X X X X  <- first input integer
        * X X X X  <- second input integer
          -------
          X X X X  \
        X X X X     \ O(n^2) intermediate operations:
      X X X X       / O(n^2) elementary  multiplications
    X X X X        /    + O(n^2) elementary additions
  ---------------
  X X X X X X X X  <-  result of length 2n
```

- **Can we do it faster than in n^2 many steps??**

- Take the two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad A = \underbrace{\overbrace{XX\dots X}^{A_1}}_{n/2\ bits}\underbrace{\overbrace{XX\dots X}^{A_0}}_{n/2\ bits}$$

$$B = B_1 2^{\frac{n}{2}} + B_0$$

- $A_1 = \text{MoreSignificantPart}(A); \quad A_0 = \text{LessSignificantPart}(A);$

- AB can now be calculated as follows:

$$\begin{aligned} AB &= A_1B_1 2^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}} + A_0B_0 \\ &= A_1B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0 \end{aligned}$$

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow \text{MoreSignificantPart}(A);$
        $A_0 \leftarrow \text{LessSignificantPart}(A);$
        $B_1 \leftarrow \text{MoreSignificantPart}(B);$
        $B_0 \leftarrow \text{LessSignificantPart}(B);$
        $U \leftarrow A_0 + A_1;$
        $V \leftarrow B_0 + B_1;$
        $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$
        $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1);$
        $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V});$
        return $W\,2^n + (Y - X - W)\,2^{n/2} + X$
    end if
end function

The Karatsuba trick

- How many multiplications does this take? (addition is in linear time!)
- Recurrence: $$T(n) = 3\,T\left(\frac{n}{2}\right) + c\,n$$

$$a = 3; \quad b = 2; \quad f(n) = c\,n; \quad n^{\log_b a} = n^{\log_2 3}$$

- since $1.5 < \log_2 3 < 1.6$ we have

$$f(n) = c\,n = O(n^{\log_2 3 - \varepsilon}) \quad \text{for any} \quad 0 < \varepsilon < \log_2 3 - 1$$

- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$$

without going through the messy calculations!

Generalizing Karatsuba's algorithm

Can we do better if we break the numbers in more than two pieces?

Lets try breaking the numbers A, B into 3 pieces:

$$A = \underbrace{XXX\ldots XX}_{k \text{ bits of } A_2}\underbrace{XXX\ldots XX}_{k \text{ bits of } A_1}\underbrace{XXX\ldots XX}_{k \text{ bits of } A_0}$$

$$A = \quad A_2 2^{2k} \quad + \quad A_1 2^k \quad + \quad A_0$$

$$B = \quad B_2 2^{2k} \quad + \quad B_1 2^k \quad + \quad B_0$$

$$AB = A_2 B_2\, 2^{4k} + (A_2 B_1 + A_1 B_2) 2^{3k} + (A_2 B_0 + A_1 B_1 + A_0 B_2) 2^{2k} + \\ + (A_1 B_0 + A_0 B_1) 2^k + A_0 B_0$$

The Karatsuba trick

$$AB = A_2B_2\,2^{4k} + (A_2B_1 + A_1B_2)2^{3k} + (A_2B_0 + A_1B_1 + A_0B_2)2^{2k} + \\ + (A_1B_0 + A_0B_1)2^{k} + A_0B_0$$

- we need only 5 coefficients:

$$\begin{aligned} C_4 &= A_2B_2 \\ C_3 &= A_2B_1 + A_1B_2 \\ C_2 &= A_2B_0 + A_1B_1 + A_0B_2 \\ C_1 &= A_1B_0 + A_0B_1 \\ C_0 &= A_0B_0 \end{aligned}$$

- Can we get these with 5 multiplications only?
- Should we perhaps look at

$$(A_2 + A_1 + A_0)(B_2 + B_1 + B_0) = \\ A_0B_0 + A_1B_0 + A_2B_0 + A_0B_1 + A_1B_1 + A_2B_1 + \\ + A_0B_2 + A_1B_2 + A_2B_2 \quad ???$$

- Not clear how to get $C_0 - C_4$ with 5 multiplications only ...

The Karatsuba trick: slicing into 3 pieces

- We now look for a method for getting these coefficients without any guesswork!

  Let
$$A = A_2\, 2^{2k} + A_1\, 2^k + A_0$$
$$B = B_2\, 2^{2k} + B_1\, 2^k + B_0$$

- We form naturally corresponding polynomials:
$$P_A(x) = A_2\, x^2 + A_1\, x + A_0$$
$$P_B(x) = B_2\, x^2 + B_1\, x + B_0$$

- Their product $P_C(x) = P_A(x)P_B(x)$ is of degree 4;
- we need 5 values to uniquely determine the product.
- We choose **the smallest possible** 5 **integer values**, i.e., -2, -1, 0, 1, 2.
- Thus, we compute
$$P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$$
$$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$

The Karatsuba trick: slicing into 3 pieces

- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$\begin{aligned}
P_A(-2) &= A_2(-2)^2 + A_1(-2) + A_0 = & 4A_2 - 2A_1 + A_0\\
P_A(-1) &= A_2(-1)^2 + A_1(-1) + A_0 = & A_2 - A_1 + A_0\\
P_A(0) &= A_2 0^2 + A_1 0 + A_0 = & A_0\\
P_A(1) &= A_2 1^2 + A_1 1 + A_0 = & A_2 + A_1 + A_0\\
P_A(2) &= A_2 2^2 + A_1 2 + A_0 = & 4A_2 + 2A_1 + A_0
\end{aligned}$$

- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

$$\begin{aligned}
P_B(-2) &= B_2(-2)^2 + B_1(-2) + B_0 = & 4B_2 - 2B_1 + B_0\\
P_B(-1) &= B_2(-1)^2 + B_1(-1) + B_0 = & B_2 - B_1 + B_0\\
P_B(0) &= B_2 0^2 + B_1 0 + B_0 = & B_0\\
P_B(1) &= B_2 1^2 + B_1 1 + B_0 = & B_2 + B_1 + B_0\\
P_B(2) &= B_2 2^2 + B_1 2 + B_0 = & 4B_2 + 2B_1 + B_0
\end{aligned}$$

The Karatsuba trick: slicing into 3 pieces

- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$ we can now compute

$$\begin{aligned} P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2) \end{aligned}$$

$$\begin{aligned} P_C(-1) &= P_A(-1)P_B(-1) \\ &= (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(0) &= P_A(0)P_B(0) \\ &= A_0B_0 \end{aligned}$$

$$\begin{aligned} P_C(1) &= P_A(1)P_B(1) \\ &= (A_0 + A_1 + A_2)(B_0 + B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(2) &= P_A(2)P_B(2) \\ &= (A_0 + 2A_1 + 4A_2)(B_0 + 2B_1 + 4B_2) \end{aligned}$$

Thus, it takes 5 **large integer multiplications** to obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$

The Karatsuba trick: slicing into 3 pieces

- A polynomial $P_C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ is uniquely determined by its values at 5 distinct inputs x_1, x_2, x_3, x_4, x_5 and we have in a matrix form

$$\begin{pmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 & (x_1)^4 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 & (x_2)^4 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 & (x_3)^4 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 & (x_4)^4 \\ 1 & x_5 & (x_5)^2 & (x_5)^3 & (x_5)^4 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} C_0 + C_1x_1 + C_2(x_1)^2 + C_3(x_1)^3 + C_4(x_1)^4 \\ C_0 + C_1x_2 + C_2(x_2)^2 + C_3(x_2)^3 + C_4(x_2)^4 \\ C_0 + C_1x_3 + C_2(x_3)^2 + C_3(x_3)^3 + C_4(x_3)^4 \\ C_0 + C_1x_4 + C_2(x_4)^2 + C_3(x_4)^3 + C_4(x_4)^4 \\ C_0 + C_1x_5 + C_2(x_5)^2 + C_3(x_5)^3 + C_4(x_5)^4 \end{pmatrix}$$

$$= \begin{pmatrix} P_C(x_1) \\ P_C(x_2) \\ P_C(x_3) \\ P_C(x_4) \\ P_C(x_5) \end{pmatrix}$$

The Karatsuba trick: slicing into 3 pieces

- For the product $P_C(x) = P_A(x)P_B(x)$ we have

$$\begin{pmatrix} 1 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 \\ 1 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\ 1 & 0^1 & 0^2 & 0^3 & 0^4 \\ 1 & 1^1 & 1^2 & 1^3 & 1^4 \\ 1 & 2^1 & 2^2 & 2^3 & 2^4 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}$$

$$= \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

The Karatsuba trick: slicing into 3 pieces

- Since

$$\begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

we have

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix}^{-1} \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ -\frac{1}{24} & \frac{2}{3} & -\frac{5}{4} & \frac{2}{3} & -\frac{1}{24} \\ -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

The Karatsuba trick: slicing into 3 pieces

- From

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ -\frac{1}{24} & \frac{2}{3} & -\frac{5}{4} & \frac{2}{3} & -\frac{1}{24} \\ -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

- If we do the multiplications we obtain

$$\begin{aligned}
C_0 &= P_C(0) \\
C_1 &= \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12} \\
C_2 &= -\frac{P_C(-2)}{24} + \frac{2P_C(-1)}{3} - \frac{5P_C(0)}{4} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{24} \\
C_3 &= -\frac{P_C(-2)}{12} + \frac{P_C(-1)}{6} - \frac{P_C(1)}{6} + \frac{P_C(2)}{12} \\
C_4 &= \frac{P_C(-2)}{24} - \frac{P_C(-1)}{6} + \frac{P_C(0)}{4} - \frac{P_C(1)}{6} + \frac{P_C(2)}{24}
\end{aligned}$$

- Thus, from the five values $P_C(-2)$, $P_C(-1)$, $P_C(0)$, $P_C(1)$, $P_C(2)$ of $P_C(x) = P_A(x)P_B(x)$ we get the five coefficients C_0, C_1, C_2, C_3, C_4 of $P_C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ in **linear time**.

The Karatsuba trick: slicing into 3 pieces

- With the coefficients C_0, C_1, C_2, C_3, C_4 obtained, we can now form the polynomial
$$P_C(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4$$
- We can now compute
$$P_C(2^k) = C_0 + C_1 2^k + C_2 2^{2k} + C_3 2^{3k} + C_4 2^{4k}$$
in linear time, because computing $P_C(2^k)$ involves only binary shifts of the coefficients plus $O(k)$ additions.
- Here is the complete algorithm:

function $\text{MULT}(A, B)$

  obtain A_0, A_1, A_2 and B_0, B_1, B_2 such that $A = A_2\, 2^{2\,k} + A_1\, 2^k + A_0;\quad B = B_2\, 2^{2\,k} + B_1\, 2^k + B_0;$

  form polynomials $P_A(x) = A_2 x^2 + A_1 x + A_0;\quad P_B(x) = B_2 x^2 + B_1 x + B_0;$

$$
\begin{array}{ll}
P_A(-2) \leftarrow 4A_2 - 2A_1 + A_0 & P_B(-2) \leftarrow 4B_2 - 2B_1 + B_0 \\
P_A(-1) \leftarrow A_2 - A_1 + A_0 & P_B(-1) \leftarrow B_2 - B_1 + B_0 \\
P_A(0) \leftarrow A_0 & P_B(0) \leftarrow B_0 \\
P_A(1) \leftarrow A_2 + A_1 + A_0 & P_B(1) \leftarrow B_2 + B_1 + B_0 \\
P_A(2) \leftarrow 4A_2 + 2A_1 + A_0 & P_B(2) \leftarrow 4B_2 + 2B_1 + B_0
\end{array}
$$

$$
\begin{array}{ll}
P_C(-2) \leftarrow \text{MULT}(P_A(-2), P_B(-2)); & P_C(-1) \leftarrow \text{MULT}(P_A(-1), P_B(-1)); \\
P_C(0) \leftarrow \text{MULT}(P_A(0), P_B(0)); & \\
P_C(1) \leftarrow \text{MULT}(P_A(1), P_B(1)); & P_C(2) \leftarrow \text{MULT}(P_A(2), P_B(2))
\end{array}
$$

$$
\begin{array}{l}
C_0 \leftarrow P_C(0); \qquad C_1 \leftarrow \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12} \\
C_2 \leftarrow -\frac{P_C(-2)}{24} + \frac{2P_C(-1)}{3} - \frac{5P_C(0)}{4} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{24} \\
C_3 \leftarrow -\frac{P_C(-2)}{12} + \frac{P_C(-1)}{6} - \frac{P_C(1)}{6} + \frac{P_C(2)}{12} \\
C_4 \leftarrow \frac{P_C(-2)}{24} - \frac{P_C(-1)}{6} + \frac{P_C(0)}{4} - \frac{P_C(1)}{6} + \frac{P_C(2)}{24}
\end{array}
$$

  form $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$; compute
  $P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$

  return $P_C(2^k)$

end function

The Karatsuba trick: slicing into 3 pieces

- How fast is this algorithm?

- We have replaced a multiplication of two n bit numbers with 5 multiplications of $n/3$ bit numbers with an overhead of additions, shifts and the similar, all doable in linear time $c\,n$; thus,

$$T(n) = 5T\left(\frac{n}{3}\right) + c\,n$$

- We now apply the Master Theorem:
we have $a = 5,\ \ b = 3$, so we consider $n^{\log_b a} = n^{\log_3 5} \approx n^{1.465\ldots}$

- Clearly the first case applies and we get $T(n) = O(n^{1.47})$;

- Recall that the original Karatsuba algorithm runs in time $n^{\log_2 3} \approx n^{1.58} > n^{1.47}$.

- Thus, we got a significantly faster algorithm.

- Then why not slice numbers into even larger number of slices? Maybe we can get even faster algorithm?

- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $n + 1$ many equal slices...

Generalizing Karatsuba's algorithm

The general case - slicing input numbers A, B into $n+1$ many slices

- For simplicity let A, B have $(n+1)k$ bits; (k can be arbitrarily large)
- Slice A, B into $n+1$ pieces each:

$$\begin{aligned} A = \quad & A_n 2^{kn} \quad + \quad A_{n-1} 2^{k(n-1)} + \quad \cdots + \quad A_0 \\ B = \quad & B_n 2^{kn} \quad + \quad B_{n-1} 2^{k(n-1)} + \quad \cdots + \quad B_0 \end{aligned}$$

- We form the naturally corresponding polynomials:

$$\begin{aligned} P_A(x) = \quad & A_n x^n \quad + \quad A_{n-1} x^{n-1} + \quad \cdots + \quad A_0 \\ P_B(x) = \quad & B_n x^n \quad + \quad B_{n-1} x^{n-1} + \quad \cdots + \quad B_0 \end{aligned}$$

  - coefficients satisfy $A_i, B_i < 2^k$
  - $A = P_A(2^k); \quad B = P_B(2^k)$
  - $AB = P_A(2^n) P_B(2^n) = (P_A(x) \cdot P_B(x)) |_{x=2^k}$

Generalizing Karatsuba's algorithm

$$AB = P_A(2^n)P_B(2^n) = (P_A(x) \cdot P_B(x))\,|_{x=2^k}$$

Strategy:

- figure out how to multiply polynomials fast to obtain

$$P_C(x) = P_A(x) \cdot P_B(x);$$

- evaluate $P_C(2^k)$.
- Note that $P_C(x) = P_A(x) \cdot P_B(x)$ is of degree $2n$:

$$P_C(x) = \sum_{j=0}^{2n} C_j x^j$$

Generalizing Karatsuba's algorithm

Example:

$$
\begin{aligned}
(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\
a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
+(a_0b_1 + a_1b_0)x + a_0b_0
\end{aligned}
$$

In general: for

$$
P_A(x) = A_nx^n + A_{n-1}x^{n-1} + \cdots + A_0
$$
$$
P_B(x) = B_nx^n + B_{n-1}x^{n-1} + \cdots + B_0
$$

setting $A_i = 0$ and $B_i = 0$ for $n < i \leq 2n$ we have

$$
P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \left(\sum_{i=0}^{j} A_iB_{j-i} \right) x^j = \sum_{j=0}^{2n} C_j\, x^j
$$

We need to find the coefficients $C_j = \sum_{i=0}^{j} A_iB_{j-i}$ without performing $(n+1)^2$ many multiplications

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n+1$ distinct input values for x:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

- If $P_A(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_0$, we can write in matrix form:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix}. \tag{1}$$

- It can be shown that, if x_i are all distinct then this matrix is invertible.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_n)$ the coefficients $A_0, A_1, \ldots, A_n$ are uniquely determined:

$$\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}^{-1} \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix} \tag{2}$$

- Equations (1) and (2) show how we can commute between:
  1. a representation of a polynomial $P_A(x)$ via its coefficients $A_n, A_{n-1}, \ldots, A_0$, i.e. $P_A(x) = A_n x^n + \ldots + A_1 x + A_0$
  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

Coefficient vs value representation of polynomials- ctd.

Commuting between a representation of a polynomial $P_A(x)$ via its coefficients $A_n, A_{n-1}, \ldots, A_0$, i.e. $P_A(x) = A_n x^n + \ldots + A_1 x + A_0$ and a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

is done via the following two matrix multiplications, with matrices made up from constants:

$$\begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix}.$$

$$\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}^{-1} \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix}$$

Thus, for fixed input values $x_0, \ldots, x_n$ this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

- Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

1. convert them into value representation at $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_{2n}, P_A(x_{2n}))\}$$

$$P_B(x) \leftrightarrow \{(x_0, P_B(x_0)), (x_1, P_B(x_1)), \ldots, (x_{2n}, P_B(x_{2n}))\}$$

2. multiply them point by point using $2n+1$ multiplications only:

$$P_A(x)P_B(x) \leftrightarrow \{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\}$$

3. Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

Fast multiplication of polynomials - ctd.

- What values should we choose for $x_0, x_1, \ldots, x_{2n}$??
- Key idea: use $2n+1$ smallest possible integer values!

$$\{-n,\ -(n-1),\ \ldots\ ,\ -1,\ 0,\ 1,\ \ldots\ ,\ n-1,\ n\}$$

- Thus, we want to find the values

$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \cdots + A_0 : \quad -n \leq m \leq n,$$

$$P_B(m) = B_n m^n + B_{n-1} m^{n-1} + \cdots + B_0 : \quad -n \leq m \leq n.$$

- Recall: $n+1$ is the number of slices we split the input numbers A, B.

- Thus, n is constant; only coefficients A_i and B_i are large - they are the slices of A and B.
- Values:

$$(-m)^n,\ (-m)^{n-1},\ \cdots\ ,\ (-m)^2,\ -m,\ m,\ m^2,\ \ldots\ m^n$$

for $0 \leq m \leq n$ are all constants which do not depend on A or B!

Fast multiplication of polynomials - ctd.

- Multiplication of a large number with k bits by a constant d can be done in linear time because it is reducible to $d-1$ additions:

$$d \cdot A = \underbrace{A + A + \ldots + A}_{d}$$

- Thus, all the values

$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \cdots + A_0 : \quad -n \leq m \leq n,$$

$$P_B(m) = B_n m^n + B_{n-1} m^{n-1} + \cdots + B_0 : \quad -n \leq m \leq n.$$

can be found in time linear in the number of bits of the input numbers!

- We now perform $2n-1$ multiplications of large numbers to obtain

$$P_A(-n)P_B(-n), \ \ldots\ , \ P_A(-1)P_B(-1), \ P_A(0)P_B(0), \ P_A(1)P_B(1), \ \ldots\ , \ P_A(n)P_B(n)$$

- For $P_C(x) = P_A(x)P_B(x)$ these are $2n+1$ many values of $P_C(x)$:

$$P_C(-n), \ \ldots\ , \ P_C(-1), \ P_C(0), \ P_C(1), \ \ldots\ , \ P_C(n)$$

Fast multiplication of polynomials - ctd.

$$
\begin{aligned}
P_C(-n) &= P_A(-n) \cdot P_B(-n)\\
P_C(-n+1) &= P_A(-n+1) \cdot P_B(-n+1)\\
&\dots\\
P_C(0) &= P_A(0) \cdot P_B(0)\\
&\dots\\
P_C(n-1) &= P_A(n-1) \cdot P_B(n-1)\\
P_C(n) &= P_A(n) \cdot P_B(n)
\end{aligned}
$$

Since $P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \cdots + C_0$, we have:

$$
\begin{aligned}
C_{2n}(-n)^{2n} + C_{2n-1}(-n)^{2n-1} + \cdots + C_0 &= P_C(-n)\\
C_{2n}(-(n-1))^{2n} + C_{2n-1}(-(n-1))^{2n-1} + \cdots + C_0 &= P_C(-(n-1))\\
&\dots\\
C_{2n}(n-1)^{2n} + C_{2n-1}(n-1)^{2n-1} + \cdots + C_0 &= P_C(n-1)\\
C_{2n}n^{2n} + C_{2n-1}n^{2n-1} + \cdots + C_0 &= P_C(n)
\end{aligned}
$$

This is just a system of linear equations, that can be solved for $C_0, C_1, \dots, C_{2n}$:

Fast multiplication of polynomials - ctd.

$$\begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix}.$$

But the inverse matrix also involves only constants depending on n only;

Thus the coefficients C_i can be obtained in linear time.

So herre is the algorithm we have just described:

function $\textsc{Mult}(n, A, B)$
  if $|A| = |B| = 1$ **then return** AB
  else
    obtain $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ such that

$$A = A_n 2^{n\,k} + A_{n-1}2^{(n-1)\,k} + \ldots + A_0$$

$$B = B_n 2^{n\,k} + B_{n-1}2^{(n-1)\,k} + \ldots + B_0$$

    form polynomials

$$P_A(x) = A_n x^n + A_{n-1}x^{(n-1)} + \ldots + A_0$$

$$P_B(x) = B_n x^n + B_{n-1}x^{(n-1)} + \ldots + B_0$$

    for $m = -n$ to $m = n$ **do**
      compute $P_A(m)$ and $P_B(m)$;
      $P_C(m) \leftarrow \textsc{Mult}(n, P_A(m)P_B(m))$
    end for
    compute $C_0,\ C_1, \ldots\ C_{2n}$ via

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} 1 & -n & (-n)^2 & \ldots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \ldots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \ldots & n^{2n} \end{pmatrix}^{-1} \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix}.$$

    form $P_C(x) = C_{2n}x^{2n} + \ldots + C_0$ and compute $P_C(2^k)$
    return $P_C(2^k)$
  end if
end function

How fast is our algorithm?

- For each m such that $-n \leq m \leq n$, the value of $P_A(m)$ is a sum of $n+1$ values;
- Each value is a product of a k-bit number A_j with a constant m^j, $-n \leq m \leq n$:

$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \ldots A_0$$

- So $|P_A(m)| \leq (n+1) \cdot 2^k \cdot n^n$.
- The number of bits of each $P_A(m)$ for $-n \leq m \leq n$ is

$$\log_2\left((n+1)2^k n^n\right) = k + \log_2\left((n+1) \cdot n^n\right)$$

- $\log((n+1) \cdot n^n)$ is constant; Thus,
- **Each $P_A(m)$ has $k + s$ bits, where s is a constant independent of k.**

Generalizing Karatsuba's algorithm

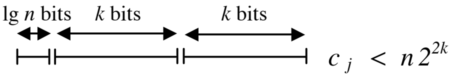
- Recall that

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)} x^j = \sum_{j=0}^{2n} \underbrace{C_j} x^j \qquad \text{where} \qquad C_j = \sum_{i=0}^{j} A_i B_{j-i}$$

- How big are C_j? Since $A_i, B_i < 2^k$, we have $A_i B_{j-i} < 2^{2k}$ and

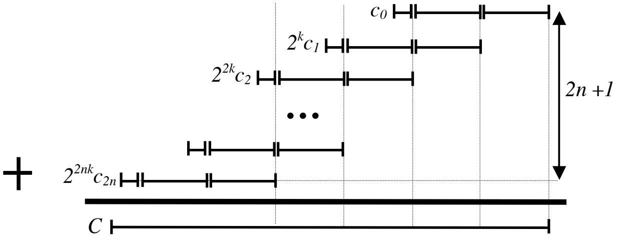
$$C_j = \sum_{i=0}^{j} A_i B_{j-i} < j 2^{2k} \leq (n+1) 2^{2k}$$

- Thus, C_j **has at most** $2k + \log_2(n+1)$ **many bits**



How fast is our algorithm?

- We compute $C = A\,B$ as $C = P_C(2^k) = \sum_{j=0}^{2n} C_j 2^{kj}$
- C is obtained as a sum of binary shifts of $2n+1$ numbers C_j;
- each C_j has $2k + \lceil \log_2(n+1) \rceil$ bits;



- n is constant: it is the number of pieces we slice input numbers;
- Thus, **evaluation of $P_C(2^k)$ takes $O(k)$ many steps.**

How fast is our algorithm?

- We have reduced a multiplication of two $k(n+1)$ digit numbers to $2n+1$ multiplications of $k+s$ digit numbers plus a linear overhead (of additions splitting the numbers atc.)
- Thus, we get the following recurrence for the complexity of $\textsc{Mult}(n, P_A(m)P_B(m))$:

$$T((n+1)k) = (2n+1)T(k+s) + c\,k$$

- Let $N = (n+1)k$. Then for another constant d

$$T(N) = (2n+1)\,T\left(\frac{N}{n+1} + s\right) + d\,N$$

- Since s is constant, its impact can be neglected.
- Since $\log_b a = \log_{n+1}(2n+1) > 1$,

$$f(N) = c\,N = O\left(N^{\log_{n+1}(2n+1)-\varepsilon}\right)$$

- Thus, with $a = 2n+1$ and $b = n+1$ the first case of the Master Theorem applies;
- we get:

$$T(N) = \Theta\left(N^{\log_b a}\right) = \Theta\left(N^{\log_{n+1}(2n+1)}\right)$$

- Note that

$$N^{\log_{n+1}(2n+1)} < N^{\log_{n+1} 2(n+1)} = N^{\log_{n+1} 2+\log_{n+1}(n+1)}$$
$$= N^{1+\log_{n+1} 2} = N^{1+\frac{1}{\log_2(n+1)}}$$

- Thus, by choosing a sufficiently large n, we can get a run time arbitrarily close to linear time!

- How large does n have to be, in order to to get an algorithm which runs in time $N^{1.1}$?

$$N^{1.1} = N^{1+\frac{1}{\log_2(n+1)}} \quad \rightarrow \quad \frac{1}{\log_2(n+1)} = \frac{1}{10} \quad \rightarrow \quad n+1 = 2^{10}$$

- Thus, we would have to slice input numbers into $2^{10} = 1024$ pieces.

- We would have to evaluate polynomials $P_A(x)$ and $P_B(x)$ at values up to $n = 2^{10} - 1$, which involves computing

$$n^n = (2^{10} - 1)^{2^{10}-1} = 1.27 \times 10^{3079}$$

- Thus, while evaluations of $P_A(x)$ and $P_B(x)$ can be all done in linear time $T(n) = c\,n$, the constant c is huge;
- the algorithm is not practical at all!
- The moral is: **asymptotic estimates are useless in practice if the size of the constants hidden by the O-notation are not estimated and found to be reasonably small!!!**
- Can we avoid explosion in the size of x^n needed for evaluation of a polynomial of degree n? Are there numbers $x_0, x_1, \ldots, x_n$ such that the size of x_i^n does not grow uncontrollably?
- Answer: YES; they are the complex numbers z_i lying on the unit circle, i.e., such that $|z_i| = 1$!

Digression: Convolution

- Let $a = \langle A_n, A_{n-1}, \ldots, A_1, A_0 \rangle$ and $b = \langle B_n, B_{n-1}, \ldots B_1, B_0 \rangle$ be two sequences;
- pad them with n $0's$ to length $2n+1$:

$$\langle \underbrace{0, 0, \ldots, 0}_{n}, A_n, A_{n-1}, \ldots, A_1, A_0 \rangle$$

$$\langle \underbrace{0, 0, \ldots, 0}_{n}, B_n, B_{n-1}, \ldots, B_1, B_0 \rangle$$

i.e., we set $A_i = B_i = 0$ for $n < i \leq 2n$.

Then the sequence $$a * b = \left\{ \sum_{i=0}^{j} A_i B_{j-i} \right\}_{j=0}^{2n}$$

is called *the (linear) convolution* of the sequences A and B.

$$a * b = \langle A_n B_n,\ A_{n-1} B_n + A_n B_{n-1},\ A_{n-2} B_n + A_{n-1} B_{n-1} + A_n B_{n-2}, \\ \ldots, A_2 B_0 + A_1 B_1 + A_0 B_2,\ A_1 B_0 + A_0 B_1,\ A_0 B_0 \rangle$$