Algorithms: COMP3121/3821/9101/9801

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LECTURE 3: FAST LARGE INTEGER MULTIPLICATION

Basics revisited: how do we multiply two numbers?

- The primary school algorithm:

```
          X X X X  <- first input integer
        * X X X X  <- second input integer
          -------
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        X X X X     \ O(n^2) intermediate operations:
      X X X X       / O(n^2) elementary  multiplications
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- **Can we do it faster than in n^2 many steps??**

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$$\begin{aligned} AB &= A_1 B_1 2^n + (A_1 B_0 + A_0 B_1) 2^{\frac{n}{2}} + A_0 B_0 \\ &= A_1 B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1 B_1 - A_0 B_0) 2^{\frac{n}{2}} + A_0 B_0 \end{aligned}$$

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow \mathrm{MoreSignificantPart}(A);$
        $A_0 \leftarrow \mathrm{LessSignificantPart}(A);$
        $B_1 \leftarrow \mathrm{MoreSignificantPart}(B);$
        $B_0 \leftarrow \mathrm{LessSignificantPart}(B);$
        $U \leftarrow A_0 + A_1;$
        $V \leftarrow B_0 + B_1;$
        $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0);$
        $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1);$
        $Y \leftarrow \textsc{Mult}(\mathrm{U}, \mathrm{V});$
        return $W\,2^n + (Y - X - W)\,2^{n/2} + X$
    end if
end function

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- since $1.5 < \log_2 3 < 1.6$ we have

$$f(n) = c\,n = O(n^{\log_2 3 - \varepsilon}) \quad \text{for any} \quad 0 < \varepsilon < \log_2 3 - 1$$

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- Thus, the first case of the Master Theorem applies.
- Consequently,

$$T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$$

without going through the messy calculations!

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$$AB = A_2B_2\,2^{4k} + (A_2B_1 + A_1B_2)2^{3k} + (A_2B_0 + A_1B_1 + A_0B_2)2^{2k} + \\ + (A_1B_0 + A_0B_1)2^k + A_0B_0$$

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- Should we perhaps look at

$$\begin{aligned} (A_2 + A_1 + A_0)(B_2 + B_1 + B_0) = \\ A_0B_0 + A_1B_0 + A_2B_0 + A_0B_1 + A_1B_1 + A_2B_1 + \\ + A_0B_2 + A_1B_2 + A_2B_2 \quad ??? \end{aligned}$$

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- Not clear how to get $C_0 - C_4$ with 5 multiplications only ...

The Karatsuba trick: slicing into 3 pieces

- We now look for a method for getting these coefficients without any guesswork!

  Let
$$A = A_2\, 2^{2k} + A_1\, 2^k + A_0$$
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- We form naturally corresponding polynomials:
$$P_A(x) = A_2\, x^2 + A_1\, x + A_0$$
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- we need 5 values to uniquely determine the product.
- We choose **the smallest possible** 5 **integer values**, i.e., -2, -1, 0, 1, 2.
- Thus, we compute
  $$P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$$
  $$P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$$

The Karatsuba trick: slicing into 3 pieces

- For $P_A(x) = A_2x^2 + A_1x + A_0$ we have

$$
\begin{aligned}
P_A(-2) &= A_2(-2)^2 + A_1(-2) + A_0 = & 4A_2 - 2A_1 + A_0 \\
P_A(-1) &= A_2(-1)^2 + A_1(-1) + A_0 = & A_2 - A_1 + A_0 \\
P_A(0) &= A_2 0^2 + A_1 0 + A_0 = & A_0 \\
P_A(1) &= A_2 1^2 + A_1 1 + A_0 = & A_2 + A_1 + A_0 \\
P_A(2) &= A_2 2^2 + A_1 2 + A_0 = & 4A_2 + 2A_1 + A_0
\end{aligned}
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\end{aligned}
$$

- Similarly, for $P_B(x) = B_2x^2 + B_1x + B_0$ we have

$$
\begin{aligned}
P_B(-2) &= B_2(-2)^2 + B_1(-2) + B_0 = & 4B_2 - 2B_1 + B_0 \\
P_B(-1) &= B_2(-1)^2 + B_1(-1) + B_0 = & B_2 - B_1 + B_0 \\
P_B(0) &= B_2 0^2 + B_1 0 + B_0 = & B_0 \\
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P_B(2) &= B_2 2^2 + B_1 2 + B_0 = & 4B_2 + 2B_1 + B_0
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- Having obtained $P_A(-2), P_A(-1), P_A(0), P_A(1), P_A(2)$ and $P_B(-2), P_B(-1), P_B(0), P_B(1), P_B(2)$ we can now compute

$$\begin{aligned} P_C(-2) &= P_A(-2)P_B(-2) \\ &= (A_0 - 2A_1 + 4A_2)(B_0 - 2B_1 + 4B_2) \end{aligned}$$

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$$\begin{aligned} P_C(1) &= P_A(1)P_B(1) \\ &= (A_0 + A_1 + A_2)(B_0 + B_1 + B_2) \end{aligned}$$

$$\begin{aligned} P_C(2) &= P_A(2)P_B(2) \\ &= (A_0 + 2A_1 + 4A_2)(B_0 + 2B_1 + 4B_2) \end{aligned}$$

Thus, it takes 5 **large integer multiplications** to obtain $P_C(-2), P_C(-1), P_C(0), P_C(1), P_C(2)$

The Karatsuba trick: slicing into 3 pieces

- A polynomial $P_C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ is uniquely determined by its values at 5 distinct inputs x_1, x_2, x_3, x_4, x_5 and we have in a matrix form

$$\begin{pmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 & (x_1)^4 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 & (x_2)^4 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 & (x_3)^4 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 & (x_4)^4 \\ 1 & x_5 & (x_5)^2 & (x_5)^3 & (x_5)^4 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} C_0 + C_1x_1 + C_2(x_1)^2 + C_3(x_1)^3 + C_4(x_1)^4 \\ C_0 + C_1x_2 + C_2(x_2)^2 + C_3(x_2)^3 + C_4(x_2)^4 \\ C_0 + C_1x_3 + C_2(x_3)^2 + C_3(x_3)^3 + C_4(x_3)^4 \\ C_0 + C_1x_4 + C_2(x_4)^2 + C_3(x_4)^3 + C_4(x_4)^4 \\ C_0 + C_1x_5 + C_2(x_5)^2 + C_3(x_5)^3 + C_4(x_5)^4 \end{pmatrix}$$

$$= \begin{pmatrix} P_C(x_1) \\ P_C(x_2) \\ P_C(x_3) \\ P_C(x_4) \\ P_C(x_5) \end{pmatrix}$$

The Karatsuba trick: slicing into 3 pieces

- For the product $P_C(x) = P_A(x)P_B(x)$ we have

$$\begin{pmatrix} 1 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 \\ 1 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\ 1 & 0^1 & 0^2 & 0^3 & 0^4 \\ 1 & 1^1 & 1^2 & 1^3 & 1^4 \\ 1 & 2^1 & 2^2 & 2^3 & 2^4 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}$$

$$= \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

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- Since

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we have

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix}^{-1} \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

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i.e.,

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ -\frac{1}{24} & \frac{2}{3} & -\frac{5}{4} & \frac{2}{3} & -\frac{1}{24} \\ -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} P_C(-2) \\ P_C(-1) \\ P_C(0) \\ P_C(1) \\ P_C(2) \end{pmatrix}$$

The Karatsuba trick: slicing into 3 pieces

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- If we do the multiplications we obtain

$$\begin{aligned}
C_0 &= P_C(0) \\
C_1 &= \frac{P_C(-2)}{12} - \frac{2P_C(-1)}{3} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{12} \\
C_2 &= -\frac{P_C(-2)}{24} + \frac{2P_C(-1)}{3} - \frac{5P_C(0)}{4} + \frac{2P_C(1)}{3} - \frac{P_C(2)}{24} \\
C_3 &= -\frac{P_C(-2)}{12} + \frac{P_C(-1)}{6} - \frac{P_C(1)}{6} + \frac{P_C(2)}{12} \\
C_4 &= \frac{P_C(-2)}{24} - \frac{P_C(-1)}{6} + \frac{P_C(0)}{4} - \frac{P_C(1)}{6} + \frac{P_C(2)}{24}
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- Thus, from the five values $P_C(-2)$, $P_C(-1)$, $P_C(0)$, $P_C(1)$, $P_C(2)$ of $P_C(x) = P_A(x)P_B(x)$
we get the five coefficients C_0, C_1, C_2, C_3, C_4 of $P_C(x) = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$ in **linear time**.

The Karatsuba trick: slicing into 3 pieces

- With the coefficients C_0, C_1, C_2, C_3, C_4 obtained, we can now form the polynomial
$$P_C(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4$$

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- We can now compute

$$P_C(2^k) = C_0 + C_1 2^k + C_2 2^{2k} + C_3 2^{3k} + C_4 2^{4k}$$

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- Here is the complete algorithm:

function $\textsc{Mult}(A, B)$

obtain A_0, A_1, A_2 and B_0, B_1, B_2 such that $A = A_2\, 2^{2\,k} + A_1\, 2^k + A_0$; $\quad B = B_2\, 2^{2\,k} + B_1\, 2^k + B_0$;

form polynomials $P_A(x) = A_2 x^2 + A_1 x + A_0$; $\quad P_B(x) = B_2 x^2 + B_1 x + B_0$;

$$
\begin{aligned}
P_A(-2) &\leftarrow 4A_2 - 2A_1 + A_0 & \qquad P_B(-2) &\leftarrow 4B_2 - 2B_1 + B_0\\
P_A(-1) &\leftarrow A_2 - A_1 + A_0 & P_B(-1) &\leftarrow B_2 - B_1 + B_0\\
P_A(0) &\leftarrow A_0 & P_B(0) &\leftarrow B_0\\
P_A(1) &\leftarrow A_2 + A_1 + A_0 & P_B(1) &\leftarrow B_2 + B_1 + B_0\\
P_A(2) &\leftarrow 4A_2 + 2A_1 + A_0 & P_B(2) &\leftarrow 4B_2 + 2B_1 + B_0
\end{aligned}
$$

$$
\begin{aligned}
&P_C(-2) \leftarrow \textsc{Mult}(P_A(-2), P_B(-2)); & \qquad &P_C(-1) \leftarrow \textsc{Mult}(P_A(-1), P_B(-1));\\
&P_C(0) \leftarrow \textsc{Mult}(P_A(0), P_B(0)); & &\\
&P_C(1) \leftarrow \textsc{Mult}(P_A(1), P_B(1)); & &P_C(2) \leftarrow \textsc{Mult}(P_A(2), P_B(2))
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form $P_C(x) = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0$; compute
$P_C(2^k) = C_4 2^{4k} + C_3 2^{3k} + C_2 2^{2k} + C_1 2^k + C_0$

return $P_C(2^k)$

end function

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$$T(n) = 5T\left(\frac{n}{3}\right) + c\,n$$

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- The answer is, in a sense, BOTH yes and no, so lets see what happens if we slice numbers into $n + 1$ many equal slices...

Generalizing Karatsuba's algorithm

The general case - slicing input numbers A, B into $n + 1$ many slices

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- For simplicity let A, B have $(n+1)k$ bits; (k can be arbitrarily large)
- Slice A, B into $n+1$ pieces each:

$$\begin{aligned} A &= A_n 2^{kn} + A_{n-1}2^{k(n-1)} + \cdots + A_0 \\ B &= B_n 2^{kn} + B_{n-1}2^{k(n-1)} + \cdots + B_0 \end{aligned}$$

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Generalizing Karatsuba's algorithm

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Generalizing Karatsuba's algorithm

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$$P_C(x) = \sum_{j=0}^{2n} C_j x^j$$

Generalizing Karatsuba's algorithm

Example:

$$
\begin{aligned}
&(a_3x^3 + a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0) = \\
&\quad a_3b_3x^6 + (a_2b_3 + a_3b_2)x^5 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
&\qquad +(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
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setting $A_i = 0$ and $B_i = 0$ for $n < i \leq 2n$ we have

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \left(\sum_{i=0}^{j} A_iB_{j-i} \right) x^j = \sum_{j=0}^{2n} C_j\, x^j$$

We need to find the coefficients $C_j = \sum_{i=0}^{j} A_iB_{j-i}$ without performing $(n+1)^2$ many multiplications

Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree n is uniquely determined by its values at any $n+1$ distinct input values for x:

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

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$$\begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix}. \tag{1}$$

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- It can be shown that, if x_i are all distinct then this matrix is invertible.

Coefficient vs value representation of polynomials - ctd.

- Thus, if all x_i are distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_n)$ the coefficients $A_0, A_1, \ldots, A_n$ are uniquely determined:

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- Equations (1) and (2) show how we can commute between:
  1. a representation of a polynomial $P_A(x)$ via its coefficients $A_n, A_{n-1}, \ldots, A_0$, i.e. $P_A(x) = A_n x^n + \ldots + A_1 x + A_0$

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  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

Coefficient vs value representation of polynomials- ctd.

Commuting between a representation of a polynomial $P_A(x)$ via its coefficients $A_n, A_{n-1}, \ldots, A_0$, i.e. $P_A(x) = A_n x^n + \ldots + A_1 x + A_0$ and a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

is done via the following two matrix multiplications, with matrices made up from constants:

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$$\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}^{-1} \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix}$$

Thus, for fixed input values $x_0, \ldots, x_n$ this switch between the two kinds of representations is done in **linear time**!

Our strategy to multiply polynomials fast:

- Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

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3. Convert such value representation of $P_C(x) = P_A(x)P_B(x)$ back to coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

Fast multiplication of polynomials - ctd.

- What values should we choose for $x_0, x_1, \ldots, x_{2n}$??

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- Thus, we want to find the values

$$P_A(m) = A_n m^n + A_{n-1} m^{n-1} + \dots + A_0 : \quad -n \le m \le n,$$

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- Thus, n is constant; only coefficients A_i and B_i are large - they are the slices of A and B.
- Values:

$$(-m)^n,\ (-m)^{n-1},\ \cdots\ ,\ (-m)^2,\ -m,\ m,\ m^2,\ \dots\ m^n$$

for $0 \le m \le n$ are all constants which do not depend on A or B!

Fast multiplication of polynomials - ctd.

- Multiplication of a large number with k bits by a constant d can be done in linear time because it is reducible to $d-1$ additions:

$$d \cdot A = \underbrace{A + A + \ldots + A}_{d}$$

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- We now perform $2n-1$ multiplications of large numbers to obtain

$$P_A(-n)P_B(-n),\ \ldots\ ,\ P_A(-1)P_B(-1),\ P_A(0)P_B(0),\ P_A(1)P_B(1),\ \ldots\ ,\ P_A(n)P_B(n)$$

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- For $P_C(x) = P_A(x)P_B(x)$ these are $2n+1$ many values of $P_C(x)$:

$$P_C(-n), \ \ldots \ , \ P_C(-1), \ P_C(0), \ P_C(1), \ \ldots \ , \ P_C(n)$$

Fast multiplication of polynomials - ctd.

$$\begin{aligned}
P_C(-n) &= P_A(-n) \cdot P_B(-n) \\
P_C(-n+1) &= P_A(-n+1) \cdot P_B(-n+1) \\
&\dots \\
P_C(0) &= P_A(0) \cdot P_B(0) \\
&\dots \\
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P_C(n) &= P_A(n) \cdot P_B(n)
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Fast multiplication of polynomials - ctd.

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P_C(n) &= P_A(n) \cdot P_B(n)
\end{aligned}$$

Since $P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \dots + C_0$, we have:

Fast multiplication of polynomials - ctd.

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$$
\begin{aligned}
C_{2n}(-n)^{2n} + C_{2n-1}(-n)^{2n-1} + \cdots + C_0 &= P_C(-n) \\
C_{2n}(-(n-1))^{2n} + C_{2n-1}(-(n-1))^{2n-1} + \cdots + C_0 &= P_C(-(n-1)) \\
&\dots \\
C_{2n}(n-1)^{2n} + C_{2n-1}(n-1)^{2n-1} + \cdots + C_0 &= P_C(n-1) \\
C_{2n}n^{2n} + C_{2n-1}n^{2n-1} + \cdots + C_0 &= P_C(n)
\end{aligned}
$$

This is just a system of linear equations, that can be solved for $C_0, C_1, \ldots, C_{2n}$:

Fast multiplication of polynomials - ctd.

$$\begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

Fast multiplication of polynomials - ctd.

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But the inverse matrix also involves only constants depending on n only;

Fast multiplication of polynomials - ctd.

$$\begin{pmatrix} 1 & -n & (-n)^2 & \dots & (-n)^{2n} \\ 1 & -(n-1) & (-(n-1))^2 & \dots & (-(n-1))^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & \dots & n^{2n} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{2n} \end{pmatrix} = \begin{pmatrix} P_C(-n) \\ P_C(-(n-1)) \\ \vdots \\ P_C(n) \end{pmatrix},$$

i.e.,

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Thus the coefficients C_i can be obtained in linear time.

So herre is the algorithm we have just described:

function MULT(n, A, B)
  if $|A| = |B| = 1$ **then return** AB
  else
    obtain $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ such that

$$A = A_n 2^{n\,k} + A_{n-1} 2^{(n-1)\,k} + \ldots + A_0$$

$$B = B_n 2^{n\,k} + B_{n-1} 2^{(n-1)\,k} + \ldots + B_0$$

    form polynomials

$$P_A(x) = A_n x^n + A_{n-1} x^{(n-1)} + \ldots + A_0$$

$$P_B(x) = B_n x^n + B_{n-1} x^{(n-1)} + \ldots + B_0$$

    for $m = -n$ to $m = n$ **do**
      compute $P_A(m)$ and $P_B(m)$;
      $P_C(m) \leftarrow$ MULT($n, P_A(m) P_B(m)$)
    end for
    compute $C_0,\ C_1, \ldots\ C_{2n}$ via

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    form $P_C(x) = C_{2n} x^{2n} + \ldots + C_0$ and compute $P_C(2^k)$
    return $P_C(2^k)$
  end if
end function

How fast is our algorithm?

- For each m such that $-n \leq m \leq n$, the value of $P_A(m)$ is a sum of $n+1$ values;
- Each value is a product of a k-bit number A_j with a constant m^j, $-n \leq m \leq n$:

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- **Each $P_A(m)$ has $k+s$ bits, where s is a constant independent of k.**

Generalizing Karatsuba's algorithm

- Recall that

$$P_A(x) \cdot P_B(x) = \sum_{j=0}^{2n} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)} x^j = \sum_{j=0}^{2n} \underbrace{C_j} \, x^j \qquad \text{where} \qquad C_j = \sum_{i=0}^{j} A_i B_{j-i}$$

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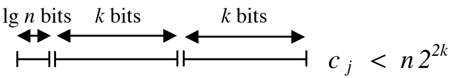

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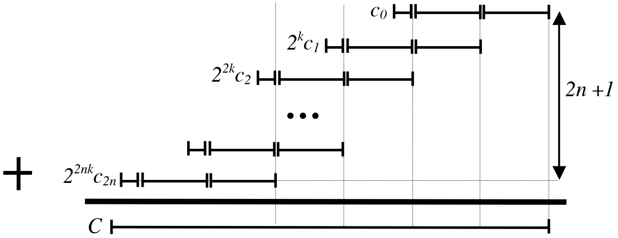

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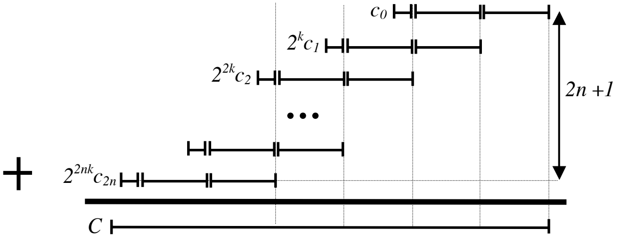

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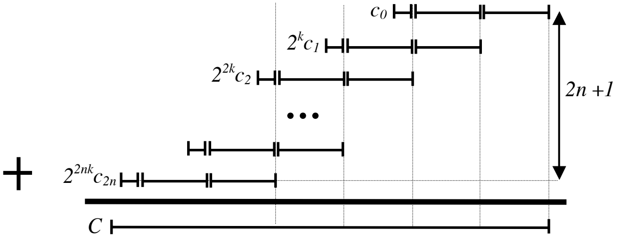

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- Thus, **evaluation of $P_C(2^k)$ takes $O(k)$ many steps.**

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- Thus, with $a = 2n+1$ and $b = n+1$ the first case of the Master Theorem applies;
- we get:

$$T(N) = \Theta\left(N^{\log_b a}\right) = \Theta\left(N^{\log_{n+1}(2n+1)}\right)$$

- Note that

$$N^{\log_{n+1}(2n+1)} < N^{\log_{n+1} 2(n+1)} = N^{\log_{n+1} 2+\log_{n+1}(n+1)}$$

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- Thus, we would have to slice input numbers into $2^{10} = 1024$ pieces.

- We would have to evaluate polynomials $P_A(x)$ and $P_B(x)$ at values up to $n = 2^{10} - 1$, which involves computing

$$n^n = (2^{10} - 1)^{2^{10}-1} = 1.27 \times 10^{3079}$$

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- Answer: YES; they are the complex numbers z_i lying on the unit circle, i.e., such that $|z_i| = 1$!

Digression: Convolution

- Let $a = \langle A_n, A_{n-1}, \ldots, A_1, A_0 \rangle$ and $b = \langle B_n, B_{n-1}, \ldots B_1, B_0 \rangle$ be two sequences;
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$$a * b = \langle A_n B_n,\ A_{n-1} B_n + A_n B_{n-1},\ A_{n-2} B_n + A_{n-1} B_{n-1} + A_n B_{n-2},$$
$$\ldots, A_2 B_0 + A_1 B_1 + A_0 B_2,\ A_1 B_0 + A_0 B_1,\ A_0 B_0 \rangle$$