# Algorithms: COMP3121/3821/9101/9801

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LECTURE 4: FAST FOURIER TRANSFORM

# Fast multiplication of polynomials

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- We saw that in this case we have

$$P_A(x) \cdot P_B(x) \quad = \quad \sum_{j=0}^{2n} \left( \sum_{i=0}^{j} A_i B_{j-i} \right) x^j$$

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- Note that the indices of $A_i$ and $B_{j-i}$ in the $j^{th}$ term all sum up to $j$.

# Coefficient vs value representation of polynomials

- Every polynomial $P_A(x)$ of degree $n$ is uniquely determined by its values at any $n+1$ distinct input values for $x$:

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- If $P_A(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_0$, we can write in matrix form:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} P_A(x_0) \\ P_A(x_1) \\ \vdots \\ P_A(x_n) \end{pmatrix}. \tag{1}$$

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- It can be shown that if $x_i$ are all distinct, then this matrix is invertible.

# Coefficient vs value representation of polynomials - ctd.

- Thus, if all $x_i$ are distinct, given any values $P_A(x_0), P_A(x_1), \ldots, P_A(x_n)$ the coefficients $A_0, A_1, \ldots, A_n$ are uniquely determined:

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  2. a representation of a polynomial $P_A(x)$ via its values

$$P_A(x) \leftrightarrow \{(x_0, P_A(x_0)), (x_1, P_A(x_1)), \ldots, (x_n, P_A(x_n))\}$$

# Our strategy to multiply polynomials fast:

- Given two polynomials of degree at most $n$,

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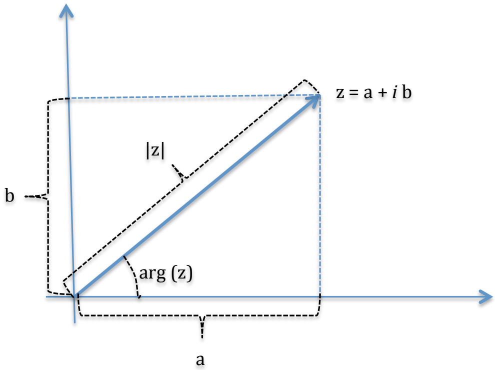
**Key Question:** What values should we take for $x_0, \ldots, x_{2n}$ to avoid "explosion" of size when we evaluate $x_i^n$ while computing $P_A(x_i) = A_n x_i^n + \ldots + A_0$?

# Complex numbers revisited

Complex numbers $z = a + ib$ can be represented using their *modulus* $|z| = \sqrt{a^2 + b^2}$ and their *argument*, $\arg(z)$, which is an angle taking values in $(-\pi, \pi]$ and satisfying:

$$z = |z|e^{i\arg(z)} = |z|(\cos\arg(z) + i\sin\arg(z)),$$

see figure below.



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Recall that

$$z^n = \left(|z|e^{i\ \arg(z)}\right)^n = |z|^n e^{i\,n\arg(z)} = |z|^n(\cos(n\arg(z)) + i\sin(n\arg(z))),$$

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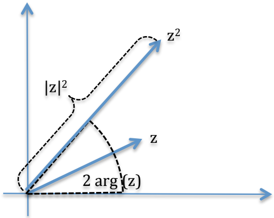

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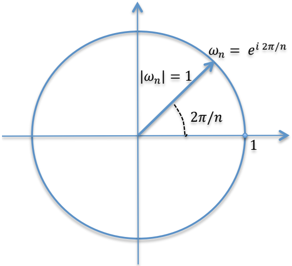


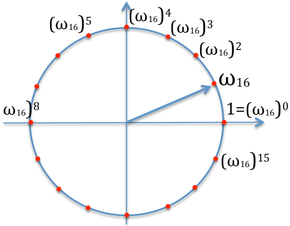


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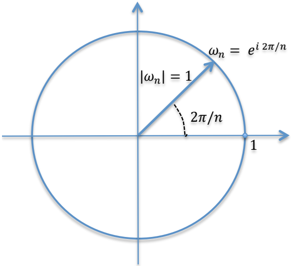


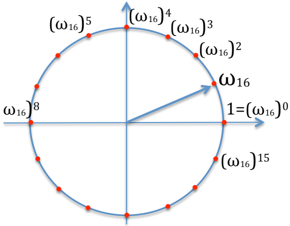


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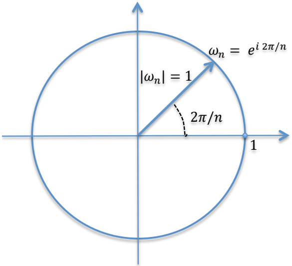


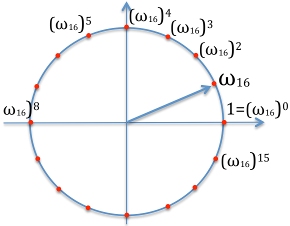


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- A root of unity $\omega$ of order $n$ is "primitive" not if it is uncivilised, but
- if all other roots of unity (of the same order) can be obtained as its powers $\omega^k$.

# Complex roots of unity

- For $\omega_n = e^{i\,2\pi/n}$

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- Since $\omega_n^k$ are roots of unity for $k = 0, 1, \ldots, n-1$ and there are exactly $n$ roots of unity of order $n$ (i.e., solutions to the equation $x^n - 1 = 0$) we get that every root of unity of order $n$ is of the form $\omega_n^k$.
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- For any power $m$ of a root of unity $\omega_n^k$ we have $(\omega_n^k)^m = \omega_n^{k\,m}$
- if $k\,m > n$ then for some integers $p \geq 1$ and $0 \leq l < n$ we have $k\,m = p\,n + l$ (i.e., $k\,m = l \mod n$) and thus $\omega_n^{k\,m} = \omega_n^{p\,n+l} = \omega_n^{p\,n}\omega_n^l = (\omega_n^n)^p\omega_n^l = \omega_n^l$.

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- So in the set of all roots of unity of order $n$, i.e., $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ we can multiply any two elements or raise an element to any power without going out of this set.

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- Note that this is not true for addition, i.e., the sum of two roots of unity is NOT another root of unity!

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- squares of the roots of unity of order $2n$ are just the roots of unity of order $n$.

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- The sequence of values $\langle P_A(1), P_A(\omega_{n+1}), P_A(\omega_{n+1}^2), \ldots, P_A(\omega_{n+1}^n) \rangle$, is called **the Discrete Fourier Transform (DFT)** of the sequence $A = \langle A_0, A_1, \ldots, A_n \rangle$.

# New way for fast multiplication of polynomials

- To multiply two polynomials of degree (at most) $n$ we will evaluate them at the roots of unity of order $2n+1$ (instead of at $-n, \ldots, -1, 0, 1, \ldots, n$ as in Karatsuba's method)

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- we then use the inverse transformation for DFT, called IDFT, to recover the coefficients of the product polynomial from its values at these roots of unity.

# New way for fast multiplication of polynomials

$$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1}x^{n-1} \qquad P_B(x) = B_0 + B_1 x + \ldots + B_{n-1}x^{n-1}$$

$$\Downarrow \text{ DFT} \qquad \Downarrow \text{ DFT}$$

$$\{P_A(1), P_A(\omega_{2n+1}), P_A(\omega_{2n+1}^2), \ldots, P_A(\omega_{2n+1}^{2n})\}; \quad \{P_B(1), P_B(\omega_{2n+1}), P_B(\omega_{2n+1}^2), \ldots, P_B(\omega_{2n+1}^{2n})\}$$

$$\Downarrow \text{ multiplication}$$

$$\{P_A(1)P_B(1), \quad P_A(\omega_{2n+1})P_B(\omega_{2n+1}), \ldots, P_A(\omega_{2n+1}^{2n})P_B(\omega_{2n+1}^{2n})\}$$

$$\Downarrow \text{IDFT}$$

$$P_C(x) = \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i}\right)}_{C_j} x^j = \sum_{j=0}^{2n} C_j x^j = P_A(x) \cdot P_B(x)$$

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- Thus, since $k$ ranges from 0 to $2n$, we would have to do $O(n^2)$ multiplications.
- Can we do it faster??
- This is precisely what the **Fast Fourier Transform (FFT)** does; it computes all of the values $P_A(\omega_{2n+1}^k)$ in $\mathbf{O(n \log n)}$ time.

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    - <u>Exercise:</u> show that for every $n$ which is not a power of two the smallest power of 2 larger than $n$ is smaller than $2n$.
    - *Hint:* consider $n$ in binary. How many bits does the nearest power of two have?

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&= A_0 + A_2x^2 + A_4(x^2)^2 + \ldots + A_{n-2}(x^2)^{\frac{n}{2}-1} \\
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- Note that the degree of the polynomials $A^0(y)$ and $A^1(y)$ is **half** of the degree of the polynomial $P_A(x)$.

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# Fast multiplication of polynomials

- **Problem of size $n$:**
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$$\begin{aligned}
P_A(\omega_n^{\frac{n}{2}+m}) &= A^0((\omega_n^{\frac{n}{2}+m})^2) + \omega_n^{\frac{n}{2}+m} A^1((\omega_n^{\frac{n}{2}+m})^2) \\
&= A^0(\omega_n^{n+2m}) + \omega_n^{\frac{n}{2}}\,\omega_n^m A^1(\omega_n^{n+2m}) \\
&= A^0(\omega_n^n\omega_n^{2m}) + \omega_{2\frac{n}{2}}^{\frac{n}{2}}\,\omega_n^m A^1(\omega_n^n\omega_n^{2m}) \\
&= A^0(\omega_n^{2m}) + \omega_2\,\omega_n^m A^1(\omega_n^{2m}) \\
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- Compare this with $P_A(\omega_n^m) = A^0((\omega_n^m)^2) + \omega_n^m A^1((\omega_n^m)^2)$

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- So we can replace evaluations of

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- We can now write a pseudo-code for our FFT algorithm:

# FFT algorithm

**function** FFT$(A)$<br>
    $n \leftarrow \text{length}[A]$<br>
    **if** $n = 1$ **then return** $A$<br>
    **else**<br>
        $A^{[0]} \leftarrow (A_0, A_2, \dots A_{n-2});$<br>
        $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1});$<br>
        $y^{[0]} \leftarrow FFT(A^{[0]});$<br>
        $y^{[1]} \leftarrow FFT(A^{[1]});$<br>
        $\omega_n \leftarrow e^{i\frac{2\pi}{n}};$<br>
        $\omega \leftarrow 1;$<br>
        **for** $k = 0$ to $k = \frac{n}{2} - 1$ **do**;<br>
            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]};$<br>
            $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$<br>
            $\omega \leftarrow \omega \cdot \omega_n;$<br>
        **end for**<br>
        **return** $y$<br>
    **end if**<br>
**end function**

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- The Master Theorem gives $T(n) = \Theta(n \log n)$.

# Recall our strategy for fast multiplication of polynomials

- Evaluation of a polynomial $P_A(x) = A_0 + A_1x + \ldots + A_{n-1}x^{n-1}$ at roots of unity $\omega_n^k$ of order $n$ can be represented in the matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \ldots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} \tag{3}$$

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- The FFT is just a method replacing this matrix-vector multiplication taking $n^2$ many multiplications with an $n \log n$ procedure;
- From $P_A(1) = P_A(\omega_n^0), \;\; P_A(\omega_n), \;\; P_A(\omega_n^2), \ldots, P_A(\omega_n^{n-1})$, we get the coefficients from

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \ldots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} \tag{4}$$

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\begin{pmatrix}
1 & 1 & 1 & \dots & 1 \\
1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\
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\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)}
\end{pmatrix}^{-1}
=
$$

$$
\frac{1}{n}
\begin{pmatrix}
1 & 1 & 1 & \dots & 1 \\
1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\
1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\
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1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)}
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To see this, note that if we compute the product

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$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \frac{1-\omega_n^{(i-j)n}}{1-\omega_n^{i-j}} = \frac{1-(\omega_n^n)^{i-j}}{1-\omega_n^{i-j}} = \frac{1-1}{1-\omega_n^{i-j}} = 0$$

So,

$$\begin{pmatrix} 1 & \omega_n^i & \omega_n^{2\cdot i} & \dots & \omega_n^{i\cdot(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-j} \\ \omega_n^{-2j} \\ \vdots \\ \omega_n^{-(n-1)j} \end{pmatrix} = \sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \begin{cases} n & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{5}$$

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

- We now have

$$
\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =
$$

$$
= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}
$$

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$$

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$$

back to the coefficient form

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1} x^{n-1}$$

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1})\rangle$$

back to the coefficient form

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1} x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1})\rangle$$

back to the coefficient form

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1} x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:(1) the root of unity $\omega_n$ is replaced by $\omega_n^{-1} = e^{-i\frac{2\pi}{n}}$,

- We now have

$$
\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =
$$

$$
= \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}
$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1})\rangle$$

back to the coefficient form

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1} x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:(1) the root of unity $\omega_n$ is replaced by $\omega_n^{-1} = e^{-i\frac{2\pi}{n}}$, and that (2) the resulting values are divided by $n$.

# IFFT algorithm

<u>Inverse Fourier Transform:</u>

**function** IFFT$(A)$
    $n \leftarrow \text{length}[A]$
    **if** $n = 1$ **then return** $A$
    **else**
        $A^{[0]} \leftarrow (A_0, A_2, \ldots A_{n-2})$;
        $A^{[1]} \leftarrow (A_1, A_3, \ldots A_{n-1})$;
        $y^{[0]} \leftarrow FFT(A^{[0]})$;
        $y^{[1]} \leftarrow FFT(A^{[1]})$;
        $\omega_n \leftarrow e^{-i\frac{2\pi}{n}}$; $\Leftarrow$ different from FFT
        $\omega \leftarrow 1$;
        **for** $k = 0$ to $k = \frac{n}{2} - 1$ **do**;
            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
            $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
            $\omega \leftarrow \omega \cdot \omega_n$;
        **end for**
        **return** $\frac{y}{n}$; $\Leftarrow$ different from FFT
    **end if**
**end function**

# Interpretation of DFT

- We have followed the textbook (CLRS);
- however, what CLRS calls DFT, namely, the sequence

$$\langle P_A(\omega_n^0), P_A(\omega_n^1), P_A(\omega_n^2), \ldots, P_A(\omega_n^{n-1})\rangle$$

  is usually considered the Inverse Discrete Fourier Transform (IDFT) of the sequence of the coefficients

$$\langle A_0, A_1, A_2, \ldots, A_{n-1}\rangle$$

  of the polynomial $P_A(x)$;
- $$\langle P_A(\omega_n^0), P_A(\omega_n^{-1}), P_A(\omega_n^{-2}), \ldots, P_A(\omega_n^{-(n-1)})\rangle$$

  is considered the "forward operation" i.e., the DFT.
- taking this as the "forward operation" has an important conceptual advantage and is used more often than the textbook's choice.

# Interpretation of DFT

- Another "tweak" of DFT: note that

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

$$= n \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

implies:

# Interpretation of DFT

$$
\begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \vdots & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{n}} & \frac{\omega_n}{\sqrt{n}} & \frac{\omega_n^{2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{n-1}}{\sqrt{n}} \\
\frac{1}{\sqrt{n}} & \frac{\omega_n^{2}}{\sqrt{n}} & \frac{\omega_n^{2\cdot 2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{2\cdot(n-1)}}{\sqrt{n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{\sqrt{n}} & \frac{\omega_n^{n-1}}{\sqrt{n}} & \frac{\omega_n^{2(n-1)}}{\sqrt{n}} & \vdots & \frac{\omega_n^{(n-1)(n-1)}}{\sqrt{n}}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \vdots & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{n}} & \frac{\omega_n^{-1}}{\sqrt{n}} & \frac{\omega_n^{-2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-(n-1)}}{\sqrt{n}} \\
\frac{1}{\sqrt{n}} & \frac{\omega_n^{-2}}{\sqrt{n}} & \frac{\omega_n^{-2\cdot 2}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-2\cdot(n-1)}}{\sqrt{n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{\sqrt{n}} & \frac{\omega_n^{-(n-1)}}{\sqrt{n}} & \frac{\omega_n^{-2(n-1)}}{\sqrt{n}} & \vdots & \frac{\omega_n^{-(n-1)(n-1)}}{\sqrt{n}}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & \dots & 0 \\
0 & 1 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1
\end{pmatrix}
$$

Thus, these two matrices are inverses of each other.

# Interpretation of DFT

- This motivates us to "tweak" the definition of DFT:
- Given a sequence of numbers $(A_0, A_1, \ldots, A_{n-1})$ the Discrete Fourier Transform of this sequence is the sequence of the values of the polynomial

$$A^*(x) = \frac{1}{\sqrt{n}}\left(A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}\right)$$

for $x = \omega_n^{-k}$ for $k = 0, \ldots, n-1$; i.e., the sequence of values $A^*(\omega_n^{-k})$:

$$A^*(\omega_n^{-k}) = \frac{1}{\sqrt{n}}\left(A_0(\omega_n^{-k})^0 + A_1(\omega_n^{-k})^1 + \ldots + A_{n-1}(\omega_n^{-k})^{n-1}\right)$$

- Given a sequence of numbers $(A_0, A_1, \ldots, A_{n-1})$ the **Inverse Discrete Fourier Transform** of this sequence is the sequence of the values of the same polynomial

$$A^*(x) = \frac{1}{\sqrt{n}}\left(A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}\right)$$

but for $x = \omega_n^{k}$ for $k = 0, \ldots, n-1$; i.e., the sequence of values $A^*(\omega_n^{k})$

$$A^*(\omega_n^{k}) = \frac{1}{\sqrt{n}}\left(A_0(\omega_n^{k})^0 + A_1(\omega_n^{k})^1 + \ldots + A_{n-1}(\omega_n^{k})^{n-1}\right)$$

# Interpretation of DFT

**function** FFT($A$)
  $n \leftarrow \text{length}[A]$
  **if** $n = 1$ **then return** $A$
  **else**
    $A^{[0]} \leftarrow (A_0, A_2, \dots A_{n-2})$;
    $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1})$;
    $y^{[0]} \leftarrow FFT(A^{[0]})$;
    $y^{[1]} \leftarrow FFT(A^{[1]})$;
    $\omega_n \leftarrow e^{-i\frac{2\pi}{n}}$;
    $\omega \leftarrow 1$;
    **for** $k = 0$ to $k = \frac{n}{2} - 1$ **do**;
      $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
      $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
      $\omega \leftarrow \omega \cdot \omega_n$;
    **end for**
    **return** $\frac{y}{\sqrt{n}}$;
  **end if**
**end function**

**function** IFFT($A$)
  $n \leftarrow \text{length}[A]$
  **if** $n = 1$ **then return** $A$
  **else**
    $A^{[0]} \leftarrow (A_0, A_2, \dots A_{n-2})$;
    $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1})$;
    $y^{[0]} \leftarrow FFT(A^{[0]})$;
    $y^{[1]} \leftarrow FFT(A^{[1]})$;
    $\omega_n \leftarrow e^{i\frac{2\pi}{n}}$;
    $\omega \leftarrow 1$;
    **for** $k = 0$ to $k = \frac{n}{2} - 1$ **do**;
      $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
      $y_{\frac{n}{2}+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
      $\omega \leftarrow \omega \cdot \omega_n$;
    **end for**
    **return** $\frac{y}{\sqrt{n}}$;
  **end if**
**end function**

# Interpretation of DFT

- *scalar product* (also called *dot* product) of two vectors with real coordinates, $\vec{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\vec{y} = (y_0, y_1, \ldots, y_{n-1})$, denoted by $\langle \vec{x}, \vec{y} \rangle$ is defined as

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=0}^{n-1} x_i y_i$$

- If the coordinates of our vectors are complex numbers, then the scalar product of such two vectors is defined as

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=0}^{n-1} x_i \overline{y_i}$$

where $\overline{z}$ denotes the complex conjugate of $z$, i.e., $\overline{a + i\,b} = a - i\,b$.

# Interpretation of DFT

- Note that

$$\overline{\omega_n^k} = \overline{e^{i\frac{2\pi k}{n}}} = \overline{\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}} = \cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n}$$
$$= \cos\frac{-2\pi k}{n} + i\sin\frac{-2\pi k}{n} = e^{-i\frac{2\pi k}{n}} = \omega_n^{-k}$$

- Thus, what we had before,

$$\begin{pmatrix} 1 & \omega_n^k & \omega_n^{2\cdot k} & \dots & \omega_n^{k\cdot(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-m} \\ \omega_n^{-2m} \\ \vdots \\ \omega_n^{-(n-1)m} \end{pmatrix} = \sum_{j=0}^{n-1} \omega_n^{(k-m)j} = \begin{cases} n & \text{if } \; k = m \\ 0 & \text{if } \; k \neq m \end{cases} \tag{6}$$

simply means that for $k \neq m$ vectors $\left(1, \;\; \omega_n^k, \;\; \omega_n^{2\cdot k}, \;\; \dots, \;\; \omega_n^{k\cdot(n-1)}\right)$ and $\left(1, \;\; \omega_n^m, \;\; \omega_n^{2\cdot m}, \;\; \dots, \;\; \omega_n^{m\cdot(n-1)}\right)$ are orthogonal.

# Interpretation of DFT

- If we define $$\vec{e_k} = \frac{1}{\sqrt{n}} \left( \omega_n^{k\cdot 0},\ \ \omega_n^{k\cdot 1},\ \omega_n^{k\cdot 2},\ \ \ldots\ \ ,\omega_n^{k\cdot(n-1)} \right)$$
  then $$\|\vec{e_k}\| = \sqrt{\langle \vec{e_k}, \vec{e_k} \rangle} = 1$$
  and $$\langle \vec{e_k}, \vec{e_m} \rangle = 0 \quad \text{for} \quad k \neq m$$

- thus, the set of vectors $\{\vec{e_0}, \vec{e_1}, \ldots, \vec{e_n}\}$ is an orthonormal base of the vector space of all complex valued sequences of length $n$.
- Let $\vec{A} = (A_0, A_1, A_2, \ldots, A_{n-1})$; then for the DFT of this sequence we have

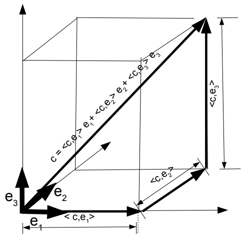
$$\begin{aligned}
\frac{1}{\sqrt{n}} P_A(\omega_n^{-k}) &= \frac{A_0}{\sqrt{n}}(\omega_n^{-k})^0 + \frac{A_1}{\sqrt{n}}(\omega_n^{-k})^1 + \frac{A_2}{\sqrt{n}}(\omega_n^{-k})^2 + \ldots + \frac{A_{n-1}}{\sqrt{n}}(\omega_n^{-k})^{n-1} \\
&= A_0 \frac{(\overline{\omega_n^k})^0}{\sqrt{n}} + A_1 \frac{(\overline{\omega_n^k})^1}{\sqrt{n}} + A_2 \frac{(\overline{\omega_n^k})^2}{\sqrt{n}} + \ldots + A_{n-1} \frac{(\overline{\omega_n^k})^{n-1}}{\sqrt{n}} \\
&= \left\langle (A_0, A_1, A_2, \ldots, A_{n-1}), \left( \frac{(\omega_n^k)^0}{\sqrt{n}}, \frac{(\omega_n^k)^1}{\sqrt{n}}, \frac{(\omega_n^k)^2}{\sqrt{n}}, \ldots, \frac{(\omega_n^k)^{n-1}}{\sqrt{n}} \right) \right\rangle \\
&= \langle \vec{A}, \vec{e_k} \rangle
\end{aligned}$$

- Thus, the DFT of a vector $\vec{A}$ is simply the sequence of projections of $\vec{A}$ onto the base vectors $\vec{e_k}$, $(k = 0, \ldots, n-1)$.

# Interpretation of DFT

- In an $n$-dimensional vector space $V$ with an orthonormal base $\mathbf{B}$ every vector $\vec{A}$ can be represented as a linear combination of the base vectors with coefficients equal to the projections of $\vec{A}$ onto the base vectors, i.e., the scalar product $\langle \vec{A}, e_k \rangle$:

$$\vec{A} = \langle \vec{A}, \vec{e_0} \rangle \vec{e_0} + \langle \vec{A}, \vec{e_1} \rangle \vec{e_1} + \ldots + \langle \vec{A}, \vec{e}_{n-1} \rangle \vec{e}_{n-1}$$




Representing vector $c$ as a linear combination of the basis vectors $e_1, e_2, e_3$ with projections as coefficients

# Interpretation of DFT

- Thus, in our case

$$\begin{aligned}\vec{A} &= \langle \vec{A}, \vec{e_0} \rangle \vec{e_0} + \langle \vec{A}, \vec{e_1} \rangle \vec{e_1} + \ldots + \langle \vec{A}, \vec{e}_{n-1} \rangle \vec{e}_{n-1} \\ &= \frac{P_A(\omega_n^0)}{\sqrt{n}} \vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}} \vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}} \vec{e}_{n-1}\end{aligned}$$

- Looking at the $k^{th}$ cordnate of both the left and the right side we get

$$A_k = \frac{P_A(\omega_n^0)}{\sqrt{n}} \frac{(\omega_n^0)^k}{\sqrt{n}} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}} \frac{(\omega_n^1)^k}{\sqrt{n}} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}} \frac{(\omega_n^{n-1})^k}{\sqrt{n}} \quad (7)$$

$$= \frac{P_A(\omega_n^0)}{\sqrt{n}} \frac{(\omega_n^k)^0}{\sqrt{n}} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}} \frac{(\omega_n^k)^1}{\sqrt{n}} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}} \frac{(\omega_n^k)^{n-1}}{\sqrt{n}} \quad (8)$$

- $A_k$ is obtained evaluating the polynomial

$$\frac{P_A(\omega_n^0)}{\sqrt{n}} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}} \frac{x}{\sqrt{n}} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}} \frac{x^{n-1}}{\sqrt{n}}$$

at $x = \omega_n^k$, which is exactly what the Inverse Discrete Fourier Transform is.

# Interpretation of DFT

- Let us denote the usual orthonormal base of $\mathbb{C}^n$ by $\mathcal{B}$:

$\vec{f_0} = (1,0,0,0,\ldots,0),\ \vec{f_1} = (0,1,0,0,\ldots,0),\ \vec{f_2} = (0,0,1,0,\ldots,0),\ \vec{f}_{n-1} = (0,0,0,0,\ldots,1)$

and by $\mathcal{F}$ the base $\mathcal{F} = \{\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_{n-1}\}$ where $\vec{e_k} = \left(\frac{1}{\sqrt{n}}, \quad \frac{\omega_n^{k\cdot 1}}{\sqrt{n}}, \ \frac{\omega_n^{k\cdot 2}}{\sqrt{n}}, \quad \ldots \quad, \frac{\omega_n^{k\cdot(n-1)}}{\sqrt{n}}\right)$.

- then

$$\vec{A} = (A_0, A_1, A_2, \ldots, A_{n-1})_{\mathcal{B}} = A_0\vec{f_0} + A_1\vec{f_1} + A_2\vec{f_2} + \ldots + A_{n-1}\vec{f}_{n-1}$$

and also

$$\vec{A} = (P_A(\omega_n^0), P_A(\omega_n^1), A_2, \ldots, P_A(\omega_n^{n-1}))_{\mathcal{F}} = \frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n}$$

- DFT is just change of base operation: it transforms the sequence of coordinates

$$(A_0, A_1, A_2, \ldots, A_{n-1})_{\mathcal{B}}$$

in the base $\mathcal{B}$ of vector $A$ into the sequence

$$\left(\frac{P_A(\omega_n^0)}{\sqrt{n}}, \frac{P_A(\omega_n^{-1})}{\sqrt{n}}, \ldots, \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\right)_{\mathcal{F}}$$

of the coordinates in the base $\mathcal{F}$;

# Interpretation of DFT

- The $k^{th}$ coordinate $\frac{P_A(\omega_n^{-k})}{\sqrt{n}}$ is obtained by projecting vector

$$\vec{A} = (A_0, A_1, A_2, \ldots, A_{n-1})_{\mathcal{B}} = A_0 \vec{f_0} + A_1 \vec{f_1} + A_2 \vec{f_2} + \ldots + A_{n-1} \vec{f}_{n-1}$$

onto the corresponding base vector $e_k = ((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1}) \ \in \mathcal{F}$.

# Interpretation of DFT

- The $k^{th}$ coordinate $\frac{P_A(\omega_n^{-k})}{\sqrt{n}}$ is obtained by projecting vector

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  onto the corresponding base vector $e_k = ((\omega_n^k)^0, (\omega_n^k)^1, \dots, (\omega_n^k)^{n-1}) \in \mathcal{F}$.

- Recall that the $k^{th}$ coordinate $A_k$ of $\vec{A}$ in the usual base $\mathcal{B}$ was obtained by looking at the $k^{th}$ coordinate of

$$\vec{A} = \frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \dots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n-1}$$

# Interpretation of DFT

- The $k^{th}$ coordinate $\frac{P_A(\omega_n^{-k})}{\sqrt{n}}$ is obtained by projecting vector

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onto the corresponding base vector $e_k = ((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1}) \ \in \mathcal{F}$.

- Recall that the $k^{th}$ coordinate $A_k$ of $\vec{A}$ in the usual base $\mathcal{B}$ was obtained by looking at the $k^{th}$ coordinate of

$$\vec{A} = \frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n-1}$$

i.e., by projecting $\frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n-1}$ onto the corresponding base vector $\vec{f_k} = (\underbrace{0, 0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0)$.

# Interpretation of DFT

- The $k^{th}$ coordinate $\frac{P_A(\omega_n^{-k})}{\sqrt{n}}$ is obtained by projecting vector

$$\vec{A} = (A_0, A_1, A_2, \ldots, A_{n-1})_{\mathcal{B}} = A_0\vec{f_0} + A_1\vec{f_1} + A_2\vec{f_2} + \ldots + A_{n-1}\vec{f}_{n-1}$$

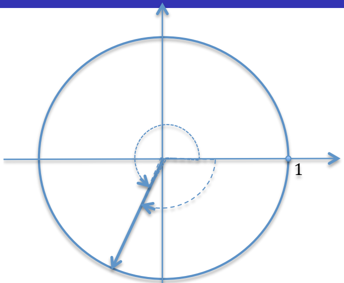
  onto the corresponding base vector $e_k = ((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1}) \in \mathcal{F}$.

- Recall that the $k^{th}$ coordinate $A_k$ of $\vec{A}$ in the usual base $\mathcal{B}$ was obtained by looking at the $k^{th}$ coordinate of

$$\vec{A} = \frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n-1}$$

  i.e., by projecting $\frac{P_A(\omega_n^0)}{\sqrt{n}}\vec{e_0} + \frac{P_A(\omega_n^{-1})}{\sqrt{n}}\vec{e_1} + \ldots + \frac{P_A(\omega_n^{-(n-1)})}{\sqrt{n}}\vec{e}_{n-1}$ onto the corresponding base vector $\vec{f_k} = (\underbrace{0, 0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0)$.

- Thus, the Inverse Discrete Fourier Transform (IDFT) simply transforms the sequence of the coordinates of $\vec{A}$ in the base $\mathcal{F}$ back to the sequence of coordinates of $\vec{A}$ in base $\mathcal{B}$, i.e., into $(A_0, A_1, A_2, \ldots, A_{n-1})_{\mathcal{B}}$




$$(\omega_{2n+1})^k = e^{i\frac{2\pi k}{2n+1}} = e^{-i\frac{2\pi(2n+1-k)}{2n+1}}; \quad k > n$$

- Note that by replacing $n$ with $2n+1$ we get

$$A_k = \frac{P_A(\omega_{2n+1}^0)}{\sqrt{2n+1}}\frac{(\omega_{2n+1}^k)^0}{\sqrt{2n+1}} + \frac{P_A(\omega_{2n+1}^1)}{\sqrt{2n+1}}\frac{(\omega_{2n+1}^k)^1}{\sqrt{2n+1}} + \ldots + \frac{P_A(\omega_{2n+1}^{2n})}{\sqrt{2n+1}}\frac{(\omega_{2n+1}^k)^{2n}}{\sqrt{2n+1}}$$

$$= \frac{P_A(\omega_{2n+1}^k)}{2n+1}e^{i\frac{2\pi\,k\cdot 0}{2n+1}} + \frac{P_A(\omega_{2n+1}^1)}{2n+1}e^{i\frac{2\pi\,k\cdot 1}{2n+1}} + \frac{P_A(\omega_{2n+1}^2)}{2n+1}e^{i\frac{2\pi\,k\cdot 2}{2n+1}} + \ldots + \frac{P_A(\omega_{2n+1}^{2n})}{2n+1}e^{i\frac{2\pi\,k\cdot 2n}{2n+1}}$$

$$= \sum_{j=-n}^{n}\frac{P_A(\omega_{2n+1}^j)}{2n+1}e^{i\frac{2\pi\,k\cdot j}{2n+1}} = \sum_{j=-n}^{n}\left|\frac{P_A(\omega_{2n+1}^j)}{2n+1}\right|e^{i\arg(P_A(\omega_{2n+1}^j))}e^{i\frac{2\pi\,k\cdot j}{2n+1}}$$

# Interpretation of DFT

- Thus,

$$A_k = \sum_{j=-n}^{n} \left| \frac{P_A(\omega_{2n+1}^j)}{2n+1} \right| e^{i \arg(P_A(\omega_{2n+1}^j))} e^{i \frac{2\pi\ k \cdot j}{2n+1}}$$

$$= \sum_{j=-n}^{n} \left| \frac{P_A(\omega_{2n+1}^j)}{2n+1} \right| e^{i \left( \frac{2\pi\ k \cdot j}{2n+1} + \arg(P_A(\omega_{2n+1}^j)) \right)}$$

- If we let

$$a(t) = \sum_{j=-n}^{n} \left| \frac{P_A(\omega_{2n+1}^j)}{2n+1} \right| e^{i \left( \frac{2\pi\ t \cdot j}{2n+1} + \arg(P_A(\omega_{2n+1}^j)) \right)}$$

$$= \sum_{j=-n}^{n} \left| \frac{P_A(\omega_{2n+1}^j)}{2n+1} \right| \left( \cos\left( \frac{2\pi\ \cdot j}{2n+1} t + \arg(P_A(\omega_{2n+1}^j)) \right) + i \sin\left( \frac{2\pi\ \cdot j}{2n+1} t + \arg(P_A(\omega_{2n+1}^j)) \right) \right)$$

then $A_k = a(k)$. Thus the sequence $\langle A_0, A_1, \ldots, A_{2n} \rangle$ has been represented as a linear combination of samples of sinusoids of frequencies $\frac{2\pi k}{2n+1}$ for $k = -n$ to $k = n$.

- If $\vec{A}$ is a real vector, the imaginary part of $a(t)$ cancel out because $P_A(\omega_{2n+1}^{-j}) = \overline{P_A(\omega_{2n+1}^j)}$ and we get a real valued interpolation signal $a(t)$.

# Interpretation of DFT

- Thus, we get

$$a(t) = 2\sum_{j=0}^{n}\left|\frac{P_A(\omega_{2n+1}^{j})}{2n+1}\right|\cos\left(\frac{2\pi \cdot j}{2n+1}t + \arg(P_A(\omega_{2n+1}^{j}))\right)$$

  and again $A_k = a(k)$. Thus the sequence $\langle A_0, A_1, \ldots, A_{2n}\rangle$ has been represented as a linear combination of samples of sinusoids of frequencies $\frac{2\pi k}{2n+1}$ for $k = -n$ to $k = n$.

- In essence we have approximated the signal with a linear combination of pure harmonic oscillations of frequencies $\frac{2\pi k}{2n+1}$ with amplitudes $2\left|\frac{P_A(\omega_{2n+1}^{j})}{2n+1}\right|$ and phase shifts $\arg(P_A(\omega_{2n+1}^{j}))$.