The Subspace Correction Method

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Introduction

What is subspace correction method?

A framework for various iterative methods.

Jacobi and Gauss-Seidel iterations, domain decomposition methods, multigrid methods and so on, can be viewed as two basic types:

- parallel subspace correction(PSC)
- successive subspace correction(SSC)

with different space decompositions and subspace corrections.

Why bother to do it?

A feature of this framework is that a quite general abstract convergence theory can be established.

Subspace Decomposition

A decomposition of V consists of a number of subspaces $V_i \subset V$ such that

$$V = \sum_{i=1}^{J} V_i$$

For each i, we define Π_i' and $\Pi_i^* : V \mapsto V_i$ by

$$(\Pi_i' v, v_i) := (v, v_i) \quad (\Pi_i^* v, v_i)_A := (v, v_i)_A \quad v \in V, v_i \in V_i$$

and $A_i : V_i \mapsto V_i$ by

$$(A_i u_i, v_i) := (A u_i, v_i) \quad u_i, v_i \in V_i$$

By definition, we get

$$A_i \Pi_i^* = \Pi_i' A$$

For $Au = f$, so we get

Subspace equation

$$A_i u_i = f_i, \text{ where } u_i = \Pi_i^* u \text{ and } f_i = \Pi_i' f$$

The subspace equation will be in general solved approximately. To describe this, we define

$$R_i : V_i \mapsto V_i$$

which approximates A_i^{-1} in certain sense.

Example

$$\mathbb{R}^n = \sum_{i=1}^{n} \mathrm{span}\{e_i\}$$

where e_i is the ith column of identity matrix. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$

$$A_i = a_{ii} \qquad \Pi_i' y = y_i e_i$$

where y_i is the ith component of y.

PSC and SSC

Consider residual equation $Ae = r^{\text{old}}$, and solve it in subspaces $A_i \hat{e}_i = \Pi_i' r^{\text{old}}$ and get

$$\hat{e}_i = R_i \Pi_i' r^{\text{old}}$$

We can update solution in two ways:

- PSC (Parallel):

$$u^{\text{new}} = u^{\text{old}} + \sum_{i=1}^{J} \hat{e}_i$$

- SSC (Successive):

$$u^{(1)} = u^{\text{old}} + R_1 \Pi_1'(f - Au^{\text{old}})$$

$$u^{(2)} = u^{(1)} + R_2 \Pi_2'(f - Au^{(1)})$$

$$\cdots\cdots$$

$$u^{\text{new}} = u^{(\text{J}-1)} + R_J \Pi_J'(f - Au^{(J-1)})$$

Example

PSC/SSC with subspace decomposition

$$\mathbb{R}^n = \sum_{i=1}^{n} \mathrm{span}\{e_i\}$$

and exact subspace solver

$$R_i = A_i^{-1}$$

is Jacobi/Gauss-Seidel iterations.

"Subspace" correction via auxiliary spaces

Subspace decomposition is not available in some cases.

Space decomposition via auxiliary spaces

Select a sequence of spaces $V_1, \cdots, V_J$, each of them is related to V by a linear map:

$$\Pi_i : V_i \mapsto V$$

Again, we assume,

$$V = \sum_{i=1}^{J} \Pi_i V_i$$

Auxiliary space: $\underset{\sim}{V} = V_1 \times V_2 \times \cdots \times V_J$
Define that $\Pi = (\Pi_1, \Pi_2, \cdots, \Pi_J) : \underset{\sim}{V} \mapsto V$,

$$\Pi \underset{\sim}{v} = \sum_{i=1}^{J} \Pi_i v_i, \qquad \underset{\sim}{v} = (v_1, v_2, \cdots, v_J)$$

which is surjective.
For linear system: $Au = f$, $u \in V$

$$Au = f \Longleftrightarrow A\Pi \underset{\sim}{u} = f \Longleftrightarrow \Pi' A \Pi \underset{\sim}{u} = \Pi' b$$

We get the expanded system,

$$\underset{\sim}{A} \underset{\sim}{u} = \underset{\sim}{f}$$

where

$$\underset{\sim}{A} = \Pi' A \Pi, \quad \underset{\sim}{f} = \Pi' f$$

Iterative methods for expanded system

Theorem

Iterative methods for $\utilde{A}\utilde{u} = \utilde{f}$:

$$\utilde{u}^m = \utilde{u}^{m-1} + \utilde{B}(\utilde{f} - \utilde{A}\utilde{u}^{m-1})$$

is equivalent to the iterative method for $Au = f$:

$$u^m = u^{m-1} + B(f - Au^{m-1})$$

with

$$u^m = \Pi \utilde{u}^m \qquad B = \Pi \utilde{B} \Pi'$$

Furthermore, if $\utilde{B}$ is SPD, then B is also SPD, and

$$(B^{-1}v, v) = \inf_{\Pi \utilde{v} = v} (\utilde{B}^{-1}\utilde{v}, \utilde{v})$$

Subspace solver

Assume that each V_i is equipped with an inner product $a_i(\cdot, \cdot)$.

$$A_i \curlywedge A$$

The following identity holds:

$$\Pi_i' A = A_i P_i$$

Subspace solver

Assume that each V_i is equipped with an inner product $a_i(\cdot,\cdot)$. We define A_i by

$$(A_i u_i, v_i) := a_i(u_i, v_i), \forall u_i, v_i \in V_i$$

and define Π_i' be the adjoint of Π_i,

$$(\Pi_i' f, v_i) := (f, \Pi v_i), \forall f \in V', v_i \in V_i$$

and define $P_i = \Pi_i^* : V \mapsto V_i$ be the adjoint of Π with respect to A-products.

$$(P_i u, v_i)_{A_i} := (u, \Pi_i v_i)_A, \forall u \in V, v_i \in V_i$$

$$A_i \curlywedge A$$

The following identity holds:

$$\Pi_i' A = A_i P_i$$

Subspace correction

if u is the solution of $Au = f$, then

$$A_i u_i = f_i$$

where

$$u_i = P_i u, f_i = \Pi_i' f$$

This equation may be regarded as the restriction to V_i. Assume each A_i has an approximate inverse or preconditioner:

$$R_i : {V_i}' \mapsto V_i$$

Algorithm 1 Parallel subspace correction

Require: $u^0 \in V$

1. **repeat**
2. $\quad v \leftarrow u^k$
3. $\quad$ **for** i=1:J **do**
4. $\quad\quad v \leftarrow v + \Pi_i R_i \Pi'(f - Au^k)$
5. $\quad$ **end for**
6. $\quad u^{k+1} \leftarrow v$
7. **until** Convergence

It can be written as

$$u^{k+1} = u^k + B_{psc}(f - Au_k)$$

where

$$B_{psc} = \sum_{i=1}^{J} \Pi_i R_i \Pi_i'$$

Algorithm 2 Successvie subspace correction

Require: $u^0 \in V$

1. **repeat**
2. $v \leftarrow u^k$
3. **for** i=1:J **do**
4. $v \leftarrow v + \Pi_i R_i \Pi'(f - Av)$
5. **end for**
6. $u^{k+1} \leftarrow v$
7. **until** Convergence

It can be written as

$$u^{k+1} = u^k + B_{ssc}(f - Au_k)$$

where B_{ssc} is defined by
$I - B_{ssc}A = (I - T_J)(I - T_{J-1}) \cdots (I - T_1)$ and $T_i = \Pi_i R_i \Pi_i' A$

PSC and SSC in the view from expanded system

Theorem

The PSC for $Au = f$ is equivalent to the modified Jacobi method for $\utilde{A}\utilde{u} = \utilde{f}$:

$$\utilde{u}^m = \utilde{u}^{m-1} + \utilde{R}(\utilde{f} - \utilde{A}\utilde{u}^{m-1})$$

The SSC for $Au = f$ is equivalent to the modified GS method for $\utilde{A}\utilde{u} = \utilde{f}$:

$$\utilde{u}^m = \utilde{u}^{m-1} + (\utilde{R}^{-1} + \utilde{L})^{-1}(\utilde{f} - \utilde{A}\utilde{u}^{m-1})$$

Convergence analysis for SSC

Theorem

Assume that $\bar{\underset{\sim}{R}} = \underset{\sim}{R}' + \underset{\sim}{R} - \underset{\sim}{R}'\underset{\sim}{D}\underset{\sim}{R}$ *is SPD, Then for SSC*

$$\bar{\underset{\sim}{B}}^{-1} = (\underset{\sim}{I} + \underset{\sim}{R}'\underset{\sim}{U})'\bar{\underset{\sim}{R}}^{-1}(\underset{\sim}{I} + \underset{\sim}{R}'\underset{\sim}{U})$$

$$= \underset{\sim}{A} + (\underset{\sim}{D} + \underset{\sim}{U} - \underset{\sim}{R}^{-1})'\underset{\sim}{R}\bar{\underset{\sim}{R}}^{-1}\underset{\sim}{R}'(\underset{\sim}{D} + \underset{\sim}{U} - \underset{\sim}{R}^{-1})$$

XZ identity: general cases

Theorem

The SSC is convergent if each subspace solver is convergent:

$$\|I - BA\|_A^2 = 1 - \frac{1}{c_1} = 1 - \frac{1}{1 + c_0}$$

where

$$c_1 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|v_i + R_i' A_i P_i \sum_{j=i+1}^{J} \Pi_j v_j\|_{\bar{R}_i^{-1}}^2$$

and

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|R_i'(A_i P_i \sum_{j=i}^{J} \Pi_j v_j - R_i^{-1} v_i\|_{\bar{R}_i^{-1}}^2$$

XZ identity: special cases

Theorem

If each subspace solver is exact, i.e. $R_i = A_i^{-1}$, *it holds that*

$$\|I - BA\|_A^2 = 1 - \frac{1}{c_1} = 1 - \frac{1}{1 + c_0}$$

where

$$c_1 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i}^{J} \Pi_j v_j\|_{A_i}^2$$

and

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i+1}^{J} \Pi_j v_j\|_{A_i}^2$$

Remarks:

- PSC convergence?
  Jacobi method is not convergent for all SPD problems.
- PSC/SSC is often used as preconditioner.

Linear iteration as preconditioner

A sufficient condition for convergence:

$$k = \|I - BA\|_A < 1$$

which means, when B is symmetric,

$$\kappa(BA) \leq \frac{1+k}{1-k}$$

Then if B is used as preconditioner for CG,

$$\delta = \frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1} \leq \frac{\sqrt{\frac{1+k}{1-k}} - 1}{\sqrt{\frac{1+k}{1-k}} + 1} = \frac{1 - \sqrt{1-k^2}}{k} < k$$