线性代数 A 答疑

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命题 7.6.2(矩阵理论苏育才等) 设 n 阶方阵 A 的最小多项式为

$$m(\lambda) = (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_s)^{k_s},$$

则

$$g(\lambda) = \sum_{i=1}^{s} m_i(\lambda)\left(a_{i0} + a_{i1}(\lambda-\lambda_i) + a_{i2}(\lambda-\lambda_i)^2 + \cdots a_{ik_i}(\lambda-\lambda_i)^{k_i-1}\right)$$

满足如下 Hermite 插值条件

$$g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), \quad i = 1,\cdots,s,\ j = 0,\cdots,k_i-1,$$

其中

$$m_i(\lambda) = m(\lambda)/(\lambda-\lambda_i)^{k_i},$$
$$a_{ij} = \frac{1}{j!}\left(\frac{f(\lambda)}{m_i(\lambda)}\right)^{(j)}\Bigg|_{\lambda=\lambda_i}.$$

证明 只需验证插值条件即可. 注意到, 当 $j \le k_i - 1$ 时,

$$g^{(j)}(\lambda)\Big|_{\lambda=\lambda_i} = \left[m_i(\lambda)\left(a_{i0} + a_{i1}(\lambda-\lambda_i) + a_{i2}(\lambda-\lambda_i)^2 + \cdots a_{ik_i}(\lambda-\lambda_i)^{k_i-1}\right)\right]^{(j)}\Big|_{\lambda=\lambda_i}$$

只需验证上式等于 $f^{(j)}(\lambda_i)$ 即可.

对 j 进行数学归纳. 当 $j=0$ 时，易见成立. 当 $j=1$ 时,

$$\begin{aligned}
g^{(1)}(\lambda_i) &= m_i(\lambda_i)^{(1)}a_{i0} + m_i(\lambda_i)a_{i1} \\
&= \left(m_i(\lambda)^{(1)}\frac{f(\lambda)}{m_i(\lambda)} + m_i(\lambda)\left(\frac{f(\lambda)}{m_i(\lambda)}\right)^{(1)}\right)\Bigg|_{\lambda=\lambda_i} \\
&= f(\lambda_i)^{(1)}.
\end{aligned}$$

假设 $j<k$ 时成立, 其中 $k \le k_i - 1$. 当 $j = k$ 时

$$\begin{aligned}
g^{(j)}(\lambda_i) &= \sum_{p=0}^{j} C_j^p m_i(\lambda_i)^{(p)}(j-p)!a_{i(j-p)} \\
&= \left(\sum_{p=0}^{j} C_j^p m_i(\lambda)^{(p)}\left(\frac{f(\lambda)}{m_i(\lambda)}\right)^{(j-p)}\right)\Bigg|_{\lambda=\lambda_i} \\
&= f(\lambda_i)^{(j)}.
\end{aligned}$$

证毕.