2.1 $G\{k^3\}$

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

$$\left(\frac{1}{1-x}\right)' = 1+2x+3x^2+\cdots$$

$$x\left(\frac{1}{1-x}\right)' = x+2x^2+3x^3+\cdots$$

$$\left(x\left(\frac{1}{1-x}\right)'\right)' = 1+2^2x+3^2x^2+\cdots$$

$$x\left(x\left(\frac{1}{1-x}\right)'\right)' = x+2^2x^2+3^2x^3+\cdots$$

$$\left(x\left(x\left(\frac{1}{1-x}\right)'\right)'\right)' = 1+2^3x+3^3x^2+\cdots$$

$$\therefore G(x) = x\left(x\left(x\left(\frac{1}{1-x}\right)'\right)'\right)' = \frac{x(x^2+4x+1)}{(x-1)^4}$$

2.2

$$G(x) = \frac{1}{6}\left[3\times2\times1+4\times3\times2x+\cdots+(n+3)(n+2)(n+1)x^n\right]$$

$$\therefore 令\ F(x) = \int G(x)\,dx = \frac{1}{6}\left[3\times2x+4\times3x^2+\cdots+(n+3)(n+2)x^{n+1}\right]$$

$$令\ H(x) = \int F(x)\,dx = \frac{1}{6}\left[3x^2+4x^3+\cdots+(n+3)x^{n+2}\right]$$

$$令\ I(x) = \int H(x)\,dx = \frac{1}{6}\left[x^3+x^4+\cdots+x^{n+3}\right]$$

$$\because \frac{1}{1-x} = 1+x+\cdots+x^n$$

$$\therefore x^3+x^4+\cdots+x^{n+3} = \left(\frac{1}{1-x}-1-x-x^2\right) = \frac{x^3}{1-x}$$

$$\therefore G(x) = \frac{1}{6}\left(\frac{x^3}{1-x}\right)''' = \frac{1}{(1-x)^4}$$

2.3 $G(x)=\frac{A}{1-9x}+\frac{B}{1+6x}$

$$\begin{cases} A+B=3 \\ 6A-9B=78 \end{cases} \Rightarrow \begin{cases} A=7 \\ B=-4 \end{cases}$$

$\therefore G(x)=\frac{7}{1-9x}-\frac{4}{1+6x}$

$63+24$

$\because \frac{1}{1-x}=1+x+x^2+\dots+x^n$

$$\therefore \begin{cases} \frac{7}{1-9x}=7[1+9x+(9x)^2+\dots+(9x)^n] \\ -\frac{4}{1+6x}=-4[1+(-6x)+(-6x)^2+\dots+(-6x)^n] \end{cases}$$

相加得 $\{a_n\}=\{3,97,\dots,[7\times9^n-4\times(-6)^n]\}$

$a_n=7\times9^n-4\times(-6)^n$

2.4 $G(x) = \dfrac{A}{1-8x} + \dfrac{B}{1+7x}$

$$\begin{cases} A+B=3 \\ 7A-8B=-9 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=2 \end{cases}$$

$$\therefore G(x) = \frac{1}{1-8x} + \frac{2}{1+7x}$$

由 $\dfrac{1}{1-x} = 1+x+\dots+x^n$

$$\begin{cases} \dfrac{1}{1-8x} = 1+8x+\dots+(8x)^n \\ \dfrac{2}{1+7x} = 2(1+(-7x)+\dots+(-7x)^n \end{cases}$$

相加得 $\{a_n\} = \{3, -6, \dots (8^n+2\times(-7)^n)\}$

$$a_n = 8^n + 2\times(-7)^n$$

2.5 (1) $F_{2n} = F_{2n-1} + F_{2n-2}$ ①

$F_{2n-1} = F_{2n-2} + F_{2n-3}$ ②

$F_{2n-2} = F_{2n-3} + F_{2n-4}$ ③

①式变化得 $F_{2n} = 2F_{2n-2} + F_{2n-3}$ ④

$\therefore G_n - 3G_{n-1} + G_{n-2}$

$= F_{2n} - 3F_{2n-2} + F_{2n-4}$

代入④式.

可得原式 $= F_{2n-3} - F_{2n+2} + F_{2n-4}$

$= 0$

得证

(2) $H(x) = G_0 + G_1 x + G_2 x^2 + \cdots + G_n x^n \longrightarrow$ ⑤

$3xH(x) = 3G_0 x + 3G_1 x^2 + \cdots + 3G_{n-1} x^n + 3G_n x^{n+1} \longrightarrow$ ⑥

$x^2 H(x) = G_0 x^2 + \cdots + G_{n-2} x^n + G_{n-1} x^{n+1} + G_n x^{n+2} \longrightarrow$ ⑦

由 $G_n - 3G_{n-1} + G_{n-2} = 0$ 得.

⑤-⑥+⑦ 得 $(1-3x+x^2)H(x) = G_0 + (G_1 - 3G_0)x$.

且 $G_n = F_{2n}$ $G_0 = F_0 = 0$ $G_1 = F_2 = 1$

$$\therefore H(x) = \frac{x}{1-3x+x^2}$$

$$H(x)=\frac{A}{\frac{3+\sqrt{5}}{2}-x}+\frac{B}{\frac{3-\sqrt{5}}{2}-x}$$

$$\begin{cases} A+B=1 \\ \frac{3-\sqrt{5}}{2}A+\frac{3+\sqrt{5}}{2}B=0 \end{cases} \Rightarrow \begin{cases} A=\frac{5+3\sqrt{5}}{10} \\ B=\frac{5-3\sqrt{5}}{10} \end{cases}$$

$$\therefore H(x)=\frac{2}{(3+\sqrt{5})}\times\frac{1}{1-\frac{3-\sqrt{5}}{2}x}+\frac{2}{(3-\sqrt{5})}\times\frac{1}{1-\frac{3+\sqrt{5}}{2}x}$$

由 $\frac{1}{1-x}=1+x+x^2+\cdots+x^n$

$$H(x)=-\frac{\sqrt{5}}{5}\sum_{i=0}^{\infty}\left(\frac{3-\sqrt{5}}{2}x\right)^i+\frac{\sqrt{5}}{5}\sum_{i=0}^{\infty}\left(\frac{3+\sqrt{5}}{2}x\right)^i$$

$$=\frac{\sqrt{5}}{5}\sum_{i=0}^{\infty}\left[\left(\frac{3+\sqrt{5}}{2}\right)^i-\left(\frac{3-\sqrt{5}}{2}\right)^i\right]x^i$$

2.6 $G(x)=1+0\cdot x+2x^2+0x^3+\cdots$

$$G(x)=1+2x^2+3x^4+\cdots \quad ①$$

$$x^2G(x)=x^2+2x^4+\cdots \quad ②$$

①−②得 $(1-x^2)G(x)=1+x^2+x^4+\cdots$

由 $\frac{1}{1-x}=1+x+x^2+\cdots$ ③

$$\frac{1}{1+x}=1-x+x^2-\cdots \quad ④$$

③+④得 $\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=1+x^2+x^4\cdots$ 即 $\frac{1}{1-x^2}=1+x^2+x^4+\cdots$ ⑤

又将 ⑤代入 $(1-x^2)G(x) = 1+x^2+x^4 \cdots$

$$(1-x^2)G(x) = \frac{1}{1-x^2}$$

$$\therefore G(x) = \frac{1}{(1-x^2)^2}$$

2.7 $G = 1+2x^2+3x^4+\cdots+(n+1)x^{2n}$ ①

$x^2 G = \quad x^2+2x^4+\cdots+nx^{2n}+(n+1)x^{2n+2}$ ②

①-②得 $(1-x^2)G = 1+x^2+x^4+\cdots+x^{2n+2}$

由2.6题推出过 $1+x^2+x^4+\cdots = \dfrac{1}{1-x^2}$ 代入得.

$$(1-x^2)G = \frac{1}{1-x^2}$$

$$(1-x^2)^2 G = 1$$

2.8 (1) $G(x) = 1+x^2+x^4+\cdots = \dfrac{1}{1-x^2}$

(2) $G(x) = -x-x^3-x^5-\cdots = -x(1+x^2+x^4+\cdots) = \dfrac{x}{x^2-1}$

(3) $G(x) = 1-x+x^2-x^3\cdots$

$$= (1+x^2+x^4+\cdots) - x(1+x^2+x^4+\cdots)$$

$$= \frac{1}{1-x^2} + \frac{x^2}{1-x^2}$$

$$= \frac{1-x-x^2}{1-x^2}$$

2.9 $G = 1+3x+6x^2+10x^3+\cdots+C_{n+2}^{2}x^n$ ①

$xG = \quad x+3x^2+6x^3+\cdots+C_{n+1}^{2}x^n+C_{n+2}^{2}x^{n+1}$ ②

(1) ①-②得 $(1-x)G = 1+2x+3x^2+4x^3+\cdots+(n+1)x^n$

(2) $G = \sum_{k=0}^{n} C_{k+2}^{2}x^n = \sum_{k=0}^{n} C_{k+2}^{k}x^n = \frac{1}{(1-x)^3}$

$\therefore (1-x)^2G = \frac{1}{1-x} = 1+x^2+x^3+\cdots+x^n.$

(3) $\because (1-x)^3G = 1 \quad \therefore G = \frac{1}{(1-x)^3}$

2.10 $H = 1+4x+10x^2+20x^3+\cdots+C_{n+3}^{3}x^n$ ①

$xH = \quad x+4x^2+10x^3+\cdots+C_{n+2}^{3}x^n+C_{n+3}^{3}x^{n+1}$ ②

(1) ①-②得 $(1-x)H = 1+3x+6x^2+10x^3+\cdots$

$= \sum_{n=0}^{\infty}\binom{n+2}{2}x^n$

(2) $H = \sum_{k=0}^{n} C_{k+3}^{3}x^n$

$= \sum_{k=0}^{n} C_{k+3}^{k}x^n = \frac{1}{(1-x)^4}$

2.18 (1) $a_n - 6a_{n-1} + 8a_{n-2} = 0$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots \quad ①$$

$$-6xG(x) = \quad -6a_0 x - 6a_1 x - \cdots \quad ②$$

$$8x^2 G(x) = \quad 8a_0 x^2 + \cdots \quad ③$$

①+②+③得 $(1-6x+8x^2)G(x) = a_0 + (a_1 - 6a_0)x$

$$G(x) = \frac{a_0 + (a_1 - 6a_0)x}{(1-2x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-4x}$$

$$\therefore \begin{cases} A + B = a_0 \\ (-2B-4A)x = (a_1 - 6a_0)x \end{cases} \qquad 2B = a_1 - 2a_0$$

$$\Rightarrow \begin{cases} A = \dfrac{4a_0 - a_1}{2} \\ B = \dfrac{-2a_0 + a_1}{2} \end{cases}$$

$$\therefore G(x) = A \cdot \sum_{n=0}^{\infty} (2x)^n + B \sum_{n=0}^{\infty} (4x)^n$$

$$= (A \cdot 2^n + B \cdot 4^n) \sum_{k=0}^{n} x^n$$

其中 A. B

$$\therefore \{a_n\} = A \cdot 2^n + B \cdot 4^n$$

(2) $a_n + 14a_{n-1} + 49a_{n-2} = 0$

$G = a_0 + a_1x + 2x^2 + \cdots$ ①

$14xG = \quad 14a_0x + \cdots$ ②

$49x^2G = \quad 49x^2 \cdots$ ③

$\frac{1}{(1-x)^2}$

①+②+③得 $(1+14x+49x^2)G = a_0 + (a_1+14a_0)x$

$(1-x)^2$

$G = \frac{a_0+(a_1+14a_0)x}{(1+7x)^2} = \frac{A}{1+7x} + \frac{B}{(1+7x)^2}$

$(k+1)$

$\begin{cases} A+B = a_0 \\ 7A = a_1+14a_0 \end{cases} \Rightarrow A = \frac{14a_0+a_1}{7}$

$B = -\frac{1}{7}(a_1+7a_0)$

$\therefore G = A\sum_{n=0}^{\infty}(-7)^n x^n + B\sum_{n=0}^{\infty}(n+1)(-7x)^n$

$= \left[(-7)^n A + B(n+1)(-7)^n\right]\sum_{n=0}^{\infty} x^n$

$= (-7)^n\left[A+B(n+1)\right]\sum_{n=0}^{\infty} x^n$

$\therefore \{a_n\} = (-7)^n\left[A+B(n+1)\right]$

(3) $a_n - 9a_{n-2} = 0$

$G = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ ①

$-9x^2 G = \quad -9a_0 x^2 - 9a_1 x^3 + \cdots$ ②

①+②得 $(1-9x^2)G = a_0 + a_1 x$

$$G = \frac{a_0 + a_1}{1-9x^2} = \frac{A}{1-3x} + \frac{B}{1+3x}$$

$$\begin{cases} A+B = a_0 \\ 3A-3B = a_1 \end{cases} \quad \therefore \begin{cases} A = \dfrac{3a_0+a_1}{6} \\ B = \dfrac{3a_0-a_1}{6} \end{cases}$$

$$\therefore G = A\sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (-3)^n x^n$$

$$= [A \cdot 3^n + B \cdot (-3)^n] x^n$$

$\therefore \{a_n\} = A \cdot 3^n + B \cdot (-3)^n$ 其中 A.B 分别为

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(4) $a_n - 6a_{n-1} - 7a_{n-2} = 0$

$G = a_0 + a_1 x + a_2 x^2 + \cdots$ ①

$-6xG = \quad -6a_0 x - 6a_1 x^2 + \cdots$ ②

$-7x^2 G = \quad -7a_0 x^2 + \cdots$ ③

①+②+③得 $(1-6x-7x^2)G = a_0 + (a_1 - 6a_0)x$

$$G = \frac{a_0 + (a_1 - 6a_0)x}{(1-7x)(1+x)} = \frac{A}{1-7x} + \frac{B}{1+x}$$

$$\therefore \begin{cases} A+B = a_0 \\ A-7B = a_1 - 6a_0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{a_1+a_0}{8} \\ B = \dfrac{7a_0-a_1}{8} \end{cases}$$

$$\therefore G = [A \cdot 7^n + B(-1)^n] \sum_{n=0}^{\infty} x^n$$

$\therefore \{a_n\} = A \cdot 7^n + B(-1)^n$

其中 A.B 分别为

① $a_n - 12a_{n-1} + 36a_{n-2} = 0$

$G = a_0 + a_1x + a_2x^2 + \cdots$ ①

$-12xG = \quad -12a_0x - 12a_1x^2 - \cdots$ ②

$36x^2G = \quad 36x^2 + \cdots$ ③

①+②+③得 $(1-12x+36x^2)G = a_0 + (a_1 - 12a_0)x$

$$G = \frac{a_0 + (a_1 - 12a_0)x}{(1-6x)^2} = \frac{A}{1-6x} + \frac{B}{(1-6x)^2}$$

$$\begin{cases} A + B = a_0 \\ -6A = a_1 - 12a_0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{12a_0 - a_1}{6} \\ B = \dfrac{-6a_0 + a_1}{6} \end{cases}$$

$\therefore G = A\sum_{n=0}^{\infty}(6x)^n + B\sum_{n=0}^{\infty}(n+1)(6x)^n$

$= [A\cdot 6^n + B(n+1)6^n]\sum_{n=0}^{\infty}x^n$

$\therefore \{a_n\} = [A + B(n+1)]\cdot 6^n$

其中 A. B 分别为

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② $a_n - 25a_{n-2} = 0$

$G = a_0 + a_1x + a_2x^2 + \cdots$ ①

$-25x^2G = \quad -25a_0x^2 - \cdots$ ②

①+②得 $(1-25x^2)G = a_0 + a_1x$

$$G = \frac{a_0 + a_1x}{(1-5x)(1+5x)} = \frac{A}{1-5x} + \frac{B}{1+5x}$$

$$\begin{cases} A + B = a_0 \\ 5(A - B) = a_1 \end{cases} \quad A = \frac{5a_0 + a_1}{10} \quad B = \frac{5a_0 - a_1}{10}$$

$\therefore G = [A\cdot(5)^n + B(-5)^n]\sum_{n=0}^{\infty}x^n$

$\therefore \{a_n\} = A\cdot 5^n + B\cdot(-5)^n$ 其中. A.B分别为