Theory of Computation, Fall 2023
Assignment 7 Solutions

Q1. For any alphabet Σ, Σ^* is countable. Since a language L is a subset of Σ^*, L must be countable.

Q2. The power set of $\{1\}^*$ is uncountable, and the set of all Turing machines is coutable. Therefore, some subset of $\{1\}^*$ cannot be decided by any Turing machine.

Q3. Suppose the there's some Turing machine M_H that decides H. We show that there is a Turing machine M^* that decides A. Since A is recursively enumerable, there is some Turing machine M_A the semidecides A. That is M_A halts on a string w if and only if $w \in A$. Therefore, to decide whether $w \in A$ or not, it suffices to decide whether M_A halts on w, and this can be done by M_H. More precisely, the following Turing machine M^* decides A.

$M =$ on input w :
1. run M_H on "M_A""w"
2. return the result of M_H

Q4. Let $A = \{$"M" : "M" is a Turing machine that halts on some input$\}$. Let B be the language in the question. In class, we have proved that A is not recursive. To prove that B is not recursive, it suffices to reduce A to B. Suppose that there is a Turing machine M_B that decides B. Then we can construct the following Turing machine M_A to decide A.

$M_A =$ on input "M":
1. Construct a Turing machine M_{all} that halts on every input as follows.

$$\begin{aligned} M_{\text{all}} \ &= \text{on input } x \\ &\ \ \text{1. halt} \end{aligned}$$

2. run M_B on "M""M_{all}""1"
3. return the result of M_B

This completes the proof.

Q5. We can conclude that A is also recursive enumerable. Since B is recursive enumerable, there is a Turing machine M_B that semidecides B. We can construct the following Turing machine M_A.

$M_A =$ on input "x":
1. compute $f(x)$
2. run M_B on $f(x)$

M_A halts on x if and only if M_B halts on $f(x)$ if and only if $f(x) \in B$ if and only if $x \in A$. So M_A semidecides A.

Q6. Let f be a reduction from A to B. By definition, for any $x \in \Sigma^*$, $x \in A$ if and only if $f(x) \in B$. In other words, $x \in \overline{A}$ if and only if $f(x) \in \overline{B}$. So f is also a reduction to $\overline{A}$ to $\overline{B}$.

Q7. Let M_1 and M_2 be Turing machines that semidecides A and $\overline{A}$, respectively. Consider the following Tuirng machine M.

$M =$ on input w :
1. run M_1 and M_2 on w in parallel
2. If M_1 halts, accept w, and if M_2 halts, reject w.

If $w \in A$, M_1 will halt on w, and M will accept w. Otherwise, M_2 will halts on w, and M will reject w. So M decides A. A is recursive.

Q8. Since $A \leq \overline{A}$, by the conclusion of Q6, we have that $\overline{A} \leq A$. Since A is recursively enumerable, by the conclusion of Q5, $\overline{A}$ is also recursively enumerable. Then by the conclusion of Q7, A is recursive.