Theory of Computation, Fall 2023
Assignment 5 Solutions

Q1. A proof using CFG: Since L is context-free, there is a CFG $G = (V, \Sigma, S, R)$ that generates L. Consider the CFG $G' = (V, \Sigma, S, R')$ where

$$R' = \{(A, u^R) \mid (A, u) \in R\}.$$

It is easy to see that G' generates L^R. Thus L^R is context-free.

A proof using PDA: Since L is context-free, there is a PDA $P = (K, \Sigma, \Gamma, \Delta, s, F)$ that accepts L.

Consider the following PDA $P' = (K', \Sigma, \Gamma, \Delta', s', F')$ where

$$\begin{aligned}
&K' = K \cup \{s'\} \text{ (assuming } s' \text{ is a new state not in } K)\\
&F' = \{s\}\\
&\Delta' = \{((p, c, \alpha), (q, \beta)) \mid ((q, c, \beta), (p, \alpha)) \in \Delta\} \cup \{((s', e, e), (f, e)) \mid f \in F\}
\end{aligned}$$

P' accepts L^R. Thus L^R is context-free.

Q2. Suppose, for the sake of contradiction, that L is context-free. Let p be the pumping length given by the pumping theorem. Consider $2 = 1^p 3^p 2^p 4^p \in L$. By pumping theorem, w can be divided into $w = uvxyz$ so that the followings hold.

1. $uv^i xy^i z \in L$ for any $i \geq 0$
2. $|v| + |y| > 0$,
3. $|vxy| \leq p$.

Since $|vxy| \leq p$, it is easy to observe that vy cannot have both 1 and 2, nor can it have both 3 and 4. Addtionally, vy contains at least one symbol. Therefore, in uv^0xy^0z, either $\#1 \neq \#2$ or $\#3 \neq \#4$, so $uv^0xy^0z \notin L$. Contradiction. This completes the proof.

Q3. Since A is context-free and B is regular,

There is some PDA $P_A = (K_A, \Sigma, \Gamma, \Delta_A, s_A, F_A)$ accepting A, and there is some NFA $M_B = (K_B, \Sigma, \Delta_B, s_B, F_B)$ accepting B.

Construct a new PDA $P_\cap = (K_\cap, \Sigma, \Gamma, \Delta_\cap, s_\cap, F_\cap)$ as follows.

$$\begin{aligned}
K_\cap &= K_A \times K_B\\
s_\cap &= (s_A, s_B)\\
F_\cap &= F_A \times F_B\\
\Delta_\cap &= \{(((q_A, q_B), c, \alpha), ((p_A, p_B), \beta)) \mid ((q_A, c, \alpha), (p_A, \beta)) \in \Delta_A \wedge ((q_B, c), p_B) \in \Delta_B\}
\end{aligned}$$

$P_\cap$ accepts $A \cap B$, so $A \cap B$ is context-free.

Q4. (a) Suppose that A is context-free. The conclusion of Q3 implies that $A \cap a^*b^*c^* = \{a^n b^n c^n \mid n \geq 0\}$ is also context-free. However, we have proved in class that $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free. As a consequence, A cannot be context-free.

(b) Consider the following two languages, where $\#a$ is the number of a's in w, and $\#b$ and $\#c$ are defined similarly.

$$A_1 = \{w \in \{a, b, c\}^* \mid \#a \neq \#b\}$$
$$A_2 = \{w \in \{a, b, c\}^* \mid \#a \neq \#c\}$$

It is easy to show that both A_1 and A_2 are context-free. We also observe that $A = A_1 \cup A_2$, so A is context-free.