The Note for An Integral Operator

Define the following two correlation functions in space and time of a solution trajectory,

$$K(x,y) := \int_0^T u(x,s)u(y,s)ds, \quad (x,y) \in \Omega \times \Omega,$$
$$G(s,t) := \int_\Omega u(x,t)u(x,s)dx, \quad (s,t) \in [0,T] \times [0,T], \tag{27}$$

where $K(x,y)$ and $G(s,t)$ define two symmetric semi-positive compact integral operators on $L^2(\Omega)$ and $L^2[0,T]$ respectively. They have the same non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_j \geq \ldots$ with $\lambda_j \to 0$ as $j \to \infty$. Their normalized eigenfunctions form an orthonormal basis in $L^2(\Omega)$ and $L^2[0,T]$. Define V_K^k and V_G^k to be the linear space spanned by their k leading eigenfunctions of $K(x,y)$ and $G(s,t)$ respectively, which provides the best k-dimensional linear spaces that approximate the family of functions $u(\cdot,t)$ (in $L^2(\Omega)$) and $u(x,\cdot)$ (in $L^2([0,T])$). We have

$$\int_0^T \|u(\cdot,t) - P_{V_K^k}u(\cdot,t)\|^2_{L^2(\Omega)}dt = \int_\Omega \|u(x,\cdot) - P_{V_G^k}u(x,\cdot)\|^2_{L^2[0,T]}dx = \sum_{j=k+1}^{\infty} \lambda_j, \tag{28}$$

where $P_{V_K^k}$ and $P_{V_G^k}$ denotes the projection operator to $V_K^k \subset L^2(\Omega)$ and $V_G^k \subset L^2[0,T]$.

Note: 我们记该算子为$(\Phi f)(x) := \int_\Omega K(x,y)f(y)dy$，$\Phi$的特征向量为$\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots$，与其特征值$\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$对应。

> $K(x,y)$ **is a symmetric semi-positive compact integral operators on** $L^2(\Omega)$

Symmetric Integral Operator

$K(x,y) := \int_0^T u(x,t)u(y,t)dt$，显然有$K(x,y) = K(y,x)$，所以 Φ 是对称积分算子。

Compact

算子有界

$$\begin{aligned}
\int_\Omega [\Phi(f)(x)]^2 dx &= \int_\Omega [\int_\Omega K(x,y)f(y)dy]^2 dx \\
&\leq \int_\Omega [\int_\Omega K^2(x,y)dy][\int_\Omega f^2(y)dy]dx \quad \text{(Cauchy–Schwarz inequality)} \\
&= [\int_{\Omega^2} K^2(x,y)dxdy][\int_\Omega f^2(y)dy] \quad \text{(Fubini's theorem)} \\
&= ||K(x,y)||_{L^2(\Omega^2)} * ||f(x)||_{L^2(\Omega)}
\end{aligned}$$

所以有$||\Phi|| \leq ||K(x,y)|| = \int_{\Omega^2 \times [0,t]} u(x,t)u(y,t)dtdxdy \leq \infty$。第二个不等号是因为 Ω 和 $[0,T]$ 的测度有限，$u(x,t)$ 连续。

紧算子

取 $L^2(\Omega)$ 的标准正交基 $\{\varphi_i(x)\}_{i=1}^\infty$，则有$\{\varphi_i(x)\varphi_j(y)\}_{i,j\geq 1}$ 构成 $L^2(\Omega^2)$ 的标准正交基。则有

$$K(x,y) = \sum_{i,j=1} a_{i,j}\varphi_i(x)\varphi_j(y), \sum_{i,j=1}^\infty |a_{i,j}| < \infty$$

定义 $K_n(x,y) := \sum_{i,j=1}^n a_{i,j}\varphi_i(x)\varphi_j(y)$，有

$$||\Phi - \Phi_n|| \leq ||K(x,y) - K_n(x,y)||_{L^2(\Omega^2)} = \sum_{i\geq orj\geq n}^\infty |a_{i,j}|^2 \to 0 (n \to \infty)$$

所以 Φ 为紧算子。

Normalized Eigenfunctions of Φ Form an Orthonormal Basis in $L^2(\Omega)$

因为特征值递减到零，由`Hilbert-Schmidt theorem`知，其特征向量$\{\varphi_i(x)\}_{i=1}^\infty$构成$L^2(\Omega)$的一组标准正交基。

Semi-positive

$$\lambda_i\varphi_i(x) = \int_\Omega K(x,y)\varphi_i(y)dy$$
$$\int_\Omega \lambda_i\varphi_i(x)\varphi_i(x)dx = \int_{\Omega^2} K(x,y)\varphi_i(y)dy\varphi_i(x)dx$$

对于左手边，因为 $\varphi_i(x)$ 构成 $L^2(\Omega)$ 的标准正交基，所以 $\mathrm{LH} = \lambda_i$，对于右手边

$$\begin{aligned}
\mathrm{RH} &= \int_{\Omega^2} [\int_0^T u(x,s)u(y,s)ds]\varphi_i(x)\varphi(y)dxdy \\
&= \int_0^T (\int_\Omega u(x,s)\varphi_i(x)dx)(\int_\Omega u(y,s)\varphi_i(y)dy)ds \\
&= \int_0^T (\int_\Omega u(x,s)\varphi_i(x)dx)^2 dt \geq 0
\end{aligned}$$

所以

$$\lambda_i = \mathrm{LH} = \mathrm{RH} \geq 0$$

Proof for (28)

> $\int_0^T ||u(\cdot,t) - P_{V_K^k}u(\cdot,t)||^2_{L^2(\Omega)}dt = \sum_{j=k+1}^\infty \lambda_j \quad (28)$

1. **$k = 0$ 时**

   因为 $\{\varphi_i(x)\}_{i=1}^\infty$ 是 $L^2(\Omega)$ 的一组正交基，所以记 $u(x,t) = \sum_{i=1}^\infty c_i(t)\varphi_i(x)$，其中$c_i(t) = \int_\Omega u(x,t)\varphi_i(x)dx$。

$$\begin{aligned}
\mathrm{LH} &= \int_{\Omega\times[0,T]} u^2(x,t)dxdt \\
&= \int_{\Omega\times[0,T]} \sum_{i,j=1} c_i(t)\varphi_i(x)c_j(t)\varphi_j(x)dxdt \\
&= \sum_{i,j} \int_\Omega \varphi_i(x)\varphi_j(x)dx \int_0^T c_i(t)c_j(t)dt \\
&= \sum_i \int_0^T c_i^2(t)dt \quad (\varphi_i(x) \text{ 正交})
\end{aligned}$$

$$\begin{aligned}
\int_0^T c_i^2(t)dt &= \int_0^T (\int_\Omega u(x,t)\varphi_i(x)dx)(\int_\Omega u(y,t)\varphi_i(y)dy)dt \\
&= \int_{\Omega^2} (\int_0^T u(x,t)u(y,t)dt)\varphi_i(x)\varphi_i(y)dxdy \\
&= \int_{\Omega^2} K(x,y)\varphi_i(x)\varphi_i(y)dxdy \\
&= \int_\Omega [\int_\Omega K(x,y)\varphi_i(x)dx]\varphi_i(y)dy \\
&= \int_\Omega \lambda_i\varphi_i(y)\varphi_i(y)dy \\
&= \lambda_i
\end{aligned}$$

所以有

$$\mathrm{LH} = \sum_{i=1}^\infty \int_0^T c_i^2(t)dt = \sum_{i=1}^\infty \lambda_i$$

2. **对任意的的 k**

   $u(\cdot,t) - P_{V_K^k}u(\cdot,t) = \sum_{i=k+1}^\infty c_i(t)\,\varphi_i(x),$

由上推导，易知不等式 (28) 成立。