Scientific Computing - Homework 4

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1 Page 208-211, 4.13, 4.19, 4.20, 4.24, 4.25, 4.27, 4.31, 4.32 (Scientific Computing: An Introductory Survey)

1.1 Page 208, 4.13

Suppose λ_0 is the eigenvalue of A with the biggest absolute value and x is the corresponding eigenvector, then $Ax = \lambda_0 x$ and $\|A\| \geqslant \dfrac{\|Ax\|}{\|x\|} = |\lambda_0| \cdot \dfrac{\|x\|}{\|x\|} = |\lambda_0| = \rho(A)$

1.2 Page 209, 4.19

Suppose λ is an eigenvalue of A and $x \neq 0$ is the eigenvector of λ, then $Ax = \lambda x$, $A^2 x = \lambda^2 x = Ax = \lambda x$, then $(\lambda^2 - \lambda)x = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0, 1$.
Hence A only has eigenvalues 0 and 1.

1.3 Page 209, 4.20

(a) A is Hermitian matrix, hence λ and μ are real.
$Ax = \lambda x \Rightarrow y^H A x = \lambda y^H x$,
$Ay = \mu y \Rightarrow y^H A^H = \mu y^H \Rightarrow y^H A^H x = \mu y^H x$.
Since A is Hermitian matrix, then $A = A^H \Rightarrow \lambda y^H x = \mu y^H x \Rightarrow (\lambda - \mu) y^H x = 0$.
$\lambda \neq \mu \Rightarrow y^H x = 0$.
(b) $Ax = \lambda x \Rightarrow y^H A x = \lambda y^H x$.
$y^H A = \mu y^H \Rightarrow y^H A x = \mu y^H x = \lambda y^H x \Rightarrow (\lambda - \mu) y^H x = 0$.
Since $\lambda \neq \mu$, then $y^H x = 0$.
(c) A is unitarily similar to upper triangular matrix. Then there exists unitary matrix U s.t.

$$U^H A U = \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix}$$

here λ is a simple eigenvalue of A and λ is not the eigenvalue of B. Then

$$U^H A U e_1 = \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix} e_1 = \lambda e_1$$

Meanwhile,

$$U^H A^H U = \begin{bmatrix} \overline{\lambda} & 0 \\ * & B^H \end{bmatrix}$$

Suppose z is the eigenvector of $U^H A^H U$ corresponding to eigenvalue $\overline{\lambda}$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}^{n-1}$.
If $z_1 = 0$, then

$$\begin{bmatrix} \overline{\lambda} & 0 \\ * & B^H \end{bmatrix} z = \begin{bmatrix} 0 \\ B^H z_2 \end{bmatrix} = \overline{\lambda} z = \begin{bmatrix} 0 \\ \overline{\lambda} z_2 \end{bmatrix}$$

hence $B^H z_2 = \overline{\lambda} z_2$, $z_2 \neq 0$, $\overline{\lambda}$ is an eigenvalue of B^H, which means λ is an eigenvalue of B, contradiction.
So $z_1 \neq 0$ and $(Ue_1)^H Uz = e_1^H z = z_1 \neq 0$.
Since $A(Ue_1) = \lambda(Ue_1)$, $A^H(Uz) = \overline{\lambda}(Uz)$ and λ is a simple eigenvalue, then $\exists\ a \neq 0, b \neq 0$, s.t. $x = aUe_1$ and $y = bUz$. Then $y^H x = a\overline{b} e_1^H z \neq 0$.

1.4 Page 209, 4.24

(a) By rank factorization $A = BC$, where $B \in \mathbb{R}^{n\times r},\ C \in \mathbb{R}^{r\times n}$, r is the rank of A,
Hence $B,\ C^T \in \mathbb{R}^{n\times 1}$, $A = B(C^T)^T = uv^T$, where $u = B,\ v = C^T$ are nonzero real vectors.
(b) $Au = uv^T u = (v^T u)u = (u^T v)u$, hence $u^T v$ is an eigenvalue of A.
(c) Since A is of rank one, then the kernel of matrix A is of rank $n-1$.
The kernel of A is $\{v \in \mathbb{R}^n, Av = 0 = 0v\}$, then the other $n-1$ eigenvalues of A are 0.
(d) Only 1 iteration is needed since A has only 1 nonzero eigenvalue.

1.5 Page 210, 4.25

From 4.24, $1 + u^T v$ is an eigenvalue of $I + uv^T$ and the other eigenvalues of $I + uv^T$ are all 1.
Hence $\det(I + uv^T) = 1 + u^T v$.

1.6 Page 210, 4.27

(a) $\rho(A) < 1$, then $I - A$ has no zero eigenvalue, $\det(I - A) \neq 0$, $I - A$ is nonsingular.
(b) Suppose $\sigma_i, i = 1, \cdots, n$ are the eigenvalues of A, x_i are the corresponding eigenvectors, $-1 < \sigma_i < 1$,
Then $Ax_i = \sigma_i x_i \Rightarrow (I - A)^{-1} x_i = \dfrac{1}{1-\sigma_i} x_i$.
Since $\Big(\sum\limits_{k=0}^{\infty} A^k\Big) x_i = \sum\limits_{k=0}^{\infty} A^k x_i = \sum\limits_{k=0}^{\infty} \sigma_i^k x_i = \Big(\sum\limits_{k=0}^{\infty} \sigma_i^k\Big) x_i = \dfrac{1}{1-\sigma_i} x_i,\ i = 1, 2, \cdots, n,$
Then $(I - A)^{-1} = \sum\limits_{k=0}^{\infty} A^k$.

1.7 Page 210, 4.31

(a) Suppose x is an eigenvector of λ, then $Qx = \lambda x$, $x^T Q^T = \lambda x^T$.
$Q^T Q = I \Rightarrow x^T x = x^T Q^T Q x = \lambda^2 x^T x$, hence $\lambda^2 = 1$, $|\lambda| = 1$.
(b) The singular values of an orthogonal matrix are 1.

1.8 Page 210, 4.32

(a) From 4.24, 1 is a nonzero eigenvalue of $\dfrac{vv^T}{v^T v}$ and the other eigenvalues are all 0.
Hence the eigenvalues of $H = I - 2\dfrac{vv^T}{v^T v}$ are -1 of multiplicity 1 and 1 of multiplicity $n-1$.
(b) $c \pm si$.

2 If pivot should be used in the QR decomposition, how would you do in the QR iteration?

Suppose the QR factorization using pivoting of A_{k-1} is $A_{k-1}P_k = Q_kR_k$, then $A_{k-1} = Q_kR_kP_k^T$, $A_k = R_kP_k^TQ_k$.

3 If a Martix is large and sparse, how would you find its largest 10 eigenvalues and eigenvectors?

Suppose the biggest 10 eigenvalues of matrix A are $\lambda_1, \cdots, \lambda_{10}$.
Use power iteration to compute the biggest eigenvalue λ_1 and corresponding eigenvector x_1.
Find vector u_1 s.t. $u_1^Tx_1 = \lambda_1$, then $A - x_1u_1^T$ has eigenvalues $0, \lambda_2, \lambda_3, \cdots, \lambda_{10}$.
Use power iteration on $A - x_1u_1^T$ to compute λ_2 and the corresponding eigenvector x_2.
Find vector u_2 s.t. $u_2^Tx_2 = \lambda_2$, then $A - x_1u_1^T - x_2u_2^T$ has eigenvalues $0, 0, \lambda_3, \lambda_4, \cdots, \lambda_{10}$.
Repeat this procedure and finally we get $\lambda_1, \lambda_2, \cdots, \lambda_{10}$.

4 Computer problem (in C or C++): Write the functions to achieve (1) Arnoldi iteration and Lanczos iteration; (2) QR iteration for Hessenburg matrix to find the eigenvalue decomposition. Test these algorithms for a few matrix.

4.1 Arnoldi iteration

First we generate a random matrix A:

$$A = \begin{bmatrix} 7.82637\times10^{-6} & 0.131538 & 0.75605 & 0.45865 & 0.532767 \\ 0.218959 & 0.0470446 & 678865 & 0.679296 & 0.934693 \\ 0.383502 & 0.519416 & 0.830965 & 0.0345721 & 0.0534616 \\ 0.5297 & 0.671149 & 00769819 & 0.383416 & 0.0668422 \\ 0.417486 & 0.686773 & 0.588977 & 0.930436 & 0.846167 \end{bmatrix}$$

We use Arnoldi iteration on A and we get Q, H s.t. $Q^TAQ = H$, where Q is a orthonormal matrix and H is a Hessenburg matrix:

$$Q = \begin{bmatrix} 0.448804 & 0.0424156 & -0.64389 & 0.386592 & 0.482418 \\ 0.0783299 & 0.91907 & 0.0124376 & -0.356336 & 0.148476 \\ 0.556967 & -0.282615 & -0.229148 & -0.710305 & -0.229944 \\ 0.354322 & -0.169652 & 0.66466 & -0.0734495 & 0.631274 \\ 0.59723 & 0.211798 & 0.301611 & 0.462215 & -0.542077 \end{bmatrix},$$

$$H = \begin{bmatrix} 1.91564 & 0.625495 & 0.212651 & -0.625938 & 0.332493 \\ 1.22833 & -0.121908 & 0.636661 & 0.0940421 & -0.168139 \\ 0 & 0.517447 & 0.0106548 & 0.298476 & 0.543828 \\ 0 & 0 & 0.461565 & 0.217729 & 0.0345409 \\ 0 & 0 & 0 & 0.0288847 & 0.0854853 \end{bmatrix}$$

Now we use QR iteration on H and finally we get the eigenvalues: $2.2688, -0.713998, 0.704934, -0.228789, 0.0766534$ and the eigenvector matrix:

$$\begin{bmatrix} 0.348176 & 0.460235 & -0.301638 & 0.538806 & 0.534472 \\ 0.500202 & 0.422534 & 0.0693567 & -0.745785 & 0.101279 \\ 0.305592 & -0.384406 & -0.834774 & -0.120082 & -0.218125 \\ 0.30077 & -0.6753 & 0.271603 & -0.0725091 & 0.611949 \\ 0.666875 & -0.0764973 & 0.365497 & 0.365808 & -0.531058 \end{bmatrix}$$

4.2 Lanczos iteration

First we generate a symmetric matrix A:

$$A = \begin{bmatrix} 0.649894 & 0.851725 & 0.717294 & 0.75354 & 0.613836 \\ 0.851725 & 1.21141 & 0.972604 & 1.00657 & 0.767806 \\ 0.717294 & 0.972604 & 2.06925 & 1.38739 & 1.5804 \\ 0.75354 & 1.00657 & 1.38739 & 1.68572 & 1.69407 \\ 0.613836 & 0.767806 & 1.5804 & 1.69407 & 1.88082 \end{bmatrix}$$

We use Lanczos iteration on A and we get Q, H s.t. $Q^T AQ = H$, where Q is a orthonormal matrix and H is a tridiaginal matrix:

$$Q = \begin{bmatrix} 0.640091 & -0.30058 & -0.148201 & 0.369508 & -0.584325 \\ 0.535938 & -0.00908986 & -0.567368 & -0.386954 & 0.490966 \\ 0.184543 & 0.572412 & 0.136456 & -0.601791 & -0.507459 \\ 0.0333746 & 0.751921 & -0.29672 & 0.580494 & 0.09211 \\ 0.517575 & 0.128561 & 0.74126 & 0.120847 & 0.389253 \end{bmatrix},$$

$$H = \begin{bmatrix} 3.41504 & 2.78875 & 0 & 0 & 0 \\ 2.78875 & 2.8195 & 0.570511 & 0 & 0 \\ 0 & 0.570511 & 0.674893 & 0.0575025 & 0 \\ 0 & 0 & 0.0575025 & 0.397974 & 0.253327 \\ 0 & 0 & 0 & 0.253327 & 0.189688 \end{bmatrix}$$

Now we use QR iteration on H and finally we get the eigenvalues: $5.94962, 0.93993, 0.564872, 0.0394495, 0.0032184$ and the eigenvector matrix:

$$\begin{bmatrix} 0.260309 & -0.483333 & 0.0544787 & -0.801079 & -0.232234 \\ 0.34802 & -0.716685 & 0.0262144 & 0.422135 & 0.431694 \\ 0.529616 & 0.16742 & -0.813526 & 0.0622987 & -0.160536 \\ 0.507936 & 0.0847219 & 0.495219 & 0.326951 & -0.618615 \\ 0.52214 & 0.466418 & 0.298795 & -0.26324 & 0.592664 \end{bmatrix}$$

5 Page 248-250, 5.5, 5.10;

5.1 Page 248 5.5

(a) In secant method, $x_{k+1} = x_k - f(x_k)\dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} = \dfrac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$.

(b) The advantage of the formula in (a) is that it has smaller computational cost and it can avoid the rounding error when $|x_k - x_{k-1}| < \epsilon_{mach}$.

5.2 Page 249 5.10

$f(t_1, t_2) = \begin{bmatrix} t_1^2 - t_2^2 \\ 2t_1t_2 - 1 \end{bmatrix}$, $J(t) = \begin{bmatrix} 2t_1, & -2t_2 \\ 2t_2, & 2t_1 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

After one iteration,

$x_1 = x_0 - J(x_0)^{-1}f(x_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

6 Computer problem (in C or C++): Write the functions to achieve Newton's method and Broyden's method. Test these methods for a few equation systems.

(1) We test Newton's method and Broyden's method on $f(x_1, x_2) = \begin{bmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 - 1 \end{bmatrix}$ and we get the solution $x_1 = x_2 = 0.707107$.

(2) We solve $f(x_1, x_2) = \begin{bmatrix} x_1 + 2x_2 - 2 \\ x_1^2 + 4x_2^2 - 4 \end{bmatrix}$.

For Newton's method, we get $x_1 = -8.08053e - 17$, $x_2 = 1$.

For Broyden's method, we get $x_1 = -2.12844e - 13, x_2 = 1$.