Analysis 1A - Tutorial 5 2022

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Infinite Limits

Before we cover any material from this week, its worth discussing the use of ∞ as a limit, especially when applying the Algebra of Limits. The main thing to note is that expressions such as $\infty - \infty$ and $\frac{\infty}{\infty}$ don't really make a lot of sense, for example:

Give examples of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, both diverging to ∞, for which:
a) $(a_n - b_n)_{n\in\mathbb{N}}$ diverges to ∞,
b) $(a_n - b_n)_{n\in\mathbb{N}}$ diverges to $-\infty$,
c) $(a_n - b_n)_{n\in\mathbb{N}}$ converges to 0.

Solution:
a) If we take $a_n = 2n$ and $b_n = n$, we see that $a_n - b_n = n$, and $(n)_{n\in\mathbb{N}}$ diverges to ∞. But if we tried to apply the algebra of limits to this result, it would suggest that $\infty - \infty = \infty$.

b) If we take $a_n = n$ and $b_n = 2n$, we see that $a_n - b_n = -n$, and $(-n)_{n\in\mathbb{N}}$ diverges to ∞. But, again, if we tried to apply AoL to this result, it would suggest that $\infty - \infty = -\infty$. Immediately this conflicts with the answer to a)!

c) Finally, if we take $a_n = b_n = n$, we see that $a_n - b_n = 0$, and $(0)_{n\in\mathbb{N}}$ is a convergent sequence — it converges to 0. Applying AoL to this result suggests that $\infty - \infty = 0$. What a), b) and c) demonstrate is that you can't consistently define $\infty - \infty$.

(Note: if you take the same sequences from a), b) and c) and divide them, you can see why $\frac{\infty}{\infty}$ isn't consistently define-able either).

Lecture Recap

Backwards to how it was discussed in the tutorial, we begin with some special types of sequences.

Monotonic Sequences

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Then (a_n) is:

- increasing if $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$,
- strictly increasing if $a_n < a_{n+1} \quad \forall n \in \mathbb{N}$,
- decreasing if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$,
- strictly decreasing if $a_n > a_{n+1} \quad \forall n \in \mathbb{N}$,

If one of these things is happening, then the sequence is called monotone . For a given sequence (a_n), the two main ways of checking monotonicity are by considering $a_{n+1} - a_n$ and/or $\frac{a_{n+1}}{a_n}$. The second of these methods is especially useful when you're dealing with powers of n, such as for the sequence (b_n) in Exercise Sheet 5, Question 2.

A useful theorem for these sequences is that *a bounded, monotone sequence converges*[1] In fact, if a sequence $(a_n)_{n\in\mathbb{N}}$ is increasing, then it converges to the supremum of the set of a_n values, and if it is decreasing, then it converges to the infimum of the set of a_n values.

The Sandwich Theorem/Pinching Theorem/Squeeze Theorem

This is a way of finding the limit of a sequence if you can find two other sequences to 'trap' it with. It's quite a good method for rational functions and proving statements about n-th roots.

Sandwich Theorem: Suppose that $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are real sequences. If $a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$, and $\exists L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\lim_{n\to\infty} b_n = L$.

There's also a slight modification to this theorem, called the 'Bitten Sandwich Theorem'.

Bitten Sandwich Theorem: Suppose that $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are real sequences. If $\exists N \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n \; \forall n \geq N$, and $\exists L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\lim_{n\to\infty} b_n = L$.

[1] If you're interested, this statement is completely equivalent to the completeness axiom.

This just says that as long as after some $N \in \mathbb{N}$, b_n is trapped between sequences a_n and c_n that share a common limit, then all three sequences will share that common limit.

More on Infinite Limits

We can also make the idea of a sequence getting increasingly more positive (or more negative) more precise via the idea of divergence to $\pm\infty$. We present the definition for 'positive' ∞ here.

A real sequence (a_n) *diverges to* ∞ as $n \to \infty$ if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M$.

There is also a corresponding version of the 'algebra of limits' for divergence to $\pm\infty$ (see below). This version has been stolen from an old set of lecture notes (ones from 2016 to be precise!), so some of these results may not appear in the current lecture notes. In any case, it's a good idea to convince yourself that these are true! Also, for (v), note that we require $x_n \neq 0$, otherwise the sequence $(1/x_n)$ will not be defined.

1.20 Theorem (Algebra of infinite limits)
Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be real sequences.
(i) If $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$ then $x_n + y_n \to \infty$ as $n \to \infty$;
(ii) if $x_n \to \infty$ as $n \to \infty$ and $c > 0$ then $cx_n \to \infty$ as $n \to \infty$;
(iii) if $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$ then $x_n y_n \to \infty$ as $n \to \infty$;
(iv) $x_n \to \infty$ as $n \to \infty$ if and only if $-x_n \to -\infty$ as $n \to \infty$
(v) if $x_n \to \infty$ as $n \to \infty$ then $1/x_n \to 0$ as $n \to \infty$;
(vi) if $x_n \to 0$ as $n \to \infty$ and $x_n > 0$ for all $n \in \mathbb{N}$ then $1/x_n \to \infty$ as $n \to \infty$

Hints

Here are the hints for Exercise Sheet 5

Q1. Try applying one of the tests for monotonicity. For the limit, if you need to use any theorems anywhere, state them!

Q2. If $a_n \neq 0$ $\forall n \in \mathbb{N}$, then this would just be an application of the algebra of (infinite) limits! Since you don't know if this is the case, you'll need to use the definition again for this question. You'll end up with an inequality of the form $a_n^2 > g(\epsilon)$, where g is a rational function of ϵ, at some point in your solution. Take cases on the sign of the numerator of $g(\epsilon)$ to find the required N in the definition of limit.

Q3. This is similar to the supremum question we did in tutorials. If you need a refresher on the argument involved, look in the solutions to Exercise Sheet 4. Try adapting this for infima!

Q4. This is similar to tutorial question 3 off Exercise Sheet 5. Try a few terms of the sequence to get a feel for what's happening first. Note that you're not explicitly told to find the limit, but it's really worth doing if you can!