

# Cauchy Condensation Example

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# 1 Example involving logarithms

## Example 1.1.

Prove that

$$\sum_{n=4}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))}$$

diverges.

## Solution.

For  $n \geq 4$ , define

$$a_n = \frac{1}{n \ln(n) \ln(\ln(n))},$$

and for  $k \geq 2$ , define

$$b_k := 2^k a_{2^k} = \frac{2^k}{2^k \ln(2^k) \ln(\ln(2^k))}.$$

Then, by properties of logarithms,

$$b_k = \frac{1}{k \ln(2) \ln(k \ln(2))} = \frac{1}{k \ln(2) [\ln(k) + \ln(\ln(2))]}.$$

Also, for  $k \geq 2$ , we know that (under the assumption that  $\ln : (0, \infty) \rightarrow \mathbb{R}$  is an increasing function),

$$\ln(k) \geq \ln(2) \geq \ln(\ln(2)).$$

Hence

$$b_k \geq \frac{1}{2 \ln(2) k \ln(k)} =: c_k.$$

From lectures (or by applying the Cauchy condensation test again to  $c_k$ ), we know that  $\sum_{k=2}^{\infty} c_k$  diverges. Hence, by comparison (as  $c_k \geq 0$ ), we find that  $\sum_{k=2}^{\infty} b_k$  diverges. Finally, by the Cauchy condensation test, we conclude that

$$\sum_{n=4}^{\infty} a_n = \sum_{n=4}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))} \quad \text{diverges.}$$

It turns out that this example is a special case of what is known as a **generalised Bertrand series**, and it's quite surprising how general we can make this example! See [here](#) for details!