Analysis 1A — Tutorial 2

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1 Lecture Recap

1.1 Induction

This week first involves a little bit more about statements, namely those that involve the natural numbers $\mathbb{N}$. Suppose we have a statement $P(n)$ which depends on a natural number n. If we want to prove this true for all natural numbers, we use the principle of *mathematical induction*.

Firstly, let $\Lambda \subseteq \mathbb{N}$ be the set of all $n \in \mathbb{N}$ such that $P(n)$ is true. Then, to prove something by induction, we use the following procedure:

1. Show $1 \in \Lambda$.
2. Assume $k \in \Lambda$ for some $k \in \mathbb{N}$. Prove $k + 1 \in \Lambda$.

If these two steps are satisfied, then $\Lambda = \mathbb{N}$, and $P(n)$ holds for all natural numbers n. You may come across different styles of inductive proofs from different lecturers at Bath, but as long as you write everything logically, these are fine for this course too!

1.2 Field Axioms

As you might know from experience, natural numbers won't get us very far in maths. So instead, we turn to studying the real numbers ($\mathbb{R}$). But before we do, we need to know how these numbers behave under certain operations. This is where the *field axioms* come in. There's a long list of them in Section 2.1 of the lecture notes , so they're not repeated here in full. However, we can summarise[1] them as follows:

- **Addition**: On $\mathbb{R}$, addition is *associative* and *commutative*, an *additive identity* exists, and *additive inverses* exist.
- **Multiplication**: On $\mathbb{R} \setminus \{0\}$, multiplication is *associative* and *commutative*, a *multiplicative identity* exists, and *multiplicative inverses* exist.
- Multiplication *distributes* over addition.

Try matching the properties here to the numbered axioms in the lecture notes!

[1] Once you've learned some group theory, we can obscure everything behind more definitions. The first two bullet points can be stated as: $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$ are abelian groups. The third bullet point remains the same.

1.3 Order Axioms

As if the first 9 field axioms weren't enough, there are 5 *order axioms* you need to know. As these are useful for the problem sheet, they are presented below. Namely, for $x, y, z \in \mathbb{R}$:

- $x \leq y$ or $y \leq x$.
- If $x \leq y$ and $y \leq x$, then $x = y$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$.
- If $x \leq y$, then $x + z \leq y + z$.
- If $x \leq y$ and $z \geq 0$, then $xz \leq yz$.

1.4 Set Bounds

You might be interested to know that at this stage, despite the fact we have specified 14 axioms, these still aren't unique to the real numbers! For example, the exact same axioms also apply to the set of rational numbers $\mathbb{Q}$. Luckily, we only need one more axiom to complete our description of the real numbers. Unfortunately, there are a few definitions we need first...

Definition 1.1. Let $S \subseteq \mathbb{R}$. Then $M \in \mathbb{R}$ is an *upper bound* for S if for all $x \in S$, $x \leq M$. In this case, we say S is *bounded above*.

Definition 1.2. Let $S \subseteq \mathbb{R}$. Then $m \in \mathbb{R}$ is a *lower bound* for S if for all $x \in S$, $x \geq m$. In this case, we say that S is bounded below.

Definition 1.3. A set S is *bounded* if it is both *bounded above and below*. Equivalently, S is *bounded* if there exists $m, M \in \mathbb{R}$ such that for all $x \in S$, $m \leq x \leq M$.

Thinking of upper bounds for a moment, if we have one, we could ask if there is a smaller number which also bounds the set from above. You might also be tempted to ask what the 'best' upper bound on a set could be, such that no smaller number will bound the set from above. This leads to the ideas of *suprema and infima*:

Definition 1.4. Let $S \in \mathbb{R}$. A number $T \in \mathbb{R}$ is said to be the *supremum* of S if it is an *upper bound* for S, and for any other upper bound M, $T \leq M$. Here, we write $T = \sup(S)$.

Definition 1.5. Let $S \in \mathbb{R}$. A number $t \in \mathbb{R}$ is said to be the *infimum* of S if it is a *lower bound* for S, and for any other lower bound m, $t \geq m$. Here, we write $t = \inf(S)$.

For example, if we consider the set $S = (-1, 2] = \{x | -1 < x \leq 2\}$, we can see that example upper and lower bounds are $M = 3$ and $m = -2$ respectively, so the set is bounded. Its supremum and infimum are $\sup(S) = 2$ and $\inf(S) = -1$.

However, note that the supremum lies inside S, whereas the infimum does not lie inside S. This also tells us that the maximum element of S is 2, whereas S has no minimum element!

1.5 The Completeness Axiom

Finally, we are ready to state the required 15^{th} axiom! As the title suggests, this is known as the *Completeness Axiom*. It says that *every non-empty set S in $\mathbb{R}$ that is bounded above has a supremum.*[2]

Loosely, this axiom ensures that there are no 'gaps' in the real number line. For some more (precise) information, see this link.

1.6 Archimedean Postulate

To finish, we mention one result which will become very useful when studying sequences in the next few weeks. This is the *Archimedean Postulate*, and says that the set of natural numbers is unbounded above. In maths terms:

$$\forall x \in \mathbb{R}, \exists N \in \mathbb{N} \quad \text{such that} \quad N > x.$$

2 Hints

As per last week, here's the hints section of this document.

H1. Recall that a number N is a multiple of 3 if there exists $j \in \mathbb{Z}$ such that $N = 3j$. The proof is then similar to tutorial question 1.

H2. Try and get the problems into the form of one of the order axioms. Make sure to state each axiom you use, when you use it!

H3. This is a similar procedure to tutorial question 3. Splitting the fraction up will help!

H4. The induction should be straightforward. To find the formula you need to prove, have you seen a way of rewriting $\binom{n}{10}$ recently? (Have a look at the proof of the binomial theorem).

H5. Think back to the definitions, and use them to construct your proof of this result.

[2]In the lecture notes, it also states that 'Every non-empty set of real numbers that is bounded below has an infimum.' But you can deduce this from the supremum result by considering the set $-S := \{-x | x \in S\}$.