Analysis 1B — Tutorial 10

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Introduction

Here is the material to accompany the 10th Analysis 1B Tutorial on the 24th April. Alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to Christian Jones (caj50). To return to the homepage, click here.

1 Lecture Recap

This week is all about making our lives easier! Firstly, we're going to see a criterion for determining whether a function is integrable, and then we're going to see that quite a large class of functions are integrable! Finally, we're going to prime ourselves to develop a well-known result — the fundamental theorem of calculus — which links differentiation and integration.

1.1 The Cauchy Criterion for Integrability

Recall the definition of the (Riemann) integral:

Definition 1.1 (Riemann Integral)**.**
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is **Riemann integrable** if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

If this happens, then the (Riemann) integral of f is defined to be the common value, and given the notation $\int_a^b f$.

Note that for a function to be integrable, we require both the upper and lower Riemann integrals to exist and be equal. These were defined as follows:

Definition 1.2 (Lower and Upper Riemann Integrals)**.**

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then:

- The **lower Riemann integral** is
$$\underline{\int_a^b} f := \sup \left\{ L(f, P) \middle| \; P \text{ is a subdivision of } [a.b] \right\}.$$
- The **upper Riemann integral** is
$$\overline{\int_a^b} f := \inf \left\{ U(f, P) \middle| \; P \text{ is a subdivision of } [a.b] \right\}.$$

To actually find these values, we need to consider **every** possible subdivision P of the domain $[a.b]$. Doing this practically is near impossible, except in very rare cases[1]. What we would really like is a way of determining integrability from only a selection of partitions. It shouldn't come as a surprise by now that such a method exists, and it's due to — you guessed it — Cauchy![2]

Proposition 1.1 (Cauchy Criterion for Integrability)**.**
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a subdivision P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon.$

So, why is this formulation useful? Due to Archimedes principle, we now only have to consider regularly spaced subdivisions P_n of $[a, b]$ to determine integrability! In particular, these subdivisions are given by

$$P_n = \{x_0, \ldots, x_n\}, \quad x_i = a + \frac{i(b-a)}{n}.$$

This criterion also gives us the following theorem:

Theorem 1.2.
Let $f : [a, b] \to \mathbb{R}$. Then

- If f is monotonic, then it is integrable.
- If f is continuous, then it is integrable.

[1] Such as the function f being constant, for example.
[2] Result number six on the 'named after Cauchy' counter!

So, using the Cauchy criterion, we have determined that a large class of functions are integrable! However, to prove the second part of this theorem, we require a (slightly) stronger version of continuity.

1.2 Uniform Continuity

Recall the definition of (standard) continuity:

Definition 1.3 (Continuity)**.**
Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$. Then f is continuous on D if

$$\forall c \in D \; \forall \epsilon > 0 \; \exists \delta = \delta(\epsilon, c) > 0 \text{ s.t. } \forall x \in D, \; |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

In this definition, the 'distance' δ away from c you can be for $f(x)$ to stay within ϵ of $f(c)$ depends on both the choice of ϵ, and where you are in the domain D, i.e. your choice of c. If instead, your choice of δ remains the same no matter where you are in D, then f is said to be **uniformly continuous**.

Definition 1.4 (Uniform Continuity)**.**
Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$. Then f is uniformly continuous on D if

$$\forall \epsilon > 0 \; \exists \delta = \delta(\epsilon) > 0 \text{ s.t. } \forall x, y \in D, \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

ADD DIAGRAM/GIF FROM WIKIPEDIA HERE (BECAUSE ITS GOOD)

From this, we see by fixing y in the definition of uniform continuity, we deduce that if a function is uniformly continuous, it is automatically continuous! In fact, in a particular case, the reverse also holds true:

Proposition 1.3.
Let $f : [a, b] \to \mathbb{R}$. Then f is continuous if and only if it is uniformly continuous.

1.2.1 Other forms of Continuity

Whilst less relevant to this course, there are versions of continuity which are stronger still! The first we will mention here is known as Hölder continuity.

Definition 1.5 (Hölder Continuity)**.**
Let $f : [a, b] \to \mathbb{R}$. Then f is said to be α-Hölder continuous if

$$\sup_{x,y \in [a,b], x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.$$

The set of all α-Hölder continuous functions from $[a, b]$ is denoted by $C^{0,\alpha}([a, b])$.

Ok, this definition looks quite scary, so a visual is probably quite welcome here. See Figure ?? for details.

You've already shown in a previous problem sheet that if $\alpha > 1$, then the only α-Hölder continuous functions are constant. Another important class of Hölder continuous functions occurs when $\alpha = 1$. This is a case you're also likely to have come across in the problem sheets:

Definition 1.6 (Lipschitz Continuity)**.**
Let $f : [a, b] \to \mathbb{R}$. Then f is said to be Lipschitz continuous if $\exists L > 0$ such that $\forall x, y \in [a, b]$:

$$|f(x) - f(y)| < L|x - y|.$$

Again, this is something we can visualise (see Figure ??). Quite handily, if we introduce $\delta = \epsilon/L$, we see that a Lipschitz continuous function satisfies the definition of uniform continuity, and so is also continuous!

1.2.2 Continuity and Differentiability

You may remember that if a function $f : I \to \mathbb{R}$ is differentiable on an open interval $I \subseteq \mathbb{R}$, then it is continuous on I. However, we cannot strengthen this result in the way you might expect. Namely, it is **not** true that differentiability implies either Lipschitz or uniform continuity.

Example 1.1.

To see why, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. We know that the derivative function is $f' : \mathbb{R} \to \mathbb{R}$ given by $f'(x) = 2x$. However, f is not uniformly continuous on $\mathbb{R}$.

To prove this, we consider the negation of the definition, i.e. we seek $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, and $|f(x) - f(y)| \geq \epsilon_0$.

Try $\epsilon_0 = 1$. Then

$$|f(x) - f(y)| \geq 1 \Leftrightarrow |x + y| \geq \frac{1}{\delta}.$$

Looking only at positive values of x, y (which we can do since we are searching for x and y in this problem), our two constraints suggest we try $x = \frac{1}{2\delta}$ and $y = x + \frac{\delta}{2}$. Then

$$|x - y| = \frac{\delta}{2} < \delta, \text{ and } |x + y| = \frac{1}{\delta} + \frac{\delta}{2} \geq \frac{1}{\delta}.$$

This shows that f is not uniformly continuous (and is; therefore, also not Lipschitz continuous)

However, all hope is not lost. In fact, using the Mean Value Theorem, we can recover a result linking differentiability and continuity!

Proposition 1.4.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If f is differentiable on (a, b) with bounded derivative, i.e. $\exists L > 0$ such that $|f'(x)| < L \; \forall x \in (a, b)$, then f is uniformly continuous.

2 Hints

As per usual, here's where you'll find the problem sheet hints!

1) Have a look back at the example we did in tutorials — this one is pretty similar! Calculate both the lower and upper Riemann sums, and make sure to justify the main steps of your argument. Another thing, feel free to quote the values of

$$\sum_{i=1}^{n} i^k, \quad k = 0, 1, 2, \ldots$$

as 'standard results', although being able to prove them is good practice too!

2) First of all, this is an 'if and only if', so there are two things to prove! The '$\Leftarrow$' direction should be fairly straightforward. For the '$\Rightarrow$' direction, what do you get from considering $U(f, P) - L(f, P)$?

3) Try the function

$$h : [0, 1] \to \mathbb{R}, \quad h(x) = xf(x),$$

where f is as given in the question. Make sure to prove continuity at zero first. To prove that h is not integrable, show that the lower and upper Riemann integrals cannot be the same. For any subdivision P, finding $L(h, P)$ should be OK. For $U(h, P)$, try and compare it to an upper Riemann sum of a function that you know the integral of.