Analysis 1A — Tutorial 3

Christian Jones: University of Bath

October 2023

Contents

Introduction

Here is the material to accompany the Analysis Tutorial in Week 3. Alternative formats can be downloaded by clicking the download icon at the top of the page. As usual, send comments and corrections to Christian Jones (caj50). To return to the homepage, click here.

1 Lecture Recap

1.1 Suprema and Infima

There's still a little bit of material to cover regarding the supremum and infimum of a set. To begin, we re-cover the definitions from last week.

Definition 1.1 (Supremum)**.** Let $S \in \mathbb{R}$. A number $T \in \mathbb{R}$ is said to be the supremum of S if it is an upper bound for S, and for any other upper bound M, $T \leq M$. Here, we write $T = \sup(S)$.

Definition 1.2 (Infimum)**.** Let $S \in \mathbb{R}$. A number $t \in \mathbb{R}$ is said to be the infimum of S if it is a lower bound for S, and for any other lower bound m, $t \geq m$. Here, we write $t = \inf(S)$.

There's also two results from last week's notes that you didn't reach in lectures either, so we (re)state these below too.

Completeness Axiom. Every non-empty set S in $\mathbb{R}$ that is bounded above has a supremum.

Proposition 1.1 (Archimedian Postulate)**.** *We have that* $\forall x \in \mathbb{R}, \exists N \in \mathbb{N}$ *such that* $N > x$. *In other words, the set of natural numbers* $\mathbb{N}$ *is unbounded above.*

It also turns out that there's an alternative characterisation of suprema and infima which can be very useful, especially if the members of a set aren't indexed by natural numbers.

Proposition 1.2. *Let* $S \subseteq \mathbb{R}$. *Then a number* $T \in \mathbb{R}$ *is the supremum of* S, *denoted* $\sup(S)$ *if:*

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s > T - \epsilon.$$

Proposition 1.3. *Let* $S \subseteq \mathbb{R}$. *Then a number* $t \in \mathbb{R}$ *is the infimum of* S, *denoted* $\inf(S)$ *if:*

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s < t + \epsilon.$$

As an example, take the set $S = (-1, 2] = \{x \,|\, -1 < x \leq 2\}$, and fix some $\epsilon > 0$. Then, if we take $s_1 = 2 - \epsilon/2$ and $s_2 = -1 + \epsilon/2$, we see that

- s_1 and s_2 are in the set S,
- $s_1 > 2 - \epsilon$, and
- $s_2 < -1 + \epsilon$.

Hence, as ϵ was arbitrary, the alternative characterisation of suprema and infima says that $\sup(S) = 2$ and $\inf(S) = -1$.

1.2 Inequalities

Inequalities come up everywhere in maths! For example, they can be used in statistics for estimation (Markov/Chebyshev inequalities), they can be used as constraints in optimisation problems (see Section 3.1 of this Wikipedia link.), and quite famously appear in Quantum Mechanics. In this latter case, we have the Heisenberg Uncertainty Principle, and this inequality states that you can't simultaneously know the position and momentum of a quantum particle, such as an electron.

Most of the inequalities in this course will be based on the absolute value, which is defined as follows:

Definition 1.3 (Absolute Value)**.** For $x \in \mathbb{R}$, the absolute value of x is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases} = \max\{x, -x\}.$$

The absolute value has the following properties:

Proposition 1.4. *For $x, y \in \mathbb{R}$:*

$$x \leq |x|, \quad -x \leq |x|, \quad |-x| = |x| \quad \textit{and} \quad |xy| = |x||y|.$$

Now we come on to what I consider to be the most important thing in this course.

Theorem 1.5 (Triangle Inequalities)**.** *For $x, y \in \mathbb{R}$:*

- $|x + y| \leq |x| + |y|$, *and*
- $||x| - |y|| \leq |x - y|$.

The first of these is known as the **Triangle Inequality**, and the second is the **Reverse Triangle Inequality**. Why do I think this is so important? This will come up in almost any course you take at university that uses analysis! If you're studying vector calculus, fluid mechanics, statistics, probability, or anything that's not abstract algebra, there's guaranteed to be a proof or technique which involves an inequality of this form! So if you only learn one result from Analysis 1A, make it this one.

Finally, there's one more inequality to mention — the binomial inequality.

Proposition 1.6 (Binomial Inequality)**.** *We have $\forall n \in \mathbb{N}_0$ (i.e. all the natural numbers with 0), and $\forall x \geq -1$,*

$$(1 + x)^n \geq 1 + nx.$$

2 Hints

As per usual, here's where you'll find the problem sheet hints!

1. The techniques involved in this one are similar to those used in Tutorial Question 1 (and there were some questions on this last week too!)
2. i) Firstly, note that this is an **if and only if** statement, so there are two things to prove! You can get most of the way between both statements by completing the square on the polynomial. For the "$\Rightarrow$" direction, remember square numbers are non-negative. For the "$\Leftarrow$" direction, make a specific choice of x.
   ii) Follow the hint here, and be careful when collecting terms together.
3. Take cases on x.
4. Without loss of generality (WLOG), consider $x \geq y$ (otherwise you can just swap them), and consider $|\sqrt{x} - \sqrt{y}|^2$. On expanding, try and find a bound for the 'middle' term.
5. Solve the modulus equation, and then use your solutions to formulate simultaneous equations for c and r.
6. You should only need the definitions given in lectures to solve this question. Make sure to write things logically!