

# Analysis 1A — Supplementary Paper 2020

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## Introduction

Here are the solutions to the past paper discussed in the revision session on 9th January 2023. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click [here](#).

## Question 1

**Question.** For each of the following concepts, give an example that satisfies the definition and an example that does not. (You need not give any proofs.)

- a) A Cauchy sequence.
- b) A decreasing sequence.
- c) A sequentially continuous function.
- d) A conditionally convergent series.
- e) An interval.

*Solution.* a) An example is the sequence  $(a_n)_{n \in \mathbb{N}}$ , where  $a_n = \frac{1}{n}$ . A non-example is the sequence  $(b_n)_{n \in \mathbb{N}}$  where  $b_n = n$ .

b) An example is the sequence  $(a_n)_{n \in \mathbb{N}}$ , where  $a_n = \frac{1}{n}$ . A non-example is the sequence  $(b_n)_{n \in \mathbb{N}}$  where  $b_n = n$ .

c) An example is the function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by  $f(x) = x$ . A non-example is the function  $g : [0, 1] \rightarrow \mathbb{R}$ , where

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 0.5, \\ 1 & \text{if } 0.5 \leq x \leq 1. \end{cases}$$

d) An example is the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ . A non-example is  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ .

e) An example of an interval is the set  $S_1 = (0, 1)$ . A non-example is the set  $S_2 = (-1, 0) \cup (1, 2)$ .

(If question 1 is like this in the exam, examples which can be used in more than one part will help you save time!)

## Question 2

**Question.** The following statements paraphrase theorems, corollaries, propositions, or lemmas from the lectures. Identify them by their names.

a) Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence. If  $\sup_{n \in \mathbb{N}} |a_n| < \infty$ , then there exists a sequence  $(n_k)_{k \in \mathbb{N}}$  such that

$$\forall k \in \mathbb{N} : (n_k \in \mathbb{N}) \wedge (n_{k+1} > n_k)$$

and there exists  $B \in \mathbb{R}$  such that

$$\forall \epsilon > 0 \exists K \in \mathbb{N} \forall k \geq K : |a_{n_k} - B| < \epsilon.$$

b) Suppose that  $(a_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$  are two real sequences such that

$$s_n = \sum_{k=1}^{\infty} (-1)^k a_k$$

for all  $n \in \mathbb{N}$ . If  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{N}$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ , then  $(s_n)_{n \in \mathbb{N}}$  converges.

c)  $\forall x \in \mathbb{R} \exists k \in \mathbb{N} : k > x$ .

d) Let  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  be two sequences such that

$$\forall n \in \mathbb{N} : x_n \leq x_{n+1} \leq y_{n+1} \leq y_n.$$

Then, there exists  $a \in \mathbb{R}$  such that

$$\forall n \in \mathbb{N} : x_n \leq a \leq y_n.$$

e) Suppose that  $a \in [-1, \infty)$  and  $k \in \mathbb{N}_0$ . Then

$$1 + ka \leq (1 + a)^k.$$

*Solution.* a) This is the *Bolzano-Weierstrass* theorem.

b) This is the *Leibniz alternating series test* for series.

c) This is the *Archimedian Postulate*.

d) This is the *Nested Intervals Theorem*.

e) This is the *Binomial inequality*.

## Question 3

**Question.** Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence and  $L \in \mathbb{R}$ .

a) Show that  $a_n \rightarrow \infty$  if and only if  $-a_n \rightarrow -\infty$  as  $n \rightarrow \infty$ .

b) Assuming that  $\lim_{n \rightarrow \infty} a_n = L$ , show that  $(|a_n|)_{n \in \mathbb{N}}$  does *not* diverge to  $\infty$ .

c) i) Use the growth factor test to show that

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^2} = 0.$$

You may use without proof that  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$  exists.

ii) Show that there exists  $N \in \mathbb{N}$  such that

$$n! \leq n^n \leq \left(\frac{n!}{100}\right)^2,$$

for all  $n \in \mathbb{N}$ .

In the following questions (d) and (e), you may use any result from the lectures without proof.

d) Find

$$\lim_{n \rightarrow \infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}).$$

e) Show that

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = 1.$$

*Solution.* a) We have that

$$\begin{aligned} a_n \rightarrow \infty &\iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } a_n \geq M, \\ &\iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } -a_n \leq -M. \end{aligned}$$

Setting  $K = -M$  in this last statement gives

$$\begin{aligned} a_n \rightarrow \infty &\iff \forall K \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } -a_n \leq K, \\ &\iff -a_n \rightarrow -\infty, \end{aligned}$$

as required.

b) We claim that  $\lim_{n \rightarrow \infty} |a_n| = |L|$ . To this end, fix  $\epsilon > 0$ . Since  $\lim_{n \rightarrow \infty} a_n = L$ , we know that there exists  $N \in \mathbb{N}$  such that

$$|a_n - L| < \epsilon \quad \forall n \geq N.$$

Now, for all  $n \geq N$ ,

$$\begin{aligned} ||a_n| - |L|| &\leq |a_n - L|, \quad (\text{by the reverse triangle inequality}) \\ &< \epsilon \end{aligned}$$

Hence, since  $\epsilon$  was arbitrary, we conclude that  $\lim_{n \rightarrow \infty} |a_n| = |L|$ . In particular,  $(|a_n|)_{n \in \mathbb{N}}$  does not diverge to  $\infty$ .

c) i) Setting  $b_n = \frac{n^n}{(n!)^2}$  for  $n \in \mathbb{N}$ , we see that  $b_n \geq 0$ , and

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} \\ &= (n+1) \frac{(n+1)^n}{n^n} \left( \frac{n!}{(n+1)!} \right)^2 \\ &= (n+1) \left( 1 + \frac{1}{n} \right)^n \frac{1}{(n+1)^2} \\ &= \frac{1/n}{1 + 1/n} \left( 1 + \frac{1}{n} \right)^n.\end{aligned}$$

Since  $\frac{1}{n} \rightarrow 0$ , and  $\left(1 + \frac{1}{n}\right)^n \rightarrow e$  as  $n \rightarrow \infty$ , we find by the algebra of limits that

$$\frac{b_{n+1}}{b_n} \rightarrow 0 \cdot e = 0 \text{ as } n \rightarrow \infty.$$

Since  $0 < 1$ , we find by the growth factor test that

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^2} = 0,$$

as required.

ii) By part i) and the definition of convergence, we know that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$ ,

$$\left| \frac{n^n}{(n!)^2} - 0 \right| = \frac{n^n}{(n!)^2} \leq \left( \frac{1}{100} \right)^2.$$

Also, note that since  $n! \leq n^n$  for all  $n \in \mathbb{N}$ ,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2}.$$

Hence, for all  $n \geq N$ ,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2} \leq \left( \frac{1}{100} \right)^2 \iff n! \leq n^n \leq \left( \frac{n!}{100} \right)^2,$$

as required.

d) First, note that via completing the square,

$$\begin{aligned}\sqrt{n}(\sqrt{n+1} - \sqrt{n}) &= \frac{\sqrt{n}(n+1-n)}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.\end{aligned}\tag{*}$$

Now, we claim that

$$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1.$$

To show this, we fix  $\epsilon > 0$  and consider for  $n \in \mathbb{N}$ :

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| = \frac{1 + \frac{1}{n} - 1}{\sqrt{1 + \frac{1}{n}} + 1} \leq \frac{1}{n}.$$

We then have that

$$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

Hence, for any  $n \geq N$ ,

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1.$$

Returning to (\*) and applying the algebra of limits, we find that as  $n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{1}{1+1} = \frac{1}{2}.$$

e) Since  $2 > 1$ , we write  $\sqrt[n]{2} = 1 + x_n$ , where  $x_n \geq 0$ . This gives

$$\begin{aligned} 2 &= (1 + x_n)^n \geq 1 + nx_n \quad (\text{by the binomial inequality}) \\ &\geq 1. \end{aligned}$$

Rearranging, we find

$$0 \leq x_n \leq \frac{1}{n}.$$

Now, since  $0 \rightarrow 0$  and  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ ,  $x_n \rightarrow 0$  by the sandwich theorem. Hence, by the algebra of limits,

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = \lim_{n \rightarrow \infty} (1 + x_n) = 1 + 0 = 1,$$

as required.

## Question 4

**Question.** In this question, you may use any result from the lectures without proof.

a) Let  $a \in (0, 1)$ . Using the theorem on the Cauchy product of series, or otherwise, show that

$$\left( \sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} (n+1)a^n.$$

b) Find the radii of convergence of the following power series.

i)

$$\sum_{n=0}^{\infty} x^n.$$

ii)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}.$$

iii)

$$\sum_{n=0}^{\infty} \left( \frac{2\pi}{13} \right)^n \cos \left( \frac{2\pi n}{13} \right) x^n.$$

c) Say whether or not the following series converge and explain your reasoning.

i)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

ii)

$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2}.$$

iii)

$$\sum_{n=1}^{\infty} \frac{1}{n \log(n)}.$$

*Solution.* a) Recall that for  $a \in (0, 1)$ , the sum is a geometric series with

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

As  $|a^n| = a^n \forall n \in \mathbb{N}_0$ , this series is absolutely convergent, so the Cauchy multiplication theorem gives that

$$\left( \sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} c_n,$$

where for  $n \in \mathbb{N}_0$ ,

$$c_n = \sum_{k=0}^n a^k a^{n-k} = \sum_{k=0}^n a^n = (n+1)a^n.$$

Hence

$$\left( \sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} (n+1)a^n,$$

as required.

b) i) Writing

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} a_n x^n,$$

with  $a_n = 1$ , we calculate the radius of convergence,  $R$ , as

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{1}{1} = 1.$$

ii) Writing

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n} = \sum_{n=1}^{\infty} b_n x^n,$$

with  $b_n = \frac{1}{n^n}$ , we calculate

$$\limsup_{n \rightarrow \infty} |b_n|^{1/n} = \limsup_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Hence, by Cauchy-Hadamard, the radius of convergence  $R$  for this power series is  $R = \infty$ .

iii) Writing

$$\sum_{n=0}^{\infty} \left( \frac{2\pi}{13} \right)^n \cos \left( \frac{2\pi n}{13} \right) x^n = \sum_{n=0}^{\infty} c_n x^n,$$

with  $c_n = \left( \frac{2\pi}{13} \right)^n \cos \left( \frac{2\pi n}{13} \right)$ , we calculate

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} = \limsup_{n \rightarrow \infty} \frac{2\pi}{13} \left| \cos \left( \frac{2\pi n}{13} \right) \right|^{1/n}.$$

Setting  $d_n = \frac{2\pi}{13} \left| \cos \left( \frac{2\pi n}{13} \right) \right|^{1/n}$ , we claim that  $\limsup_{n \rightarrow \infty} d_n = \frac{2\pi}{13}$ . First, as for all  $y \in \mathbb{R}$ ,  $|\cos(y)| \leq 1$ , we see that  $d_n \leq \frac{2\pi}{13}$ . This means that

$$\limsup_{n \rightarrow \infty} d_n \leq \frac{2\pi}{13}.$$

Moreover, taking the subsequence  $(d_{n_k})_{k \in \mathbb{N}}$ , where  $n_k = 13k$ , we see that as  $k \rightarrow \infty$ ,

$$d_{13k} = \frac{2\pi}{13} \left| \cos(2\pi k) \right|^{\frac{1}{13k}} = \frac{2\pi}{13} \cdot 1 \rightarrow \frac{2\pi}{13} \quad (\text{by AoL.})$$

So,  $\limsup_{n \rightarrow \infty} d_n \geq \frac{2\pi}{13}$ , from which we conclude that  $\limsup_{n \rightarrow \infty} d_n = \frac{2\pi}{13}$ . Hence, by Cauchy-Hadamard, the radius of convergence  $R$  for this power series is given by

$$R = \left( \limsup_{n \rightarrow \infty} d_n \right)^{-1} = \frac{13}{2\pi}.$$

c) i) Recall from lectures that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{diverges if } \alpha \leq 1, \\ \text{converges if } \alpha > 1. \end{cases}$$

Since  $\sqrt{n} = n^{1/2}$ , and  $\frac{1}{2} < 1$ , we know that the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges.

ii) First, note that

$$\frac{n}{2^n + n^2} \leq \frac{n}{2^n}. \quad (**)$$

Now, setting  $x_n = \frac{n}{2^n}$ , we find

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)2^n}{2^{n+1}n} = \frac{1}{2} \left( 1 + \frac{1}{n} \right).$$

Hence, by the algebra of limits,

$$\frac{|x_{n+1}|}{|x_n|} \rightarrow \frac{1}{2} < 1,$$

as  $n \rightarrow \infty$ . So, by d'Alembert's ratio test, we find that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \text{ converges.}$$

Finally, by the comparison test as applied to (\*\*), we conclude that

$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2} \text{ converges.}$$

iii) Setting  $y_n = \frac{1}{n \log(n)}$ , we define for  $k \geq 1$ ,

$$z_k := 2^k y_{2^k} = \frac{2^k}{2^k \log(2^k)} = \frac{1}{k \log(2)}.$$

Using the result stated in part i), we know that as  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges,  $\sum_{k=1}^{\infty} z_k$  diverges. Hence, by the Cauchy condensation test, the given series  $\sum_{n=1}^{\infty} y_n$  diverges.