Cauchy Condensation Example

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1 Example involving logarithms

Example 1.1.

Prove that

$$\sum_{n=4}^{\infty} \frac{1}{n\ln(n)\ln(\ln(n))}$$

diverges.

Solution.

For $n \geq 4$, define

$$a_n = \frac{1}{n\ln(n)\ln(\ln(n))},$$

and for $k \geq 2$, define

$$b_k := 2^k a_{2^k} = \frac{2^k}{2^k\ln(2^k)\ln(\ln(2^k))}.$$

Then, by properties of logarithms,

$$b_k = \frac{1}{k\ln(2)\ln(k\ln(2))} = \frac{1}{k\ln(2)[\ln(k)+\ln(\ln(2))]}.$$

Also, for $k \geq 2$, we know that (under the assumption that $\ln : (0,\infty) \to \mathbb{R}$ is an increasing function),

$$\ln(k) \geq \ln(2) \geq \ln(\ln(2))).$$

Hence

$$b_k \geq \frac{1}{2\ln(2)k\ln(k)} =: c_k.$$

From lectures (or by applying the Cauchy condensation test again to c_k), we know that $\sum_{k=2}^{\infty} c_k$ diverges. Hence, by comparison (as $c_k \geq 0$), we find that $\sum_{k=2}^{\infty} b_k$ diverges. Finally, by the Cauchy condensation test, we conclude that

$$\sum_{n=4}^{\infty} a_n = \sum_{n=4}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))} \quad \text{diverges.}$$

It turns out that this example is a special case of what is known as a **generalised Bertrand series**, and it's quite surprising how general we can make this example! See here for details!