Analysis 1A — Tutorial 4

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1 Lecture Recap

1.1 Sequences and Convergence

Firstly, to discuss anything this week, we need to introduce the idea of a sequence.

Definition 1.1. A *sequence* of real numbers is a function

$$\begin{aligned} a : \mathbb{N} &\longrightarrow \mathbb{R}, \\ n &\longmapsto a_n. \end{aligned}$$

Since this notation can get kind of annoying, we instead denote a sequence by $(a_n)_{n\in\mathbb{N}}$. If it's clear from the context what set we're indexing over, we can even just simply write a sequence as (a_n).

Now, this gives us an infinitely long list of real numbers, and sometimes its interesting to look at the 'long-term' behaviour of these lists. This gives rise to the idea of convergence.

Definition 1.2. A sequence (a_n) *converges* to a real number L as $n \longrightarrow \infty$, written as either $a_n \longrightarrow L$, or $\lim_{n\to\infty} a_n = L$ if

$$\forall \epsilon > 0,\ \exists N = N(\epsilon) \in \mathbb{N}, \text{ such that } \forall n \geq N,\ |a_n - L| < \epsilon.$$

Loosely speaking, this says that no matter how close you want the sequence to get to L, you will always be able to find some point in the sequence after which all points in the sequence will be as close to L as you wanted. For an example of this, have a look at this Desmos link. For $\epsilon = 0.5$ and $L = 3$, you can see that every member of the sequence after the 11^{th} lies within a strip of width 2ϵ around L. Have a go at messing with the value of ϵ!

Something else we can mention for the definition is its *negation*. Specifically, a sequence (a_n) does not converge to L if

$$\exists\ \epsilon_0 > 0, \text{ such that } \forall N \in \mathbb{N}, \exists n \geq N \text{ such that } |a_n - L| \geq \epsilon_0.$$

1.2 Useful Sequences

Some (straightforward) results from using the definition include

- $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.
- For a real number c: $c \longrightarrow c$ as $n \longrightarrow \infty$.
- For $q \in \mathbb{R}$ with $|q| < 1$: $q^n \longrightarrow 0$ as $n \longrightarrow \infty$.

1.3 Two Useful Theorems

Theorem 1.1 (Preservation of Non-Strict Inequalities)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences and $L, M \in \mathbb{R}$ such that $a_n \to L$ and $b_n \to M$ as $n \to \infty$. If $a_n \le b_n$ $\forall n \in \mathbb{N}$, then $L \le M$.*

There are two good uses for this theorem. The first says that non-negative sequences should have non-negative limits — which you might expect — and secondly, it gives us this second theorem.[1]

Theorem 1.2 (Uniqueness of Limits)**.** *If $(a_n)_{n\in\mathbb{N}}$ is convergent with $a_n \to L$ and $a_n \to M$ as $n \to \infty$, then $L = M$.*

1.4 Bounded Sequences

Much like sets, we can formulate a definition which allows us to 'trap' sequences.

Definition 1.3. A sequence (a_n) is bounded if there exists $M \in \mathbb{R}$ such that $|a_n| \le M$.

If you prefer to think diagramatically, this says we can trap the sequence within a strip of width $2M$ around 0. More importantly, this leads to the idea that *all convergent sequences are bounded*. Note that this is equivalent to saying that if a sequence is not bounded, then it is not convergent.

1.5 Algebra of Limits

Using the definition to prove all limits would be an incredibly boring way to go through this course. Luckily, there are a few general results we can prove which make our lives so much easier. This is known as the *algebra of limits* (AoL).

Theorem 1.3 (Algebra of Limits)**.** *Let $A, B, c \in \mathbb{R}$ and let (a_n) and (b_n) be sequences with $a_n \to A$ and $b_n \to B$ as $n \to \infty$. Then:*

1. $\lim_{n\to\infty}(a_n + b_n) = A + B$,
2. $\lim_{n\to\infty}(ca_n) = cA$,
3. $\lim_{n\to\infty}(a_n b_n) = AB$,
4. *If $b_n \ne 0$ $\forall n \in \mathbb{N}$ and $B \ne 0$,* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

2 Hints

As per usual, here's where you'll find the problem sheet hints!

H1. Use the definition! Try and follow a similar format to what we did in tutorials. Make sure to write things logically, and ensure that you've satisfied each part of the definition.

[1] Feel free to ignore this footnote, but there are areas of maths where limits are not unique. This is usually in the realm of topology, which you can take in Year 3 (MA30055). Luckily for us, everything behaves nicely, and our limits are unique.

H2i). The hint on the sheet will certainly help (can you see the difference of two squares trick here?) The definition is probably the best way to go here. Remember that making a positive denominator smaller will also make the fraction bigger too!

H2ii). Feel free to use AoL here, but make sure to justify why you can use it!

H3. This one is a bit tricky. Firstly, what do you get if you factorise $x^3 - y^3$? Next, you'll want to use the fact that $\lim_{n\to\infty} a_n = 1$ twice — once to introduce an ϵ into the problem, and again to find a point in the sequence after which all of the a_n are positive. Combining all this information should help you prove the required result.