Analysis 1A — Tutorial 3

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Contents

Introduction

Here is a version of Tutorial Question 3 off of Problem Sheet 3 with an alternative solution for part c). Parts a) and b) are included for completeness.

Example 0.1 (PS3 Question 3)**.** a) Show that

$$2xy \leq x^2 + y^2, \ \ \forall x, y \in \mathbb{R},$$

and that equality holds only if $x = y$. b) Show that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \leq \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \ \ \forall x, y > 0.$$

c) Prove that

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| \leq |x-y| \ \ \forall x, y \in \mathbb{R}.$$

Part a)

Solution. We have that for any $x, y \in \mathbb{R}$,

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2,$$

from which rearranging gives

$$2xy \leq x^2 + y^2.$$

Now,

$$2xy = x^2 + y^2 \ \Leftrightarrow \ 0 = (x-y)^2 \ \Leftrightarrow \ x - y = 0 \ \Leftrightarrow \ x = y.$$

So equality holds *if and only if* $x = y$.

Part b)

Solution. Consider the second inequality first. Note that since $x, y > 0$, $\sqrt{x}, \sqrt{y} > 0$, and so

$$x + y \leq x + 2\sqrt{x}\sqrt{y} + y = (\sqrt{x} + \sqrt{y})^2.$$

Square rooting this result gives us that

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}.$$

Next, we have that

$$\begin{aligned}\left(\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}}\right)^2 &= \frac{x}{2} + 2\sqrt{\frac{x}{2}}\sqrt{\frac{y}{2}} + \frac{y}{2}, \\ &\leq \frac{x}{2} + 2\left(\frac{x}{2} + \frac{y}{2}\right) + \frac{y}{2} \quad \text{(by part a)}, \\ &= x + y.\end{aligned}$$

Again, square rooting gives us that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \leq \sqrt{x+y}.$$

Part c)

Solution. Firstly for $x = -y$,

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| = 0 \leq |-2y| = |x-y|.$$

For $x \neq -y$, we have

$$\begin{aligned} |\sqrt{1+x^2} - \sqrt{1+y^2}| &= \frac{1 + |x^2 - (1+y^2)|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \qquad (*) \\ &= \frac{|x^2 - y^2|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \\ &= \frac{|x+y||x-y|}{\sqrt{1+x^2} + \sqrt{1+y^2}}. \end{aligned}$$

Now,

$$|x| \leq \sqrt{1+x^2}, \text{ and } |y| \leq \sqrt{1+y^2}.$$

(This can be seen by squaring both sides of each inequality)

So,

$$\begin{aligned} |x+y| &\leq |x| + |y| \quad \text{(by the triangle inequality)} \\ &\leq \sqrt{1+x^2} + \sqrt{1+y^2}, \\ \Leftrightarrow \frac{1}{|x+y|} &\geq \frac{1}{\sqrt{1+x^2} + \sqrt{1+y^2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\sqrt{1+x^2} - \sqrt{1+y^2}| &= \frac{|x+y||x-y|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \\ &\leq \frac{|x-y||x+y|}{|x+y|}, \qquad (**) \\ &= |x-y|, \end{aligned}$$

as required!

You might have a few questions about this:

Q1) Why is 3c) done in this way?

A1) It's an alternative way to the one in the model solutions, but I think it's good because it uses some techniques that are useful for the sequences part of the course (e.g. the triangle inequality and step (*)).

Q2) Where on Earth did the case $x = -y$ come from?

A2) If you look at (**), this expression doesn't work if $x = -y$, so you need to consider this separately. It's not an obvious case until you actually reach (**), but once you realise it, it's an easy thing to add to the start of your solution.

Q3) What about (*)? Where does this come from?

A3) Recall for $a, b \in \mathbb{R}$,

$$(a-b)(a+b) = a^2 - b^2.$$

Taking $a = \sqrt{1+x^2}$, and $b = \sqrt{1+y^2}$, we have that

$$\sqrt{1+x^2} - \sqrt{1+y^2} = \frac{(1+x^2) - (1+y^2)}{\sqrt{1+x^2} + \sqrt{1+y^2}}.$$