MA10207 - Liminf and Limsup Examples

Below are three different examples of finding the liminf and limsup of a sequence, with different methods used in each case.

Example 1 (PS7)

Let $a_n = (-1)^n \dfrac{2n}{1+3n} = (-1)^n \dfrac{2}{3} \dfrac{1}{1/3n + 1}$

Splitting into odd and even cases:

$$a_n = \begin{cases} \dfrac{2}{3} \dfrac{1}{1/3n + 1} & \text{for } n \text{ even} \\[2ex] \dfrac{-2}{3} \dfrac{1}{1/3n + 1} & \text{for } n \text{ odd.} \end{cases}$$

Note that $a_{2j-1} \leq 0 \leq a_{2j}$ $\forall j \in \mathbb{N}$. Also note that $(a_{2j-1})_{j \in \mathbb{N}}$ is a decreasing sequence and $(a_{2j})_{j \in \mathbb{N}}$ is an increasing sequence [Try showing this!]. Moreover, $|a_n| \leq 2/3$ $\forall n \in \mathbb{N}$, so $(a_n)_{n \in \mathbb{N}}$ is bounded.

Now, fix $k \in \mathbb{N}$. Then:

$$\begin{aligned} \sup_{n \geq k} a_n &= \sup_{2j \geq k} a_{2j} && (\text{by } \quad) \\ &= \lim_{j \to \infty} a_{2j} && (\text{since } (a_{2j})_{j \in \mathbb{N}} \text{ is a bounded increasing sequence}) \\ &= 2/3 && (\text{by AoL}) \end{aligned}$$

$\therefore$ As $k \longrightarrow \infty$, $\sup_{n \geq k} a_n = \dfrac{2}{3} \longrightarrow \dfrac{2}{3}$. $\therefore \limsup_{n \to \infty} a_n = \dfrac{2}{3}$

2) Similarly, fixing $k \in \mathbb{N}$:

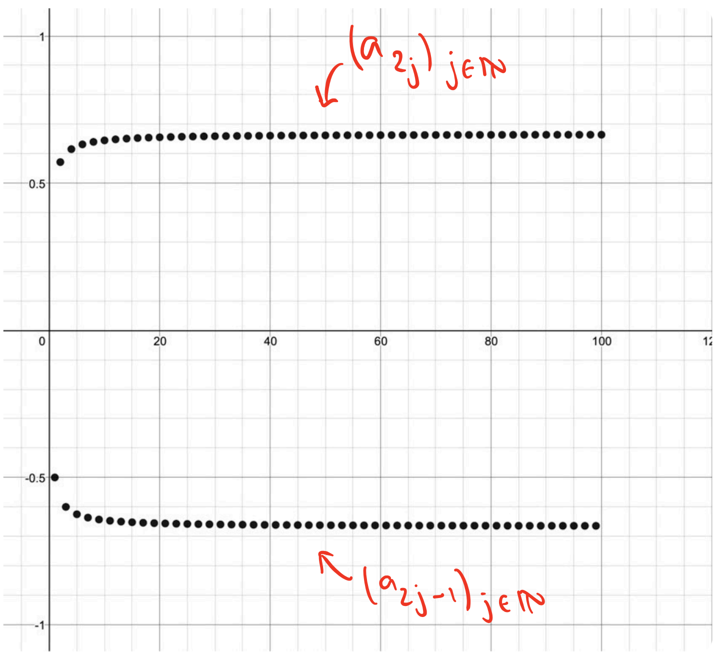
$$\inf_{n \geq k} a_n = \inf_{2j-1 \geq k} a_{2j-1} \quad (\text{by } \quad)$$

$$= \lim_{j \to \infty} a_{2j-1} \quad (\text{since } (a_{2j-1})_{j \in \mathbb{N}} \text{ is a bounded decreasing sequence})$$

$$= -2/3 \quad (\text{by AoL})$$

$\therefore$ As $k \to \infty$, $\inf_{n \geq k} a_n = -\frac{2}{3} \longrightarrow -\frac{2}{3}$ $\quad \therefore \liminf_{n \to \infty} a_n = -\frac{2}{3}$

Graphically, the sequence $(a_n)_{n \in \mathbb{N}}$ looks like:



First 100 terms of (a_n)

Example 2

Let $a_n = \frac{1}{n^2} - (-1)^n + 2$

Again, note that $a_{2j} \leq 2 \leq a_{2j-1} \ \forall j \in \mathbb{N}$. But note that this time, both $(a_{2j})_{j\in\mathbb{N}}$ and $(a_{2j-1})_{j\in\mathbb{N}}$ are decreasing sequences!

In this case, the argument in example 1 will only work for $\liminf\limits_{n\to\infty} a_n$

⌜We have $\liminf\limits_{n\to\infty} a_n = 1$ ← Try using the argument in Ex 1 to show this⌟

For $\limsup\limits_{n\to\infty} a_n$, we have to, therefore, look towards the start of the sequence. So, fix $k \in \mathbb{N}$.

Then, $\sup\limits_{n\geq k} a_n = \sup\limits_{2j-1\geq k} a_{2j-1}$ (by $\quad$)

$$= \begin{cases} a_k & \text{if } k \text{ is odd} \\ a_{k+1} & \text{if } k \text{ is even} \end{cases}$$

⌜This is because $(a_{2j-1})_{n\in\mathbb{N}}$ is a decreasing sequence⌟

$$= \begin{cases} 1/k^2 + 3 & \text{if } k \text{ is odd} \\ 1/(k+1)^2 + 3 & \text{if } k \text{ is even.} \end{cases}$$

In both cases, as $k \to \infty$, $\sup\limits_{n\geq k} a_n \longrightarrow 3$, so $\limsup\limits_{n\to\infty} a_n = 3$

Example 3

Let $a_n = \cos\left(\frac{n\pi}{3}\right) + \frac{(-1)^n}{n}$.

Note that this time, you can't split a_n up into two monotonic subsequences, so neither of the methods in the previous example work. So we need to be crafty.

It's always handy to have an idea of what the liminf and limsup should be. Since $|\cos(n\pi/3)| \leq 1 \ \forall n \in \mathbb{N}$ and $(-1)^n/n$ is convergent, we (hopefully) would guess that:

$$\limsup_{n\to\infty} a_n = 1 \quad ; \quad \liminf_{n\to\infty} a_n = -1$$

How do we show these? Recall that the limsup is the largest limit of any subsequence of $(a_n)_{n\in\mathbb{N}}$.

Take $n_j = 6j$.

Then:

$$a_{n_j} = \cos\left(\frac{6j\pi}{3}\right) + \frac{(-1)^{6j}}{6j} = 1 + \frac{1}{6j} \longrightarrow 1 \quad \text{as } j \to \infty.$$

So $\lim_{j\to\infty} a_{n_j} = 1$, hence $\limsup_{n\to\infty} a_n \geq 1$.

To show that $\limsup_{n\to\infty} a_n \leq 1$, recall that

Try proving this!

$$\limsup_{n\to\infty}(b_n + c_n) \leq \limsup_{n\to\infty} b_n + \limsup_{n\to\infty} c_n$$

for bounded sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$.

Taking $b_n = \cos\left(\frac{n\pi}{3}\right)$ and $c_n = \frac{(-1)^n}{n}$ we have that

$\limsup\limits_{n\to\infty} b_n = 1$ (See PS7) and $\limsup\limits_{n\to\infty} c_n = 0$ (as $(c_n)_{n\in\mathbb{N}}$ is convergent)

$$\therefore \quad \limsup_{n\to\infty} a_n = \limsup_{n\to\infty} (b_n + c_n) \leq 1 + 0 = 1$$

$$\therefore \quad \text{and} \quad \Rightarrow \quad \limsup_{n\to\infty} a_n = 1.$$

Have a go at proving that $\liminf\limits_{n\to\infty} a_n = -1$. You'll need:

- liminf is smallest limit of any subsequence of $(a_n)_{n\in\mathbb{N}}$
- $\liminf\limits_{n\to\infty} (b_n + c_n) \geq \liminf\limits_{n\to\infty} b_n + \liminf\limits_{n\to\infty} c_n$.