Alternative Chain Rule Proof

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Introduction

Here is an alternative proof of the chain rule which uses the ϵ-δ definition of the limit. It's quite involved, but its a great example if you want more practice with these types of arguments. The typed version here is based off of one presented by Adrian Hill, who previously lectured this course.

The Chain Rule

Theorem 0.1 (Chain Rule)**.** *Let $g:(a,b)\to\mathbb{R}$ and $f:(A,B)\to\mathbb{R}$ be such that $g\left((a,b)\right)\subseteq(A,B)$. Assume that g is differentiable at c and f is differentiable at $g(c)$. Then the composition $f\circ g$ is differentiable at c with*

$$(f\circ g)'(c) = f'\left(g(c)\right)g'(c).$$

Proof. Firstly, note that using the definition of limit, we can recast this problem into the following form: given $\epsilon>0$, we seek $\delta>0$ such that

$$|x-c|<\delta \Rightarrow |(f\circ g)(x) - [(f\circ g)(c) + f'(g(c))g'(c)(x-c)]| \leq \epsilon|x-c| \tag{*}$$

Now, fix $\epsilon>0$, and let $\eta_1,\eta_2>0$ be determined later. As f is differentiable at $g(c)$, $\exists\theta_1>0$ such that

$$|y-g(c)|<\theta_1 \Rightarrow |f(y) - [f(g(c)) + f'(g(c))(y-g(c))]| \leq \eta_1|y-g(c)| \tag{1}$$

Also, as g is differentiable at c, $\exists\theta_2>0$ such that

$$\begin{aligned} |x-c|<\theta_2 &\Rightarrow |g(x) - [g(c) + g'(c)(x-c)]| \leq \eta_2|x-c|, \qquad (2)\\ &\Rightarrow |g(x)-g(c)| \leq \left(\eta_2 + |g'(c)|\right)|x-c|. \qquad (3)\end{aligned}$$

So, if $|x-c|<\delta$ for

$$\delta = \min\left\{\theta_2, \frac{\theta_1}{\eta_2+|g'(c)|}\right\}, \tag{4}$$

then using (3), we find that

$$|x-c|<\delta \Rightarrow |g(x)-g(c)| < \left(\eta_2+|g'(c)|\right)\delta \leq \theta_1.$$

Substituting $y=g(x)$ into (1) then gives for $|x-c|<\delta$,

$$|f(g(x)) - [f(g(c)) + f'(g(c))(g(x)-g(c))]| \leq \eta_1|g(x)-g(c)|$$

Adding and subtracting $f'(g(c))g'(c)(x-c)$ yields

$$|f(g(x)) - [f(g(c)) + f'(g(c))g'(c)(x-c)] - f'(g(c))\left[g(x)-g(c)-g'(c)(x-c)\right]| \leq \eta_1|g(x)-g(c)|.$$

Applying the reverse triangle inequality and rearranging,

$$\begin{aligned} |f(g(x)) - [f(g(c)) + f'(g(c))g'(c)(x-c)]| &\leq |f'(g(c))|\,|g(x)-g(c)-g'(c)(x-c)| \\ &\quad + \eta_1|g(x)-g(c)|, \\ &\leq \eta_2\,|f'(g(c))|\,|x-c| + \eta_1(\eta_2+|g'(c)|)|x-c|. \end{aligned}$$

So, if

$$\eta_2 = \frac{\epsilon}{2\left(|f'(g(c))|+1\right)} \quad \text{and} \quad \eta_1 = \frac{\epsilon}{2\left(\eta_2+|g'(c)|\right)},$$

then this final inequality implies that for $|x-c|<\delta$,

$$|(f\circ g)(x) - [(f\circ g)(c) + f'(g(c))g'(c)(x-c)]| \leq \left(\frac{\epsilon}{2}+\frac{\epsilon}{2}\right)|x-c| = \epsilon|x-c|$$

Hence, provided θ_1 and θ_2 are respectively defined for η_1 and η_2 by (1) and (2), and δ is defined by (4), we find that (*) is satisfied, and the result follows. $\square$