Cauchy Condensation Test II

In the tutorial, I got halfway through the following example:
Prove that

$$\sum_{n=4}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))}$$

converges.

And up to a point, what I wrote down makes sense:

Answer?:
For $n \geq 4$, define

$$a_n = \frac{1}{n \ln(n) \ln(\ln(n))},$$

and for $k \geq 2$, define

$$b_k := 2^k a_{2^k} = \frac{2^k}{2^k \ln(2^k) \ln(\ln(2^k))}.$$

Now, by properties of logarithms:

$$b_k = \frac{1}{k \ln(2) \ln(k \ln(2))} = \frac{1}{k \ln(2)[\ln(k) + \ln(\ln(2))]}$$

Also, for $k \geq 2$, we know that (under the assumption that ln is an increasing function)

$$\ln(k) \geq \ln(2) \geq \ln(\ln(2)).$$

Hence,

$$b_k \geq \frac{1}{2k \ln(2) \ln(\ln(2))} := c_k.$$

Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, we must have that $\sum_{k=2}^{\infty} c_k$ diverges, from which $\sum_{k=2}^{\infty} b_k$ diverges (by comparison). Hence by the Cauchy condensation test, the original sum diverges.

Those of you in the tutorials will remember that in my calculations, I struggled to reach this conclusion properly. This is because there's an error here somewhere! In fact, there's two, but one of them is massively more important than the other.

You'll be pleased to know that I've found it, but I'm interested to know if you can spot it too, and correct/finish the question! We'll discuss this at the start of next week's tutorial.