Vectors, Vector Calculus and Mechanics — Past Paper 2017

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Contents

Introduction

Here are the solutions to the past paper discussed in the revision session on XXth May 2024. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click here.

Section A

Question 1

Question. In the triangle OAB, we set $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C is located at the midpoint of the side AB. The point D divides the side OB in the ratio $2:1$, i.e. $OD = \frac{2}{3}OB$.

The lines AD and OC cross at X. Using vectors, show that $AX = \frac{3}{5}AD$ and find the ratio in which the point X divides OC.

Solution. TBD

Question 2

Question. Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$.

a) Find the lengths of $\mathbf{a}$ and $\mathbf{b}$.

b) Find the cosine of the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

c) Find a unit vector $\hat{\mathbf{c}}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}, \mathbf{b}, \hat{\mathbf{c}}$ form a right-handed system.

Solution. TBD

Question 3

Question. State the expansion formula for the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Use it to prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot \{(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})\} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2 \,,$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the scalar triple product.

State, but do not prove, the results of vector algebra which you use.

Solution. TBD

Question 4

Question. a) Let $T(x, y)$ be a differentiable function of x and y, and let (a, b) be a point on the xy-plane. Define the gradient vector $\nabla T(a, b)$, and prove that the rate of change of $T(x, y)$ at (a, b) in the direction of the unit vector $\mathbf{u}$ is given by

$$D_{\mathbf{u}}T(a, b) = \mathbf{u} \cdot \nabla T(a, b).$$

b) Find the gradient vector of the function

$$T(x, y) = 3x^2 + 2y^2 + 6$$

at the point $(2, 1)$, and find the equation of the tangent plane to the 3D surface $z = T(x, y)$ at the point $(2, 1, c)$ where $c = T(2, 1)$.

Solution. TBD

Question 5

Question. In planar polar coordinates $(r.\theta)$, the position vector of a particle at time t can be written as

$$\mathbf{x}(t) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$$

where r, θ are functions of t.

a) Express the radial and angular unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ in terms of $\mathbf{i}, \mathbf{j}$. Derive expressions for $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$, and hence show that

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

and

$$\ddot{\mathbf{x}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}_\theta.$$

b) Find the acceleration vector of a particle which travels in a circular orbit of radius a, at a constant angular speed ω.

Solution. TBD

Section B

Question 6

Question. a) Prove that the shortest distance from a point A with position vector $\mathbf{a}$, to the plane Π defined by $\mathbf{r}\cdot\mathbf{n} = d$ is

$$h = \frac{|d - \mathbf{a}\cdot\mathbf{n}|}{\|\mathbf{n}\|},$$

and find the position vector of the point on Π which achieves this distance.

b) Let $\mathbf{a}, \mathbf{b}, \mathbf{l}, \mathbf{m}$ be given vectors, and let $\mathbf{r} = \mathbf{a} + \lambda\mathbf{l}$ and $\mathbf{r} = \mathbf{b} + \mu\mathbf{m}$ be two non-parallel and non-intersecting lines L_1 and L_2 respectively.

   i) Find the vector equations of two parallel planes Π_1 and Π_2, such that Π_1 contains L_1 and Π_2 contains L_2.

   ii) Find the distance between the two planes Π_1 and Π_2.

Solution. TBD

Question 7

Question. a) By either finding the Complementary Function and Particular Integral, or using the Integrating Factor method, solve the vector differential equation

$$\ddot{\mathbf{x}} + k\dot{\mathbf{x}} = \mathbf{h}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

where k is a non-zero scalar and $\mathbf{h}$ is a constant vector. Show that the solution can be written as

$$\mathbf{x} = \mathbf{x}_0 + \frac{1}{k^2}\left(\mathrm{e}^{-kt} + kt - 1\right)\mathbf{h}.$$

b) A food parcel of mass m is dropped from a helicopter hovering at a height D directly above a target on horizontal ground. There is a constant horizontal crosswind of velocity u, and the air resistance is proportional to the relative velocity of the parcel with respect to the wind, with coefficient μ. Derive the vector differential equation of motion of the parcel, and hence find its position vector $\mathbf{x}(t)$.

c) How far away from the target will the food parcel land?

Solution. TBD

Question 8

Question. A particle of mass m moves under the action of a central "planetary" force

$$\mathbf{F} = -\mu m \|\dot{\mathbf{x}}\|^2 \frac{\mathbf{x}}{r^2}$$

where $r = \|\mathbf{x}\|$ and $\mu > 0$ is a constant.

a) Prove that the motion takes place in a plane.

b) Using polar coordinates (r, θ) on the plane, show that the equations of motion are

$$r^2\dot{\theta} = h, \quad \ddot{r} + (\mu - 1)\frac{h^2}{r^3} + \mu\frac{\dot{r}^2}{r} = 0,$$

where h is a constant.

Note: You may use without proof the formulae for velocity and acceleration in polar coordinates, given in Question 5(a).

c) The particle is projected radially outwards from the surface of a "planet" of radius R, with initial speed v_0. Show that $h = 0$ in part (b).

Integrate the second equation of motion in part (b) to obtain

$$\ln(\dot{r}) = -\mu \ln(r) + c$$

where c is a constant of integration. Hence deduce that for all $v_0 > 0$ the particle will never fall back to the "planet".

Hint: $\frac{d}{dr}(\ln(r)) = \frac{1}{r}\frac{dr}{dt}$.

Solution. TBD