Analysis IA - Problem Sheet 3

2 a) Solve the inequality

$$\frac{1}{x} + \frac{1}{x+1} > 0 \qquad (*)$$

To solve this problem, we take cases on x. Note that the inequality doesn't make sense for $x = 0$ or $x = -1$.

Case I: $x > 0$

We have that

$$\frac{1}{x} + \frac{1}{x+1} > 0 \iff \frac{2x+1}{x(x+1)} > 0$$

$$\iff 2x+1 > 0 \quad (\text{as } x > 0 \text{ and } x+1 > 0)$$

$$\iff x > -1/2$$

So the solution set is $(0, \infty) \cap (-1/2, \infty) = (0, \infty)$ for this case.

Case II: $-1 < x < 0$

We have that

$$\frac{1}{x} + \frac{1}{x+1} > 0 \iff \frac{2x+1}{x(x+1)} > 0$$

$$\iff 2x+1 < 0 \quad (\text{as } x < 0 \text{ but } x+1 > 0)$$

$$\iff x < -1/2$$

So the solution set is $(-1, 0) \cap (-\infty, -1/2) = (-1, -1/2)$ for this case.

<u>Case III: $x < -1$</u>

We have that

$$\frac{1}{x} + \frac{1}{x+1} > 0 \iff \frac{2x+1}{x(x+1)} > 0$$

$$\iff 2x+1 > 0 \quad \text{(as } x < 0 \text{ and } x+1 < 0\text{)}$$

$$\iff x > -1/2$$

In this case, the solution set is $(-\infty, -1) \cap (-1/2, \infty) = \emptyset$

Therefore, the complete set of solutions to $(*)$ is $(-1, -1/2) \cup (0, \infty)$

<u>Question 3</u>

a) Show that $2xy \leq x^2 + y^2 \quad \forall x, y \in \mathbb{R}$, and that equality holds only if $x = y$

<u>Proof</u>

Fix $x, y \in \mathbb{R}$.

We have that

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

$$\iff 2xy \leq x^2 + y^2.$$

Since x and y were arbitrary, $2xy \leq x^2 + y^2 \ \forall x, y \in \mathbb{R}$, as required.

Now,

$$2xy = x^2 + y^2 \iff 0 = (x-y)^2 \quad \text{(Using previous calc.)}$$
$$\iff 0 = x - y$$
$$\iff x = y$$

$\therefore$ Equality holds if and only if $x = y$

Note that we have proved 'if and only if', which is a stronger result than the 'only if' required in the question!

b) Show that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \underset{(1)}{\leq} \sqrt{x+y} \underset{(2)}{\leq} \sqrt{x} + \sqrt{y} \quad \forall x, y > 0$$

<u>Proof</u>

Consider inequality (2). We have for any $x, y > 0$

$$x + y \leq x + 2\sqrt{x}\sqrt{y} + y \quad (\text{as } \sqrt{x}, \sqrt{y} > 0)$$
$$= (\sqrt{x} + \sqrt{y})^2$$

Square-rooting gives

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$$

(as the square root function is an increasing function)

For inequality (1), we have for any $x, y > 0$:

$$\left(\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}}\right)^2 = \frac{x}{2} + 2\sqrt{\frac{x}{2}}\sqrt{\frac{y}{2}} + \frac{y}{2}.$$

Applying the result from 3a) to the middle term yields

$$\left(\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{2}}\right)^2 \leqslant \frac{x}{2}+\frac{x}{2}+\frac{y}{2}+\frac{y}{2}$$
$$= x+y$$

Square-rooting gives the result, i.e. $\forall x, y > 0$:

$$\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{2}} \leqslant \sqrt{x+y}$$

c) Prove that

$$\left|\sqrt{1+x^2}-\sqrt{1+y^2}\right| \leqslant |x-y| \qquad \forall x, y \in \mathbb{R}$$

Proof

First, for $x=-y$:

$$\left|\sqrt{1+x^2}-\sqrt{1+y^2}\right| = 0 \leqslant |-2y| = |x-y|$$

so the inequality is true.

For $x \neq -y$, we have:

$$\left|\sqrt{1+x^2}-\sqrt{1+y^2}\right| = \frac{|1+x^2-1-y^2|}{\sqrt{1+x^2}+\sqrt{1+y^2}} \qquad (*)$$

$$= \frac{|x^2-y^2|}{\sqrt{1+x^2}+\sqrt{1+y^2}}$$

$$\Rightarrow \quad \left| \sqrt{1+x^2} - \sqrt{1+y^2} \right| = \frac{|x-y|\,|x+y|}{\sqrt{1+x^2}+\sqrt{1+y^2}}$$

Now, $|x| \leqslant \sqrt{1+x^2}$ and $|y| \leqslant \sqrt{1+y^2}$.

So,

$$|x+y| \leqslant |x| + |y| \qquad \text{(by Triangle Inequality)}$$

$$\leqslant \sqrt{1+x^2} + \sqrt{1+y^2}$$

$$\Leftrightarrow \quad \frac{1}{|x+y|} \geqslant \frac{1}{\sqrt{1+x^2}+\sqrt{1+y^2}}$$

$$\therefore \quad \left| \sqrt{1+x^2} - \sqrt{1+y^2} \right| = \frac{|x-y|\,|x+y|}{\sqrt{1+x^2}+\sqrt{1+y^2}}$$

$$\leqslant \frac{|x-y|\,|x+y|}{|x+y|} \qquad (**)$$

$$= |x-y| \qquad \text{as required.}$$

You might have a few questions about 3c):

Q. Why is 3c) done this way?

A. It's an alternative way to the one in the model solutions. But I think it's good because it uses some techniques that are useful for the sequences part of the course (e.g. the triangle inequality and step (*)).

Q. Where on Earth did the case $x = -y$ come from?

A. If you look at (**), this expression doesn't work if $x = -y$. So you need to consider this separately. It's not an obvious case until you reach (**), but once you realise it, its an easy thing to add to the start of your answer.

Q. What about (*)? Where does this come from?

Recall for $a, b \in \mathbb{R}$

$$(a+b)(a-b) = a^2 - b^2$$

Taking $a = \sqrt{1+x^2}$, $b = \sqrt{1+y^2}$, we have that

$$\sqrt{1+x^2} - \sqrt{1+y^2} = \frac{(1+x^2)-(1+y^2)}{\sqrt{1+x^2}+\sqrt{1+y^2}}$$