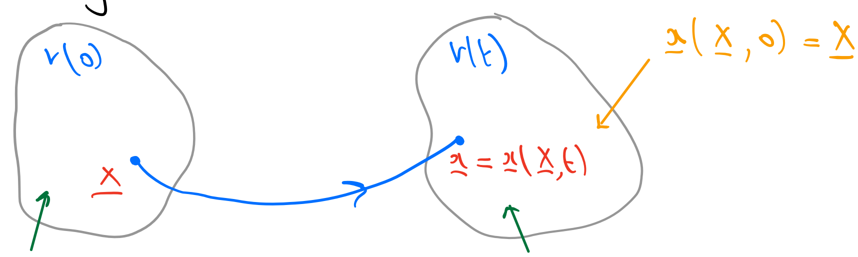
Viscous Fluid Dynamics – Revision Notes

Kinematics

Two coordinate systems:



Lagrangian: · label fluid particles
· great for applying conservation principles!

Eulerian: · label fixed points in space.
· used in fixed laboratory frame
· great for calculations!

Time derivatives:

- Eulerian (holding $\underline{x}$ fixed): $\dfrac{\partial}{\partial t} = \left.\dfrac{\partial}{\partial t}\right|_{\underline{x}}$

velocity in Eulerian

- Material derivative (holding $\underline{X}$ fixed): $\dfrac{D}{Dt} = \left.\dfrac{\partial}{\partial t}\right|_{\underline{X}} = \dfrac{\partial}{\partial t} + (\underline{u} \cdot \nabla)$

Jacobian of motion

- $\underline{X} \longmapsto \underline{x}(\underline{X}, t)$ continuous and injective.

$$J(\underline{X}, t) = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \varepsilon_{ijk} \frac{\partial x_1}{\partial X_i} \frac{\partial x_2}{\partial X_j} \frac{\partial x_3}{\partial X_k}$$

Properties:

- $J(\underline{X}, 0) = 1$ (as $\underline{x}(\underline{X}, 0) = \underline{X}$)
- Elemental volume: $dV_{\underline{x}} = dx_1 \, dx_2 \, dx_3 = J(\underline{X}, t) \, dX_1 \, dX_2 \, dX_3 = J(\underline{X}, t) \, dV_{\underline{X}}$

Euler's Identity: $\frac{DJ}{Dt} = J(\nabla \cdot \underline{u})$

Derivation: Use definition of J and the product rule. Recall $Dx_\alpha / Dt = u_\alpha$. Use the chain rule to write DJ/Dt as a sum of Jacobians. Finally, note that a determinant is zero if it has repeated rows.

Reynolds' Transport Theorem

- $f(\underline{x}, t)$ any continuously differentiable property of the fluid.
- $V(t)$ a material volume $(= \underline{x}(V(0), t))$, boundary $\partial V(t)$

$$\frac{d}{dt} \int_{V(t)} f(\underline{x}, t)\, dV_{\underline{x}} = \int_{V(t)} \frac{Df}{Dt} + f(\nabla \cdot \underline{u})\, dV_{\underline{x}}$$

$$= \int_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f\underline{u})\, dV_{\underline{x}}$$

flux of f across boundary

$$= \int_{V(t)} \frac{\partial f}{\partial t}\, dV_{\underline{x}} + \int_{\partial V(t)} f\, \underline{u} \cdot \underline{n}\, dS$$

Derivation: (Equality 1) change integral into Lagrangian variables, differentiate under integral sign, then convert back. (Equality 2) Def^n of Df/Dt and $\nabla \cdot (f\underline{u}) = \nabla f \cdot \underline{u} + f(\nabla \cdot \underline{u})$. (Equality 3) Divergence Theorem!

Pointwise conservation of mass:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \underline{u}) = 0 \qquad (\text{Continuity equation, } \rho(\underline{x}, t) \text{ is density})$$

Derivation: Apply R.T.T with $f = \rho$. As $V(t)$ arbitrary, apply localisation principle.

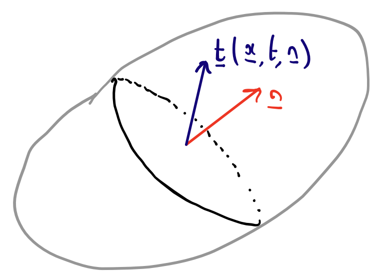
A fluid is incompressible when $\frac{D\rho}{Dt} = 0$. Equivalently, from above $\Rightarrow \nabla \cdot \underline{u} = 0$

R.T.T. Corollary! Assuming conservation of mass holds; continuously differentiable f and ρ.

$$\frac{d}{dt}\int_{V(t)} \rho f \, dV_{\underline{x}} = \int_{V(t)} \rho \frac{Df}{Dt} \, dV_{\underline{x}}$$

Kinematics

For a plane in a fluid with unit normal $\underline{n}$, the stress vector $\underline{t}(\underline{x},t,\underline{n})$ is the force per unit area (stress) that the fluid towards which $\underline{n}$ points exerts on the plane at a point $\underline{x}$ and at time t. $\longleftarrow$ Cauchy - Euler hypothesis.



Cauchy Stress Tensor: $\sigma = (\sigma_{ij}) \quad i,j = 1,2,3$

$\sigma_{ij}(\underline{x},t)$ is the component of stress in the x_i - direction exerted on a surface with normal in the x_j - direction by the fluid towards which $\underline{e}_j$ points.

Representation:

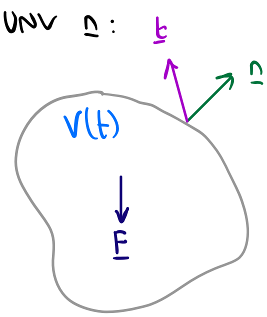
$$\sigma_{ij}(\underline{x},t) = \underline{e}_i \cdot \underline{t}(\underline{x},t,\underline{e}_j)$$

$$\underline{t}(\underline{x},t,\underline{e}_j) = \underline{e}_i \, \sigma_{ij}(\underline{x},t)$$

diagonal components are called normal stresses
off-diagonal ones are called shear stresses.

Newton's Second and Third Laws

For a volume $V(t)$ with boundary $\partial V(t)$ and OPUNV $\underline{n}$:



Forces on $V(t)$:

① Body Forces:

- act on each point in $V(t)$
- due to external cause (e.g. gravity)

② Surface Traction:

- Represents stress exerted by fluid outside of $V(t)$ on fluid inside $V(t)$ via $\partial V(t)$

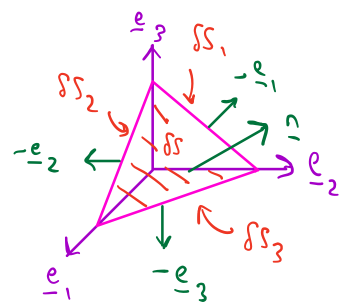
(N2L) $$\underbrace{\frac{d}{dt}\int_{V(t)} \rho \underline{u}\, dV_{\underline{x}}}_{\text{Rate of change of linear momentum}} = \underbrace{\int_{V(t)} \rho \underline{F}\, dV_{\underline{x}}}_{\text{Total body force } (\underline{F} = \text{body force per unit mass})} + \underbrace{\int_{\partial V(t)} \underline{t}\, dS}_{\text{Total surface force on } \partial V(t)}$$

(N3L) $$\underline{t}(\underline{x}, t, -\underline{n}) = -\underline{t}(\underline{x}, t, \underline{n}) \qquad \forall \underline{x} \in \partial V(t)$$

Cauchy's Stress Theorem

$$\underline{t}(\underline{x}, t, \underline{n}) = \sigma_{ij}(\underline{x}, t)\, n_j\, \underline{e}_i$$

(may be written as $\underline{t}(\underline{n}) = \sigma_{ij} n_j \underline{e}_i$)



Derivation: Take infinitesimal tetrahedron (see picture).

Apply divergence theorem to $\underline{e}_j$ to show $\delta S_i = n_i \delta S$.

Use R.T.T on (N2L) and rearrange for $\int \underline{t}\, dS$

Find traction vector on each face.

Take leading order approximation of integrals: $\sum_{k=1}^{3} \underline{t}(-\underline{e}_k)\delta S_k + \underline{t}(\underline{n})\delta S = \left(\frac{D\underline{u}}{Dt} - \underline{F}\right)\rho\, \delta V.$

Rearrange for $\underline{t}(\underline{n})$, take $\delta V \to 0$ (why does $\delta V/\delta S \to 0$?) Apply N3L.

Balance Laws

① Conservation of linear momentum:

$$\frac{d}{dt}\int_{V(t)} \rho \underline{u}\, dV_{\underline{x}} = \int_{V(t)} \rho \underline{F}\, dV_{\underline{x}} + \int_{\partial V(t)} \underline{t}\, dS \qquad (+)$$

$$\Leftrightarrow \rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i \qquad i = 1, 2, 3 \quad \text{(pointwise)}$$

Derivation: Use component form of (†). Apply R.T.T and C.S.T. Then apply divergence theorem to ∂V integral. Rearrange, apply localisation principle.

② Conservation of angular momentum (around origin):

$$\frac{d}{dt}\int_{V(t)} \underline{x} \wedge \rho \underline{u} \, dV_{\underline{x}} = \int_{V(t)} \underline{x} \wedge \rho \underline{F} \, dV_{\underline{x}} + \int_{\partial V(t)} \underline{x} \wedge \underline{t} \, dS \qquad (\text{H})$$

$\Longleftrightarrow \quad \sigma_{ij} = \sigma_{ji} \quad \forall i, j = 1, 2, 3$ (stress tensor is symmetric)

Derivation: Use component form of (H). Apply R.T.T and C.S.T. Then apply divergence theorem to ∂V integral. Rearrange, use CoLM (pointwise), and apply localisation principle to get $\varepsilon_{ilk}\sigma_{kl} = 0$ (indices may vary). Multiply by ε_{imr}.

Newtonian Constitutive Law

To solve equations for CoM and CoLM, need to decide how σ_{ij} depends on velocity and pressure.

$\Rightarrow$ Constitutive relation! (Cannot deduce this $\to$ need experiments!)

Here, suppose that $\sigma_{ij} = \underset{\text{pressure}}{-p}\delta_{ij} + \tau_{ij}$ ← deviatoric stress tensor, due to viscosity.

For a Newtonian fluid:

(A) τ_{ij} is a linear function of the velocity gradients

(B) relation between τ_{ij} and velocity gradients is isotropic (no preferred direction)

$$\Rightarrow \quad \tau_{ij} = \lambda(\overbrace{\nabla\cdot\underline{u}}^{e_{kk}})\delta_{ij} + 2\mu e_{ij} \leftarrow e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

bulk viscosity (λ) — dynamic viscosity (μ) — rate of strain tensor (e_{ij})

Derivation (outline): (A) $\Rightarrow \tau_{ij} = A_{ij\alpha\beta}\, \partial u_\alpha / \partial u_\beta$. As τ_{ij} and $\partial u_\alpha / \partial x_\beta$ are CT2 $\Rightarrow A_{ij\alpha\beta}$ is a CT4. (quotient rule). (B) $\Rightarrow A_{ij\alpha\beta}$ is isotropic. $\Rightarrow A_{ij\alpha\beta} = \lambda \delta_{ij}\delta_{\alpha\beta} + \mu \delta_{i\alpha}\delta_{j\beta} + \mu' \delta_{i\beta}\delta_{j\alpha}$.
As $\sigma_{ij} = \sigma_{ji} \Rightarrow A_{ij\alpha\beta} = A_{ji\alpha\beta}$. Use this to show $\mu = \mu'$. Insert $A_{ij\alpha\beta}$ into τ_{ij}.

Hydrodynamic pressure: $p = -\frac{1}{3}\sigma_{kk}$

The full constitutive law : $\sigma_{ij} = -p\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)$

Derivation: Calculate σ_{kk}, and use hydrodynamic pressure to show $\lambda = -2/3\, \mu$
For an incompressible fluid: $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$

Incompressible Navier – Stokes Equations (Cartesian Form)

Body force $\underline{F}$, velocity $\underline{u}$, density ρ, pressure p

$$\begin{cases} \rho \dfrac{Du_i}{Dt} = -\dfrac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \rho F_i & i = 1, 2, 3 \\ \nabla \cdot \underline{u} = 0 \end{cases}$$

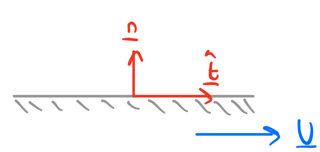
Derivation: Apply incompressible assumption to CoM equation. Also insert Newtonian σ_{ij} into CoLM equation, and use $\partial u_i / \partial x_i = 0$.

For other coordinate systems, see formula sheet!

Boundary Conditions for Navier - Stokes

① Solid surface S, normal $\underline{n}$, tangent vector $\hat{\underline{t}}$, velocity $\underline{U}$

$$\left.\begin{aligned} \text{no-flux:} \quad & \underline{u}\cdot\underline{n} = \underline{U}\cdot\underline{n} \\ \text{no-slip:} \quad & \underline{u}\cdot\hat{\underline{t}} = \underline{U}\cdot\hat{\underline{t}} \end{aligned}\right\} \quad \underline{u} = \underline{U}$$

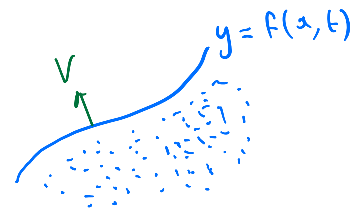


③ Free surface $y = f(x,t)$, $\underline{u} = (u,v)$

A) Kinematic condition: $\dfrac{D}{Dt}\big(y - f(x,t)\big) = 0$

use def^n

$$\Leftrightarrow \quad v = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} \qquad (*)$$



"particles on surface stay on surface"

The kinematic condition is also equivalent to $\underline{u}\cdot\underline{n} = V$ ← outward normal velocity.

Proof: Parametrise surface as $\underline{r}(x) = (x, f(x,t))$. Find unit tangent $\hat{\underline{t}}$ and normal $\underline{n}$. Use $(*)$ to calculate $\underline{u}\cdot\underline{n}$, and then calculate $V = \frac{d\underline{r}}{dt}\cdot\underline{n}$ to show equivalence.

B) Stress-balance: On $y = f(x,t)$ $\underline{t}(\underline{n}) = \underline{t}_a$ for $\underline{t}_a$ the external stress.

e.g. $\underline{t}(\underline{n}) = -p_a\underline{n}$ for exposure to air.

Vorticity

Measure of local rotation: $\underline{\omega} = \nabla \wedge \underline{u}$.

For conservative body forces $(\underline{F} = -\nabla\chi)$:

$$\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} + \nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right) = -\nu\nabla\wedge\underline{\omega} + \underline{F} \qquad (\bullet)$$

Derivation: Use vector identities in Navier-Stokes (e.g. $(\underline{u}\cdot\nabla)\underline{u} = \frac{1}{2}\nabla|\underline{u}|^2 + (\nabla\wedge\underline{u})\wedge\underline{u}$.

Vorticity Transport Equation:

kinematic viscosity $\nu = \mu/\rho$

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} - (\underline{\omega}\cdot\nabla)\underline{u} = \nu\nabla^2\underline{\omega}$$

Derivation: Take curl of momentum equation $(\bullet)$. Use vector identities e.g. $\nabla\wedge(\underline{\omega}\wedge\underline{u}) = \ldots$

2D: $\underline{\omega} = \omega\underline{k}$ satisfies $\dfrac{D\omega}{Dt} = \nu\nabla^2\omega$

convection of vorticity →

↖ diffusion of vorticity.

← vorticity may not be conserved! It can be generated at boundaries!

Conservative form of 2D VTE

$$\frac{\partial \omega}{\partial t} + \nabla \cdot \underline{Q} = 0 \quad \text{for} \quad \underline{Q} = \underbrace{\omega \underline{u}}_{\text{Convection}} - \underbrace{\nu \nabla \omega}_{\text{diffusion}} \quad \text{(vorticity flux)}$$

Derivation: As flow is incompressible, can add $\omega(\nabla \cdot \underline{u})$ to LHS of 2D V.T.E. Use a vector identity on LHS, and write $\nabla^2 \omega = \nabla \cdot (\nabla \omega)$. Rearrange.

Note! On rigid boundary $\underline{Q} \cdot \underline{n} = \cancel{\omega \underline{u} \cdot \underline{n}}^{\,0} - \nu \dfrac{\partial \omega}{\partial n}$ ← need this to vanish to conserve vorticity (boundary can be a source/sink)

Conservation of Energy

For material volume $V(t)$ with boundary $\partial V(t)$ and outward normal $\underline{n}$

$$\underbrace{\frac{d}{dt}\int_{V(t)} \underbrace{\rho}_{\text{density}} \underbrace{c_v}_{\text{Specific heat capacity}} \underbrace{T}_{\text{Temperature}} + \frac{1}{2}\rho|\underline{u}|^2 \, dV}_{\text{Total internal energy (K.E + P.E)}} = \underbrace{\int_{\partial V(t)} \underbrace{\underline{q}}_{\text{Heat flux}} \cdot (-\underline{n}) \, dS}_{\text{Heat flux through boundary (conduction)}} + \underbrace{\int_{\partial V(t)} \underline{t}(\underline{n}) \cdot \underline{u} \, dS}_{\text{Rate of work done by traction}} + \underbrace{\int_{V(t)} \rho \underline{F} \cdot \underline{u} \, dV}_{\text{Rate of work done by body forces}}$$

Fourier's Law: $\underline{q} = -k \nabla T$ ← Thermal conductivity (k)

Energy equation: $\rho c_v \dfrac{DT}{Dt} = k \nabla^2 T + \Phi \;, \quad \underbrace{\Phi}_{\text{dissipation.}} = \sigma_{ij} \dfrac{\partial u_i}{\partial x_j}$

↓

(assuming c_v, k are constant)

Derivation: Use summation convention in CoE. Apply Fourier's law and RTT. Apply divergence theorem to $\partial V(t)$ integrals and rearrange. Apply localisation principle and pointwise CoLM to obtain energy equation.

Alternative form of dissipation:

$$\Phi = \frac{\mu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \quad \longleftarrow \; >0, \text{ so fluid deformation always increases temperature!}$$

Derivation: Apply Newtonian constitutive law version of σ_{ij} to definition of Φ.

Remember that i, j are dummy indices, and flow is incompressible!

↳ so $\partial u_m / \partial x_m = 0$

Unidirectional Flows - Cartesian

Choose x-axis in direction of flow, i.e. $\underline{u} = u(x, y, z, t)\, \underline{i}$

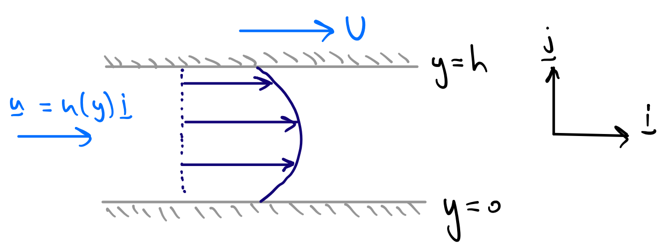
General setup (for zero body force!!)

- $\underline{u} = u(y, z, t)\, \underline{i}$ (independent of x!)
- u satisfies $\dfrac{\partial u}{\partial t} = \nu \left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \right) - \dfrac{G(t)}{\rho}$, $\quad G(t) = \dfrac{\partial p}{\partial x}$ ← applied pressure gradient.

Derivation: Substitute $\underline{u} = u(x, y, z, t)\, \underline{i}$ into Navier-Stokes (split equations into components!)
Use incompressibility to show u is independent of x. Use $\underline{j}$, $\underline{k}$ equations to show $p = p(x, t)$
Use these relations in $\underline{i}$ equation to write as $-\partial p/\partial x = \dots$ Here RHS indep. of y and z,
LHS indep of x $\Rightarrow$ $-\partial p/\partial x$ is function of time only, $G(t)$.

Examples

(A) Flow in a pipe



- Lower plate at rest
- Upper plate moving to right, speed U.
- $\partial p / \partial x = G$ constant.
- 1D steady unidirectional flow $\underline{u} = u(y)\, \underline{i}$

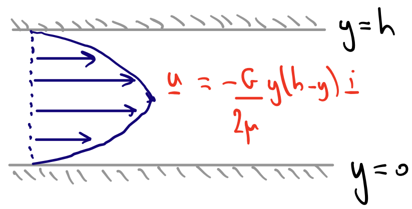
No flux + no-slip $\Rightarrow$

On $y = 0$, $u(0) = 0$

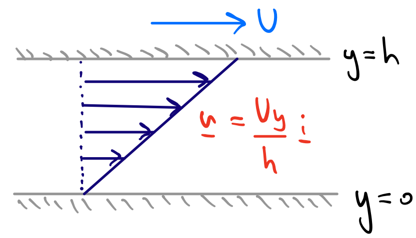
On $y = h$, $u(h) = U$

Equation: $\mu \dfrac{d^2u}{dy^2} = G \Rightarrow u(y) = -\dfrac{G}{2\mu} y(h-y) + \dfrac{Uy}{h}$

Special Cases:

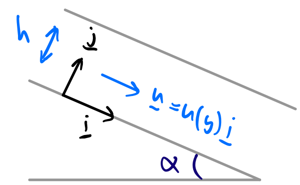


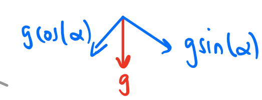
Poiseuille Flow: $U = 0$



Couette flow: $G = 0$

(B) Flow down inclined plane





- Plane at angle α to horizontal
- 1D steady unidirectional flow $\underline{u} = u(y)\,\underline{i}$
- $\underline{g} = g\sin(\alpha)\,\underline{i} - g\cos(\alpha)\,\underline{j}$ ← body forces: watch out

Equations: $\underline{i}: \quad 0 = -\dfrac{\partial p}{\partial x} + \mu \dfrac{d^2u}{dy^2} + \rho g \sin(\alpha)$

$\underline{j}: \quad 0 = -\dfrac{\partial p}{\partial y} - \rho g \cos(\alpha)$

BCs:

No flux + no-slip: On $y=0$, $u(0)=0$

Stress-balance: On $y=h$, $\underline{t} = -p_a \underline{n}$

$\Leftrightarrow \mu \, du/dy = 0$, $p = p_a$ (use σ_{ij}!)

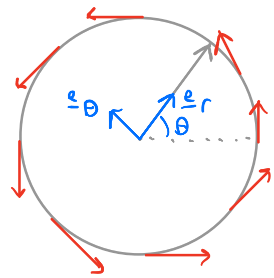
↑ atmospheric pressure.

$\Rightarrow u(y) = \dfrac{-G}{2\mu} y(2h-y)$

$p = p_a + \rho g (h-y)\cos(\alpha)$

Volume flux: $Q = \displaystyle\int_S \underline{u}\cdot\underline{n}\, dS = \int_0^h u\, dy = \dfrac{\rho g h^3 \sin(\alpha)}{3\mu}$

Unidirectional Flow - Circular



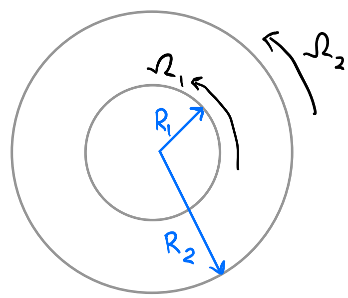
Set $\underline{u} = u_\theta(r,t)\,\underline{e}_\theta$, $u_r = u_z = 0$

(N-S) equations become:

$$\rho \frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}, \quad \rho \frac{\partial u_\theta}{\partial t} = \mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)$$

Derivation: Substitute $\underline{u} = u_\theta \underline{e}_\theta$ into component cylindrical N-S equations. Use $\underline{e}_z$ equation to show $p = p(r,\theta,t)$. Rewrite $\underline{e}_\theta$ equation to show $\partial p/\partial\theta = P_0(r,t)$ for some P_0. Integrate up, and use that p should be a single-valued function of position to show $P_0 = 0$. This gives $p = p(r,t)$ and required equations.

Example: Steady flow between rotating cylinders



- Cylinders of radii R_1, R_2 with respective angular speeds Ω_1, Ω_2.

Equation: $r^2 \dfrac{d^2 u_\theta}{dr^2} + r\dfrac{du_\theta}{dr} - u_\theta = 0 \qquad R_1 < r < R_2$

Invariant under scaling $r = \alpha\tilde{r}$, so let $z = \ln(r)$ to solve.

BCs: No-slip: On $r = R_1$, $u_\theta(R_1) = \Omega_1 R_1$

On $r = R_2$, $u_\theta(R_2) = \Omega_2 R_2$

Non-dimensionalisation

- Reduces number of parameters in the model, and identifies what controls solution behaviour.

e.g. Navier-Stokes with zero body force.

$$\nabla\cdot\underline{u} = 0 \ ; \quad \rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$$

Step 1: Scale ALL independent and dependent variables.

Introduce dimensionless variables:

external pressure.

$$\underline{x} = L\,\underline{\bar{x}}, \quad \underline{u} = U\,\underline{\bar{u}}, \quad t = \frac{L}{U}\,\bar{t}, \quad p = [p]\,\bar{p} + p_0$$

L: typical length scale (e.g. size of an obstacle)

U: typical flow speed (e.g. far field speed)

$[p]$: no "obvious" scaling - use equation to find.

Step 2: Insert into N-S equations:

$$\Rightarrow \quad 0 = \nabla\cdot\underline{u} = \frac{U}{L}\,\bar{\nabla}\cdot\underline{\bar{u}} \quad \Rightarrow \quad \bar{\nabla}\cdot\underline{\bar{u}} = 0$$

$$\Rightarrow \quad \frac{\rho U^2}{L}\left(\frac{\partial \underline{\bar{u}}}{\partial \bar{t}} + (\underline{\bar{u}}\cdot\bar{\nabla})\,\underline{\bar{u}}\right) = -\frac{[p]}{L}\,\bar{\nabla}\bar{p} + \frac{\mu U}{L^2}\,\bar{\nabla}^2\underline{\bar{u}}$$

Step 3: Choose scaling for p

Consider the Reynolds number Re

$$Re = \frac{[\text{inertia terms}]}{[\text{viscous terms}]} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho U L}{\mu} = \frac{LU}{\nu}$$

This is dimensionless! ratio of forces

If $Re \gg 1$, choose $[p] = \rho U^2$ (inertia scaling)

If $Re \ll 1$, choose $[p] = \mu U/L$ (viscous scaling)

If $Re = O(1)$, either scaling works.

Step 4: What if I choose the wrong scaling?

Answer: rescale $\bar{p}$!

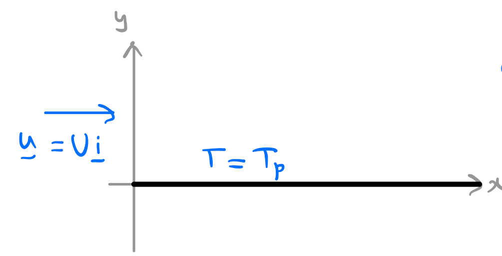
e.g. if inertia scaling wrongly chosen, scale $\bar{p}$ by $Re\,\bar{p} = \tilde{p}$

Useful! $Re = \dfrac{L^2/\nu}{L/U} = \dfrac{\text{timescale for vorticity diffusion}}{\text{timescale for vorticity convection}}$

Two flows are dynamically similar if they satisfy the same dimensionless problem.
(i.e. same geometry, governing equations, B.Cs and dimensionless parameters)

High Reynolds Number - Thermal Boundary Layer

- 2D steady heat convection - conduction
- inviscid fluid, velocity $\underline{u} = U\underline{i}$
- Far field temperature $T = T_\infty$
- On $y = 0$, $x > 0$: plate at $T = T_p$



$T \to T_\infty$ as $x^2 + y^2 \to \infty$

Equation:

$$U \frac{\partial T}{\partial x} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad \kappa = \frac{k}{\rho c_v} \leftarrow \text{thermal diffusivity}$$

Dimensionless problem:

Set $x = L\bar{x}$, $y = L\bar{y}$, $T = T_\infty + (T_p - T_\infty)\bar{T}$

we don't know this (L) from geometry, so introduce arbitrary one.

$$\Rightarrow \begin{cases} \dfrac{\partial \bar{T}}{\partial \bar{x}} = \varepsilon \left(\dfrac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right) & \leftarrow \varepsilon = \dfrac{1}{Pe} = \dfrac{\kappa}{LU} \ll 1 \\ \text{On } \bar{y} = 0,\ \bar{x} > 0 \quad \bar{T} = 1 & \text{(Plate condition)} \\ \text{As } \bar{x}^2 + \bar{y}^2 \to \infty, \quad \bar{T} \to 0 & \text{(Far field condition)} \end{cases}$$

Péclet Number: $Pe = \dfrac{LU}{\kappa} = \dfrac{L^2/\kappa}{L/U} = \dfrac{\text{diffusion timescale}}{\text{convection timescale}}$

Exact solution (dropping $\bar{\ }$'s)

Similarity solution $T(x,y) = f(\eta)$ for $\eta = \left(\dfrac{(x^2+y^2)^{1/2} - x}{2\varepsilon} \right)^{1/2}$

Sub into problem

$$f'' + 2\eta f' = 0, \quad f(0) = 1, \quad f(\infty) = 0 \quad \Rightarrow T(x,y) = \operatorname{erfc}(\eta) \quad \to \eta = \text{constant are parabolae (isotherms)}$$

Boundary layer analysis (dropping $\bar{\ }$s)

Step 1: Find the outer solution (away from plate)

- In $x = O(1)$, $y = O(1)$ pose $T(x,y) \sim T_0(x,y) + \varepsilon T_1(x,y)$ as $\varepsilon \to 0$.
- Insert expansion into equation; compare terms of $O(\varepsilon^0)$.
- Solve $O(\varepsilon^0)$ system and apply far-field $\Rightarrow T_0 = 0$
- Continue for $O(\varepsilon^1)$... Explain why $T_j = 0 \ \forall j \in \mathbb{N}_0$
- NOTE: This expansion does not satisfy plate condition — suggests boundary layer!

Step 2: Find the inner solution

- Scale relevant variables: $x = x$, $y = \delta Y$, $T = T$ for $\delta = \delta(\varepsilon) \ll 1$

 (Don't scale x as we want solution for $x = O(1)$. Don't scale T as $O(1)$ on plate)

- Find the gauge δ:

↳ Insert variables into dimensionless problem, find dominant balance (relation between coefficients that gives a nontrivial balance involving both convection and conduction). Here, $\varepsilon/\delta^2 = 1$

$\Rightarrow \delta = \varepsilon^{1/2} = Pe^{-1/2}$

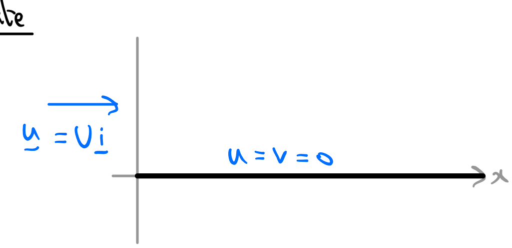
- Using δ, find problem in x and Y variables.

↳ Instead of far-field condition, impose matching condition $T(x,Y) \to 0$ as $Y \to \infty$

- In $x = O(1)$, $Y = O(1)$ pose $T(x,Y) \sim T_0(x,Y) + \varepsilon^{1/2} T_1(x,Y) + \dots$ as $\varepsilon \to 0$.
- Find leading order problem ($O(\varepsilon^0)$) and solve (similarity solution!)

High Reynolds Number - Steady Flow Past Plate

- 2D steady incompressible viscous flow
- Far field velocity $\underline{u} = U\underline{i}$
- Zero body force
- On $y = 0$, $x > 0$, $u = v = 0$ ($\underline{u} = (u,v)$)



Under usual scalings, the dimensionless problem is (without $\bar{\ }$'s)

inertial!

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(Eq 1)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \text{(Eq 2)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(Eq 3)}$$

Plate condition: On $x>0,\ y=0$ $\quad u = v = 0$ (BC 1)

Far-field: As $x^2+y^2 \to \infty$ $\quad u \to 1,\ v \to 0$ (BC 2)

Streamfunction formulation

Flow incompressible $\Rightarrow \exists\, \psi = \psi(x,y)$ s.t. $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$ (Uses Eq 3)

$\Rightarrow$ Problem is: $\dfrac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \varepsilon \nabla^4 \psi$ $\quad (\varepsilon = 1/Re)$

Plate condition: On $x>0,\ y=0$: $\psi = \dfrac{\partial \psi}{\partial y} = 0$ ($\psi = 0$ WLOG, as can absorb any non-zero constant into ψ)

Far Field: As $x^2+y^2 \to \infty$, $\psi \sim y$

Derivation: Insert ψ relations into (Eq 1), (Eq 2), (BC 1), (BC 2). Diff (Eq 1) wrt y and diff (Eq 2) wrt x to eliminate pressure.

Boundary layer analysis (similar method to thermal one):

Outer region: $\psi \sim y$ (inviscid flow) $\leftarrow$ does not satisfy plate condition!
$(x = O(1),\ y = O(1))$

Inner region: Scale $x = x$, $y = \delta Y$, $\psi = \theta \Psi$ for $\theta, \delta \ll 1$.

- Insert into streamfunction equation, plate condition and matching condition $\lim_{y \to 0} \psi_{outer} \sim y$ to get $\theta = \delta = \varepsilon^{1/2}$. (Need to expand Jacobian and ∇^4!)
- Write out problem for x, Y, Ψ (equation, plate + matching conditions)
- Pose $\Psi \sim \Psi_0 + \ldots$

↳ Leading order equations:
$$\frac{\partial \Psi_0}{\partial Y}\frac{\partial^3 \Psi_0}{\partial Y^2 \partial x} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^3 \Psi_0}{\partial Y^3} = \frac{\partial^4 \Psi_0}{\partial Y^4}$$

(Prandtl's Boundary Layer Equations)

$$\Psi_0 = \frac{\partial \Psi_0}{\partial Y} = 0 \qquad \text{(Plate condition)}$$

$$\Psi_0 \sim Y \qquad \text{(Matching condition)}$$

Blasius' Similarity Solution

Seek solution of form $\Psi_0 = x^{1/2} f(\eta)$ $\eta = Y x^{-1/2}$

Solves system if f satisfies Blasius' Equation:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0 \;;\; f(0) = 0,\; f'(0) = 0,\; f'(\infty) = 1$$

Derivation: Integrate first streamfunction up wrt y. Show arbitrary function of x is 0. Substitute $\Psi_0 = x^{1/2} f(\eta)$ into system. See appendix on how to derive form of solution.

BVP → IVP: Set $f''(0) = C$

- Rescale problem by letting $\eta = \xi / a$, $f(\eta) = bF(\xi)$
- Substitute into ODE → show that it is invariant under scaling if $a = b$.
- Use this scaling to find IVP for F, taking $a^3 = C$.
- Solve numerically to find $\dot{F}(\infty)$. Use this and BCs for f to calculate a, b and C

↳ $a = b = (\dot{F}(\infty))^{-1/2}$; $C = (\dot{F}(\infty))^{-3/2}$

Boundary layer analysis using velocity and pressure

Return to (Eq 1-3 + BC 1-2)

Outer region: Expand u, v and p. Find leading order equations, and show $u_0 = 1$, $v_0 = 0$ and (by Bernoulli's Theorem) $p_0 = \text{constant}$. (Note, at leading order, flow is inviscid)

Inner region: You know the method! Note that scalings are: $x = x$, $y = \delta Y$, $u = u$, $v = \delta V$, $p = p$
$\Rightarrow \delta = \varepsilon^{1/2}$. Matching condition is $u \to 1$, $v \to 0$, $P \to p_0$. Can use a streamfunction here!

Variable External Flow

Similar procedure to constant external flow (previous example), but inner matching condition is: As $Y \to \infty$ $x > 0$, $\dfrac{\partial \Psi_0}{\partial Y} \sim U_s(x)$.

non. uniform slip velocity

For outer solution, use Bernoulli's Theorem to show $\dfrac{\partial p_0}{\partial x} = -U_s(x) U_s'(x)$.

Falkner-Skan Problem

$U_s(x) = x^m$. $\leftarrow$ arises when modelling flow around a corner.

Seek similarity solution of form: $\dfrac{\Psi_0}{x^{(1+m)/2}} = f(\eta)$ $\quad \eta = \dfrac{Y}{x^{(1-m)/2}}$

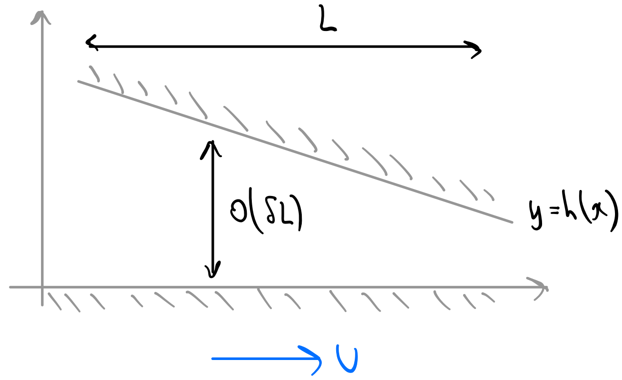
$\Downarrow$

$$(F\text{-}S): \begin{cases} f''' + \dfrac{1+m}{2} f f'' + m\left(1 - (f')^2\right) = 0 \\ f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \end{cases}$$

Low Reynolds Number Flow

Lubrication Theory - Slider Bearing

- 2D bearing, top part (slider) is fixed.
- Plane $y=0$: constant velocity $\underline{u} = U\underline{i}$
- $\delta \ll 1$
- $\underline{u} = (u, v)$, steady flow.



Problem:

$$\nabla \cdot \underline{u} = 0, \quad \rho(\underline{u} \cdot \nabla)\underline{u} = -\nabla p + \mu \nabla^2 \underline{u}$$

No slip: $\begin{cases} \text{On } y = 0 : & u = U, \; v = 0 \\ \text{On } y = h(x) : & u = v = 0 \end{cases}$

Dimensionless problem: Use incompressibility condition to see this Due to δ in length scaling

Viscous scalings: $x = L\bar{x}$, $y = \delta L \bar{y}$, $u = U\bar{u}$, $v = \delta U \bar{v}$, $p = (\mu U / \delta^2 L)\bar{p}$, $h = \delta L \bar{h}$

For $\delta \ll 1$ and the reduced Reynolds number, $Re\,\delta^2 \ll 1$, we have at leading order:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

$$-\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0$$

$$-\frac{\partial \bar{p}}{\partial \bar{y}} = 0$$

On $\bar{y} = \bar{h}(\bar{x})$: $\bar{u} = \bar{v} = 0$; On $\bar{y} = 0$, $\bar{u} = 1$, $\bar{v} = 0$

This is the lubrication model.

Equations (dropping $\bar{\ }$'s) $\Rightarrow$:

① $p = p(x)$

② $u = \frac{1}{2} p'(x) y (y - h(x)) + 1 - \frac{y}{h(x)}$

How to find p? Useful trick: integrate u across gap, differentiate w.r.t. x and incompressibility equation!

$$\Rightarrow \frac{d}{dx} \int_0^{h(x)} u(x,y)\,dy = \dots = 0$$

Remember to use definition of u!

Reynold's Lubrication Equation: $\frac{1}{6} \frac{d}{dx}\left(h^3 \frac{dp}{dx} \right) = \frac{dh}{dx}$ $\leftarrow$ Use me to find p given h!

Need 2 BCs!!

Useful tip: integrate up, divide by h^3 then sub given h in!

Stress! (Every exam season. Ever)

Dimensional Stress Tensor:

$$(\sigma_{ij}) = \begin{pmatrix} -p + 2\mu \, \partial u/\partial x & \mu(\partial u/\partial y + \partial v/\partial x) \\ \mu(\partial u/\partial y + \partial v/\partial x) & -p + 2\mu \, \partial v/\partial y \end{pmatrix}$$

$$= \frac{\mu U}{L} \begin{pmatrix} -\bar{p}/\delta^2 + 2\, \partial \bar{u}/\partial \bar{x} & 1/\delta \, \partial \bar{u}/\partial \bar{y} + \delta \, \partial \bar{v}/\partial \bar{x} \\ 1/\delta \, \partial \bar{u}/\partial \bar{y} + \delta \, \partial \bar{v}/\partial \bar{x} & -\bar{p}/\delta^2 + 2\, \partial \bar{v}/\partial \bar{y} \end{pmatrix}$$

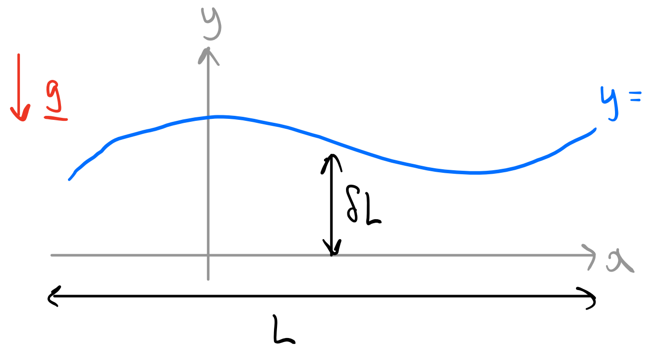
Inward normal to $y = h(x)$: $\underline{n} = (n_1, n_2) = \frac{(\delta \bar{h}', -1)}{(1 + \delta^2 h'^2)^{1/2}}$

needed for Cauchy stress vector! ($\underline{t} = (t_1, t_2)$)

We have: $O(1/\delta) = t_1 << t_2 = O(1/\delta^2)$ $\leftarrow$ vertical stress is much greater than horizontal stress!

Thin Films with Free Surfaces

body force term! → $\underline{g}$



$y = h(x,t)$ initially unknown!

Problem: $\nabla \cdot \underline{u} = 0$; $\rho \dfrac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} + \rho \underline{g}$!!

BCs:

- No slip: On $y=0$: $u = v = 0$
- Kinematic: On $y=h$: $\dfrac{\partial h}{\partial t} = v - u\dfrac{\partial h}{\partial x}$
- Stress: On $y=h$, $\underline{t} = -p_a \underline{n}$ (p_a = external / atmospheric pressure)

$$\underline{n} = \frac{\left(-\partial h/\partial x,\ 1\right)}{\sqrt{1+\left(\partial h/\partial x\right)^2}}$$

Dimensionless Problem:

Viscous scalings: $x = L\bar{x}$, $y = \delta L \bar{y}$, $u = U\bar{u}$, $v = \delta U \bar{v}$, $t = L/U\, \bar{t}$

$$p = \frac{\mu U}{\delta^2 L}\bar{p}, \quad p_a = \frac{\mu U}{\delta^2 L}\bar{p}_a, \quad \sigma_{ij} = \frac{\mu U}{\delta^2 L}\bar{\sigma}_{ij}, \quad h = \delta L \bar{h}$$

$\Rightarrow$

$$\delta^2 Re \frac{D\bar{u}}{D\bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \delta^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

$$\delta^4 Re \frac{D\bar{v}}{D\bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{y}} + \delta^2 \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \delta^4 \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - \rho g \frac{\delta^3 L^2}{\mu U}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

The "straightforward" bit.

BCs: On $\bar{y}=0$, $\bar{u}=0=\bar{v}$

$$\text{On } \bar{y}=\bar{h}: \begin{cases} \dfrac{\partial \bar{h}}{\partial \bar{t}} = \bar{v}, \; \bar{u}\dfrac{\partial \bar{h}}{\partial \bar{x}} \\ \bar{p} = \bar{p}_a + O(\delta^2), \; \dfrac{\partial \bar{u}}{\partial \bar{y}} + O(\delta^2) = 0 \end{cases}$$

also "straightforward"

Where do the stress balance BCs come from?

$\underline{t} = -p_a \underline{n} \iff t_i = \sigma_{ij} n_j = -p_a n_i$. To non-dimensionalise, we need to find:

- n_i in terms of $\bar{\ }$ variables. Use binomial expansion to show $n_1 = -\delta \partial\bar{h}/\partial\bar{x} + O(\delta^3)$ and $n_2 = 1 + O(\delta^2)$
- $\bar{\sigma}_{ij}$ (Calculate stress tensor, then write in terms of $\bar{\ }$ variables)
- $\bar{t}_i = \bar{\sigma}_{ij} n_j$ and equate to $-\bar{p}_a n_i$. Use these relationships to get BCs.

KEY ASSUMPTION HERE: Assume gravity drives flow, so $U = \dfrac{\rho g \delta^3 L^2}{\mu}$

For $\delta \ll 1$ and the reduced Reynolds number, $Re\,\delta^2 \ll 1$, we have at leading order:

$$0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \qquad \text{(Eq 1)}$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}} - 1 \qquad \text{(Eq 2)}$$

$$0 = \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \qquad \text{(Eq 3)}$$

On $\bar{y}=0$: $\bar{u}=\bar{v}=0$; on $\bar{y}=\bar{h}$: $\dfrac{\partial \bar{h}}{\partial t} = \bar{v} - \bar{u}\dfrac{\partial \bar{h}}{\partial \bar{x}}$, $\bar{p}=\bar{p}_a$, $\dfrac{\partial \bar{u}}{\partial \bar{y}} = 0$

Solutions (dropping ~s)

(Eq 2) + BCs $\Rightarrow$ $p = p_a + h - y$

(Eq 1) + BCs $\Rightarrow$ $u = \frac{1}{2}\frac{\partial p}{\partial x} y(y-2h)$

Using our trick on (Eq 3) + Kinematic condition

$$\Rightarrow \quad \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y)\,dy = -\frac{\partial h}{\partial t}$$

Use equation for u!

Thin Film / Porous Medium Equation:

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)$$

Use equation for p!

Use me (a non-linear heat equation) to find $h(x,t)$! 2 BCs!

Useful Vector Results

- If a flow is incompressible, $\nabla \cdot \underline{u} = 0$

  $\Rightarrow$ $\exists$ vector potential $\underline{A}$ s.t. $\underline{u} = \nabla \wedge \underline{A}$

- Biharmonic operator in 2D:

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}$$

Finding a similarity solution

Aim: Find a group of scalings for which the PDE is invariant under.

1. Scale all independent and dependent variables.
2. Insert scaled variables into PDE + BCs
3. Find relations between the scalings that make the PDE + BC invariant. Use these relations to write the scaling transformation in terms of the "free" parameters.
4. Use the dependent variable scaling transformation to pick a value (can vary) for the free parameter(s). This should be done such that at least one argument of the dependent variable is fixed. This also motivates the structure of the solution.

Example: Solve $\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial^3 \Psi}{\partial y^3}$ (+)

with $\Psi = \frac{\partial \Psi}{\partial y} = 0$ on $y = 0,\ x > 0$ (++)

and $\Psi \sim y$ as $y \to \infty$ (+++)

1. Introduce $x = \alpha \hat{x}$, $y = \beta \hat{y}$, $\Psi = \gamma \hat{\Psi}$ for some $\alpha, \beta, \gamma > 0$
2. Substituting gives:

(+) $\frac{\gamma^2}{\alpha\beta^2}\frac{\partial \hat{\Psi}}{\partial \hat{y}}\frac{\partial^2 \hat{\Psi}}{\partial \hat{x}\partial \hat{y}} - \frac{\gamma^2}{\alpha\beta^2}\frac{\partial \hat{\Psi}}{\partial \hat{x}}\frac{\partial^2 \hat{\Psi}}{\partial \hat{y}^2} = \frac{\gamma}{\beta^3}\frac{\partial^3 \hat{\Psi}}{\partial \hat{y}^3}$

(++) $\gamma \hat{\Psi} = 0$ and $\frac{\gamma}{\beta}\frac{\partial \hat{\Psi}}{\partial \hat{y}} = 0$ on $\beta\hat{y} = 0,\ \alpha\hat{x} > 0$

(+++) $\gamma\hat{\Psi} \sim \beta\hat{y}$ as $\beta\hat{y} \to \infty$

3. From step 2, we see that the scalings must satisfy:

(†) $\gamma = \dfrac{\alpha}{\beta}$

(††) $\gamma = \beta$

Note (††) is already invariant.

$\therefore$ The scaling group is:

$$x = \alpha \hat{x}, \quad y = \alpha^{1/2} \hat{y}, \quad \Psi = \alpha^{1/2} \hat{\Psi} \quad \text{for any } \alpha > 0.$$

4. Now, $\Psi(x, y) = \alpha^{1/2} \hat{\Psi}(\hat{x}, \hat{y})$

$$= \alpha^{1/2} \hat{\Psi}\left(\frac{x}{\alpha}, \frac{y}{\alpha^{1/2}}\right)$$

As we solve on the upper half plane, $x > 0$. We can choose $\alpha = x$ to "eliminate" an argument of $\hat{\Psi}$.

$$\Rightarrow \Psi(x, y) = x^{1/2} \hat{\Psi}\left(1, \frac{y}{x^{1/2}}\right)$$

We see that Ψ is a "scaling" multiplied by a "single variable" function. So, we seek a similarity solution of the form:

$$\Psi(x, y) = x^{1/2} f(\eta) \quad \text{for } \eta = \frac{y}{x^{1/2}}$$