

# Analysis 1B — Integral Test

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# Introduction

We've reached the end of the course! However, despite their prominence in Analysis 1A, we didn't really say much about infinite series. So, to finish off this semester, I wanted to give you a test for series convergence which we can develop using the theory of integration. This is non-examinable, but the method might come in useful for future courses! Furthermore, the examples here may serve as good practice for unseen exam questions.

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## 1 The Test

**Theorem 1.1** (Integral Test for Series).

Suppose  $(a_n)_{n \in \mathbb{N}}$  is a real sequence. Suppose also that a function  $f$  is positive and decreasing on  $[1, \infty)$  and that  $f(n) = a_n$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the limit

$$\int_1^{\infty} f := \lim_{A \rightarrow \infty} \int_1^A f$$

exists.

**Proof.** Note that the existence of  $\lim_{A \rightarrow \infty} \int_1^A f$  is equivalent (by linearity of integration) to the convergence of the series

$$\sum_{n=1}^{\infty} \int_n^{n+1} f = \int_1^2 f + \int_2^3 f + \int_3^4 f + \dots$$

Now, since  $f$  is decreasing, for each  $n \in \mathbb{N}$ , we can use the subdivision  $P_n = \{n, n+1\}$  of the intervals  $[n, n+1]$  to find

$$f(n+1) \leq \int_n^{n+1} f \leq f(n) \tag{*}$$

Applying the comparison test to the left hand side of (\*) shows that if  $\sum_{n=1}^{\infty} \int_n^{n+1} f$  exists, then  $\sum_{n=1}^{\infty} a_{n+1}$  (and hence  $\sum_{n=1}^{\infty} a_n$ ) also exists. This proves that

$$\lim_{A \rightarrow \infty} \int_1^A f \text{ exists} \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Finally, applying the comparison test to the right hand side of (\*) shows that if  $\sum_{n=1}^{\infty} a_n$  exists then  $\sum_{n=1}^{\infty} \int_n^{n+1} f$  also exists. This proves the remaining statement, i.e.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \lim_{A \rightarrow \infty} \int_1^A f \text{ exists.}$$

□

Note that we can replace 1 with any  $N \in \mathbb{N}$  in this theorem (such as in the lower series/integral limit), and the resulting modified version of the test still works.

## 2 Example

Providing a result without any practical uses is a bit pointless. So here's an example of this theorem in action! The question(s) here are taken from the textbook '**Calculus**' by Michael Spivak.

### Question.

a) Show that  $\int_1^{\infty} e^y/y^y \, dy$  exists, by considering the series  $\sum_{n=1}^{\infty} (e/n)^n$ .

b) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$$

converges, by using the integral test. **Hint: use an appropriate substitution and part (a).**

c) Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

diverges, by using the integral test. **Hint: Use the same substitution as in part (b), and show directly that the resulting integral diverges.**

## 2.1 Solutions

### Solution (Part a).

Firstly, setting  $a_n = (e/n)^n$ , we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{e^{n+1} n^n}{(n+1)^{n+1} e^n} = e \cdot \frac{1}{n+1} \cdot \left( 1 - \frac{1}{n+1} \right).$$

Taking  $n \rightarrow \infty$ , the algebra of limits gives that as  $n \rightarrow \infty$

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow 0,$$

so by d'Alembert's ratio test, the series  $\sum_{n=1}^{\infty} (e/n)^n$  is convergent.

Now, define  $f : [1, \infty) \rightarrow \mathbb{R}$  by  $f(y) = e^y/y^y$ . Note that  $f$  is strictly decreasing on  $[1, \infty)$  and for each  $n \in \mathbb{N}$ ,  $f(n) = a_n = (e/n)^n$ . Hence, by the integral test, the integral  $\int_1^{\infty} e^y/y^y dy$  exists, as required.

### Solution (Part b).

Consider the function  $f : [2, \infty) \rightarrow \mathbb{R}$  given by

$$f(x) = \frac{1}{(\ln(x))^{\ln(x)}}.$$

Setting  $y = \ln(x)$ , we find that

$$\int_2^{\infty} f(x) dx = \lim_{A \rightarrow \infty} \int_{\ln(2)}^{\ln(A)} \frac{1}{y^y} e^y dy,$$

which exists by part a). Now, for all  $n \geq 2$ , we have that  $f(n) = \frac{1}{(\ln(n))^{\ln(n)}}$ . Also, by the chain rule, we find that on  $(2, \infty)$ ,

$$f'(x) = -\frac{\ln(\ln(x)) + 1}{x \ln(x)^{\ln(x)}},$$

which is always negative, so  $f$  is decreasing on  $[2, \infty)$ . Hence, by the integral test, we find that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$$

converges.

### Solution (Part c).

Consider the function  $f : [2, \infty) \rightarrow \mathbb{R}$  given by

$$f(x) = \frac{1}{(\ln(x))^{\ln(\ln(x))}}.$$

By differentiating, we can show that  $f$  is strictly decreasing on  $[2, \infty)$ , so we can apply the integral test to this function.

Now, setting  $y = \ln(x)$  we have that (if it exists),

$$\int_2^\infty f(x) dx = \lim_{A \rightarrow \infty} \int_{\ln(2)}^{\ln(A)} \frac{1}{y^{\ln(y)}} e^y dy.$$

By rules of exponentials, we can rewrite the integrand as

$$\frac{e^y}{y^{\ln(y)}} = e^{y \left(1 - \frac{\ln^2(y)}{y}\right)}.$$

Writing  $y = e^z$ , we know that (by e.g. the growth factor test)

$$\lim_{y \rightarrow \infty} \frac{\ln^2(y)}{y} = \lim_{z \rightarrow \infty} \frac{z^2}{e^z} = 0.$$

So by the definition of convergence at  $\infty$  (see Problem Sheet 3), we know that  $\exists M \in [\ln(2), \infty)$  such that for all  $y > M$ ,

$$\left| e^{-\frac{\ln^2(y)}{y}} - 1 \right| < \frac{1}{2}.$$

Rearranging and multiplying by  $e$ , we find  $\forall y > M$ ,

$$\frac{e}{2} < e^{1 - \frac{\ln^2(y)}{y}} < \frac{3e}{2},$$

from which raising everything to the power of  $y$  yields

$$\left(\frac{e}{2}\right)^y < e^{y \left(1 - \frac{\ln^2(y)}{y}\right)} < \left(\frac{3e}{2}\right)^y.$$

Finally, by properties of the integral, we have that  $\forall y > M$ , and large enough  $A$ ,

$$\int_M^{\ln(A)} \frac{e^y}{y^{\ln(y)}} dy > \int_M^{\ln(A)} \left(\frac{e}{2}\right)^y dy.$$

Using the fundamental theorem of calculus, we can evaluate the right hand integral to obtain

$$\int_M^{\ln(A)} \frac{e^y}{y^{\ln(y)}} dy > \frac{1}{1 - \ln(2)} \left[ \left(\frac{e}{2}\right)^{\ln(A)} - \left(\frac{e}{2}\right)^M \right].$$

This right hand side of this inequality diverges as  $A \rightarrow \infty$ , and since  $\int_{\ln(2)}^M \frac{e^y}{y^{\ln(y)}} dy$  is finite, the original improper integral  $\int_2^\infty f(x) dx$  also diverges. Hence, by the integral test, the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

diverges.