Analysis 1B — Integral Test

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Introduction

We've reached the end of the course! However, despite their prominance in Analysis 1A, we didn't really say much about infinite series. So, to finish off this semester, I wanted to give you a test for series convergence which we can develop using the theory of integration. This is non-examinable, but the method might come in useful for future courses! Furthermore, the examples here may serve as good practice for unseen exam questions.

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1 The Test

Theorem 1.1 (Integral Test for Series)**.** *Suppose $(a_n)_{n\in\mathbb{N}}$ is a real sequence. Suppose also that a function f is positive and decreasing on $[1,\infty)$ and that $f(n) = a_n$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the limit*

$$\int_1^{\infty} f := \lim_{A\to\infty} \int_1^A f$$

exists.

Proof. Note that the existence of $\lim_{A\to\infty} \int_1^A f$ is equivalent (by linearity of integration) to the convergence of the series

$$\sum_{n=1}^{\infty} \int_n^{n+1} f = \int_1^2 f + \int_2^3 f + \int_3^4 f + \dots.$$

Now, since f is decreasing, for each $n \in \mathbb{N}$, we can use the subdivision $P_n = \{n, n+1\}$ of the intervals $[n, n+1]$ to find

$$f(n+1) \leq \int_n^{n+1} f \leq f(n) \tag{*}$$

Applying the comparison test to the left hand side of (*) shows that if $\sum_{n=1}^{\infty} \int_n^{n+1} f$ exists, then $\sum_{n=1}^{\infty} a_{n+1}$ (and hence $\sum_{n=1}^{\infty} a_n$) also exists. This proves that

$$\lim_{A\to\infty} \int_1^A f \;\text{ exists} \implies \sum_{n=1}^{\infty} a_n \;\text{ converges.}$$

Finally, applying the comparison test to the right hand side of (*) shows that if $\sum_{n=1}^{\infty} a_n$ exists then $\sum_{n=1}^{\infty} \int_n^{n+1} f$ also exists. This proves the remaining statement, i.e.

$$\sum_{n=1}^{\infty} a_n \;\text{ converges} \implies \lim_{A\to\infty} \int_1^A f \;\text{ exists.}$$

□

Note that we can replace 1 with any $N \in \mathbb{N}$ in this theorem (such as in the lower series/integral limit), and the resulting modified version of the test still works.

2 Example

Providing a result without any practical uses is a bit pointless. So here's an example of this theorem in action! The question(s) here are taken from the textbook *'Calculus'* by Michael Spivak.

Question. a) Show that $\int_1^\infty \mathrm{e}^y/y^y\, dy$ exists, by considering the series $\sum_{n=1}^\infty (\mathrm{e}/n)^n$.

b) Show that

$$\sum_{n=2}^\infty \frac{1}{(\ln(n))^{\ln(n)}}$$

converges, by using the integral test. *Hint: use an appropriate substitution and part (a).*

c) Show that

$$\sum_{n=2}^\infty \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

diverges, by using the integral test. *Hint: Use the same substitution as in part (b), and show directly that the resulting integral diverges.*

2.1 Solutions

Solution (Part a). Firstly, setting $a_n = (\mathrm{e}/n)^n$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\mathrm{e}^{n+1}n^n}{(n+1)^{n+1}\mathrm{e}^n} = \mathrm{e}\cdot\frac{1}{n+1}\cdot\left(1-\frac{1}{n+1}\right).$$

Taking $n\to\infty$, the algebra of limits gives that as $n\to\infty$

$$\left|\frac{a_{n+1}}{a_n}\right| \to 0,$$

so by d'Alembert's ratio test, the series $\sum_{n=1}^\infty (\mathrm{e}/n)^n$ is convergent.

Now, define $f:[1,\infty)\to\mathbb{R}$ by $f(y) = \mathrm{e}^y/y^y$. Note that f is strictly decreasing on $[1,\infty)$ and for each $n\in\mathbb{N}$, $f(n) = a_n = (\mathrm{e}/n)^n$. Hence, by the integral test, the integral $\int_1^\infty \mathrm{e}^y/y^y\, dy$ exists, as required.

Solution (Part b). Consider the function $f:[2,\infty)\to\mathbb{R}$ given by

$$f(x) = \frac{1}{(\ln(x))^{\ln(x)}}.$$

Setting $y = \ln(x)$, we find that

$$\int_2^\infty f(x)\, dx = \lim_{A\to\infty}\int_{\mathrm{e}^2}^{\mathrm{e}^A} \frac{1}{y^y}\mathrm{e}^y\, dy,$$

which exists by part a). Now, for all $n\geq 2$, we have that $f(n) = \frac{1}{(\ln(n))^{\ln(n)}}$. Also, by the chain rule, we find that on $(2,\infty)$,

$$f'(x) = -\frac{\ln(\ln(x))+1}{x\ln(x)^{\ln(x)}},$$

which is always negative, so f is decreasing on $[2,\infty)$. Hence, by the integral test, we find that the series

$$\sum_{n=2}^\infty \frac{1}{(\ln(n))^{\ln(n)}}$$

converges.

Solution (Part c). Consider the function $f:[2,\infty)\to\mathbb{R}$ given by

$$f(x) = \frac{1}{(\ln(x))^{\ln(\ln(x))}}.$$

By differentiating, we can show that f is strictly decreasing on $[2,\infty)$, so we can apply the integral test to this function.

Now, setting $y = \ln(x)$ we have that (if it exists),

$$\int_2^{\infty} f(x)\, dx = \lim_{A\to\infty} \int_{e^2}^{e^A} \frac{1}{y^{\ln(y)}} e^y \, dy.$$

By rules of exponentials, we can rewrite the integrand as

$$\frac{e^y}{y^{\ln(y)}} = e^{y\left(1-\frac{\ln^2(y)}{y}\right)}.$$

Writing $y = e^z$, we know that (by e.g. the growth factor test)

$$\lim_{y\to\infty} \frac{\ln^2(y)}{y} = \lim_{z\to\infty} \frac{z^2}{e^z} = 0.$$

So by the definition of convergence at ∞ (see Problem Sheet 3), we know that $\exists M \in [e^2, \infty)$ such that for all $y > M$,

$$\left| e^{-\frac{\ln^2(y)}{y}} - 1 \right| < \frac{1}{2}.$$

Rearranging and multiplying by e, we find $\forall y > M$,

$$\frac{e}{2} < e^{1-\frac{\ln^2(y)}{y}} < \frac{3e}{2},$$

from which raising everything to the power of y yields

$$\left(\frac{e}{2}\right)^y < e^{y\left(1-\frac{\ln^2(y)}{y}\right)} < \left(\frac{3e}{2}\right)^y.$$

Finally, by properties of the integral, we have that $\forall y > M$, and large enough A,

$$\int_M^{e^A} \frac{e^y}{y^{\ln(y)}} \, dy > \int_M^{e^A} \left(\frac{e}{2}\right)^y \, dy.$$

Using the fundamental theorem of calculus, we can evaluate the right hand integral to obtain

$$\int_M^{e^A} \frac{e^y}{y^{\ln(y)}} \, dy > \frac{1}{1-\ln(2)} \left[\left(\frac{e}{2}\right)^{e^A} - \left(\frac{e}{2}\right)^M \right].$$

This right hand side of this inequality diverges as $A \to \infty$, and since $\int_{e^2}^{M} \frac{e^y}{y^{\ln(y)}} \, dy$ is finite, the original improper integral $\int_2^{\infty} f(x)\, dx$ also diverges. Hence, by the integral test, the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

diverges.