Analysis 1B — Further Integral Examples

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Introduction

Hi everyone! Since Problem Sheet 11 is being covered in Revision Week, I thought it might be more useful for you to have a few extra questions (and solutions) related to those on the sheet. As usual, alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to Christian Jones (caj50). To return to the homepage, click here.

1 Example 1

Example 1.1.

Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x < 0, \\ 1 & \text{if} \quad 0 \leq x \leq 1. \end{cases}$$

Using the Cauchy criterion, prove that f is integrable.

Solution.

If you weren't asked to use the Cauchy criterion, the easy way of doing this is by using the theorem that says that a monotone function is integrable. Here's how you would do it via the Cauchy criterion:

For some $\delta \in (0, 1]$, consider the subdivision

$$P_\delta = \{-1, -\delta, \delta, 1\} = \{x_0, x_1, x_2, x_3\}.$$

Then,

$$L(f, P_\delta) = \sum_{i=1}^{3} \inf_{[x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}),$$

$$= 0(-\delta - -1) + 0(\delta - -\delta) + 1(1 - \delta),$$

$$= 1 - \delta.$$

Also,

$$U(f, P_\delta) = \sum_{i=1}^{3} \sup_{[x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}),$$

$$= 0(-\delta - -1) + 1(\delta - -\delta) + 1(1 - \delta),$$

$$= 1 + \delta.$$

Hence,

$$U(f, P_\delta) - L(f, P_\delta) = 1 + \delta - (1 - \delta) = 2\delta.$$

Now, fix $\epsilon > 0$. We have that $2\delta < \epsilon$ when $\delta < \epsilon/2$. So, taking $\delta = \epsilon/4$, for example, we find that

$$P_\delta = \left\{-1, -\frac{\epsilon}{4}, \frac{\epsilon}{4}, 1\right\}$$

is such that $U(f, P_\delta) - L(f, P_\delta) < \epsilon$. Therefore, by the Cauchy criterion, f is integrable.

This idea of 'cutting out the discontinuity' by introducing a δ comes in really handy in a few areas of maths. It's also really handy for dealing with improper integrals, for example

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

2 Example 2

Example 2.1.

Consider $f : [1, 4] \to \mathbb{R}$ given by $f(x) = \frac{1}{2\sqrt{x}}$. Find $\int_1^4 f$ by using the Fundamental Theorem of Calculus.

Solution.

Let $F : (0, 5) \to \mathbb{R}$ be defined by $F(x) = \sqrt{x}$. We know that F is differentiable on $(0, 5)$, and $\forall x \in [1, 4]$:

$$F'(x) = \frac{1}{2\sqrt{x}} = f(x).$$

Hence, F is a primitive for f. Moreover, f is continuous on $[1,4]$, so it is integrable. Hence, by the FTC:

$$\int_1^4 f = F(4) - F(1) = \sqrt{4} - \sqrt{1} = 1.$$

3 Example 3

Example 3.1.

Let $f : [0, \pi/2] \to \mathbb{R}$ be defined by $f(t) = \mathrm{e}^{-t^2}$, and let $b : (0, \pi/2) \to (0, \pi/2)$ be defined by $b(x) = x\sin(x)$. Find

$$\frac{d}{dx}\int_0^{b(x)} f(t)\,dt.$$

Solution.

First, let

$$G(b) = \int_0^b f(t)\,dt,$$

where $b = b(x) = x\sin(x)$. By the second Fundamental Theorem of Calculus, as f is continuous on $[0, \pi/2]$, then G is a primitive for f in terms of b, with

$$f(b(x)) = \frac{dG}{db}(b(x)) \quad \forall x \in (0, \pi/2).$$

Hence, by the chain rule:

$$\frac{d}{dx}\int_0^{b(x)} f(t)\,dt = \left.\frac{dG}{db}\right|_{b=b(x)} \frac{db}{dx},$$

$$= f(b(x))b'(x) \quad \text{(by second FTC)},$$

$$= \exp\left(-x^2\sin^2(x)\right)(x\sin(x))'.$$

Finally, by the product rule, we obtain for any $x \in (0, \pi/2)$

$$\frac{d}{dx}\int_0^{b(x)} f(t)\,dt = (\sin(x) + x\cos(x))\exp\left(-x^2\sin^2(x)\right).$$