Further Examples — Limits Inferior and Superior

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Overview

In this document, you'll find three different examples of finding the limit superior ($\limsup$) and limit inferior ($\liminf$) of a sequence, with different methods used in each case.

0.1 Example 1

Example 0.1.

Consider the sequence $(a_n)_n$ defined by

$$a_n = (-1)^n \frac{2n}{1+3n}.$$

Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Solution Firstly, note that we can rewrite each a_n as

$$a_n = (-1)^n \frac{2}{3} \frac{1}{\frac{1}{3n}+1}.$$

Splitting into odd and even cases, we obtain

$$a_n = \begin{cases} \frac{2}{3}\frac{1}{\frac{1}{3n}+1} & \text{for } n \text{ even,} \\ \frac{2}{3}\frac{-1}{\frac{1}{3n}+1} & \text{for } n \text{ odd.} \end{cases}$$

Note that for $j \in \mathbb{N}$, $a_{2j-1} \leq 0 \leq a_{2j}$. Also note that $(a_{2j-1})_j$ is a decreasing sequence and $(a_{2j})_j$ is an increasing sequence [Try showing these!] Moreover, $|a_n| \leq \frac{2}{3} \forall n \in \mathbb{N}$, so $(a_n)_n$ is bounded.

Now, fix $k \in \mathbb{N}$. We have:

$$\sup_{k \geq n} a_n = \sup_{2j \geq k} a_{2j}, \text{ (since only 'even' elements are non-negative.)}$$

$$= \lim_{j \to \infty} a_{2j}, \text{ (since } (a_{2j})_j \text{ is a bounded increasing sequence),}$$

$$= \frac{2}{3} \text{ (by AoL)}$$

0.2 Series

It might look like we're done with sequences, but in the grand scheme of things, we're only really getting started. Since with each sequence $(a_n)_{n \in \mathbb{N}}$, we have an infinite list of real numbers, we might consider trying to manipulate them in some way. One way we can do this is by adding them together, which leads to the notion of a **series**.

Definition 0.1 (Series)**.**

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. Then

$$\sum_{n=1}^{\infty} a_n$$

is called a series for $(a_n)_{n\in\mathbb{N}}$.

Much like with sequences, we have an analogous version of convergence for a series:

Definition 0.2 (Series Convergence and Partial Sums)**.**

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(S_N)_{N\in\mathbb{N}}$ converges, where

$$S_N := \sum_{n=1}^{N} a_n$$

is the N th partial sum. If $S_N \to \ell$ as $N \to \infty$, we define

$$\ell = \sum_{n=1}^{\infty} a_n.$$

If $(S_N)_{N\in\mathbb{N}}$ diverges to $\pm\infty$, we say that the corresponding series

$$\sum_{n=1}^{\infty} a_n = \pm\infty.$$

Finally, if $(S_N)_{N\in\mathbb{N}}$ doesn't converge to a limit, we say that the series diverges without limit.

0.2.1 Algebra of Series

By applying the algebra of limits to the sequences of partial sums, we can deduce some handy results.

Theorem 0.1 (Algebra of Series)**.**

Let $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ be convergent series, and let $\alpha, \beta \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.$$

0.2.2 Some Other Useful Results

Firstly, we can relate the size of the terms of a series to the overall sum.

Proposition 0.1.

Let $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ be real series. If $a_n \leq b_n \forall n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Secondly, we have a **necessary** condition for convergence of a series.

Proposition 0.2.

Let $\Sigma_{n=1}^{\infty} a_n$ be a convergent series. Then $a_n \to 0$ as $n \to \infty$.

Note that the converse of this theorem **does not** hold (think of the sum $\Sigma_{n=1}^{\infty} \frac{1}{n}$). However, the contrapositive is very good at showing that a series does not converge!

Proposition 0.3.

Let $\Sigma_{n=1}^{\infty} a_n$ be a series. If $a_n \not\to 0$ as $n \to \infty$, then $\Sigma_{n=1}^{\infty} a_n$ does not converge.

1 Hints

As per usual, here's where you'll find the problem sheet hints!

- [H1.] Try using a similar argument to the one used in tutorial question 1 (i.e. use the fact that the sequence can be split into odd and even cases to your advantage)

- [H2.] For this question, think about what it means for a series to be convergent. You'll also want to split the terms of the series up in some way. (Think of tutorial question 3a.)
- [H3.] For the first part, think induction. The only other thing I'll say is to make sure you state all the main results you use!