Analysis 1A - Problem Sheet 2 (2019)

4. ~~a. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that A is bounded below and find $\inf(A)$.~~

b. **Homework** Let $B = \{\frac{2n-1}{n+1} : n \in \mathbb{N}\}$. Show that B is bounded above and find $\sup(B)$.

Method 1 - Contradiction

First, note for $n \in \mathbb{N}$: $\dfrac{2n-1}{n+1} = \dfrac{2n+2-3}{n+1} = 2 - \dfrac{3}{n+1} < 0$ as $\dfrac{3}{n+1} > 0$

So B is bounded above by 2.

$\therefore$ By the completeness axiom, as $B \neq \emptyset$, $\sup(B)$ exists and $\sup(B) \leq 2$. $(*)$

Suppose for $\times$ that $\sup(B) < 2$.

Now, for any $x < 2$, $2 - \dfrac{3}{n+1} > x \iff n+1 > \dfrac{3}{2-x} \iff n > \dfrac{3}{2-x} - 1$

Taking $x = \sup(B)$ and applying Archimedes Postulate, $\exists N \in \mathbb{N}$ s.t.

$$N > \frac{3}{2-\sup(B)} - 1$$

$$\iff 2 - \frac{3}{N+1} > \sup(B),$$ which is a contradiction as $2 - \dfrac{3}{N+1} \in B$.

Hence, $\sup(B) \geq 2$, and by $(*)$, $\sup(B) = 2$

Method 2 - Alternative Characterisation of sup

First, note for $n \in \mathbb{N}$: $\dfrac{2n-1}{n+1} = \dfrac{2n+2-3}{n+1} = 2 - \dfrac{3}{n+1} < 0$ as $\dfrac{3}{n+1} > 0$

So B is bounded above by 2.

$\therefore$ By the completeness axiom, as $B \neq \emptyset$, $\sup(B)$ exists and $\sup(B) \leq 2$. $(*)$

Now, we claim that $\sup(B) = 2$.

Fix $\varepsilon > 0$, then for $n \in \mathbb{N}$:

$$2 - \frac{3}{n+1} > 2 - \varepsilon$$

$$\Leftrightarrow \quad \varepsilon > \frac{3}{n+1}$$

$$\Leftrightarrow \quad n\varepsilon > 3 - \varepsilon$$

$$\Leftrightarrow \quad n > \frac{3-\varepsilon}{\varepsilon}$$

Now, by Archimedes Postulate, $\exists N \in \mathbb{N}$ s.t $N > \dfrac{3-\varepsilon}{\varepsilon}$

$$\Rightarrow 2 - \frac{3}{N+1} > 2 - \varepsilon$$

Take $b = 2 - \dfrac{3}{N+1} \in B$. Since $\varepsilon > 0$ was arbitrary, we have that

$\forall \varepsilon > 0, \exists b \in B$ s.t. $b > 2 - \varepsilon$. So by the alternative characterisation of suprema, (Theorem 2.1) $\sup(B) = 2$.

Method 3 - Limits

Note that this doesn't work in general, but it is quicker when you can use it. It relies on the following theorem (which we'll cover later):

A bounded, increasing sequence (b_n) is convergent, and its limit is
$$\lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\}$$

Define $b_n = \dfrac{2n-1}{n+1}$ for $n \in \mathbb{N}$.

Step 1 - Show (b_n) is bounded above

First, note for $n \in \mathbb{N}$: $b_n = \dfrac{2n+2-3}{n+1} = 2 - \dfrac{3}{n+1} < 0$ as $\dfrac{3}{n+1} > 0$

So B is bounded above by 2, and by completeness axiom, $\sup(B)$ exists.

Step 2 - Show (b_n) is increasing (i.e. $b_{n+1} \geq b_n \ \forall n \in \mathbb{N}$)

We have for $n \in \mathbb{N}$, $b_{n+1} - b_n = 2 - \dfrac{3}{n+2} - \left(2 - \dfrac{3}{n+1}\right)$

$$= \frac{3(n+2) - 3(n+1)}{(n+1)(n+2)}$$

$$= \frac{3}{(n+1)(n+2)}$$

$$\geq 0$$

So (b_n) is increasing.

$\therefore$ By the above theorem, (b_n) converges and by the Algebra of Limits:

$$\sup(B) = \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(2 - \frac{3/n}{1+1/n}\right) = 2$$ (as expected!)