Vectors, Vector Calculus and Mechanics — Past Paper 2017

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Contents

Introduction

Here are the solutions to the past paper discussed in the revision session on XXth May 2024. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click here.

Section A

Question 1

Question.
In the triangle OAB, we set $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C is located at the midpoint of the side AB. The point D divides the side OB in the ratio $2 : 1$, i.e. $OD = \frac{2}{3}OB$.

The lines AD and OC cross at X. Using vectors, show that $AX = \frac{3}{5}AD$ and find the ratio in which the point X divides OC.

Solution.
TBD

Question 2

Question.
Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$.

a) Find the lengths of $\mathbf{a}$ and $\mathbf{b}$.

b) Find the cosine of the acute angle between $\mathbf{a}$ and $\mathbf{b}$.

c) Find a unit vector $\hat{\mathbf{c}}$ which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}, \mathbf{b}, \hat{\mathbf{c}}$ form a right-handed system.

Solution.

TBD

Question 3

Question.

State the expansion formula for the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Use it to prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot \{(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})\} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2,$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the scalar triple product.

State, but do not prove, the results of vector algebra which you use.

Solution.

TBD

Question 4

Question.

a) Let $T(x, y)$ be a differentiable function of x and y, and let (a, b) be a point on the xy-plane. Define the gradient vector $\nabla T(a, b)$, and prove that the rate of change of $T(x, y)$ at (a, b) in the direction of the unit vector $\mathbf{u}$ is given by

$$D_{\mathbf{u}} T(a, b) = \mathbf{u} \cdot \nabla T(a, b).$$

b) Find the gradient vector of the function

$$T(x, y) = 3x^2 + 2y^2 + 6$$

at the point $(2,1)$, and find the equation of the tangent plane to the 3D surface $z = T(x,y)$ at the point $(2,1,c)$ where $c = T(2,1)$.

Solution.

TBD

Question 5

Question.

In planar polar coordinates $(r.\theta)$, the position vector of a particle at time t can be written as

$$\mathbf{x}(t) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$$

where r, θ are functions of t.

a) Express the radial and angular unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ in terms of $\mathbf{i}, \mathbf{j}$. Derive expressions for $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$, and hence show that

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

and

$$\ddot{\mathbf{x}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}_\theta.$$

b) Find the acceleration vector of a particle which travels in a circular orbit of radius a, at a constant angular speed ω.

Solution.

TBD

Section B

Question 6

Question.

a) Prove that the shortest distance from a point A with position vector $\mathbf{a},$ to the plane Π defined by $\mathbf{r} \cdot \mathbf{n} = d$ is

$$h = \frac{|d - \mathbf{a} \cdot \mathbf{n}|}{\|\mathbf{n}\|},$$

and find the position vector of the point on Π which achieves this distance.

b) Let $\mathbf{a}, \mathbf{b}, \mathbf{l}, \mathbf{m}$ be given vectors, and let $\mathbf{r} = \mathbf{a} + \lambda \mathbf{l}$ and $\mathbf{r} = \mathbf{b} + \mu \mathbf{m}$ be two non-parallel and non-intersecting lines L_1 and L_2 respectively.

i) Find the vector equations of two parallel planes Π_1 and $\Pi_2,$ such that Π_1 contains L_1 and Π_2 contains $L_2.$

ii) Find the distance between the two planes Π_1 and $\Pi_2.$

Solution.

TBD

Question 7

Question.

a) By either finding the Complementary Function and Particular Integral, or using the Integrating Factor method, solve the vector differential equation

$$\ddot{\mathbf{x}} + k\dot{\mathbf{x}} = \mathbf{h}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

where k is a non-zero scalar and $\mathbf{h}$ is a constant vector. Show that the solution can be written as

$$\mathbf{x} = \mathbf{x}_0 + \frac{1}{k^2}\left(\mathrm{e}^{-kt} + kt - 1\right)\mathbf{h}.$$

b) A food parcel of mass m is dropped from a helicopter hovering at a height D directly above a target on horizontal ground. There is a constant horizontal cross-wind of velocity $u,$ and the air resistance is proportional to the relative velocity of the parcel with respect to the wind, with coefficient $\mu.$ Derive the vector differential equation of motion of the parcel, and hence find its position vector $\mathbf{x}(t).$

c) How far away from the target will the food parcel land?

Solution.

TBD

Question 8

Question.

A particle of mass m moves under the action of a central "planetary" force

$$\mathbf{F} = -\mu m \|\dot{\mathbf{x}}\|^2 \frac{\mathbf{x}}{r^2}$$

where $r = \|\mathbf{x}\|$ and $\mu > 0$ is a constant.

a) Prove that the motion takes place in a plane.

b) Using polar coordinates (r, θ) on the plane, show that the equations of motion are

$$r^2\dot{\theta} = h, \quad \ddot{r} + (\mu - 1)\frac{h^2}{r^3} + \mu\frac{\dot{r}^2}{r} = 0,$$

where h is a constant.

Note: You may use without proof the formulae for velocity and acceleration in polar coordinates, given in Question 5(a).

c) The particle is projected radially outwards from the surface of a "planet" of radius $R,$ with initial speed $v_0.$ Show that $h = 0$ in part (b).

Integrate the second equation of motion in part (b) to obtain

$$\ln(\dot{r}) = -\mu \ln(r) + c$$

where c is a constant of integration. Hence deduce that for all $v_0 > 0$ the particle will never fall back to the "planet".

Hint: $\frac{d}{dr}\left(\ln(r)\right) = \frac{1}{r}\frac{dr}{dt}$.

Solution.

TBD