

# Analysis 1A — Tutorial 3

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## Contents

<b>Introduction</b>	<b>1</b>
Part a) . . . . .	1
Part b) . . . . .	1
Part c) . . . . .	2

## Introduction

Here is a version of Tutorial Question 3 off of Problem Sheet 3 with an alternative solution for part c). Parts a) and b) are included for completeness.

**Example 0.1** (PS3 Question 3). a) Show that

$$2xy \leq x^2 + y^2, \quad \forall x, y \in \mathbb{R},$$

and that equality holds only if  $x = y$ . b) Show that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \leq \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \quad \forall x, y > 0.$$

c) Prove that

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| \leq |x-y| \quad \forall x, y \in \mathbb{R}.$$

### Part a)

*Solution.* We have that for any  $x, y \in \mathbb{R}$ ,

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2,$$

from which rearranging gives

$$2xy \leq x^2 + y^2.$$

Now,

$$2xy = x^2 + y^2 \Leftrightarrow 0 = (x-y)^2 \Leftrightarrow x-y=0 \Leftrightarrow x=y.$$

So equality holds *if and only if*  $x = y$ .

### Part b)

*Solution.* Consider the second inequality first. Note that since  $x, y > 0$ ,  $\sqrt{x}, \sqrt{y} > 0$ , and so

$$x + y \leq x + 2\sqrt{x}\sqrt{y} + y = (\sqrt{x} + \sqrt{y})^2.$$

Square rooting this result gives us that

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}.$$

Next, we have that

$$\begin{aligned} \left(\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}}\right)^2 &= \frac{x}{2} + 2\sqrt{\frac{x}{2}}\sqrt{\frac{y}{2}} + \frac{y}{2}, \\ &\leq \frac{x}{2} + 2\left(\frac{x}{2} + \frac{y}{2}\right) + \frac{y}{2} \quad (\text{by part a}), \\ &= x + y. \end{aligned}$$

Again, square rooting gives us that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \leq \sqrt{x+y}.$$

## Part c)

*Solution.* Firstly for  $x = -y$ ,

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| = 0 \leq |-2y| = |x - y|.$$

For  $x \neq -y$ , we have

$$\begin{aligned} |\sqrt{1+x^2} - \sqrt{1+y^2}| &= \frac{|1+x^2 - (1+y^2)|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \\ &= \frac{|x^2 - y^2|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \\ &= \frac{|x+y||x-y|}{\sqrt{1+x^2} + \sqrt{1+y^2}}. \end{aligned} \tag{*}$$

Now,

$$|x| \leq \sqrt{1+x^2}, \text{ and } |y| \leq \sqrt{1+y^2}.$$

(This can be seen by squaring both sides of each inequality)

So,

$$\begin{aligned} |x+y| &\leq |x| + |y| \quad (\text{by the triangle inequality}) \\ &\leq \sqrt{1+x^2} + \sqrt{1+y^2}, \\ \Leftrightarrow \frac{1}{|x+y|} &\geq \frac{1}{\sqrt{1+x^2} + \sqrt{1+y^2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\sqrt{1+x^2} - \sqrt{1+y^2}| &= \frac{|x+y||x-y|}{\sqrt{1+x^2} + \sqrt{1+y^2}}, \\ &\leq \frac{|x-y||x+y|}{|x+y|}, \\ &= |x-y|, \end{aligned} \tag{**}$$

as required!

You might have a few questions about this:

**Q1)** Why is 3c) done in this way?

A1) It's an alternative way to the one in the model solutions, but I think it's good because it uses some techniques that are useful for the sequences part of the course (e.g. the triangle inequality and step (\*)).

**Q2)** Where on Earth did the case  $x = -y$  come from?

A2) If you look at (\*\*), this expression doesn't work if  $x = -y$ , so you need to consider this separately. It's not an obvious case until you actually reach (\*\*), but once you realise it, it's an easy thing to add to the start of your solution.

**Q3)** What about (\*)? Where does this come from?

A3) Recall for  $a, b \in \mathbb{R}$ ,

$$(a-b)(a+b) = a^2 - b^2.$$

Taking  $a = \sqrt{1+x^2}$ , and  $b = \sqrt{1+y^2}$ , we have that

$$\sqrt{1+x^2} - \sqrt{1+y^2} = \frac{(1+x^2) - (1+y^2)}{\sqrt{1+x^2} + \sqrt{1+y^2}}.$$