

# Analysis 1B — Epsilon-Delta Example

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# Introduction

Here is an extra example of finding the limit of a function using the definition. This should hopefully give you a guide to the techniques required, and how much detail you should put in your solutions.

## 1 Worked Example

### Question.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that  $\lim_{x \rightarrow 1} f(x) = 1$ .

### Solution.

Fix  $\epsilon > 0$ , and suppose that  $0 < |x - 1| < \delta$  for some  $\delta > 0$  to be chosen later. Without loss of generality, suppose that  $\delta \leq 1$  (this deals with the different definition of  $f$  at  $x = 0$ ). Then

$$\begin{aligned} |f(x) - 1| &= \left| \frac{1}{x^2} - 1 \right|, \\ &= \left| \frac{1 - x^2}{x^2} \right|, \\ &= \frac{|x - 1||x + 1|}{|x|^2}. \end{aligned}$$

Now, by the triangle inequality, we have that

$$|x + 1| = |x - 1 + 2| \leq |x - 1| + 2.$$

Also, by the reverse triangle inequality,

$$|x| = |x - 1 + 1| \geq 1 - |x - 1|.$$

So, if  $\delta \leq \frac{1}{2}$ , we obtain  $|x + 1| < \frac{5}{2}$ ,  $|x| > \frac{1}{2}$ , and

$$|f(x) - 1| < \frac{5/2|x - 1|}{(1/2)^2} = 10|x - 1| < 10\delta.$$

Hence, if  $\delta = \min\{1, 1/2, \epsilon/10\}$ , we find that

$$0 < |x - 1| < \delta \implies |f(x) - 1| < \epsilon.$$

Therefore, as  $\epsilon$  was arbitrary, we conclude that  $\lim_{x \rightarrow 1} f(x) = 1$ .