

Analysis 1A — Tutorial 4

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Introduction

Here is the material to accompany the Analysis Tutorial in Week 4. Alternative formats can be downloaded by clicking the download icon at the top of the page. As usual, send comments and corrections to [Christian Jones \(caj50\)](#). To return to the homepage, click [here](#).

1 Lecture Recap

1.1 Suprema and Infima

Hopefully by now you're getting more familiar with the ideas of set bounds, especially the idea of suprema/infima! It also turns out that there's an alternative characterisation of suprema and infima which can be very useful, especially if the members of a set aren't indexed by natural numbers. Technically, this is something that came up in last week's lectures, but is more relevant to this week's problem sheet (see Tutorial Question 1).

Proposition 1.1. *Let $S \subseteq \mathbb{R}$. Then a number $T \in \mathbb{R}$ is the supremum of S , denoted $\sup(S)$ if:*

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s > T - \epsilon.$$

Proposition 1.2. *Let $S \subseteq \mathbb{R}$. Then a number $t \in \mathbb{R}$ is the infimum of S , denoted $\inf(S)$ if:*

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s < t + \epsilon.$$

As an example, take the set $S = (-1, 2] = \{x \mid -1 < x \leq 2\}$, and fix some $\epsilon > 0$. Then, if we take $s_1 = 2 - \epsilon/2$ and $s_2 = -1 + \epsilon/2$, we see that

- s_1 and s_2 are in the set S ,
- $s_1 > 2 - \epsilon$, and
- $s_2 < -1 + \epsilon$.

Hence, as ϵ was arbitrary, the alternative characterisation of suprema and infima says that $\sup(S) = 2$ and $\inf(S) = -1$.

1.2 Complex Numbers

Up until this point, we have only looked at numbers which are subsets of the real numbers (\mathbb{R}). However, to steal an archetypal example, how do we solve an equation like $x^2 + 1 = 0$? We can't do this in the reals, so we 'invent' a solution by defining $i := \sqrt{-1}$. This gives us a new set of numbers, namely

$$\mathbb{C} := \{a + ib \mid a, b \in \mathbb{R}\},$$

and call members of this set **complex numbers**. There is a ton of theory on these numbers, which is left for the lecture notes.¹ Instead, we just repeat a list of important definitions, and one important result.

Definition 1.1 (Complex Numbers). Given a complex number $z = a + ib$, we define the following quantities:

- i) The real part of z is $\mathcal{R}(z) := a$.
- ii) The imaginary part of z is $\mathcal{I}(z) := b$.
- iii) The complex conjugate of z is $\bar{z} := a - ib$.
- iv) The modulus of z is defined to be $|z| := \sqrt{a^2 + b^2}$.

¹We could even spend the entire semester studying complex analysis instead of real analysis, but that adds a whole other layer of — for want of a better word — complexity. Having two dimensions to deal with instead of one causes a whole raft of issues!

Just like the absolute value on the real numbers, the modulus defines a *distance* on the complex numbers.² In particular for complex numbers z_1 and z_2 , the quantity $|z_1 - z_2|$ tells us how ‘far apart’ the two numbers are. Plotting these numbers on an Argand diagram, we see that $|z_1 - z_2|$ gives us the distance of the straight line joining z_1 and z_2 .

Since the modulus defines a distance on \mathbb{C} , it had better obey the triangle inequality. Luckily for us, it does!

Proposition 1.3 (Triangle Inequality). For $z_1, z_2 \in \mathbb{C}$,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

1.3 Sequences and Convergence

We’re finally onto the first main topic of this course! To discuss anything from this point, we need to introduce the idea of a sequence.

Definition 1.2 (Sequence). A sequence of real/complex numbers is a function

$$\begin{aligned} a : \mathbb{N} &\longrightarrow X, \\ n &\longmapsto a_n, \end{aligned}$$

where X is either \mathbb{R} or \mathbb{C} respectively.

Since this notation can get kind of annoying, we instead denote a sequence by $(a_n)_{n \in \mathbb{N}}$. If it’s clear from the context what set we’re indexing over, we can even just simply write a sequence as $(a_n)_n$.

Now, this gives us an infinitely long list of real numbers, and sometimes its interesting to look at the ‘long-term’ behaviour of these lists. This gives rise to the idea of convergence.

Definition 1.3 (Sequence Convergence). A sequence $(a_n)_{n \in \mathbb{N}}$ converges to a real number L as $n \longrightarrow \infty$, written as either $a_n \longrightarrow L$, or $\lim_{n \rightarrow \infty} a_n = L$ if

$$\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \text{ such that } \forall n \geq N, |a_n - L| < \epsilon.$$

Loosely speaking, this says that no matter how close you want the sequence to get to L , you will always be able to find some point in the sequence after which all points in the sequence will be as close to L as you wanted. For an example of this, have a look at this [Desmos link](#). For $\epsilon = 0.5$ and $L = 3$, you can see that every member of the sequence after the 11th lies within a strip of width 2ϵ around L . Have a go at messing with the value of ϵ !

Something else we can mention for the definition is its *negation*. Specifically, a sequence $(a_n)_n$ does not converge to L if

$$\exists \epsilon_0 > 0, \text{ such that } \forall N \in \mathbb{N}, \exists n \geq N \text{ such that } |a_n - L| \geq \epsilon_0.$$

We can say the exact same things for the convergence of complex sequences without much effort too. We just need to remember to change the distance from the absolute value to the modulus!

1.3.1 Useful Sequences

Since there’s no point in having a definition without using it, it’s a great idea to obtain some (straight-forward) results:

- As $n \longrightarrow \infty$:

$$\frac{1}{n} \longrightarrow 0.$$
- For a real number c : as $n \longrightarrow \infty$,

$$c \longrightarrow c.$$
- For $q \in \mathbb{R}$ with $|q| < 1$: as $n \longrightarrow \infty$

$$q^n \longrightarrow 0.$$

²The more technical term is that the modulus defines a *metric* on \mathbb{C} . You’ll see more of this next year, but if you’re interested, see the section on metrics in this document!

1.4 Metric Spaces (Non-examinable)

Have you noticed that both the absolute value on the real numbers and the modulus on the complex numbers behave suspiciously alike? They even have the same notation! This is because they're both examples of *metrics* on a set:

Definition 1.4 (Metric Space). A metric space (X, d) consists of a set X together with a function $d : X \times X \rightarrow \mathbb{R}$ satisfying $\forall x, y, z \in X$:

- i) Non-negativity: $d(x, y) \geq 0$,
- ii) $d(x, y) = 0 \Leftrightarrow x = y$,
- iii) Symmetry: $d(x, y) = d(y, x)$,
- iv) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

As we've seen, $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ both define metric spaces. The existence of this definition suggests that we can define the distance between two members of a set in different ways! For example, thinking about the complex numbers $X = \mathbb{C}$, with complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we could instead look at:

- The discrete metric

$$d(z_1, z_2) = \begin{cases} 1 & \text{if } z_1 \neq z_2, \\ 0 & \text{otherwise} \end{cases}$$

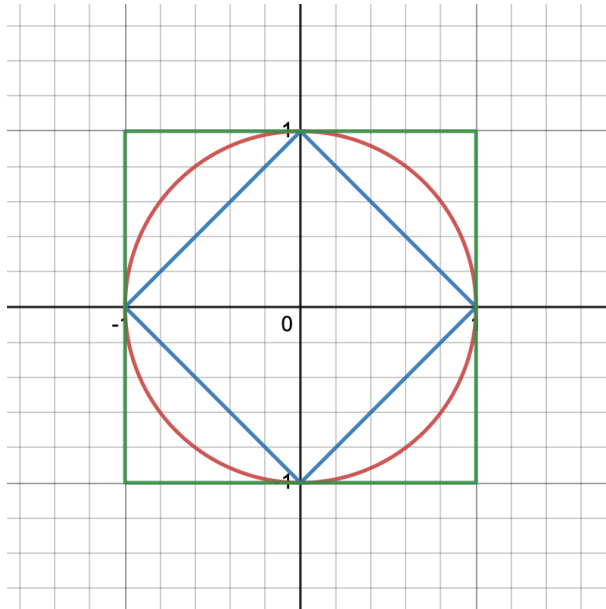
- The p -norm:

$$d(z_1, z_2) = (|a_1 - a_2|^p + |b_1 - b_2|^p)^{1/p}, \quad 1 \leq p < \infty.$$

- The ∞ -norm:

$$d(z_1, z_2) = \max\{|a_1 - a_2|, |b_1 - b_2|\}.$$

These last two are called norms because they satisfy some additional properties, but don't worry about these until next year! It's an interesting thing to look at what the unit circle looks like under the 1-norm (blue), 2-norm (red) and ∞ -norm (green). The complex numbers used here are $z_1 = x + iy$ and $z_2 = 0$:



We can even look at defining distances on other sets. For example, consider the set $X = C^0([a, b])$, which is the set of all continuous functions $f : [a, b] \rightarrow \mathbb{R}$.³ One way we can define the distance between two functions f and g is to consider evaluating them at all members of the domain, and finding what the maximum deviation between the two are. This gives rise to a metric $d : X \times X \rightarrow \mathbb{R}$, namely

$$d(f, g) := \max_{x \in [a, b]} |f(x) - g(x)|.$$

Have a go at proving that this is a metric!

³For the time being, think of these functions as ones that you can draw without lifting your pen off the page.

1.4.1 Convergence

In a similar way, have you noticed that the definitions of convergence for a real/complex sequence are pretty much identical? Again, this is because we can generalise the definition of convergence to metric spaces!

Definition 1.5 (Convergence). Let (X, d) be a metric space, let $a \in X$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in X . Then $(a_n)_{n \in \mathbb{N}}$ converges to a if

$$\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}, \text{ such that } \forall n \geq N, d(a_n, a) < \epsilon.$$

This is a huge generalisation of Definition 1.3, and you'll meet it properly in Analysis 2A next year!

2 Hints

As per usual, here's where you'll find the problem sheet hints!

1. Use the definition! Try and follow a similar format to what we did in tutorials. Make sure to write things logically, and ensure that you've satisfied each part of the definition.
2. Again, this is an exercise in using the definition of convergence.
3. Without loss of generality, assume that $|z_1| \geq |z_2|$. The advice here is that if you ever get stuck in Analysis, either add 0, or multiply by 1 in a clever way. Try and do one of these tricks to allow you to apply the triangle inequality.
4.
 - a) This is an *if and only if* statement, so there are two things to prove! Try and manipulate the definition of convergence for one side of the \Leftrightarrow statement to resemble the other.
 - b) Recall what it means for a complex number to be in modulus-argument form.
 - c) Definition again, and follow the hint on the sheet.
 - d) I'll leave this to you! Think simple!
5. For a) and b), De Moivre's Theorem will come in handy! For c), when bounding, remember that if you make a denominator smaller, the whole fraction will get bigger.