Sharp Corner Singularity of the White–Metzner Model

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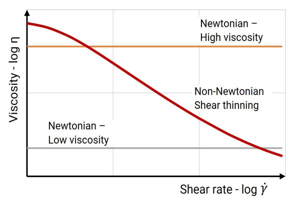
Introduction

Viscoelastic Fluids

- Many materials we encounter have viscoelastic properties.
- Characterised by a *relaxation time* — quantifies 'memory' of deformation history.
- Focus here on *polymer melts* — thermoplastics heated to temperatures above melting points.

Cabrera, X., (2020)



Østergård, A. L., (2020)

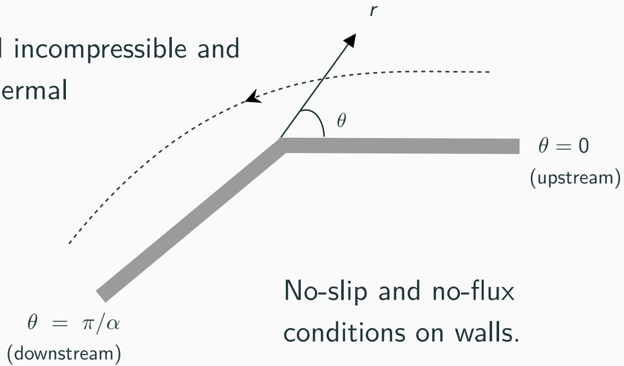
AlexRaths, (2012)

White–Metzner Flow

Problem Setup

Planar, steady flow around corner of angle $\theta = \pi/\alpha$. ($1/2 \leq \alpha < 1$)

Fluid incompressible and isothermal



No-slip and no-flux conditions on walls.

Aim: Apply asymptotic techniques to study White–Metzner fluids in region of corner apex.

Mathematics taken from Chaffin et. al. (2021), Evans, J.D. and Jones, C.A. (2024)

Governing Equations

Mass and momentum conservation:

$$\nabla \cdot \mathbf{u} = 0, \quad (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{T}.$$

White–Metzner relation:

$$\boldsymbol{T} + \dot{\gamma}^{q-1} \underbrace{\left[(\mathbf{u} \cdot \nabla)\,\boldsymbol{T} - (\nabla \mathbf{u})\,\boldsymbol{T} - \boldsymbol{T}\,(\nabla \mathbf{u})^T\right]}_{\overset{\nabla}{\boldsymbol{T}}} = \dot{\gamma}^{n-1} \underbrace{\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)}_{2\boldsymbol{D}}.$$

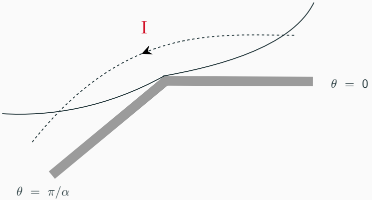
Shear rate: $\dot{\gamma} := \sqrt{2\boldsymbol{D} : \boldsymbol{D}}$

Decompose $\boldsymbol{T}$ into *natural stress variables*:

$$\boldsymbol{T} = -\dot{\gamma}^{n-q}\boldsymbol{I} + T_{uu}\mathbf{u}\mathbf{u}^T + T_{uw}\left(\mathbf{u}\mathbf{w}^T + \mathbf{w}\mathbf{u}^T\right) + T_{ww}\mathbf{w}\mathbf{w}^T.$$

White, J.L. and Metzner, A.B. (1963).

Asymptotics



Near to corners (but away from walls):

- Assume $\overset{\nabla}{\boldsymbol{T}} = \mathbf{0}$ and $\nabla p = \nabla \cdot \boldsymbol{T}$.
- Outer solution (Hinch; Evans):
  As $r \to 0$:

$$\psi \sim \frac{C_0}{\alpha^m} r^{m\alpha} \sin^m(\alpha\theta) \quad \boldsymbol{T} \sim T_{uu}(\psi)\mathbf{u}\mathbf{u}^T, \quad T_{uu,uw,ww} = d_i \left(\frac{\psi}{C_0}\right)^{m_i},$$

  with m, m_i, C_0, d_i to be determined.
- Stress singularity $\boldsymbol{T} = O\left(r^{-2(1-\alpha)}\right)$.
- Cannot match to shear behaviour near walls! Need boundary layers.

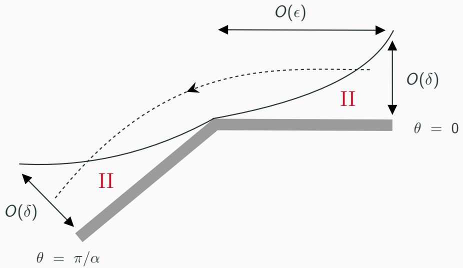
Hinch, E.J. (1993), Evans, J.D. (2008)

Asymptotics

- Focus on upstream layer $(x, y > 0)$.
- Rescale to look close to the corner: $x = \epsilon X,\ y = \delta Y$, with $0 < \delta \ll \epsilon \ll 1$.
- Dominant balance in WM equations *determines* m, m_i! Also gives

$$\delta = \epsilon^{2-\alpha+\frac{(q-n)(1-\alpha)}{n+q}}.$$

- Same results apply to the downstream layer by reorientation of axes!



Similarity Solutions

- BL analysis gives system of leading order PDEs.
- Scalings suggest layer similarity solutions in the variable

$$\chi = X^{-a} Y, \quad a = \frac{n + (3 - 2\alpha)q}{n + q}.$$

- System of PDEs in $(\psi, T_{uu}, T_{uw}, T_{ww})$ reduces to four ODEs in $(f, t_{uu}, t_{uw}, t_{ww})$!
- Boundary + Matching conditions:

$$\begin{aligned} &\text{at } \chi = 0: && f = f' = 0, && \\ &\text{as } \chi \to \infty: && f \sim C_0 \chi^m, && t_{uu} \sim d_1 \chi^{m m_1}, \\ & && t_{uw} \sim d_2 \chi^{m m_2}, && t_{ww} \sim d_3 \chi^{m m_3}. \end{aligned}$$

- Note far-field contains four unknown constants! (m, m_i known)

Similarity ODEs

Momentum:

$$0 = 2(1-\alpha)p_0 - bff' t'_{uu} + t'_{uw} + [(bm_1 + b - a)\left(f'\right)^2 - bff'']t_{uu}.$$

White–Metzner:

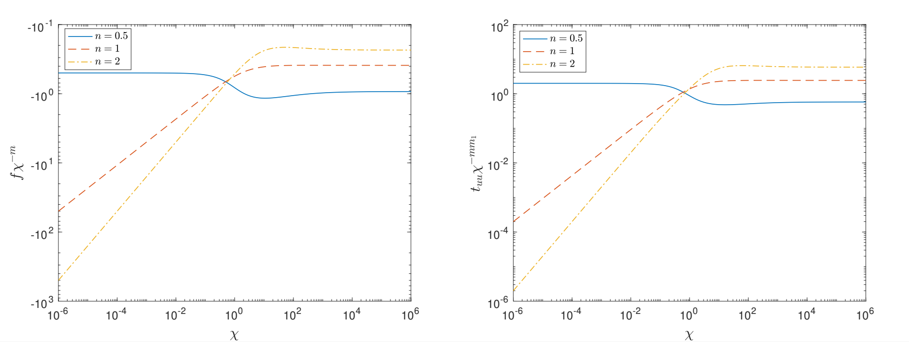
$$\begin{aligned}
&- bft'_{uu} + (bm_1 f' + |f''|^{1-q})t_{uu} - 2\frac{f''}{(f')^2}t_{uw} = 0,\\
&- bft'_{uw} + (bm_2 f' + |f''|^{1-q})t_{uw} - \frac{f''}{(f')^2}t_{ww} = 0,\\
&- bft'_{ww} + (bm_3 f' + |f''|^{1-q})t_{ww} = \\
&\qquad (f')^2|f''|^{n-q}\left[|f''|^{1-q} - (n-q)\left[(2a-b)f' + b\frac{ff''}{|f''|^2}f'''\right]\right].
\end{aligned}$$

Here, $b = (\alpha - 1 + a)m$.

Numerics

Upstream Numerics

Streamfunction (f) and stress (t_{uu}) profiles, scaled by far-field.

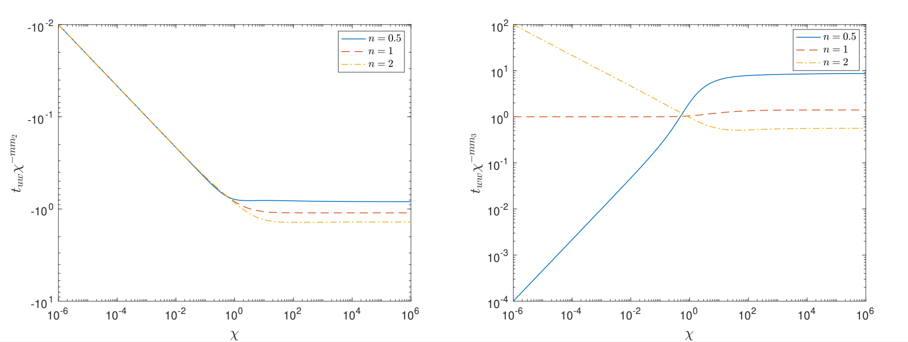


Parameters: $n = q, p_{0u} = 1, f_{2u} = -1, \alpha = 2/3$.

Solved using `ode15s` on domain $[10^{-6}, 10^{6}]$.

Upstream Numerics

Stress component (t_{uw}, t_{ww}) profiles, scaled by far-field.

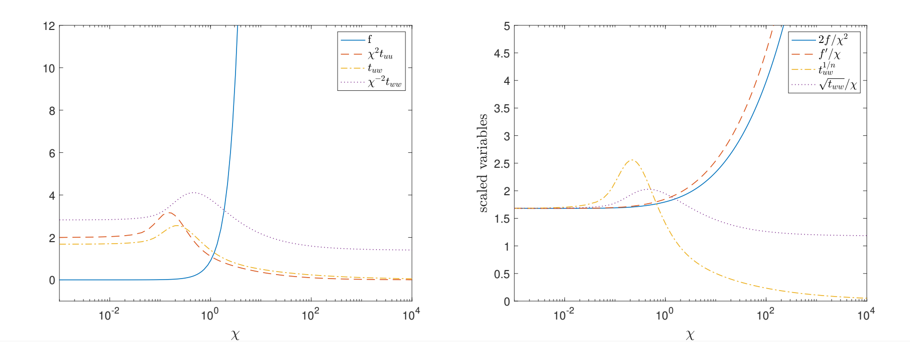


Parameters: $n = q, p_{0u} = 1, f_{2u} = -1, \alpha = 2/3$.

Solved using `ode15s` on domain $[10^{-6}, 10^{6}]$.

Downstream Numerics

Downstream variables $(f, t_{uu}, t_{uw}, t_{ww})$ and estimation of wall shear rate (right).

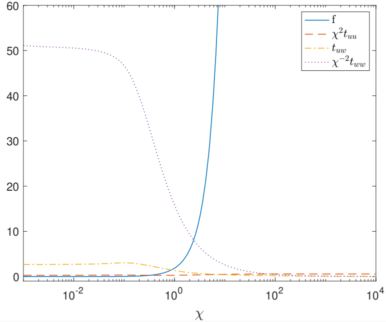


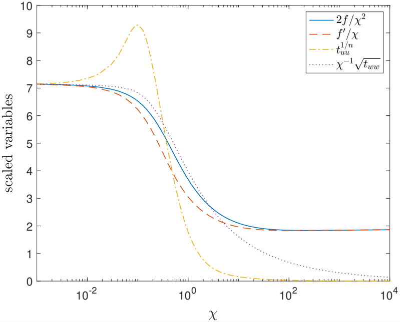
Parameters: $n = q = 1,\ p_{0u} = 1,\ f_{2u} = -1,\ \alpha = 2/3.$

Solved using `bvp4c` on domain $[10^{-3}, 10^{4}]$.

Downstream Numerics

Downstream variables $(f, t_{uu}, t_{uw}, t_{ww})$ and estimation of wall shear rate (right).





Parameters: $n = q = \frac{1}{2}$, $p_{0u} = 1$, $f_{2u} = -1$, $\alpha = 2/3$.
Solved using `bvp4c` on domain $[10^{-3}, 10^{4}]$.

Summary

For the WM model, we have

- (Re)formulated the equations in natural stress variables.
- Shown the existence of a three-region solution structure for re-entrant corner flow in natural stresses.
- Completed the flow description near to the apex via solvable similarity solutions in the upstream and downstream regions.

Thanks for Listening!

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Outer Solution:
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Outer Solution Natural Stress:
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Extra Slides

Outer Solution

- Equation $\overset{\nabla}{\boldsymbol{T}} = \mathbf{0}$ solved by $\boldsymbol{T} = T_{uu}(\psi)\mathbf{u}\mathbf{u}^T$.
- Introducing $\mathbf{v} := T_{uu}^{1/2}\mathbf{u}$, mass and momentum equations becomes

$$\nabla \cdot \mathbf{v} = 0, \quad (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla p.$$

- Incomp. means $\nabla^2 \psi^* = 0$ at leading order (ψ^* streamfn. for $\mathbf{v}$).
- Leading order solution: $\psi^* \sim C r^{\alpha} \sin(\alpha\theta)$.
- Since $\mathbf{u} \,||\, \mathbf{v}$, streamfn. for $\mathbf{u}$, $\psi^* \propto \psi^{1/m}$, $T_{uu} \propto \psi^{2(1/m-1)}$.
- Based on leading order $\boldsymbol{T}$ behaviour, assume remaining stress variables are power law at leading order.

Boundary Layer PDEs

Momentum:

$$0 = -\frac{\partial \bar{p}}{\partial X} + \left(\bar{\mathbf{u}} \cdot \bar{\nabla}\right)\left(\bar{T}_{uu}\bar{u}\right) + \frac{\partial \bar{T}_{uw}}{\partial Y},$$
$$0 = \frac{\partial \bar{p}}{\partial Y},$$

WM Stresses:

$$\bar{T}_{uu} + \left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\left(\bar{\mathbf{u}} \cdot \bar{\nabla}\right)\bar{T}_{uu} + 2\left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\bar{T}_{uw}\frac{\partial}{\partial Y}\left(\frac{1}{\bar{u}}\right) = 0,$$
$$\bar{T}_{uw} + \left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\left(\bar{\mathbf{u}} \cdot \bar{\nabla}\right)\bar{T}_{uw} + \left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\bar{T}_{ww}\frac{\partial}{\partial Y}\left(\frac{1}{\bar{u}}\right) = 0,$$
$$\bar{T}_{ww} + \left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\left(\bar{\mathbf{u}} \cdot \bar{\nabla}\right)\bar{T}_{ww} = \bar{u}^2\left(\left|\frac{\partial \bar{u}}{\partial Y}\right|^{q-1}\left(\bar{\mathbf{u}} \cdot \bar{\nabla}\right)\left|\frac{\partial \bar{u}}{\partial Y}\right|^{n-q} + \left|\frac{\partial \bar{u}}{\partial Y}\right|^{n-q}\right)$$