# Analysis 1A — Supremum Example

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October 2022

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# Question

Here is an example of finding the supremum of a set taken from an old problem sheet, together with three possible methods to find it.

**Example 0.1.**

Let
$$B = \left\{ \frac{2n-1}{n+1} \,\middle|\, n \in \mathbb{N} \right\}.$$
Show that $B$ is bounded above and find $\sup(B)$.

# Method 1 — Contradiction

**Solution.**

First, note for $n \in \mathbb{N}$:
$$\frac{2n-1}{n+1} = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence $B$ is bounded above by $2$. Therefore, by the completeness axiom, as $B \neq \emptyset$, $\sup(B)$ exists and $\sup(B) \leq 2$.

Next, suppose for contradiction that $\sup(B) < 2$. Now, for

any $x < 2$,

$$2 - \frac{3}{n+1} > x \Leftrightarrow n + 1 > \frac{3}{2-x} \Leftrightarrow n > \frac{3}{2-x} - 1.$$

Taking $x = \sup(B)$ and applying Archimedes Postulate, $\exists N \in \mathbb{N}$ such that

$$N > \frac{3}{2 - \sup(B)} - 1,$$

$$\Leftrightarrow 2 - \frac{3}{N+1} > \sup(B),$$

which is a contradiction as $2 - \frac{3}{N+1} \in B$. Hence $\sup(B) \geq 2$, and by combining our found inequalities, $\sup(B) = 2$.

# Method 2 — Alternative Characterisation of Suprema

**Solution.**

First, note for $n \in \mathbb{N}$:

$$\frac{2n-1}{n+1} = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence $B$ is bounded above by $2$. Therefore as $B \neq \emptyset$, the completeness axiom says that $\sup(B)$ exists.

We claim that $\sup(B) = 2$. **Fix** $\epsilon > 0$. **Then, for** $n \in \mathbb{N}$:

$$2 - \frac{3}{n+1} > 2 - \epsilon,$$

$$\Leftrightarrow \epsilon > \frac{3}{n+1},$$

$$\Leftrightarrow n\epsilon > 3 - \epsilon,$$

$$\Leftrightarrow n > \frac{3 - \epsilon}{\epsilon}.$$

Now, by Archimedes Postulate, $\exists N \in \mathbb{N}$ such that $N > \frac{3-\epsilon}{\epsilon}$, from which

$$2 - \frac{3}{N+1} > 2 - \epsilon.$$

At this stage, take $b = 2 - \frac{3}{N+1} \in B$. Since $\epsilon > 0$ was arbitrary, we have that $\forall \epsilon > 0, \exists b \in B$ such that $b > 2 - \epsilon$. So, by the alternative characterisation of suprema (Theorem 2.1), $\sup(B) = 2$.

# Method 3 — Limits

Note that this doesn't work in general, but it might be quicker when you can use it. It relies on the following theorem (which we'll eventually cover):

**Theorem 0.1.**
A bounded, increasing sequence $(b_n)_{n\in\mathbb{N}}$ is convergent, and its limit is given by

$$\lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\}.$$

**Solution.**
Define $b_n = \frac{2n-1}{n+1}$ for $n \in \mathbb{N}$.

**Step 1 — Show $(b_n)_n$ is bounded above**:

First, note for $n \in \mathbb{N}$:

$$b_n = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence $B$ is bounded above by $2$. Therefore, as $B \neq \emptyset$, the completeness axiom says that $\sup(B)$ exists.

**Step 2 — Show $(b_n)_n$ is increasing (i.e. show $b_{n+1} \geq$**

$b_n \ \forall n \in \mathbb{N}$ **):**

We have for $n \in \mathbb{N}$,

$$b_{n+1} - b_n = 2 - \frac{3}{n+2} - \left(2 - \frac{3}{n+1}\right),$$

$$= \frac{3(n+2) - 3(n+1)}{(n+1)(n+2)},$$

$$= \frac{3}{(n+1)(n+2)},$$

$$\geq 0.$$

So $(b_n)$ is increasing. Hence, by the above theorem, $(b_n)$ converges, and by the **Algebra of Limits**,

$$\sup(B) = \lim_{n \to \infty} b_n = \lim_{n \to \infty} \left(2 - \frac{\frac{3}{n}}{1 + \frac{1}{n}}\right) = 2,$$

as expected!