# Sharp Corner Singularity of the White–Metzner Model

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# Viscoelastic?

- Many materials we encounter have viscoelastic properties.
  - Viscosity: measure of resistance to flow.
  - Elasticity: ability of a material to return to its original state after removal of deforming forces.
- Characterised by a *relaxation time* — quantifies 'memory' of deformation history.

iStock, (2021)

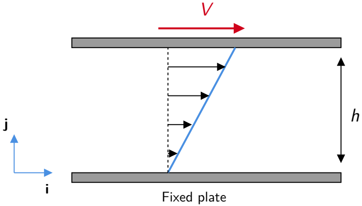
Cabrera, X., (2020)

Janine, (2009)

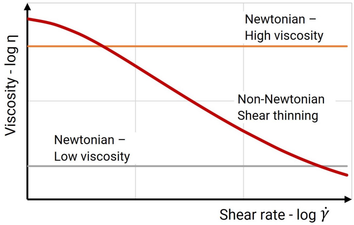
# Properties in Shearing Flow

- In (simple) shear, viscosity $\mu$ should vary with shear rate $\dot{\gamma}$.

$$\mathbf{u} = (\dot{\gamma} y, 0, 0)^T, \quad \dot{\gamma} = \frac{V}{h},$$
$$\psi = \frac{1}{2}\dot{\gamma} y^2.$$




- Material is called:
  - Shear thinning: $\mu$ decreases with increasing $\dot{\gamma}$.
  - Shear thickening: $\mu$ increases with increasing $\dot{\gamma}$.




Østergård, A. L., 2020.

# Viscoelastic Modelling

## Overview

So, how do we model these materials?

Conservation of mass and momentum:

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \nabla \cdot \boldsymbol{T}, \quad \boldsymbol{T}^T = \boldsymbol{T}.$$

How do we specify $\boldsymbol{T}$?

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How do we specify $\boldsymbol{T}$?

$$\boldsymbol{T} + \lambda(\dot{\gamma}) \overset{\nabla}{\boldsymbol{T}} = 2\mu(\dot{\gamma})\, \boldsymbol{D}$$

This is the *White-Metzner* model!

White and Metzner, (1963)

# Viscoelastic Modelling

## The White–Metzner (WM) Model

$$\boldsymbol{T} + \lambda\left(\dot{\gamma}\right) \overset{\nabla}{\boldsymbol{T}} = 2\mu\left(\dot{\gamma}\right) \boldsymbol{D}$$

Involves an *upper convected derivative*:

$$\overset{\nabla}{\boldsymbol{T}} := \underbrace{\frac{\partial \boldsymbol{T}}{\partial t}}_{\text{change at } \mathbf{x} \in \mathbb{R}^3} + \underbrace{(\mathbf{u} \cdot \nabla)\boldsymbol{T}}_{\text{motion of fluid}} \underbrace{-(\nabla \mathbf{u})\boldsymbol{T} - \boldsymbol{T}(\nabla \mathbf{u})^{T}}_{\text{rotation and stretching of frame}} .$$

Oldroyd, J. G., (1950)

# Viscoelastic Modelling

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Introduce shear rate dependence *phenomenologically* into relaxation time and viscosity.

Suppose dependence is of power-law form:

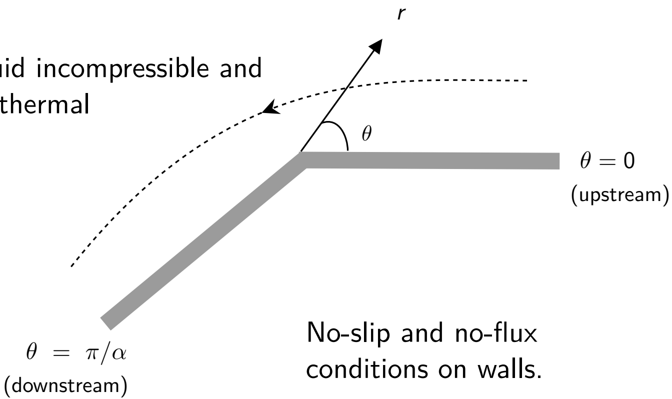
$$\lambda(\dot\gamma) = \lambda_0 \dot\gamma^{q-1}, \quad \mu(\dot\gamma) = \mu_0 \dot\gamma^{n-1},$$

with $0 < \lambda_0, \mu_0, n, q < \infty$

# Problem Setup

Planar, steady flow around corner of angle $\theta = \pi/\alpha$. $(1/2 \leq \alpha < 1)$

Fluid incompressible and isothermal




No-slip and no-flux conditions on walls.

Aim: Apply asymptotic techniques to study WM fluids in region of corner apex.

Mathematics taken from Chaffin et. al. (2021), Evans and Jones (2024)

# Governing Equations

Dimensionless equations:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathrm{Re}(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{T},$$
$$\boldsymbol{T} + \mathrm{Wi}\dot{\gamma}^{q-1} \overset{\nabla}{\boldsymbol{T}} = 2\dot{\gamma}^{n-1}\boldsymbol{D}.$$

Obtain Reynolds $(\mathrm{Re})$ and Weissenberg $(\mathrm{Wi})$ numbers:

$$\mathrm{Re} = \frac{[\text{inertial forces}]}{[\text{viscous forces}]}; \quad \mathrm{Wi} = \frac{[\text{elastic forces}]}{[\text{viscous forces}]}.$$

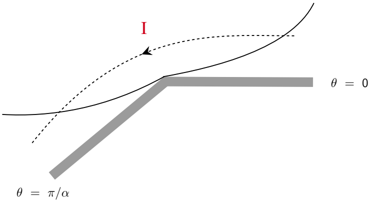
Decompose $\boldsymbol{T}$ into either *Cartesian* or *natural stress variables*:

$$\begin{aligned} \boldsymbol{T} &= T_{11}\mathbf{i}\mathbf{i}^T + T_{12}\left(\mathbf{i}\mathbf{j}^T + \mathbf{j}\mathbf{i}^T\right) + T_{22}\mathbf{j}\mathbf{j}^T, \\ &= -\dot{\gamma}^{n-q}\boldsymbol{I} + T_{uu}\mathbf{u}\mathbf{u}^T + T_{uw}\left(\mathbf{u}\mathbf{w}^T + \mathbf{w}\mathbf{u}^T\right) + T_{ww}\mathbf{w}\mathbf{w}^T. \end{aligned}$$

# Asymptotics

## Region I: Outer

Assume elasticity dominates $(\mathrm{Wi} = \infty)$:




- Leads to $\overset{\nabla}{\boldsymbol{T}} = 0$ and $\nabla p = \nabla \cdot \boldsymbol{T}$.
- Obtain a *stretching solution*: $\boldsymbol{T} = T_{uu}\mathbf{u}\mathbf{u}^T$.
- 'Velocity' $\mathbf{v} := T_{uu}^{1/2}\mathbf{u}$ governed by Euler equations

$$(\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla p.$$

- Find leading order streamfunction $\psi$ and stresses as

$$\psi \sim \frac{C_0}{\alpha^m} r^{m\alpha} \sin^m(\alpha\theta) \qquad T_{uu,uw,ww} \sim d_i \left(\frac{\psi}{C_0}\right)^{mm_i}$$

with $m, m_i, C_0, d_i$ to be determined.

# Asymptotics
## Viscometric Flow

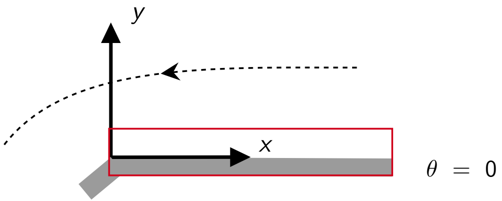
- Flow is simple shear to leading order near the walls

$$\text{No-slip/no-flux} \implies \psi = O(y^2) \text{ as } y \to 0.$$

- Cannot match outer stresses to wall behaviour:

$$\text{Outer:}\, T_{uw} = O\left(y^{mm_2}\right), \quad \text{At wall:}\, T_{uw} = O(1).$$

- Matching fails because we neglected viscosity — need *boundary layers* to recapture these effects.



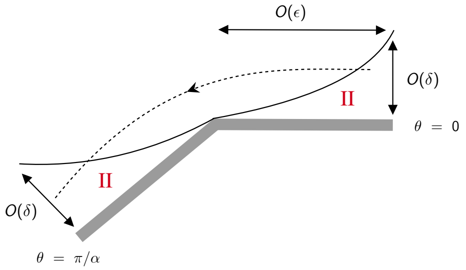
# Asymptotics

## Region II: Boundary Layers

- Focus on upstream layer $(x, y > 0)$.
- Rescale to look close to the corner: $x = \epsilon X,\ y = \delta Y$, with $0 < \delta \ll \epsilon \ll 1$.
- Dominant balance in WM equations *determines m*! Also gives

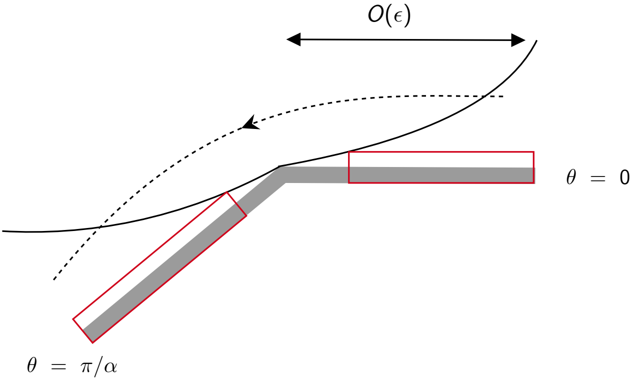
$$\delta = \epsilon^{2-\alpha+\frac{(q-n)(1-\alpha)}{n+q}}.$$

- Same results apply to the downstream layer by reorientation of axes!



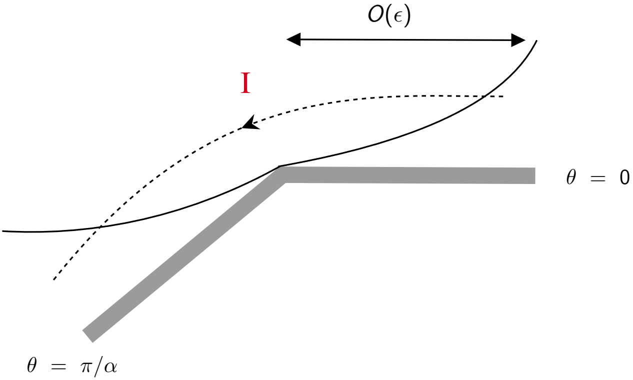
# Asymptotics

## Solution Structure



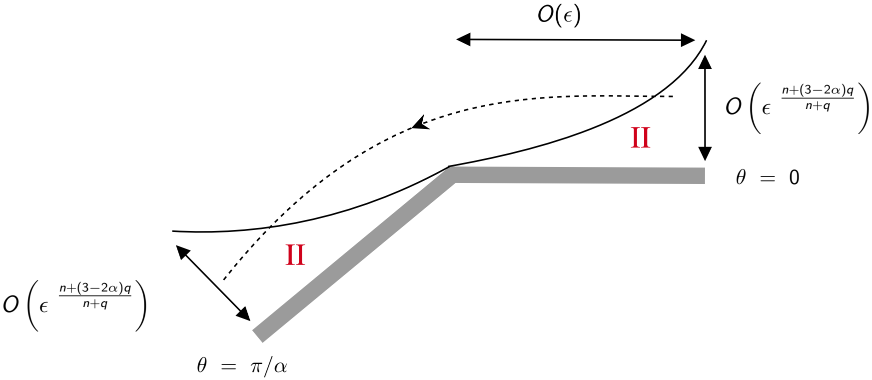

1. Flow near walls is (to leading order) simple shear.

# Asymptotics
## Solution Structure




2. Fluid behaves elastically as it flows round the corner (Region I).

# Asymptotics
## Solution Structure




3. Viscous effects remain important in Regions II. Reduced PDEs determine velocities/stresses.

# Similarity Solutions

- BL scalings suggest layer similarity solutions in the variable

$$\chi = X^{-a} Y, \quad a = \frac{n + (3 - 2\alpha)q}{n + q}.$$

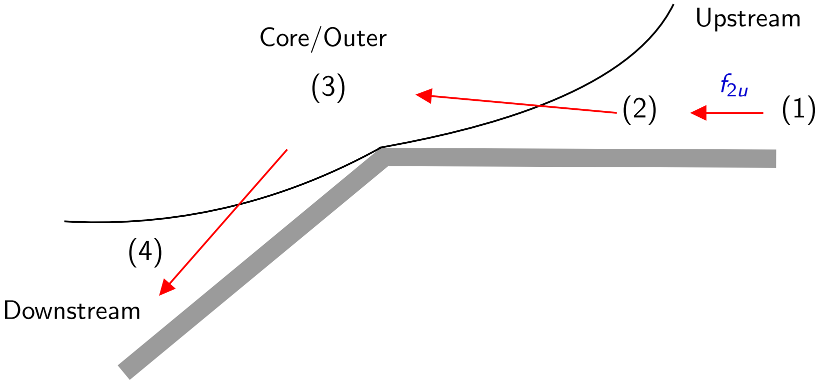
- System of PDEs in $(\psi, T_{uu}, T_{uw}, T_{ww})$ reduces to four ODEs in $(f, t_{uu}, t_{uw}, t_{ww})$!
- Boundary + Matching conditions:

$$\begin{aligned} &\text{at } \chi = 0: && f = f' = 0, && \\ &\text{as } \chi \to \infty: && f \sim C_0 \chi^{m}, && t_{uu} \sim d_1 \chi^{m m_1}, \\ & && t_{uw} \sim d_2 \chi^{m m_2}, && t_{ww} \sim d_3 \chi^{m m_3}. \end{aligned}$$

- Note far-field contains four unknown constants! ($m$, $m_i$ known)

# Solving the Equations

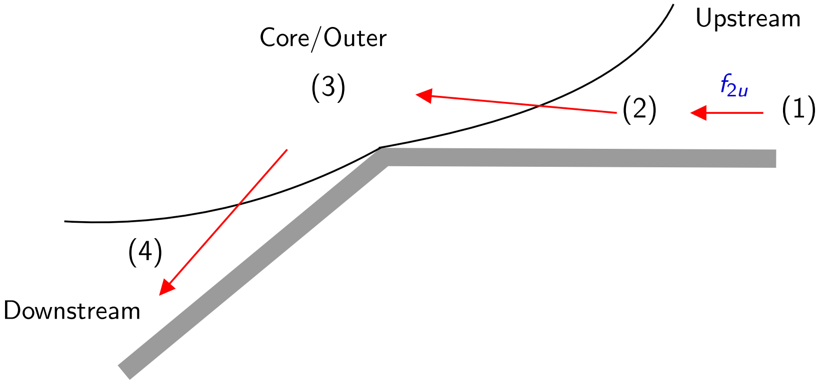
## Numerical Procedure




1. Specify flow parameters (wall shear rate $f_{2u}$ and $n$, $q$, $Wi$, $\alpha$, $p_{0u}$).

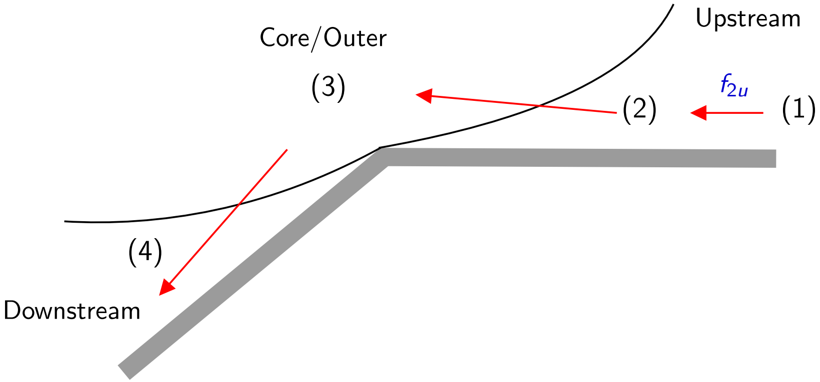
# Solving the Equations

## Numerical Procedure



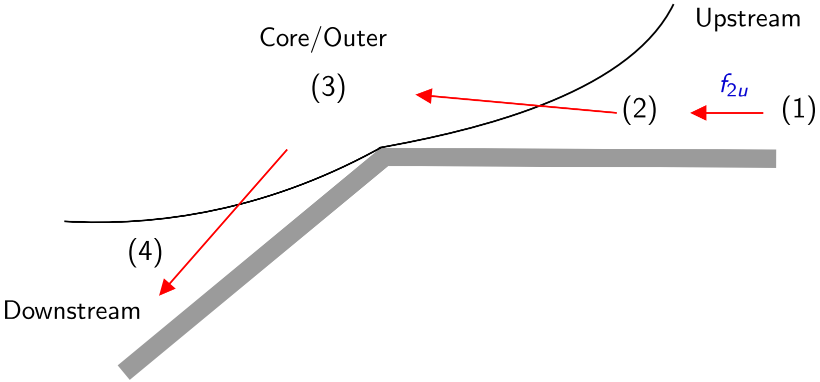

2. Solve similarity equations in upstream region as an IVP ($f_{2u}$ determines initial conditions).

# Solving the Equations
## Numerical Procedure




3. Scale IVP solution by far-field behaviour to find $C_0, d_1, d_2, d_3$ (determines outer solution).

# Solving the Equations
## Numerical Procedure

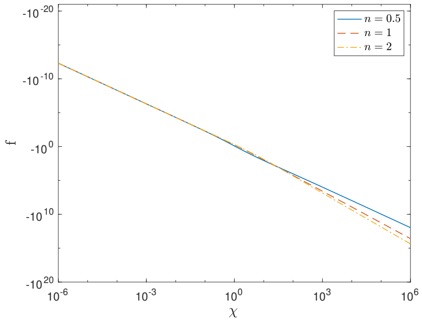


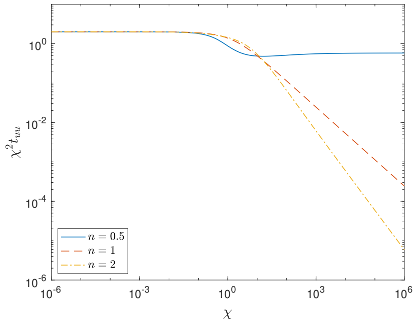

4. Solve in downstream region via a BVP (specify outer solution and wall shear as conditions)

# Solving the Equations

## Upstream Numerics

Streamfunction ($f$) and stress ($t_{uu}$) profiles.

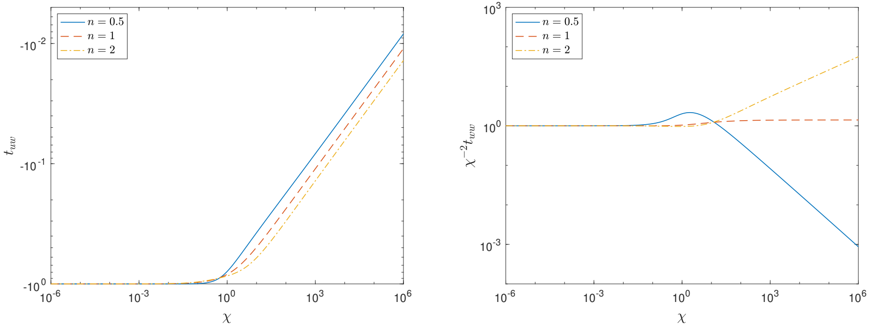



Parameters: $n = q, \mathrm{Wi} = 1, p_{0u} = 1, f_{2u} = -1, \alpha = 2/3$.
Solved using `ode15s` on domain $[10^{-6}, 10^{6}]$.

# Solving the Equations
## Upstream Numerics

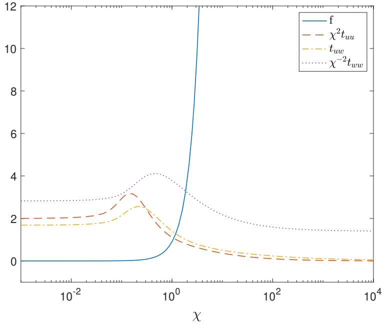
Stress component ($t_{uw}$, $t_{ww}$) profiles.

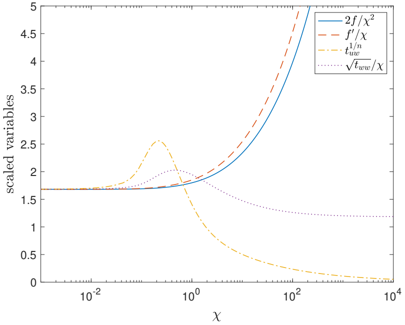



Parameters: $n = q, \mathrm{Wi} = 1, p_{0u} = 1, f_{2u} = -1, \alpha = 2/3.$
Solved using `ode15s` on domain $[10^{-6}, 10^{6}]$.

# Solving the Equations
## Downstream Numerics

Downstream variables $(f, t_{uu}, t_{uw}, t_{ww})$ and estimation of wall shear rate (right).



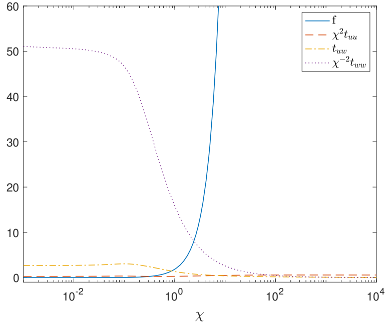



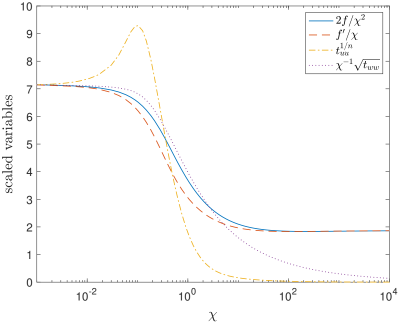

Parameters: $n = q = 1$, $p_{0u} = 1$, $f_{2u} = -1$, $\alpha = 2/3$. Solved using `bvp4c` on domain $[10^{-3}, 10^4]$.

# Solving the Equations

## Downstream Numerics

Downstream variables $(f, t_{uu}, t_{uw}, t_{ww})$ and estimation of wall shear rate (right).




Parameters: $n = q = \frac{1}{2}$, $p_{0u} = 1$, $f_{2u} = -1$, $\alpha = 2/3$. Solved using `bvp4c` on domain $[10^{-3}, 10^{4}]$.

# Summary

For the WM model, we have

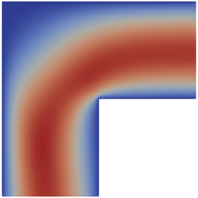
- (Re)formulated the equations in natural stress variables.
- Shown the existence of a three-region solution structure for re-entrant corner flow in natural stresses.
- Completed the flow description near to the apex via solvable similarity solutions in the upstream and downstream regions.

# Future Work

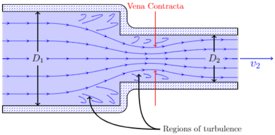
## Full Numerical Verification

- Perform *volume-of-fluid* simulations for WM in OpenFOAM (+rheoTool).
- Simulate WM in re-entrant corner geometries:

L-shaped domain

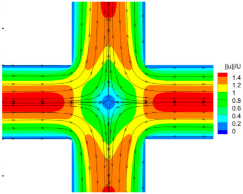


Contraction flow




Dusling, K. (n.d.)

Cross slot




Yuan et. al. (2020)

# Thank you for listening!

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