

Analysis 1A — Supplementary Paper 2020

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Contents

Introduction	1
1 Question 1	1
2 Question 2	1
3 Question 3	2
4 Question 4	4

Introduction

Here are the solutions to the past paper discussed in the revision session on XXth January 2023. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click [here](#).

1 Question 1

Example 1.1. For each of the following concepts, give an example that satisfies the definition and an example that does not. (You need not give any proofs.)

- a) A Cauchy sequence.
- b) A decreasing sequence.
- c) A sequentially continuous function.
- d) A conditionally convergent series.
- e) An interval.

Solution. a) An example is the sequence $(a_n)_{n \in \mathbb{N}}$, where $a_n = \frac{1}{n}$. A non-example is the sequence $(b_n)_{n \in \mathbb{N}}$ where $b_n = n$.

b) An example is the sequence $(a_n)_{n \in \mathbb{N}}$, where $a_n = \frac{1}{n}$. A non-example is the sequence $(b_n)_{n \in \mathbb{N}}$ where $b_n = n$.

c) An example is the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x$. A non-example is the function $g : [0, 1] \rightarrow \mathbb{R}$, where

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 0.5, \\ 1 & \text{if } 0.5 \leq x \leq 1. \end{cases}$$

d) An example is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. A non-example is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

e) An example of an interval is the set $S_1 = (0, 1)$. A non-example is the set $S_2 = (-1, 0) \cup (1, 2)$.

(If question 1 is like this in the exam, examples which can be used in more than one part will help you save time!)

2 Question 2

Example 2.1. The following statements paraphrase theorems, corollaries, propositions, or lemmas from the lectures. Identify them by their names.

a) Let $(a_n)_{n \in \mathbb{N}}$ be a real sequence. If $\sup_{n \in \mathbb{N}} |a_n| < \infty$, then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N} : (n_k \in \mathbb{N}) \wedge (n_{k+1} > n_k)$$

and there exists $B \in \mathbb{R}$ such that

$$\forall \epsilon > 0 \exists K \in \mathbb{N} \forall k \geq K : |a_{n_k} - B| < \epsilon.$$

b) Suppose that $(a_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ are two real sequences such that

$$s_n = \sum_{k=1}^{\infty} (-1)^k a_k$$

for all $n \in \mathbb{N}$. If $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$, then $(s_n)_{n \in \mathbb{N}}$ converges.

c) $\forall x \in \mathbb{R} \exists k \in \mathbb{N} : k > x$.

d) Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be two sequences such that

$$\forall n \in \mathbb{N} : x_n \leq x_{n+1} \leq y_{n+1} \leq y_n.$$

Then, there exists $a \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N} : x_n \leq a \leq y_n.$$

e) Suppose that $a \in [-1, \infty)$ and $k \in \mathbb{N}_0$. Then

$$1 + ka \leq (1 + a)^k.$$

Solution. a) This is the *Bolzano-Weierstrass* theorem.

b) This is the *Leibniz alternating series test* for series.

c) This is the *Archimedian Postulate*.

d) This is the *Nested Intervals Theorem*.

e) This is the *Binomial inequality*.

3 Question 3

Example 3.1. Let $(a_n)_{n \in \mathbb{N}}$ be a real sequence and $L \in \mathbb{R}$.

a) Show that $a_n \rightarrow \infty$ if and only if $-a_n \rightarrow -\infty$ as $n \rightarrow \infty$.

b) Assuming that $\lim_{n \rightarrow \infty} a_n = L$, show that $(|a_n|)_{n \in \mathbb{N}}$ does *not* diverge to ∞ .

c) i) Use the growth factor test to show that

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^2} = 0.$$

You may use without proof that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ exists.

ii) Show that there exists $N \in \mathbb{N}$ such that

$$n! \leq n^n \leq \left(\frac{n!}{100}\right)^2,$$

for all $n \in \mathbb{N}$.

In the following questions (d) and (e), you may use any result from the lectures without proof.

d) Find

$$\lim_{n \rightarrow \infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}).$$

e) Show that

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = 1.$$

Solution. a) We have that

$$\begin{aligned} a_n \rightarrow \infty &\iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } a_n \geq M, \\ &\iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } -a_n \leq -M. \end{aligned}$$

Setting $K = -M$ in this last statement gives

$$\begin{aligned} a_n \rightarrow \infty &\iff \forall K \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } a_n \leq K, \\ &\iff -a_n \rightarrow -\infty, \end{aligned}$$

as required.

b) We claim that $\lim_{n \rightarrow \infty} |a_n| = |L|$. To this end, fix $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} a_n = L$, we know that there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \quad \forall n \geq N.$$

Now, for all $N \geq \mathbb{N}$,

$$\begin{aligned} ||a_n| - |L|| &\leq |a_n - L|, \quad (\text{by the reverse triangle inequality}) \\ &< \epsilon \end{aligned}$$

Hence, since ϵ was arbitrary, we conclude that $\lim_{n \rightarrow \infty} |a_n| = |L|$. In particular, $(|a_n|)_{n \in \mathbb{N}}$ does not diverge to ∞ .

c) i) Setting $b_n = \frac{n^n}{(n!)^2}$ for $n \in \mathbb{N}$, we see that $b_n \geq 0$, and

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} \\ &= (n+1) \frac{(n+1)^n}{n^n} \left(\frac{n!}{(n+1)!} \right)^2 \\ &= (n+1) \left(1 + \frac{1}{n} \right)^n \frac{1}{(n+1)^2} \\ &= \frac{1/n}{1 + 1/n} \left(1 + \frac{1}{n} \right)^n.\end{aligned}$$

Since $\frac{1}{n} \rightarrow 0$, and $\left(1 + \frac{1}{n}\right)^n \rightarrow e$ as $n \rightarrow \infty$, we find by the algebra of limits that

$$\frac{b_{n+1}}{b_n} \rightarrow 0 \cdot e = 0 \text{ as } n \rightarrow \infty.$$

Since $0 < 1$, we find by the growth factor test that

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^2} = 0,$$

as required.

ii) By part i) and the definition of convergence, we know that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left| \frac{n^n}{(n!)^2} - 0 \right| = \frac{n^n}{(n!)^2} \leq \left(\frac{1}{100} \right)^2.$$

Also, note that since $n! \leq n^n$ for all $n \in \mathbb{N}$,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2}.$$

Hence, for all $n \geq N$,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2} \leq \left(\frac{1}{100} \right)^2 \iff n! \leq n^n \leq \left(\frac{n!}{100} \right)^2,$$

as required.

d) First, note that via completing the square,

$$\begin{aligned}\sqrt{n}(\sqrt{n+1} - \sqrt{n}) &= \frac{\sqrt{n}(n+1-n)}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.\end{aligned}\tag{*}$$

Now, we claim that

$$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1.$$

To show this, we fix $\epsilon > 0$ and consider for $n \in \mathbb{N}$:

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| = \frac{1 + \frac{1}{n} - 1}{\sqrt{1 + \frac{1}{n}} + 1} \leq \frac{1}{n}.$$

We then have that

$$\frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

Hence, for any $n \geq N$,

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1.$$

Returning to (*) and applying the algebra of limits, we find that as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{1}{1+1} = \frac{1}{2}.$$

e) Since $2 > 1$, we write $\sqrt[n]{2} = 1 + x_n$, where $x_n \geq 0$. This gives

$$2 = (1 + x_n)^n \geq 1 + nx_n \geq 1 \text{ by the binomial inequality.}$$

Rearranging, we find

$$0 \leq x_n \leq \frac{1}{n}.$$

Now, since $0 \rightarrow 0$ and $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, $x_n \rightarrow 0$ by the sandwich theorem. Hence, by the algebra of limits,

$$\lim_{n \rightarrow \infty} \sqrt[n]{2} = \lim_{n \rightarrow \infty} (1 + x_n) = 1 + 0 = 1,$$

as required.

4 Question 4

Example 4.1. In this question, you may use any result from the lectures without proof.

a) Let $a \in (0, 1)$. Using the theorem on the Cauchy product of series, or otherwise, show that

$$\left(\sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} (n+1)a^n.$$

b) Find the radii of convergence of the following power series.

i)

$$\sum_{n=0}^{\infty} x^n.$$

ii)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}.$$

iii)

$$\sum_{n=0}^{\infty} \left(\frac{2\pi}{13} \right)^n \cos \left(\frac{2\pi n}{13} \right) x^n.$$

c) Say whether or not the following series converge and explain your reasoning.

i)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

ii)

$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2}.$$

iii)

$$\sum_{n=1}^{\infty} \frac{1}{n \log(n)}.$$

Solution. a) Recall that for $a \in (0, 1)$, the sum is a geometric series with

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

As $|a^n| = a^n \forall n \in \mathbb{N}_0$, this series is absolutely convergent, so the Cauchy multiplication theorem gives that

$$\left(\sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} c_n,$$

where for $n \in \mathbb{N}_0$,

$$c_n = \sum_{k=0}^n a^k a^{n-k} = \sum_{k=0}^n a^n = (n+1)a^n.$$

Hence

$$\left(\sum_{n=0}^{\infty} a^n \right)^2 = \sum_{n=0}^{\infty} (n+1)a^n,$$

as required.

b) i) Writing

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} a_n x^n,$$

with $a_n = 1$, we calculate the radius of convergence, R , as

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n \rightarrow \infty} \frac{1}{1} = 1.$$

ii) Writing

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n} = \sum_{n=1}^{\infty} b_n x^n,$$

with $b_n = \frac{1}{n^n}$, we calculate

$$\limsup_{n \rightarrow \infty} |b_n|^{1/n} = \limsup_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Hence, by Cauchy-Hadamard, the radius of convergence R for this power series is $R = \infty$.

iii) Writing

$$\sum_{n=0}^{\infty} \left(\frac{2\pi}{13} \right)^n \cos \left(\frac{2\pi n}{13} \right) x^n = \sum_{n=0}^{\infty} c_n x^n,$$

with $c_n = \left(\frac{2\pi}{13} \right)^n \cos \left(\frac{2\pi n}{13} \right)$, we calculate

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} = \limsup_{n \rightarrow \infty} \frac{2\pi}{13} \left| \cos \left(\frac{2\pi n}{13} \right) \right|^{1/n}.$$

Setting $d_n = \frac{2\pi}{13} \left| \cos \left(\frac{2\pi n}{13} \right) \right|^{1/n}$, we claim that $\limsup_{n \rightarrow \infty} d_n = \frac{2\pi}{13}$. First, as for all $y \in \mathbb{R}$, $|\cos(y)| \leq 1$, we see that $d_n \leq \frac{2\pi}{13}$. This means that

$$\limsup_{n \rightarrow \infty} d_n \leq \frac{2\pi}{13}.$$

Moreover, taking the subsequence $(d_{n_k})_{k \in \mathbb{N}}$, where $n_k = 13k$, we see that as $k \rightarrow \infty$,

$$d_{13k} = \frac{2\pi}{13} \left| \cos(2\pi k) \right|^{\frac{1}{13k}} = \frac{2\pi}{13} \cdot 1 \rightarrow \frac{2\pi}{13} \quad (\text{by AoL.})$$

So, $\limsup_{n \rightarrow \infty} d_n \geq \frac{2\pi}{13}$, from which we conclude that $\limsup_{n \rightarrow \infty} d_n = \frac{2\pi}{13}$. Hence, by Cauchy-Hadamard, the radius of convergence R for this power series is given by

$$R = \left(\limsup_{n \rightarrow \infty} d_n = \frac{2\pi}{13} \right)^{-1} = \frac{13}{2\pi}.$$

c) i) Recall from lectures that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{diverges if } \alpha \leq 1, \\ \text{converges if } \alpha > 1. \end{cases}$$

Since $\sqrt{n} = n^{1/2}$, and $\frac{1}{2} < 1$, we know that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

ii) First, note that

$$\frac{n}{2^n + n^2} \leq \frac{n}{2^n}. \quad (**)$$

Now, setting $x_n = \frac{n}{2^n}$, we find

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)2^n}{2^{n+1}n} = \frac{1}{2} \left(1 + \frac{1}{n} \right).$$

Hence, by the algebra of limits,

$$\frac{|x_{n+1}|}{|x_n|} \rightarrow \frac{1}{2} < 1,$$

as $n \rightarrow \infty$. So, by d'Alembert's ratio test, we find that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \text{ converges.}$$

Finally, by the comparison test as applied to (**), we conclude that

$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2} \text{ converges.}$$

iii) Setting $y_n = \frac{1}{n \log(n)}$, we define for $k \geq 1$,

$$z_k := 2^k y_{2^k} = \frac{2^k}{2^k \log(2^k)} = \frac{1}{k \log(2)}.$$

Using the result stated in part i), we know that as $\sum_{k=1}^{\infty} k$ diverges, $\sum_{k=1}^{\infty} z_k$ diverges. Hence, by the Cauchy condensation test, the given series $\sum_{n=1}^{\infty} y_n$ diverges.