

Analysis 1A — Tutorial 3

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1 Lecture Recap

1.1 Suprema/Infima

There's still a little bit of material to cover regarding the supremum and infimum of a set. To begin, we re-cover the definitions from last week.

Definition 1.1. Let $S \in \mathbb{R}$. A number $T \in \mathbb{R}$ is said to be the *supremum* of S if it is an *upper bound* for S , and for any other upper bound M , $T \leq M$. Here, we write $T = \sup(S)$.

Definition 1.2. Let $S \in \mathbb{R}$. A number $t \in \mathbb{R}$ is said to be the *infimum* of S if it is a *lower bound* for S , and for any other lower bound m , $t \geq m$. Here, we write $t = \inf(S)$.

It also turns out that there's an alternative characterisation of suprema and infima, which turns out to be very useful, especially if the members of a set aren't indexed by natural numbers. Letting $S \subseteq \mathbb{R}$ be a set, we have that

- A number $T \in \mathbb{R}$ is the *supremum* of S , denoted $\sup(S)$ if:

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s > T - \epsilon.$$

- Similarly, a number $t \in \mathbb{R}$ is the *infimum* of S , denoted $\inf(S)$ if:

$$\forall \epsilon > 0, \exists s \in S \text{ such that } s < t + \epsilon.$$

As an example, take the set $S = (-1, 2] = \{x \mid -1 < x \leq 2\}$, and fix some $\epsilon > 0$. Then, if we take $s_1 = 2 - \epsilon/2$ and $s_2 = -1 + \epsilon/2$, we see that

1. s_1 and s_2 are in the set S ,
2. $s_1 > 2 - \epsilon$, and
3. $s_2 < -1 + \epsilon$.

Hence, as ϵ was arbitrary, the alternative characterisation of suprema and infima says that $\sup(S) = 2$ and $\inf(S) = -1$.

1.2 Inequalities

Inequalities come up everywhere in maths! For example, they can be used in statistics for estimation (Markov/Chebyshev inequalities), they can be used as constraints in optimisation problems (see Section 3.1 of [this Wikipedia link](#).), and quite famously appear in Quantum Mechanics. This latter case is known as the [Heisenberg Uncertainty Principle](#),

and states that you can't simultaneously know the position and momentum of a quantum particle, such as an electron.

Most of the inequalities in this course will be based on the absolute value, which is defined as follows:

Definition 1.3. For $x \in \mathbb{R}$, the *absolute value* of x is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases} = \max\{x, -x\}.$$

This absolute value has the following properties — for $x, y \in \mathbb{R}$:

$$x \leq |x|, \quad -x \leq |x|, \quad |-x| = |x| \quad \text{and} \quad |xy| = |x||y|.$$

Now we come on to what I consider to be the most important thing in this course — the *triangle inequality*. For all $x, y \in \mathbb{R}$:

$$|x + y| \leq |x| + |y|.$$

Why do I think this is so important? This will come up in almost any course you take at university that uses analysis! If you're studying vector calculus, fluid mechanics, statistics, probability, or anything that's not abstract algebra, there's guaranteed to be a proof or technique which involves an inequality of this form! So if you only learn one thing from Analysis 1, make it this.

There's also a *reverse triangle inequality*, which states for $x, y \in \mathbb{R}$:

$$||x| - |y|| \leq |x - y|.$$

This will also come in handy, especially for the rest of this unit!

Finally, there's one more inequality to mention — the binomial inequality. This says that $\forall n \in \mathbb{N}_0$ (i.e. all the natural numbers with 0), and $\forall x \geq -1$,

$$(1 + x)^n \geq 1 + nx.$$

2 Hints

As per usual, here's where you'll find the problem sheet hints!

H1. Take cases on x .

H2. You should only need the definitions given in lectures to solve this question. Make sure to write things logically!

H3. Without loss of generality (WLOG), consider $x \geq y$ (otherwise you can just swap them), and consider $|\sqrt{x} - \sqrt{y}|^2$. On expanding, try and find a bound for the 'middle' term.

H4. Solve the modulus equation, and then use your solutions to formulate simultaneous equations for c and r .