Analysis 1A — Supremum Example

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Contents

Question

Here is an example of finding the supremum of a set taken from an old problem sheet, together with three possible methods to find it.

Example 0.1. Let

$$B = \left\{ \frac{2n-1}{n+1} | n \in \mathbb{N} \right\}.$$

Show that B is bounded above and find $\sup(B)$.

Method 1 — Contradiction

Solution. First, note for $n \in \mathbb{N}$:

$$\frac{2n-1}{n+1} = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence B is bounded above by 2. Therefore, by the completeness axiom, as $B \neq \emptyset$, $\sup(B)$ exists and $\sup(B) \leq 2$.

Next, suppose for contradiction that $\sup(B) < 2$. Now, for any $x < 2$,

$$2 - \frac{3}{n+1} > x \Leftrightarrow n+1 > \frac{3}{2-x} \Leftrightarrow n > \frac{3}{2-x} - 1.$$

Taking $x = \sup(B)$ and applying Archimedes' Postulate, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} N &> \frac{3}{2-\sup(B)} - 1, \\ \Leftrightarrow 2 - \frac{3}{N+1} &> \sup(B), \end{aligned}$$

which is a contradiction as $2 - \frac{3}{N+1} \in B$. Hence $\sup(B) \geq 2$, and by combining our found inequalities, $\sup(B) = 2$.

Method 2 — Alternative Characterisation of Suprema

Solution. First, note for $n \in \mathbb{N}$:

$$\frac{2n-1}{n+1} = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence B is bounded above by 2. Therefore as $B \neq \emptyset$, the completeness axiom says that $\sup(B)$ exists.

We claim that $\sup(B) = 2$. Fix $\epsilon > 0$. Then, for $n \in \mathbb{N}$:

$$\begin{aligned} 2 - \frac{3}{n+1} &> 2 - \epsilon, \\ \Leftrightarrow \epsilon &> \frac{3}{n+1}, \\ \Leftrightarrow n\epsilon &> 3 - \epsilon, \\ \Leftrightarrow n &> \frac{3-\epsilon}{\epsilon}. \end{aligned}$$

Now, by Archimedes' Postulate, $\exists N \in \mathbb{N}$ such that $N > \frac{3-\epsilon}{\epsilon}$, from which

$$2 - \frac{3}{N+1} > 2 - \epsilon.$$

At this stage, take $b = 2 - \frac{3}{N+1} \in B$. Since $\epsilon > 0$ was arbitrary, we have that $\forall \epsilon > 0, \exists b \in B$ such that $b > 2 - \epsilon$. So, by the alternative characterisation of suprema (Theorem 2.1), $\sup(B) = 2$.

Method 3 — Limits

Note that this doesn't work in general, but it might be quicker when you can use it. It relies on the following theorem (which we'll eventually cover):

Theorem 0.1. *A bounded, increasing sequence $(b_n)_{n\in\mathbb{N}}$ is convergent, and its limit is given by*

$$\lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\}.$$

Solution. Define $b_n = \frac{2n-1}{n+1}$ for $n \in \mathbb{N}$.

Step 1 — Show $(b_n)_n$ is bounded above:

First, note for $n \in \mathbb{N}$:

$$b_n = \frac{2n+2-3}{n+1} = 2 - \frac{3}{n+1} < 2.$$

Hence B is bounded above by 2. Therefore, as $B \neq \emptyset$, the completeness axiom says that $\sup(B)$ exists.

Step 2 — Show $(b_n)_n$ is increasing (i.e. show $b_{n+1} \geq b_n \ \forall n \in \mathbb{N}$):

We have for $n \in \mathbb{N}$,

$$\begin{aligned} b_{n+1} - b_n &= 2 - \frac{3}{n+2} - \left(2 - \frac{3}{n+1}\right), \\ &= \frac{3(n+2) - 3(n+1)}{(n+1)(n+2)}, \\ &= \frac{3}{(n+1)(n+2)}, \\ &\geq 0. \end{aligned}$$

So (b_n) is increasing. Hence, by the above theorem, (b_n) converges, and by the *Algebra of Limits*,

$$\sup(B) = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(2 - \frac{\frac{3}{n}}{1+\frac{1}{n}}\right) = 2,$$

as expected!