Analysis 1B — Epsilon-Delta Example

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Introduction

Here is an extra example of finding the limit of a function using the definition. This should hopefully give you a guide to the techniques required, and how much detail you should put in your solutions.

1 Worked Example

Question. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $\lim_{x \to 1} f(x) = 1$.

Solution. Fix $\epsilon > 0$, and suppose that $0 < |x - 1| < \delta$ for some $\delta > 0$ to be chosen later. Without loss of generality, suppose that $\delta \leq 1$. Then

$$\begin{aligned} |f(x) - 1| &= \left| \frac{1}{x^2} - 1 \right|, \\ &= \left| \frac{1 - x^2}{x^2} \right|, \\ &= \frac{|x-1||x+1|}{|x|^2}. \end{aligned}$$

Now, by the triangle inequality, we have that

$$|x + 1| = |x - 1 + 2| \leq |x - 1| + 2.$$

Also, by the reverse triangle inequality,

$$|x| = |x - 1 + 1| \geq 1 - |x - 1|.$$

So, if $\delta \leq \frac{1}{2}$, we obtain $|x + 1| < \frac{5}{2}$, $|x| > \frac{1}{2}$, and

$$|f(x) - 1| < \frac{5/2|x-1|}{(1/2)^2} = 10|x - 1| < 10\delta.$$

Hence, if $\delta = \min\{1, 1/2, \epsilon/10\}$, we find that

$$0 < |x - 1| < \delta \Longrightarrow |f(x) - 1| < \epsilon$$

Finally, since ϵ was arbitrary, we conclude that $\lim_{x \to 1} f(x) = 1$, as required.