Analysis 1B — Tutorial 5

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Introduction

Here is the material to accompany the 5th Analysis 1B Tutorial on the 6th March. Alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to Christian Jones (caj50). To return to the homepage, click here.

1 Lecture Recap

After what was mainly revision last week, we're moving onto some new stuff again! It turns out there's still a bit we can say about continuity, especially on compact intervals. Finally, we're going to look at differentiation, which gives us a way of describing how fast a function changes.

1.1 Inverse Functions

A particularly useful class of functions we may be interested in are known as invertible. These functions $f : A \to B$ provide a way of moving between sets A and B (and back again) without losing any information about A and B. Before we talk about them in more detail, it's worth recalling some definitions:

Definition 1.1 (Injectivity, Surjectivity and Bijectivity)**.** Let $f : A \to B$ be a function.

- If $\forall x, y \in B$ with $x \neq y$, $f(x) = f(y) \implies x = y$, then f is said to be injective.
- If $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$, then f is surjective.
- If f is both injective and surjective, then it is called bijective.

In words, bijectivity means that for a function $f : A \to B$, every element in the codomain B is mapped to by a unique element in the domain A. These bijective functions are said to be invertible, that is, there exists an inverse function $f^{-1} : B \to A$ such that $f^{-1} \circ f$ and $f \circ f^{-1}$ produce the identity maps on A and B respectively.

Now that we have these definitions, we can say something about the continuity of inverse functions:

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ be a non-empty*[1] *interval, and let $f : I \to \mathbb{R}$ be continuous on I. Assume that f is strictly increasing*[2] *(or strictly decreasing) on I. Then:*

- *$J := f(I)$ is an interval,*
- *$f : I \to J$ is bijective, and*
- *$f^{-1} : J \to I$ is continuous on J.*

You've seen an example of this theorem in action in the lectures. This is repeated below, as we're going to use it to prove a powerful result regarding sequences.

Example 1.1. Consider the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x.$$

Firstly, note that $\mathbb{R}$ is a non-empty interval. Now, using some results from Semester 1, we know that

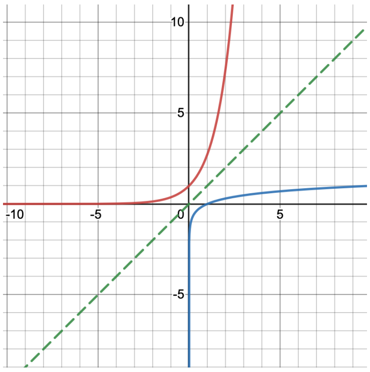
- exp is continuous and strictly increasing on $\mathbb{R}$, and
- $\exp(\mathbb{R}) = (0, \infty)$.

[1] This is so we can talk about surjectivity.

[2] In other words, for all $x, y \in I$ with $x < y$, $f(x) < f(y)$.

Therefore, exp satisfies the hypotheses of the above theorem, and so $\exp : \mathbb{R} \to (0, \infty)$ is a bijection, with continuous inverse. This inverse function is the well-known *natural logarithm* $\ln : (0, \infty) \to \mathbb{R}$, where $x = \ln(y) \iff y = \exp(x)$.

We can plot the graphs of $y = \exp(x)$ (in red), and $y = \ln(x)$ (in blue) to visually see that Theorem 1.1 works. Also note that to plot the graph of an inverse function, we only need to reflect the graph of the original function through the line $y = x$ (dashed green line).



Now that we have this example, we can easily calculate another large class of sequence limits:

Proposition 1.2. *Let $(a_n)_n$ and $(b_n)_n$ be real sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. If $a_n^{b_n} \in \mathbb{R}\ \ \forall n \in \mathbb{N}$, and $a > 0$, then*

$$\lim_{n\to\infty} a_n^{b_n} = \left(\lim_{n\to\infty} a_n\right)^{\lim_{n\to\infty} b_n}.$$

Proof

Proof. Since $\lim_{n\to\infty} a_n = a > 0$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|a_n - a| < \frac{a}{2} \iff \frac{a}{2} < a_n < \frac{3a}{2}.$$

In particular, $a_n > 0$ for all $n \geq N$.

Now, for $n \geq N$,

$$\begin{aligned} a_n^{b_n} &= \exp\left(\ln\left(a_n^{b_n}\right)\right) \quad (\text{as } a_n > 0), \\ &= \exp\left(b_n \ln\left(a_n\right)\right) \quad (\text{properties of } \ln). \end{aligned}$$

So as $n \to \infty$, we have that as both exp and ln are continuous (Example 1.1),

$$\begin{aligned} a_n^{b_n} &\to \exp\left(b \ln(a)\right) \\ &= \exp\left(\ln\left(a^b\right)\right), \\ &= a^b. \end{aligned}$$

□

1.2 Weierstrass Extremal Theorem

Much like the Intermediate Value Theorem, we can obtain some special continuity results when our functions are defined on compact (i.e. closed and bounded) intervals. One of the main results from this week is stated below:

Theorem 1.3 (Weierstrass Extremal Theorem (WET))**.** *Let $a, b \in \mathbb{R}$ with $a < b$, and let*[3] *$f \in C^0([a, b])$. Then:*

[3]Recall that $C^0([a, b])$ is the set of continuous functions mapping from the set $[a, b]$.

1. *f is bounded:*
$$\exists M > 0 \ \ s.t. \ \ |f(x)| \leq M \ \ \forall x \in [a, b].$$

2. *f attains its bounds:*
$$\exists p, q \in [a, b] \ \ s.t \ \ \forall x \in [a, b], f(q) \leq f(x) \leq f(p).$$

This last point states that if $f \in C^0([a, b])$,

$$\sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x) \ \text{ and } \ \inf_{x\in[a,b]} f(x) = \min_{x\in[a,b]} f(x).$$

So in fact, what this theorem tells us is that for a function defined on a compact interval, we have some control on its growth, and we know that the function has a maximum and minimum value! This can be seen pictorally in Figure 1.

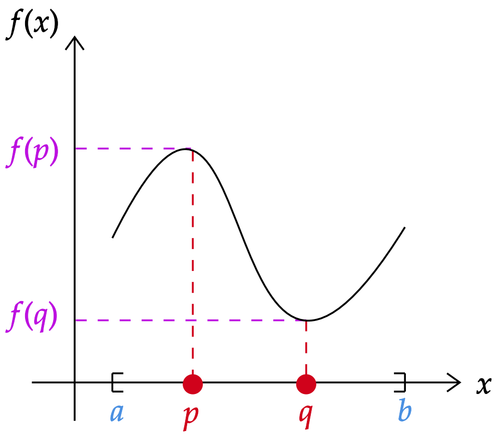


Figure 1: This function f is continuous on $[a, b]$, so by the Weierstrass Extremal Theorem, f is bounded on $[a, b]$. Also, we see that there exist points p and q in the domain at which f achieves its maximum and minimum values.

1.3 Differentiation

While functions are very good at describing physical quantities such as temperature, density or momentum we can usually gain more insight into these variables by studying how fast they change at a given position or time. Mathematically, we study rates of change using derivatives, which again relies on the ideas behind limits!

Definition 1.2 (Derivative)**.** Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ is an open set, and let $c \in D$. Then, if $\exists L \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(c + h) - f(c)}{h} = L,$$

we say that f is differentiable at c, and call L the derivative of f at c.

We can note a few things here:

- Firstly, if this L exists, we write it as $f'(c)$ to make it clear that its a derivative.

- We require D to be open, so that we can actually take limits! If, for example, $D = [-1, 2]$, we could attempt to define the derivative at any point in the interior of D, $D^\circ = (-1, 2)$, but we couldn't define the derivative at $x = -1$ or $x = 2$.[4]
- Substituting $x = c + h$ into the definition gives us an equivalent definition: f is differentiable at c if there exists $L \in \mathbb{R}$ such that
$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L.$$

One quick result we obtain from this definition is the following:

Proposition 1.4. *If a function $f : D \to \mathbb{R}$ is differentiable at a point c, then it is continuous at c.*

The contrapositive of this is very useful for ruling functions out: if a function is **not** continuous, it is not differentiable. As a final remark, or warning, **continuity does not imply differentiability**! To see this, think of either $f(x) = |x|$ at $x = 0$, or look up the Weierstrass function.

2 Hints

As per usual, here's where you'll find the problem sheet hints!

1) This one is largely similar to the one that was covered in tutorials — you just need to be a bit more careful when verifying the hypothesis of the theorem involving inverse functions. When proving bijectivity, you can use results from tutorial question 1 to help too!

2) i) The question you're trying to answer here is does $\max_{[a,x]} f(x)$ exist?
   ii) For $x \leq y$, $[a, x] \subseteq [a, y]$.
   iii) This is a bit tricky[5]. Consider the case $f(c) < g(c)$ first, and use inertia to show that $\exists \delta > 0$ such that $g(x) = g(c)$ on some interval. For the case $f(c) = g(c)$, recall that continuity of f says that for any $\epsilon > 0$, there exists a $\delta > 0$ such that
$$|x - c| < \delta \implies -\frac{\epsilon}{2} < f(x) - f(c) < \frac{\epsilon}{2}.$$
   Using each side of this inequality in turn, the definition of g, and part ii), you need to show that for $|x - c| < \delta$, we have
$$g(x) - g(c) > -\frac{\epsilon}{2} \quad \text{and} \quad g(x) \leq g(c) + \frac{\epsilon}{2}.$$
   Combine these inequalities to then prove continuity of g.

[4] There is nothing stopping us; however, trying to define *left* and *right derivatives* at these points, i.e. we could search for
$$\lim_{x \to -1^+} \frac{f(-1 + h) - f(-1)}{h} \quad \text{or} \quad \lim_{x \to 2^-} \frac{f(2 + h) - f(2)}{h}.$$

[5] Alternatively, you could try and find left and right limits at the point c, using (some variations of) a result from Problem Sheet 3. Note that this way involves three main cases: $c = a$, $c = b$, or c is in (a, b). (There's also a fourth case when $a = b$ and f is defined at a single point, but then g is automatically continuous.)