Analysis 1A — Supplementary Paper 2020

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Contents

Introduction

Here are the solutions to the past paper discussed in the revision session on 9th January 2023. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click here.

Question 1

Question.
For each of the following concepts, give an example that satisfies the definition and an example that does not. (You need not give any proofs.)

a) A Cauchy sequence.
b) A decreasing sequence.
c) A sequentially continuous function.
d) A conditionally convergent series.
e) An interval.

Solution. a) An example is the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \frac{1}{n}$. A non-example is the sequence $(b_n)_{n\in\mathbb{N}}$ where $b_n = n$.

b) An example is the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \frac{1}{n}$. A non-example is the sequence $(b_n)_{n\in\mathbb{N}}$ where $b_n = n$.

c) An example is the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$. A non-example is the function $g : [0,1] \to \mathbb{R}$, where

$$g(x) = \begin{cases} 0 \text{ if } 0 \leq x < 0.5, \\ 1 \text{ if } 0.5 \leq x \leq 1. \end{cases}$$

d) An example is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. A non-example is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

e) An example of an interval is the set $S_1 = (0,1)$. A non-example is the set $S_2 = (-1,0) \cup (1,2)$.

(If question 1 is like this in the exam, examples which can be used in more than one part will help you save time!)

Question 2

Question.

The following statements paraphrase theorems, corollaries, propositions, or lemmas from the lectures. Identify them by their names.

a) Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If $\sup_{n\in\mathbb{N}}|a_n| < \infty$, then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}:\ (n_k \in \mathbb{N}) \wedge (n_{k+1} > n_k)$$

and there exists $B \in \mathbb{R}$ such that

$$\forall \epsilon > 0\, \exists K \in \mathbb{N}\, \forall k \geq K:\ |a_{n_k} - B| < \epsilon.$$

b) Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ are two real sequences such that

$$s_n = \sum_{n=1}^{\infty}(-1)^k a_k$$

for all $n \in \mathbb{N}$. If $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ and $a_n \to 0$ as $n \to \infty$, then $(s_n)_{n\in\mathbb{N}}$ converges.

c) $\forall x \in \mathbb{R}\, \exists k \in \mathbb{N}:\ k > x.$

d) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences such that

$$\forall n \in \mathbb{N} : x_n \leq x_{n+1} \leq y_{n+1} \leq y_n.$$

Then, there exists $a \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}:\ x_n \leq a \leq y_n.$$

e) Suppose that $a \in [-1\, \infty)$ and $k \in \mathbb{N}_0$. Then

$$1 + ka \leq (1 + a)^k.$$

Solution. a) This is the **Bolzano-Weierstrass** theorem.

b) This is the **Leibniz alternating series test** for series.

c) This is the **Archimedian Postulate**.

d) This is the **Nested Intervals Theorem**.

e) This is the **Binomial inequality**.

Question 3

Question.

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence and $L \in \mathbb{R}$.

a) Show that $a_n \to \infty$ if and only if $-a_n \to -\infty$ as $n \to \infty$.

b) Assuming that $\lim_{n\to\infty} a_n = L$, show that $(|a_n|)_{n\in\mathbb{N}}$ does **not** diverge to ∞.

c) i) Use the growth factor test to show that

$$\lim_{n\to\infty} \frac{n^n}{(n!)^2} = 0.$$

You may use without proof that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists.

ii) Show that there exists $N \in \mathbb{N}$ such that

$$n! \le n^n \le \left(\frac{n!}{100}\right)^2,$$

for all $n \in \mathbb{N}$.

In the following questions (d) and (e), you may use any result from the lectures without proof.

d) Find

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}).$$

e) Show that

$$\lim_{n\to\infty} \sqrt[n]{2} = 1.$$

Solution. a) We have that

$$a_n \to \infty \iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } a_n \geq M,$$

$$\iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } -a_n \leq -M.$$

Setting $K = -M$ in this last statement gives

$$a_n \to \infty \iff \forall K \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } -a_n \leq K,$$

$$\iff -a_n \to -\infty,$$

as required.

b) We claim that $\lim_{n\to\infty}|a_n| = |L|$. To this end, fix $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, we know that there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \ \forall n \geq N.$$

Now, for all $n \geq N$,

$$||a_n| - |L|| \leq |a_n - L|, \text{ (by the reverse triangle inequality)}$$

$$< \epsilon$$

Hence, since ϵ was arbitrary, we conclude that $\lim_{n\to\infty}|a_n| = |L|$. In particular, $(|a_n|)_{n\in\mathbb{N}}$ does not diverge to ∞.

c) i) Setting $b_n = \frac{n^n}{(n!)^2}$ for $n \in \mathbb{N}$, we see that $b_n \geq 0$, and

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^{n+1}}{((n+1)!)^2}\frac{(n!)^2}{n^n}$$

$$= (n+1)\frac{(n+1)^n}{n^n}\left(\frac{n!}{(n+1)!}\right)^2$$

$$= (n+1)\left(1+\frac{1}{n}\right)^n \frac{1}{(n+1)^2}$$

$$= \frac{1/n}{1+1/n}\left(1+\frac{1}{n}\right)^n.$$

Since $\frac{1}{n} \to 0$, and $\left(1+\frac{1}{n}\right)^n \to \mathrm{e}$ as $n \to \infty$, we find by the algebra of limits that

$$\frac{b_{n+1}}{b_n} \to 0 \cdot \mathrm{e} = 0 \text{ as } n \to \infty.$$

Since $0 < 1$, we find by the growth factor test that

$$\lim_{n\to\infty} \frac{n^n}{(n!)^2} = 0,$$

as required.

ii) By part i) and the definition of convergence, we know that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{n^n}{(n!)^2} - 0\right| = \frac{n^n}{(n!)^2} \leq \left(\frac{1}{100}\right)^2.$$

Also, note that since $n! \leq n^n$ for all $n \in \mathbb{N}$,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2}.$$

Hence, for all $n \geq N$,

$$\frac{1}{n!} \leq \frac{n^n}{(n!)^2} \leq \left(\frac{1}{100}\right)^2 \iff n! \leq n^n \leq \left(\frac{n!}{100}\right)^2,$$

as required.

d) First, note that via completing the square,

$$\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}(n+1-n)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}. \tag{*}$$

Now, we claim that

$$\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1.$$

To show this, we fix $\epsilon > 0$ and consider for $n \in \mathbb{N}$:

$$\left|\sqrt{1+\frac{1}{n}} - 1\right| = \frac{1+\frac{1}{n}-1}{\sqrt{1+1/n}+1} \leq \frac{1}{n}.$$

We then have that

$$\frac{1}{n} < \epsilon \Longleftrightarrow n > \frac{1}{\epsilon}.$$

Hence, for any $n \geq N,$

$$\left|\sqrt{1+\frac{1}{n}}-1\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Therefore,

$$\lim_{n\to\infty}\sqrt{1+\frac{1}{n}} = 1.$$

Returning to (*) and applying the algebra of limits, we find that as $n \to \infty$,

$$\lim_{n\to\infty}\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = \frac{1}{1+1} = \frac{1}{2}.$$

e) Since $2 > 1$, we write $\sqrt[n]{2} = 1 + x_n,$ where $x_n \geq 0$. This gives

$$2 = (1+x_n)^n \geq 1 + nx_n \text{ (by the binomial inequality)}$$

$$\geq 1.$$

Rearranging, we find

$$0 \leq x_n \leq \frac{1}{n}.$$

Now, since $0 \to 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$, $x_n \to 0$ by the sandwich theorem. Hence, by the algebra of limits,

$$\lim_{n\to\infty}\sqrt[n]{2} = \lim_{n\to\infty}(1+x_n) = 1 + 0 = 1,$$

as required.

Question 4

Question.

In this question, you may use any result from the lectures without proof.

a) Let $a \in (0,1)$. Using the theorem on the Cauchy product of series, or otherwise, show that

$$\left(\sum_{n=0}^{\infty} a^n\right)^2 = \sum_{n=0}^{\infty}(n+1)a^n.$$

b) Find the radii of convergence of the following power series.

i)
$$\sum_{n=0}^{\infty} x^n.$$

ii)
$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}.$$

iii)
$$\sum_{n=0}^{\infty} \left(\frac{2\pi}{13}\right)^n \cos\left(\frac{2\pi n}{13}\right) x^n.$$

c) Say whether or not the following series converge and explain your reasoning.

i)
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

ii)
$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2}.$$

iii)
$$\sum_{n=1}^{\infty} \frac{1}{n \log(n)}.$$

Solution. a) Recall that for $a \in (0,1)$, the sum is a geometric series with

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

As $|a^n| = a^n \, \forall n \in \mathbb{N}_0$, this series is absolutely convergent, so the Cauchy multiplication theorem gives that

$$\left(\sum_{n=0}^{\infty} a^n\right)^2 = \sum_{n=0}^{\infty} c_n,$$

where for $n \in \mathbb{N}_0,$

$$c_n = \sum_{k=0}^{n} a^k a^{n-k} = \sum_{k=0}^{n} a^n = (n+1)a^n.$$

Hence

$$\left(\sum_{n=0}^{\infty} a^n\right)^2 = \sum_{n=0}^{\infty} (n+1)a^n,$$

as required.

b) i) Writing

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} a_n x^n,$$

with $a_n = 1$, we calculate the radius of convergence, R, as

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{1}{1} = 1.$$

ii) Writing

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n} = \sum_{n=1}^{\infty} b_n x^n,$$

with $b_n = \frac{1}{n^n}$, we calculate

$$\limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} \frac{1}{n} = 0.$$

Hence, by Cauchy-Hadamard, the radius of convergence R for this power series is $R = \infty$.

iii) Writing

$$\sum_{n=0}^{\infty} \left(\frac{2\pi}{13}\right)^n \cos\left(\frac{2\pi n}{13}\right) x^n = \sum_{n=0}^{\infty} c_n x^n,$$

with $c_n = \left(\frac{2\pi}{13}\right)^n \cos\left(\frac{2\pi n}{13}\right)$, we calculate

$$\limsup_{n\to\infty} |c_n|^{1/n} = \limsup_{n\to\infty} \frac{2\pi}{13} \left|\cos\left(\frac{2\pi n}{13}\right)\right|^{1/n}.$$

Setting $d_n = \frac{2\pi}{13} \left|\cos\left(\frac{2\pi n}{13}\right)\right|^{1/n}$, we claim that $\limsup_{n\to\infty} d_n = \frac{2\pi}{13}$. First, as for all $y \in \mathbb{R}$, $|\cos(y)| \leq 1$, we see that $d_n \leq \frac{2\pi}{13}$. This means that

$$\limsup_{n\to\infty} d_n \leq \frac{2\pi}{13}.$$

Moreover, taking the subsequence $(d_{n_k})_{k\in\mathbb{N}}$, where $n_k = 13k$, we see that as $k \to \infty$,

$$d_{13k} = \frac{2\pi}{13} |\cos(2\pi k)|^{\frac{1}{13k}} = \frac{2\pi}{13} \cdot 1 \to \frac{2\pi}{13} \text{ (by AoL.)}$$

So, $\limsup_{n\to\infty} d_n \geq \frac{2\pi}{13}$, from which we conclude that $\limsup_{n\to\infty} d_n = \frac{2\pi}{13}$. Hence, by Cauchy-Hadamard, the radius of convergence R for this power series is given by

$$R = \left(\limsup_{n\to\infty} d_n\right)^{-1} = \frac{13}{2\pi}.$$

c) i) Recall from lectures that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \text{diverges if } \alpha \leq 1, \\ \text{converges if } \alpha > 1. \end{cases}$$

Since $\sqrt{n} = n^{1/2}$, and $\frac{1}{2} < 1$, we know that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

ii) First, note that

$$\frac{n}{2^n + n^2} \leq \frac{n}{2^n}. \qquad (**)$$

Now, setting $x_n = \frac{n}{2^n}$, we find

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)2^n}{2^{n+1}n} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

Hence, by the algebra of limits,

$$\frac{|x_{n+1}|}{|x_n|} \to \frac{1}{2} < 1,$$

as $n \to \infty$. So, by d'Alembert's ratio test, we find that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \text{ converges.}$$

Finally, by the comparison test as applied to (**), we conclude that

$$\sum_{n=1}^{\infty} \frac{n}{2^n + n^2} \text{ converges.}$$

iii) Setting $y_n = \frac{1}{n \log(n)}$, we define for $k \geq 1$,

$$z_k := 2^k y_{2^k} = \frac{2^k}{2^k \log(2^k)} = \frac{1}{k \log(2)}.$$

Using the result stated in part i), we know that as $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\sum_{k=1}^{\infty} z_k$ diverges. Hence, by the Cauchy condensation test, the given series $\sum_{n=1}^{\infty} y_n$ diverges.