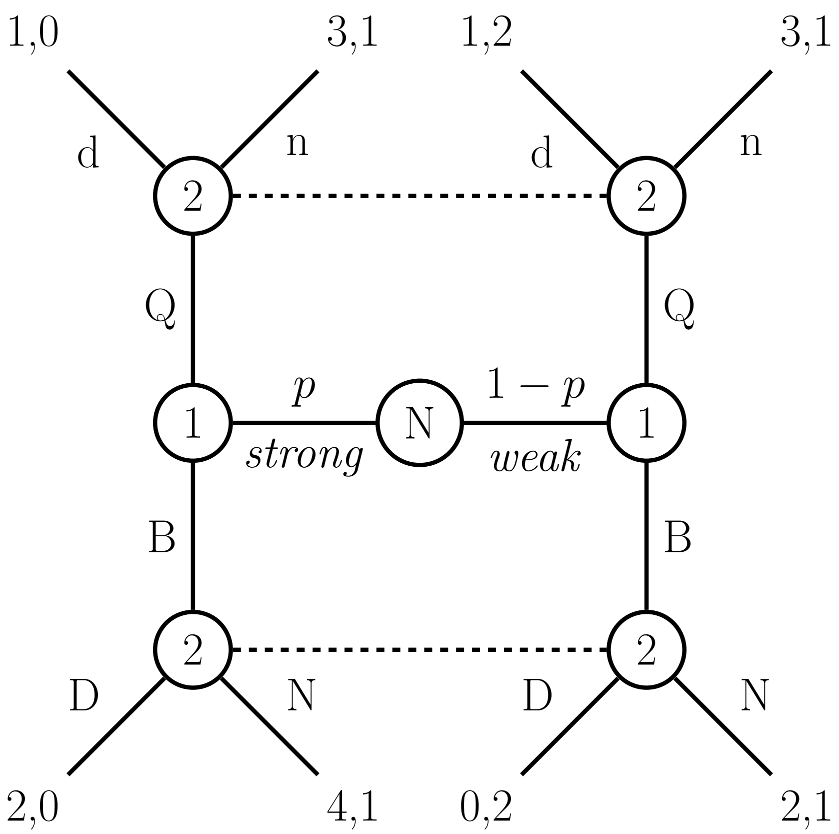
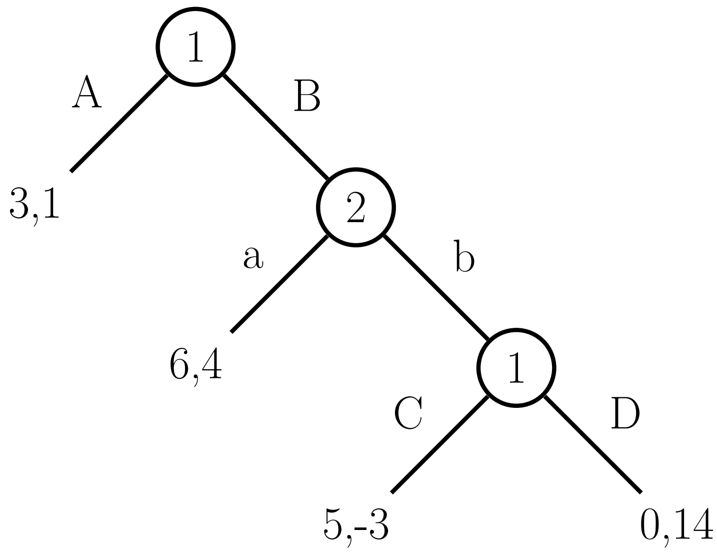
ECON3014 Topic 5:
Extensive games with incomplete information
Osborne: Ch 9-10

- Consider an extensive form game (possibly with simultaneous moves) with one extra player: **nature**
- Nature is not strategic: it follows a predetermined, and often mixed, strategy
- Nature is fictional: it is a way to model uncertainty about the structure of the game (the same as in Bayesian games)

Example:



Mixed and behavioral strategies



In the example above:

- A set of pure strategies for player 1: $\{AC, AD, BC, BD\}$
- A mixed strategy is a vector of probabilities $\sigma = (\sigma_{AC}, \sigma_{AD}, \sigma_{BC}, \sigma_{BD})$
- A **behavioral** strategy is two vectors of probabilities: $(\sigma_A, \sigma_B), (\sigma_C, \sigma_D)$. Randomization between A and B is independent from randomization between C and D

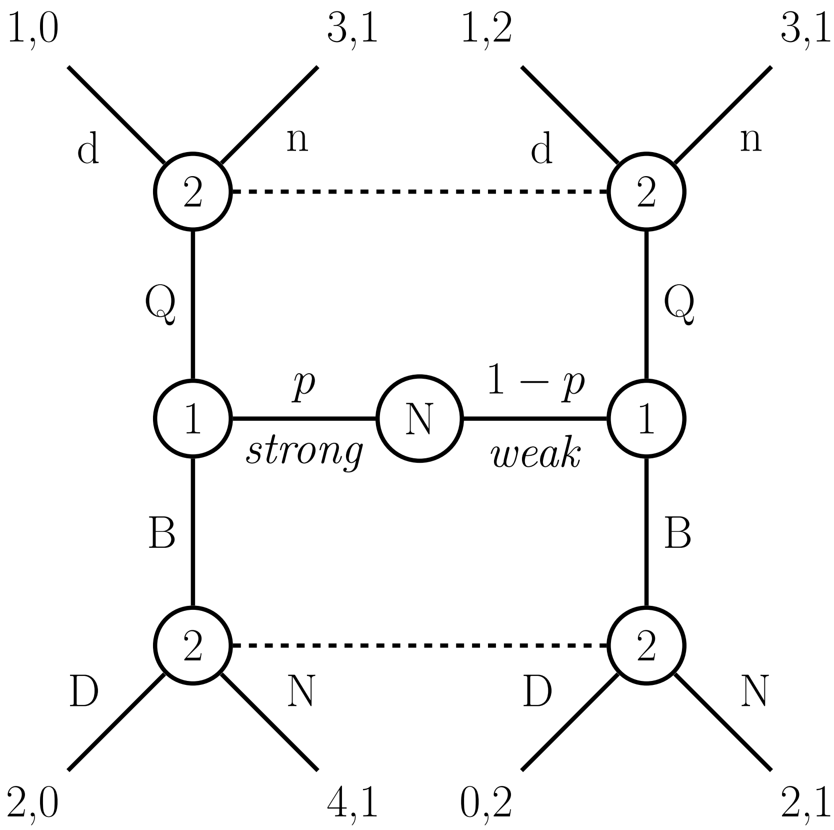
With perfect recall mixed and behavioral strategies are equivalent.

We will use behavioral strategies because they are more convenient.

Formally, a **behavioral strategy** of player i is

$$\sigma_i : X_i \to \Delta(A) \text{ satisfying } supp(\sigma_i(x)) \subset A(x)$$

Beliefs



Consider any nonsingleton information set I_i. Player i, who makes a decision at I, forms a belief about his exact location within the information set I_i.

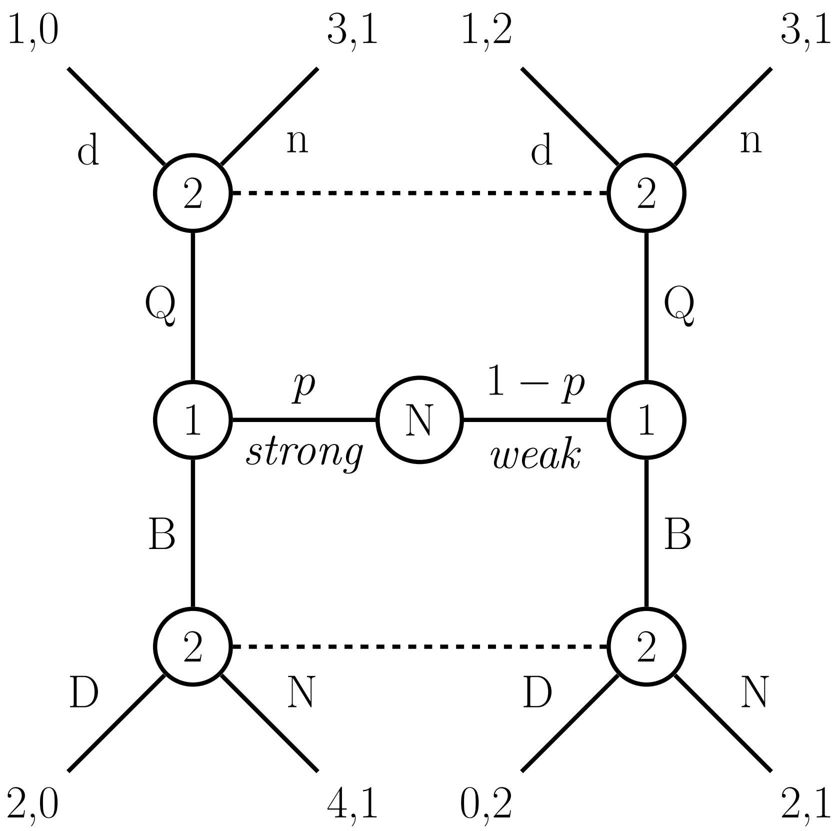
Formally a **belief** of player i at I_i is a probability distribution $\mu \in \Delta(I_i)$. A **belief system** is a set of beliefs, one per each (nonsingleton) information set.

Interpretation: if player i finds himself inside the information set I_i he believes that history $x \in I_i$ was played with probability $\mu(x)$.

Assessments

In extensive games with incomplete information we are interested in both players strategies and their beliefs.

An **assessment** in an extensive game is a pair that consists of a profile of behavioral strategies and a belief system.



Weak sequential equilibrium

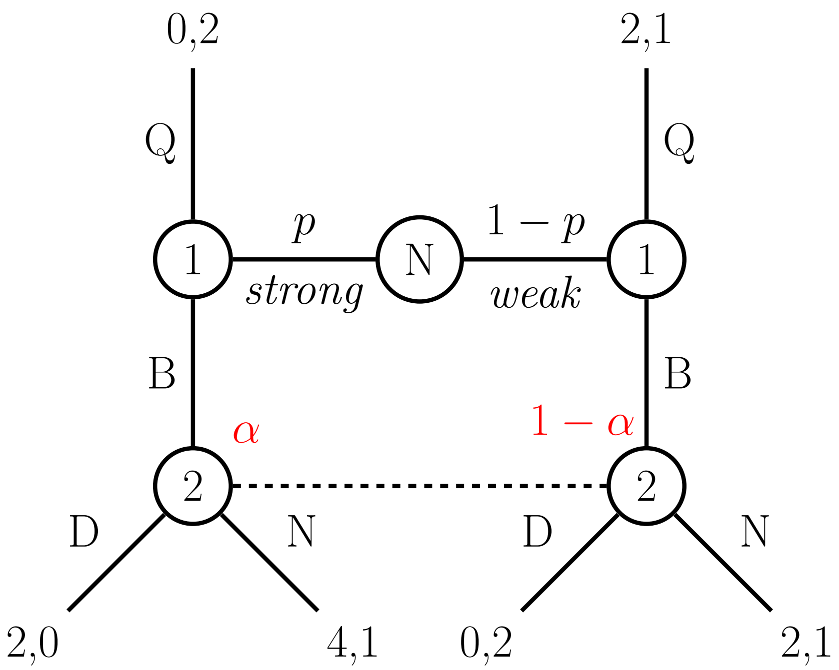
A **weak sequential equilibrium** is an assessment that satisfies two conditions: (i) sequential rationality and (ii) consistency of beliefs with strategies.

Sequential rationality: Each player's strategy is optimal given her beliefs and the strategies of her opponents.

Consistency of beliefs: Beliefs are determined by **Bayes rule** whenever possible (i.e. in all information sets that are reached with positive probability according to strategies of players).

Bayes rule

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

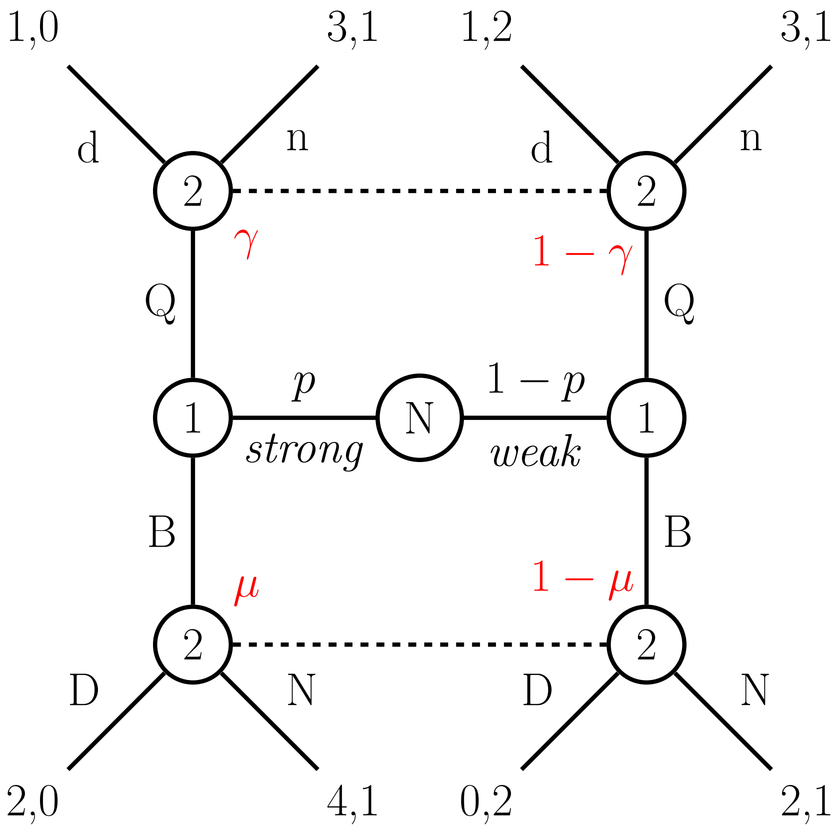


How to check for consistency:

$$\alpha \overset{C}{=} \Pr(str. \mid B) = \frac{\Pr(str. \cap B)}{\Pr(B)} = \frac{\Pr(str. \cap B)}{\Pr(str. \cap B) + \Pr(weak \cap B)}$$

...whenever possible: if the information set is not reached (reached with probability zero), any belief on this information set is consistent.

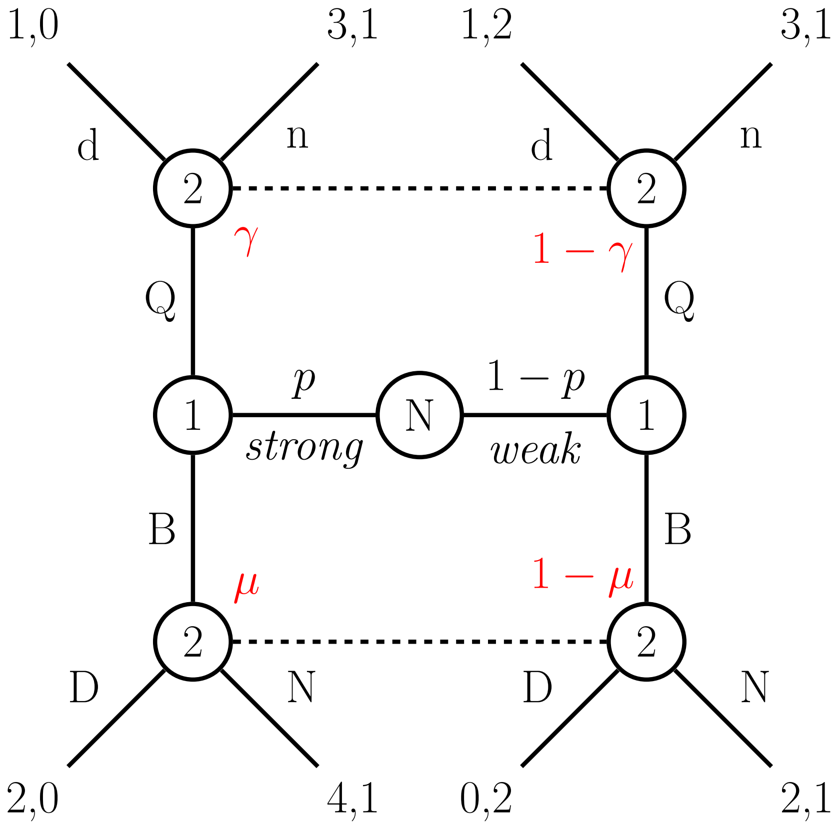
Beer-quiche game: Looking for equilibria



	$\mu < 0.5 : D$	$\mu = 0.5 : D \sim N$	$\mu > 0.5 : N$
$\gamma < 0.5 : d$	case 1	case 6	case 2
$\gamma = 0.5 : d \sim n$	DIY	case 5	DIY
$\gamma > 0.5 : n$	case 3	DIY	case 4

Cases 1 and 4

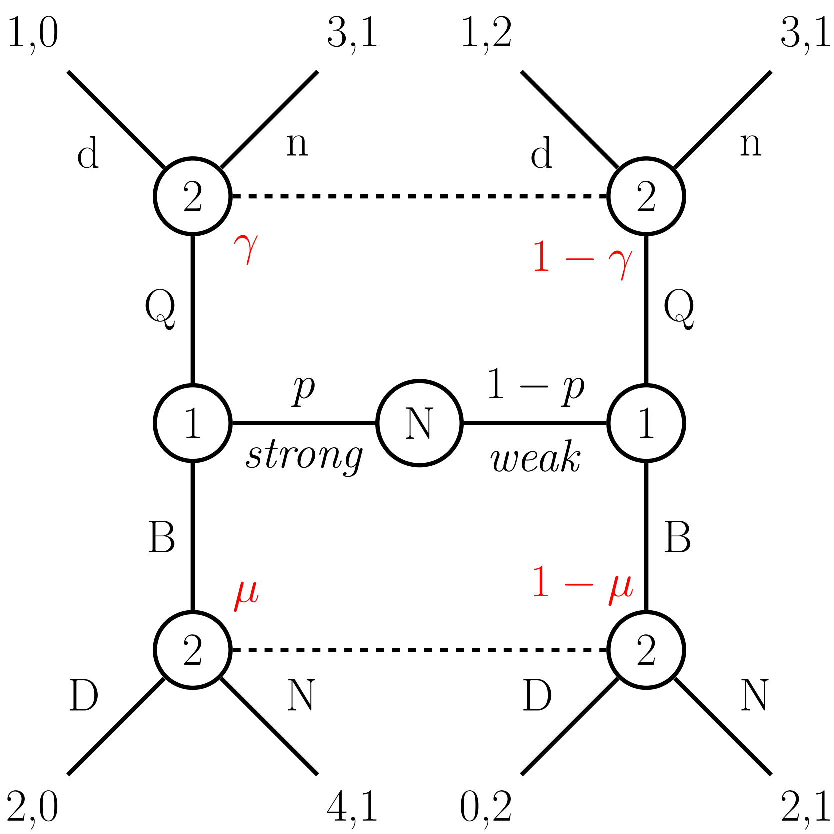
Case 1:



$\mu = 1$ – not possible!

In case 4, similarly, you get that $\gamma = 0$ – not possible.

Case 2



$s(str.) = B, s(weak) = B.$

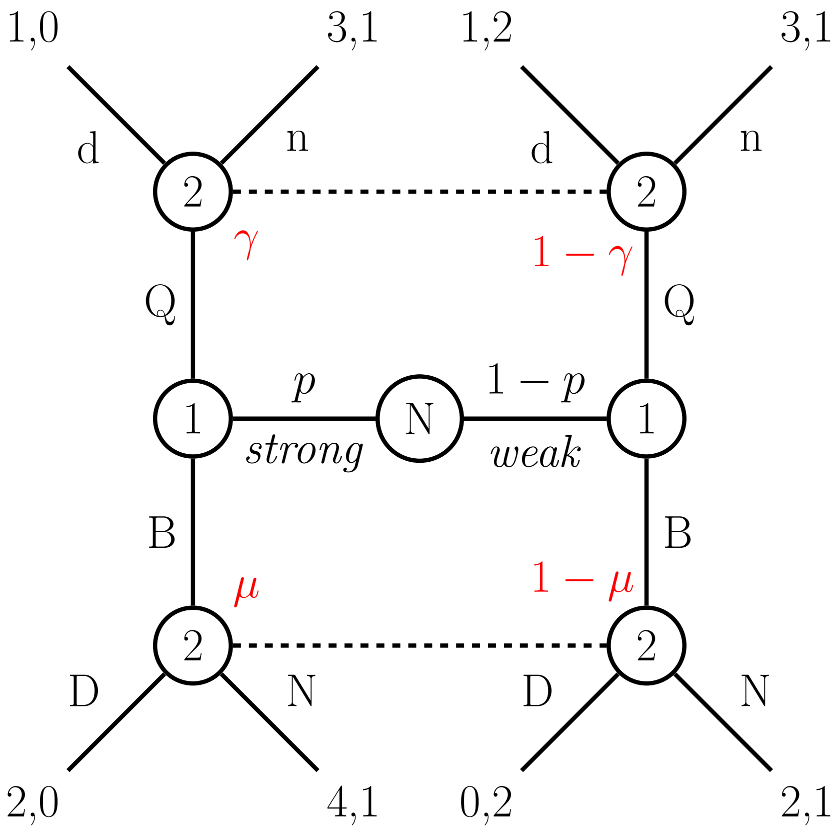
Consistency requires that $\mu = p$. As long as $p > 0.5$, and $\gamma < 0.5$ this assessment is a sequential equilibrium.

Note that γ is a free parameter because there is no consistency requirement for it: no one plays Q in this equilibrium.

Properties: Both types select the same actions, hence action is not informative about the type: player 2's posterior belief is the same as his prior belief $(p, 1-p)$.

Equilibria in which no additional information about the types is revealed to uninformed player are called **pooling** equilibria.

Case 3

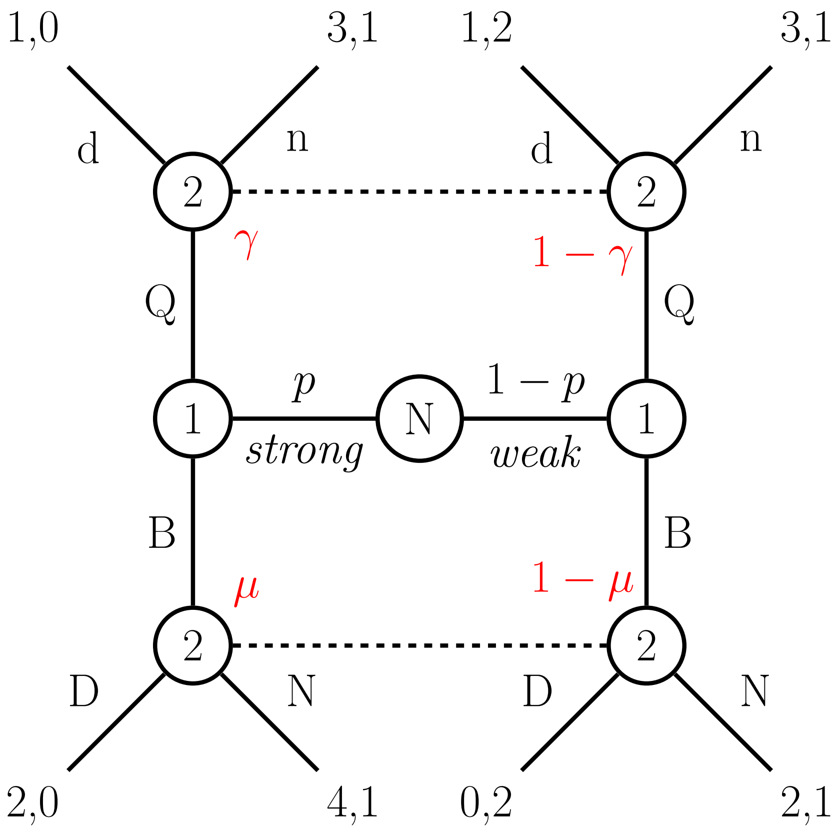


Another pooling equilibrium

$s(str.) = Q, s(weak) = Q.$

Consistency requires that $\gamma = p$. As long as $p > 0.5$, and $\mu < 0.5$ this assessment is a sequential equilibrium.

Case 5



Since $\mu = 0.5$ and $\gamma = 0.5$, consistency implies that both types of player 1 are mixing. Also case 5 implies that player 2 is mixing after observing any history.

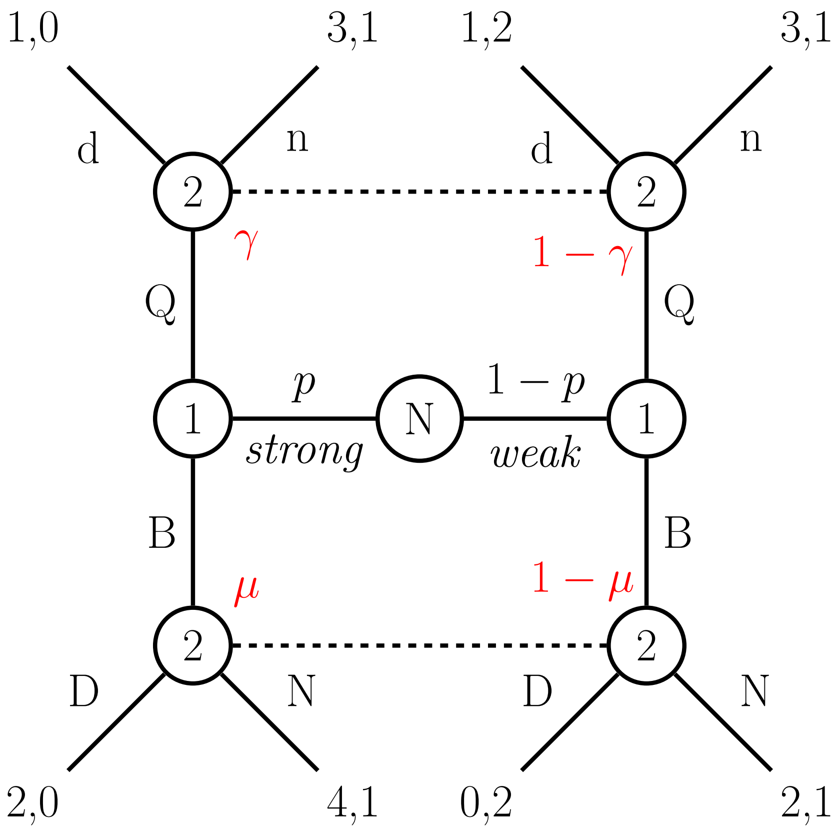
Suppose player 2 plays D with probability θ and d with probability δ.

Then player 1's indifference conditions are

$$2\theta + 4(1-\theta) = \delta + 3(1-\delta)$$
$$0\theta + 2(1-\theta) = \delta + 3(1-\delta)$$

No such θ and δ exist, hence case 5 cannot be an equilibrium.

Case 6



Player 2 is indifferent between playing D and N. Let him play D with probability α and N with a remaining prob.

Player 1 strictly prefers to play B if he is strong.

Suppose he plays B with prob. β if he is weak.

Consistency:

$$0.5 = \mu = \frac{\Pr(B \cap str.)}{\Pr(B)} = \frac{\Pr(B \cap str.)}{\Pr(B \cap str.) + \Pr(B \cap weak)} =$$
$$\frac{\Pr(B \mid str.)\Pr(str.)}{\Pr(B \mid str.)\Pr(str.) + \Pr(B \mid weak)\Pr(weak)}$$

or

$$\beta = \frac{p}{1-p}$$

Case 6 (cont.)

We have to make sure that $\beta \in [0,1]$. This holds if $p \in [0, 0.5]$.

Weak type is mixing if he is indifferent between B and Q:

$$2(1-\alpha) = 1 \text{ or } \alpha = 0.5$$

Note that consistency also implies $\gamma = 0$ which is compatible with case 6 ($\gamma < 0.5$).

Properties:

In this equilibrium player 2 obtains some extra information which allows him to make a more precise judgment or inference of player 1's type. In particular he can say for sure that player 1 is weak if player 1 plays Q. He still cannot distinguish perfectly between weak and strong type when he sees B played.

Such equilibria are called **partially separating** equilibria.

An equilibrium is called **separating** if uninformed agent can make precise inference (can guess without making a mistake) of his opponent's type. There are no separating equilibria in this particular game.

Signaling games

Consider a labor market.

- single (continuum) agent with ability θ
- ability θ is drawn from a binomial distribution: θ_L with prob p and θ_H with prob $(1-p)$
- agent's productivity is θ
- labor market is competitive: firms pay their employees their expected productivity (and collect zero profits)

Benchmark

In equilibrium firms cannot distinguish between high ability and low ability employees, hence the wage is uniform:

$$w = \mathbb{E}[\theta]$$

Education

- Before joining the labor force, agents can obtain some education
- Agents choose how many years to spend studying: $e \in \mathbb{R}_+$
- Education is costly: $c(e, \theta) = \frac{e}{\theta}$ (easily generalizable)
- Note, that marginal costs of education are lower for high productivity agents
- Education is a pure waste: has no effect on productivity

Pooling equilibria: benchmark

Benchmark equilibrium remains:

- $w = \mathbb{E}[\theta] = p\theta_L + (1-p)\theta_H$
- both types obtain $e = 0$
- beliefs:
  - one information set is reached with probability 1 ($e = 0$): $\Pr(\theta = \theta_L \mid e = 0) = p$
  - if $e > 0$ beliefs are arbitrary as long as seq. rationality holds. Example: $\Pr(\theta = \theta_L \mid e > 0) = 1$
  - is it possible to have $\Pr(\theta = \theta_L \mid e > 0) = 0$ with the strategy profile above? what is $\max\{\Pr(\theta = \theta_L \mid e > 0)\}$?

Pooling equilibria: the role of beliefs

There are other pooling eq. driven by special beliefs:

- $w = \mathbb{E}[\theta] = p\theta_L + (1-p)\theta_H$
- both types obtain $e = e^P$
- beliefs:
  - one information set is reached with probability 1 ($e = e^P$): $\Pr(\theta = \theta_L \mid e = 0) = p$
  - if $e \neq e^P$ beliefs are arbitrary as long as seq. rationality holds. Example: $\Pr(\theta = \theta_L \mid e > 0) = 1$
  - $\Pr(\theta = \theta_L \mid e \neq e^P) = 1$ implies that agents should e^P over any other level of education.

$$p\theta_L + (1-p)\theta_H - \frac{e^P}{\theta} \geq w(e) - \frac{e}{\theta}$$

Note, that since $\Pr(\theta = \theta_L \mid e \neq e^P) = 1$, for any $e \neq e^P : w(e) = \theta_L$. Therefore, for all $e \neq e^P$ and for all θ

$$p\theta_L + (1-p)\theta_H - \theta_L \geq \frac{e^P - e}{\theta}$$

This set of inequalities implies that

$$e^P \leq \theta_L(\theta_H - \theta_L)(1-p)$$

Separating equilibria

Agents signal their ability to firms using education:

- Let low productivity agents choose e_L and high prod. agents choose $e_H : e_H \neq e_L$
- Consistency implies that firms can distinguish between high and low ability by looking at education

$$w(e_L) = \theta_L \text{ and } w(e_H) = \theta_H$$

- History $e \notin \{e_L, e_H\}$ is reached with prob. 0 therefore beliefs there are free: for now assume that $\Pr(\theta_H \mid e \notin \{e_L, e_H\}) = 0$, hence for all $e \notin \{e_L, e_H\} : w(e) = \theta_L$

Sequential rationality for the agents:

- Low productivity:

$$w(0) \geq w(e \neq e_H) - \frac{e}{\theta_L}$$

  hence $e_L = 0$.

- Low type does not want to pretend to be a high type and vice versa:

$$\theta_L \geq \theta_H - \frac{e_H}{\theta_L}$$
$$\theta_H - \frac{e_H}{\theta_H} \geq \theta_L$$

  or

$$e_H \in [(\theta_H - \theta_L)\theta_L, (\theta_H - \theta_L)\theta_H]$$

Equilibria: graphic representation

Cheap talk

Strategic transmission of information. The model of experts and policymakers.

- Two agents: an advisor (or an expert) and a policymaker
- Nature chooses state of the world $x \sim U[0, 1]$
- The advisor knows the state of the world and the policymaker does not.
- The advisor can send a message $m \in [0, 1]$ to the policymaker
- The policymaker observes the message and chooses a policy $y \in [0, 1]$
- Policymaker's payoff:

$$u_p(x, y) = -(x - y)^2$$

- Adisor's payoff:

$$u_a(x, y) = -(x - y - b)^2$$

  where b is his bias (the larger the b the larger the conflict of interest between A and P).

Babbling equilibria

The advisor will send a meaningful message only if he thinks that the policymaker pays attention to his messages. There are always an equilibria in which policymaker does not pay attention to messages and the advisor sends arbitrary messages:

- The advisor sends a random message that does not depend on the state of the world: for example $m \mid x \sim U[0,1]$
- The policymaker's posterior beliefs coincide with his prior: $\Pr\{x < z \mid m\} = z$.
- The policymaker chooses best policy maximizing his expected payoff:

$$\max_y \{ \int_0^1 -(x-y)^2 dx \}$$

or

$$y^*(m) = \frac{1}{2}$$

Separating equilibrium does not exist

Perfect communication in this model is impossible: the advisor always wants to lower his recommendation, but the policymaker makes upward corrections.

Formally:

- By contradiction, assume that separating equilibrium exists. Let $\mu(x)$ be an eq. strategy of the advisor
- Since the eq. is separating, $\mu(x)$ is invertible. WLOG assume $\mu(x)$ strictly increasing.
- The policymaker implements $y(m) = x = \mu^{-1}(m)$.
- The advisor's favorite policy is $x - b$, hence he is stricly better off sending $\hat{m} = \mu(x - b)$ instead of his eq. action $m = \mu(x)$, therefore it cannot be an equilibrium.
- Question: is sending $\mu(x - b)$ an equilibrium?

Partially separating equilibrium

Some communication is possible in equilibrium. Suppose the policy-maker only pays attention to two messages m_1 and m_2 (for example: x is high or low)

- Let the advisors strategy be $\mu(x) = m_1$ if $x < x^*$ and $\mu(x) = m_2$ otherwise.
- If message m_1 is received, policymaker thinks that $x < x^*$ and implements the policy that maximizes

$$\max_y \{ \int_0^{x^*} -(x-y)^2 dx \},$$

i.e.

$$y(m_1) = \frac{x^*}{2}$$

- If message m_2 is received, policymaker thinks that $x \geq x^*$ and implements the policy that maximizes

$$\max_y \{ \int_{x^*}^{1} -(x-y)^2 dx \},$$

i.e.

$$y(m_2) = \frac{1+x^*}{2}$$

Partial separation: advisor's incentives

for any $x < x^*$:

$$-\left(x - \frac{x^*}{2} - b\right)^2 \geq -\left(x - \frac{1 + x^*}{2} - b\right)^2$$

and for any $x \geq x^*$:

$$-\left(x - \frac{x^*}{2} - b\right)^2 \leq -\left(x - \frac{1 + x^*}{2} - b\right)^2$$

Since both functions are cont., it must be that

$$-\left(x^* - \frac{x^*}{2} - b\right)^2 \leq -\left(x^* - \frac{1 + x^*}{2} - b\right)^2,$$

i.e.

$$x^* = \frac{1}{2} + 2b$$

It is an equilibrium only if b is relatively small: $b \leq 1/4$

Partially separating equilibrium: graphic representation

Model of advertising

- Firm, consumer
- Firm can be either of type H (with prob. π) or of type L (with prob. $1 - \pi$)
- Type is unobservable by a consumer
- Firm of type T produced a product that has a value v_T at a cost c_T
- $V_H > V_L > c_H > c_L$
- Two periods
- Firm sets a price p each period and an advertising expenditures E in the first period
- Both prices and expenditures are observable
- Consumer chooses whether to buy at each period
- Consumer can learn the value once the product is consumed

Timing

- Nature draws the type of the firms
- The firm observes its type and chooses $(p_1, E) \in \mathbb{R}^2_+$
- The consumer observes (p_1, E) and makes a decision to buy or not
- If the consumer decided not to buy, the game ends with payoffs $(-E, 0)$
- If the consumer bought the product, the quality is revealed and the game progresses to the second period
- The firm sets p_2
- The consumer observes p_2 and decides to buy or not

The second period

- Quality is known
- Consumer buys iff

$$v - p_2 \geq 0$$

- The firm sets a price $p_{2T} = v_T$ such that the consumer purhcases the product
- Firms profit in the second period is $p_{2T} - c_T$

The first period

Separating equilibrium:

- Say firm of type T chooses (p_{1T}, E_T)
- Consumer's beliefs (consistent):
  - $\Pr(H \mid p_{1H}, E_H) = 1$
  - $\Pr(H \mid (p_{1T}, E_T) \neq (p_{1H}, E_H)) = 0$
- Consumer always buys on eq. path (after histories that are reached in eq. with positive prob.):
  - $p_{1H} \leq v_H$
  - $p_{1L} \leq v_L$

Seq. rationality for the firm

- Can the firm set the highest acceptable prices, i.e. $p_{1H} = v_H$ and $p_{1L} = v_L$? Type L firm can and type H firm in general cannot. Precise answer depends on the consumers beliefs.

- Type L will not spend money on advertising: $E_L = 0$.

- Type H incentive compatibility (IC):

$$p_{1H} + v_H - 2c_H - E_H \geq v_L + v_H - 2c_H$$

- Type L IC:

$$2v_L - 2c_L \geq p_{1H} + v_L - 2c_L - E_H$$

- Combine the two together:

$$E_H = p_1 H - v_L$$

Intuition: the firm "burns" money to convince the consumer that it is type H. The amount of money burnt is exactly the difference in period 1 profits for type L and H

What if $v_L < c_L$

- Type L cannot make profit in the second period hence does not sell.
- Type H incentive compatibility (IC):

$$p_{1H} + v_H - 2c_H - E_H \geq 0$$

- Type L IC:

$$0 \geq p_{1H} - c_L - E_H$$

If we combine all inequalities together:

$$\begin{aligned}
&p_{1H} \leq v_H \\
&E_H \geq p_{1H} - c_L \\
&E_H \leq p_{1H} + v_H - 2c_H \\
&E_H \geq 0, p_{1H} \geq 0
\end{aligned}$$

Separating equilibria: graphic representation

$$
\begin{aligned}
&p_{1H} \leq v_H \\
&E_H \geq p_{1H} - c_L \\
&E_H \leq p_{1H} + v_H - 2c_H \\
&E_H \geq 0, p_{1H} \geq 0
\end{aligned}
$$