PHY 3650: Quantum Information Processing

Spring 2024

Homework Set #6 - Due on April 8, 2024

1. **(30)** Build a "black-box" (oracle) operator $\hat{U}_f$ that implements the relation

$$\hat{U}_f \, |x\rangle^{(2)} \otimes |y\rangle = |x\rangle^{(2)} \otimes |f(x) \oplus y\rangle$$

where $|x\rangle^{(2)}$ represents a two-qubit computational basis state and $|y\rangle$ a single-qubit state, also in the computational basis. For this problem, the two-bit function $f(x)$ is defined as

$$f(x) = \begin{cases} 1, & x = 2 \\ 0, & x \neq 2 \end{cases}.$$

Hint: Find the unitary operator $\hat{U}_f$ in the 3-qubit computational basis. What does it do?

2. **(30)** Reformulate the E91 protocol to employ the $|B_{00}\rangle$ Bell state.

3. [Adapted from C. P. Williams, Explorations in Quantum Computing, 2nd edition.] The BB84 protocol employs single, non-entangled qubits to perform QKD, while the E91 protocol is based on entangled qubit pairs. Despite this physical difference, they are formally similar.

    a) **(30)** Verify that the following identity holds for arbitrary one-qubit gates $\hat{U}_1$ and $\hat{U}_2$:

    $$(\hat{U}_1 \otimes \hat{U}_2) \, |B_{00}\rangle = \hat{I} \otimes (\hat{U}_2 \, \hat{U}_1{}^T) \, |B_{00}\rangle,$$

    where $|B_{00}\rangle$ is the first Bell state and the superscript T denotes transposition. *Hint*: represent operators and vectors in the 2-qubit computational basis.

    b) **(10)** Explain how this identity can be used to find a connection between B84 and E91.