1 Numerical methods for ODES

Considering the 2-step Simpson's method:

$$y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1}1 + f_{n+2}) \tag{1}$$

and implement it to solve the test equation:

$$\begin{cases} \text{y'=5y} \\ \text{y(0)=1} \end{cases}$$

with $h = 0,02$ and $T = t_N = 10$.

Than we had to compute the second initial value both with Fordward Euler method:

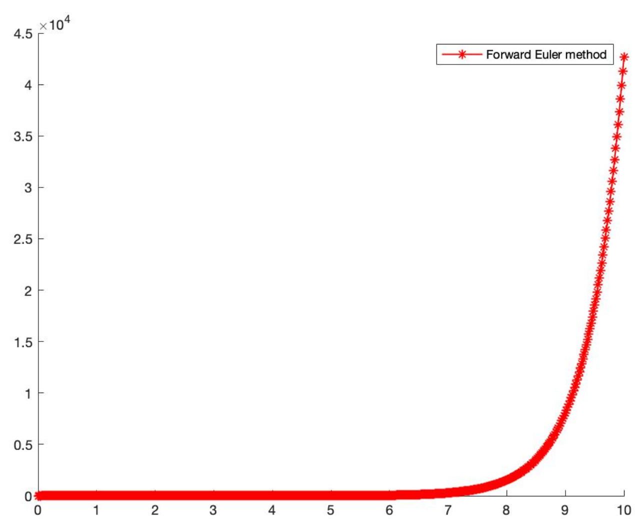
$$y_{n+1} = y_n + hf_n \tag{2}$$

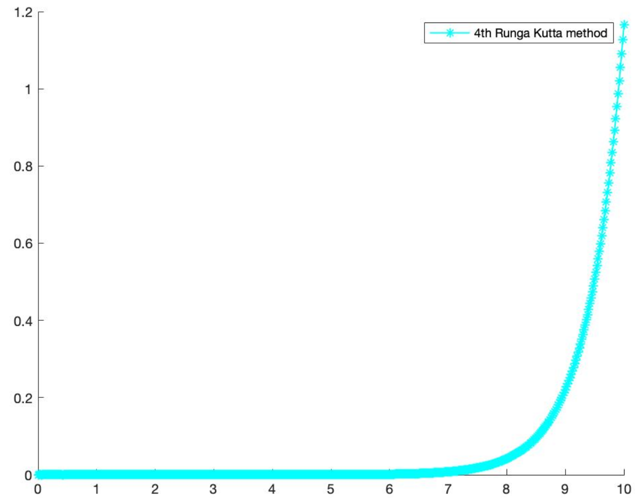
and with 4th order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{3}$$

where:
$K_1 = f_n$
$K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1)$
$K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2)$
$K_4 = f(x_{n+1}, y_n + hK_3)$

In the following images there are the graphs of the error with both methods:





As we can see the Simpson's method is not stable, the errors in both graphs diverge.
It is important to notice that the graph calculated with Fordward Euler deverge faster than the one calculated with Runge-Kutta. We may notice that in the first case we reach an order of 10^4, while in the second case we have an order of 10^0.

Now for the last request e) we consider a new test equation:

$$\begin{cases} \text{y'=5y} \\ \text{y(0)=}\alpha \end{cases}$$

And we must find the value of α for which the method is stable in exact arithmetic.

So we start from the Simpson's method define as:

$$y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1}1 + f_{n+2})$$

and because of the condition of the test equation we can find y_0 and y_1:

$$\begin{cases} \text{y}_0 = \alpha \\ \text{y}_1 = \alpha - h5\alpha = (1-5h)\alpha \end{cases}$$

Now we can define y_{n+2} as:

$$y_{n+2} = y_n + \frac{h}{3}(-5y_n - 20y_{n+1} - 5y_{n+2})$$

So

$$y_{n+2}\Big(1 + \frac{5}{3}h\Big) = -\frac{20}{3}hy_{n+1} + \Big(1 + \frac{5}{3}h\Big)y_n$$

We know that the solution is:

$$y_n = C_1\lambda_1^n + C_2\lambda_2^n$$

Than the characteristic equation will be:

$$\lambda^2 = \frac{(-\frac{20}{3}h)}{(1+\frac{5}{3}h)} + \frac{(1-\frac{5}{3}h)}{(1+\frac{5}{3}h)}$$

then:

$$\lambda^2 = \lambda(-\frac{20}{3}h)(\frac{3}{3+5h}) + (\frac{3-5h}{3})(\frac{3}{3+5h})$$

then:

$$\lambda^2 + \tfrac{20h}{(3+5h)}\lambda - \tfrac{3-5h}{3+5h} = 0$$

and so:

$$\lambda_1 = -\tfrac{10h+\sqrt{75h^2+9}}{(3+5h)}$$

$$\lambda_2 = -\tfrac{10h-\sqrt{75h^2+9}}{(3+5h)}$$

then for $h = 0,02$, according with the theory, we obtain:

$$\begin{cases} \lambda_1 \simeq 0,9 \\ \lambda_2 \simeq -1,04 \end{cases}$$

Indeed the condition of Z_1 and Z_2 are satisfied: $\begin{cases} |Z_1| < 1 \\ |Z_2| > 1 \end{cases}$

Know for stability we have the condition : $y_n \to 0 \Rightarrow C_2 = 0$

$$\Rightarrow \begin{cases} y_0 = C_1 + C_2 = \alpha \\ y_1 = (1-5h)\alpha = C_1\lambda_1 + C_2\lambda_2 \end{cases} \quad \Rightarrow \quad C_1 = \alpha - C_2 \Rightarrow$$

$$\Rightarrow \quad (1-5h)\alpha = (\alpha - C_2)\lambda_1 + C_2\lambda_2 \quad \Rightarrow \quad C_2(\lambda_1 - \lambda_2) = (1-5h)\alpha - \alpha\lambda_1$$

Finally we have $\quad C_2 = 0 \quad \Rightarrow \quad (1-5h)\alpha - \alpha\lambda_1 = 0 \quad \Rightarrow$

$$\Rightarrow \quad \alpha(1 - 5h - \lambda_1) = 0 \quad \Rightarrow \quad \alpha = 0.$$

In conclusion, for the calcolus above, we can say that we have stability for $\alpha = 0$.

2

Now we have to compute the differential Cauchy Problem:

$$\begin{cases} \text{y'(t)} = \text{-10y}^2(t) \\ \text{y(0)} = 1 \end{cases}$$

$$\text{with } t \in (0,2) \text{ and } h = 2^{-k}, k = (5, ..., 10).$$

Using the 4-stage Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{4}$$

where:
$K_1 = f_n$
$K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1)$
$K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2)$
$K_4 = f(x_{n+1}, y_n + hK_3)$

Subsequently there is the plot (loglog plot) of every value of h with the final error associate at each h. We may notice, in accordance with the theory, that the error reduces as h decreases.

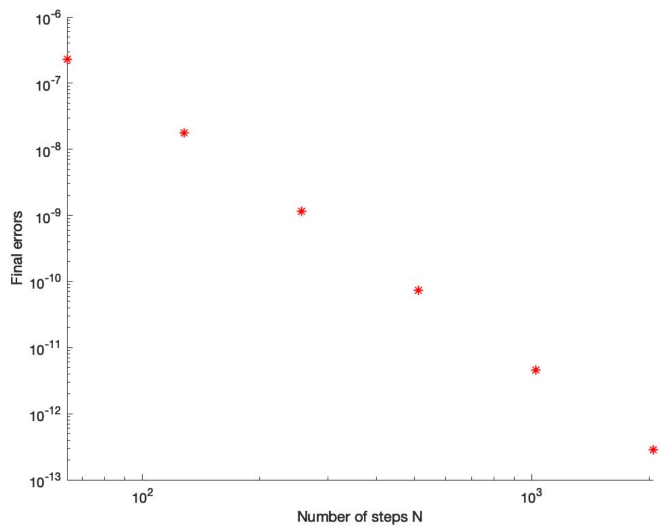


Table of errors depending on h	
h: Step-size	**Errors**
$h = 2^{-5}$	2,292 * 10^{-7}
$h = 2^{-6}$	1,786 * 10^{-8}
$h = 2^{-7}$	1,602 * 10^{-9}
$h = 2^{-8}$	7,313 * 10^{-11}
$h = 2^{-9}$	4,580 * 10^{-12}
$h = 2^{-10}$	2,864 * 10^{-13}

3

we have to compute a Backward differentiation formulas, BDF, obtained writing the ODE on a point t_{n+k} as

$$y'(t_{n+k}) = f(t_{n+k}, y_{n+k}).$$

Then the unkonw function y is interpolated at the left hand side of $y' = f(t, y)$ using the nodes $t_n, ..., t_{n+k}$ and then differentieted.
We proceed by generalizing the formula BDF and than obtain:

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h\beta_k f(t_{n+k}, y(t_{n+k})) \tag{5}$$

The polynomial must interpolate the function in this three points:

1) $\rightarrow (t_n, y_n)$
2) $\rightarrow (t_{n+1}, y_{n+1})$
3) $\rightarrow (t_{n+2}, y_{n+2})$

Consequently the polynomial $P(t)$ is:

$$P(t) = \left(y_n \frac{(t-t_{n+1})(t-t_{n+2})}{2h^2}\right) + \left((y_{n+1} \frac{(t-t_n)(t-t_{n+2})}{-h^2}\right) + \left(y_{n+2} \frac{(t-t_n)(t-t_{n+1})}{2h^2}\right)$$

Now computing the derivative and evalueting it in the point (t_{n+2}) we obtain:

$$P'(t) = \left(\frac{y_n}{2h} - \frac{2(y_{n+1})}{h} + \frac{3(y_{n+2})}{2h}\right)$$

Now we are able to write the BDF2 formula (considering $P'(t_{n+2}) = f(t_{n+2}, y(t_{n+2}))$):

$$y_{n+2} = \frac{4}{3}y_{n+1} - \frac{y_n}{3} + \frac{2}{3}hf(t_{n+2}, y(t_{n+2}))$$

Now for discuss the local troncation error of BDF2 formula, we start from the definition of the interpolation error E(t):

$$E(t) = \frac{(t-t_{n+k})...(t-t_n)}{(k+1)!} f^{(k+1)}(\eta(t))$$

And since we know from the exercise that $K = 2$ we continue and obtain:

$$E(t) = \frac{(t-t_{n+2})(t-t_{n+1})(t-t_n)}{(2+1)!} f^{(2+1)}(\eta(t))$$

Now we continuing by deriving and putting the absolute value: $|E'(t)|$ and this is equal to:

$$|E'(t)| = \Bigg|\left(\frac{(t-t_{n+1})(t-t_n)f^{(3)}(\eta(t))}{6}\right) + \left(\frac{(t-t_{n+2})(t-t_n)f^{(3)}(\eta(t))}{6}\right) + \left(\frac{(t-t_{n+2})(t-t_{n+1})f^{(3)}(\eta(t))}{6}\right) + \left(\frac{(t-t_{n+2})(t-t_{n+1})(t-t_n)f^{(4)}(\eta(t))(\eta'(t))}{6}\right)\Bigg|$$

Now we solve $|E'(t)|$ in $t = t_n$ and $t = t_{n+2}$ for find the maximum, so we obtain:

$$(1) \rightarrow |E'(t_n)| = \left|\frac{h^2 f^{(3)}(\eta(t))}{3}\right|$$

$$(2) \rightarrow |E'(t_{n+2})| = \left|\frac{h^2 f^{(3)}(\eta(t_{n+2}))}{3}\right|$$

From this we are able to compute the local truncation error, and it is:

$$T_L = \left|\frac{h^2 max(f^{(3)}(\eta(t_n)), f^{(3)}(\eta(t_{n+2})))}{3}\right|$$

Finally we must prove that BDF2 is absolutely stable in the real interval $\overline{h} \in (-\infty, 0)$. We proceed studying the characteristic polynomial of this method applied to the test equation:

$$t^2(3 - 2\overline{h}) - 4t + 1 = 0 \tag{6}$$

with roots t_1, t_2 define as:

$$t_1 = \frac{2+\sqrt{1+\overline{h}}}{3-2\overline{h}}$$

$$t_2 = \frac{2-\sqrt{1+\overline{h}}}{3-2\overline{h}}$$

Therefore regard to stability we can say that if $\overline{h} \in [-\frac{1}{2}, 0[$ the solution $t_1, t_2 \in R$ and $|t_1|, |t_2| < 1$, so for this interval we can conclude that the method is absolutely stable. In the case $\overline{h} \in [-\infty, -\frac{1}{2}]$ the solution $t_1, t_2 \in C$ but also in this case $|t_1|, |t_2| < 1$ so we can conclude that this method is absolutely stable for $\overline{h} \in (-\infty, 0)$.

4

We have to solve the linear system of ODEs:

$$\begin{cases} \text{y'(t) = -Ay(t)} \\ \text{y(0) = [1,1,...,1]}^T \end{cases}$$

To compute the absolute region of stability of 4-th order Runge Kutta method we focused on the equation:

$$y_{n+1} = \Big(1 + \overline{h} + \frac{\overline{h}^2}{2} + \frac{\overline{h}^3}{6} + \frac{\overline{h}^4}{24}\Big) y_n \tag{7}$$

Now we pose the condition $\left|1+\overline{h}+\frac{\overline{h}^2}{2}+\frac{\overline{h}^3}{6}+\frac{\overline{h}^4}{24}\right| < 1$ to find the interval of absolute stability.

From which we obtain: $\overline{h} \in (-2,78;0)$.

For the interval of absolute stability of 4-th order Runge Kutta method we consider the value τ as:

$$\tau = \left|\tfrac{2,78}{2\lambda}\right|$$

from which we obtain the interval of stability define as:

$$h \in (-\infty, -\tau)$$

To solve the problem we used:

→ Ode45 with $t_0 = 0$ and $T = 0,1$
→ Crank Nicolson method with $h \in [10^{-3}, 10^{-4}, 10^{-5}]$
→ BDF3 method with $h \in [10^{-3}, 10^{-4}, 10^{-5}]$.

All the result are summarised in the following table:

Final Result			
METHOD:	**NUMBER OF STEP:**	**ERROR:**	**CPU TIME: (sec)**
ODE45	9945	1.1554e-05	7.742
C-N Method	100	3.7841e-05	2.0068
C-N Method	1000	1.4389e-07	11.3295
C-N Method	10000	6.2363e-09	91.4022
BDF3	100	3.7661e-07	1.2337
BDF3	1000	5.3514e-09	7.3357
BDF3	10000	4.9899e-09	64.9988

$(result - computed - with - file - exact - solution)$

As we can see as if h decreased, the errors decreased (in accordance with the theory).

5

we have to solve the Lotka-Volterra equations,a first-order nonlinear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\begin{cases} \text{x'(t)} = \text{x(t)}(\alpha - \beta y(t)) \\ \text{y'(t)} = \text{y(t)}(\gamma x(t) - \delta) \\ \text{x}(0) = \text{x}_0 \\ \text{y}(0) = \text{y}_0 \end{cases}$$

y is the number of predator and x the number of pray, y' and x' are the instantly growth rates of the two populations, $\alpha, \beta, \gamma, \delta$ are parameters and t is the time.
We had to Solve the Lotka-Volterra equation with parameters $\alpha = 0,2$; $\beta = 0,01$; $\gamma = 0,07$; $\delta = 0,004$ by a 4th order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{8}$$

with:
$K_1 = f_n$
$K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1)$
$K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2)$
$K_4 = f(x_{n+1}, y_n + hK_3)$

using as initial conditions $x_0 = 19$; $y_0 = 22$, $t_0 = 0$; $T = 300$ and a stepsize $h = 10^-3$.

We may notice that the two functions interact with each other and that the method must be applied to both populations in each step.

In the following picture there is a plot of the numerical solution of x(t) and y(t):