Control of Linear Systems with Saturating Actuators

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***Abstract*— This paper deals with the design of linear systems with saturating actuators where the actuator limitations have to be incorporated *a priori* into control design. We take a semiglobal approach to solve some of the central control problems for such systems. These problems include stabilization, input-additive disturbance rejection, and robust stabilization in the presence of matched nonlinear uncertainties. We develop further the semiglobal design technology which was initiated in our earlier work [7] and utilize it to deal with these control problems.**

I. Introduction

RECENTLY, there has been a renewed interest in the study of linear systems subject to input saturation since this phenomenon is a common feature of control systems. Several significant results have emerged. These results pertain primarily to the problem of global and semiglobal stabilization. In regard to global stabilization, it was shown in [2] and [15] that, in general, global asymptotic stabilization of linear systems subject to input saturation cannot be achieved using linear feedback laws. On the other hand, it was shown in [14] that a linear system subject to input saturation can be globally asymptotically stabilized by nonlinear feedback if and only if the system in the absence of saturation is asymptotically null controllable with bounded controls. (This condition, as shown in [12] and [13], is equivalent to the system being stabilizable in the usual linear sense and having open-loop poles in the closed left half plane.) A nested feedback design technology for designing nonlinear globally asymptotically stabilizing feedback laws was proposed in [17] for a chain of integrators of length n and was fully generalized in [16].

In the recent work of [4] and [5], the notion of semiglobal stabilization of linear systems subject to input saturation was introduced. The semiglobal framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an *a priori* given (arbitrarily large) bounded set. In [4] and [5] it was shown that, under the appropriate conditions, both for discrete time and continuous time one can achieve semiglobal stabilization of linear systems subject to input saturation using linear feedback laws. In [4] and [5], a low-gain design technology was proposed to construct semiglobal stabilizing controllers. These low-gain control laws were constructed in such a way that the control input does not saturate for any *a priori* given arbitrarily large bounded set of initial conditions. The low-gain design method given in [4] and [5] is based on the eigenstructure assignment and is referred to as a direct method design for low-gain controllers. Later on, however, algebraic Riccati equation (ARE)-based methods utilizing H_2 and H_∞ optimal control theory for designing low-gain controllers were also proposed independently in [9] and [19].

In a recent paper [7] (see also [6]), we have introduced yet another design technology, the so-called low-and-high gain design technique, for a chain of integrators subject to input saturation. This design technique was basically conceived for semiglobal control problems beyond stabilization and was related to performance issues such as semiglobal stabilization with enhanced utilization of the available control capacity of the system and semiglobal disturbance rejection.

This paper is a continuation of our earlier efforts in [7], and its goal is twofold: first, to establish the low-and-high gain design technique for general linear systems subject to input saturation, and second, to utilize this design technique to provide a solution to problems of semiglobal asymptotic stabilization, robust semiglobal stabilization for a class of matched uncertainties, and semiglobal disturbance rejection. The proposed low-and-high gain control laws are composite control laws. Namely, they are composed by adding a low-gain control law and a high-gain control law. The design is sequential. First, a low-gain control law is designed using the technique of [4], [9], and [19]. Then, utilizing an appropriate Lyapunov function for the closed-loop system under this low-gain control law, a high-gain control law is constructed. Both low-gain and high-gain controllers are equipped with tuning parameters. The roles of the low-gain and the high-gain controllers are completely separated. The role of the low-gain control law is to ensure, independent from the high-gain controller, 1) the asymptotic stability of the equilibrium of the closed-loop system and 2) that the domain of attraction of the closed-loop system contains an *a priori* given bounded set. In fact, the tuning parameter in the low-gain controller can be tuned to increase the domain of attraction of the equilibrium of the closed-loop system to include any *a priori* given (arbitrarily large) bounded set. On the other hand, the role of the high-gain controller is to achieve performance beyond stabilization, such as disturbance rejection, robustness, and enhancing the utilization of the control capacity. Again, this performance is achieved by appropriate choice of the tuning parameter of the high-gain controller. We should also emphasize that the

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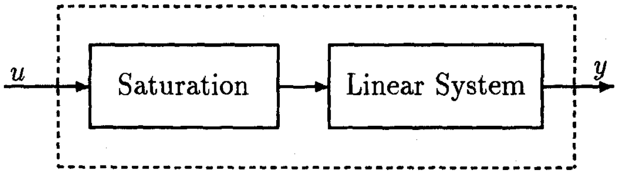

Fig. 1. Linear system subject to input saturation.

low-and-high gain design is, in general, a semiglobal design technology.

In the design of low-and-high gain control laws, one can utilize either the ARE-based method or direct method in the design of their low-gain components. This leads to two methods of low-and-high gain control design which we referred to as the ARE-based and the direct method. The ARE-based method is more compact and elegant, while the direct method seems to be numerically superior. In this paper, we have chosen the ARE-based method as the primary method to establish our results. We will not engage ourselves with the numerical aspects of these methods, and we have left them as a future research problem.

This paper is organized as follows. In Section II, we pose the problems to be solved. In Section III we present a low-and-high gain state-feedback design technique which leads to our state feedback results in Section IV. In Section V we give the low-and-high gain output feedback design which leads to our output feedback results of Section VI. Finally, we draw some concluding remarks on our current work in Section VII.

Throughout the paper, A' denotes the transpose of the matrix A, $\lambda(A)$ denotes the set of eigenvalues of the matrix A, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote, respectively, the minimal and maximal eigenvalue of the positive definite matrix P, I_r denotes the identity matrix of dimension $r \times r$, and I denotes the identity matrix of appropriate dimension. $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. The open left half s-plane is denoted by $\mathbb{C}^-$. A function $f: \mathcal{W} \to \mathbb{R}_+$ is said to be positive definite on $\mathcal{W}_0 \subset \mathcal{W}$ if $f(x)$ is strictly positive for all $x \in \mathcal{W}_0$. For a continuous function $V: \mathbb{R}^n \to \mathbb{R}_+$, a level set $L_V(c)$ is defined as $L_V(c) := \{x \in \mathbb{R}^n : V(x) \leq c\}$.

II. Definitions and Problem Statements

We consider a class of nonlinear systems which are obtained by cascading linear systems with memory-free input nonlinearities of saturation type

$$\Sigma_0: \begin{cases} \dot{x} = Ax + B\sigma(u) \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurement output, and $\sigma: \mathbb{R}^m \to \mathbb{R}^m$ is a saturation function defined as follows.

Definition 1: A function $\sigma: \mathbb{R}^m \to \mathbb{R}^m$ is called a saturation function if

1) $\sigma(u)$ is decentralized, i.e., $\sigma(u) = [\sigma_1(u_1), \cdots, \sigma_m(u_m)]'$,
2) σ_i is locally Lipschitz,
3) $s\sigma_i(s) > 0$ whenever $s \neq 0$,
4) $\min\left\{\lim_{s\to 0^+} \frac{\sigma_i(s)}{s}, \lim_{s \to 0^-} \frac{\sigma_i(s)}{s}\right\} > 0$,
5) $\liminf_{|s|\to\infty} |\sigma_i(s)| > 0$.

Remark 1:

1) Graphically, the saturation function resides in the first and third quadrants and there exist $\Delta > 0$ and $k > 0$ such that the saturation function lies in the nonlinear sector between the vertical axis and the graph $(s, k\,\text{sat}_\Delta(s))$, where $\text{sat}_\Delta(s) = \text{sign}(s)\min\{|s|, \Delta\}$, i.e.,

$$s[\sigma_i(\alpha s) = k\,\text{sat}_\Delta(s)] \geq 0, \qquad \forall\, \alpha \geq 1. \tag{2}$$

Since our designs will require knowledge of (a lower bound on) k, we can, without loss of generality, assume that $k = 1$. Otherwise, if $k < 1$, we can redefine B to be kB and redefine σ to be $k^{-1}\sigma$ so that the new function σ lies between the vertical axis and the graph $(s, \text{sat}_\Delta(s))$.

2) Since σ_i is locally Lipschitz, there exists a function $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each i

$$\|\sigma_i(s+d) - \sigma_i(s)\| \leq \delta(\|d\|)\|d\| \qquad \forall\, s: \|s\| \leq \Delta. \tag{3}$$

Remark 2: It follows directly from Definition 1 that the functions $\sigma(t) = t$, $\arctan(t)$, $\tanh(t)$, and the standard saturation function $\sigma(t) = \text{sign}(t)\min\{|t|, 1\}$ are all saturation functions as defined in Definition 1.

Definition 2: The set of all saturation functions that satisfy the properties (2) and (3) with a fixed constant Δ, $k = 1$ and a function δ is denoted by $\mathcal{S}(\Delta, \delta)$.

We also make the following standing assumptions on the system Σ_0.

Assumption 1: The pair (A, B) is asymptotically null controllable with bounded controls, i.e.,

1) The eigenvalues of A have nonpositive real part.
2) The pair (A, B) is stabilizable.

Assumption 2: The pair (C, A) is detectable.

Clearly, the assumptions on stabilizability and detectability are necessary. The requirement that all eigenvalues have nonpositive part is also necessary for semiglobal stabilization. Without such an assumption on the open-loop eigenvalues, interesting local results are still possible and will be pointed out later.

We will be interested in controllers which guarantee (possibly robust) semiglobal stabilization or disturbance attenuation with respect to input additive uncertainties and disturbances. We denote the system Σ_0 with these perturbations as Σ_{ud}

$$\Sigma_{ud}: \begin{cases} \dot{x} = Ax + B\sigma(u + g(x, t)) \\ y = Cx \end{cases} \tag{4}$$

where $g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ represents both the uncertainties and the disturbances.

Regarding the uncertain element g, we only require knowing an upper bound on its norm. More specifically, we make the following assumption.

Assumption 3: The uncertain element $g(x, t)$ is piecewise continuous in t and locally Lipschitz in x, and its norm is bounded by a known function

$$\|g(x, t)\| \le g_0(\|x\|) + D_0, \qquad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{5}$$

where D_0 is a known positive constant and the known function $g_0(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and satisfies

$$g_0(0) = 0. \tag{6}$$

We will be interested in finding controllers that achieve semiglobal results independent of the precise $\sigma \in \mathcal{S}(\Delta, \delta)$ and independent of the precise g that satisfies Assumption 3. To state the problems we will solve, we make the preliminary definition.

Definition 3: The data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$ is said to be admissible for state feedback if Δ is a strictly positive real number, $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz with $g_0(0) = 0$, D_0 is a nonnegative real number, $\mathcal{W}$ is a bounded subset of $\mathbb{R}^n$, and $\mathcal{W}_0$ is a subset of $\mathbb{R}^n$ which contains the origin as an interior point.

The main state feedback problem we will consider is the following.

Problem 1: Given the data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$, admissible for state feedback, find a feedback gain matrix F such that, for all $\sigma \in \mathcal{S}(\Delta, \delta)$ and all $g(x, t)$ satisfying Assumption 3 with (g_0, D_0), the closed-loop system Σ_{ud} with the control $u = Fx$ satisfies the following conditions:

1) If $D_0 = 0$, the point $x = 0$ is locally uniformly asymptotically stable and $\mathcal{W}$ is contained in its basin of attraction.
2) If $D_0 > 0$, every trajectory starting from $\mathcal{W}$ enters and remains in $\mathcal{W}_0$ after some finite time.

Remark 3: Corresponding to specific values for (g_0, D_0), this problem is given special names. For the case when $g_0 \equiv 0$ and $D_0 = 0$, this is called the semiglobal stabilization by state feedback problem. When $g_0 \not\equiv 0$ but $D_0 = 0$, this is called the robust semiglobal stabilization by state feedback problem. When $g_0 \equiv 0$ but $D_0 > 0$, this is called the semiglobal disturbance rejection by state feedback problem. When $g_0 \not\equiv 0$ and $D_0 > 0$, this is called the robust semiglobal disturbance rejection by state feedback problem. Since the choice of F depends on (g_0, D_0), the solution to Problem 1 is automatically adapted to the appropriate special problem.

Definition 4: The data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$ is said to be admissible for output feedback if Δ is a strictly positive real number, $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz with $g_0(0) = 0$, D_0 is a nonnegative real number, $\mathcal{W}$ is a bounded subset of $\mathbb{R}^{2n}$ and $\mathcal{W}_0$ is a subset of $\mathbb{R}^{2n}$ which contains the origin as an interior point.

The main output feedback problem we will consider is the following.

Problem 2: Given the data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$, admissible for output feedback, find matrices (A_c, B_c, C_c) such that, for all $\sigma \in \mathcal{S}(\Delta, \delta)$ which are uniformly bounded over $\mathcal{S}(\Delta, \delta)$ and all $g(x, t)$ satisfying Assumption 3 with (g_0, D_0), the closed-loop system Σ_{ud} with the compensator

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y, & x_c \in \mathbb{R}^n \\ u = C_c x_c \end{cases}$$

satisfies the following conditions:

1) If $D_0 = 0$, the point $(x, x_c) = (0, 0)$ is locally asymptotically stable and $\mathcal{W}$ is contained in its basin of attraction.
2) If $D_0 > 0$, every trajectory starting from $\mathcal{W}$ enters and remains in $\mathcal{W}_0$ after some finite time.

Remark 4: The comments of Remark 3 apply again, this time for output feedback.

Remark 5: The condition that σ be uniformly bounded over $\mathcal{S}(\Delta, \delta)$ is a condition that is used to guarantee that the compensator can be chosen to be linear. If this property does not hold, it turns out that the problem has a solution if one allows a nonlinear compensator. The approach is then to saturate the control term of the compensator outside of the domain of interest so that, effectively, σ is bounded. This idea was first introduced in [1] and later exploited for very general nonlinear output feedback problems in [21] and [20]. For a further discussion of this idea, see Remark 12.

III. Low-and-High Gain State Feedback Design—An ARE-Based Method

In this section, we provide an algorithm for the construction of low-and-high gain state feedback control laws for the plant Σ_0.

As we stated in the introduction, the low-and-high gain state feedback law is a composite control law. Namely, it is composed by adding together a low-gain control law and a high-gain control law. The design is sequential. First a low-gain control law is designed, and then a high-gain control law is constructed.

Our algorithm for the design of low-and-high gain state feedback laws is divided into three steps. Steps 1 and 2 deal, respectively, with the design of the low-gain control and the high-gain control. In Step 3, the low-and-high gain control is composed by adding together the low-gain and the high-gain control designed in Steps 1 and 2.

Step 1—Low-Gain Design: We start by choosing a continuous function $Q : (0, 1] \to \mathbb{R}^{n \times n}$ such that $Q(\epsilon)$ is positive definite for each $\epsilon \in (0, 1]$ and

$$\lim_{\epsilon \to 0} Q(\epsilon) = 0. \tag{7}$$

A simple choice is $Q(\epsilon) = \epsilon I$.

Next, we form the following ARE:

$$A'P(\epsilon) + P(\epsilon)A - P(\epsilon)BB'P(\epsilon) + Q(\epsilon) = 0. \tag{8}$$

We have the following lemma regarding (8). This lemma was first proved in [9]. The proof is short, and for completeness, we repeat it here.

Lemma 1: Assume (A, B) is stabilizable and A has all its eigenvalues in the closed left-half plane. Then, for all $\epsilon > 0$, there exists a unique matrix $P(\epsilon) > 0$ which solves the ARE (8). Moreover

$$\lim_{\epsilon \to 0} P(\epsilon) = 0. \tag{9}$$

Proof: The existence of a unique positive definite solution $P(\epsilon)$ for all $\epsilon > 0$ has been established in [23]. The same paper established that for $\epsilon = 0$ there is a unique solution $P(0) = 0$ for which $A - BB'P(0)$ has all eigenvalues in the closed left half plane. Continuity of the solution of the ARE for $\epsilon = 0$ (in other words, that $P(\epsilon) \to P(0) = 0$ as $\epsilon \to 0$) has been established in [22].

The low-gain state feedback control law is then formed as

$$u_L = F_L(\epsilon)x \tag{10}$$

where

$$F_L(\epsilon) := -B'P(\epsilon) \tag{11}$$

and where $P(\epsilon)$ is the positive definite solution of (8). We refer to the control law (10) as a low-gain state feedback control law and ϵ as the low-gain parameter since, in view of Lemma 1, one can make the norm of $F_L(\epsilon)$ arbitrarily small by choosing ϵ sufficiently small.

Step 2—High-Gain Design: We form the high-gain state feedback control law as

$$u_H = F_H(\epsilon, \rho)x \tag{12}$$

where

$$F_H(\epsilon, \rho) := -\rho B'P(\epsilon), \qquad \rho \geq 0 \tag{13}$$

and where $P(\epsilon)$ is the same as in Step 1 and is the positive definite solution of (8). We refer to the nonnegative parameter ρ as the high-gain parameter.

Step 3—Low-and-High-Gain Design: The family of parameterized low-and-high gain state feedback laws, denoted by $\Sigma_{LH}^S(\epsilon, \rho)$, is simply formed by adding together the low-gain control and the high-gain control as designed in the previous steps. Namely

$$\Sigma_{LH}^S(\epsilon, \rho)\colon\ u = u_L + u_H = F_{LH}(\epsilon, \rho)x \tag{14}$$

where

$$F_{LH}(\epsilon, \rho) = F_L(\epsilon) + F_H(\epsilon, \rho) = -(1+\rho)B'P(\epsilon). \tag{15}$$

Remark 6: The ARE-based low-and-high gain state feedback design is actually an optimal design for the linear system (A, B) in the absence of input saturation with appropriately chosen Q_n and R_n. More specifically, choosing $R_n = I/(1+\rho)$ and $Q_n = Q + \rho PBB'P$, it is easy to verify that P is the solution to the new ARE

$$A'P + PA - PBR_n^{-1}B'P + Q_n = 0 \tag{16}$$

and hence $u = -(1+\rho)B'Px = -B'R_n^{-1}Px$ and is an optimal control.

IV. State Feedback Results

Theorem 1: Let Assumption 1 hold. Given the data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$, admissible for state feedback, there exists $\epsilon^*(\Delta, \mathcal{W})$ and, for each $\epsilon \in (0, \epsilon^*]$, there exists $\rho^*(\epsilon, \Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$ such that, for $\epsilon \in (0, \epsilon^*]$ and $\rho \geq p^*$, the low-and-high gain feedback matrix of $\Sigma_{LH}^S(\epsilon, \rho)$ solves Problem 1. Moreover, if $D_0 = 0$, then ρ^* is independent of $\mathcal{W}_0$; if, in addition, $g_0 \equiv 0$ then $\rho^* \equiv 0$.

Remark 7 (Infinite Gain Margin): We note that the low-and-high gain control law which solves Problem 1 has infinite gain margin. That is, one can increase the gain of the controller arbitrarily and the resulting closed-loop system is guaranteed to satisfy the property required in the solution of Problem 1. This follows from the fact that the effect of increasing the controller gain is the same as the effect of increasing ρ.

Remark 8 (Phase Margin): Note that the control law given in Theorem 1 works for all $\sigma \in \mathcal{S}(\Delta, \delta)$ and all g satisfying Assumption 3 with (g_0, D_0). In particular, consider the case where $\sigma(s) = s$, $g \equiv 0$, and $D_0 = 0$. In this case, it is straightforward to show that for each $\epsilon > 0$ and each $\rho \geq 0$, the closed-loop system is globally exponentially stable and, moreover, in view of Remark 6, the control law has $\pm$ 60° phase margin. It is a very interesting open question to see whether such an impressive phase margin is preserved for other members of $\mathcal{S}(\Delta, \delta)$.

Remark 9: The freedom in choosing the high-gain parameter ρ arbitrarily large can be employed to achieve full utilization of the available control capacity. In particular, by increasing ρ, we can increase the utilization of the available control capacity. In fact, as shown in [7], as $\rho \to \infty$, $\sigma(u)$ approaches a bang-bang control.

Remark 10: For the case when $\rho = 0$, the control law $\Sigma_{LH}^S(\epsilon, \rho)$ reduces to the low-gain control law given earlier in [9] and [19].

To prove the state feedback results, we will need the following lemma.

Lemma 2: Given $(\Delta, \delta, g_0, D_0)$, a subset of admissible data for state feedback, let $\epsilon \in (0, 1]$ and c be a strictly positive real number such that, using the notation $P := P(\epsilon)$, we have

$$\|B'P^{\frac{1}{2}}z\| \leq \Delta \qquad \forall\, z \in \{z \in \mathbb{R}^n : \|z\| \leq c\}. \tag{17}$$

Define

$$F = c\sqrt{\lambda_{\min}(P)^{-1}} \tag{18}$$

$$M = \sup_{s \in (0, F]} \left\{\frac{g_0(s)}{s}\right\}, \qquad N = \max_{s \in [0, D_0 + MF]} \delta(2s) \tag{19}$$

$$\begin{aligned} \rho_1^* &:= \rho_1^*(\epsilon) = 16mM^2N\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)} \\ \rho_2^* &:= \rho_2^*(\epsilon) = 16mD_0^2N\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\frac{2}{c^2} \end{aligned} \tag{20}$$

and

$$\rho^* := \rho^*(\epsilon) = \max\{\rho_1^*, \rho_2^*\}. \tag{21}$$

Assume $\rho \geq \rho^*$. For the system Σ_{ud} where g satisfies Assumption 3 with (g_0, D_0) and with the control $\Sigma_{LH}^S(\epsilon, \rho)$, the function $V(x) = x'Px$ satisfies the following conditions:

1) If $\rho^* = 0$, then

$$x \in \{x \in \mathbb{R}^n : 0 < V(x) \leq c^2\} \Rightarrow \dot{V} < 0. \tag{22}$$

2) If $\rho^* > 0$, then

$$x \in \left\{x \in \mathbb{R}^n : \frac{\rho_2^*}{\rho}\frac{c^2}{2} < V(x) \leq c^2\right\} \Rightarrow \dot{V} < 0. \tag{23}$$

Proof: Defining $z = P^{\frac{1}{2}}x$ and using Σ_{ud}, we have

$$\dot{z} = P^{\frac{1}{2}}AP^{-\frac{1}{2}}z + P^{\frac{1}{2}}B\sigma\Big(u + g\Big(P^{-\frac{1}{2}}z,\, t,\, d(t)\Big)\Big). \tag{24}$$

$\Sigma_{HL}^S(\epsilon, \rho)$ becomes

$$u = -(1+\rho)B'P^{\frac{1}{2}}z \doteq (1+\rho)u_L. \tag{25}$$

Consider the function $V = z'z$ and its derivative in the set $L_V(c^2)$. Using (8) and the definition of u_L we have

$$\begin{aligned}\dot{V} = &-z'\Big(P^{-\frac{1}{2}}QP^{-\frac{1}{2}} + P^{\frac{1}{2}}BB'P^{\frac{1}{2}}\Big)z\\ &-2u_L'\Big[\sigma\Big((1+\rho)u_L + g\Big(P^{-\frac{1}{2}}z,\, t,\, d(t)\Big)\Big) - u_L\Big].\end{aligned} \tag{26}$$

Using u_L^i and g_i to denote the ith component of the vectors u_L and g, respectively, it follows from (2) and (17) that, for all $z \in L_V(c^2)$

$$|\rho u_L^i| \geq |g_i| \Rightarrow -2u_L^i[\sigma_i((1+\rho)u_L^i + g_i) - u_L^i] \leq 0. \tag{27}$$

To see this, first note that from (17), $z \in L_V(c^2)$ implies $u_L^i = \mathrm{sat}_\Delta(u_L^i)$. Next, define $\alpha' = g_i/\rho u_L^i$ when $\rho u_L^i \neq 0$. For the case considered in (27), $|\alpha'| \leq 1$ and $(1+\rho)u_L^i + g_i = (1+\rho+\alpha'\rho)u_L^i =: \alpha u_L^i$. Since $\alpha \geq 1$, (27) follows from (2).

In particular, if $\rho^* = 0$, then $g \equiv 0$ and point 1) of the lemma follows. In addition, for $\rho \geq \rho^* > 0$, it follows from (3) that:

$$\begin{aligned}|\rho u_L^i| \leq |g_i| \Rightarrow &-2u_L^i[\sigma_i((1+\rho)u_L^i + g_i) - u_L^i]\\ &= -2u_L^i[\sigma_i((1+\rho)u_L^i + g_i)\\ &\quad - \sigma_i(u_L^i) + \sigma_i(u_L^i) - \mathrm{sat}_\Delta(u_L^i)]\\ &\leq \frac{4|g_i|^2\delta(2|g_i|)}{\rho}\\ &\leq \frac{4(2D_0^2 + 2M^2\lambda_{\min}(P)^{-1}\|z\|^2)N}{\rho}.\end{aligned} \tag{28}$$

Hence, we can conclude, for all $z \in L_V(c^2)$

$$\begin{aligned}\dot{V} \leq &-z'\Big(P^{-\frac{1}{2}}QP^{-\frac{1}{2}} + P^{\frac{1}{2}}BB'P^{\frac{1}{2}}\Big)z\\ &+\frac{1}{\rho}[8mD_0^2N + 8m\lambda_{\min}(P)^{-1}M^2N\|z\|^2]\\ \leq &-\left[0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} - \frac{1}{\rho}8mM^2N\lambda_{\min}(P)^{-1}\right]\|z\|^2\\ &-\left[0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\|z\|^2 - \frac{1}{\rho}8mD_0^2N\right].\end{aligned} \tag{29}$$

Since $\rho \geq \rho_1^*$, we get, for all $z \in L_V(c^2)$

$$\dot{V} \leq -0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\left(\|z\|^2 - \frac{\rho_2^*}{\rho}\frac{c^2}{2}\right). \tag{30}$$

Since $z'z = x'Px$, we get point 2) of the lemma.

Proof: Proof of Theorem 1: Let c be a strictly positive real number such that

$$c^2 \geq \sup_{x\in\mathcal{W},\epsilon\in(0,1]} x'P(\epsilon)x. \tag{31}$$

The right-hand side is well defined since $\lim_{\epsilon\to 0} P(\epsilon) = 0$ and $\mathcal{W}$ is bounded. Let $\epsilon^* \in (0, 1]$ be such that (17) is satisfied for each $\epsilon \in (0, \epsilon^*]$. Such an ϵ^* exists as a result of Lemma 1. Moreover, ϵ^* depends only on $\mathcal{W}$ and Δ. Fix $\epsilon \in (0, \epsilon^*]$.

Consider the case where $D_0 = 0$. Then ρ_2^* defined in (20) is equal to zero. So, if $\rho \geq \rho_1^*$, it follows from point 2) of the lemma that the point $x = 0$ is locally asymptotically stable with basin of attraction containing the set $\mathcal{W}$. Notice also that ρ^* is independent of $\mathcal{W}_0$. Moreover, if we also have $g_0 \equiv 0$, then $\rho_1^* = 0$.

Now consider the case where $D_0 > 0$. Let $\nu(\epsilon)$ be a strictly positive real number such that, with $V = x'P(\epsilon)x$

$$L_V(\nu) \subset \mathcal{W}_0. \tag{32}$$

Such a strictly positive real number exists because $\mathcal{W}_0$ has the origin as an interior point and $P(\epsilon) > 0$. It then follows from the lemma that if we set

$$\rho^* = \max\left\{\rho_1^*,\, \rho_2^*,\, \frac{\rho_2^* c^2}{2\nu}\right\} \tag{33}$$

then we get

$$x \in \{x \in \mathbb{R}^n : \nu < V(x) \leq c^2\} \Rightarrow \dot{V} < 0. \tag{34}$$

By the choices of c and ν, the solutions which start in $\mathcal{W}$ enter and remain in the set $\mathcal{W}_0$ after some finite time. □

Remark 11: As can be seen from the proof of Theorem 1, when there are open-loop eighenvalues in the open right-half plane, we do not get that $P \to 0$. Instead, we can fix Q to be positive definite which fixes $P > 0$. Now we can pick c such that (17) holds. Then with any $\mathcal{W} = L_V(c^2)$ (instead of arbitrarily large), the same results hold. This remark also applies to the output feedback results.

V. Low-and-High Gain Output Feedback Design

A. Preliminaries—Special Coordinate Basis

In this section we recapitulate the special coordinate basis (scb) theorem of [11] which has a distinct feature of explicitly displaying the finite and infinite zero structure of a given linear time invariant system. Such a special coordinate is instrumental in our high-gain observer design. A software package in the Matlab environment to generate the scb for any given system is given in [8].

Theorem 2 (Special Coordinate Basis): Consider the linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{35}$$

where the state vector $x \in \mathbb{R}^n$, output vector $y \in \mathbb{R}^p$, and input vector $u \in \mathbb{R}^m$. Without loss of generality, assume that both B and C are of maximal rank. Then there exist nonsingular transformations Γ_S, Γ_O, and Γ_I, and integers m_b and m_f, and integer indexes r_i, $i = 1$ to m_b and q_i, $i = 1$ to m_f, such that

$$x = \Gamma_S \overline{x}, \qquad y = \Gamma_O \overline{y}, \qquad u = \Gamma_I[\overline{u}', \overline{v}']'$$

$$\overline{x} = [\overline{x}_a', \overline{x}_b', \overline{x}_c', \overline{x}_f']'$$

$$\overline{x}_b = [\overline{x}_{b1}', \overline{x}_{b2}', \cdots, \overline{x}_{bm_b}']', \quad \overline{x}_{bi} = [\overline{x}_{bi1}, \overline{x}_{bi2}, \cdots, \overline{x}_{bir_i}]'$$

$$\overline{x}_f = [\overline{x}_{f1}', \overline{x}_{f2}', \cdots, \overline{x}_{fm_f}']', \quad \overline{x}_{fi} = [\overline{x}_{fi1}, \overline{x}_{fi2}, \cdots, \overline{x}_{fiq_i}]'$$

$$\overline{y} = [\overline{y}_b', \overline{y}_f']', \qquad \overline{y}_b = [\overline{y}_{b1}, \overline{y}_{b2}, \cdots, \overline{y}_{bm_b}]'$$

$$\overline{y}_f = [\overline{y}_{f1}, \overline{y}_{f2}, \cdots, \overline{y}_{fm_f}]'$$

$$\overline{u} = [\overline{u}_1, \overline{u}_2, \cdots, \overline{u}_{m_f}]'$$

$$\dot{\overline{x}}_a = A_{aa}\overline{x}_a + L_{ab}\overline{y}_b + L_{af}\overline{y}_f \tag{36}$$

$$\dot{\overline{x}}_c = A_{cc}\overline{x}_c + L_{cb}\overline{y}_b + L_{cf}\overline{y}_f + B_c[E_{ca}\overline{x}_a + \overline{v}] \tag{37}$$

and for $i = 1$ to m_b

$$\dot{\overline{x}}_{bi} = A_{r_i}\overline{x}_{bi} + L_{bib}\overline{y}_b + L_{bif}\overline{y}_f \tag{38}$$

$$\overline{y}_{bi} = C_{r_i}\overline{x}_{bi} = \overline{x}_{bi1}. \tag{39}$$

For $i = 1$ to m_f

$$\dot{\overline{x}}_{fi} = A_{q_i}\overline{x}_{fi} + L_i\overline{y}_f + B_{q_i}[\overline{u}_i + E_{ia}\overline{x}_a + E_{ib}\overline{x}_b + E_{ic}\overline{x}_c + E_{if}\overline{x}_f] \tag{40}$$

$$\overline{y}_{fi} = C_{q_i}\overline{x}_{fi} = \overline{x}_{fi1}. \tag{41}$$

Here the states $\overline{x}_a$, $\overline{x}_b$, $\overline{x}_c$, and $\overline{x}_f$ are, respectively, of dimensions n_a, n_b, n_c, and n_f with

$$n_b = \sum_{i=1}^{m_b} r_i, \qquad n_f = \sum_{i=1}^{m_f} q_i$$

while $\overline{x}_{bi}$ is of dimension r_i for $i = 1$ to m_b and $\overline{x}_{fi}$ is of dimension q_i for $i = 1$ to m_f. The control vectors $\overline{u}$ and $\overline{v}$ are, respectively, of dimension m_f and $m_v = m - m_f$ while the output vectors $\overline{y}_b$ and $\overline{y}_f$ are, respectively, of dimension m_b and m_f with $m_b + m_f = p$. Also, for an integer $r \geq 1$

$$A_r = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}, \qquad B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_r = [1 \quad 0].$$

(Obviously for the case when $r = 1$, $A_r = 0$, $B_r = 1$, and $C_r = 1$.) Furthermore, the pair (A_c, B_c) is controllable. Also, the last row of each L_i is identically zero.

Proof: The proof follows from [13, Theorem 2.1]. From the proof, we observe that when the state $\overline{x}_b$ is nonexistent, $\Gamma_O = I$. Dually, when the state $\overline{x}_c$ is nonexistent, $\Gamma_I = I$. □

In what follows, we state some important properties of the scb which are pertinent to our present work. The proofs of these properties can be found in [11].

Property 1: The given system (A, B, C) is right invertible if and only if $\overline{x}_b$ and hence $\overline{y}_b$ are nonexistent, left-invertible if and only if $\overline{x}_c$ and hence $\overline{v}$ are nonexistent, and invertible if and only if both $\overline{x}_b$ and $\overline{x}_c$ are nonexistent.

Property 2: Invariant zeros of the system (A, B, C) are the eigenvalues of A_{aa}.

Property 3: The pair (A, B) is controllable (stabilizable) if and only if $(A_{\text{con}}, B_{\text{con}})$ is controllable (stabilizable), where

$$A_{\text{con}} = \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \qquad B_{\text{con}} = \begin{bmatrix} L_{af} \\ L_{bf} \end{bmatrix}$$

$$C_b = \begin{bmatrix} C_{r_1} & 0 & \cdots & 0 \\ 0 & C_{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{r_{m_b}} \end{bmatrix}, \qquad L_{bf} = \begin{bmatrix} L_{b1f} \\ L_{b2f} \\ \vdots \\ L_{bm_bf} \end{bmatrix}$$

and

$$A_{bb} = \begin{bmatrix} A_{r_1} & 0 & \cdots & 0 \\ 0 & A_{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_{r_{m_b}} \end{bmatrix} + \begin{bmatrix} L_{b1b}C_b \\ L_{b2b}C_b \\ \vdots \\ L_{bm_bb}C_b \end{bmatrix}.$$

In particular, if the system (A, B, C) is right invertible, then it is controllable (stabilizable) if and only if (A_{aa}, L_{af}) is controllable (stabilizable).

Similarly, the pair (A, C) is observable (detectable) if and only if $(A_{\text{obs}}, C_{\text{obs}})$ is observable (detectable), where

$$A_{\text{obs}} = \begin{bmatrix} A_{aa} & 0 \\ B_cE_{ca} & A_{cc} \end{bmatrix}, \qquad C_{\text{obs}} = [E_a \quad E_c]$$

$$E_a = [E_{1a}' \quad E_{2a}' \quad \cdots \quad E_{m_fa}']'$$

$$E_c = [E_{1c}' \quad E_{2c}' \quad \cdots \quad E_{m_fc}']'.$$

In particular, if the system (A, B, C) is left invertible, then it is observable (detectable) if and only if (A_{aa}, E_a) is observable (detectable).

B. Low-and-High Gain Output Feedback Design

We construct a family of parameterized low-and-high gain output feedback control laws, denoted by $\Sigma_{LH}^{O1}(\epsilon, \rho, l)$. This family of control laws have observer-based structure and are constructed by utilizing the high-gain observer as developed in [10] to implement the low-and-high gain state feedback laws constructed previously. To utilize the high-gain observer, we make the following assumption.

Assumption 4: The linear system represented by (A, B, C) is left invertible and minimum phase.

This family of parameterized high-gain observer based low-and-high gain output feedback control laws takes the form of

$$\Sigma_{LH}^{O1}(\epsilon, \rho, l): \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(l)(y - C\hat{x}) \\ u = F_{LH}(\epsilon, \rho)\hat{x} \end{cases} \tag{42}$$

where $L(l)$ is the high-gain observer gain and l is referred to as the high-gain observer parameter. The high-gain observer gain $L(l)$ is constructed in the following three steps.

Step 1: By Assumption 4, the linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ u = Cx \end{cases} \tag{43}$$

is left invertible. By Theorem 2, there exist a nonsingular state transformation and output transformation

$$x = \Gamma_s \overline{x}, \qquad y = \Gamma_o \overline{y}$$

such that

$$\begin{aligned}
\overline{x} &= [\overline{x}_a', \overline{x}_b', \overline{x}_f']' \\
\overline{x}_b &= [\overline{x}_{b1}', \overline{x}_{b2}', \cdots, \overline{x}_{bp-m}']', \qquad \overline{x}_{bi} = [\overline{x}_{bi1}, \overline{x}_{bi2}, \cdots, \overline{x}_{bir_i}]' \\
\overline{x}_f &= [\overline{x}_{f1}', \overline{x}_{f2}', \cdots, \overline{x}_{fm}']', \qquad \overline{x}_{fi} = [\overline{x}_{fi1}, \overline{x}_{fi2}, \cdots, \overline{x}_{fiq_i}]' \\
\overline{y} &= [\overline{y}_b', \overline{y}_f']', \qquad \overline{y}_b = [\overline{y}_{b1}, \overline{y}_{b2}, \cdots, \overline{y}_{bp-m}]' \\
\overline{y}_f &= [\overline{y}_{f1}, \overline{y}_{f2}, \cdots, \overline{y}_{fm}]' \\
u &= [u_1, u_2, \cdots, u_m]' \\
\dot{\overline{x}}_a &= A_{aa}\overline{x}_a + L_{ab}\overline{y}_b + L_{af}\overline{y}_f
\end{aligned} \tag{44}$$

and for $i = 1$ to $p - m$

$$\dot{\overline{x}}_{bi} = A_{r_i}\overline{x}_{bi} + L_{bib}\overline{y}_b + L_{bif}\overline{y}_f \tag{45}$$

$$\overline{y}_{bi} = C_{r_i}\overline{x}_{bi} = \overline{x}_{bi1} \tag{46}$$

for $i = 1$ to m

$$\dot{\overline{x}}_{fi} = A_{q_i}\overline{x}_{fi} + L_i\overline{y}_f + B_{q_i}[u_i + E_{ia}\overline{x}_a + E_{ib}\overline{x}_b + E_{if}\overline{x}_f] \tag{47}$$

$$\overline{y}_{fi} = C_{q_i}\overline{x}_{fi} = \overline{x}_{fi1} \tag{48}$$

where, for an integer $r \geq 1$

$$A_r = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}, \qquad B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_r = [1 \quad 0].$$

Step 2: For $i = 1$ to $p - m$, choose $L_{bi} \in \mathbb{R}^{r_i \times 1}$ such that

$$\lambda(A_{r_i}^c) \in \mathbb{C}^-, \qquad A_{r_i}^c := A_{r_i} - L_{bi}C_{r_i}.$$

Note that the existence of such an L_{bi} is guaranteed by the special structure of the matrix pair (A_{r_i}, C_{r_i}).

Similarly, for $i = 1$ to m, choose $L_{fi} \in \mathbb{R}^{q_i \times 1}$ such that

$$\lambda(A_{q_i}^c) \in \mathbb{C}^-, \qquad A_{q_i}^c := A_{q_i} - L_{fi}C_{q_i}.$$

Again, the existence of such an L_{fi} is guaranteed by the special structure of the matrix pair (A_{q_i}, C_{q_i}).

Step 3: For any $l \in (0, 1]$, define a matrix $L(l) \in \mathbb{R}^{n \times p}$ as

$$L(l) = \Gamma_s \begin{bmatrix} L_{ab} & L_{af} \\ L_{bb} + L_b(l) & L_{bf} \\ 0 & L_{ff} + L_f(l) \end{bmatrix} \Gamma_o^{-1} \tag{49}$$

where

$$L_{bb} = \begin{bmatrix} L_{b1b} \\ L_{b2b} \\ \vdots \\ L_{bp-mb} \end{bmatrix}, \quad L_{bf} = \begin{bmatrix} L_{b1f} \\ L_{b2f} \\ \vdots \\ L_{bmf} \end{bmatrix}, \quad L_{ff} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix}$$

$$L_b(l) = \begin{bmatrix} S_{r_1}(l)L_{b1} & 0 & \cdots & 0 \\ 0 & S_{r_2}(l)L_{b2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{r_{p-m}}(l)L_{bp-m} \end{bmatrix}$$

$$L_f(l) = \begin{bmatrix} S_{q_1}(l)L_{f1} & 0 & \cdots & 0 \\ 0 & S_{q_2}(l)L_{f2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{q_m}(l)L_{fm} \end{bmatrix}$$

and where for any integer $r \geq 1$

$$S_r(l) = \begin{bmatrix} l & 0 & \cdots & 0 \\ 0 & l^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l^r \end{bmatrix}.$$

VI. Output Feedback Results

Theorem 3: Let Assumptions 1 and 4 hold. Given the data $(\Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$, admissible for the output feedback problem, there exists $\epsilon^*(\Delta, \mathcal{W})$, for each $\epsilon \in (0, \epsilon^*]$ there exists $\rho^*(\epsilon, \Delta, \delta, g_0, D_0, \mathcal{W}, \mathcal{W}_0)$, and for each $\epsilon \in (0, \epsilon^*]$, $\rho \geq \rho^*$ there exists $l^*(\epsilon, \rho)$, such that, for $\epsilon \in (0, \epsilon^*]$, $\rho \geq \rho^*$, and $l \geq l^*(\epsilon, \rho)$, the matrices of the high-gain observer based low-and-high gain output feedback control law $\Sigma_{LH}^{O1}(\epsilon, \rho, l)$, as given by (42), solve Problem 2. Moreover, if $D_0 = 0$, then ρ^* is independent of $\mathcal{W}_0$; if, in addition, $g_0 \equiv 0$, then $\rho^* \equiv 0$.

This result is obtained by utilizing high-gain observers which motivated Assumption 4. High-gain observers are not needed when $g \equiv 0$ and ρ is chosen equal to zero. This result is covered, for example, in [19] and is not repeated here.

To prove the output feedback results we will use two lemmas. Let Σ_{ude} represent a system of the form

$$\begin{aligned} \dot{x} &= Ax + B[\sigma(u + g(x + Te, t, d(t))) + Ee] \\ \dot{e} &= A_o e \end{aligned} \tag{50}$$

where $x \in \mathbb{R}^n$, $e \in \mathbb{R}^m$. Assume A_o is Hurwitz and let P_o be the positive definite solution to the Lyapunov equation

$$A_o'P_o + P_oA_o = -I. \tag{51}$$

Also, let

$$\tau = \sqrt{\lambda_{\max}(E'E)}, \qquad \kappa = \sqrt{\lambda_{\max}(T'T)}. \tag{52}$$

Lemma 3: Given $(\Delta, \delta, g_0, D_0)$, a subset of admissible data for output feedback, let $\epsilon \in (0, 1]$ and c be a strictly positive real number such that, using the notation $P := P(\epsilon)$, we have

$$\|B'P^{\frac{1}{2}}z\| \leq \Delta \qquad \forall z \in \{z \in \mathbb{R}^n : \|z\| \leq \sqrt{c^2+1}\}. \tag{53}$$

Define

$$\gamma = \frac{\max\{1, (\tau^2+1)\lambda_{\max}(P_o)\}}{\min\left\{1, \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\right\}} \tag{54}$$

$$F = \sqrt{c^2+1}(\sqrt{\lambda_{\min}(P)^{-1}} + \kappa\sqrt{[(\tau^2+1)\lambda_{\min}(P_o)]^{-1}}) \tag{55}$$

$$M = \sup_{s\in(0,F]}\left\{\frac{g_0(s)}{s}\right\}, \qquad N = \max_{s\in[0,D_0+MF]} \delta(2s) \tag{56}$$

$$\rho_1^* := \rho_1^*(\epsilon) = 32mM^2N\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)} \tag{57}$$

$$\rho_2^* := \rho_2^*(\epsilon) = 32mM^2N\kappa^2, \qquad \rho_3^* := \rho_3^*(\epsilon) = \frac{32mD_0^2N}{\gamma(c^2+1)} \tag{58}$$

and

$$\rho^* := \rho^*(\epsilon) = \max\{\rho_1^*, \rho_2^*, \rho_3^*\}. \tag{59}$$

Assume $\rho \geq \rho^*$. For the system Σ_{ude} where g satisfies Assumption 3 with (g_0, D_0) and with the control $\Sigma_{LH}^S(\epsilon, \rho)$, there exists a continuous function $\psi: \mathbb{R}^n \times \mathbb{R}^m$ such that the function

$$V(x, e) = x'Px + (\tau^2+1)e'P_oe \tag{60}$$

satisfies $\dot{V} \leq -\psi(x, e)$ and

1) If $\rho^* = 0$, then
$$(x', e')' \in L_V(c^2+1) \Rightarrow \psi(x, e) \geq 0.5\gamma V. \tag{61}$$

2) If $\rho^* > 0$, then
$$(x', e')' \in L_V(c^2+1) \Rightarrow \psi(x, e) \geq 0.5\gamma\left(V - \frac{\rho_3^*}{\rho}\frac{c^2+1}{2}\right). \tag{62}$$

Proof: Defining $z = P^{\frac{1}{2}}x$ and using Σ_{ude}, we have

$$\begin{aligned}\dot{z} =& P^{\frac{1}{2}}AP^{-\frac{1}{2}}z\\ &+ P^{\frac{1}{2}}B\left[\sigma\left(u + g\left(P^{-\frac{1}{2}}z + Te, t, d(t)\right)\right) + Ee\right]\\ \dot{e} =& A_oe.\end{aligned} \tag{63}$$

$\Sigma_{HL}^S(\epsilon, \rho)$ becomes

$$u = -(1+\rho)B'P^{\frac{1}{2}}z \doteq (1+\rho)u_L. \tag{64}$$

Consider the function V defined in (60) and its derivative in the set $L_V(c^2+1)$. Using (8) and the definition of u_L we have

$$\begin{aligned}\dot{V} =& -z'\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}} + P^{\frac{1}{2}}BB'P^{\frac{1}{2}}\right)z - 2u_L'\Big[\sigma\Big((1+\rho)u_L\\ &+ g\left(P^{-\frac{1}{2}}z + Te, t, d(t)\right)\Big) - u_L\Big]\\ &+ 2u_L'Ee - (\tau^2+1)e'e\\ \leq& -z'P^{-\frac{1}{2}}QP^{-\frac{1}{2}}z - 2u_L'\Big[\sigma\Big((1+\rho)u_L\\ &+ g\left(P^{-\frac{1}{2}}z + Te, t, d(t)\right)\Big) - u_L\Big] - e'e.\end{aligned} \tag{65}$$

The existence of ψ follows from the bound on g. Now, using u_L^i and g_i to denote the ith component of the vectors, u_L and g, respectively, it follows from (2) and (53) that, for all $(z', e')' \in L_V(c^2+1)$:

$$|\rho u_L^i| \geq |g_i| \Rightarrow -2u_L^i[\sigma_i((1+\rho)u_L^i + g_i) - u_L^i] \leq 0. \tag{66}$$

See the discussion following (27).

In particular, if $\rho^* = 0$, then $g \equiv 0$ and point 1 of the lemma follows. In addition, for $\rho \geq \rho^* > 0$, it follows from (3) that:

$$\begin{aligned}|\rho u_L^i| \leq |g_i|&\\ \Rightarrow& -2u_L^i[\sigma_i((1+\rho)u_L^i + g_i) - u_L^i]\\ =& -2u_L^i[\sigma_i((1+\rho)u_L^i + g_i) - \sigma_i(u_L^i) + \sigma_i(u_L^i)\\ &- \mathrm{sat}_\Delta(u_L^i)]\\ \leq& \frac{4|g_i|^2\delta(2|g_i|)}{\rho}\\ \leq& \frac{4\left(2D_0^2 + 2M^2\|P^{-\frac{1}{2}}z + Te\|^2\right)N}{\rho}.\end{aligned} \tag{67}$$

Hence, we can conclude for all $(z', e')' \in L_V(c^2+1)$

$$\begin{aligned}\dot{V} \leq& -z'P^{-\frac{1}{2}}QP^{-\frac{1}{2}}z - e'e + \frac{1}{\rho}8mD_0^2N\\ &+ \frac{1}{\rho}16mM^2N[(\lambda_{\min}(P))^{-1}\|z\|^2 + \kappa^2\|e\|^2]\\ \leq& -\left[0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} - \frac{1}{\rho}16mM^2N\lambda_{\min}(P)^{-1}\right]\|z\|^2\\ &- \left[0.5 - \frac{1}{\rho}16mM^2N\kappa^2\right]\|e\|^2\\ &- 0.5\|e\|^2 - 0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\|z\|^2 + \frac{1}{\rho}8mD_0^2N.\end{aligned} \tag{68}$$

Since $\rho \geq \max\{\rho_1^*, \rho_2^*\}$, we get, for all $(z', e')' \in L_V(c^2+1)$

$$\begin{aligned}\dot{V} \leq& -0.5\|e\|^2 - 0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\|z\|^2 + \frac{1}{\rho}8mD_0^2N\\ \leq& -0.5\gamma V + \frac{1}{\rho}8mD_0^2N\\ \leq& -0.5\gamma\left(V - \frac{\rho_3^*}{\rho}\frac{c^2+1}{2}\right).\end{aligned} \tag{69}$$

□

The following lemma is adapted from [21].

Lemma 4: Consider the nonlinear system

$$\dot{z} = f(z, e, t), \qquad z \in \mathbb{R}^n \tag{70}$$
$$\dot{e} = lAe + g(z, e, t), \qquad e \in \mathbb{R}^m \tag{71}$$

where $l > 0$ and A is a Hurwitz matrix. Assume that for the system

$$\dot{z} = f(z, 0, t)$$

there exists a neighborhood $\mathcal{W}_1$ of the origin in $\mathbb{R}^n$ and a $\mathcal{C}^1$ function $V_1: \mathcal{W}_1 \to \mathbb{R}_+$ which is positive definite on $\mathcal{W}_1\backslash\{0\}$ and proper on $\mathcal{W}_1$ and satisfies

$$\frac{\partial V_1}{\partial z}f(z, 0, t) \leq -\psi_1(z)$$

where $\psi_1(z)$ is continuous on $\mathcal{W}_1$ and positive definite on $\{z: \nu_1 < V_1(z) \le c_1+1\}$ for some nonnegative real number $\nu_1 < 1$ and some real number $c_1 \ge 1$. Also assume that there exist positive real numbers α and β and a bounded function γ with $\gamma(0)=0$ satisfying

$$\left.\begin{aligned}\|f(z,e,t)-f(z,0,t)\| &\le \gamma(\|e\|)\\ \|g(z,e,t)\| &\le \alpha\|e\|+\beta\end{aligned}\right\},\quad \forall (z,e,t)\in L_{V_1}(c_1+1)\times\mathbb{R}^m\times\mathbb{R}_+. \tag{72}$$

Let c_2 be a class $\mathcal{K}_\infty$ function satisfying

$$\lim_{l\to\infty}\frac{l}{c_2^4(l)}=\infty. \tag{73}$$

Let P solve the Lyapunov equation $A'P+PA=-I$. Define the function

$$V(z,e):=c_1\frac{V_1(z)}{c_1+1-V_1(z)}+c_2(l)\frac{\ln(1+e'Pe)}{c_2(l)+1-\ln(1+e'Pe)} \tag{74}$$

and the set

$$\mathcal{W}:=\{z: V_1(z)<c_1+1\}\times\{e: \ln(1+e'Pe)<c_2(l)+1\}.$$

Then, for $l>0$, $V:\mathcal{W}\to\mathbb{R}_+$ is positive definite on $\mathcal{W}\backslash\{0\}$ and proper on $\mathcal{W}$. Furthermore, for any $\nu_2\in(0,1)$, there exists an $l^*(\nu_2)>0$ such that, for all $l\in[l^*(\nu_2),\infty)$, the derivative of V along the trajectories of (70) and (71) satisfies

$$\dot V\le-\psi(z,e)$$

where $\psi(z,e)$ is positive definite on $\{(z,e): \nu_1+\nu_2\le V(z,e)\le c_1^2+c_2^2(l)+1\}$.

Proof: Proof of Theorem 3: For the system Σ_{ud} under the family of low-and-high gain output feedback laws $\Sigma_{LH}^{O1}(\epsilon,\rho,l)$, the closed-loop system takes the form of

$$\dot x = Ax+B\sigma(u+g(x,t,d(t))) \tag{75}$$
$$\dot{\hat x} = A\hat x+Bu+L(l)(y-C\hat x) \tag{76}$$
$$u=F_{LH}(\epsilon,\rho)\hat x. \tag{77}$$

Recall that Γ_S and Γ_O are the state and output transformation that take the system into its scb form. Partition the state $\bar x=\Gamma_S^{-1}x$ and $\bar{\hat x}=\Gamma_S^{-1}\hat x$ as

$$\bar x=[\bar x_a',\bar x_b',\bar x_f']',\qquad \bar{\hat x}=[\bar{\hat x}_a',\bar{\hat x}_b',\bar{\hat x}_f']'$$

where $\bar x_a,\bar{\hat x}_a\in\mathbb{R}^{n_a}$, $\bar x_b,\bar{\hat x}_b\in\mathbb{R}^{n_b}$, $\bar x_f,\bar{\hat x}_f\in\mathbb{R}^{n_f}$. We then perform a state transformation as follows:

$$\tilde x=\Gamma_S[\bar{\hat x}_a',\bar x_b',\bar x_f']',\qquad \tilde e=[\tilde e_a',\tilde e_b',\tilde e_f']'$$

where $\tilde e_a=\bar x_a-\bar{\hat x}_a$, $\tilde e_b=S_b(l)(\bar x_b-\bar{\hat x}_b)$, $\tilde e_f=S_f(l)(\bar x_f-\bar{\hat x}_f)$, and where

$$S_b(l)=\begin{bmatrix} l^{r_1}S_{r_1}^{-1}(l) & 0 & \cdots & 0\\ 0 & l^{r_2}S_{r_2}^{-1}(l) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l^{r_{p-m}}S_{r_{p-m}}^{-1}(l)\end{bmatrix}$$

and

$$S_f(l)=\begin{bmatrix} l^{q_1}S_{q_1}^{-1}(l) & 0 & \cdots & 0\\ 0 & l^{q_2}S_{q_2}^{-1}(l) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & l^{q_m}S_{q_m}^{-1}(l)\end{bmatrix}.$$

Denoting $\tilde e_{bf}=[\tilde e_b',\tilde e_f']'$, we write the closed-loop system (75)–(77) in the new state $\tilde x$, $\tilde e_a$, and $\tilde e_{bf}$ as

$$\dot{\tilde x}=A\tilde x+B[\sigma(u+g(\tilde x+\Gamma_{Sa}\tilde e_a,t,d(t)))+E_a\tilde e_a] \tag{78}$$
$$\dot{\tilde e}_a=A_{aa}\tilde e_a \tag{79}$$
$$\dot{\tilde e}_{bf}=lA_{bf}\tilde e_{bf}+B_{bf}[\sigma(u+g(\tilde x+\Gamma_{Sa}\tilde e_a,t,d(t)))-u+E_{bf}S_{bf}^{-1}(l)\tilde e_{bf}+E_a\tilde e_a] \tag{80}$$
$$u=F_{LH}(\epsilon,\rho)[\tilde x-\Gamma_{Sbf}S_{bf}^{-1}(l)\tilde e_{bf}] \tag{81}$$

where

$$A_{bf}=\operatorname{diag}\{A_{r_1}^c,A_{r_2}^c,\cdots,A_{r_{p-m}}^c,A_{q_1}^c,A_{q_2}^c,\cdots,A_{q_m}^c\}$$

$$B_{bf}=[0,\operatorname{diag}\{B_{q_1},B_{q_2},\cdots,B_{q_m}\}']'$$

$$E_{bf}=\begin{bmatrix}E_{1b} & E_{1f}\\ E_{2b} & E_{2f}\\ \vdots & \vdots\\ E_{mb} & E_{mf}\end{bmatrix}$$

$$S_{bf}(l)=\operatorname{diag}\{S_b(l),S_f(l)\}$$

and Γ_{Sa} and Γ_{Sbf} are defined through the following partitioning:

$$\Gamma=[\Gamma_{Sa},\Gamma_{Sbf}],\quad \Gamma_{Sa}\in\mathbb{R}^{n\times n_a},\quad \Gamma_{Sbf}\in\mathbb{R}^{n_a\times(n_b+n_f)}.$$

We note here that A_{bf} is Hurwitz since $A_{r_i}^c$'s and $A_{q_i}^c$'s are all Hurwitz.

This system is in the form (70) and (71) of Lemma 4, with $(\tilde x',\tilde e_a')'=z$ and $\tilde e_{bf}=e$. To ensure the assumptions of Lemma 4, we will apply Lemma 3. To that end, we set $\tilde e_{bf}=0$ in the closed-loop system equations. Equations (78) and (79) then become

$$\dot{\tilde x}=A\tilde x+B[\sigma(u+g(\tilde x+\Gamma_{Sa}\tilde e_a,t,d(t)))+E_a\tilde e_a] \tag{82}$$
$$\dot{\tilde e}_a=A_{aa}\tilde e_a \tag{83}$$
$$u=F_{LH}(\epsilon,\rho)\tilde x \tag{84}$$

which is in the form of (50). By Theorem 2 and Assumption 4, A_{aa} is Hurwitz. Let $P_a>0$ be such that

$$A_{aa}^TP_a+P_aA_{aa}=-I. \tag{85}$$

Following Lemma 3, we define

$$V_1(\tilde x,\tilde e_a)=\tilde x'P\tilde x+(\tau^2+1)\tilde e_a'P_a\tilde e_a \tag{86}$$

where $\tau=\sqrt{\lambda_{\max}(E_a'E_a)}$. Let the real number $c_1\ge1$ be such that

$$c_1^2\ge\sup_{(x,\hat x)\in\mathcal{W},\epsilon\in(0,1]}V_1(\tilde x,\tilde e_a). \tag{87}$$

Such a c_1 exists since $\tilde x$ and $\tilde e_a$ are both independent of l, $\lim_{\epsilon\to0}P(\epsilon)=0$ and the set $\mathcal{W}$ is bounded. Let $\epsilon^*\in(0,1]$

be such that (17) is satisfied for each $\epsilon \in (0, \epsilon^*]$. Such an ϵ^* exists as a result of Lemma 1. Moreover, ϵ^* depends only on $\mathcal{W}$ and Δ. Fix $\epsilon \in (0, \epsilon^*]$.

Consider the case where $D_0 = 0$. Then ρ_3^* defined in (58) is equal to zero. So, if $\rho \geq \max\{\rho_1^*, \rho_2^*\}$, it follows from point 2) of Lemma 3 that $\dot{V}_1 \leq -\psi_1(\tilde{x}, \tilde{e}_a)$, where

$$(\tilde{x}, \tilde{e}_a) \in \{(\tilde{x}, \tilde{e}) \in \mathbb{R}^n \times \mathbb{R}^m : 0 < V_1(\tilde{x}, \tilde{e}_a) \leq c_1^2 + 1\} \Rightarrow -\psi_1(\tilde{x}, \tilde{e}_a) < 0. \quad (88)$$

Notice that $\max\{\rho_1^*, \rho_2^*\}$ is independent of $\mathcal{W}_0$. Moreover, $\max\{\rho_1^*, \rho_2^*\} = 0$ when we also have $g_0 \equiv 0$.

Now consider the case where $D_0 > 0$. Let P_{bf} satisfy the Lyapunov equation

$$A_{bf}' P_{bf} + P_{bf} A_{bf} = -I \quad (89)$$

and let

$$V_3 = \tilde{e}_{bf}' P_{bf} e_{bf}. \quad (90)$$

Observe that, from the definition of $S_r(l)$, if we restrict our attention to the case where $l \geq 1$ we have that there exists $k > 0$ such that, for each $r \geq 0$

$$|(\tilde{x}', \tilde{e}_a', e_{bf}')'| \leq r \Rightarrow |(x', \hat{x}')'| \leq kr. \quad (91)$$

Now, since we have not yet specified l, the level sets of V_3, expressed in the original coordinates, will depend on l. Nevertheless, we can pick $\nu < 1$, a strictly positive real number such that, for all $l \geq 1$

$$L_{V_1}(\nu) \times L_{V_3}(\exp(\nu) - 1) \subset \mathcal{W}_0. \quad (92)$$

Such a strictly positive real number exists because $\mathcal{W}_0$ has the origin as an interior point, $P(\epsilon)$, P_a, and P_{bf} are all positive definite, and (91) holds. It then follows from Lemma 3 that if we set:

$$\rho^* = \max\left\{\rho_1^*, \rho_2^*, \rho_3^*, \frac{2\rho_3^*(c_0^2+1)}{\nu}\right\} \quad (93)$$

then we get $\dot{V}_1 \leq -\psi_1(\tilde{x}, \tilde{e}_a)$, where

$$(\tilde{x}, \tilde{e}_a) \in \left\{(\tilde{x}, \tilde{e}_a) \in \mathbb{R}^n \times \mathbb{R}^m : \frac{\nu}{4} < V_1(\tilde{x}, \tilde{e}_a) \leq c_1^2 + 1\right\} \Rightarrow -\psi_1(\tilde{x}, \tilde{e}_a) < 0. \quad (94)$$

Henceforth $\rho \geq \rho^*(\epsilon)$ will be fixed. To this point we have that the first assumption of Lemma 4 is satisfied with $\mathcal{W}_1 = \mathbb{R}^{n+n_a}$, V_1, ψ_1, $\nu_1 = \frac{\nu}{2}$ and c_1. For the case where $D_0 > 0$, ν has been specified. For the case where $D_0 = 0$, ν is still arbitrary.

Concerning the conditions (72), they follow readily from a comparison of (78)–(81), with (70) and (71) and the fact that σ is locally Lipschitz and uniformly bounded.

Remark 12: If σ were not uniformly bounded, we could redefine u so that, with $\tilde{e}_{bf} = 0$, it saturated outside of the set $L_{V_1}(c_1^2 + 1)$. This would not change any of the properties deduced thus far and would induce the properties in (72). Of course, this action would make the compensator nonlinear, and the location of the saturation would be a function of (a subset of) the amissible data. This idea was first introduced in [1] and later exploited for very general nonlinear output feedback problems in [19] and [18].

Now we choose $c_2(l) = \ln(1 + \lambda_{\max}(P_{bf}) R^2 l^{2(n_b+n_f)})$, where R is such that $(x, \hat{x}) \in \mathcal{W}$ implies $\|\overline{x}_b - \bar{\hat{x}}_b\| \leq R/2$ and $\|\overline{x}_f - \bar{\hat{x}}_f\| \leq R/2$. The function $c_2(l)$ is obviously of class $\mathcal{K}_\infty$. Furthermore, it satisfies $\ln(1+\tilde{e}_{bf}(0)' P_{bf} \tilde{e}_{bf}(0)) \leq c_2(l)$ and

$$\lim_{l \to \infty} \frac{l}{c_2^4(l)} = \infty.$$

We then define the Lyapunov function

$$V_2(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf}) = \frac{c_1 V_1(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf})}{c_1 + 1 - V_1(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf})} + \frac{c_2(l) \ln(1 + \tilde{e}_{bf}' P_{bf} \tilde{e}_{bf})}{c_2(l) + 1 - \ln(1 + \tilde{e}_{bf}' P_{bf} \tilde{e}_{bf})} \quad (95)$$

and the set

$$\mathcal{W}_2 = \{(\tilde{x}, \tilde{e}_a) : V_1(\tilde{x}, \tilde{e}_a) < c_1 + 1\} \times \{\tilde{e}_{bf} : \ln(1 + \tilde{e}_{bf}' P_{bf} \tilde{e}_{bf}) < c_2(l) + 1\}. \quad (96)$$

It then follows from Lemma 4 that for all $l > 0$, $V_2 : \mathcal{W}_2 \to \mathbb{R}_+$ is positive definite on $\mathcal{W}_2 \backslash \{0\}$ and proper on $\mathcal{W}_2$. Furthermore, there exists an $l_1^*(\epsilon, \rho, \nu) \geq 1$, such that, for all $l \geq l_1^*(\epsilon, \rho, \nu)$

$$\dot{V}_2 \leq -\psi_2(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf}) \quad (97)$$

where $\psi_2(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf})$ is continuous on $\mathcal{W}_2$ and is positive definite on $\mathcal{W}_3 := \{(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf}), \frac{\nu}{2} \leq V_2(\tilde{x}, \tilde{e}_a, \tilde{e}_{bf}) \leq c_1^2 + c_2^2(l) + 1\}$.

Finally

$$(x, \hat{x}) \in \mathcal{W} \Rightarrow V_2 \leq c_1^2 + c_2^2(l) \quad (98)$$

and

$$V_2 \leq \frac{\nu}{2} \Rightarrow \begin{cases} V_1 \leq \nu \\ V_3 \leq \exp(\nu) - 1 \end{cases} \Rightarrow (x, \hat{x}) \in \mathcal{W}_0. \quad (99)$$

This establishes the result for the case where $D_0 > 0$.

For the case where $D_0 = 0$, recall that ν is arbitrary. So the result follows if there exists a neighborhood $\mathcal{A}$ of the origin and a positive real number l_2^* such that, for all $l \geq l_2^*$, the origin of (78)–(81) is uniformly locally asymptotically stable with basin of attraction containing $\mathcal{A}$. But this is just a standard singular perturbation result, since the origin of the $(\tilde{x}, \tilde{e}_a)$ subsystem is locally exponentially stable. For example, one could just follow the calculations in the proof of [5, Theorem 8.3] using the Lyapunov function candidate $V_1 + V_3$. The function V_1 has the appropriate properties since point 1) of Lemma 3 holds. □

VII. Concluding Remarks

In this paper, we have employed feedback laws with a low-and-high gain structure to solve several semiglobal control problems for linear systems asymptotically null controllable with bounded controls where the input is subject to magnitude saturation. In particular, we have showed that robust semiglobal disturbance rejection can be achieved with linear state feedback. Moreover, with a minimum phase and left invertibility assumption, the same result can be achieved with output feedback.

While we have focused on linear systems asymptotically null controllable with bounded controls, it should be clear from the technical Lemmas 2 and 3 that the same type of results can be achieved locally, without a vanishing domain of attraction, for stabilizable linear systems (without requiring that the open-loop eigenvalues are in the closed left-half plane.)

We have also focused our attention on the use of linear control laws. This framework imposed some assumptions on the saturation characteristics that would not otherwise be necessary. For example, if the properties in Definition 1 only hold for u sufficiently small, then u itself can be saturated so that the composition of the original nonlinearity and the saturation satisfy the properties in Definition 1. This, of course, would then be a nonlinear control law.

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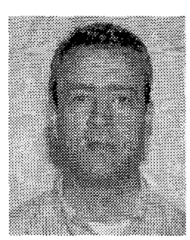

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