TAWAS Calculations

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This document will give a detailed explanation of the calculations made in *TAWAS*. The code begins by calculating solutions to the wave equation for a torsion Alfvén wave propagating in a plasma with negligible plasma beta and in a force-free axisymmetric magnetic field with no azimuthal component embedded in a high density divergent tube structure. From this the velocity perturbation v and magnetic field perturbation b are calculated and from these solutions we calculate the wave energy flux across magnetic surfaces at different heights. All variables are calculated over a uniform grid in radius r and height z.

Equilibrium Quantities

The code begins by calculating the characteristic Alfvén speed from the given parameters,

$$V_0 = \frac{B_0}{\mu_0 \rho_0}$$

The curvilinear field orientated coordinates ϕ and ψ are calculated,

$$\begin{aligned}
\phi = &\quad - H \exp(-z/H) J_0(r/H) \\
\psi = &\quad r \exp(-z/H) J_1(r/H)
\end{aligned}$$

The value of ψ at the boundary of the flux tube, ψ_b, is calculated,

$$\psi_b = \frac{r_0^2}{2H}$$

The magnetic field components B_r and B_z are calculated as well as the total magnetic field strength B,

$$\begin{aligned}
B_r = &\quad B_0 \exp(-z/H) J_1(r/H) \\
B_z = &\quad B_0 \exp(-z/H) J_0(r/H) \\
B = &\quad \sqrt{B_r^2 + B_Z^2}
\end{aligned}$$

The density profile and Alfvén speed are calculated,

$$\hat{\rho}(\psi) = \begin{cases} \frac{\rho_0}{\zeta}(1 + (\zeta - 1)(1 - (\psi/\psi_b))^2), & \text{if } \psi \leq \psi_b \\ 0, & \text{if } \psi > \psi_b \end{cases}$$

$$\rho(\psi, z) = \quad \hat{\rho}(\psi)\exp(-\alpha z/H)$$

$$V_A = \quad \frac{B}{\sqrt{\mu_0 \rho}}$$

Field Line Plot

If the plotting is set to 1 or 2, a contour of the density with field lines overlaid is now plotted.

For ϕ the grid lines plotted are: $\phi = \phi_{min} + (\phi_{max} - \phi_{min})\sqrt{\frac{l}{n-1}}$

For ψ the grid lines plotted are: $\psi = \psi_{min} + (\psi_{max} - \psi_{min})\left(\frac{l}{n-1}\right)^2$

for $0 \leq l < n$, where n is the number of gridlines.

Calculation of h

The variable h is now calculated,

$$h \quad = \quad \int \frac{V_0 B_0}{V_A B}\, d\phi \quad = \quad \int h_{int}\, d\phi$$

We begin by calculating h_{int}

$$h_{int} \quad = \quad \frac{V_0 B_0}{V_A B}$$

In order to calculate h at each point (i, j) we need to integrate h_{int} with respect to ϕ along the field lines from $j = 0$ to that point. The field lines correspond to lines on which $\psi = const.$, these lines do not coincide with the grid points. We therefore need to use interpolation to find the correct values for h_{int} and ϕ along the field lines.

For each height index j_1 between zero and the height index of our point j we find the two closest points either side of the field line with radial indices p_1 and p_2 such that $\psi(p_1, j_1) < \psi(i, j) <$

$\psi(p_2, j_1)$. To find h_{int} and ϕ for the point at this height on the field line we perform simple linear interpolation,

$$\begin{aligned}
\phi_{j_1} &= \phi(p_1, j_1) + (\phi(p_2, j_1) - \phi(p_1, j_1)) \frac{\psi(i, j) - \psi(p_1, j_1)}{\psi(p_2, j_1) - \psi(p_1, j_1)} \\
h_{int, j_1} &= h_{int}(p_1, j_1) + (h_{int}(p_2, j_1) - h_{int}(p_1, j_1)) \frac{\psi(i, j) - \psi(p_1, j_1)}{\psi(p_2, j_1) - \psi(p_1, j_1)}
\end{aligned}$$

Finally we integrate h_{int} with respect to ϕ to calculate the value for h.

Calculation for $\partial h / \partial \psi$

Next $\partial h / \partial \psi$ is calculated, to calculate the $\partial h / \partial \psi$ at each point (i,j) we must take the partial derivative of h with respect to ψ along lines of magnetic potential. These lines correspond to lines on which $\phi = const.$ as these lines do not coincide with the grid points we must use interpolation again to find the correct values for h and ψ along the lines of magnetic potential.

For radial indices either side of our point i_1 and i_2 such that $i_1 < i < i_2$ we identify the height indices either side of the line of magnetic potential passing through our point. We label these p_{11} and p_{12} for i_1 such that $\phi(i_1, p_{11} < \phi(i, j) < \phi(i_1, j_{12})$, and, p_{21} and p_{22} for i_2 such that $\phi(i_2, p_{21} < \phi(i, j) < \phi(i_2, j_{22})$.

We then use linear interpolation to find h and ψ for the points where the line of magnetic potential coincides with i_1 and i_2. We label these values h_{i1} and ψ_{i1} for i_1, and, h_{i2} and ψ_{i2} for i_2

$$\begin{aligned}
h_{i_1} &= h(i_1, p_{11}) + (h(i_1, p_{12}) - h(i_1, p_{11})) \frac{\phi(i, j) - \phi(i_1, p_{11})}{\phi(i_1, p_{12}) - \phi(i_1, p_{11})} \\
\psi_{i_1} &= \psi(i_1, p_{11}) + (\psi(i_1, p_{12}) - \psi(i_1, p_{11})) \frac{\phi(i, j) - \phi(i_1, p_{11})}{\phi(i_1, p_{12}) - \phi(i_1, p_{11})} \\
h_{i_2} &= h(i_2, p_{21}) + (h(i_2, p_{22}) - h(i_2, p_{21})) \frac{\phi(i, j) - \phi(i_2, p_{21})}{\phi(i_2, p_{22}) - \phi(i_2, p_{21})} \\
\psi_{i_2} &= \psi(i_2, p_{21}) + (\psi(i_2, p_{22}) - \psi(i_2, p_{21})) \frac{\phi(i, j) - \phi(i_2, p_{21})}{\phi(i_2, p_{22}) - \phi(i_2, p_{21})}
\end{aligned}$$

Finally we take the partial derivative with a central difference calculation using these values,

$$\frac{\partial h}{\partial \psi} = \frac{h_{i_2} - h_{i_1}}{\psi_{i_2} - \psi_{i_1}}$$

For points on or near the boundaries we cannot calculate $\partial h / \partial \psi$ directly using a central difference. Instead we linearly extrapolate the values from the nearest two grid points within the domain. This approach is well justified based on visual inspection of $\partial h / \partial \psi$ over a wide range of parameters.

Calculation of Dissipation Υ and Reflection R Coefficients

The dissipation and reflection coefficients are very simply calculated,

$$\Upsilon \quad = \quad \frac{\nu\omega^2}{2G^2V_0^3}\left(\frac{\partial h}{\partial\psi}\right)^2 \quad = \quad \frac{\nu\omega^2}{2V_0^3}\frac{r^2}{H^2}\sqrt{\frac{\rho}{\rho_0}}\left(\frac{\partial h}{\partial\psi}\right)^2$$

$$R \quad = \quad \frac{V_0 G}{2\omega H^2}\partial_\phi\left(\frac{r^2B^2}{B_0^2}\partial_\phi G\right) \quad = \quad \frac{V_0}{2\omega H^2}\frac{B_0^2}{B^2}\sqrt{\frac{\rho_0}{\rho}}\left[\frac{\psi^2}{r^2}\left(\frac{H^2}{r^2}-\frac{\alpha}{4}\right)-\frac{\alpha(4-\alpha)\phi^2}{16H^2}\right]$$

In the limit of $r = 0$,

$$R \quad = \quad \frac{V_0 G}{2\omega H^2}\partial_\phi\left(\frac{r^2B^2}{B_0^2}\partial_\phi G\right) \quad = \quad \frac{V_0}{2\omega H^2}\sqrt{\frac{\rho_0}{\rho}}\left(\frac{2-\alpha}{4}\right)^2$$

Calculation of Damping Factor D and Damping Envelope Q

We now wish to calculate the damping factor through integration,

$$D \quad = \quad \int \Upsilon - iR\, d\phi \quad = \quad \int \Upsilon\, d\phi - i\int R\, d\phi \quad = \quad D_\Upsilon - iD_R$$

In order to calculate D at each point (i, j) we need to integrate with respect to ϕ along the field lines from $j = 0$ to that point. We therefore need to use interpolation again to find the correct values for Υ, R and ϕ along the field lines.

However when we find values for Υ and R along the field lines we encounter a problem. The scale of ϕ is of the order H, that is 10^7, whilst the scale of Υ and R is of the order 10^{-10}, this is illustrated in fig. 1 which shows the graphs of Υ and R against ϕ for a typical field line within the flux tube.

The large difference in these two scales causes errors during numerical integration. To avoid this issue we reformulate our integrals as so,

$$X \;=\; \ln(-\phi) \quad \Rightarrow \quad dX \;=\; -\frac{d\phi}{\phi} \quad \Rightarrow \quad d\phi \;=\; -\phi dX \;=\; -\phi d(\ln(-\phi))$$

so we have,

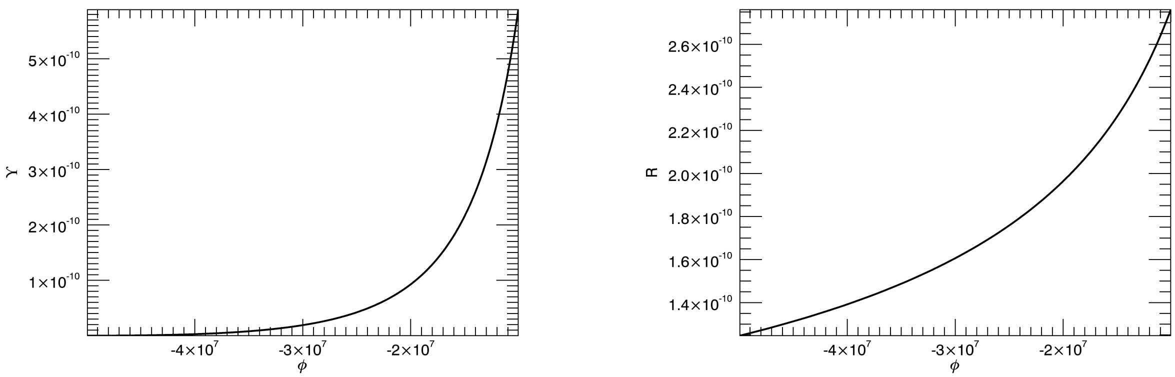

Figure 1: Plots of Υ and R against ϕ along a typical field line within the central flux tube.

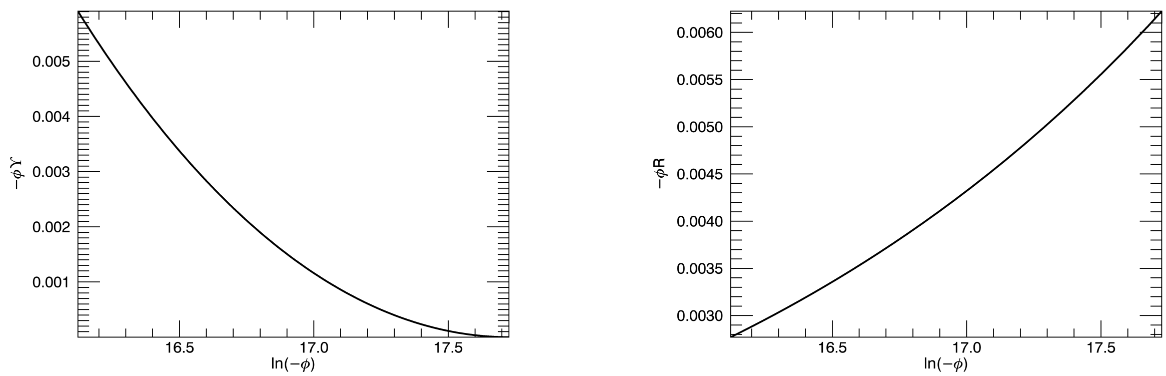

Figure 2: Plots of $-\phi\Upsilon$ and $-\phi R$ against $\ln(-\phi)$ along a typical field line within the central flux tube.

$$
\begin{aligned}
D_\Upsilon &= \int \Upsilon \, d\phi &&= \int -\phi\Upsilon \, d(\ln(-\phi)) &&= \int \Upsilon_{int} \, d\phi_{int} \\
D_R &= \int R \, d\phi &&= \int -\phi R \, d(\ln(-\phi)) &&= \int R_{int} \, d\phi_{int}
\end{aligned}
$$

This reduces the scale of our differential, now ϕ_{int}, to around unity and increases the scale of our integrands, now Υ_{int} and R_{int} to around 10^{-3} as shown in fig. 2, which shows the graphs of Υ_{int} and R_{int} against ϕ_{int} for a typical field line within the flux tube. This improves the accuracy of our numerical integration.

In our code therefore we do not interpolate to find Υ, R and ϕ along the field lines but rather to find $\Upsilon_{int} = -\phi\Upsilon$, $R_{int} = -\phi R$ and $\phi_{int} = \ln(-\phi)$ along the field lines.

For each height index j_1 between zero and the height index of our point j we find the two closest points either side of the field line with radial indices p_1 and p_2 such that $\psi(p_1, j_1) < \psi(i, j) < \psi(p_2, j_1)$. To find Υ_{int}, R_{int} and ϕ_{int} for the point at this height on the field line we perform simple linear interpolation,

$$\phi_{int,j_1} = \phi_{int}(p_1,j_1) + (\phi_{int}(p_2,j_1) - \phi_{int}(p_1,j_1))\frac{\psi(i,j)-\psi(p_1,j_1)}{\psi(p_2,j_1)-\psi(p_1,j_1)}$$

$$\Upsilon_{int,j_1} = \Upsilon_{int}(p_1,j_1) + (\Upsilon_{int}(p_2,j_1) - \Upsilon_{int}(p_1,j_1))\frac{\psi(i,j)-\psi(p_1,j_1)}{\psi(p_2,j_1)-\psi(p_1,j_1)}$$

$$R_{int,j_1} = R_{int}(p_1,j_1) + (R_{int}(p_2,j_1) - R_{int}(p_1,j_1))\frac{\psi(i,j)-\psi(p_1,j_1)}{\psi(p_2,j_1)-\psi(p_1,j_1)}$$

We then integrate to find D_Υ and D_R for each point across the grid. Finally we combine our arrays for D_Υ and D_R to find $D = D_\Upsilon - iD_R$. Now we can calculate our damping envelope, Q is very simply,

$$Q = e^{-(D_\Upsilon - iD_R)} = e^{-D}$$

Calculation of Driving Function $A_0(\psi)$

Before we calculate the velocity and magnetic field perturbations we define the driving driving by the function $A_0(\psi)$ and calculate it over the grid,

$$A_0(\psi) = \begin{cases} \left(\frac{\hat{\rho}}{\rho_0}\right)^{1/4}\left(\frac{\psi}{\psi_b}\right)^{1/2}\left(1-\frac{\psi}{\psi_b}\right) & \text{if } \psi \le \psi_b \\ 0, & \text{if } \psi > \psi_b \end{cases}$$

Calculation and plotting of Velocity Perturbation v

We can now calculate the velocity perturbation.

$$v = ru = u_0 Q A_0 \left(\frac{\rho_0}{\rho}\right)^{1/4} \exp\left(i\omega\left(\frac{h}{V_0} - t\right)\right) = u_0 W \exp\left(i\omega\left(\frac{h}{V_0} - t\right)\right)$$

The general wave envelope W, velocity perturbation envelope v_{env} and velocity perturbation v are calculated as,

$$W = QA_0\left(\frac{\rho_0}{\rho}\right)^{1/4}$$

$$v_{env} = v_0 W$$

$$v = v_{env}\exp\left(i\omega\left(\frac{h}{V_0} - t\right)\right)$$

If the plotting is set to 1 or 3 then a shaded surface and contour of the velocity perturbation are now plotted.

Calculation and plotting of Magnetic Field Perturbation b

The envelope for the magnetic field perturbation b_{env} is given by,

$$b_{env} = -u_0 B\left[\frac{iWB}{\omega B_0}\left(\frac{1}{Q}\partial_\phi Q - \frac{1}{4\rho}\partial_\phi \rho - \frac{1}{r}\partial_\phi r\right) - \frac{W}{V_A}\right]$$

We can now consider our particular forms for Q, ρ, r and z to find their derivatives,

$$\frac{\partial Q}{\partial \phi} = \left(iR - \Upsilon\right) Q$$

$$\frac{\partial \rho}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\rho_0 \exp\left(-\frac{\alpha z}{H}\right)\right) = -\frac{\alpha}{H}\rho_0 \exp\left(-\frac{\alpha z}{H}\right)\frac{\partial z}{\partial \phi} = -\frac{\alpha}{H}\rho\frac{\partial z}{\partial \phi}$$

$$\frac{\partial r}{\partial \phi} = \frac{B_0^2 \psi}{B^2 r}, \qquad \frac{\partial z}{\partial \phi} = -\frac{B_0^2 \phi}{B^2 H}$$

Subbing these back into b_{env} gives,

$$\begin{aligned}
b_{env} &= -u_0 B\left[\frac{iWB}{\omega B_0}\left(\frac{1}{Q}\left(iR - \Upsilon\right) Q + \frac{1}{4\rho}\frac{\alpha\rho}{H}\partial_\phi z - \frac{1}{r}\partial_\phi r\right) - \frac{W}{V_A}\right] \\
&= -u_0 B\left[\frac{iWB}{\omega B_0}\left(iR - \Upsilon - \frac{\alpha\phi B_0^2}{4H^2B^2} - \frac{\psi B_0^2}{r^2 B^2}\right) - \frac{W}{V_A}\right] \\
&= -u_0 B\left[\frac{iW}{\omega}\left((iR - \Upsilon)\frac{B}{B_0} - \frac{\alpha\phi B_0}{4H^2 B} - \frac{\psi B_0}{r^2 B}\right) - \frac{W}{V_A}\right]
\end{aligned}$$

This last equation is used to calculate b_{env} across the domain. With b_{env} calculated we move on the calculate the magnetic field perturbation b,

$$b = -b_{env}\exp\left(i\omega\left(\frac{h}{V_0} - t\right)\right)$$

If the plotting is set to 1 or 3 then a shaded surface and contour of the magnetic field perturbation are now plotted.

WKB Parameter for Reflection σ

For the WKB approximation to hold both the viscous daming and reflectie damping of the wave must be weak. That is, the length scale over which this damping occurs must be much larger than the wavelength of the Alfvén wave. The condition that wave reflection is weak is expressed by the inequality,

$$\sigma \quad = \quad \frac{\rho_0}{\rho}\left[\frac{\psi^2}{r^2}\left(\frac{H^2}{r^2}-\frac{\alpha}{4}\right)-\frac{\alpha\phi^2}{16H^2}\left(4-\alpha\right)\right] \quad \ll \quad \frac{\omega H^2}{V_0^2}$$

we can calculate σ over the domain. Along the $z-$axis it can be shown that,

$$\lim_{r\to 0}\sigma \quad = \quad \frac{\rho_0}{\rho}\frac{B^2}{B_0^2}\left[\frac{2-\alpha}{4}\right]^2$$

We then print the maximum value of σ and $\omega H/V_0$ to check whether the inequality holds.

Wave Energy Density Calculations

We calculate the wave energy flux Π across magnetic surfaces defined by ϕ = constant. Each surface is uniquely identified by its height of intersection with the z-axis, hence we can define the energy flux as a function of height z. We use the following formula to calculate $\Pi(z)$,

$$\Pi(z) \;=\; -\frac{1}{\mu_0}\int_{\Sigma_b} B\langle vb\rangle\; d\Sigma \;=\; \pi\frac{HB_0}{\mu_0}\int_0^{\psi_b} v_p b_p\; d\psi$$

We begin by calculating the wave energy flux density,

$$\epsilon \;=\; \frac{\pi B_0 H}{\mu_0} v_p b_p$$

If the plotting is set to 1 or 4 then a contour of ϵ is now plotted.

To find the wave energy flux Π we need to integrate ϵ with respect to ψ across magnetic surfaces defined by $\phi = const.$. We define each magnetic surface by the height at which it intersect the $z-$axis.

We begin by defining the lowest magnetic surface that can fit inside the domain. We label the height at which this surface intersects the $z-$axis as z_0. Using the definition for ϕ we have,

$$
\begin{aligned}
\phi(r_0, 0) &= \phi(0, z_0) \\
-HJ_0\left(\frac{r_0}{H}\right) &= -He^{-z_0/H} \\
z_0 &= -H\ln\left(J_0\left(\frac{r_0}{H}\right)\right)
\end{aligned}
$$

With that defined we loop over each height index k for which $z > z_0$ and calculate the wave energy flux for each corresponding surface. These surfaces do not coincide with the grid points, we therefore need to use interpolation to find the correct values for ϵ and ψ along the magnetic surfaces.

We begin by finding the height z at which the surface intersects the $z-$axis, this is $z_{en} = kz/z_{max}$. Then we find the value of ϕ for the surface, $\phi_1 = -He^{-z_{en}/H}$.

Now for each radial index i between zero and the tube boundary ψ_b we find the two closest points above and below the magnetic surface with height indices p_1 and p_2 such that $\phi(i, p_1) < \phi_1 < \phi(i, p_2)$. To find ϵ and ψ for the point at this radius on the magnetic surface we perform simple linear interpolation,

$$
\begin{aligned}
\psi_{int} &= \psi(i, p_1) + (\psi(i, p_2) - \psi(i, p_1))\frac{\phi_1 - \phi(i, p_1)}{\phi(i, p_2) - \phi(i, p_1)} \\
\epsilon_{int} &= \epsilon(i, p_1) + (\epsilon(i, p_2) - \epsilon(i, p_1))\frac{\phi_1 - \phi(i, p_1)}{\phi(i, p_2) - \phi(i, p_1)}
\end{aligned}
$$

During this process the loop continuously checks whether the calculated value $\psi_{int} < \psi_b$ and also save the position of each point across the magnetic surface. If the plotting is set to 1 or 4 then each magnetic surface line will be drawn onto the contour of ϵ.

For each magnetic surface the total wave energy flux Π is calculated by integrating ϵ_{int} with respect to ψ_{int}. The wave energy flux Π as well as the intersection height z_{en} for each surface are saved in arrays.

If the plotting is set to 1 or 4 then the total wave energy flux $\Pi(z)$ will be plotted against the intersection heights of the magnetic surfaces z_{en}.

Outputs

The code now saves 2D arrays for the values of: B_r, B_z, ρ, ϕ, ψ, h, W, v_{env}, b_{env}, v, b and σ across the domain.

As well as 1D arrays for z_{en}, z_{en} in units of the scale height H, $\Pi(z)$ and $\Pi(z)$ normalized by the lowest magnetic surface (i.e. $\Pi(z)/\Pi(z_0)$)