Analytical Solution & Benchmarking

$$\frac{dP}{dt} = -b\left(P + \frac{\Delta P_a}{2}\right) - b\left(P - \frac{\Delta P_a'}{2}\right), \quad \Delta P_a' = \begin{cases} \Delta P_a & \text{if } t < t_{MAR} \\ \Delta P_a + \Delta P_{MAR} & \text{otherwise} \end{cases}$$

Assume that $t < t_{MAR}$ for simplicity.

$$\Rightarrow \frac{dP}{dt} = -b\left(P + \frac{\Delta P_a}{2}\right) - b\left(P - \frac{\Delta P_a}{2}\right)$$

$$\frac{dP}{dt} = -2bP$$

$$\int \frac{1}{P}\, dP = -\int 2b\, dt$$

Constant

$$\ln|P| = -2bt + d_1$$

$$P = e^{-2bt + d_1}$$

$$P = e^{-2bt} \cdot e^{d_1}$$

$$P = Ae^{-2bt}$$

$h, a, b, g, P_{surf}, P_a = [1, 1, 1, 0, 1, 1]$

To simplify this even further, let $A, b = 1$.

$$\Rightarrow P = e^{-2t}$$

$$M_0 \frac{dc}{dt} = -n_{stock}(t-\tau)\, b'(t-\tau)(P - \Delta P_{surf}) + b_c\left(P - \frac{\Delta P_a'}{2}\right)c, \quad b'(t) = \begin{cases} b_1 & \text{if } t < t_c \\ \alpha b_1 & \text{otherwise} \end{cases}$$

Assume that $t < t_{MAR}$ & $t < t_c$ once again for simplicity.

Also assume that $-n_{stock}(t-\tau)$ is a constant.

$$\Rightarrow M_0 \frac{dc}{dt} = -n_{stock} \cdot b_1 (P - \Delta P_{surf}) + b_c\left(P - \frac{\Delta P_a}{2}\right)c$$

$$\frac{dc}{dt} = \frac{-n_{stock} \cdot b_1}{M_0}(P - \Delta P_{surf}) + \frac{b_c}{M_0}\left(P - \frac{\Delta P_a}{2}\right)$$

Let $\frac{n_{stock} \cdot b_1}{M_0} = g$ and $\frac{b_c}{M_0} = h$.

$$\Rightarrow \frac{dc}{dt} = -g(P - \Delta P_{surf}) + h\left(P - \frac{\Delta P_a}{2}\right)c$$

Assume $n_{stock} = 0$ (i.e. $g = 0$) and $h, \Delta P_a = 1$.

$$\Rightarrow \frac{dc}{dt} = c\,(e^{-2t} - 0.5)$$

Derivative function to be used for benchmarking

$$\int \frac{1}{c}\, dc = \int (e^{-2t} - 0.5)\, dt$$

Constant

$$\ln|c| = \frac{-e^{-2t}}{2} - 0.5t + d_2$$

$$c = e^{-\frac{1}{2}e^{-2t} - 0.5t + d_2}$$

$$c = e^{-\frac{1}{2}e^{-2t}} \cdot e^{-0.5t} \cdot e^{d_2}$$

$$c = Ae^{-\frac{1}{2}e^{-2t}} \cdot e^{-0.5t}$$

Let $A = 1$:

$$\Rightarrow c = e^{-\frac{1}{2}e^{-2t}} \cdot e^{-0.5t}$$

Simplified analytical solution for concentration to be used for benchmarking

$$c(0) = e^{-0.5}$$

As $t \to \infty$, $c(t) \to 0$, thus the steady state solution is at $c(t) = 0$.