Second Order Differential Equations
Hooke's Law without Friction

(1) The force equation for a spring governed by Hooke's Law is

$$F = ma = -ky$$
$$my'' = -ky$$
$$y'' = \frac{-k}{m}y$$
$$y'' = -y \text{ if } k=m$$

Based on your work from C&P107, differential equations of this form, $y'' = -y$ such that $k=m$, can have the general solution $y = \sin(t)$ or the general solution $y = \cos(t)$ depending on the initial conditions.

(1a) Show that the linear combination , $y = A\cos(t) + B\sin(t)$, of these two general solutions is also a solution.

(1b) Find the constants A and B given the initial conditions

$$y(0) = 5 \text{ in and } y'(0) = 0\ \frac{\text{in}}{\text{sec}}$$

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(1c) Graph this particular solution.

(1d) Rewrite your solution in this form: $y = \sqrt{A^2 + B^2}\,\sin\left(t + \arctan\left(\frac{A}{B}\right)\right)$

Does this function's graph match your graph?

Second Order Differential Equations
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(2) The force equation for a spring governed by Hooke's Law is

$$F = ma = -ky$$
$$my'' = -ky$$
$$y'' = \frac{-k}{m}y$$
$$y'' + \frac{k}{m}y = 0, \text{ if } k \text{ and } m \text{ are not equal}$$
$$y'' + \omega^2 y = 0$$

(2a) Show that the general solution of a differential equation of the form $y'' + \omega^2 y = 0$ is $y = A\cos(\omega t) + B\sin(\omega t)$.

(2b) Find the constants ω, A and B given that $\omega^2 = \frac{k}{m} = 4$ and the initial conditions $y(0) = 1$ in and $y'(0) = -6\,\frac{\text{in}}{\text{sec}}$.

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(2c) Graph this particular solution.

(2d) Rewrite your solution in this form: $y = \sqrt{A^2 + B^2}\,\sin\left(\omega t + \arctan\left(\frac{A}{B}\right)\right)$

Does this function's graph match your graph?

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(3) A brick of mass 3 kg hangs from the end of a spring. When the brick is at rest, the spring is stretched 2 cm. Now, you perturb the system by stretching the spring an additional 5 cm and then releasing the brick.

(3a) Calculate the spring constant k (Hint: mg = ks).
(3b) Use your value of k and the given mass to write a differential equation modeling the motion of the brick.
(3c) Find the general solution to this differential equation.
(3d) Use the given initial conditions to find the particular solution.

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(4) A pendulum of length l feet makes an angle x radians with the vertical. When x is small, the motion of the pendulum can be modeled as:

$$x'' = \frac{-g}{l}x$$

where g is the acceleration due to gravity.

(4a) Solve this differential equation given the initial conditions: $x(0) = 0$ and $x'(0) = v_0$. Interpret this solution.
(4b) Solve this differential equation given the initial conditions: $x(0) = x_0$ and $x'(0) = 0$. Interpret this solution.
(4c) How is your solution affected if x_0 is increased?
(4d) How is your solution affected if l is increased?

Second Order Differential Equations
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(5) ***Given*** $\frac{d^2y}{dx^2} = xy$, $y(0) = 1$, $y'(0) = -1$

(5a) Use a MacLaurin series to find a sixth degree Taylor Polynomial solving the given second order ordinary differential equation.

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(5b) Let $P_4(x)$ be the fourth degree Taylor Polynomial approximating $y=f(x)$. Find the roots of $y = P_4(x)$.

(5c) Let $P_4(x)$ be the fourth degree Taylor Polynomial approximating $y=f(x)$. Find the absolute max of $y = P_4(x)$.

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(5d) Complete the following table and make a sketch of $y=P_4(x)$.

x	$P_4(x)$	$\lvert P_4(x) - f(x) \rvert$
-2		
$-\sqrt{2}$		
-1		
$\frac{-1}{\sqrt{2}}$		
0		
$\frac{1}{\sqrt{2}}$		
1		
$\sqrt{2}$		
2		

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(6) The equation for the charge Q(t) on a capacitor in a circuit with inductance L satisfies the differential equation:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0.$$

NB: [Q, L, C, t] = Coulomb, Henry, Farad, and second

$$I = \frac{dQ}{dt} \text{ is the current measured in Amperes}$$

L and C are constants

(6a) Given L=36 and C=9, find the particular solution for Q(t) when Q(0) = 6 and I(0) = Q'(0) = 0.

(6b) Suppose that Q(0)=0 and I(0)=Q'(0)=4 and the maximum charge is $2\sqrt{2}$. Find the capacitance if the inductance is 10.

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Teacher's notes:

Do word problems from exercises in section 11.10 of Hughes-Hallett's Calculus: Single Variable 4th ed. © 2005 from Wiley (p. 586).