Odds & Ends

(1) Complex Powers and Euler's Identity

Theorem

The set of Complex Numbers is closed under the operations +, –, *, / & ^.

Lemma 1	***If $z_1, z_2 \in C$, then $z_1 + z_2 \in C$.***
Lemma 2	***If $z_1, z_2 \in C$, then $z_1 - z_2 \in C$.***
Lemma 3	***If $z_1, z_2 \in C$, then $z_1 * z_2 \in C$.***
Lemma 4	***If $z_1, z_2 \in C$, then $z_1 / z_2 \in C$.***
Lemma 5	***If $z_1, z_2 \in C$, then $z_1 \wedge z_2 \in C$.***

Your task is to demonstrate that the given Theorem is true. The Theorem is true if and only if all five Lemmas are valid. We have already seen that the first four lemmas are correct. What about the fifth?

Remember that C includes all Real numbers of the form a + 0i and all Imaginary numbers of the form 0 + bi as well as any Complex number of the form a + bi such that $a, b \in \Re$. So to see that $z_1 \wedge z_2 \in C$ we have to analyze nine cases where z_1 is of the form $a_1 + 0i$, $0 + b_1i$ or $a_1 + b_1i$ and z_2 is of the form $a_2 + 0i$, $0 + b_2i$ or $a_2 + b_2i$

Case 1 $(a_1 + 0i)\wedge(a_2 + 0i) = a_1\wedge a_2$

For example: 2^3 = 8.

Case 2 $(0 + b_1i)\wedge(a_2 + 0i) = b_1\wedge a_2$

For example (2i)^3 = –8i.

Case 3 $(a_1 + b_1i)\wedge(a_2 + 0i) = (a_1 + b_1i)\wedge a_2$

For example: If $(1 + 1i) = \sqrt{2}\ \mathrm{cis}(45°)$,

then $(1 + 1i)\wedge 3 = \sqrt{2^3}\ \mathrm{cis}(135°) = -2 + 2i$.

The previous three cases demonstrate that $z_1 \wedge z_2 \in C$ when z_2 is of the form $a_2 + 0i \in \Re$. What happens when $z_2 \notin \Re$? Well, don't forget that z_1 can be written in several forms. For instance, $z_1 = a_1 + b_1i = r\ \mathrm{cis}(\theta) = r\ e^{i\theta}$ too. How would you calculate i^i? You can convert the base to $r\ e^{i\theta}$ form, where θ is measured in radians, then raise to the power of i and convert back to a + bi form.

If $i = 1\ \mathrm{cis}(\frac{\pi}{2}) = e\wedge(\frac{i\pi}{2})$, then $i^i = e\wedge(\frac{-\pi}{2}) \approx 0.208 + 0i \in \Re \subseteq C$. Note that taking roots is no different. If $\sqrt[i]{i} = i\wedge(\frac{1}{i})$, then $\sqrt[i]{i} = e\wedge(\frac{\pi}{2}) \approx 4.810 + 0i \in \Re \subseteq C$ too! Investigate the remaining 6 cases using this process. On the next two pages, find each power $z_1 \wedge z_2$ exactly.

Odds & Ends

(1) Complex Powers

(1a) Case 4

$z_1 \wedge z_2 = (a_1 + 0i)\wedge(0 + b_2 i)$
Let $z_1 = 2 + 0i$ and $z_2 = 0 + 3i$
Hint: $z_1 \wedge z_2 \approx -0.487 + 0.873i$

(1b) Case 5

$z_1 \wedge z_2 = (0 + b_1 i)\wedge(0 + b_2 i)$
Let $z_1 = 0 + 2\sqrt{3}\, i$ and $z_2 = 0 + 3i$
Hint: $z_1 \wedge z_2 \approx -0.007 - 0.005i$

(1c) Case 6

$z_1 \wedge z_2 = (a_1 + b_1 i)\wedge(0 + b_2 i)$
Let $z_1 = 2 + 2\sqrt{3}\, i$ and $z_2 = 0 + 3i$
Hint: $z_1 \wedge z_2 \approx -0.023 - 0.037i$

Odds & Ends

(1) Complex Powers

(1d) Case 7
$z_1 \wedge z_2 = (a_1 + 0i)\wedge(a_2 + b_2 i)$
Let $z_1 = 2 + 0i$ and $z_2 = 3 + 3i$
Hint: $z_1 \wedge z_2 \approx -3.896 + 6.987i$

(1e) Case 8
$z_1 \wedge z_2 = (0 + b_1 i)\wedge(a_2 + b_2 i)$
Let $z_1 = 0 + 2\sqrt{3}\,i$ and $z_2 = 3 + 3i$
Hint: $z_1 \wedge z_2 \approx -0.206 + 0.311i$

(1f) Case 9
$z_1 \wedge z_2 = (a_1 + b_1 i)\wedge(a_2 + b_2 i)$
Let $z_1 = 2 + 2\sqrt{3}\,i$ and $z_2 = 3 + 3i$
Hint: $z_1 \wedge z_2 \approx 1.454 + 2.353i$

(2) Escape Velocity and Differential Equations

An object of mass m is thrown upward from the surface of the Earth with initial velocity v_0. You are to calculate the value of v_o, the Escape Velocity, with which the object can escape the pull of gravity and never return to the Earth. Since the object is moving far from the surface of the Earth, you need to take into account the variation of gravity with respect to altitude. If the acceleration due to gravity at Sea Level is g, and R is the radius of the Earth, then the gravitational force F acting on the object of mass m at an altitude h above the surface of the Earth is approxiamated by:

$$F = \frac{mgR^2}{(R+h)^2}.$$

(2a) Suppose v is the velocity of the object (measured upward) at time t. Use Newton's Law of Motion to show that:

$$\frac{dv}{dt} = \frac{-gR^2}{(R+h)^2}.$$

(2b) Rewrite this equation with h instead of t as the independent variable using the chain rule $\frac{dv}{dt} = \frac{dv}{dh}\frac{dh}{dt}$ and show that:

$$v\frac{dv}{dh} = \frac{-gR^2}{(R+h)^2}.$$

(2) Escape Velocity and Differential Equations

(2c) Solve the last Differential Equation.

(2d) Use your General Solution to find the Escape Velocity, the smallest value of v_0 such that v is never zero.

(3) Escape Velocity, Improper Integrals and the Work–Energy Thm

The Work–Energy Thm states that the amount of work done in a closed system (affected by no other forces) will impart potential energy to said system.

(3a) Use $F = \dfrac{mgR^2}{(R + h)^2}$ to calculate the work done to move a rocket an infinite distance from the surface of the Earth.

(3b) Assume that the system has zero total energy before take off and has zero total energy at the end of the flight. By conservation of energy, all the Potential Energy is used up in the Kenetic Energy of flight.

$$PE_0 + KE_0 = PE_\infty + KE_\infty = 0.$$

Therefore, if $PE_0 = 0$ (assuming the rocket is already moving) and $KE_\infty = 0$ (rocket stops at end of flight) then $KE_0 = PE_\infty$. Let PE_∞ be the amount of energy you calculated in part (3a). Let $KE_0 = \dfrac{mv_0^2}{2}$ and find v_0 the escape velocity.

(4) Uniform Circular Motion

Consider the following trajectory with displacement vector $\overline{r}$ (t):

$$\overline{r(t)} = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ ct \end{pmatrix}$$

where R, ω and c are positive Real constants.

(4a) Calculate the velocity vector $\overline{v(t)}$ and the acceleration vector $\overline{a(t)}$.

(4b) Find the magnitude of the velocity vector and the acceleration vector.

(4c) This trajectory is not circular, describe this trajectory.

(4) Uniform Circular Motion

(4d) Let c=0. Show that this new trajectory is now an example of Uniform Circular Motion such that $\overline{a(t)} = \frac{\overline{v(t)}^{\,2}}{R}$.

(5) Impulse–Momentum Theorem

As you know, the accumulation of Force over Distance, that is the Definite Integral of a Force function from s_1 to s_2, measures Work:

$$\int_{s_1}^{s_2} F\,ds = W.$$

Consider the accumulation of Force over time, that is the Definite Integral of a Force function from t_1 to t_2, measures Impulse. The Impulse–Momentum Theorem states that this Impulse results in a change in Momentum:

$$\int_{t_1}^{t_2} F\,dt = m\Delta v.$$

Note that thisis just a restatement of Newton's Second Law of Motion:

$$\Sigma F = \frac{d(mv)}{dt}.$$

(5a) A Rocket is flying through space such that all external forces on the rocket $\Sigma F = 0$. Apply the chain rule to $\frac{d(mv)}{dt}$.where m and v vary over time. Show that

$$m\frac{dv}{dt} = -v\frac{dm}{dt}$$

which is a restatement of Newton's Third Law of Motion.

(5) Impulse–Momentum Theorem

(5b) Let $m\frac{dv}{dt}$ represent the change in the rocket's momentum and $v\frac{dm}{dt}$ represent the change in propellant momemtum. Show that:

$$\int dv = -v_p \int \frac{dm}{m}$$

where v_p is the constant velocity of the propellant and m is the mass of the rocket.

(5c) Find the General Solution, v = f(m), for this Differential Equation.

(5d) Find the Particular Solution given that $v(m_i) = 0$ where m_i = mass of rocket + mass of propellant initially.

(5e) Show that the Rocket Equation is:

$$v = v_p \ln(1 + \frac{m_p}{m_r})$$

where m_p = mass of propellant and m_r = mass of rocket without fuel.

(6) Kepler's Laws of Planetary Motion

Kepler's First Law states that planet orbits are elliptical. However the proof of this law yields a conic section:

$$R(\theta) = \frac{R_0}{1 + e\cos(\theta)}.$$

(6a) Graph $R(\theta)$ when $R_0 = 1$ and $e = \frac{1}{2}$.

(6b) What kind of orbit does have you just modeled?

(6) Kepler's Laws of Planetary Motion

(6c) Graph $R(\theta)$ when $R_0 = 1$ and $e = 1$

(6d) What kind of orbit does have you just modeled?

(6) Kepler's Laws of Planetary Motion

(6e) Graph $R(\theta)$ when $R_0 = 1$ and $e = \frac{3}{2}$.

(6f) What kind of orbit does have you just modeled?

Teacher's notes:

No classroom notes are needed for this take home.