New Differential Models
More Ballistic Trajectories and Terminal Velocity

Consider the velocity of a bullet shot off the top of Mount Everst, the largest mountain on the Earth such that:

Approximated

Initial Conditions:

$$\overline{a}\,(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \frac{m}{s^2}$$

$$\overline{v}\,(0) = \begin{pmatrix} 720\cos(0°) \\ 720\sin(0°) \end{pmatrix} \frac{m}{s}$$

$$\overline{r}\,(0) = \begin{pmatrix} 0 \\ 8842 \end{pmatrix} m.$$

Remember that:

$$\overline{a}\,(t) = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} \frac{m}{s^2}$$

You will be calculating:

$$\overline{v}\,(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \frac{m}{s}$$

New Differential Models
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(1) Consider the following differential equation modeling the horizontal component of the velocity.

Model 1: $m\,x'' = -k\,x'$

In the given differential equation, m is the mass of the bullet in kilograms, x is the horizontal displacement of the bullet in meters and k is a positive constant of proportionality.

(1a) Interpret the meaning of this model with respect to Newton's 2^{nd} Law of Motion.

(1b) Based on your interpretation, draw the force diagram described by this model.

(1c) Rewrite the given differential equation in terms of v, the horizontal component of the velocity, instead of x, the horizontal component of the displacement.

(1d) Solve your new differential model for $v = f(t)$ analytically if possible.

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In the given differential equation, m is the mass of the bullet in kilograms, x is the horizontal displacement of the bullet in meters and k is a positive constant of proportionality.

(1e) Let m=.1 and k=.01, what are the units of k?
(1f) Let m=.1 and k=.01, graph the velocity function.
(1g) Classify this velocity model as Exponential Decay or Exponential Approach or Neither.

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(2) Consider the following differential equation modeling the horizontal component of the velocity.

$$\textbf{Model 2:} \quad m\, x'' = -k\sqrt{x'}$$

In the given differential equation, m is the mass of the bullet in kilograms, x is the horizontal displacement of the bullet in meters and k is a positive constant of proportionality.

(2a) Interpret the meaning of this model with respect to Newton's 2^{nd} Law of Motion.
(2b) Based on your interpretation, draw the force diagram described by this model.
(2c) Rewrite the given differential equation in terms of v, the horizontal component of the velocity, instead of x, the horizontal component of the displacement.
(2d) Solve your new differential model for v = f(t) analytically if possible.
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New Differential Models
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(3) Consider the following differential equation modeling the vertical component of the velocity.

Model 3: m y" = m g – k y'

In the given differential equation, g is the acceleration due to gravity in $\frac{m}{s^2}$, m is the mass of the bullet in kilograms, y is the vertical displacement of the bullet in meters and k is a positive constant of proportionality.

(3a) Interpret the meaning of this model with respect to Newton's 2nd Law of Motion.
(3b) Based on your interpretation, draw the force diagram described by this model.
(3c) Rewrite the given differential equation in terms of v, the vertical component of the velocity, instead of y, the vertical component of the displacement.
(3d) Solve your new differential model for v = f(t) analytically if possible.

New Differential Models

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(3f) Let m=.1 and k=.01, graph the velocity function.

(3g) Classify this velocity model as Exponential Decay, Exponential Approach or Neither.

New Differential Models
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(4) Consider the following differential equation modeling the vertical component of the velocity.

Model 4: $m\,y'' = m\,g - k\,(y')^2$

In the given differential equation, g is the acceleration due to gravity in $\frac{m}{s^2}$***, m is the mass of the bullet in kilograms, y is the vertical displacement of the bullet in meters and k is a positive constant of proportionality.***

(4a) Interpret the meaning of this model with respect to Newton's 2nd Law of Motion.

(4b) Based on your interpretation, draw the force diagram described by this model.

(4c) Rewrite the given differential equation in terms of v, the vertical component of the velocity, instead of y, the vertical component of the displacement.

(4d) Solve your new differential model for $v = f(t)$ analytically if possible.

New Differential Models
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In the given differential equation, g is the acceleration due to gravity in $\frac{m}{s^2}$***, m is the mass of the bullet in kilograms, y is the vertical displacement of the bullet in meters and k is a positive constant of proportionality.***

(4e) Let m=.1 and k=.01, what are the units of k?
(4f) Let m=.1 and k=.01, graph the velocity function.
(4g) Classify this velocity model as Exponential Decay, Exponential Approach or Neither.

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(5) Calculate the Terminal Velocity for each of these models.

(5a) Find $\lim_{t \to \infty} v(t)$ for Model 1. Interpret this result wrt terminal velocity.

(5b) Find $\lim_{t \to \infty} v(t)$ for Model 2. Interpret this result wrt terminal velocity.

(5c) Find $\lim_{t \to \infty} v(t)$ for Model 3. Interpret this result wrt terminal velocity.

(5d) Find $\lim_{t \to \infty} v(t)$ for Model 4. Interpret this result wrt terminal velocity.

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Teacher's notes:

Review:

Exponential Growth and Decay
Newton's Law of Heating and Cooling (Exponential Approach)
Logistic Growth and Decay

Do word problems from exercises in section 11.6 of Hughes-Hallett's
Calculus: Single Variable 4th ed. © 2005 from Wiley.