Odds & Ends

(3) Escape Velocity, Improper Integrals and the Work–Energy Thm

The Work–Energy Thm states that the amount of work done in a closed system (affected by no other forces) will impart potential energy to said system.

(3a) Use $F = \dfrac{mgR^2}{(R + h)^2}$ to calculate the work done to move a rocket an infinite distance from the surface of the Earth.

(3b) Assume that the system has zero total energy before take off and has zero total energy at the end of the flight. By conservation of energy, all the Potential Energy is used up in the Kinetic Energy of flight.

$$PE_0 + KE_0 = PE_\infty + KE_\infty = 0.$$

Therefore, if $PE_0 = 0$ (assuming the rocket is already moving) and $KE_\infty = 0$ (rocket stops at end of flight) then $KE_0 = PE_\infty$. Let PE_∞ be the amount of energy you calculated in part (3a). Let $KE_0 = \dfrac{mv_0^2}{2}$ and find v_0 the escape velocity.

(4) Uniform Circular Motion

Consider the following trajectory with displacement vector $\overline{r}$ (t):

$$\overline{r(t)} = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ ct \end{pmatrix}$$

where R, ω and c are positive Real constants.

(4a) Calculate the velocity vector $\overline{v(t)}$ and the acceleration vector $\overline{a(t)}$.

(4b) Find the magnitude of the velocity vector and the acceleration vector.

(4c) This trajectory is not circular, describe this trajectory.

(4) Uniform Circular Motion

(4d) Let c=0. Show that this new trajectory is now an example of Uniform Circular Motion such that the magnitude of $\overline{a(t)} = \frac{\overline{v(t)}^{\,2}}{R}$.

(5) Impulse–Momentum Theorem

As you know, the accumulation of Force over Distance, that is the Definite Integral of a Force function from s_1 to s_2, measures Work:

$$\int_{s_1}^{s_2} F\,ds = W.$$

Consider the accumulation of Force over time, that is the Definite Integral of a Force function from t_1 to t_2, which measures Impulse. The Impulse–Momentum Theorem states that this Impulse results in a change in Momentum:

$$\int_{t_1}^{t_2} F\,dt = m\Delta v.$$

Note that this is just a restatement of Newton's Second Law of Motion:

$$\Sigma F = \frac{d(mv)}{dt}.$$

(5a) A Rocket is flying through space such that all external forces on the rocket $\Sigma F = 0$. Apply the chain rule to $\frac{d(mv)}{dt}$ where m and v vary over time. Show that

$$m\frac{dv}{dt} = -v\frac{dm}{dt}$$

which is a restatement of Newton's Third Law of Motion.

(5) Impulse–Momentum Theorem

(5b) Let $m\frac{dv}{dt}$ represent the change in the rocket's momentum and $v\frac{dm}{dt}$ represent the change in propellant momemtum. Show that:

$$\int dv = -v_p \int \frac{dm}{m}$$

where v_p is the constant velocity of the propellant and m is the mass of the rocket.

(5c) Find the General Solution, $v = f(m)$, for this Differential Equation.

(5d) Find the Particular Solution given that $v(m_i) = 0$ where m_i = mass of rocket + mass of propellant initially.

(5e) Show that the Rocket Equation is:

$$v = v_p \ln(1 + \frac{m_p}{m_r})$$

where m_p = mass of propellant and m_r = mass of rocket without fuel.