Odds & Ends

(1) Complex Powers and Euler's Identity

Theorem

The set of Complex Numbers is closed under the operations +, –, *, / & ^.

Lemma 1	*If $z_1, z_2 \in C$, then $z_1 + z_2 \in C$.*
Lemma 2	*If $z_1, z_2 \in C$, then $z_1 - z_2 \in C$.*
Lemma 3	*If $z_1, z_2 \in C$, then $z_1 * z_2 \in C$.*
Lemma 4	*If $z_1, z_2 \in C$, then $z_1 / z_2 \in C$, $\lvert z_2 \rvert > 0$.*
Lemma 5	*If $z_1, z_2 \in C$, then z_1 ^ $z_2 \in C$.*

Your task is to demonstrate that the given Theorem is true. The Theorem is true if and only if all five Lemmas are valid. We have already seen that the first four lemmas are correct. What about the fifth?

Remember that C includes all Real numbers of the form a + 0i and all Imaginary numbers of the form 0 + bi as well as any Complex number of the form a + bi such that $a, b \in \Re$. So to see that z_1 ^ $z_2 \in C$ we have to analyze nine cases where z_1 is of the form $a_1 + 0i$, $0 + b_1i$ or $a_1 + b_1i$ and z_2 is of the form $a_2 + 0i$, $0 + b_2i$ or $a_2 + b_2i$

Case 1 $(a_1 + 0i)$^$(a_2 + 0i) = a_1$^a_2
For example: 2^3 = 8.

Case 2 $(0 + b_1i)$^$(a_2 + 0i) = (b_1i)$^a_2
For example (2i)^3 = –8i.

Case 3 $(a_1 + b_1i)$^$(a_2 + 0i) = (a_1 + b_1i)$^$a_2$
For example: If $(1 + 1i) = \sqrt{2}\,\text{cis}(45°)$,
then $(1 + 1i)$^$3 = \sqrt{2^3}\,\text{cis}(135°) = -2 + 2i$.

The previous three cases demonstrate that z_1 ^ $z_2 \in C$ when z_2 is of the form $a_2 + 0i \in \Re$. What happens when $z_2 \notin \Re$? Well, don't forget that z_1 can be written in several forms. For instance, $z_1 = a_1 + b_1i = r\,\text{cis}(\theta) = r\,e^{i\theta}$ too. How would you calculate i^i? You can convert the base to $r\,e^{i\theta}$ form, where θ is measured in radians, then raise to the power of i and convert back to a + bi form.

If $i = 1\,\text{cis}(\frac{\pi}{2}) = e$^$(\frac{i\pi}{2})$, then $i^i = e$^$(\frac{-\pi}{2}) \approx 0.208 + 0i \in \Re \subseteq C$. Note that taking roots is no different. If $\sqrt[i]{i} = i$^$(\frac{1}{i})$, then $\sqrt[i]{i} = e$^$(\frac{\pi}{2}) \approx 4.810 + 0i \in \Re \subseteq C$ too! Investigate the remaining 6 cases using this process. On the next two pages, find each power z_1 ^ z_2 exactly.

(1) Complex Powers

(1a) Case 4

z_1 ^ z_2 = $(a_1 + 0i)$^$(0 + b_2 i)$
Let $z_1 = 2 + 0i$ and $z_2 = 0 + 3i$
Hint: z_1 ^ $z_2 \approx -0.487 + 0.873i$

(1b) Case 5

z_1 ^ z_2 = $(0 + b_1 i)$^$(0 + b_2 i)$
Let $z_1 = 0 + 2\sqrt{3}\,i$ and $z_2 = 0 + 3i$
Hint: z_1 ^ $z_2 \approx -0.007 - 0.005i$

(1c) Case 6

z_1 ^ z_2 = $(a_1 + b_1 i)$^$(0 + b_2 i)$
Let $z_1 = 2 + 2\sqrt{3}\,i$ and $z_2 = 0 + 3i$
Hint: z_1 ^ $z_2 \approx -0.023 - 0.037i$

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(1) Complex Powers

(1d) Case 7
z_1 ^ z_2 = $(a_1 + 0i)$^$(a_2 + b_2 i)$
Let $z_1 = 2 + 0i$ and $z_2 = 3 + 3i$
Hint: z_1 ^ $z_2 \approx -3.896 + 6.987i$

(1e) Case 8
z_1 ^ z_2 = $(0 + b_1 i)$^$(a_2 + b_2 i)$
Let $z_1 = 0 + 2\sqrt{3}\, i$ and $z_2 = 3 + 3i$
Hint: z_1 ^ $z_2 \approx -0.206 + 0.311i$

(1f) Case 9
z_1 ^ z_2 = $(a_1 + b_1 i)$^$(a_2 + b_2 i)$
Let $z_1 = 2 + 2\sqrt{3}\, i$ and $z_2 = 3 + 3i$
Hint: z_1 ^ $z_2 \approx 1.454 + 2.353i$

(2) Escape Velocity and Differential Equations

An object of mass m is thrown upward from the surface of the Earth with initial velocity v_0. You are to calculate the value of v_o, the Escape Velocity, with which the object can escape the pull of gravity and never return to the Earth. Since the object is moving far from the surface of the Earth, you need to take into account the variation of gravity with respect to altitude. If the acceleration due to gravity at Sea Level is g, and R is the radius of the Earth, then the gravitational force F acting on the object of mass m at an altitude h above the surface of the Earth is approximated by:

$$F = \frac{mgR^2}{(R + h)^2}.$$

(2a) Suppose v is the velocity of the object (measured upward) at time t. Use Newton's Law of Motion to show that:

$$\frac{dv}{dt} = \frac{-gR^2}{(R + h)^2}.$$

(2b) Rewrite this equation with h instead of t as the independent variable using the chain rule $\frac{dv}{dt} = \frac{dv}{dh}\frac{dh}{dt}$ and show that:

$$v\frac{dv}{dh} = \frac{-gR^2}{(R + h)^2}.$$

(2) Escape Velocity and Differential Equations

(2c) Solve the last Differential Equation.

(2d) Use your General Solution to find the Escape Velocity, the smallest value of v_0 such that v is never zero.