CΔLCULUS

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Presentation 1

Contents

Introduction

Welcome!

Expectations for Thursday Sessions

- Arrive on time
- Do not miss sessions (Or notify us beforehand if absolutely necessary)
- Try to pay attention
- Attempt the examples
- Make notes
- Respect the people speaking
- **ASK questions if there is anything that you do not quite understand!**

Link to docs:

1. https://docs.google.com/document/d/1QooSTfNWN7uFu0g9hI0kyp6EAMFBn0lUIsuqLFKnkkg/edit
2. https://github.com/ethanolex/Calculus-Society.git

Please also feel free to suggest any topics to cover in the future on the docs!

Also:

Follow Calculus Society on IG!

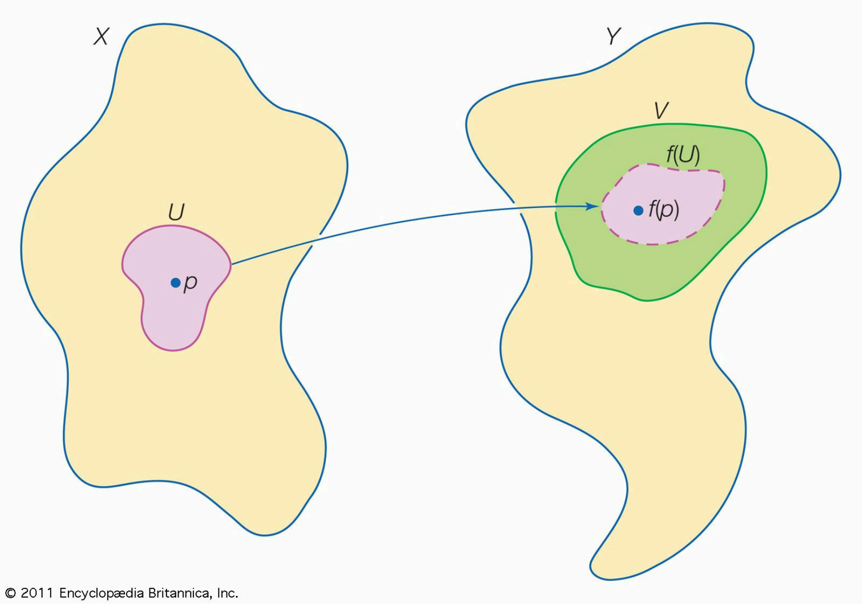
Preliminaries

Functions

We first formally understand what functions are.

Definition 1: A function f consists of a set of inputs, a set of outputs, and a rule for assigning *each input to exactly one output.*

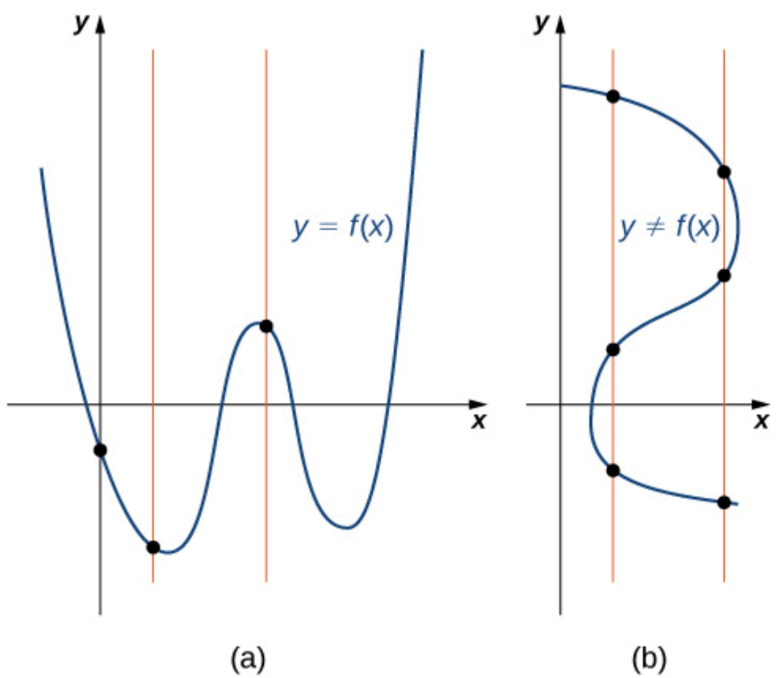
The set of inputs is called the **domain** of the function, whereas the set of outputs is called the **range** of the function.



Functions

Theorem 1 (Vertical line test): Given a function f , every vertical line that may be drawn will intersect the graph of f **no more than once.**

If any vertical line intersects the graph of f more than once, then the set of points does not represent a function.



Functions

Definition 2:

1. If $f(x) = f(-x)$ for all x in the domain of f,then f is an **even function**. An even function is symmetric about the y-axis.
2. If $f(-x) = -f(x)$ for all x in the domain of f , then f is an **odd function**. An odd function is symmetric about the origin.

Limits

Let $f(x)$ be a function defined everywhere in an open interval containing a, with the possible exception of a itself, and let L be a real number.

Definition 3: If all values of the function $f(x)$ approach L as the values of x approach the number a ($x \neq a$), then we say that the limit of $f(x)$ as x approaches a is L.

$$\lim_{x \to a} f(x) = L$$

Limits

Definition 4:

1. **Left sided limits:** If the values of the function $f(x)$ approach L as the values of x (where $x < a$) approach the number a, then
$$\lim_{x \to a^+} f(x) = L$$
2. **Right sided limits:** If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then
$$\lim_{x \to a^-} f(x) = L$$

Theorem 2:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L$$

Continuity

Finally, lets take a look at continuity

Definition 5: A function $f(x)$ is **continuous** at a point a if and only if the following three conditions are satisfied:

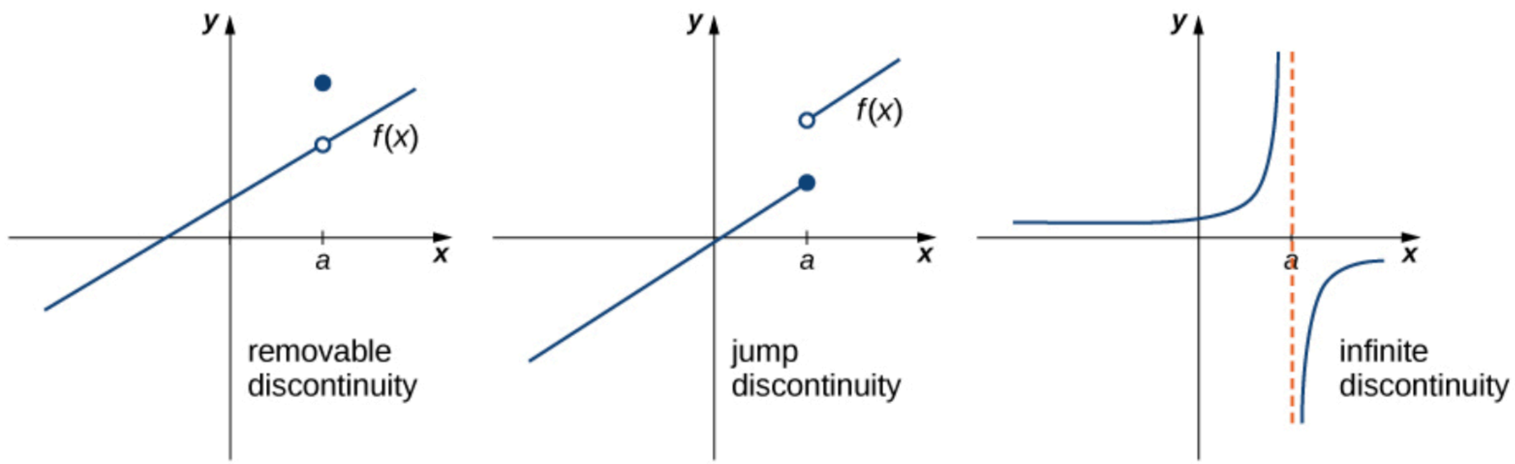
- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

A function is *discontinuous* at a point a if it fails to be continuous at a.

Continuity

Definition 6: If $f(x)$ is discontinuous at a, then:

- $f(x)$ has a *removable discontinuity* at a if $\lim_{x\to a} f(x)$ exists
- $f(x)$ has a *jump discontinuity* at a if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, but are not equal to each other
- $f(x)$ has an *infinite discontinuity* at a if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, but are not equal to each other.



Basic Principles

The derivative is defined to be the rate of change of functions.

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Definition 9: Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having gradient:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Basic Principles

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Definition 11: Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having gradient:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition 12: The derivative of $f(x)$ is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Continuity and differentiability

Theorem 3: Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

f is differentiable $\Rightarrow f$ is continuous

BUT

f is continuous $\not\Leftarrow f$ is differentiable

Counter examples:

- $f(x) = |x|$
- Weierstrass function

Proof:

Differential Calculus

Basic rules of differentiation

Theorem 4 (Contant rule): Let c be a constant. Then,

$$\frac{d}{dx}(c) = 0$$

Basic rules of differentiation

Theorem 6 (Contant rule): Let c be a constant. Then,

$$\frac{d}{dx}(c) = 0$$

Theorem 7 (Power rule): Let n be a constant. Then,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof of power rule

Basic rules of differentiation

Theorem 8: The derivative is *linear*, which means that:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

And, for k being a constant,

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$$

Basic rules of differentiation

Theorem 9: The derivative is *linear*, which means that:

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

And, for k being a constant,

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Example 2: Find the derivative of the following:

1. $y = \frac{1}{x^2}$
2. $y = \sqrt{x}$
3. $f(x) = 1$
4. $f(x) = \frac{x^4-3x^2+4}{x^2}$

Product rule, Quotient rule and Chain rule

Let $u(x)$ and $v(x)$ be functions. Then,

Theorem 10 (Product rule):

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Product rule, Quotient rule and Chain rule

Let $u(x)$ and $v(x)$ be functions. Then,

Theorem 12 (Product rule):

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Theorem 13 (Quotient rule):

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Product rule, Quotient rule and Chain rule

Example 3: Find the derivative of the following:

1. $f(x) = \frac{2x+5}{3x-4}$
2. $y = (3x+2)\sqrt{4x-1}$

Product rule, Quotient rule and Chain rule

Example 4: Find the derivative of the following:
1. $f(x) = \frac{2x+5}{3x-4}$
2. $y = (3x+2)\sqrt{4x-1}$

Answers:
1. $-\frac{23}{(3x-4)^2}$
2. $\frac{18x+1}{\sqrt{4x-1}}$

Product rule, Quotient rule and Chain rule

Theorem 14: If $y(u)$ and $u(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product rule, Quotient rule and Chain rule

Theorem 15: If $y(u)$ and $u(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 6: Find the derivative at $x = 3$:

$$y = \frac{1}{(x-2)^5}$$

Product rule, Quotient rule and Chain rule

Theorem 16: If $y(u)$ and $u(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 7: Find the derivative at $x = 3$:

$$y = \frac{1}{(x-2)^5}$$

<u>*Answer:*</u> -5

Higher order derivatives

The function $\frac{dy}{dx}$ is the *first derivative* of y with respect to x.

Definition 13: Differentiating the first derivative with respect to x gives the *second derivative* of y:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Remark 1: Note that:

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$$

Higher order derivatives

Example 8: Given that

$$y = \frac{2x^2}{x-3}$$

Find its second order derivative.

Now lets take a look at some physics applications!

Kinematics

Remember that:

- velocity is the ***rate of change*** of displacement
- acceleration is the ***rate of change*** of velocity

What does this mean?

Theorem 17 (Kinematics)**:** If the displacement $s(t)$ is a function of time:

$$v(t) = \frac{ds}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Kinematics

Example 9: A particle P is moving on the x-axis and its displacement s m/s , t s after a given instant, is given by:

$$s(t) = t^3 - \frac{1}{4}t^4$$

1. Find the *instantaneous velocity* of the particle at time $t = 2$.
2. Find the *instantaneous acceleration* of the particle at time $t = 2$.

Kinematics

Example 10: A particle P is moving on the x-axis and its displacement s m/s , t s after a given instant, is given by:

$$s(t) = t^3 - \frac{1}{4}t^4$$

1. Find the *instantaneous velocity* of the particle at time $t = 2$.
2. Find the *instantaneous acceleration* of the particle at time $t = 2$.

Answer: $v(2) = 4$ m/s ; $a(2) = 0$ m/s^2.

Derivatives of trigonometric functions

Theorem 18:

$$\frac{d}{dx}\sin x = \cos x$$

Theorem 19:

$$\frac{d}{dx}\cos x = -\sin x$$

Derivatives of trigonometric functions

Example 11: Find the derivative of $y = 5x^3 \sin x$ with respect to x.

Derivatives of trigonometric functions

Example 13: Find the derivative of $y = 5x^3 \sin x$ with respect to x.

<u>*Answer:*</u> $5x^3 \cos x + 15x^2 \sin x$

Example 14: Find the derivative of $y = \tan x$. (Hint: $\tan x = \frac{\sin x}{\cos x}$)

Derivatives of trigonometric functions

Example 15: Find the derivative of $y = 5x^3 \sin x$ with respect to x.

Answer: $5x^3 \cos x + 15x^2 \sin x$

Example 16: Find the derivative of $y = \tan x$. (Hint: $\tan x = \frac{\sin x}{\cos x}$)

Theorem 22:

$$\frac{d}{dx} \tan x = \sec^2 x$$

Derivative of exponential and logarithmic functions

Theorem 23:

$$\frac{d}{dx}e^x = e^x$$

How can this be applied to differentiate a more general exponential function of the form:

$$f(x) = b^{g(x)}$$

Any ideas?

Derivative of exponential and logarithmic functions

Theorem 24: If $f(x) = b^{g(x)}$, where $b > 0$ and $g(x)$ is a differentiable function,

$$f'(x) = b^{g(x)} g'(x) \ln b$$

(looks complicated, but no need to remember, since you know how to derive it!)

Derivative of exponential and logarithmic functions

Example 17: Find the derivative of $y = e^{x^2+2}$

Derivative of exponential and logarithmic functions

Example 19: Find the derivative of $y = e^{x^2+2}$

<u>*Answer:*</u> $2xe^{x^2+2}$

Example 20: Find the derivative of $y = 3^{x^2}$

Derivative of exponential and logarithmic functions

Example 21: Find the derivative of $y = e^{x^2+2}$

Answer: $2xe^{x^2+2}$

Example 22: Find the derivative of $y = 3^{x^2}$

Answer: $3^{x^2}(2x\ln 3)$

Derivative of exponential and logarithmic functions

Theorem 25:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Once again, how could this rule be applied to find the derivative of functions of the general form:

$$f(x) = \log_a g(x)$$

Derivative of exponential and logarithmic functions

Theorem 26: If $f(x) = \log_a g(x)$, where $b > 0$ and $g(x)$ is a differentiable function,

$$f'(x) = \frac{g'(x)}{g(x)\ln b}$$

Derivative of exponential and logarithmic functions

Theorem 27: If $f(x) = \log_a g(x)$, where $b > 0$ and $g(x)$ is a differentiable function,

$$f'(x) = \frac{g'(x)}{g(x)\ln b}$$

Example 24:

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Derivative of exponential and logarithmic functions

Theorem 28: If $f(x) = \log_a g(x)$, where $b > 0$ and $g(x)$ is a differentiable function,

$$f'(x) = \frac{g'(x)}{g(x) \ln b}$$

Example 25:

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Answer: $\frac{3}{\ln 16}$