CΔLCULUS $\oint$ciety

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Presentation 3

Contents

Welcome Back!

As we delve into deeper and more complicated concepts, it would be beneficial to:

- Try to (somewhat) pay attention
- Attempt the examples
- Make notes?
- **ASK questions if there is anything that you do not quite understand!**

Link to docs:

1. https://docs.google.com/document/d/1QooSTfNWN7uFu0g9hI0kyp6EAMFBn0lUIsuqLFKnkkg/edit

2. https://drive.google.com/drive/folders/1hhO1l8HUrkutdXbvT_XPetFJoRa5Hqlk?usp=sharing

Please also feel free to suggest any topics to cover in the future on the docs!

Also:

Follow Calulus Society on IG for interesting problems and math content!

Formal definition of limits

Theorem 1 (Epsilon delta definition)$: \forall \varepsilon > 0, \exists \delta > 0$ such that:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Formal definition of limits

Theorem 2 (Epsilon delta definition): $\forall \varepsilon > 0, \exists \delta > 0$ such that:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

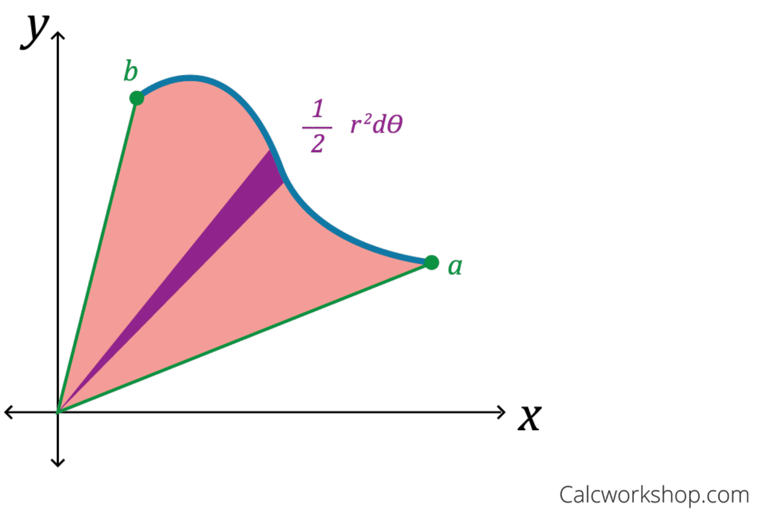
Example 2:

1. *Prove* that $\lim_{x \to 1}(3x - 1) = 2$
2. *Prove* that $\lim_{x \to 1}(2x + 1) = 3$

Polar coordinates

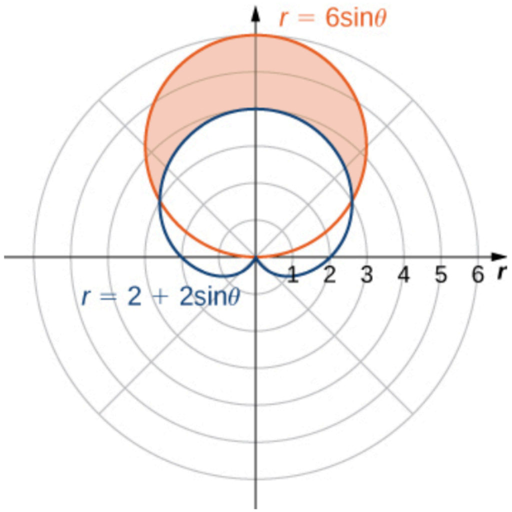
Theorem 3 (Area of curve in polar coordinates)**:** The area fo a curve $r(\theta)$ is given by:

$$\frac{1}{2}\int_{\theta_i}^{\theta_f} r^2 d\theta$$



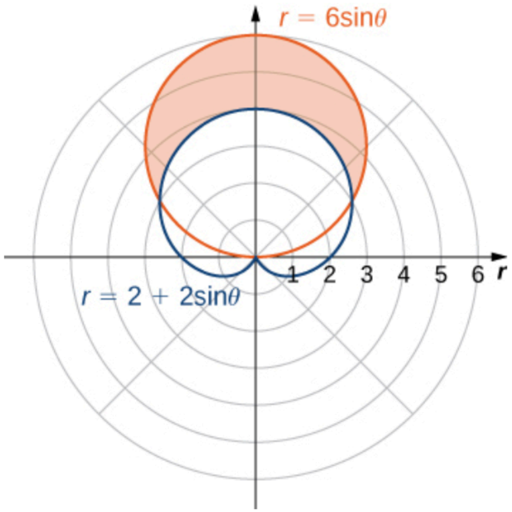
Polar coordinates

Example 3: Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.



Polar coordinates

Example 4: Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.



Ans: 4π

Complex numbers

Definition 1 (Imaginary number):

$$i = \sqrt{-1}$$

Theorem 4 (Euler's formula):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Theorem 5 (Euler's formula):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The famous Euler's identity:

$$e^{i\pi} = -1$$

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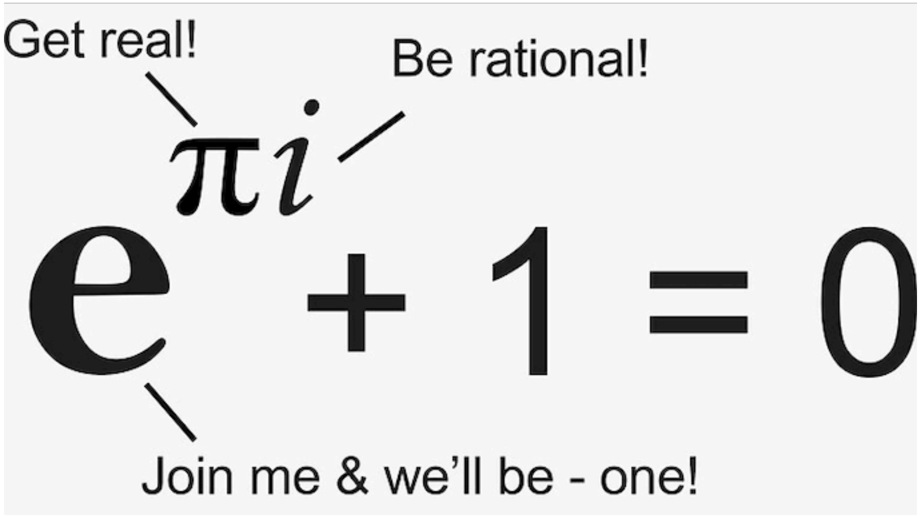

Argand Diagrams

Definition 2 (Complex number): A complex number z can be expressed in the form:

$$z = x + iy$$

where x and y are real constants.

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How can we represent complex numbers?

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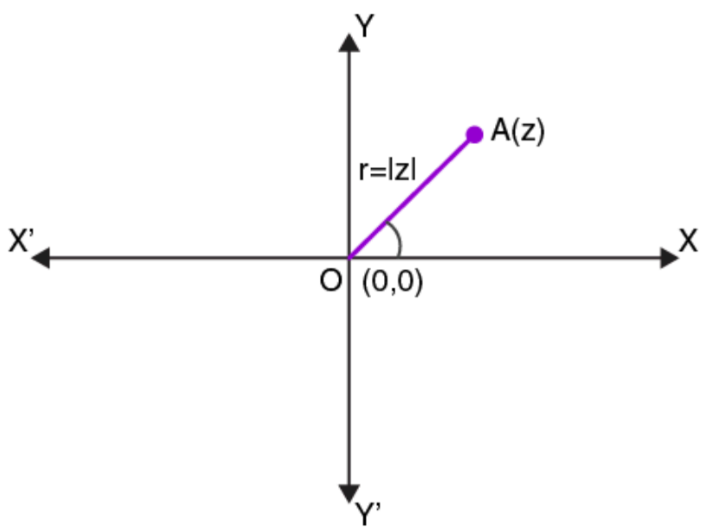
How can we represent complex numbers?



Argand Diagrams

Argand diagrams can be used to represent complex numbers, in a way similar to polar coordinates.

Theorem 7 (Argand Diagram): For a complex number $z = x + iy$,

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Argand Diagrams

Argand diagrams can be used to represent complex numbers, in a way similar to polar coordinates.

Theorem 8 (Argand Diagram): For a complex number $z = x + iy$,

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

If we are given r and θ instead, how can we determine the real and imaginary components (i.e. x and y)?

Mod-arg form

Theorem 9: A complex number $z = x + iy$ can also be expressed in the following way:

$$z = re^{i\theta}$$

where $r = |z|$ is called the *modulus* and θ is called the *argument* of z.

Mod-arg form

Theorem 10: A complex number $z = x + iy$ can also be expressed in the following way:

$$z = re^{i\theta}$$

where $r = |z|$ is called the *modulus* and θ is called the *argument* of z.

Why might this be of any use?

De Moivre's theorem

Consider a complex number $z = x + iy$. How can we determine the values of the powers of z (i.e, z^n)?

Theorem 11: (De Moivre's theorem) For a complex number $z = re^{i\theta}$,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

De Moivre's theorem

Consider a complex number $z = x + iy$. How can we determine the values of the powers of z (i.e, z^n)?

Theorem 12: (De Moivre's theorem) For a complex number $z = re^{i\theta}$,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Proof:

Checking your solution in annoying FM problems

Example 5: The quadratic equation

$$x^2 + 2x + 3 = 0$$

has roots α and β. Find the value of:

1. $\alpha^3 + \beta^3$
2. $\alpha^5 + \beta^5$

Checking your solution in annoying FM problems

Example 6: The quadratic equation

$$x^2 + 2x + 3 = 0$$

has roots α and β. Find the value of:

1. $\alpha^3 + \beta^3$
2. $\alpha^5 + \beta^5$

How can you check your answer to (1)?

Checking your solution in annoying FM problems

Example 7: The quadratic equation

$$x^2 + 2x + 3 = 0$$

has roots α and β. Find the value of:

1. $\alpha^3 + \beta^3$
2. $\alpha^5 + \beta^5$

How can you check your answer to (1)?

How could you do (2)?

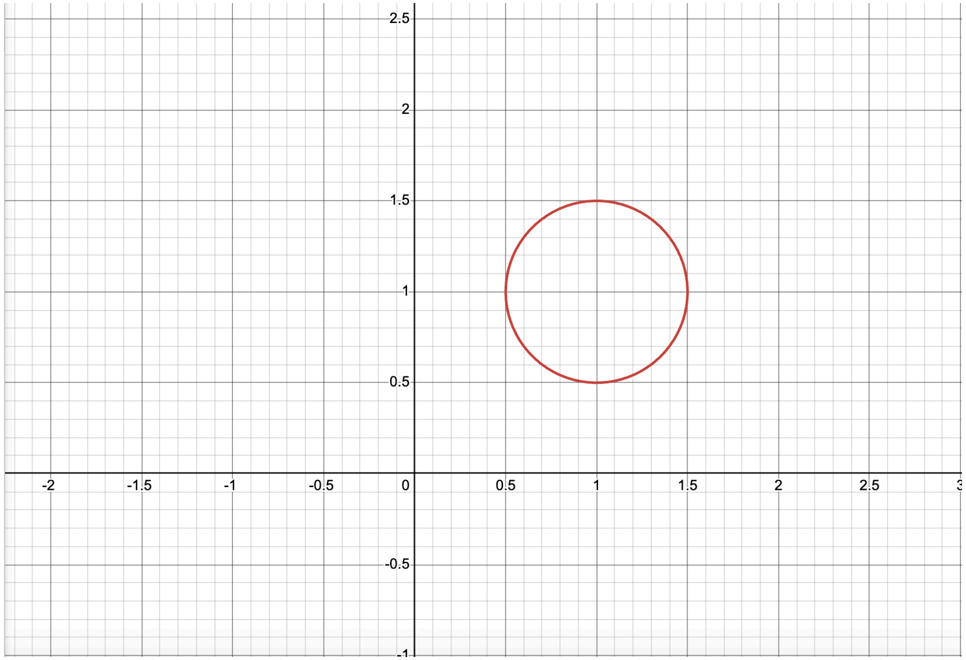
Theorem of Pappus

Theorem 13 (Theorem of Pappus): If a plane area is rotated about an axis in its plane, which does not cross the area, the **volume swept out equals the area times the distance moved by the centroid.**

Theorem of Pappus

Theorem 14 (Theorem of Pappus): If a plane area is rotated about an axis in its plane, which does not cross the area, the **volume swept out equals the area times the distance moved by the centroid.**

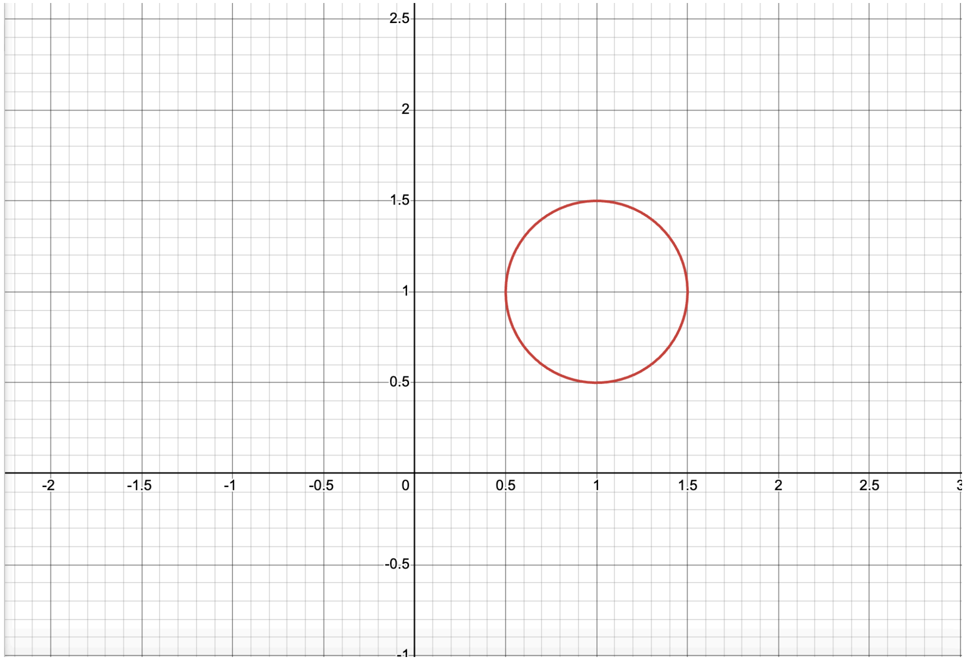
Torus:



Theorem of Pappus

Theorem 15 (Theorem of Pappus): If a plane area is rotated about an axis in its plane, which does not cross the area, the **volume swept out equals the area times the distance moved by the centroid.**

Torus:



$$V = (\pi R^2)(2\pi r)$$

Arc length

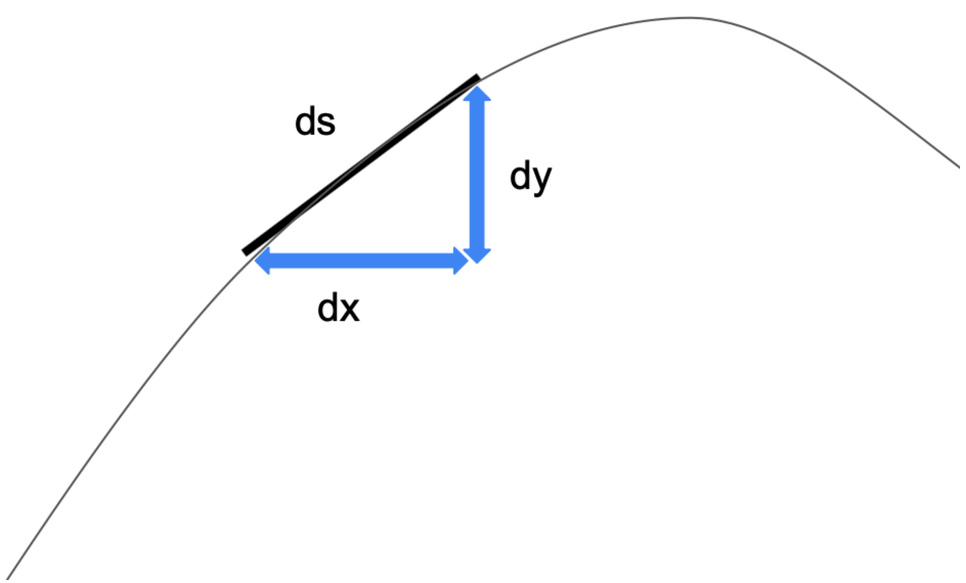
Theorem 16 (Arc length of curve):

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Arc length

Theorem 17 (Arc length of curve):

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Surface area of revolution

Theorem 18:

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Surface area of revolution

Theorem 19:

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Can you use this to determine the surface area of a sphere?

Surface area of revolution

Theorem 20:

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Can you use this to determine the surface area of a sphere?

Example 10: Find the surface area of the solid obtained by revolving the curve $y = \sqrt{x}$, $1 \leq x \leq 4$, $360°$ around the x-axis.

Surface area of revolution

Theorem 21:

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Can you use this to determine the surface area of a sphere?

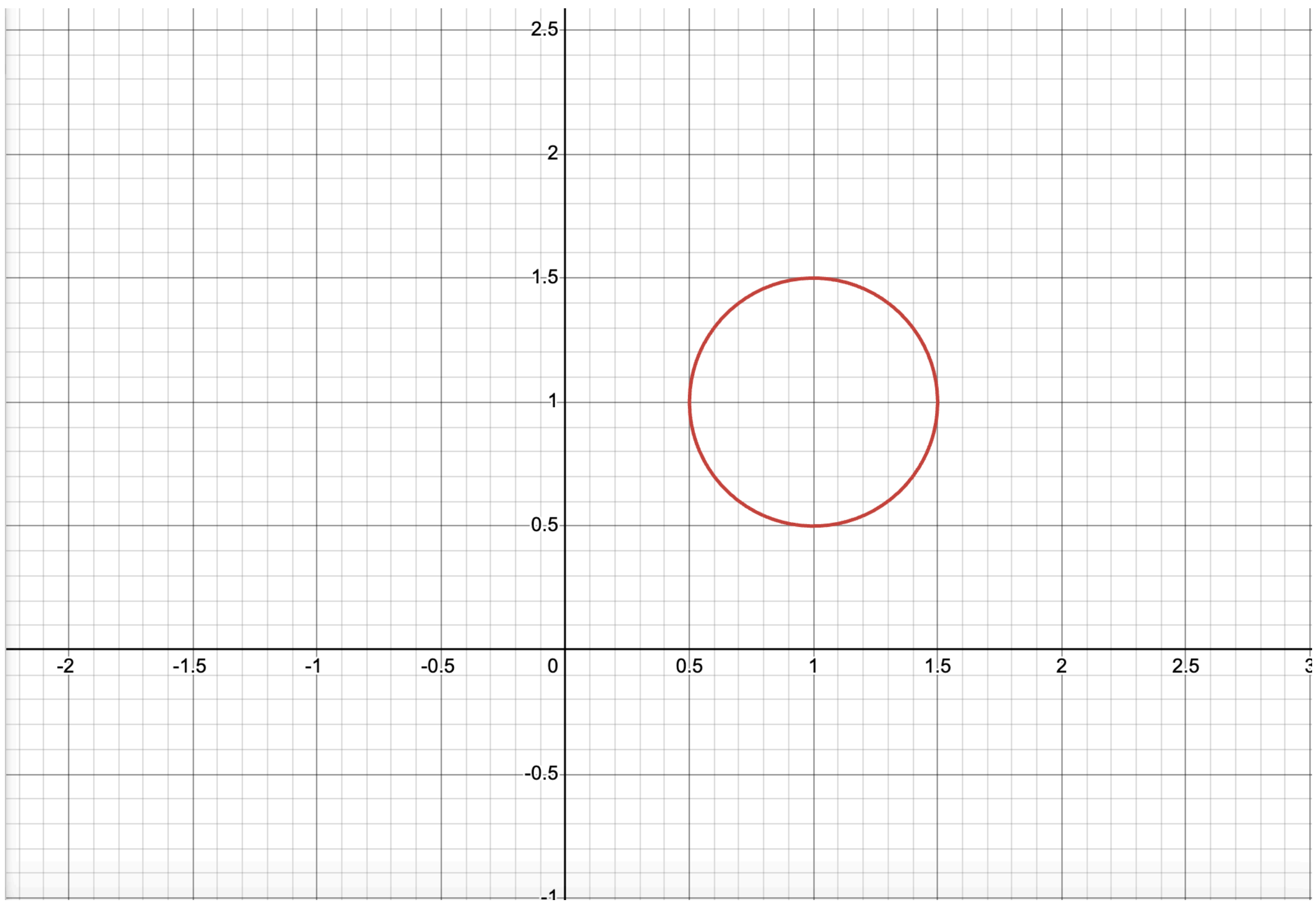
Example 11: Find the surface area of the solid obtained by revolving the curve $y = \sqrt{x}$, $1 \leq x \leq 4$, 360° around the x-axis.

Answer: 30.846 units^2

Weekly Problems

Find the volume of the solid obtained by revolving the following curve about the y-axis 360°:

$$(x-1)^2 + (y-1)^2 = \frac{1}{4}$$



u-substitution

$$\int_{-1}^{1} \sqrt{1-x^2}dx$$

1. Define a dummy variable u to be an expression in terms of the integrating variable (or vice versa), for example,

$$x = \sin u$$

2. Find dx in terms of du:

$$\frac{dx}{du} = \cos u$$

$$\Rightarrow dx = (\cos u)du$$

3. Change the integration bounds:

   When $x = 1$, $u = \frac{\pi}{2}$

   When $x = -1$, $u = -\frac{\pi}{2}$
4. Subsititute everything in:

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}(\cos u)du$$

Integrals

Example 12 (MIT Integration Bee Qualifying 2023 Q1):

$$\int x^{\frac{1}{\ln(x)}}\,dx$$

Integrals

Example 14 (MIT Integration Bee Qualifying 2023 Q1):

$$\int x^{\frac{1}{\ln(x)}} dx$$

Answer: $ex + C$

Integrals

Example 16 (MIT Integration Bee Qualifying 2023 Q1):

$$\int x^{\frac{1}{\ln(x)}} dx$$

<u>*Answer:*</u> $ex + C$

Example 17 (MIT Integration Bee Qualifying 2024 Q5):

$$\int_0^{2\pi} \cos^{-1}(\sin x) dx$$

Integrals

Example 18 (MIT Integration Bee Qualifying 2023 Q1):

$$\int x^{\frac{1}{\ln(x)}} dx$$

<u>*Answer:*</u> $ex + C$

Example 19 (MIT Integration Bee Qualifying 2024 Q5):

$$\int_0^{2\pi} \cos^{-1}(\sin x) dx$$

<u>*Answer:*</u> π^2

More random questions

Example 20:

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx$$

More random questions

Example 22:

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx$$

Answer: $e^{\tan(x)} + C$

More random questions

Example 24:

$$\int \frac{e^{\tan(x)}}{\cos^2(x)}dx$$

<u>*Answer:*</u> $e^{\tan(x)} + C$

Example 25:

$$\int_{-e^{\sqrt{\pi}+1}}^{e^{\sqrt{\pi}+1}} x^3 \sin(x) \tan(x) dx$$

More random questions

Example 26:

$$\int \frac{e^{\tan(x)}}{\cos^2(x)} dx$$

Answer: $e^{\tan(x)} + C$

Example 27:

$$\int_{-e^{\sqrt{\pi}+1}}^{e^{\sqrt{\pi}+1}} x^3 \sin(x) \tan(x) dx$$

Answer: 0 (You can thank Edward for thinking of this)

More...

Example 28 (MIT Integration Bee Qualifying 2024 Q2):

$$\int \frac{(x-1)^{\ln(x+1)}}{(x+1)^{\ln(x-1)}} dx$$

More...

Example 30 (MIT Integration Bee Qualifying 2024 Q2):

$$\int \frac{(x-1)^{\ln(x+1)}}{(x+1)^{\ln(x-1)}} dx$$

Answer: $x + C$

More...

Example 32 (MIT Integration Bee Qualifying 2024 Q2):

$$\int \frac{(x-1)^{\ln(x+1)}}{(x+1)^{\ln(x-1)}}dx$$

<u>*Answer:*</u> $x + C$

Example 33 (MIT Integration Bee Qualifying 2025 Q1):

$$\int \frac{x+\sqrt{x}}{1+\sqrt{x}}dx$$

More...

Example 34 (MIT Integration Bee Qualifying 2024 Q2):

$$\int \frac{(x-1)^{\ln(x+1)}}{(x+1)^{\ln(x-1)}} dx$$

Answer: $x + C$

Example 35 (MIT Integration Bee Qualifying 2025 Q1):

$$\int \frac{x+\sqrt{x}}{1+\sqrt{x}} dx$$

Answer: $\frac{2}{3}x^{\frac{3}{2}} + C$

More...

Example 36 (MIT Integration Bee Qualifying 2025 Q5):

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(20x)\sin(25x)dx$$

More...

Example 38 (MIT Integration Bee Qualifying 2025 Q5):

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(20x)\sin(25x)dx$$

<u>*Answer:*</u> 0

Example 39:

$$\int \cos^4 x - \sin^4 x dx$$

More...

Example 40 (MIT Integration Bee Qualifying 2025 Q5):

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(20x)\sin(25x)dx$$

<u>*Answer:*</u> 0

Example 41:

$$\int \cos^4 x - \sin^4 x dx$$

<u>*Answer:*</u> $\frac{1}{2}\sin 2x + C$

More...

Example 42 (MIT Integration Bee Qualifying 2025 Q14):

$$\int \sec^4 x - \tan^4 x dx$$

More...

Example 44 (MIT Integration Bee Qualifying 2025 Q14):

$$\int \sec^4 x - \tan^4 x dx$$

Answer: $2\tan x - x + C$

Example 45 (Gamma function):

$$\Gamma(z) = \int_0^\infty e^t t^{z-1} dt$$

Express $\Gamma(z+1)$ in terms of $\Gamma(z)$. Hint: Integrate by parts to reveal a special property!

More...

$$\int \sin^5 x dx$$

More...

$$\int \sin^5 x dx$$

<u>*Answer:*</u> $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

More...

$$\int \frac{e^x}{(1+e^x)\ln(1+e^x)}dx$$

More...

$$\int \frac{e^x}{(1+e^x)\ln(1+e^x)}dx$$

Answer: $\ln(\ln(1+e^x)) + C$

More...

$$\int \csc^2 x \tan^{2024} x dx$$

More...

$$\int \csc^2 x \tan^{2024} x dx$$

<u>*Answer:*</u> $\frac{\tan^{2023} x}{2023} + C$

More...

$$\int \sin^5 x dx$$

More...

$$\int \sin^5 x dx$$

<u>*Answer:*</u> $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

More...

$$\int_{\frac{1}{e}}^{e}\left(1-\frac{1}{x^2}\right)e^{e^{x+\frac{1}{x}}}\,dx$$

More...

$$\int_{\frac{1}{e}}^{e} \left(1 - \frac{1}{x^2}\right) e^{e^{x+\frac{1}{x}}} \, dx$$

Answer: 0

More...

$$\int x \ln x dx$$

More...

$$\int x \ln x dx$$

<u>*Answer:*</u> $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

More...

$$\int \cos^x x(\ln(\cos x) - x \tan x)dx$$

More...

$$\int \cos^x x(\ln(\cos x) - x \tan x)dx$$

Answer: $\cos^x x + C$

More...

$$\int_1^{e^e} \frac{\ln\left(x^{\ln(x^x)}\right)}{x^2}\,dx$$

More...

$$\int_1^{e^e} \frac{\ln\left(x^{\ln(x^x)}\right)}{x^2}\,dx$$

<u>*Answer:*</u> $\frac{e^3}{3}$

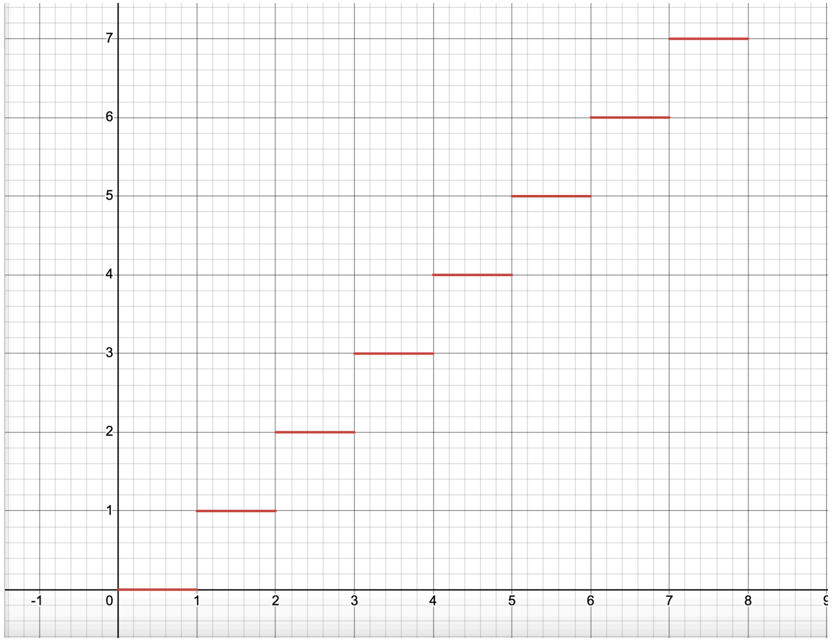
More...

$$\int_0^{100} \lfloor x \rfloor dx$$

More...

$$\int_0^{100} \lfloor x \rfloor dx$$

<u>*Answer:*</u> 5050



More...

$$\int_0^1 \left(\sum_{k=1}^{\infty} (-1)^k x^{2k} \right) dx$$

More...

$$\int_0^1 \left(\sum_{k=1}^{\infty} (-1)^k x^{2k} \right) dx$$

<u>*Answer:*</u> $\frac{\pi}{4} - 1$

More...

$$\int_1^3 \frac{x + \frac{x+\dots}{1+\dots}}{1 + \frac{x+\dots}{1+\dots}} dx$$

More...

$$\int_1^3 \frac{x + \frac{x+\dots}{1+\dots}}{1 + \frac{x+\dots}{1+\dots}} dx$$

<u>*Answer:*</u> $2\sqrt{3} - \frac{2}{3}$

MIT Integration Bee Qualifying 2025 Q5

https://openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$\text{Let } u = \cos x \Rightarrow \tfrac{du}{dx} = -(\sin x) \Rightarrow dx = -\tfrac{1}{\sin x} du$$

$$\begin{aligned}
\therefore \int \frac{\sin x}{\cos x} dx &= \int \frac{\sin x}{u} \left(-\frac{1}{\sin x}\right) du \\
&= -\int \frac{1}{u} du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C
\end{aligned}$$