回顾：① 条件概率 $p(x|y) = \frac{p(x,y)}{p(y)}$
(or 预习)

② 三大公式

1. 乘法公式　2. 全概率公式　3. Bayes.

③ 先验概率：预测

④ 后验概率：观测

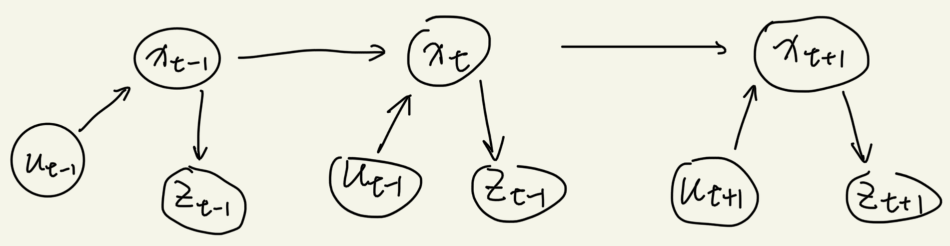
⑤ 马尔科夫假设

x：系统状态．u：控制量　z：观测量

$$p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

⑥ 动态贝叶斯网络.



贝叶斯滤波

$X_k = f(X_{k-1}) + Q_k$ ← 预测噪声

$Y_k = h(X_k) + R_k$ ← 观测噪声

$X_0, Q_1, \cdots Q_k, R_1 \cdots R_k$ 相互独立

预测：$P(X_k < x) = \sum_{u=-\infty}^{x} P(X_k = u)$

$$P(X_k = u) = \sum_{v=-\infty}^{+\infty} P(X_k = u | X_{k-1} = v) P(X_{k-1} = v)$$

$$= \sum_{v=-\infty}^{+\infty} P(X_k - f(X_{k-1}) = u - f(v) | X_{k-1} = v) P(X_{k-1} = v)$$

$$= \sum_{v=-\infty}^{+\infty} P(Q_k = u - f(v) | X_{k-1} = v) P(X_{k-1} = v)$$

X_{k-1}．Q_k 独立

$$= \sum_{v=-\infty}^{+\infty} P(Q_k = u - f(v)) P(X_{k-1} = v)$$

$$= \lim_{\varepsilon \to 0} \sum_{v=-\infty}^{+\infty} f_{Q_k}[u - f(v)] \cdot \varepsilon \cdot f_{k-1}(v) \cdot \varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} f_{Q_k}[u - f(v)] \cdot f_{k-1}(v) \cdot dv \cdot \varepsilon$$

$$\Rightarrow P(X_k < x_k) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{Q_k}[u - f(v)] \cdot f_{k-1}(v)\, du\, dv$$

$$\Rightarrow f_k^-(x_k) = \int_{-\infty}^{+\infty} f_{Q_k}[u - f(v)] \cdot f_{k-1}(v)\, dv \text{（两边求导）}$$

观测：
$$f_{Y_k|X_k}(Y_k | x_k) = \lim_{\varepsilon \to 0} \frac{P(y_k < Y_k < y_k + \varepsilon | X_k = x_k)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{P(y_k - h(x_k) < \underbrace{Y_k - h(x_k)}_{R_k} < y_k - h(x_k) + \varepsilon | X_k = x_k)}{\varepsilon}$$

R_k．X_k 独立

$$= \lim_{\varepsilon \to 0} \frac{P(y_k - h(x_k) < R_k < y_k - h(x_k) + \varepsilon)}{\varepsilon}$$

$$= f_{R_k}[y_k - h(x_k)]$$

$$\Rightarrow f_k(x_k) = \frac{f_k^-(x_k) \cdot f_{Y_k|X_k}(Y_k | x_k)}{\int_{-\infty}^{+\infty} f_k^-(x_k) \cdot f_{Y_k|X_k}(Y_k | x_k)\, dx_k}$$

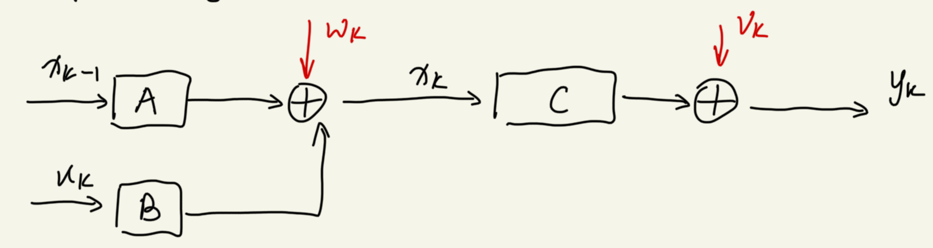
$$x = E(X_k)$$

卡尔曼滤波．　条件：线性高斯

观测与估计的权重化

状态方程：$x_k = A x_{k-1} + B u_k + \underbrace{w_k}_{噪声}$

观测方程：$y_k = C x_k + v_k$



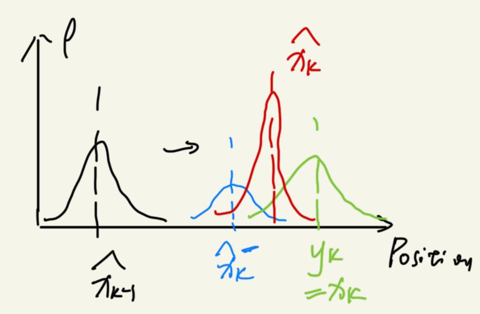
过程噪声 $w_k \sim N(0, Q_k)$　　Q．R：方差（超参数）

观测噪声 $v_k \sim N(0, R_k)$

$\hat{x}_{k-1}$ 最优估计值．后验估计．

$\hat{x}_k^-$ 先验估计．预测．

y_k 观测值．$= x_k$



使用上一次最优结果预测当前值．

同时使用观测值修正当前值．得到最优结果

观测矢量 $z_t = H \cdot x_t + v_t \Rightarrow Cov(z_t, z_t) = H$

$$\Rightarrow \hat{x}_t = p \cdot H^{-1}(z_t - v_t) + (1-p)\hat{x}_t^- = \hat{x}_t + pH^{-1}(z_t - v_t - H\hat{x}_t^-)$$

保证 z_t．x_t 维数一致．

$$= \hat{x}_t + K_t(z_t - H\hat{x}_t^-)$$

预测：

$$\hat{x}_t^- = F\hat{x}_{t-1} + Bu_{t-1} + w_t$$

协方差公式：$P_t^- = FP_{t-1}F^T + Q$

$$Cov(\hat{x}_t^-, \hat{x}_t^-) = F \cdot Cov(\hat{x}_{t-1}, \hat{x}_{t-1})F^T + \underbrace{Cov(w_t, w_t)}_{Q}$$

（噪声与信号独立）

更新．

$$\hat{x}_t = \hat{x}_t^- + K_t(z_t - H\hat{x}_t^-)$$

$$K_t = P_t^- H^T (HP_t^- H^T + R)^{-1}$$

$$P_t = (I - K_t H)P_t^-$$

设 $H = \binom{1}{0}$

$$K = (P_{t-1} + Q)(P_{t-1} + Q + R)^{-1}$$

观测权重↑，K↑ → R↓．Q↑

预测权重↑．K↓ → R↑．Q↓

ex．匀加速

$$p_i = p_{i-1} + v_i \cdot \Delta t + \frac{1}{2} a_i (\Delta t)^2$$

$$v_i = v_{i-1} + a_i \Delta t$$

$$\Rightarrow \begin{bmatrix} p_i \\ v_i \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{i-1} \\ v_{i-1} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} a_i$$

$$\hat{x}_t^- = F \cdot \hat{x}_{t-1} + B u_{t-1} + w_t$$

取值：Q．R．

传感器越厉害．R↓

模型越理想，Q↓

P_0：一般取 1．$\hat{x}_0$ 一般取 0

陀螺仪滤波．

状态：$\begin{bmatrix} angle \\ Q\_bias \end{bmatrix}$　观测：$\begin{bmatrix} new\ angle \end{bmatrix}$

陀螺仪噪声协方差 Q_angle　　角速度：newGyro

〃 〃 〃漂移噪声协 Q_gyro　　采样周期 dt．

角度测量噪声协 R_angle

① 预测当前角度．$angle_i = angle_{i-1} - Q\_bias \cdot dt + newGyro \cdot dt$

$$Q\_bias_i = Q\_bias_{i-1}$$

② 预测协方差矩阵．

$$Q = \begin{bmatrix} Q\_angle & 0 \\ 0 & Q\_gyro \end{bmatrix}$$

③．测量方程：$z(k) = H \cdot x(k) + V(k)$

（设 $H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$．把 MPU 测量角直接代入 $x(k)$．（已包含 $V(k)$））

$\Rightarrow z(k)$ (measure) $=$ new Angle

④ 卡尔曼增益

⑤ 协方差．

⑥．更新

扩展卡尔曼 → 非线性情形．

$$f(x) \approx f(x_k) + J(x_k)(x - x_k)$$

状态转移：$\theta_k = f(\theta_{k-1}) + w_k$

$$= f(\langle\theta_{k-1}\rangle) + J_f(\langle\theta_{k-1}\rangle)(\theta_{k-1} - \langle\theta_{k-1}\rangle) + w_k$$

$\langle\theta_{k-1}\rangle$：近似估计值．

观测方程：$z_k = h(\theta_k) + v_k$

$$= h(\langle\theta_k\rangle) + J_h(\langle\theta_k\rangle)(\theta_k - \langle\theta_k\rangle) + v_k$$

粒子滤波：非线性非高斯 ⇒ ☆ 贝叶斯滤波的蒙特卡洛实现

以先验分布作为重要性密度函数．

$$q(x_k | x_{k-1}^{(i)}, z_k) = p(x_k | x_{k-1}^{(i)})$$

重要性权重 $w_k^{(i)} = w_{k-1}^{(i)} p(z_k | x_k^{(i)})$

归一化重要性权值：$\tilde{w}_k^{(i)} = \frac{w_k^{(i)}}{\sum_{j=1}^{N} w_k^{(j)}}$

$$E|\hat{x}_k| = \sum_{i=1}^{N} x_k^{(i)} \tilde{w}_k^{(i)}$$