POLISHCHUK'S CONJECTURE AND KAZHDAN-LAUMON REPRESENTATIONS

CALDER MORTON-FERGUSON

Abstract. In their 1988 paper "Gluing of perverse sheaves and discrete series representations," D. Kazhdan and G. Laumon constructed an abelian category $\mathcal{A}$ associated to a reductive group G over a finite field with the aim of using it to construct discrete series representations of the finite Chevalley group $G(\mathbb{F}_q)$. The well-definedness of their construction depended on their conjecture that this category has finite cohomological dimension. This was disproven by R. Bezrukavnikov and A. Polishchuk in 2001, who found a counterexample in the case $G = SL_3$. In the same paper, Polishchuk then made an alternative conjecture: though this counterexample shows that the Grothendieck group $K_0(\mathcal{A})$ is not spanned by objects of finite projective dimension, he noted that a graded version of $K_0(\mathcal{A})$ can be thought of as a module over Laurent polynomials and conjectured that a certain localization of this module is generated by objects of finite projective dimension, and suggested that this conjecture could lead toward a proof that Kazhdan and Laumon's construction is well-defined. He proved this conjecture in Types A_1, A_2, A_3, and B_2. In the present paper, we prove Polishchuk's conjecture in full generality, and go on to prove that Kazhdan and Laumon's construction is indeed well-defined, giving a new geometric construction of discrete series representations of $G(\mathbb{F}_q)$.

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1. Introduction

In their 1988 paper [KL88], Kazhdan and Laumon described a gluing construction for perverse sheaves on the basic affine space associated to a semisimple algebraic group G split over a finite field $\mathbb{F}_q$, defining an abelian category $\mathcal{A}$ of "glued perverse sheaves" consisting of certain tuples of perverse sheaves on the basic affine space indexed by the Weyl group. They aimed to use these categories to provide a new geometric construction of discrete series representations of $G(\mathbb{F}_q)$.

Their proposal was to use $\mathcal{A}$ to construct representations as follows. First, they observed that the discrete series representations they sought to construct arise from the non-split tori $T(w)$ of G, which are indexed by the Weyl group. For each $w \in W$, they defined a category $\mathcal{A}_{w,\mathbb{F}_q}$ in a way such that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$.

They expected that the wildly infinite-dimensional representation $K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$ of $G(\mathbb{F}_q)$ admits a finite-dimensional quotient whose $T(w)$-isotypic components are the discrete series representations they sought to construct. Following the philosophy of Grothendieck's sheaf-function dictionary, Kazhdan and Laumon knew that the appropriate subspace of $K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$ by which one should take the quotient should be the kernel of a certain "Grothendieck-Lefschetz pairing" on $K_0(\mathcal{A}_{w,\mathbb{F}_q})$, which is defined in terms of the $\mathrm{Ext}^\bullet$ groups in the category $\mathcal{A}$. They then made the following conjecture and proved that it implies the well-definedness of their representations.

Conjecture 1.1 (Kazhdan-Laumon, [KL88])**.** *The category $\mathcal{A}$ has finite cohomological dimension. In other words, for any two objects A and B of $\mathcal{A}$, there is an n for which $\mathrm{Ext}^i(A, B) = 0$ whenever $i > n$.*

More than decade later, Bezrukavnikov and Polishchuk found a counterexample to this conjecture in the case $G = \mathrm{SL}_3$.

Proposition 1.2 (Bezrukavnikov-Polishchuk, Appendix to [Pol01])**.** *Conjecture 1.1 is false.*

In [Pol01], Polishchuk put forward the idea that although Conjecture 1.1 is false as stated in [KL88], it is not strictly necessary to prove the more important assertion that Kazhdan and Laumon's construction of representations is well-defined. He notes that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries the structure of a $\mathbb{Z}[v, v^{-1}]$-module using the formalism of mixed sheaves where v acts by a Tate twist, and then frames Conjecture 1.1 as a claim that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ is spanned by objects of finite projective dimension. In this situation, the Grothendieck-Lefschetz pairing defined on $K_0(\mathcal{A}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$ could be thought of as taking polynomial values in $\mathbb{Z}[v, v^{-1}]$ and then specializing at $v = q^{\frac{1}{2}}$, which is one way to see why Conjecture 1.1 would imply the well-definedness of this pairing and therefore the well-definedness of Kazhdan and Laumon's construction.

Although Proposition 1.2 shows that this is false, he instead proposes that this pairing is still well-defined if one allows it to take values in a certain localization of the ring $\mathbb{Z}[v, v^{-1}]$.

Letting $\mathcal{A}_{\mathbb{F}_q} = \mathcal{A}_{e,\mathbb{F}_q}$, i.e. the category of "Weil sheaves" in the Kazhdan-Laumon context, Polishchuk proposes the following more precise conjecture as a first step toward this goal.

Conjecture 1.3. *There exists a finite set of polynomials nonzero away from roots of unity such that the localization of $K_0(\mathcal{A}_{\mathbb{F}_q})$ at the multiplicative set generated by these polynomials is generated by objects of finite projective dimension.*

In [Pol01], Polishchuk develops a framework toward answering this conjecture, resolving it himself in Types A_2, A_2, A_3, and B_2. In our first main theorem, we use this framework along with the deeper understanding of symplectic Fourier transforms provided by [MF23] to prove this conjecture in general.

Theorem 1.4. *The Conjecture 1.3 is true. In particular, the localization of the $\mathbb{Z}[v, v^{-1}]$-module $K_0(\mathcal{A}_{\mathbb{F}_q})$ at the polynomial*

$$p(v) = \prod_{i=1}^{\ell(w_0)} \left(1 - v^{2i}\right) \tag{1}$$

is generated by objects of finite projective dimension.

As Polishchuk expected, the resolution of Conjecture 1.3 brings us very close to showing that Kazhdan and Laumon's construction of discrete series representations is well-defined. The main result of the present paper is that by using the formalism of monodromic perverse sheaves, we can prove a similar theorem which indeed completes the necessary technicalities to carry out Kazhdan and Laumon's construction in general.

Theorem 1.5. *For any character sheaf $\mathcal{L}$ of T and element $w \in W$, the localization of the $\mathbb{Z}[v, v^{-1}]$-module $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ at $p(v)$ is spanned by classes of objects of finite projective dimension in $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$.*

This monodromic approach to Kazhdan and Laumon's construction was already successfully carried out in [BP98], in which Braverman and Polishchuk explain how to carry out a well-defined version of Kazhdan and Laumon's construction in the case where $\mathcal{L}$ corresponds to a *quasi-regular* character. So one can think of the following corollary to Theorem 1.5 as a generalization of Braverman and Polishchuk's result to the case of an arbitrary character.

Corollary 1.6. *The Kazhdan-Laumon construction proposed in* [KL88] *is well-defined for monodromic sheaves corresponding to any arbitrary character.*

1.1. **Layout of the paper.** In Section 2, we explain some background on Kazhdan-Laumon categories. In Section 3, we then explain the monodromic setting required to state Theorem 1.5 and discuss the categorical center of the monodromic Hecke category, which will be a crucial tool in the proof. Then in Section 4 we use dg formalism to explain why Kazhdan-Laumon categories admit an action of this categorical center. This is followed in Section 5 by an explanation of a crucial tool in the study of Kazhdan-Laumon categories proposed by

Polishchuk [Pol01] called the canonical complex. We then complete the proofs of our results: in Section 6 we prove Theorem 1.5, and in Section 7 we recall the results of [MF23] and explain how it, combined with the previous setup, allow us to prove Polishchuk's original conjecture and establish Theorem 1.4 independently of the monodromic setting. Finally, in Section 8, we explain how to carry out the construction of Kazhdan-Laumon representations explicitly given our theorems.

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2. Preliminaries

2.1. **Background and notation.**

2.1.1. *General setup.* Let G be a split semisimple group over a finite field $\mathbb{F}_q$. Let T be a Cartan subgroup split over $\mathbb{F}_q$, B a Borel subgroup containing T, and U its unipotent radical. Let $X = G/U$ be the basic affine space associated to G considered as a variety over $\mathbb{F}_q$. Let W be the Weyl group W. We let S denote the set of simple reflections in W. Writing $q = p^m$ for some prime number p, we choose ℓ to be a prime with $\ell \neq p$.

2.1.2. *ℓ-adic sheaves, Tate twists, and Grothendieck groups.* We will work with the category $\mathrm{Perv}(G/U)$ of mixed ℓ-adic perverse sheaves on the basic affine space G/U, and more generally with the constructible derived category $D^b(G/U)$ of mixed ℓ-adic sheaves on G/U. We choose an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and work with $\mathbb{C}$ going forward. Pick a square root $q^{\frac{1}{2}}$ of q in $\mathbb{C}$ once and for all, and define the half-integer Tate twist $(\frac{1}{2})$ on $D^b(G/U)$. We then view $K_0(G/U) = K_0(D^b(G/U))$ as a $\mathbb{Z}[v, v^{-1}]$-module where v^{-1} acts by $(\frac{1}{2})$. When $\mathcal{C}$ is any category for which $K_0(\mathcal{C})$ is a $\mathbb{Z}[v, v^{-1}]$-module, we denote by $K_0(\mathcal{C}) \otimes \mathbb{C}$ the specialization $K_0(\mathcal{C}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ at $v = q^{\frac{1}{2}}$. We use $\mathbb{D}$ to denote the Verdier duality functor. We let ${}^pH^i$ be the perverse cohomology functors for any $i \in \mathbb{Z}$.

We also choose once and for all a nontrivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}_\ell}$, and let $\mathcal{L}_\psi$ be the corresponding Artin-Schrieier sheaf on $\mathbb{G}_a$.

The variety G/U comes with a natural Frobenius morphism $\mathrm{Fr} : G/U \to G/U$; we can then consider the category of Weil sheaves $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$ on G/U, i.e. sheaves $K \in \mathrm{Perv}(G/U)$ equipped with a natural isomorphism $\mathrm{Fr}^* K \cong K$.

2.1.3. *Elements of G indexed by W.* For every simple root α_s of G corresponding to the simple reflection s, we fix an isomorphism of the corresponding one-parameter subgroup $U_s \subset U$ with the additive group $\mathbb{G}_a$. This uniquely defines a homomorphism $\rho_s : SL_2 \to G$ which induces the given isomorphism of $\mathbb{G}_a$ (embedded in SL_2 as upper-triangular matrices) with U_s; then let

$$n_s = \rho_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any $w \in W$, writing a reduced word $w = s_{i_1} \dots s_{i_k}$ we set $n_w = n_{s_{i_1}} \dots n_{s_{i_k}}$, and one can check that this does not depend on the reduced word. We also define for any $s \in S$ the subtorus $T_s \subset T$ obtained from the image of the coroot $\alpha_s^\vee$ and define T_w for any $w \in W$ to be the product of all T_s ($s \in S$) for which $s \leq w$ in the Bruhat order.

2.2. Kazhdan-Laumon categories.

2.2.1. *Fourier transforms on* $\mathrm{Perv}(G/U)$*.* In [KL88] and [Pol01], to each $w \in W$ the authors assign an element of $D^b(G/U \times G/U)$ which, up to shift, is perverse and irreducible. Following [Pol01], let $X(w) \subset G/U \times G/U$ be the subvariety of pairs $(gU, g'U) \subset (G/U)^2$ such that $g^{-1}g' \in U n_w T_w U$. There is a canonical projection $\mathrm{pr}_w : X(w) \to T_w$ sending $(gU, g'U)$ to the unique $t_w \in T_w$ such that $g^{-1}g' \in U n_w t_w U$. In the case when $w = s \in S$, the morphism $\mathrm{pr}_s : X(s) \to T_s \cong \mathbb{G}_{m,k}$ extends to $\overline{\mathrm{pr}}_s : \overline{X(s)} \to \mathbb{G}_{a,k}$ and we have

$$\overline{K(s)} = (-\overline{\mathrm{pr}}_s)^* \mathcal{L}_\psi$$

and in the case of general $w \in W$

$$\overline{K(w)} = \overline{K(s_{i_1})} * \dots * \overline{K(s_{i_k})} \tag{2}$$

whenever $w = s_{i_1} \dots s_{i_k}$ is a reduced expression. One can take this as the definition of Kazhdan-Laumon sheaves, which is well-defined by the proposition below, or refer to the explicit definition of $\overline{K(w)}$ which works for all $w \in W$ at once given in [KL88] or [Pol01].

Proposition 2.1 ([KL88])**.** *The Kazhdan-Laumon sheaves $\overline{K(s)}$ for $s \in S$ under convolution satisfy the braid relations (up to isomorphism).*

For any $s \in S$, they note that the endofunctor $K \to K * \overline{K(s)}$ of $D^b(G/U)$ can be identified with a certain "symplectic Fourier-Deligne transform" defined as follows. Let P_s be the parabolic subgroup of G associated to s, and let $Q_s = [P_s, P_s]$. The map $G/U \to G/Q_s$ has all fibers isomorphic to $\mathbb{A}^2 \setminus \{(0,0)\}$, and it is shown in Section 2 of [KL88] that there exists a natural fiber bundle $\pi : V_s \to G/Q_s$ of rank 2 equipped with a G-invariant symplectic pairing which contains G/U as an open set, with inclusion $j : G/U \to V_s$ with $\pi \circ j$ being the original projection $G/U \to G/Q_s$. There is then a symplectic Fourier-Deligne transform $\tilde{\Phi}_s$ on $D^b(V_s)$ defined by

$$\tilde{\Phi}_s(K) = p_{2!}(\mathcal{L} \otimes p_1^*(K))[2](1), \tag{3}$$

where the p_i are the projections of the product $V_s \times_{G/Q_s} V_s$ on its factors, and $\mathcal{L} = \mathcal{L}_\psi(\langle , \rangle)$ is a smooth rank-1 $\overline{\mathbb{Q}_\ell}$-sheaf which is the pullback of the Artin—Schreier sheaf $\mathcal{L}_\psi$ under the morphism $\langle , \rangle$, c.f. Section 4 of [Pol01]. We then define the endofunctor Φ_s of $D^b(G/U)$ By

$$\Phi_s(K) = j^* \tilde{\Phi}_s j_! K. \tag{4}$$

Proposition 2.2 ([KL88], [Pol01])**.** *The functors Φ_s and $- * \overline{K(s)}$ are naturally isomorphic.*

For any $w \in W$, we let $\Phi_w = \Phi_{s_{i_1}} \dots \Phi_{s_{i_k}}$ where $s_{i_1} \dots s_{i_k}$ is a reduced expression for w as a product of simple reflections. The functors Φ_w are the gluing functors which Kazhdan and Laumon use to define the so-called glued categories $\mathcal{A}$.

Definition 2.3. Using the six-functor formalism, one can check that each Φ_s has a right adjoint which we call Ψ_s, following the setup of Section 1.2 of [Pol01]. The functors Ψ_s also form a braid action on $D^b(G/U)$. We then define Ψ_w similarly.

The functors Φ_w (resp. Ψ_w) are each right (resp. left) t-exact on $D^b(G/U)$ with respect to the perverse t-structure. For any $w \in W$, let $\Phi_w^\circ = {}^pH^0\Phi_w$ and $\Psi_w^\circ = {}^pH^0\Psi_w$, noting that $\Phi_w = L\Phi_w^\circ$, $\Psi_w = R\Psi_w^\circ$.

Proposition 2.4 ([Pol01], Section 4.1)**.** *For any $s \in S$, there are natural morphisms $c_s : \Phi_s^2 \to \mathrm{Id}$ and $c_s' : \mathrm{Id} \to \Psi_s^2$ satisfying the associativity conditions*

$$\Phi_s \circ c_s = c_s \circ \Phi_s : \Phi_s^3 \to \Phi_s \qquad\qquad \Psi \circ c_s' = c_s' \circ \Psi : \Psi_s \to \Psi_s^3. \tag{5}$$

Corollary 2.5. *For any $y', y \in W$, there is a natural transformation $\nu_{y',y} : \Phi_{y'}\Phi_y \to \Phi_{y'y}$.*

Proof. We go by induction on $\ell(y)+\ell(y')$. If $\ell(y'y) = \ell(y')+\ell(y)$, then $\nu_{y',y}$ is the tautological map arising from the fact that the Φ_w form a braid action. If instead $\ell(y'y) < \ell(y') + \ell(y)$, then there exists some $s \in S$ such that $y' = \tilde{y}'s$ and $y = s\tilde{y}$ for some $\tilde{y}', \tilde{y} \in W$ with $\ell(\tilde{y}'s) = \ell(\tilde{y}') + 1$ and $\ell(s\tilde{y}) = \ell(\tilde{y}) + 1$, and so we have maps

$$\Phi_{y'}\Phi_y = \Phi_{\tilde{y}'}\Phi_s^2\Phi_{\tilde{y}} \xrightarrow{\Phi_{\tilde{y}} \circ c_s \circ \Phi_{\tilde{y}}} \Phi_{\tilde{y}'}\Phi_{\tilde{y}} \xrightarrow{\nu_{\tilde{y}',\tilde{y}}} \Phi_{y'y},$$

the former coming from c_s and the latter coming from our induction hypothesis. ∎

2.2.2. *Definition of the Kazhdan-Laumon category.*

Definition 2.6 ([KL88], [Pol01])**.** The Kazhdan-Laumon category $\mathcal{A}$ has objects which are tuples $(A_w)_{w \in W}$ with $A_w \in \mathrm{Perv}(G/U)$ and equipped with morphisms

$$\theta_{y,w} : \Phi_y^\circ A_w \to A_{yw} \tag{6}$$

for every $y, w \in W$ such that the diagram

$$\begin{tikzcd}
\Phi^\circ_{y'}\Phi^\circ_y A_w \arrow[r, "\Phi_{y'}\theta_{y,w}"] \arrow[d, "\nu_{y',y}"] & \Phi^\circ_{y'}A_{yw} \arrow[d, "\theta_{y',yw}"] \\
\Phi^\circ_{y'y}A_w \arrow[r, "\theta_{y'y,w}"] & A_{y'yw}
\end{tikzcd}$$

commutes for any $y, y', w \in W$.

A morphism f between objects $(A_w)_{w\in W}$ and $(B_w)_{w\in W}$ is a collection of morphisms $f_w : A_w \to B_w$ such that

$$\begin{tikzcd}
\Phi^\circ_y A_w \arrow[r, "\Phi^\circ_y f_w"] \arrow[d, "\theta^A_{y,w}"] & \Phi^\circ_y B_w \arrow[d, "\theta^B_{y,w}"] \\
A_{yw} \arrow[r, "f_{yw}"] & B_{yw}.
\end{tikzcd}$$

It is shown in [Pol01] that this category is abelian, and that the functors $j^*_w : \mathcal{A} \to \mathrm{Perv}(G/U)$ defined by $j^*_w((A_w)_{w\in W}) = A_w$ are exact.

Remark 2.7. We could have instead asked for morphisms $A_{yw} \to \Psi^\circ_y A_w$, making reference to the functors Ψ°_y rather than the Φ°_y. Later, we will discuss an alternate and more elegant definition of $\mathcal{A}$ as coalgebras over a certain comonad on $\oplus_{w\in W}\mathrm{Perv}(G/U)$ which is assembled from the functors Ψ°_y. In Definition 2.6 though, we present the definition of the Kazhdan-Laumon category as it was originally formulated in [KL88] and later explained in more detail in [Pol01].

2.2.3. *Definition of the w-twisted categories $\mathcal{A}_{w,\mathbb{F}_q}$.* We note that the category $\mathcal{A}$ carries an action of the Weyl group W as follows. For any $w \in W$, we let $\mathcal{F}_w : \mathcal{A} \to \mathcal{A}$ be the exact functor defined by right translation of the indices in the tuple, i.e. $\mathcal{F}_w((A_y)_{y\in W}) = (A_{yw})_{y\in W}$.

Definition 2.8. For any $w \in W$, let $\mathcal{A}_{w,\mathbb{F}_q}$ be the category with objects (A, ψ_A) where $A \in \mathcal{A}$ and $\psi_A : \mathcal{F}_w\mathrm{Fr}^*A \to A$ is an isomorphism. We call these w-twisted Weil sheaves in the Kazhdan-Laumon category.

For any two such objects (A, ψ_A) and (B, ψ_B), we let $\mathrm{Hom}_{\mathcal{A}_{w,\mathbb{F}_q}}(A, B)$ be the set of morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ such that $f \circ \psi_A = \psi_B \circ \mathcal{F}_w\mathrm{Fr}^*f$.

Remark 2.9. When $w = e$ is trivial, we will write $\mathcal{A}_{\mathbb{F}_q} = \mathcal{A}_{w,\mathbb{F}_q}$. In this case, it is shown in [Pol01] that $\mathcal{A}_{\mathbb{F}_q}$ is equivalent to the category obtained by applying the Kazhdan-Laumon gluing procedure described in Definition 2.6 to the category $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$ of Weil perverse sheaves on G/U, i.e. perverse sheaves K equipped with an isomorphism $\mathrm{Fr}^*K \to K$.

2.2.4. *Adjoint functors on $D^b(\mathcal{A})$.*

Definition 2.10. For any $w \in W$, let $j^*_w : D^b(\mathcal{A}) \to D^b(G/U)$ be the functor arising from the same-named exact functor $j^*_w : \mathcal{A} \to \mathrm{Perv}(G/U)$ given by $j^*_w((A_y)_{y\in W}) = A_w$.

We then define a functor $j_{w!} : \mathrm{Perv}(G/U) \to \mathcal{A}$ by

$$j_{w!}(K) = (\Phi^\circ_{yw^{-1}} K)_{y \in W}. \tag{7}$$

One can check that the morphisms $\nu_{y',y}$ for $y, y' \in W$ introduced in Corollary 2.5 endow the tuple $(\Phi^\circ_{yw^{-1}} K)_{y\in W}$ with the structure morphisms required to define an object of $\mathcal{A}$.

Proposition 2.11 (Proposition 7.1.2, [Pol01])**.** *For any $w \in W$, there is an adjunction $(j_{w!}, j^*_w)$. Further, the left-derived functor $Lj_{w!} : D^b(G/U) \to D^b(\mathcal{A})$ has the property that*

$${}^pH^i(Lj_{w!}(K)) = (\Phi^\circ_{yw^{-1}}(K))_{y\in W}, \tag{8}$$

and there is also an adjunction $(Lj_{w!}, j^*_w)$ on derived categories.

Analogously, acting instead by the functors $\Psi_{yw^{-1}}$ in (8) defines a right-adjoint Rj_{w*} to j^*_w in the very same way, as is also explained in [Pol01].

2.2.5. *The functor ι.*

Definition 2.12. We define an endofunctor ι of $\mathcal{A}$ by

$$\iota((A_w)_{w\in W}) = (\Phi^\circ_{w_0} A_{w_0 w})_{w \in W} \tag{9}$$

with the structure morphisms described in 7.2 of [Pol01].

It is shown in loc. cit. that we can also view ι as a functor on $D^b(\mathcal{A})$; for our purposes we will only need the action of ι^2, which we will later describe as an endofunctor of $D^b(\mathcal{A})$ more conceptually in Lemma 6.2.

2.2.6. *Objects of the form $j_{w!}(A)$.*

Proposition 2.13. *For any $B \in D^b(G/U)$ and any $w \in W$, the object $j_{w!}B \in D^b(\mathcal{A})$ has finite projective dimension.*

Proof. By Proposition 7.1.2 of [Pol01], the functor $j_{w!} : D^b(G/U) \to D^b(\mathcal{A})$ is left-adjoint to the restriction functor j^*_w. Hence for any $A \in \mathcal{A}$,

$$\mathrm{Ext}^\bullet_{\mathcal{A}}(j_{w!}(B), A) = \mathrm{Ext}^\bullet_{\mathrm{Perv}(G/U)}(B, j^*_w A), \tag{10}$$

and the latter is a finite-dimensional vector space since the category $\mathrm{Perv}(G/U)$ has finite cohomological dimension. $\square$

2.3. **Grothendieck groups and localizations.**

2.3.1. *Polynomial definitions.* Fix the Weyl group W and choose w_0 its longest element.

Definition 2.14. Define $P(x,v), \tilde{P}(x,v) \in \mathbb{Z}[x,v,v^{-1}]$ by

$$P(x,v) = \prod_{i=0}^{\ell(w_0)} (x - v^{2i}) \tag{11}$$

$$\tilde{P}(x,v) = \prod_{i=1}^{\ell(w_0)} (x - v^{2i}) \tag{12}$$

and let $p(v) = \tilde{P}(1,v)$.

2.3.2. *Grothendieck group localizations.*

Definition 2.15. For any abelian category $\mathcal{C}$ such that $K_0(\mathcal{C})$ has the structure of a $\mathbb{Z}[v,v^{-1}]$-module, let $V^{\mathrm{fp}} \subset K_0(\mathcal{C})$ be the submodule spanned by all objects of finite projective dimension. Further, let $V^{\mathrm{fp}}_{p(v)}$ denote the localization of this module at $p(v)$, or equivalently at all of the linear factors $(1 - v^{2i})$ for $1 \leq i \leq \ell(w_0)$.

3. Monodromic sheaves, character sheaves, and the categorical center

3.0.1. *Rank one character sheaves on T.* We let $\mathrm{Ch}(T)$ be the category of rank one character sheaves on T; we refer to Appendix A of [Yun17] for a detailed treatment. We note that $\mathrm{Ch}(T)$ carries a natural action of the Weyl group W.

For any $\mathcal{L}$ in $\mathrm{Ch}(T)$, we let $W^\circ_{\mathcal{L}}$ be the normal subgroup of the stabilizer of $\mathcal{L}$ in W which is the Weyl group of the root subsystem of the root system of W on which $\mathcal{L}$ is trivial; see 2.4 of [LY20] for details.

3.0.2. *Monodromic version of the Kazhdan-Laumon category.* In Sections 2.1.3 and 2.3.4 of [BP98], the authors explain how to define a category $\mathrm{Perv}_{\mathcal{L}}(G/U)$ of monodromic sheaves on G/U with respect to the monodromy $\mathcal{L}$. We then let $D^b_{\mathcal{L}}(G/U) = D^b(\mathrm{Perv}_{\mathcal{L}}(G/U))$ be its derived category.

Remark 3.1. We note that in this definition, monodromic perverse sheaves are defined such that the extension of two $\mathcal{L}$-equivariant sheaves may not be $\mathcal{L}$-equivariant, only $\mathcal{L}$-monodromic. In other words, for $\mathcal{L}$ trivial, this reduces to the category of perverse sheaves on G/U with unipotent monodromy (c.f. [BY13]) on the right rather than simply to $\mathrm{Perv}(G/B)$.

We use this to define the $\mathcal{L}$-monodromic Kazhdan-Laumon category.

Definition 3.2. For any $\mathcal{L} \in \mathrm{Ch}(T)$, we define $\mathcal{A}^{\mathcal{L}}$ to be the category obtained by applying the gluing procedure in Definition 2.6 to the category $\mathrm{Perv}_{\mathcal{L}}(G/U)$. We further define $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$ in analogy to Definition 2.8.

3.1. **The monodromic Hecke category and its center.**

3.1.1. *The monodromic Hecke category.* In [Gou21], the author defines for any $\mathcal{L}$ a category $\mathcal{P}_{\mathcal{L}}$ (called $D^b_{(B)}(G/U)_t$ in loc. cit., where t is a parameter determined by $\mathcal{L}$) such that for $\mathcal{L}$ trivial, this reduces to the familiar derived category $D^b_{(B)}(G/U)$ of B-constructible sheaves on G/U. Elements of $\mathcal{P}_{\mathcal{L}}$ are $\mathcal{L}$-monodromic with respect to the right action of T, while their left monodromies may correspond to any character sheaf in the W-orbit of $\mathcal{L}$.

For any $w \in W$, this category contains monodromic versions ${}_{w\mathcal{L}}\Delta(w)_{\mathcal{L}}$ and ${}_{w\mathcal{L}}\nabla(w)_{\mathcal{L}}$ of standard and costandard sheaves; we emphasize that extensions of such objects in this category may not be $\mathcal{L}$-equivariant with respect to the right T-action even when they remain $\mathcal{L}$-monodromic, so this category also contains monodromic versions of tilting objects $\mathcal{T}(w)_{\mathcal{L}}$ for any $w \in W$.

We let ${}_{\mathcal{L}}\mathcal{P}_{\mathcal{L}}$ be the triangulated subcategory of objects generated by the standard and costandard objects corresponding to $w \in W^{\circ}_{\mathcal{L}}$, in other words, the subcategory of objects in $\mathcal{P}_{\mathcal{L}}$ which are also left-monodromic with respect to $\mathcal{L}$.

3.1.2. *Free-monodromic Hecke categories.* In [BY13], in the unipotent monodromy case where $\mathcal{L}$ is trivial, the authors define a category formed from a certain completion of $D^b_{\mathcal{L}}(G/U)$ called the category of *unipotently free-monodromic* sheaves.

In [Gou21], the case of non-unipotent monodromy was treated carefully. In loc. cit., the author defines a category $\hat{\mathcal{P}}_{\mathcal{L}}$ (which is called $\hat{D}^b_{(B)}(G/U)_t$ in loc. cit. where t is a parameter determined by $\mathcal{L}$), which is a certain completion of the category $\mathcal{P}_{\mathcal{L}}$ defined in Section 3.1.1. It inherits a monoidal structure by extension of the convolution product on $\mathcal{P}_{\mathcal{L}}$.

This category contains elements $\varepsilon_{n,\mathcal{L}}$ and $\hat{\delta}_{\mathcal{L}}$ introduced in Corollary 5.3.3 of [BT22]. The object $\hat{\delta}_{\mathcal{L}}$ is the monoidal unit for the convolution product on $\hat{\mathcal{P}}_{\mathcal{L}}$, whereas convolution with the objects $\varepsilon_{n,\mathcal{L}}$ can be thought of as a sort of projection to the subcategory of objects whose corresponding "logarithmic monodromy operator" is nilpotent of order at most n; c.f. Appendix A of [BY13] for an explanation of this perspective.

Finally, we recall that $\hat{\mathcal{P}}_{\mathcal{L}}$ contains for all $w \in W$ free-monodromic versions ${}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}}$, ${}_{w\mathcal{L}}\hat{\nabla}(w)_{w\mathcal{L}}$ of standard and costandard sheaves, c.f. [BY13] for the unipotent case, [LY20] for a description of these objects in $\mathcal{P}_{\mathcal{L}}$ for arbitrary $\mathcal{L}$, and [Gou21] for their free-monodromic versions in $\hat{\mathcal{P}}_{\mathcal{L}}$. We define the subcategory ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ analogously to the subcategory ${}_{\mathcal{L}}\mathcal{P}_{\mathcal{L}}$ of $\mathcal{P}_{\mathcal{L}}$.

3.1.3. *The center of the monodromic Hecke category.* To define the notion of categorical center which will be useful in this setting, we will closely follow the conventions of [BITV23] in this section. We begin by recalling some definitions from loc. cit.

Definition 3.3 ([BITV23])**.** Let $\mathcal{Y} = (G/U \times G/U)/T$ where T acts by the right-diagonal multiplication. Let $\mathcal{Y}^{(2)} = (G/U)^4/T^2$ where the right T^2-action on $(G/U)^4$ is defined by

$$(x_1U, x_2U, x_3U, x_4U) \cdot (t, z) = (x_1tU, x_2zU, x_3zU, x_4tU), \tag{13}$$

and is equipped with a G^2 action by

$$(g,h)\cdot(x_1U,x_2U,x_3U,x_4U)=(gx_1U,gx_2U,hx_3U,hx_4U). \tag{14}$$

Let $\mathcal{H}^{(1)}=D^b(G\backslash\mathcal{Y})$ and $\mathcal{H}^{(2)}=D^b(G^2\backslash\mathcal{Y}^{(2)})$. It is shown in [BITV23] that $\mathcal{H}^{(2)}$ is equipped with a convolution product which makes it a monoidal category, and that $\mathcal{H}^{(1)}$ is a module category over $\mathcal{H}^{(2)}$ via the "two-sided convolution" described in 2.2 of loc. cit.

Definition 3.4 ([BITV23], Definition 5.2.1)**.** Let

$$\mathcal{ZH}^{(1)}_{\mathcal{L}}=\mathrm{Fun}^{\mathrm{fd}}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}},\mathcal{H}^{(1)}_{\mathcal{L}}),$$

where $\mathrm{Fun}^{\mathrm{fd}}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}},\mathcal{H}^{(1)}_{\mathcal{L}})$ is the full subcategory of $\mathrm{Fun}_{\mathcal{H}^{(2)}_{\mathcal{L}}}(\mathcal{H}^{(1)}_{\mathcal{L}},\mathcal{H}^{(1)}_{\mathcal{L}})$ consisting of functors F such that the limit $F(\hat{\delta}_{\mathcal{L}})$ exists in $\mathcal{H}^{(1)}_{\mathcal{L}}$.

Going forward, if $Z\in\mathcal{H}^{(1)}_{\mathcal{L}}$, we will sometimes write $-*Z$ to denote the functor Z, since the functor Z is equivalent to the convolution with the object $Z(\hat{\delta}_{\mathcal{L}})$.

3.1.4. *Harish-Chandra functor and equivalence from center to character sheaves.* Following 3.2 of [BITV23], consider the diagram

$$\begin{tikzcd} & G\times G/B \arrow[dl, "\pi"'] \arrow[dr, "q"] & \\ G & & \mathcal{Y} \end{tikzcd}$$

where π is the projection and q is the quotient of the map $q':G\times G/U\to G/U\times G/U$ given by $q'(g,xU)=(xU,gxU)$ by the free right T-action, with respect to which q' is equivariant.

Definition 3.5. The Harish-Chandra transform is the functor

$$\mathfrak{hc}=q_!\circ\pi^*:D^b(G/_{\mathrm{Ad}}G)\to D^b(G\backslash\mathcal{Y}), \tag{15}$$

and is monoidal with respect to the natural convolution product on each side by [Gin89], c.f. 2.2 of [BITV23] for a detailed exposition of these convolution products.

Definition 3.6. For any $\mathcal{L}$, let $D^b_{\mathfrak{C},\mathcal{L}}(G)\subset D^b(G/_{\mathrm{Ad}}G)$ be the full triangulated subcategory with objects $\mathcal{F}$ satisfying $\mathfrak{hc}(\mathcal{F})\in\mathcal{H}^{(1)}_{\mathcal{L}}$.

Proposition 3.7 ([BITV23], Theorem 5.2.2)**.** *There is an equivalence $\tilde{a}_{\mathcal{L}}$ of semigroupal categories*

$$\tilde{a}_{\mathcal{L}}:D^b_{\mathfrak{C},\mathcal{L}}(G)\to\mathcal{ZH}^{(1)}_{\mathcal{L}} \tag{16}$$

such that $\varepsilon\circ\tilde{a}_{\mathcal{L}}=\mathfrak{hc}$ where ε is the evaluation of a functor at $\hat{\delta}$.

3.1.5. *Two-sided cells and character sheaves.* In this and subsequent sections, we use the notion of two-sided Kazhdan-Lusztig cells in the Weyl group; see e.g. [Wil03] for a clear exposition.

Definition 3.8. For any $\mathcal{L}$, let $\underline{C}_{\mathcal{L}}$ denote the set of two-sided cells in $W^\circ_{\mathcal{L}}$. For any $w \in W^\circ_{\mathcal{L}}$, we let $\underline{c}_w$ denote the corresponding cell. We let $\underline{c}_e$ be the top cell which corresponds to the identity element. There is a well-defined partial order $\leq$ on $\underline{C}_{\mathcal{L}}$ for which $\underline{c}_e$ is maximal.

Two-sided Kazhdan-Lusztig cells give a filtration on the category $D^b_{\mathfrak{C},\mathcal{L}}(G)$ (c.f. [Lus86]), and so for each $\underline{c} \in \underline{C}_{\mathcal{L}}$, there are triangulated subcategories $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\leq \underline{c}}$ and $D^b_{\mathfrak{C},\mathcal{L}}(G)_{< \underline{c}}$ of $D^b_{\mathfrak{C},\mathcal{L}}(G)$. We then define $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$ as the quotient category $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\leq \underline{c}}/D^b_{\mathfrak{C},\mathcal{L}}(G)_{< \underline{c}}$, referring to the unipotent case treated in Section 5 of [BFO12] for details. Let G_{ad} be the adjoint quotient of G; the following is a consequence of the classification of irreducible character sheaves in terms of cells given in [Lus86], c.f. Corollary 5.4 of [BFO12].

Proposition 3.9. *For any $\mathcal{L} \in \mathrm{Ch}(T)$,*

$$K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})) \cong \bigoplus_{\underline{c} \in \underline{C}_{\mathcal{L}}} K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}}) \tag{17}$$

as vector spaces. Further, for any $\underline{c} \in \underline{C}_{\mathcal{L}}$, the preimage of the subspace

$$\bigoplus_{\underline{c}' \leq \underline{c}} K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}'}) \tag{18}$$

in $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ under this isomorphism is a monoidal ideal.

3.1.6. *The big free-monodromic tilting object and $\mathbb{K}_{\mathcal{L}}$.* In Section 9.4 of [Gou21], the author defines free-monodromic tilting sheaves with general monodromy, analogous to the ones appearing in [BY13] for the case of unipotent monodromy.

Definition 3.10. Given $\mathcal{L} \in \mathrm{Ch}(T)$, let $\hat{\mathcal{T}}(w_{0,\mathcal{L}})_{\mathcal{L}}$ be the free-monodromic tilting object in ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ corresponding to the longest element $w_{0,\mathcal{L}}$ of $W^\circ_{\mathcal{L}}$. In this paper, we will denote it simply by $\hat{\mathcal{T}}_{\mathcal{L}}$ for convenience.

In [BT22], working in the case of unipotent monodromy, the authors construct from $\hat{\mathcal{T}}(w_0)$ an object they call $\mathbb{K}$, defined as $\mathbb{K} = p^* p_! \hat{\mathcal{T}}(w_0)$ where $p : U\backslash G/U \to (U\backslash G/U)/T$ where T acts on $U\backslash G/U$ by conjugation.

They also define a character sheaf Ξ whose details are explained in 1.4 of [BT22] obtained by averaging the derived pushforward of the constant sheaf on the regular locus of the unipotent variety of G to obtain an element of $D^b(G/_{\mathrm{Ad}}G)$.

Proposition 3.11 (Theorem 1.4.1, [BT22])**.** *There exists an object $\Xi \in D^b(G/_{\mathrm{Ad}}G)$ such that, if $\Xi * \hat{\delta}$ is the projection of Ξ to the derived category of unipotent character sheaves, then $\mathfrak{hc}(\Xi * \hat{\delta}) = \mathbb{K}$.*

We now define an anologue for arbitrary monodromy generalizing the unipotent case.

Definition 3.12. Let $\mathbb{K}_{\mathcal{L}} = p^* p_! \hat{\mathcal{T}}_{\mathcal{L}}$. Then by the same argument as in the proof of Theorem 1.4.1 of [BT22], $\mathfrak{hc}(\Xi * \hat{\delta}_{\mathcal{L}}) = \mathbb{K}_{\mathcal{L}}$ where $\Xi * \hat{\delta}_{\mathcal{L}}$ is the analogous projection to the derived category of character sheaves with monodromy $\mathcal{L}$. As a result, $\mathbb{K}_{\mathcal{L}}$ can be considered as an element of $\mathcal{Z}\mathcal{H}^{(1)}_{\mathcal{L}}$.

By abuse of notation, in the future when we use the convolution functor $- * \mathbb{K}_{\mathcal{L}}$, we will often identify $\mathbb{K}_{\mathcal{L}}$ with the character sheaf it comes from under the Harish-Chandra transform, e.g. as in the convolution action of character sheaves which we describe in Proposition 4.6.

3.1.7. *Fourier transform and convolution with costandard sheaves.*

Proposition 3.13. *Let $\mathcal{F} \in D^b_{\mathcal{L}}(G/U)$ where $\mathcal{L} \in Ch(T_k)$. Then for any $s \in S$,*

$$\Phi_s(\mathcal{F}) = \begin{cases} \mathcal{F} *_{\mathcal{L}} \hat{\nabla}(s)_{\mathcal{L}}(\frac{1}{2}) & s \in W^\circ_{\mathcal{L}} \\ \mathcal{F} *_{\mathcal{L}} \hat{\nabla}(s)_{s\mathcal{L}} & s \notin W^\circ_{\mathcal{L}}. \end{cases} \tag{19}$$

Proof. In the case where $s \in W^\circ_{\mathcal{L}}$, this follows for equivariant sheaves by Proposition 4.3 of [MF22], c.f. 6.3 of [Pol01], while the second case follows from a similar computation. The proof then generalizes to arbitrary extensions of $\mathcal{L}$-equivariant sheaves, and therefore to all monodromic sheaves as in the claim. □

4. Coalgebras and dg enhancements

4.1. Coalgebras over comonads and Barr-Beck for Kazhdan-Laumon categories.

4.1.1. *The Kazhdan-Laumon category as coalgebras over a comonad.* We now recall a result from [BBP02] exhibiting the Kazhdan-Laumon category $\mathcal{A}$ as the category of coalgebras over a certain comonad on the underlying category $\mathcal{B} = \mathrm{Perv}(G/U)^{\oplus W}$.

Definition 4.1. Let $\Psi : \mathcal{B} \to \mathcal{B}$ be the endofunctor defined for any $A = (A_w)_{w \in W} \in \mathcal{B}$ by

$$(\Psi^\circ A)_w = \oplus_{y \in W} \Psi^\circ_{wy^{-1}} A_y. \tag{20}$$

It has right-derived functor $\Psi = R\Psi$ given by

$$(\Psi A)_w = \oplus_{y \in W} \Psi_{wy^{-1}} A_y. \tag{21}$$

Theorem 4.2 ([BBP02])**.** *The Kazhdan-Laumon category $\mathcal{A}$ is equivalent to the category of coalgebras over the comonad $\Psi : \mathcal{B} \to \mathcal{B}$.*

4.1.2. *dg enhancements of derived categories.* Given an abelian category $\mathcal{C}$, let $C_{\mathrm{dg}}(\mathcal{C})$ be the dg category with objects complexes of sheaves and morphisms the usual complexes of maps between complexes. We define the dg derived category $D_{\mathrm{dg}}(\mathcal{C})$ to be the dg quotient of $C_{\mathrm{dg}}(\mathcal{C})$ by the full subcategory of acyclic objects; its homotopy category is the usual derived category $D(\mathcal{C})$.

The bounded dg derived category $D^b_{\mathrm{dg}}(\mathcal{C})$ is defined to be the full dg subcategory of $D_{\mathrm{dg}}(\mathcal{C})$ consisting of objects projecting to $D^b(\mathcal{C})$ when passing to the homotopy category.

4.1.3. *Barr-Beck-Lurie for dg categories applied to the Kazhdan-Laumon category.* Note that the comonad $\Psi^\circ : \mathcal{B} \to \mathcal{B}$ can be enhanced to a comonad $\Psi = R\Psi^\circ : D^b_{\mathrm{dg}}(\mathcal{B}) \to D^b_{\mathrm{dg}}(\mathcal{B})$; i.e. the right-derived functor of Ψ° has a dg enhancement (since it arises from the functors Ψ_w which are themselves defined using the six-functor formalism). We can then consider the dg category $D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ of coalgebras over the comonad Ψ. The following is an application of the Barr-Beck-Lurie theorem for dg categories.

Proposition 4.3. *The dg categories $D^b_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ})$ and $D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ are equivalent.*

Proof. We first check the conditions of the Barr-Beck-Lurie theorem for the functor $F : D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D_{\mathrm{dg}}(\mathcal{B})$ obtained from the forgetful functor $\mathcal{B}_{\Psi^\circ} \to \mathcal{B}$. This functor has a right adjoint given by the dg enhancement of the right-derived functor of the cofree coalgebra functor $\mathcal{B} \to \mathcal{B}_{\Psi^\circ}$.

Indeed, we check this adjunction explicitly: this cofree coalgebra functor sends an element $(B_w)_{w\in W} \in D^b_{\mathrm{dg}}(\mathcal{B})$ to the direct sum $\oplus_{w\in W} Rj^\circ_{w*}(A_w)$ where Rj°_{w*} is the right-adjoint to j^*_w. It is then clear from the adjunction (j^*_w, Rj°_{w*}) that the forgetful functor is its left adjoint. Now notice that in a similar way, we observe that the forgetful functor F has a left adjoint: it follows from the opposite adjunction $(j_{w!}, j^*_w)$ that the functor sending any $(B_w)_{w\in W} \in D^b_{\mathrm{dg}}(\mathcal{B})$ to $\oplus_{w\in W} j_{w!}(B_w)$ (where again $j_{w!} = Lj^\circ_{w!}$) is left adjoint to F. It then follows that F preserves all limits which exists in $D^b_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ})$, one can then show directly that F actually creates all limits.

We are now in the setup of Example 2.6 of [Gun17] (but dualizing to get coalgebras over a comonad rather than algebras over a monad as in loc. cit.), where F and its right adjoint both preserve small limits since each is itself a right adjoint. Since $D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ})$ and $D_{\mathrm{dg}}(\mathcal{B})$ are compactly-generated, the result of loc. cit. gives that the hypotheses of the Barr-Beck-Lurie theorem will be satisfies as long as F is conservative. It then only remains to check that F is conservative, but this is clear by its exactness and the fact that the forgetful functor $\mathcal{B}_{\Psi^\circ} \to \mathcal{B}$ on the abelian level is faithful.

To conclude, we then note that the equivalence $D_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D_{\mathrm{dg}}(\mathcal{B})_\Psi$ provided by the Barr-Beck-Lurie theorem then restricts to an equivalence of bounded derived categories $D^b_{\mathrm{dg}}(\mathcal{B}_{\Psi^\circ}) \to D^b_{\mathrm{dg}}(\mathcal{B})_\Psi$ since cohomology in both categories is computed in the underlying category $D(\mathcal{B})$ after forgetting the coalgebra structure. □

4.2. **The center of the monodromic Hecke category.**

4.2.1. *dg-enhancement for the monodromic Hecke category.* In 5.3 of [BITV23], the authors explain how to construct a dg enhancement of $\mathcal{H}^{(1)}_{\mathrm{mon}}$, and a similar construction works for the case of the category $\mathcal{H}^{(1)}_{\mathcal{L}}$ having non-unipotent monodromy; this will be treated in a future version of loc. cit. Going forward, when we consider the convolution action of $\mathcal{H}^{(1)}_{\mathrm{mon}}$ on $D^b(\mathrm{Perv}_{\mathcal{L}}(G/U))$, we will use the fact that this functor has a dg lift and therefore can be considered as a dg functor which preserves distinguished triangles in any of its arguments.

4.2.2. *Action of the center on the Kazhdan-Laumon dg-category.* For any $\mathcal{L} \in \mathrm{Ch}(T)$, let $\mathcal{B}_{\mathcal{L}} = \oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$.

Definition 4.4. Given $Z \in \mathcal{ZH}_{\mathcal{L}}^{(1)}$, we consider Z as a functor on the direct sum of categories $\oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$ as follows. Given any object $A = (A_w)$ in $\oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$, let $Z(A) = A * Z \in \oplus_{w\in W}\mathrm{Perv}_{w\mathcal{L}}(G/U)$ be defined such that $Z(A)_w = (A * Z)_w = A_w * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}}$.

By the definition of $\mathcal{ZH}_{\mathcal{L}}^{(1)}$, for any $B \in {}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$, the functors $- * Z * B$ and $- * B * Z$ on $\mathrm{Perv}_{\mathcal{L}}(G/U)$ are naturally isomorphic. The following lemma is a consequence of this fact.

Lemma 4.5. *For any $Z \in \mathcal{ZH}_{\mathcal{L}}^{(1)}$, there is a natural isomorphism $Z \circ \Psi \cong \Psi \circ Z$ of endofunctors of $\mathcal{A}^{\mathcal{L}}$.*

Proof. By Proposition 3.13, for any $w \in W$, there exists some Tate twist (d) such that $((Z \circ \Psi)(A))_w(d)$ is isomorphic to

$$\oplus_{y\in W}\ (\Psi_{wy^{-1}}A_y) * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{22}$$
$$\cong A_y * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{23}$$
$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{24}$$
$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} * {}_{w\mathcal{L}}\hat{\Delta}(w)_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(w^{-1})_{w\mathcal{L}} \tag{25}$$
$$\cong A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * Z * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(yw^{-1})_{w\mathcal{L}} \tag{26}$$
$$\cong \oplus_{y\in W}\Psi_{wy^{-1}}(Z(A))(d) \tag{27}$$
$$\cong ((\Psi \circ Z)(A))_w(d), \tag{28}$$

and these isomorphisms across all $w \in W$ assemble together to give a natural isomorphism $Z \circ \Psi \cong \Psi \circ Z$. A conceptual explanation for the existence of this isomorphism is explained in Lemma 11.12 of [LY20]. ☐

In the following, let $\pi_{\mathrm{ad}} : G \to G_{\mathrm{ad}}$ be the natural map from G to its adjoint quotient.

Proposition 4.6. *There is an action of the monoidal category $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on the category $D^b(\mathcal{A}^{\mathcal{L}})$, which we denote by convolution on the right.*

Proof. Given an element $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, we first pull back along π_{ad} to obtain an element in $D^b_{\mathfrak{C},\mathcal{L}}(G)$, then pass to $\mathcal{ZH}_{\mathcal{L}}^{(1)}$ via the isomorphism $\tilde{a}_{\mathcal{L}}$, obtaining a new element $\tilde{Z}$. We can then act by the usual convolution $-*\tilde{Z}$. By abuse of notation, for any element $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ we will define the endofunctor of $\mathrm{Perv}_{\mathcal{L}}(G/U)$ which we just described by the same symbol $- * Z$. We note that $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ has a dg enhancement discussed for the unipotent case in [BZN15] but analogous in general, and by Proposition 4.3 and Lemma 4.5. Further, the functors we use are built from compositions of those occuring in the six-functor formalism and therefore each has a dg enhancement as well.

For any $B \in D^b_{\mathrm{dg}}(\mathcal{B}_{\mathcal{L}})_\Psi$ and $Z \in D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, we define the new object $B * Z$ by $F(B) * Z$ as a tuple of elements of $\mathrm{Perv}_{\mathcal{L}}(G/U)$ where F is the forgetful functor. We then define the coalgebra structure map $B * Z \to \Psi \circ (B * Z)$ by the composition

$$B * Z \to \Psi(B) * Z \cong \Psi(B * Z) = R\Phi(Z(B)), \tag{29}$$

where the first map comes from the coalgebra structure on B and the subsequent isomorphism is the natural isomorphism described in Lemma 4.5. One can then check that this defines a coalgebra structure on $Z(B)$. □

We note that we get a similar action of $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on $D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ since any object Z here is a sheaf on the variety G_{ad} and therefore has a natural isomorphism $\mathrm{Fr}^*Z \to Z$; this means $- * Z$ commutes with Fr^*, and therefore the action in 4.6 also gives an action in the setting of w-twisted Weil sheaves.

5. Polishchuk's canonical complex

5.1. **Parabolic Kazhdan-Laumon categories.**

5.1.1. *Definition of the categories $\mathcal{A}^I_{w,\mathbb{F}_q}$.* We will now define certain "parabolic" analogues of the Kazhdan-Laumon category which correspond to subsets $I \subset S$. We first recall the definition of the categories $\mathcal{A}_{W_I}$ from [Pol01], build off of this definition to define the categories $\mathcal{A}^I$ which we will need in the present work, and finally define $\mathcal{A}^I_{w,\mathbb{F}_q}$ for any $w \in W$.

Definition 5.1 ([Pol01], Section 7)**.** For any $I \subset S$, the category $\mathcal{A}_{W_I}$ is the category of tuples of elements of $\mathrm{Perv}(G/U)$ indexed by W_I with morphisms and compatibilities as in Definition 2.6 but only for $w \in W_I$, $s \in I$.

For any $J \subset I \subset S$, and any right coset W_Jx of W_J in W_I, there is a restriction functor $j^{W_I *}_{W_Jx} : \mathcal{A}_{W_I} \to \mathcal{A}_{W_J}$ remembering only the tuple elements and morphisms for $w \in W_Jx$, $s \in J$. When $I = S$, we write only $j^*_{W_Jx}$, and in this case we omit superscripts similarly for all functors introduced in this section.

Proposition 5.2 ([Pol01], Proposition 7.1.1)**.** *For any $J \subset I \subset S$, and any W_Jx, the functor $j^{W_I *}_{W_Jx}$ admits a left adjoint $j^{W_I}_{W_Jx!}$.*

Definition 5.3. For $I \subset S$, let $\mathcal{A}^I$ be the category whose objects are tuples $(A_w)_{w\in W}$ equipped with morphisms $\Phi^\circ_s A_w \to A_{sw}$ for any $w \in W$, $s \in I$ satisfying the same conditions as in Definition 2.6 but only for $s \in I$. Morphisms in $\mathcal{A}^I$ are morphisms in $\mathrm{Perv}(G/U)^{\oplus W}$ satisfying the compatibilities in Definition 2.6 but only for $s \in I$.

Alternatively, $\mathcal{A}^I = \oplus_{W_I\backslash W}\mathcal{A}_{W_I}$ with a reindexing by W on the tuples in this category.

Just like $\mathcal{A}$, the category $\mathcal{A}^I$ admits an action of W by functors $\{\mathcal{F}_w\}_{w\in W}$ defined by $\mathcal{F}_w((A_y)_{y\in W}) = (A_{yw})_{w\in W}$. We define the category $\mathcal{A}^I_{w,\mathbb{F}_q}$ as in Definition Definition 2.8 but with $\mathcal{A}$ replaced by $\mathcal{A}^I$.

5.1.2. *Adjunctions between these categories.* For any $J \subset I$, there is an obvious restriction functor $j^*_{I,J} : \mathcal{A}^I \to \mathcal{A}^J$ which is the identity on objects but which remembers only the morphisms $\Phi^\circ_s A_w \to A_{sw}$ for $s \in J$. As in Proposition 2.11, there is an analogous derived version of this morphism and the following adjunction.

Proposition 5.4. *For any $J \subset I$, the functor $j^*_{I,J}$ admits a left adjoint $j_{I,J!}$. For any $A \in \mathcal{A}^J$, $j^\circ_{I,J!}(A)$ is the direct sum*

$$\bigoplus_{x \in W_J \backslash W_I} j^{W_I \circ}_{W_J x!}((A_w)_{w \in W_J x}), \tag{30}$$

where $(A_w)_{w \in W_J x}$ is considered as an element of $\mathcal{A}_{W_J}$. Further, the adjoint pair $(j^\circ_{I,J!}, j^*_{I,J})$ gives also an adjunction between $\mathcal{A}^I_{w,\mathbb{F}_q}$ and $\mathcal{A}^J_{w,\mathbb{F}_q}$.

Proof. The direct sum in (30) has the natural structure of an object of $\mathcal{A}^I$, as each object $j^{W_I \circ}_{W_J x!}(A_w)_{w \in W_J x}$ has such a structure by Proposition 5.2. We then note that for any such $A \in \mathcal{A}^J$ and any $B \in \mathcal{A}^I$,

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{A}^I}\left(\bigoplus_{x \in W_J \backslash W_I} j^{I\circ}_{W_J x!}((A_w)_{w \in W_J x}), B\right) \\
&\cong \bigoplus_{x \in W_J \backslash W_I} \operatorname{Hom}_{\mathcal{A}^I}(j^\circ_{W_J x!}((A_w)_{w \in W_J x}), B) \\
&\cong \bigoplus_{x \in W_J \backslash W_I} \operatorname{Hom}_{\mathcal{A}_{W_J x}}((A_w)_{w \in W_J x}, j^*_{W_J x} B) \\
&\cong \operatorname{Hom}_{\mathcal{A}^J}((A_w)_{w \in W}, \oplus_{x \in W_J \backslash W_I} j^*_{W_J x} B) \\
&= \operatorname{Hom}_{\mathcal{A}^J}((A_w)_{w \in W}, j^*_{I,J} B).
\end{aligned}$$

To see that the adjoint pair $(j^\circ_{I,J!}, j^*_{I,J})$ gives also an adjunction between $\mathcal{A}^I_{w,\mathbb{F}_q}$ and $\mathcal{A}^J_{w,\mathbb{F}_q}$, we note that the morphisms on both sides of the equation above which are compatible with the morphism $\psi_A : \mathcal{F}_w \mathrm{Fr}^* A \to A$ are preserved by these isomorphisms. $\square$

5.2. **Polishchuk's complex for $\mathcal{A}_{w,\mathbb{F}_q}$.**

5.2.1. *Polishchuk's canonical complex in* [Pol01]. For a fixed $A \in \mathcal{A}$ and a choice of $J \subset S$, Polishchuk writes $A(J) = j_{S-J!} j^*_{S-J} A$. In 7.1 of [Pol01], Polishchuk explains that for any $A \in \mathcal{A}$, adjunction of parabolic pushforward and pullback functors $(j_{W_I x!}, j^*_{W_I x})$ gives a canonical morphism $A(J) \to A(J')$ whenever $J \subset J'$. He then defines the complex $C_\bullet(A)$ as

$$C_{n-1} = A(S) \longrightarrow \ldots \longrightarrow C_1 = \bigoplus\nolimits_{|J|=2} A(J) \longrightarrow C_0 = \bigoplus\nolimits_{|J|=1} A(J),$$

where $n = |S|$.

Recalling the morphism $\iota : \mathcal{A} \to \mathcal{A}$ sending $(A_w)_{w\in W}$ to $(\Phi^\circ_{w_0} A_{w_0 w})_{w\in W}$, he describes natural morphisms $C_0 \to A$ and $\iota(A) \to C_{n-1}$, and shows that

$$H_i(C_\bullet(A)) = \begin{cases} A & i = 0, \\ 0 & i \neq 0, n-1, \\ \iota(A) & i = n-1. \end{cases} \tag{31}$$

He then describes a $2n$-term complex $\tilde{C}_\bullet$ formed from attaching $C_\bullet(\iota A)$ to $C_\bullet(A)$ via the maps $C_0(\iota A) \to \iota A \to C_{n-1}(A)$ with the property that

$$H_i(\tilde{C}_\bullet(A)) = \begin{cases} A & i = 0, \\ 0 & i \neq 0, 2n-1, \\ \iota^2(A) & i = 2n-1. \end{cases} \tag{32}$$

5.2.2. *Compatibility of the complex with w-twisted structure.*

Proposition 5.5. *If $A \in \mathcal{A}_{w,\mathbb{F}_q}$, then the complex $C_\bullet(A)$ is compatible with the w-twisted Weil structures, i.e. is a complex of objects in $\mathcal{A}_{w,\mathbb{F}_q}$.*

Proof. For any k, components of the map $C_k(A) \to C_{k-1}(A)$ are each maps of the form

$$j_{S-J!} j^*_{S-J} A \to j_{S-J'!} j^*_{S-J'} A \tag{33}$$

where $J' \subset J \subset S$ are such that $|J| = k$, $|J'| = k-1$, which we now describe. By adjunction the data of such a map is equivalent to a map

$$j^*_{S-J} A \to j^*_{S-J} j_{S-J'!} j^*_{S-J'} A, \tag{34}$$

and compatibility with the w-twisted structure is preserved under this adjunction. By the definition of $j_{S-J'!}$, we have that

$$j^*_{S-J} j_{S-J'!} j^*_{S-J'} A \cong \bigoplus_{x \in W_{S-J}\backslash W} j^*_{S-J} j_{W_{S-J}x!} j^*_{S-J'} A, \tag{35}$$

and one can check that the map in (33) appearing in the definition of Polishchuk's complex in [Pol01] comes in (34) from a natural injection in $\mathcal{A}^{S-J}$ from $j^*_{S-J}A$ into this direct sum defined by sending the yth tuple entry to the yth tuple entry in the direct summand corresponding to the unique x for which $y \in W_{S-J}x$. It is straightforward to check that this injection preserves w-twisted Weil structures on both sides coming from the w-twisted Weil structure on A, and therefore the map in (33) is a map in $\mathcal{A}_{w,\mathbb{F}_q}$. $\square$

5.2.3. *Parabolic canonical complexes.* We remark how that the content appearing in 5.2.1 can be generalized to provide complexes in $\mathcal{A}^I_{w,\mathbb{F}_q}$ for any $I \subset S$ with $|I| = k$ and any $w \in W$. Namely, if we fix $I \subset S$, $w \in W$, and $A \in \mathcal{A}^I_{w,\mathbb{F}_q}$, we let $A^I(J) = j^I_{S-J!} j^{I*}_{S-J} A$ whenever $J \supset S - I$, and we define the complex $C^I_\bullet(A)$ by

$$A^I(S) \longrightarrow \dots \longrightarrow \bigoplus\nolimits_{|J|=n-k+2} A^I(J) \longrightarrow \bigoplus\nolimits_{|J|=n-k+1} A^I(J),$$

indexed such that C_{k-1} is the first term and C_0 is the last term in the above. The results in Section 5.2.1 then still hold, giving a version of the canonical complex associated to an object $A \in \mathcal{A}^I_{w,\mathbb{F}_q}$.

5.2.4. *Equations in the Grothendieck group.* The following is a consequence of the fact that the full twist is central in the braid group, along with the fact that the symplectic Fourier transforms Φ_w form a braid action.

Lemma 5.6. *For any $J \subset I \subset S$ there is a natural isomorphism*

$$Lj^I_{J!} \circ \iota^2 \cong \iota^2 \circ Lj^I_{J!} \tag{36}$$

of functors from $D^b(\mathcal{A}^J)$ to $D^b(\mathcal{A}^I)$.

Recall that $n = |S|$.

Theorem 5.7. *For any $w \in W$ and $A \in \mathcal{A}_{w,\mathbb{F}_q}$ the element*

$$(\iota^2 - 1)^n [A] \in K_0(\mathcal{A}_{w,\mathbb{F}_q}) \tag{37}$$

lies in V^{fp}.

Proof. The derived category version of the canonical complex construction in combination with the observation about its homology provided in 5.2.1, we get that for any $A \in \mathcal{A}_{w,\mathbb{F}_q}$, we have the following equation in $K_0(\mathcal{A}_{w,\mathbb{F}_q})$:

$$[\iota A] + (-1)^{|I|-1}[A] = \sum_{\substack{I' \\ I \subset I' \subsetneq S}} (-1)^{|I'|} [j^I_{J!} j^{I*}_J A], \tag{38}$$

c.f. the proof of Theorem 11.5.1 in [Pol01] where the analogous equation is used in the case where $w = e$, $I = S$. By the "doubled" canonical complex $\tilde{C}_\bullet(A)$ and the description of its homology in 5.2.1, this means that $[\iota^2 A] - [A]$ is a linear combination of elements lying in the image of the functors $j^I_{J!} j^{I*}_J$.

Now by induction on the $|I|$ appearing in the equation above, it follows from Lemma 5.6 that $(\iota^2 - 1)^n [A]$ is a linear combination of elements lying in the image of the functors $j_{\varnothing !} j^*_{\varnothing}$. We have that for any $B \in \mathcal{A}^{\varnothing}_{w,\mathbb{F}_q}$,

$$j_{\varnothing !} j^*_{\varnothing} = \oplus_{y \in W} j_{y!} B_y, \tag{39}$$

and each of these direct summands has finite cohomological dimension by Proposition 2.13, completing the proof of the theorem. ∎

6. Central objects and the full twist

6.1. Cells, the big tilting object, and the full twist.

6.1.1. *The full twist.*

Definition 6.1. Let $\mathcal{L} \in \mathrm{Ch}(T)$. Then we define the element

$$\mathrm{FT}_{\mathcal{L}} = {}_{\mathcal{L}}\hat{\nabla}(w_0)_{w_0\mathcal{L}} * {}_{w_0\mathcal{L}}\hat{\nabla}(w_0)_{\mathcal{L}} \tag{40}$$

of ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$. The object $\mathrm{FT}_{\mathcal{L}}$ admits a central structure and arises comes from a character sheaf in $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ via pullback composed with the Harish-Chandra functor, see [BT22] for an explicit description of this character sheaf in the unipotent case, whose proof can be adapted similarly for arbitrary monodromy. As with $\mathbb{K}_{\mathcal{L}}$, we will identify $\mathrm{FT}_{\mathcal{L}}$ as an element of $\mathcal{Z}\mathcal{H}^{(1)}_{\mathcal{L}}$ with its underlying character sheaf in $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$, and by $- * \mathrm{FT}_{\mathcal{L}}$ we will denote the action as described in Proposition 4.6.

By the definition of $\mathrm{FT}_{\mathcal{L}}$ combined with Proposition 3.13, we obtain the following.

Lemma 6.2. *The functors $- * \mathrm{FT}_{\mathcal{L}}$ and ι^2 on $D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ are naturally isomorphic.*

6.1.2. *Action of the full twist on cells.*

Proposition 6.3. *For any $a \in K_0(D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}})$,*

$$[\mathrm{FT}_{\mathcal{L}}] * a = v^{d_{\mathcal{L}}(\underline{c})} a \tag{41}$$

Proof. We follow the same argument as in Proposition 4.1 and Remark 4.2 of [BFO12]; in other words, we begin with the observation that $[\mathrm{FT}_{\mathcal{L}}]$ acts trivially on the non-graded version of the Grothendieck group $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}})$. Continuing to follow the argument of loc. cit., we then know that for any object A in the heart of $D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$, the object $\mathrm{FT}_{\mathcal{L}} * A \in D^b_{\mathfrak{C},\mathcal{L}}(G)_{\underline{c}}$ is perverse up to shift, and furthermore has the property that $[\mathrm{FT}_{\mathcal{L}} * A] = v^d[A]$ for some d.

To compute the value of d, we can pass back along the Harish-Chandra transform and work in the category $\mathcal{H}^{(1)}_{\mathcal{L}}$. The Grothendieck ring $K_0(\mathcal{H}^{(1)}_{\mathcal{L}})$ is the monodromic Hecke algebra $\mathcal{H}_{\mathcal{L}}$. By [LY20], this is isomorphic to the usual Hecke algebra associated to the group $W^\circ_{\mathcal{L}} \subset W$, with $\mathrm{FT}_{\mathcal{L}}$ being identified with the usual full-twist $\tilde{T}^2_{w_{0,\mathcal{L}}}$. By [Lus84, 5.12.2], the full twist in the usual Hecke algebra acts on the cell subquotient module of the Hecke algebra corresponding to a cell $\underline{c}$ by the scalar $v^{d(\underline{c})}$, where $d(\underline{c})$ is described in loc. cit. Passing this fact back along the monodromic-equivariant isomorphism from [LY20], the result follows. □

Definition 6.4. For any two-sided cell $\underline{c}$, let $d_{\mathcal{L}}(\underline{c})$ be the nonnegative integer between 0 and $2\ell(w_0)$ for which the equation in Proposition 6.3 holds.

6.1.3. $\mathbb{K}_{\mathcal{L}}$ *in the top cell subquotient.*

Definition 6.5. Let $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$ be the submodule of $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))$ spanned by the image of the ideal $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$ under the action described in Proposition 4.6.

Definition 6.6. For any $\mathcal{L} \in \mathrm{Ch}(T)$, let $q_{\mathcal{L}}(v)$ be the Poincaré polynomial

$$q_{\mathcal{L}}(v) = \sum_{w \in W} (-v^2)^{\ell(w)}$$

of the group $W^\circ_{\mathcal{L}}$. Note by the Chevalley-Solomon formula that $q_{\mathcal{L}}(v)$ can be expressed as the product of some linear factors each of which is a factor of $(v^{2i} - 1)$ for some $1 \leq i \leq \ell(w_0)$.[1]

Lemma 6.7. *The multiplicity with grading of the irreducible object* ${}_{\mathcal{L}}\mathrm{IC}(e)_{\mathcal{L}}$ *in the Jordan-Holder decomposition of* $\hat{\mathcal{T}}_{\mathcal{L}}$ *is* $q_{\mathcal{L}}(v)$.

Proof. In [Yun09], Yun computes the $\mathbb{Z}[v, v^{-1}]$-graded multiplicity of any standard object $\Delta(y)$ in a filtration of $T(w)$, where $w, y \in W$ and $\Delta(y)$ and $T(w)$ are standard and tilting objects respectively in the usual Hecke category. The main equivalence of categories in [LY20] allows us to extend these results to the monodromic Hecke category by replacing W with $W^\circ_{\mathcal{L}}$, whose combinatorics in terms of tilting, standard, and irreducible objects matches exactly the combinatorics of the completed category ${}_{\mathcal{L}}\hat{\mathcal{P}}_{\mathcal{L}}$ (c.f. 9.3.3 of [Gou21] for an explicit description of the standard filtration on a tilting object in the monodromic setting).

Combining Theorem 5.3.1 of [Yun09] with the expression of standard objects in terms of irreducible objects via inverse Kazhdan-Lusztig polynomials, we compute that the multiplicity of $\hat{\mathrm{IC}(e)}_{\mathcal{L}}$ is exactly

$$\sum_{w \in W^\circ_{\mathcal{L}}} (-v^2)^{\ell(w_0) - \ell(w)} = \sum_{w \in W^\circ_{\mathcal{L}}} (-v^2)^{\ell(w)} = q_{\mathcal{L}}(v). \tag{42}$$

□

Proposition 6.8. *For any n, the element*

$$[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] - (v^2 - 1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)[\varepsilon_{n,\mathcal{L}}] \tag{43}$$

of $K_0(D^b_{\mathfrak{C}}(G_{\mathrm{ad}}))$ *lies in the subspace* $K_0(D^b_{\mathfrak{C},<\underline{c}_e}(G_{\mathrm{ad}}))$.

Proof. First note that $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}_e})$ is of rank 1 as a $\mathbb{Z}[v, v^{-1}]$-module. This means we have that the classes in this Grothendieck group of the images of $\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}$ and $\varepsilon_{n,\mathcal{L}}$ under the cell quotient map to $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{\underline{c}_e}$ are scalar multiples, so $[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] - q'(v)[\varepsilon_{n,\mathcal{L}}]$ for some $q'(v) \in \mathbb{Z}[v, v^{-1}]$.

By [BT22], $[\mathbb{K}_{\mathcal{L}}] = (v^2 - 1)^{\mathrm{rank}(T)}[\hat{\mathcal{T}}_{\mathcal{L}}]$ in the full Grothendieck group $K_0(\hat{\mathcal{P}}_{\mathcal{L}})$. Note that in the corresponding top cell subquotient module for the monodromic Hecke algebra

[1] The variable u used throughout [Pol01] is replaced in the present work by v^2.

$K_0(\mathcal{H}^{(1)}_{\mathcal{L}})$, the equation $[\hat{\mathcal{T}}_{\mathcal{L}}] - q_{\mathcal{L}}(v)[\hat{\delta}(e)]$ holds by Lemma 6.7. This means that $q'(v) = (v^2-1)^{\mathrm{rank}(T)}q_{\mathcal{L}}(v)$ is the only value for which $[\varepsilon_{n,\mathcal{L}} * \mathbb{K}_{\mathcal{L}}] - q'(v)[\varepsilon_{n,\mathcal{L}}]$ lies in a lower cell submodule of $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$, and therefore $[\mathbb{K}_{\mathcal{L}}] = (v^2-1)^{\mathrm{rank}(T)}q_{\mathcal{L}}(v)[\varepsilon_{n,\mathcal{L}}]$. □

6.1.4. *Convolution with $\mathbb{K}_{\mathcal{L}}$ for Kazhdan-Laumon objects.*

Lemma 6.9. *For any $s \in S$, the map*

$$c_s * \mathbb{K}_{\mathcal{L}} : {}_{\mathcal{L}}\hat{\nabla}(s)_{s\mathcal{L}} * {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} \to \mathbb{K}_{\mathcal{L}} \tag{44}$$

is an isomorphism.

Proof. It is enough to show that the corresponding map $\tilde{c}_s : {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} \to {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}}$ (obtained by convolving c_s with ${}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}}$) is an isomorphism. If $s\mathcal{L} \neq \mathcal{L}$, then by Lemma 3.6 in [LY20], ${}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} \cong {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}}$, and so this becomes immediate. We now consider the case when $s\mathcal{L} = \mathcal{L}$. Note that $\tilde{c}_s = (i' \circ p) * \mathbb{K}_{\mathcal{L}}$ where i' and p are as in the canonical exact sequences

$$0 \longrightarrow \hat{\mathrm{IC}}(s)_{\mathcal{L}} \xrightarrow{i} {}_{s\mathcal{L}}\hat{\nabla}(s)_{\mathcal{L}} \xrightarrow{p} \hat{\mathrm{IC}}(e)_{\mathcal{L}} \longrightarrow 0,$$

$$0 \longrightarrow \hat{\mathrm{IC}}(e)_{\mathcal{L}} \xrightarrow{i'} {}_{s\mathcal{L}}\hat{\Delta}(s)_{\mathcal{L}} \xrightarrow{p'} \hat{\mathrm{IC}}(s)_{\mathcal{L}} \longrightarrow 0.$$

It is then enough to show that $p * \mathbb{K}_{\mathcal{L}}$ and $i' * \mathbb{K}_{\mathcal{L}}$ are each isomorphisms. This follows from the fact that $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} = 0$. Indeed, $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \mathbb{K} = 0$ if and only if $\hat{\mathrm{IC}}(s)_{\mathcal{L}} * \hat{\mathcal{T}}_{\mathcal{L}} = 0$, which follows from the fact that its class in the Grothendieck is zero combined with the fact that $\hat{\mathcal{T}}_{\mathcal{L}}$ is convolution-exact, since it is tilting. □

Corollary 6.10. *For any $A \in \mathcal{A}_{w,\mathbb{F}_q}$, the object $A * \mathbb{K}_{\mathcal{L}}$ of $\mathcal{A}_{w,\mathbb{F}_q}$ has the property that for any $y, z \in W$, the composition*

$$\theta_{z^{-1},zy} \circ (\Phi^{\circ}_{z^{-1}}\theta_{z,y}) : A_y * \mathbb{K}_{\mathcal{L}} \to A_y * \mathbb{K}_{\mathcal{L}} \tag{45}$$

is an isomorphism.

Proof. By Proposition 3.13, the morphism in question can be written, up to Tate twist, as a morphism from $A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}}$ to $A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}$. In particular, it is a Tate twist of the morphism

$$\begin{array}{c}
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\big\downarrow \\
A_y * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\big\downarrow{\scriptstyle \theta_{z,y} * {}_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}}} \\
A_{zy} * {}_{zy\mathcal{L}}\hat{\Delta}(zy)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1}z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\big\downarrow \\
A_{zy} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\big\downarrow{\scriptstyle \theta_{z^{-1},zy} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}} \\
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},
\end{array}$$

where the unlabeled arrows are the isomorphisms given by the central structure on $\mathbb{K}_{\mathcal{L}}$; these extend to similar central morphisms for conjugates of of $\mathbb{K}_{\mathcal{L}}$ by standard/costandard sheaves by the same argument as in Lemma 11.12 of [LY20].

We note that since the second and third morphisms above clearly commute with these central morphisms, the above morphism agrees with the morphism

$$\begin{array}{c}
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \\
\big\downarrow \\
A_y * {}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\big\downarrow{\scriptstyle \theta_{z,y} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}} \\
A_y * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} \\
\big\downarrow{\scriptstyle \theta_{z^{-1},zy} * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}}} \\
A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},
\end{array}$$

where the first morphism is again the isomorphism coming from centrality of $\mathbb{K}_{\mathcal{L}}$ The composition of the last two morphisms in the sequence above must, by the definition of Kazhdan-Laumon categories, be equal to the morphism

$$A_y * c_z * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}},$$

where c_z is the morphism ${}_{y\mathcal{L}}\hat{\nabla}(z^{-1})_{zy\mathcal{L}} * {}_{zy\mathcal{L}}\hat{\nabla}(z)_{y\mathcal{L}} \to \mathrm{Id}$ which is obtained by applying the morphisms c_s successively for every simple reflection s in a reduced expression for z. By the functoriality of the central morphisms discussed above, they also commute with this c_z, and so the entire composition above actually agrees with the morphism

$$A_y * {}_{y\mathcal{L}}\hat{\Delta}(y)_{\mathcal{L}} * \mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * c_z,$$

By our inductive definition of c_z along with the same argument as in Lemma 6.9, $\mathbb{K}_{\mathcal{L}} * {}_{\mathcal{L}}\hat{\nabla}(y^{-1})_{y\mathcal{L}} * c_z$ is an isomorphism, and therefore $\theta_{z^{-1},zy} \circ (\Phi^{\circ}_{z^{-1}}\theta_{z,y})$ must be, too. □

Proposition 6.11. *For any $A \in \mathcal{A}_{w,\mathbb{F}_q}$, $A * \mathbb{K}_{\mathcal{L}}$ has finite projective dimension.*

Proof. We can forget the w-twisted Weil structure on $A * \mathbb{K}$ and consider it as an object in $\mathcal{A}$. In $\mathcal{A}$, the adjunction $(j_{e!}, j_e^*)$ gives a morphism

$$a : j_{e!}j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to A * \mathbb{K}_{\mathcal{L}}. \tag{46}$$

We claim that this is an isomorphism.

It is enough to show that the component morphisms $a_y : j_y^* j_{e!} j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to j_y^*(A * \mathbb{K}_{\mathcal{L}})$ are each isomorphisms. By definition, these are the structure morphisms $\theta_{y,e} : \Phi^{\circ}_y(A * \mathbb{K}_{\mathcal{L}})_e \to (A * \mathbb{K}_{\mathcal{L}})_y$.

By Corollary 6.10, the morphisms

$$\theta_{y^{-1},y} \circ (\Phi^{\circ}_{y^{-1}}\theta_{y,e}) : \Phi^{\circ}_{y-1}\Phi^{\circ}_y A_e \to A_e \tag{47}$$

$$\theta_{y,e} \circ (\Phi^{\circ}_{y^{-1}}\theta_{y^{-1},y}) : \Phi^{\circ}_y\Phi^{\circ}_{y^{-1}} A_y \to A_y \tag{48}$$

are both isomorphisms. This tells us that $\theta_{y,e}$ is both a monomorphism and an epimorphism.

This means $a : j_{e!}j_e^*(A * \mathbb{K}_{\mathcal{L}}) \to A * \mathbb{K}_{\mathcal{L}}$ is an isomorphism as we claimed. The proposition then follows since all objects in the image of $Lj_{e!}$ have finite cohomological dimension. □

6.2. **Completing the proof of Theorem 1.5.**

6.2.1. *Proof of Theorem 1.5 by the action of the full twist on cells.*

Lemma 6.12. *For any $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$, there exists some $r \geq 1$ for which*

$$P^r(\mathrm{FT}_{\mathcal{L}}, v) \cdot a = 0 \tag{49}$$

in $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$, where P^r is the polynomial for which $P^r(x, v) = P(x, v)^r$ for any x.

Further, if $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})_{<\underline{c}_e}$, then

$$\tilde{P}^r(\mathrm{FT}_{\mathcal{L}}, v) \cdot a = 0. \tag{50}$$

Proof. By and Propositions 4.3 and 4.6, the category $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ acts on $D^b_{\mathrm{dg}}(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ in a way which respects distinguished triangles, therefore giving an action of the $\mathbb{Z}[v, v^{-1}]$-algebra $D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})$ on $K_0(\mathcal{A}_{w,\mathbb{F}_q})$. It is then enough to show that there exists r for which $P^r([\mathrm{FT}_{\mathcal{L}}], v) = 0$ in $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ and that $\tilde{P}^r([\mathrm{FT}_{\mathcal{L}}], v) \cdot b = 0$ for any $b \in K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$.

Indeed, since by Proposition 6.3 and Proposition 3.9, the eigenvalues of $[\mathrm{FT}_{\mathcal{L}}] * -$ on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$ are each of the form v^{2i} for $0 \leq i \leq \ell(w_0)$ and of the form v^{2i} for $1 \leq i \leq \ell(w_0)$

on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}})_{<\underline{c}_e})$, we must only choose r to be the maximum multiplicity occuring in the characteristic polynomial of $[\mathrm{FT}_{\mathcal{L}}] * -$ on $K_0(D^b_{\mathfrak{C},\mathcal{L}}(G_{\mathrm{ad}}))$, since each degree 1 term of this characteristic polynomial is a factor of $P(x,v)$ (resp. $\tilde{P}(x,v)$) by definition of the latter. Choosing r in this way, we get that the polynomials in the lemma vanish, as desired. $\square$

Corollary 6.13. *For any $a \in K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$, $a \in V^{\mathrm{fp}}_{p(v)}$.*

Proof. Let $a \in K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})_{<\underline{c}_e}$. By Theorem 5.7, $(\iota^2-1)^n \cdot a \in V^{\mathrm{fp}}$, while by Lemmas 6.2 and 6.12, $\tilde{P}(\iota^2, v)\cdot a = 0$. We claim that this implies that $(\iota^2-1)^{n-k}\cdot a \in V^{\mathrm{fp}}_{p(v)}$ for all $0 \leq k \leq n$. Indeed, suppose for induction that $(\iota^2-1)^{n-k+1}\cdot a \in V^{\mathrm{fp}}_{p(v)}$, and let $a' = (\iota^2-1)^{n-k}\cdot a$. Then $(\iota^2-1)\cdot a' \in V^{\mathrm{fp}}_{p(v)}$ and $\tilde{P}(\iota^2,v)^r \cdot a' = 0$, so by the Euclidean algorithm, $p(v)^r \cdot a' = \tilde{P}(\iota^2, v)\cdot a'$ is in the $\mathbb{Z}[\iota^2, v, v^{-1}]$-span of $(\iota^2-1)\cdot a'$, and therefore lies in $V^{\mathrm{fp}}_{p(v)}$. Dividing by $p(v)^r$ gives $a' \in V^{\mathrm{fp}}_{p(v)}$, and so we can proceed by induction until $k = n$ where we conclude that $a \in V^{\mathrm{fp}}_{p(v)}$. $\square$

Proof of Theorem 1.5. Let $A \in \mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$. By Proposition 6.8,

$$[A * \mathbb{K}_{\mathcal{L}}] - (v^2-1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)[A] \tag{51}$$

is an element of $K_0(D^b(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}))_{<\underline{c}_e}$

By Corollary 6.13, this means $(v^2-1)^{\mathrm{rank}(T)}([A * \mathbb{K}_{\mathcal{L}}] - q_{\mathcal{L}}(v)[A]) \in V^{\mathrm{fp}}_{p(v)}$.

By Proposition 6.11, $A * \mathbb{K}_{\mathcal{L}}$ itself has finite projective dimension, and so in combination with equation (51), this means $(v^2-1)^{\mathrm{rank}(T)} q_{\mathcal{L}}(v)p(v)[A] \in V^{\mathrm{fp}}_{p(v)}$. Note that by Definition 6.6 each degree 1 factor of $q_{\mathcal{L}}(v)$ is also a factor of $p(v)$, so we can divide by $q_{\mathcal{L}}(v)$ in the localization $V^{\mathrm{fp}}_{p(v)}$ to get $[A] \in V^{\mathrm{fp}}_{p(v)}$, completing the proof. $\square$

7. Polishchuk's rationality conjecture

7.1. A general study of $K_0(\mathcal{A}_{\mathbb{F}_q})$.

7.1.1. *Polishchuk's description of $K_0(\mathcal{A}_{\mathbb{F}_q})$.* A crucial tool which we will use in the proof of Theorem 1.4 is the following description of $K_0(\mathcal{A}_{\mathbb{F}_q})$ provided by Polishchuk.

Theorem 7.1 ([Pol01], Proposition 3.4.1)**.** *The map*

$$K_0(\mathcal{A}_{\mathbb{F}_q}) \to \bigoplus_{w \in W} K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U)) \tag{52}$$

induced by the functor $\oplus_{w\in W} j_w^*$ is injective. Its image is the subset

$$\{(a_w)_{w\in W} \in K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U)) \mid \textit{for any } s \in S, w \in W, a_{sw} - \Phi_s a_w \in \mathrm{im}(\Phi_s^2 - 1)\}.$$

7.1.2. *Recalling* [MF23]. In [MF23], we study the subalgebra $\mathrm{KL}(v)$ of endomorphisms of $K_0(G/U)$ generated by the symplectic Fourier transforms $\{\Phi_s\}_{s\in S}$. By Section 2.2.1, this is the same as the subalgebra of $K_0(G/U \times G/U)$ generated under convolution by classes of Kazhdan-Laumon sheaves; we denote the generator of $\mathrm{KL}(v)$ corresponding to $w \in W$ by a_w. In this section, we use the monodromic Hecke algebras $\mathcal{H}_{\mathcal{L}}$ and $\mathcal{H}_{\mathfrak{o}}$ with the standard generators $\tilde{T}_w$ as defined in [LY20], see [MF23] for a more precise outline of our conventions.

In Section 4 of [MF23], we show that for any character sheaf $\mathcal{L}$ with W-orbit $\mathfrak{o}$, there is a surjection $\pi_{\mathcal{L}} : \mathrm{KL}(v) \to \mathcal{H}_{\mathfrak{o}}$. The following result follows from the main result of loc. cit. which explicitly identifies the algebra $\mathrm{KL}(v)$ as a subalgebra of a generic-parameter version of the Yokonuma-Hecke algebra.

Proposition 7.2 ([MF23])**.** *The following properties are satisfied by the morphisms $\{\pi_{\mathcal{L}}\}_{\mathcal{L}}$.*

(1) If $w_1, w_2 \in W$, then $\pi_{\mathcal{L}}(\mathsf{a}_{w_1}\mathsf{a}_{w_2}) = \pi_{w_2\mathcal{L}}(\mathsf{a}_{w_1})\pi_{\mathcal{L}}(\mathsf{a}_{w_2})$ in $\mathcal{H}_{\mathfrak{o}}$.

(2) If $w \in W^\circ_{\mathcal{L}}$, then $\pi_{\mathcal{L}}(\mathsf{a}_w) = \tilde{T}_w \in \mathcal{H}_{\mathfrak{o}}$.

(3) If $s \in S$ is not in $W^\circ_{\mathcal{L}}$, then $\pi_{\mathcal{L}}(\mathsf{a}_s^2) = 1$.

Finally, the morphism $\prod_{\mathcal{L}} \pi_{\mathcal{L}}$ is injective, so if $a \in \mathrm{KL}(v)$ is such that $\pi_{\mathcal{L}}(a) = 0$ for all character sheaves $\mathcal{L}$, then $a = 0$.

Lemma 7.3. *For any $\mathcal{L}$ with W-orbit $\mathfrak{o}$, the algebra morphism $\pi_{\mathcal{L}} : \mathrm{KL}(v) \to \mathcal{H}_{\mathfrak{o}}$ is such that*

$$\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) = \tilde{T}^2_{w_{0,\mathcal{L}}}, \tag{53}$$

where $w_{0,\mathcal{L}}$ is the longest element of $W^\circ_{\mathcal{L}}$.

Proof. Recall that if $s \in S$ is such that $s \notin W^\circ_{\mathcal{L}}$, then $\pi_{\mathcal{L}}(\mathsf{a}_s^2) = 1$ in $\mathcal{H}_{\mathfrak{o}}$. For induction, we claim that if $y \in W$ is such that $\ell(y) + \ell(w_{\mathcal{L},0}) = l(yw_{\mathcal{L},0})$, and if $s \in S$ such that $l(sy) < l(y)$, then $s \notin W_{y\mathcal{L}}$, i.e. $y^{-1}sy \notin W^\circ_{\mathcal{L}}$.

Indeed, if we had $y^{-1}sy \in W^\circ_{\mathcal{L}}$, then we since $w_{0,\mathcal{L}}$ dominates all elements of $W^\circ_{\mathcal{L}}$ in the Bruhat order, we could pick $z \in W$ such that $y^{-1}syz = w_{0,\mathcal{L}}$ with $\ell(y^{-1}sy) + \ell(z) = \ell(w_0, \mathcal{L})$. But then we would have $syz = yw_{0,\mathcal{L}}$ with $\ell(sy) + \ell(z) = \ell(syz) = \ell(yw_{\mathcal{L},0})$. But since $\ell(sy) < \ell(y)$ and $\ell(z) \leq \ell(w_{0,\mathcal{L}})$, this is impossible.

Now choose some y for which $yw_{0,\mathcal{L}} = w_0$ with $\ell(y) + \ell(w_{0,\mathcal{L}})$. We will show that $\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) = \tilde{T}^2_{w_{0,\mathcal{L}}}$ by induction on the length of y. Choosing $s \in S$ such that $\ell(sy) < \ell(y)$, this induction hypothesis along with the fact proved in the previous paragraph gives that

$$\pi_{\mathcal{L}}(\mathsf{a}_{w_0}^2) = \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{y^{-1}}\mathsf{a}_y\mathsf{a}_{w_{0,\mathcal{L}}}) \tag{54}$$

$$= \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}}\mathsf{a}_s^2\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) \tag{55}$$

$$= \pi_{sy\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}})\pi_{sy\mathcal{L}}(\mathsf{a}_s^2)\pi_{\mathcal{L}}(\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) \tag{56}$$

$$= \pi_{\mathcal{L}}(\mathsf{a}_{w_{0,\mathcal{L}}}\mathsf{a}_{(sy)^{-1}}\mathsf{a}_{sy}\mathsf{a}_{w_{0,\mathcal{L}}}) \tag{57}$$

$$= \pi_{\mathcal{L}}(\mathsf{a}^2_{syw_{0,\mathcal{L}}}) \tag{58}$$

$$= \tilde{T}^2_{w_0,\mathcal{L}}. \tag{59}$$

□

7.1.3. *The action of the full twist on* $\mathrm{Perv}_{\mathbb{F}_q}(G/U)$.

Proposition 7.4. *The endomorphism* ι^2 *on* $K_0(\mathcal{A})$ *satisfies* $P(\iota^2, v) = 0$.

Proof. By Theorem 7.1, it is enough to show that $P(\Phi^2_{w_0}, v) = 0$ as an endomorphism of $K_0(\mathrm{Perv}_{\mathbb{F}_q}(G/U))$.

Now recall that the endomorphism $\Phi_{w_0} : K_0(G/U) \to K_0(G/U)$ agrees with right convolution with the Kazhdan-Laumon sheaf $\overline{K(w_0)}$. Since the convolution $D^b(G/U) \times D^b(G/U \times G/U) \to D^b(G/U)$ is a triangulated functor, it is enough to show that $P(\overline{K(w_0)} * \overline{K(w_0)}, v) = 0$ in $K_0(G/U \times G/U)$.

Letting $1_{\mathcal{L}}$ be the idempotent in $\mathcal{H}_{\mathfrak{o}}$ corresponding to the $\mathcal{L}$-monodromic subalgebra, then in loc. cit. we show that

$$\pi_{\mathcal{L}}(\mathsf{a}_s) = \begin{cases} -v\tilde{T}_s^{-1}1_{\mathcal{L}} & s \in W^\circ_{\mathcal{L}} \\ -\tilde{T}_s 1_{\mathcal{L}} & s \notin W^\circ_{\mathcal{L}}. \end{cases} \tag{60}$$

It is a straightforward calculation from the above to show that $\pi_{\mathcal{L}}([K(w_0) * K(w_0)]) = v^{2\ell(y)}\tilde{T}_y^{-2}1_{\mathcal{L}} \in \mathcal{H}_{\mathcal{L}}$, where y is the longest element of $W^\circ_{\mathcal{L}}$. By the main result of [LY20], the monodromic Hecke algebra $\mathcal{H}_{\mathcal{L}}$ is isomorphic to $\mathcal{H}_{W^\circ_{\mathcal{L}}}$, with full twists on each side being identified as in Lemma 7.3. By 5.12.2 of [Lus84] which identifies the eigenvalues of the full twist in the regular representation, we have that $P(-v^{2\ell(y)}\tilde{T}_y^2, v) = 0$ in $\mathcal{H}_{\mathcal{L}}$.

Finally, we note that by Proposition 7.2, if a polynomial is satisfied by $\pi_{\mathcal{L}}([\overline{K(w_0)} * \overline{K(w_0)}])$ in each $\mathcal{H}_{\mathfrak{o}}$, then it is also satisfied in $K_0(G/U \times G/U)$; this completes the proof of the proposition. □

7.2. Completing the proof of Theorem 1.4.

7.2.1. *Proof of Theorem 1.4.* Let $a \in K_0(\mathcal{A}_{\mathbb{F}_q})$. We can write $p(v) = \tilde{P}(1, v) = \tilde{P}(x, v) + r(x, v)(x - 1)$ for some $r(x, v) \in \mathbb{Z}[x, v]$. So if

$$a_0 = \tilde{P}(\iota^2, v)a \tag{61}$$

$$a_1 = r(\iota^2, v)(\iota^2 - 1)a, \tag{62}$$

then $a_0 + a_1 = p(v)a$, so it suffices to show that $a_0, a_1 \in V^{\mathrm{fp}}_{p(v)}(\mathcal{A}_{\mathbb{F}_q})$.

First we show this for a_0. We claim that for any $w \in W$ and $s \in S$, $\Phi^2_s((a_0)_w) = (a_0)_w$. Since $a_0 = \tilde{P}(\Phi^2_{w_0}, v)a$, this will follow from the fact that for any $s \in I$, $(\Phi^2_s - 1)\tilde{P}(\Phi^2_{w_0}, v) = 0$ as an endomorphism of $K_0(G/U)$. By Proposition 7.2, this relation holds if and only if it holds after applying $\pi_{\mathcal{L}}$ for any character sheaf $\mathcal{L}$. By Lemma 7.3, this reduces to showing that if $\tilde{P}(\tilde{T}^2_{w_0}, v) \neq 0$, then $(\tilde{T}^2_s - 1)\tilde{P}(\tilde{T}^2_{w_0}, v) = 0$ for any $s \in S$. We can rephrase this as saying that if the full twist acts by the eigenvalue 1, then so does $\tilde{T}^2_s$ for every $s \in S$. Indeed, this follows from the classification of irreducible representations of Hecke algebras, and it is shown directly in 11.5.3 of [Pol01].

Now note that by Theorem 7.1, we have $(a_0)_{sw} - \Phi_s(a_0)_w = (\Phi_s^2 - 1)b$ for some b. Using the relation $(\Phi_s + v^2)(\Phi_s^2 - 1) = 0$ from Proposition 6.2.1 of [Pol01] and applying Φ_s to both sides, we then get

$$(\Phi_s^2 - 1)((a_0)_{ws} - (a_0)_w) = (\Phi_s^2 - 1)^2 b \tag{63}$$

$$= (v^4 - 1)(\Phi_s^2 - 1)b, \tag{64}$$

$$= (v^4 - 1)((a_0)_{sw} - \Phi_s(a_0)_w), \tag{65}$$

so $(v^4 - 1)((a_0)_{sw} - \Phi_s(a_0)_w) = 0$, meaning $(a_0)_{sw} = \Phi_s(a_0)_w$. This means $a_0 = j_{w!}j_w^*(a_0)$ in $K_0(\mathcal{A}_{\mathbb{F}_q})$ for any $w \in W$, and so $a_0 \in V_{p(v)}^{\mathrm{fp}}$.

Now it only remains to observe that $a_1 \in V_{p(v)}^{\mathrm{fp}}$. By the definition of a_1, $\tilde{P}(\Phi_{w_0}^2, v)a_1 = 0$. This means we can then apply the exact same argument as in the proof of Corollary 6.13 to get that $a_1 \in V_{p(v)}^{\mathrm{fp}}$. Then since a_0 and a_1 both lie in $V_{p(v)}^{\mathrm{fp}}$ and $p(v)a = a_0 + a_1$, we have that $a \in V_{p(v)}^{\mathrm{fp}}$.

8. Construction of Kazhdan-Laumon representations

8.1. **The Grothendieck-Lefschetz pairing.**

8.1.1. *The original proposal in* [KL88]. In Section 3 of [KL88], the proposed construction of representations is as follows. They begin by making Conjecture 1.1, which we now know to be false by Bezrukavnikov and Polishchuk's appendix to [Pol01]. However, for objects of finite projective dimension, one can still define the Grothendieck-Lefschetz-type pairing in the manner they describe

First, they define a Verdier duality functor $\mathbb{D} : \mathcal{A}_\psi \to \mathcal{A}_{\psi^{-1}}$, where $\mathcal{A}_\psi = \mathcal{A}$ as we have been using it throughout this paper, while $\mathcal{A}_{\psi^{-1}}$ is the same category but using the additive character ψ^{-1} instead of ψ (where ψ is the additive character we chose in Section 2.1.2).

They note that for any $A \in \mathcal{A}_{w,\mathbb{F}_q}$ and $B \in (\mathcal{A}_{\psi^{-1}})_{w,\mathbb{F}_q}$ and any $i \in \mathbb{Z}$, the isomorphisms $\psi_A : \mathcal{F}_w \mathrm{Fr}^* A \to A$ and $\psi_B : \mathcal{F}_w \mathrm{Fr}^* B \to B$ give an endomorphism $\psi_{A,B}^i$ of the vector space $\mathrm{Ext}_{\mathcal{A}}^i(A, \mathbb{D}B)$ given by the composition,

$$\mathrm{Ext}_{\mathcal{A}}^i(A, \mathbb{D}B) \to \mathrm{Ext}_{\mathcal{A}}^i(\mathcal{F}_w \mathrm{Fr}^* A, \mathcal{F}_w \mathrm{Fr}^* \mathbb{D}B) \to \mathrm{Ext}_{\mathcal{A}}^i(\mathcal{F}_w A, \mathcal{F}_w \mathbb{D}B) \to \mathrm{Ext}_{\mathcal{A}}^i(A, \mathbb{D}B)$$

where the first map arises from the morphisms ψ_A and ψ_B, the next from the canonical isomorphisms $\mathrm{Fr}^* A \to A$ and $\mathrm{Fr}^* B \to B$, and the last from the fact that $\mathcal{F}_w$ is invertible. This map is also described explicitly in 4.3.1 of [BP98].

We can then define, for A having finite projective dimension and arbitrary B, the value

$$\langle [A], [B] \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}(\psi_{A,B}^i, \mathrm{Ext}_{\mathcal{A}}^i(A, \mathbb{D}B)). \tag{66}$$

This is clearly well-defined at the level of Grothendieck groups. We will now explain how to use the result in Theorem 1.5 to extend Kazhdan and Laumon's pairing, which as of this point is only defined for objects of finite projective dimension, to the full Grothendieck group in the monodromic case.

It is a straightforward computation that for any such A and B, we have

$$\langle [A(-\tfrac{1}{2})], [B] \rangle = q^{\frac{1}{2}} \langle [A], [B] \rangle, \tag{67}$$

so the pairing is $\mathbb{Z}[v, v^{-1}]$-linear where $\mathbb{Z}[v, v^{-1}]$ acts on the target field such that v is multiplication by $q^{\frac{1}{2}}$.

8.1.2. *A pairing on* $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$. We can do the same construction on the monodromic Kazhdan-Laumon category $\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}$ and its Grothendieck group. Then using $\mathbb{Z}[v, v^{-1}]$-linearity, the above definition gives us a pairing which is well-defined on elements of V^{fp}.

Now we note that the polynomial $p(v)$ evaluated at $v = q^{\frac{1}{2}}$ is nonzero, so we can extend this pairing linearly to the localization $V^{\mathrm{fp}}_{p(v)}$. This then gives that the pairing is well-defined on all of $V^{\mathrm{fp}}_{p(v)} \otimes \mathbb{C}$, where in the tensor product we send $v \mapsto q^{\frac{1}{2}}$. But by Theorem 1.5, $V^{\mathrm{fp}}_{p(v)} \otimes \mathbb{C}$ is all of $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$, so we can indeed define the pairing on this entire vector space; we will now explain how to use this to construct the Kazhdan-Laumon representations.

8.2. **Construction of representations.** As Kazhdan and Laumon explain in [KL88], the category $\mathcal{A}_{w,\mathbb{F}_q}$ is defined so that $K_0(\mathcal{A}_{w,\mathbb{F}_q})$ carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$, where $T(w)$ is the (usually non-split) torus of G corresponding to $w \in W$, defined by

$$T(w) = \{t \in T(\overline{\mathbb{F}_q}) \mid \mathrm{Fr}^*(t) = w(t)\}. \tag{68}$$

We note that $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q})$ then also carries commuting actions of $G(\mathbb{F}_q)$ and $T(w)$ where $T(w)$ acts by its character θ which corresponds to the data of the character sheaf $\mathcal{L}$.

As we explained, the pairing $\langle , \rangle$ is well-defined on $K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C}$, and so we can define $K^{\mathcal{L}}_w$ to be its kernel. Then the Kazhdan-Laumon representation corresponding to the pairing $(T(w), \theta)$ which was originally sought in [KL88] is the vector space $V_{w,\mathcal{L}} = (K_0(\mathcal{A}^{\mathcal{L}}_{w,\mathbb{F}_q}) \otimes \mathbb{C})/K^{\mathcal{L}}_w$.

In future work, we hope to explicitly decompose this vector space into irreducibles and compute the characters of $V_{w,\mathcal{L}}$ explicitly, generalizing the work which was done in [BP98] for quasi-regular characters.

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