Chapter 8: Brownian Motion

2022-05-30

Examples

Example 8.1

For $0 < s < t$, find the distribution of $B_s + B_t$.

$$B_s + B_t = B_s + B_s + B_t - B_s = 2B_s + (B_t - B_s)$$

By independent increments, B_s and $B_t - B_s$ are independent random variables.

$$\begin{aligned} E(B_s + B_t) &= E(B_s) + E(B_t) = 0 \\ \mathrm{Var}(B_s + B_t) &= \mathrm{Var}(2B_s + (B_t - B_s)) = 4Var(B_s) + \mathrm{Var}(B_t - B_s) \\ &= 4s + t - s \\ &= 3s + t \end{aligned}$$

Therefore, $B_s + B_t \sim \mathcal{N}(0, 3s + t)$.

Example 8.2

If a standard Brownian motion particle is at position 1 at time 2, find the probability that the position is at most 3 at time 5.

By stationary increments, $B_5 - B_2 \sim B_3$.

$$P(B_5 \leq 3 \mid B_2 = 1) = P(B_5 - B_2 \leq 3 - B_2 \mid B_2 = 1) = P(B_5 - B_2 \leq 2) = P(B_3 \leq 2) = 0.876$$

Example 8.3

Find the covariance of B_s and B_t.

$$\mathrm{Cov}(B_s, B_t) = E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t)$$

Assuming that $s < t$:

$$\begin{aligned} B_t &= B_s + (B_t - B_s) \\ E(B_s B_t) &= E(B_s(B_s + (B_t - B_s))) \\ &= E(B_s^2 + B_s(B_t - B_s)) \\ &= E(B_s^2) + E(B_s(B_t - B_s)) \\ &= \mathrm{Var}(B_s) + E(B_s)E(B_t - B_s) \\ &= s + 0 = s \end{aligned}$$

By symmetry, if $t < s$, then $\mathrm{Cov}(B_s B_t) = t$. In general, $\mathrm{Cov}(B_s B_t) = \min(s, t)$.

Example 8.4

For a simple symmetric random walk, consider the maximum of the walk in the first n steps. Let $g(f) = \max_{0\le t\le 1} f(t)$.

By the invariance principle, $g(S_{nt}/\sqrt{n}) \approx g(B_t)$ for large n.

$$\lim_{n\to\infty} g\left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n\to\infty} \max_{0\le t\le 1}\left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n\to\infty} \max_{0\le k\le n}\left(\frac{S_k}{\sqrt{n}}\right)$$

This converges to $g(B_t) = \max_{0\le t\le 1}(B_t)$ in the limit.

$$\max_{0\le k\le n} S_k/\sqrt{n} = \max_{0\le t\le 1}(B_t)$$
$$\max_{0\le k\le n} S_k = \sqrt{n}\max_{0\le t\le 1}(B_t)$$

$\max_{0\le t\le 1}(B_t)$ has density $f(x) = \sqrt{2/\pi}\exp(-x^2/2)$ for $x > 0$. If $n = 10,000$ steps, the probability that a value greater than 200 is reached is:

$$P\left(\max_{0\le k\le n} S_k > 200\right) = P\left(\max_{0\le k\le n}\frac{S_k}{100} > 2\right) = P\left(\max_{0\le k\le n}\frac{S_k}{\sqrt{n}} > 2\right) = P(M > 2) = 0.0455$$

```r
# simulating a random walk with 10000 steps
n <- 10000
nsim <- 10000
sim <- replicate(nsim,
                 max(cumsum(sample(size=n, c(-1,1), replace = T))))

mean(sim)
```

```
## [1] 78.4081
```

```r
sd(sim)
```

```
## [1] 60.37541
```

```r
sim <- replicate(nsim,
                 ifelse(max(cumsum(sample(c(-1, 1), n, replace = T))) > 200, 1, 0))

mean(sim)
```

```
## [1] 0.0422
```

Example 8.5

For $a > 0$, let $X_t = B_{at}/\sqrt{a}$ for $t \ge 0$. Show that X_t is a standard Brownian motion.

The linear combination of X_k for $0 \le t_1 < \cdots < t_k$ is

$$\sum_{i=1}^{k} a_i X_{t_i} = \sum_{i=1}^{k}\frac{a_i}{\sqrt{a}} B_{at}$$

This has a univariate Normal distribution since B_{at} is a standard Brownian motion process and is thus a Gaussian process. $X_0 = 0$ since $B_0 = 0$. The mean function is:

$$E(X_t) = E(B_{at}/\sqrt{a}) = 0$$

The covariance is:

$$\begin{aligned}\mathrm{Cov}(X_s, X_t) &= \mathrm{Cov}(B_{as}/\sqrt{a}, B_{at}/\sqrt{a}) \\ &= \frac{1}{a}\mathrm{Cov}(B_{as}, B_{at}) \\ &= \frac{1}{a}\min(as, at) = min(s,t)\end{aligned}$$

Finally, because B_t is path continuous, X_t is path continuous for all $a > 0$.

Example 8.6

Let $(X_t)_{t\geq 0}$ be a Brownian motion process started at $x = 3$. Find $P(X_2 > 0)$.

Write $X_t = B_t + 3$. Then,

$$P(X_2 > 0) = P(B_2 + 3 > 0) = P(B_2 > -3) = 0.983$$

Example 8.7

A particle moves according to Brownian motion started at $x = 1$. After $t = 3$ hours, the particle is at level 1.5. Find the probability that the particle reaches level 2 sometime in the next hour.

The translated process is a Brownian motion process started at $x = 1.5$. The event that the translated process hits level 2 within the next hour is equal to the event that a standard Brownian motion process hits $x = 2 - 1.5 = 0.5$ within the next hour.

$$P(T_{0.5} < 1) = \int_0^1 f_{T_{0.5}}(t)\, dt = \int_0^1 \frac{0.5}{\sqrt{2\pi t^3}} e^{-0.5^2/2t}\, dt = 0.617$$

Example 8.8

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and Normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. for how many years can the lab be guaranteed that there is at least 90% probability that errors are less than 4 degrees?

Let B_t be the error at time t. Find t such that $P(M_t < 4) \geq 90\%$.

$$0.9 \leq P(M_t < 4) = 1 - P(M_t \geq 4) = 1 - 2P(B_t \geq 4) = 2P(B_t < 4) - 1$$

This yields

$$0.95 \leq P(B_t < 4) = P\left(Z \leq \frac{4}{\sqrt{t}}\right)$$

The 95th percentile of the standard Normal distribution is 1.645:

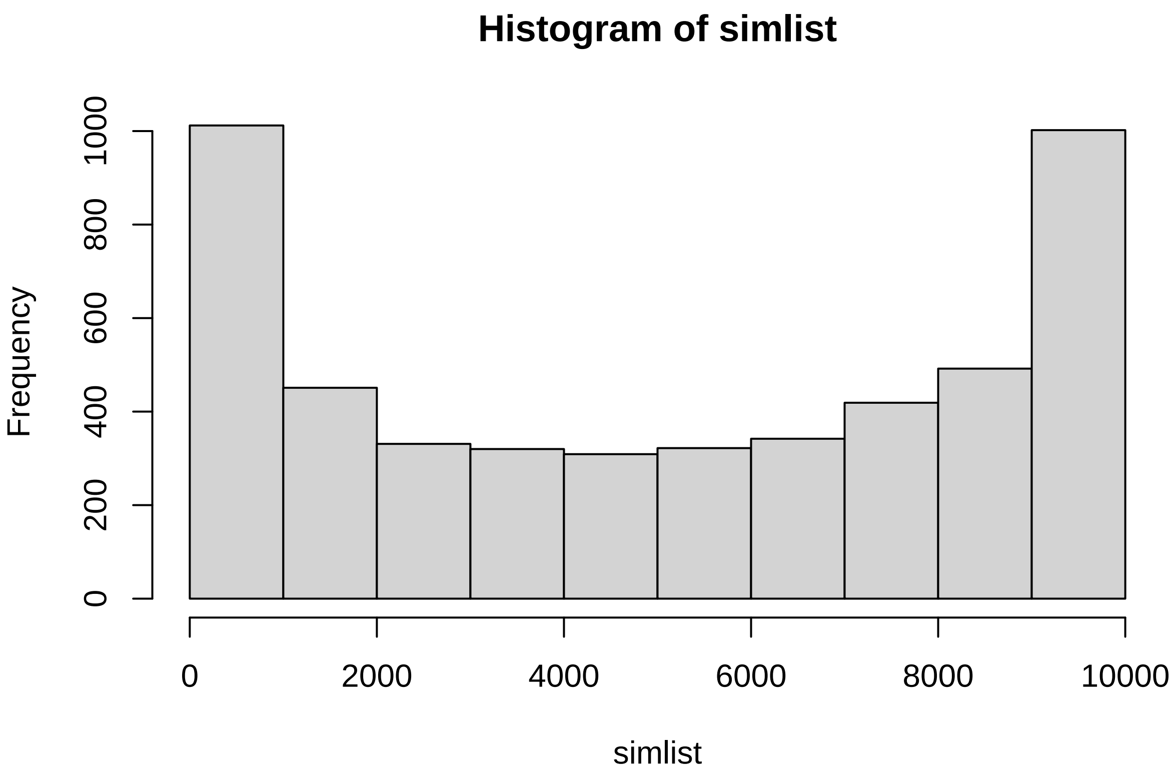
$$\begin{aligned}1.645 &= \frac{4}{\sqrt{t}} \\ t &= \left(\frac{4}{1.645}\right)^2 = 5.91\end{aligned}$$

Last Even Time

Flip a coin $n = 10,000$ times. If heads, A pays B \$1, otherwise if tails, B pays A. Let $\tilde{L}_n$ be the last time in n times that the players are tied (aka the last zero in a simple symmetric random walk on $\{0, 1, \ldots, n\}$). What is the distribution of $\tilde{L}_n$?

```
nsim <- 5000
n <- 10000
simlist <- numeric(nsim)
for (i in 1:nsim) {
  random_walk <- c(0, cumsum(sample(c(-1,1), size = n-1, replace = T)))
  simlist[i] = tail(which(random_walk == 0), 1)
}

hist(simlist)
```



The probability that the last zero occurs by time tn for $0 < t < n$ is approximately equal to the probability that the last zero for Brownian motion occurs by time t:

$$P(\tilde{L}_n \leq tn) \approx P(L_1 \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}$$

since $P(L_t \leq x) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}$ for a Brownian motion on $(0, t)$.

Example 8.10

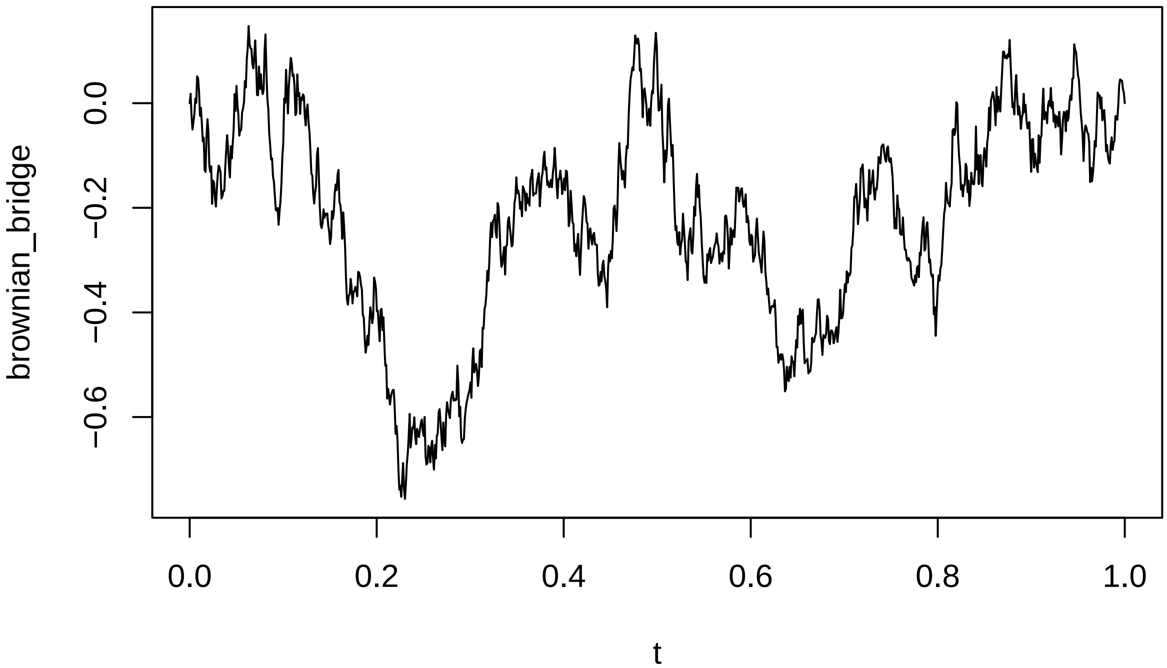
Find the probability that Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ takes values between 1 and 3 at time $t = 4$.

$$P(1 \leq X_4 \leq 3) = P(1 \leq 0.6(4) + \sqrt{0.25} B_4 \leq 3) = P(1 \leq 2.4 + 0.5 B_4 \leq 3) = P(-2.8 \leq B_4 \leq 1.2) = 0.645$$

Example 8.11

Brownian Bridge

```r
nsteps <- 1000
nsim <- 5000
t <- seq(0, 1, length = nsteps)
bm <- c(0,cumsum(rnorm(nsteps-1,0,1)))/sqrt(nsteps)
brownian_bridge <- bm - t*bm[nsteps]
plot(t, brownian_bridge, type = "l")
```



Mean and Variance of Geometric Brownian Motion

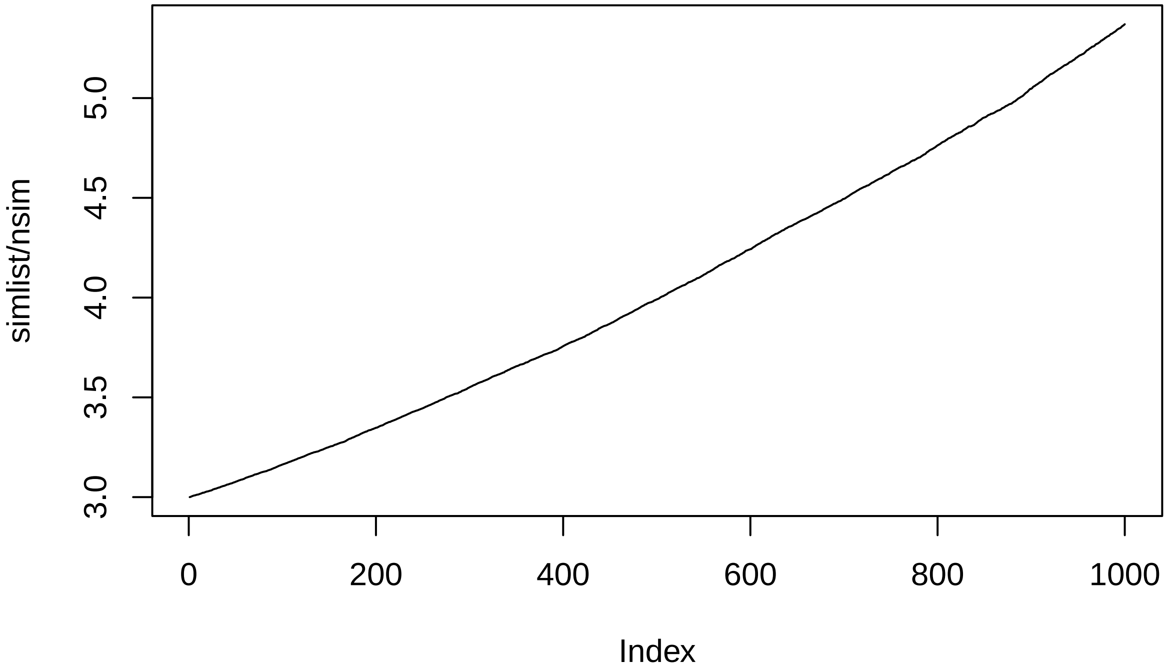
Let $G_t = G_0 e^{X_t}$ for $t \geq 0$ and $G_0 > 0$ with drift μ and variance σ^2. Then $\ln G_t = \ln G_0 + X_t$ is distributed Normally with mean $\ln G_0 + \mu t$ and variance $\sigma^2 t$.

$$
\begin{aligned}
E(G_t) &= E(G_0 e^{X_t}) \\
&= \int_{-\infty}^{\infty} G_0 e^x f_{X_t}(x) dx \\
&= G_0 \int_{-\infty}^{\infty} \exp(x) \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) dx
\end{aligned}
$$

Simulating Geometric Brownian Motion

```r
g0 <- 3
mu <- 0.4
sigma <- 0.6
nsim <- 10e3
n <- 1000
t <- seq(0, 1, length = n)
simlist <- numeric(n)
```

```r
for (i in 1:nsim) {
  bm <- cumsum(c(0, rnorm(n-1, 0, 1))) / sqrt(n)
  simlist <- simlist + g0 * exp(mu * t + sigma * bm)
}
plot(simlist / nsim, type = "l")
```



Example 8.15

Assume XYZ stock sells for \$80 and follows a geometric Brownian motion with drift 0.10 and volatility 0.50. Find the probability that in 90 days the price of XYZ will rise to ast least \$100.

$$
\begin{aligned}
P(X_{0.25} \geq 100) &= P(80e^{0.1(0.25)+0.50B_{0.25}} \geq 100) \\
&= P(e^{0.025+0.50B_{0.25}} \geq 1.25) \\
&= P(0.025 + 0.50B_{0.25} \geq \ln 1.25) \\
&= P(B_{0.25} \geq (\ln 1.25 - 0.025)/0.5) = 0.214
\end{aligned}
$$

Exercises

Exercise 8.1

Show that

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$$

satisfies the partial differential heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$$

Solution: Examining the first-order time partial differential:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\\
&= \frac{1}{\sqrt{2\pi t}}\frac{\partial}{\partial t}\exp\left(-\frac{x^2}{2t}\right) + \exp\left(-\frac{x^2}{2t}\right)\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi t}}\\
&= \frac{1}{\sqrt{2\pi t}}\frac{-x^2}{2}\frac{-1}{t^2}\exp\left(-\frac{x^2}{2t}\right) + \exp\left(-\frac{x^2}{2t}\right)\frac{1}{\sqrt{2\pi}}\frac{-1}{2}t^{-3/2}\\
&= \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[\frac{x^2}{2t^2}-\frac{1}{2t}\right]\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[\frac{x^2}{t^2}-\frac{1}{t}\right]
\end{aligned}$$

The right side, which includes a second-order in x term:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\frac{\partial^2}{\partial x^2}\exp\left(-\frac{x^2}{2t}\right)\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\frac{\partial}{\partial x}\frac{-2x}{2t}\exp\left(-\frac{x^2}{2t}\right)\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{x^2}{2t}\right)\frac{\partial}{\partial x}\frac{-x}{t} - \frac{x}{t}\frac{\partial}{\partial x}\exp\left(-\frac{x^2}{2t}\right)\right]\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\left[-\frac{1}{t}\exp\left(-\frac{x^2}{2t}\right) + \frac{x^2}{t^2}\exp\left(-\frac{x^2}{2t}\right)\right]\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)\left[\frac{x^2}{t^2}-\frac{1}{t}\right]
\end{aligned}$$

These two sides are equal, thus $f(x,t)$ satisfies the heat equation.

Exercise 8.2

For standard Brownian motion, find (a) $P(B_2 \leq 1)$

Solution: B_2 is distributed Normal with mean 0 and variance 2, so $P(B_2 \leq 1) = \Phi\left(\frac{1}{\sqrt{2}}\right) \approx 0.760$

(b) $E(B_4|B_1 = x)$

Solution: By independent increments, $B_4 - B_1$ and B_1 are independent random variables. By stationary increments, $B_4 - B_1$ is distributed the same as B_3.

$$E(B_4|B_1 = x) = E(B_4 - B_1|B_1 = x) + E(B_1|B_1 = x) = E(B_3) + x = x$$

(c) $\mathrm{Corr}(B_{t+s}, B_s)$

Solution:

$$\mathrm{Corr(B_{t+s}, B_s)} = \frac{\mathrm{Cov(B_{t+s}, B_s)}}{\sigma_{\mathrm{B_{t+s}}}\sigma_{\mathrm{B_s}}} = \frac{\mathrm{s}}{\sqrt{\mathrm{t+s}}\sqrt{\mathrm{s}}} = \sqrt{\frac{\mathrm{s}}{\mathrm{t+s}}}$$

(d) $\mathrm{Var}(B_4|B_1)$

Solution: Let X be the process from $t = 1$ to 4. By stationary increments, $X = B_4 - B_1$ has the same distribution as B_3 and by independent increments, is uncorrelated with B_1.

$$\mathrm{Var(B_4|B_1) = Var(B_1 + X|B_1) = 0 + Var(X) = 3}$$

(e) $P(B_3 \leq 5|B_1 = 2)$

Solution: By the Markov property, the Wiener process restarts at $t = 1$. Thus, we can calculate the unconditional probability for the event $W_2 \leq 3$, where W is a standard Wiener process.

$$P(B_3 \leq 5|B_1 = 2) = P(W_2 \leq 3) = \Phi\left(\frac{3}{\sqrt{2}}\right)$$

Exercise 8.3

For standard Brownian motion started at $x = -3$, find (a) $P(X_1 + X_2 > -1)$

Solution: Consider the process X_t, where $t > 0$. This is distributed equal to a standard Brownian motion process but shifted down by 3, so $X_t = B_t - 3$. Then, we can rewrite X_2 as $B_2 - 3$ which is distributed $(B_2 - B_1) + B_1 - 3$.

$$P(X_1+X_2 > -1) = P(B_1-3+(B_2-B_1)+B_1-3 > -1) = P(2B_1+(B_2-B_1)-6 > -1) = P(2B_1+(B_2-B_1) > 5)$$

By stationary intervals, $(B_2 - B_1)$ is distributed Gaussian with variance 1. $2B_1 + (B_2 - B_1)$ is distributed $2X+Y$, where X and Y are both standard Normal random variables, so $2B_1+(B_2-B_1) \sim \mathcal{N}(0, 5)$. Therefore, $P(X_1 + X_2 > -1) = 1 - \Phi(\sqrt{5})$, which is approximately 0.013.

(b) The conditional density of X_2 given $X_1 = 0$

Solution: X_2 is distributed like $X_1 + B_1$ by stationary and independent increments, so we have:

$$X_2|X_1 = 0 \Rightarrow X_1 + B_1|X_1 = 0 \Rightarrow B_1$$

The conditional density of $X_2 \mid X_1 = 0$ is a standard Normal with variance 1. Intuitively, this makes sense. By independent increments, the process on the interval $t = 1$ to $t = 2$ is a new Brownian motion process restarted at the origin.

(c) $\mathrm{Cov}(X_3, X_4)$

Solution:

$$\mathrm{Cov(X_3, X_4) = Cov(B_3 - 3, B_4 - 4) = Cov(B_3, B_4) = 3}$$

(d) $E(X_4|X_1)$

Solution: By independence and stationary increments, X_4 is distributed $X_1 + B_3$.

$$E(X_4|X_1) = E(X_1|X_1) + E(B_3) = X_1 + 0 = X_1$$

Intuitively, the most recent information we have at $t = 4$ is the value of the process at $t = 1$. The process restarts at $t = 1$, so the conditional expectation is the given value X_1. This is in line with the process's martingale property.

Exercise 8.4

In a race between Lisa and Cooper, let X_t denote the amount of time (in seconds) by which Lisa is ahead when $100t$ percent of the race has been completed. Assume that $(X_t)_{0\leq t\leq 1}$ can be modeled by a Brownian motion with drift parameter 0 and variance parameter σ^2. If Lisa is leading by $\sigma/2$ seconds after three-fourths of the race is complete, what is the probability that she is the winner?

Solution: The desired probability is $P(X_1 \geq 0|X_{0.75} = \sigma/2)$. The race process can be rewritten as $X_1 = (X_1 - X_{0.75}) + X_{0.75}$. By stationary increments, $X_1 - X_{0.75} \sim \sigma B_{0.25}$ and by independent increments, is uncorrelated with the process $X_{0.75}$.

$$\begin{aligned}
P(X_1 \geq 0|X_{0.75} = \sigma/2) &= P((X_1 - X_{0.75}) + X_{0.75} > 0|X_{0.75} = \sigma/2) \\
&= P(\sigma B_{0.25} + X_{0.75} > 0|X_{0.75} = \sigma/2) \\
&= P(\sigma B_{0.25} + \sigma/2 > 0) \\
&= P(B_{0.25} > -1/2) \\
&= P(1/2 \cdot Z > -1/2) \\
&= 1 - \Phi(-1) \\
&= \Phi(1)
\end{aligned}$$

This is approximately 0.841.

Exercise 8.5

Consider standard Brownian motion. Let $0 < s < t$. (a) Find the joint density of (B_s, B_t)

Solution: Because B_t is a standard Brownian motion process, B_s and B_t have a multivariate Normal distribution. $\vec{\mu} = (0,0)^T$ and the covariance matrix is:

$$\mathrm{Cov}(B_s, B_t) = \begin{bmatrix} s & s \\ s & t \end{bmatrix}$$

(b) Show that the conditional distribution of B_s given $B_t = y$ is Normal, with mean and variance

$$E(B_s|B_t = y) = \frac{sy}{t} \quad \text{and} \, \mathrm{Var}(B_s|B_t = y) = \frac{s(t-s)}{t}$$