Chapter 2

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Exercises

Exercise 2.1

A Markov chain has transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix} \end{array}$$

with initial distribution $\alpha = (0.2, 0.3, 0.5)$. Find the following: (a) $P(X_7 = 3|X_6 = 2)$

Solution: By time homogeneity,

$$P(X_7 = 3|X_6 = 2) = P(X_1 = 3|X_0 = 2) = P_{23} = 0.6$$

(b) $P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3)$

Solution: By the Markov property and time homogeneity,

$$\begin{aligned} P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3) &= P(X_9 = 2|X_7 = 3) \\ &= P(X_2 = 2|X_0 = 3)(P^2)_{23} \\ &= 0.54 \end{aligned}$$

(c) $P(X_0 = 3|X_1 = 1)$

Solution: By Bayes' rule,

$$\begin{aligned} P(X_0 = 3|X_1 = 1) &= \frac{P(X_1 = 1|X_0 = 3)P(X_0 = 3)}{P(X_1 = 1)} \\ &= \frac{P_{31}\alpha_3}{(\alpha P)_1} \\ &= \frac{0.3 \cdot 0.5}{0.17} \\ &= \frac{15}{17} \approx 0.88 \end{aligned}$$

(d) $E(X_2)$

Solution:

$$E(X_2) = \sum_k kP(X_2 = k) = \sum_k k(\alpha P^2)_k = 2.363$$

Exercise 2.2

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

and initial distribution $\alpha = (1/2, 0, 1/2)$. Find the following: (a) $P(X_2 = 1|X_1 = 3)$ **Solution:** $P(X_2 = 1|X_1 = 3) = P_{31} = 1/3$

(b) $P(X_1 = 3, X_2 = 1)$ **Solution:** $P(X_1 = 3, X_2 = 2) = (\alpha P)_3 P_{31}$

Exercise 2.3 NOT STARTED

Exercise 2.4

For the general two-state chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

and initial distribution $\alpha = (\alpha_1, \alpha_2)$, find the following: (a) the two-step transition matrix **Solution:**

$$\begin{aligned} P^2 &= \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \\ &= \begin{bmatrix} (1-p)^2 + pq & (1-p)p + p(1-q) \\ q(1-p) + q(1-q) & pq + (1-q)^2 \end{bmatrix} \\ &= \begin{bmatrix} (1-p)^2 + pq & p(2-p-q) \\ q(2-p-q) & (1-q)^2 + pq \end{bmatrix} \end{aligned}$$

(b) the distribution of X_1 **Solution:**

$$\begin{aligned} X_1 &= \alpha P \\ &= \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 - \alpha_1 p + \alpha_2 q & \alpha_1 p + \alpha_2 - \alpha_2 q \end{bmatrix} \end{aligned}$$

Exercise 2.5

Consider a random walk on $\{0, \ldots, k\}$, which moves left and right with respective probabilities q and p. If the walk is at 0 it transitions to 1 on the next step. If the walk is at k it transitions to $k-1$ on the next step. This is called random walk with reflecting boundaries. Assume that $k = 3, q = 1/4, p = 3/4$, and the initial distribution is uniform. For the following, use technology if needed.

(a) Exhibit the transition matrix.

Solution: The transition matrix is:

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(b) Find $P(X_7 = 1 | X_0 = 3, X_2 = 2, X_4 = 2)$.

Solution: By the Markov property, this probability depends on the most recent state, so $P = P(X_7 = 1 | X_4 = 2) = (P^3)_{21} = 19/64$.

$$
\begin{aligned}
P^3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1/4 & 0 & 3/4 & 0 \\ 0 & 7/16 & 0 & 9/16 \\ 1/16 & 0 & 15/16 & 0 \\ 0 & 1/4 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 7/16 & 0 & 9/16 \\ 7/64 & 0 & 57/64 & 0 \\ 0 & 19/64 & 0 & 45/64 \\ 1/16 & 0 & 15/16 & 0 \end{bmatrix}
\end{aligned}
$$

(c) Find $P(X_3 = 1, X_5 = 3)$.

Solution: $P(X_3 = 1, X_5 = 3) = (\alpha P^3)_1 (P^2)_{13} = 0.103$

Exercise 2.6

A tetrahedron die has four faces labeled 1,2,3, and 4. In repeated independent rolls of the die $R_0, R_1, \ldots,$ let $X_n = \max\{R_0, \ldots, R_n\}$ be the maximum value after $n+1$ rolls, for ≥ 0. (a) Give an intuitive argument for why $X_0, X_1, \ldots,$ is a Markov chain, and exhibit the transition matrix.

Solution: Each subsequent roll is independent from the previous rolls. In order to determine what X_n is, we only need the state of X_{n-1} and the current roll, which is independent of any prior states or future states. The transition matrix is:

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$