

Chapter 8: Brownian Motion

2022-05-30

Example 8.1

For $0 < s < t$, find the distribution of $B_s + B_t$.

$$B_s + B_t = B_s + B_s + B_t - B_s = 2B_s + (B_t - B_s)$$

By independent increments, B_s and $B_t - B_s$ are independent random variables.

$$\begin{aligned} E(B_s + B_t) &= E(B_s) + E(B_t) = 0 \\ \text{Var}(B_s + B_t) &= \text{Var}(2B_s + (B_t - B_s)) = 4\text{Var}(B_s) + \text{Var}(B_t - B_s) \\ &= 4s + t - s \\ &= 3s + t \end{aligned}$$

Therefore, $B_s + B_t \sim \mathcal{N}(0, 3s + t)$.

Example 8.2

If a standard Brownian motion particle is at position 1 at time 2, find the probability that the position is at most 3 at time 5.

By stationary increments, $B_5 - B_2 \sim B_3$.

$$P(B_5 \leq 3 \mid B_2 = 1) = P(B_5 - B_2 \leq 3 - B_2 \mid B_2 = 1) = P(B_5 - B_2 \leq 2) = P(B_3 \leq 2) = 0.876$$

Example 8.3

Find the covariance of B_s and B_t .

$$\text{Cov}(B_s, B_t) = E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t)$$

Assuming that $s < t$:

$$\begin{aligned} B_t &= B_s + (B_t - B_s) \\ E(B_s B_t) &= E(B_s(B_s + (B_t - B_s))) \\ &= E(B_s^2 + B_s(B_t - B_s)) \\ &= E(B_s^2) + E(B_s(B_t - B_s)) \\ &= \text{Var}(B_s) + E(B_s)E(B_t - B_s) \\ &= s + 0 = s \end{aligned}$$

By symmetry, if $t < s$, then $\text{Cov}(B_s B_t) = t$. In general, $\text{Cov}(B_s B_t) = \min(s, t)$.

Example 8.4

For a simple symmetric random walk, consider the maximum of the walk in the first n steps. Let $g(f) = \max_{0 \leq t \leq 1} f(t)$.

By the invariance principle, $g(S_{nt}/\sqrt{n}) \approx g(B_t)$ for large n .

$$\lim_{n \rightarrow \infty} g\left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} \max_{0 \leq t \leq 1} \left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} \max_{0 \leq k \leq n} \left(\frac{S_k}{\sqrt{n}}\right)$$

This converges to $g(B_t) = \max_{0 \leq t \leq 1} (B_t)$ in the limit.

$$\begin{aligned} \max_{0 \leq k \leq n} S_k / \sqrt{n} &= \max_{0 \leq t \leq 1} (B_t) \\ \max_{0 \leq k \leq n} S_k &= \sqrt{n} \max_{0 \leq t \leq 1} (B_t) \end{aligned}$$

$\max_{0 \leq t \leq 1} (B_t)$ has density $f(x) = \sqrt{2/\pi} \exp(-x^2/2)$ for $x > 0$. If $n = 10,000$ steps, the probability that a value greater than 200 is reached is:

$$P\left(\max_{0 \leq k \leq n} S_k > 200\right) = P\left(\max_{0 \leq k \leq n} \frac{S_k}{100} > 2\right) = P\left(\max_{0 \leq k \leq n} \frac{S_k}{\sqrt{n}} > 2\right) = P(M > 2) = 0.0455$$

```
# simulating a random walk with 10000 steps
n <- 10000
nsim <- 10000
sim <- replicate(nsim,
                 max(cumsum(sample(size=n, c(-1,1), replace = T))))

mean(sim)

## [1] 79.6946

sd(sim)

## [1] 60.58774

sim <- replicate(nsim,
                 ifelse(max(cumsum(sample(c(-1, 1), n, replace = T))) > 200, 1, 0))

mean(sim)

## [1] 0.0426
```

Example 8.5

For $a > 0$, let $X_t = B_{at}/\sqrt{a}$ for $t \geq 0$. Show that X_t is a standard Brownian motion.

The linear combination of X_k for $0 \leq t_1 < \dots < t_k$ is

$$\sum_{i=1}^k a_i X_{t_i} = \sum_{i=1}^k \frac{a_i}{\sqrt{a}} B_{at}$$

This has a univariate Normal distribution since B_{at} is a standard Brownian motion process and is thus a Gaussian process. $X_0 = 0$ since $B_0 = 0$. The mean function is:

$$E(X_t) = E(B_{at}/\sqrt{a}) = 0$$

The covariance is:

$$\begin{aligned}\text{Cov}(X_s, X_t) &= \text{Cov}(B_{as}/\sqrt{a}, B_{at}/\sqrt{a}) \\ &= \frac{1}{a} \text{Cov}(B_{as}, B_{at}) \\ &= \frac{1}{a} \min(as, at) = \min(s, t)\end{aligned}$$

Finally, because B_t is path continuous, X_t is path continuous for all $a > 0$.

Example 8.6

Let $(X_t)_{t \geq 0}$ be a Brownian motion process started at $x = 3$. Find $P(X_2 > 0)$.

Write $X_t = B_t + 3$. Then,

$$P(X_2 > 0) = P(B_2 + 3 > 0) = P(B_2 > -3) = 0.983$$

Example 8.7

A particle moves according to Brownian motion started at $x = 1$. After $t = 3$ hours, the particle is at level 1.5. Find the probability that the particle reaches level 2 sometime in the next hour.

The translated process is a Brownian motion process started at $x = 1.5$. The event that the translated process hits level 2 within the next hour is equal to the event that a standard Brownian motion process hits $x = 2 - 1.5 = 0.5$ within the next hour.

$$P(T_{0.5} < 1) = \int_0^1 f_{T_{0.5}}(t) dt = \int_0^1 \frac{0.5}{\sqrt{2\pi t^3}} e^{-0.5^2/2t} dt = 0.617$$

Example 8.8

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and Normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. for how many years can the lab be guaranteed that there is at least 90% probability that errors are less than 4 degrees?

Let B_t be the error at time t . Find t such that $P(M_t < 4) \geq 90\%$.

$$0.9 \leq P(M_t < 4) = 1 - P(M_t \geq 4) = 1 - 2P(B_t \geq 4) = 2P(B_t < 4) - 1$$

This yields

$$0.95 \leq P(B_t < 4) = P\left(Z \leq \frac{4}{\sqrt{t}}\right)$$

The 95th percentile of the standard Normal distribution is 1.645:

$$\begin{aligned}1.645 &= \frac{4}{\sqrt{t}} \\ t &= \left(\frac{4}{1.645}\right)^2 = 5.91\end{aligned}$$

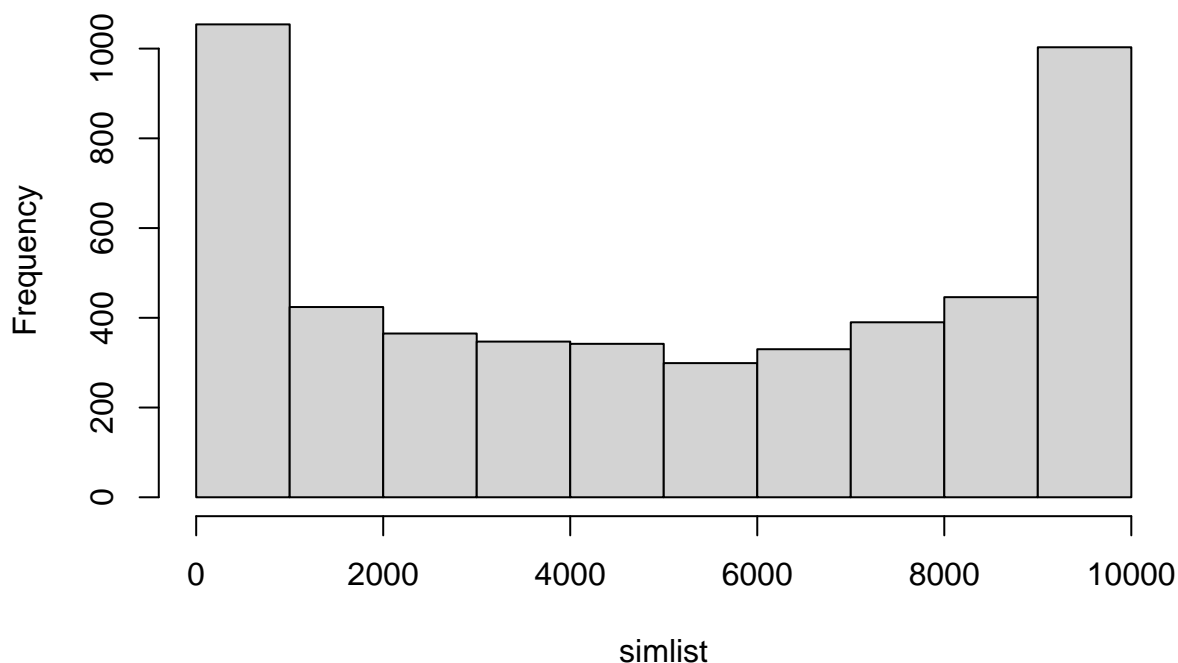
Last Even Time

Flip a coin $n = 10,000$ times. If heads, A pays B \$1, otherwise if tails, B pays A. Let \tilde{L}_n be the last time in n times that the players are tied (aka the last zero in a simple symmetric random walk on $\{0, 1, \dots, n\}$). What is the distribution of \tilde{L}_n ?

```
nsim <- 5000
n <- 10000
simlist <- numeric(nsim)
for (i in 1:nsim) {
  random_walk <- c(0, cumsum(sample(c(-1,1), size = n-1, replace = T)))
  simlist[i] = tail(which(random_walk == 0), 1)
}

hist(simlist)
```

Histogram of simlist



The probability that the last zero occurs by time tn for $0 < t < n$ is approximately equal to the probability that the last zero for Brownian motion occurs by time t :

$$P(\tilde{L}_n \leq tn) \approx P(L_1 \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}$$

since $P(L_t \leq x) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}$ for a Brownian motion on $(0, t)$.

Example 8.10

Find the probability that Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ takes values between 1 and 3 at time $t = 4$.

$$P(1 \leq X_4 \leq 3) = P(1 \leq 0.6(4) + \sqrt{0.25}B_4 \leq 3) = P(1 \leq 2.4 + 0.5B_4 \leq 3) = P(-2.8 \leq B_4 \leq 1.2) = 0.645$$

Example 8.11

Exercises

8.5

Consider standard Brownian motion. Let $0 < s < t$.

###(a)

Find the joint density of (B_s, B_t) .

Solution: Because $\{B_s = x, B_t = y\} = \{B_s = x, B_{t-s} = y - x\}$ (which are independent events by stationary and independent increments), it follows that the joint density can be factored:

$$f_{B_s, B_t}(x, y) = f_{B_s}(x) f_{B_{t-s}}(y - x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \cdot \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$

(b)

Show that the conditional distribution of B_s given $B_t = y$ is Normal with mean and variance $E(B_s | B_t = y) = \frac{sy}{t}$ and $\text{Var}(B_s | B_t = y) = \frac{s(t-s)}{t}$.

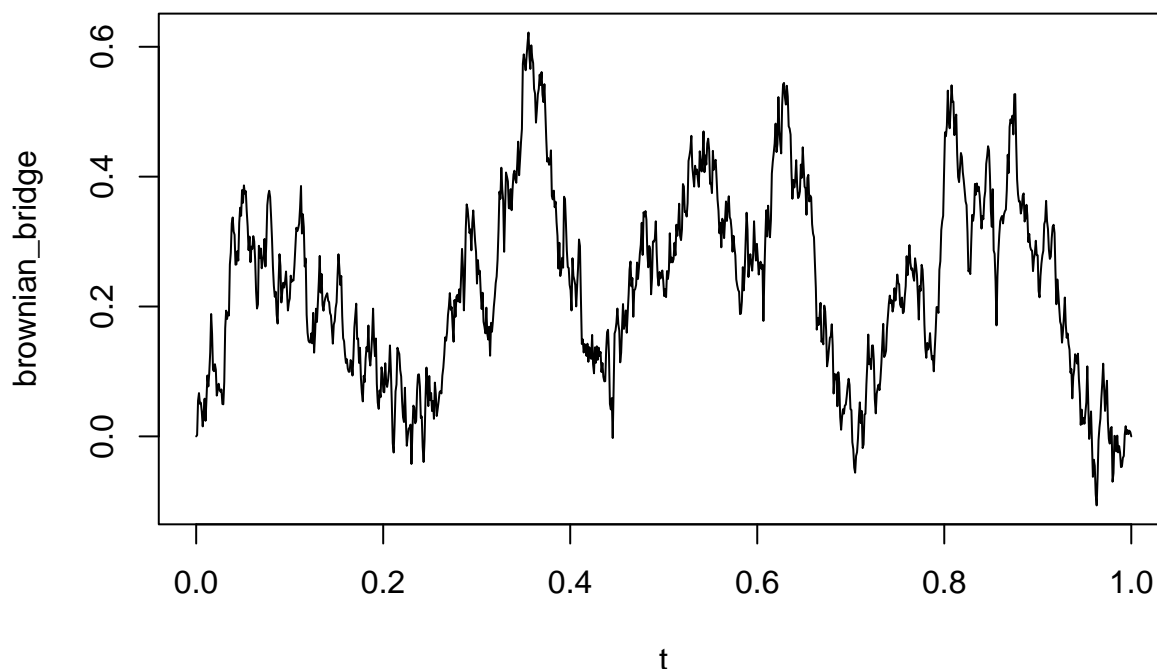
Solution:

$$\begin{aligned} E(B_s | B_t = y) &= \int_{-\infty}^{\infty} x \frac{f_{B_s, B_t}(x, y)}{f_{B_t}(y)} dx \\ &= \frac{1}{f_{B_t}(y)} \int_{-\infty}^{\infty} x f_{B_s, B_t}(x, y) dx \\ &= \left[\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) \right]^{-1} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \cdot \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right) dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(\frac{y^2}{2t}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2} \left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} \right)\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(\frac{y^2}{2t}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2} \left(\frac{x^2 t - x^2 s + y^2 s - 2xys + x^2 s}{s(t-s)} \right)\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(\frac{y^2}{2t} - \frac{y^2}{2(t-s)}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2} \left(\frac{x^2 t - 2xys}{s(t-s)} \right)\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(-\frac{y^2 s}{2t(t-s)}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2s(t-s)/t} \left(x^2 - \frac{2xys}{t} + \frac{y^2 s^2}{t^2} - \frac{y^2 s^2}{t^2} \right)\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(-\frac{y^2 s}{2t(t-s)}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2s(t-s)/t} \left(x - \frac{sy}{t} \right)^2\right] \exp\left[\frac{t}{2s(t-s)} \frac{y^2 s^2}{t^2}\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \exp\left(-\frac{y^2 s}{2t(t-s)}\right) \exp\left(\frac{y^2 s}{2t(t-s)}\right) \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2s(t-s)/t} \left(x - \frac{sy}{t} \right)^2\right] dx \\ &= \frac{1}{\sqrt{2\pi} \sqrt{s(t-s)/t}} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2s(t-s)/t} \left(x - \frac{sy}{t} \right)^2\right] dx \end{aligned}$$

This is the density of a Gaussian distribution with mean $\mu = \frac{sy}{t}$ and variance $\sigma^2 = \frac{s(t-s)}{t}$. As s approaches 0, the mean goes to 0 and the variance goes to 0. As s approaches t , the mean approaches y and the variance also approaches 0.

Brownian Bridge

```
nsteps <- 1000
nsim <- 5000
t <- seq(0, 1, length = nsteps)
bm <- c(0, cumsum(rnorm(nsteps-1, 0, 1))) / sqrt(nsteps)
brownian_bridge <- bm - t * bm[nsteps]
plot(t, brownian_bridge, type = "l")
```



Mean and Variance of Geometric Brownian Motion

Let $G_t = G_0 e^{X_t}$ for $t \geq 0$ and $G_0 > 0$ with drift μ and variance σ^2 . Then $\ln G_t = \ln G_0 + X_t$ is distributed Normally with mean $\ln G_0 + \mu t$ and variance $\sigma^2 t$.

$$\begin{aligned} E(G_t) &= E(G_0 e^{X_t}) \\ &= \int_{-\infty}^{\infty} G_0 e^x f_{X_t}(x) dx \\ &= G_0 \int_{-\infty}^{\infty} \exp(x) \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right) dx \end{aligned}$$

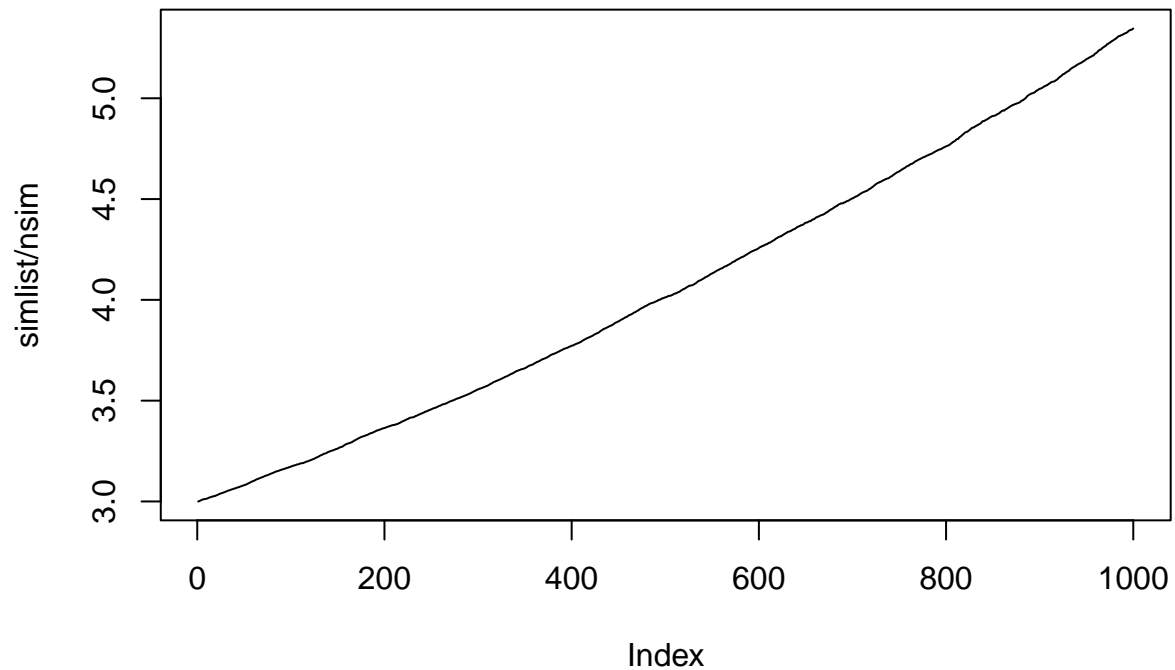
Simulating Geometric Brownian Motion

```
g0 <- 3
mu <- 0.4
sigma <- 0.6
nsim <- 10e3
n <- 1000
t <- seq(0, 1, length = n)
simlist <- numeric(n)
for (i in 1:nsim) {
  bm <- cumsum(c(0, rnorm(n-1, 0, 1))) / sqrt(n)
  simlist[i] <- g0 * exp(mu * t + sigma * bm)
```

```

    simlist <- simlist + g0 * exp(mu * t + sigma * bm)
  }
  plot(simlist / nsim, type = "l")

```



Example 8.15

Assume XYZ stock sells for \$80 and follows a geometric Brownian motion with drift 0.10 and volatility 0.50. Find the probability that in 90 days the price of XYZ will rise to at least \$100.

$$P(X_{90} \geq \$100)$$