

Chapter 8: Brownian Motion

2022-05-30

Examples

Example 8.1

For $0 < s < t$, find the distribution of $B_s + B_t$.

$$B_s + B_t = B_s + B_s + B_t - B_s = 2B_s + (B_t - B_s)$$

By independent increments, B_s and $B_t - B_s$ are independent random variables.

$$\begin{aligned} E(B_s + B_t) &= E(B_s) + E(B_t) = 0 \\ \text{Var}(B_s + B_t) &= \text{Var}(2B_s + (B_t - B_s)) = 4\text{Var}(B_s) + \text{Var}(B_t - B_s) \\ &= 4s + t - s \\ &= 3s + t \end{aligned}$$

Therefore, $B_s + B_t \sim \mathcal{N}(0, 3s + t)$.

Example 8.2

If a standard Brownian motion particle is at position 1 at time 2, find the probability that the position is at most 3 at time 5.

By stationary increments, $B_5 - B_2 \sim B_3$.

$$P(B_5 \leq 3 | B_2 = 1) = P(B_5 - B_2 \leq 3 - B_2 | B_2 = 1) = P(B_5 - B_2 \leq 2) = P(B_3 \leq 2) = 0.876$$

Example 8.3

Find the covariance of B_s and B_t .

$$\text{Cov}(B_s, B_t) = E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t)$$

Assuming that $s < t$:

$$\begin{aligned} B_t &= B_s + (B_t - B_s) \\ E(B_s B_t) &= E(B_s(B_s + (B_t - B_s))) \\ &= E(B_s^2 + B_s(B_t - B_s)) \\ &= E(B_s^2) + E(B_s(B_t - B_s)) \\ &= \text{Var}(B_s) + E(B_s)E(B_t - B_s) \\ &= s + 0 = s \end{aligned}$$

By symmetry, if $t < s$, then $\text{Cov}(B_s B_t) = t$. In general, $\text{Cov}(B_s B_t) = \min(s, t)$.

Example 8.4

For a simple symmetric random walk, consider the maximum of the walk in the first n steps. Let $g(f) = \max_{0 \leq t \leq 1} f(t)$.

By the invariance principle, $g(S_{nt}/\sqrt{n}) \approx g(B_t)$ for large n .

$$\lim_{n \rightarrow \infty} g\left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} \max_{0 \leq t \leq 1} \left(\frac{S_{nt}}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} \max_{0 \leq k \leq n} \left(\frac{S_k}{\sqrt{n}}\right)$$

This converges to $g(B_t) = \max_{0 \leq t \leq 1}(B_t)$ in the limit.

$$\begin{aligned}\max_{0 \leq k \leq n} S_k / \sqrt{n} &= \max_{0 \leq t \leq 1} (B_t) \\ \max_{0 \leq k \leq n} S_k &= \sqrt{n} \max_{0 \leq t \leq 1} (B_t)\end{aligned}$$

$\max_{0 \leq t \leq 1}(B_t)$ has density $f(x) = \sqrt{2/\pi} \exp(-x^2/2)$ for $x > 0$. If $n = 10,000$ steps, the probability that a value greater than 200 is reached is:

$$P\left(\max_{0 \leq k \leq n} S_k > 200\right) = P\left(\max_{0 \leq k \leq n} \frac{S_k}{100} > 2\right) = P\left(\max_{0 \leq k \leq n} \frac{S_k}{\sqrt{n}} > 2\right) = P(M > 2) = 0.0455$$

```
# simulating a random walk with 10000 steps
n <- 10000
nsim <- 10000
sim <- replicate(nsim,
                  max(cumsum(sample(size=n, c(-1,1), replace = T)))))

mean(sim)

## [1] 79.461
sd(sim)

## [1] 59.87315
sim <- replicate(nsim,
                  ifelse(max(cumsum(sample(c(-1, 1), n, replace = T))) > 200, 1, 0))

mean(sim)

## [1] 0.0435
```

Example 8.5

For $a > 0$, let $X_t = B_{at}/\sqrt{a}$ for $t \geq 0$. Show that X_t is a standard Brownian motion.

The linear combination of X_k for $0 \leq t_1 < \dots < t_k$ is

$$\sum_{i=1}^k a_i X_{t_i} = \sum_{i=1}^k \frac{a_i}{\sqrt{a}} B_{at}$$

This has a univariate Normal distribution since B_{at} is a standard Brownian motion process and is thus a Gaussian process. $X_0 = 0$ since $B_0 = 0$. The mean function is:

$$E(X_t) = E(B_{at}/\sqrt{a}) = 0$$

The covariance is:

$$\begin{aligned}\text{Cov}(X_s, X_t) &= \text{Cov}(B_{as}/\sqrt{a}, B_{at}/\sqrt{a}) \\ &= \frac{1}{a} \text{Cov}(B_{as}, B_{at}) \\ &= \frac{1}{a} \min(as, at) = \min(s, t)\end{aligned}$$

Finally, because B_t is path continuous, X_t is path continuous for all $a > 0$.

Example 8.6

Let $(X_t)_{t \geq 0}$ be a Brownian motion process started at $x = 3$. Find $P(X_2 > 0)$.

Write $X_t = B_t + 3$. Then,

$$P(X_2 > 0) = P(B_2 + 3 > 0) = P(B_2 > -3) = 0.983$$

Example 8.7

A particle moves according to Brownian motion started at $x = 1$. After $t = 3$ hours, the particle is at level 1.5. Find the probability that the particle reaches level 2 sometime in the next hour.

The translated process is a Brownian motion process started at $x = 1.5$. The event that the translated process hits level 2 within the next hour is equal to the event that a standard Brownian motion process hits $x = 2 - 1.5 = 0.5$ within the next hour.

$$P(T_{0.5} < 1) = \int_0^1 f_{T_{0.5}}(t) dt = \int_0^1 \frac{0.5}{\sqrt{2\pi t^3}} e^{-0.5^2/2t} dt = 0.617$$

Example 8.8

A laboratory instrument takes annual temperature measurements. Measurement errors are assumed to be independent and Normally distributed. As precision decreases over time, errors are modeled as standard Brownian motion. For how many years can the lab be guaranteed that there is at least 90% probability that errors are less than 4 degrees?

Let B_t be the error at time t . Find t such that $P(M_t < 4) \geq 90\%$.

$$0.9 \leq P(M_t < 4) = 1 - P(M_t \geq 4) = 1 - 2P(B_t \geq 4) = 2P(B_t < 4) - 1$$

This yields

$$0.95 \leq P(B_t < 4) = P\left(Z \leq \frac{4}{\sqrt{t}}\right)$$

The 95th percentile of the standard Normal distribution is 1.645:

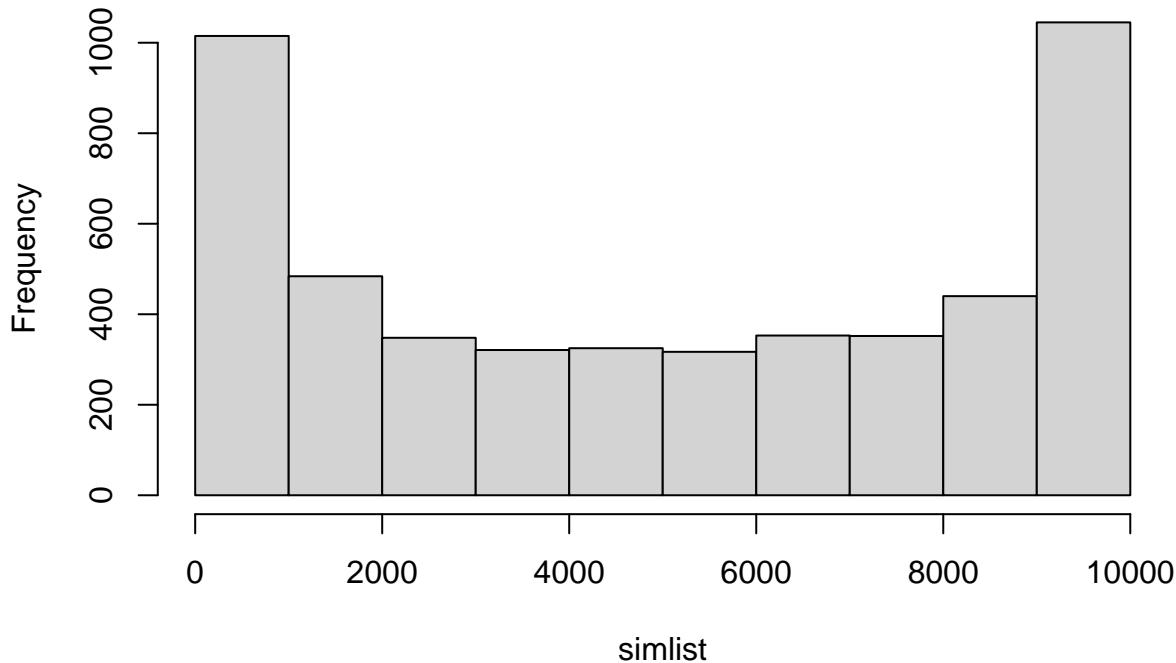
$$\begin{aligned}1.645 &= \frac{4}{\sqrt{t}} \\ t &= \left(\frac{4}{1.645}\right)^2 = 5.91\end{aligned}$$

Last Even Time

Flip a coin $n = 10,000$ times. If heads, A pays B \$1, otherwise if tails, B pays A. Let \tilde{L}_n be the last time in n times that the players are tied (aka the last zero in a simple symmetric random walk on $\{0, 1, \dots, n\}$). What is the distribution of \tilde{L}_n ?

```
nsim <- 5000
n <- 10000
simlist <- numeric(nsim)
for (i in 1:nsim) {
  random_walk <- c(0, cumsum(sample(c(-1,1), size = n-1, replace = T)))
  simlist[i] = tail(which(random_walk == 0), 1)
}
hist(simlist)
```

Histogram of simlist



The probability that the last zero occurs by time tn for $0 < t < n$ is approximately equal to the probability that the last zero for Brownian motion occurs by time t :

$$P(\tilde{L}_n \leq tn) \approx P(L_1 \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}$$

since $P(L_t \leq x) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}$ for a Brownian motion on $(0, t)$.

Example 8.10

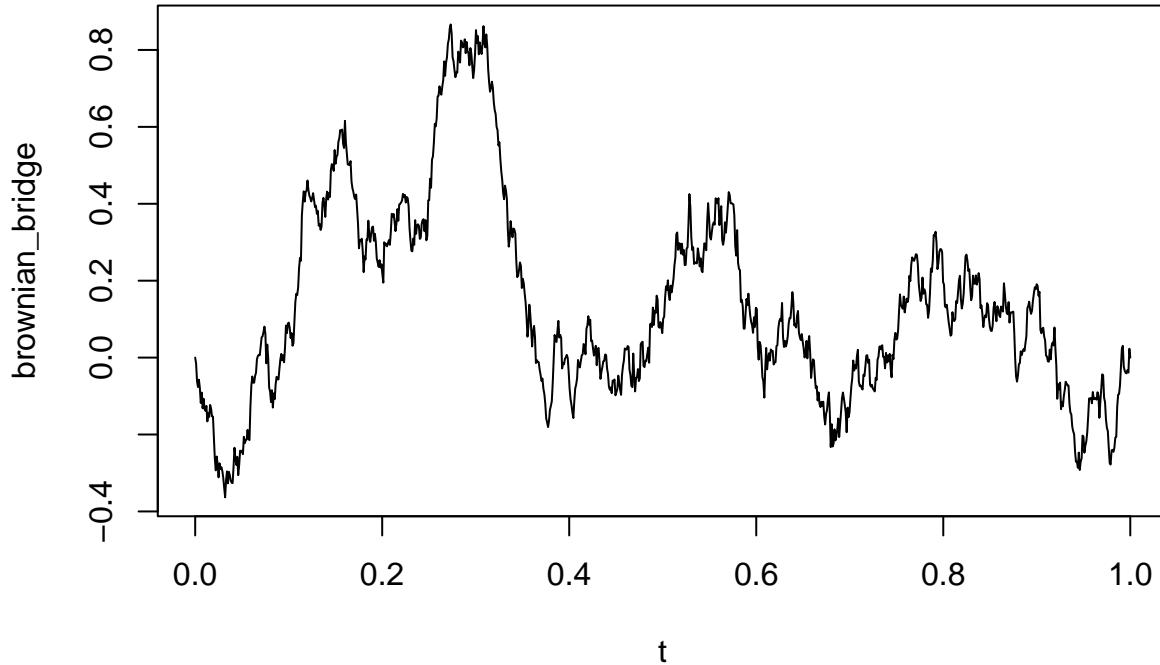
Find the probability that Brownian motion with drift parameter $\mu = 0.6$ and variance $\sigma^2 = 0.25$ takes values between 1 and 3 at time $t = 4$.

$$P(1 \leq X_4 \leq 3) = P(1 \leq 0.6(4) + \sqrt{0.25}B_4 \leq 3) = P(1 \leq 2.4 + 0.5B_4 \leq 3) = P(-2.8 \leq B_4 \leq 1.2) = 0.645$$

Example 8.11

Brownian Bridge

```
nsteps <- 1000
nsim <- 5000
t <- seq(0, 1, length = nsteps)
bm <- c(0,cumsum(rnorm(nsteps-1,0,1)))/sqrt(nsteps)
brownian_bridge <- bm - t*bm[nsteps]
plot(t, brownian_bridge, type = "l")
```



Mean and Variance of Geometric Brownian Motion

Let $G_t = G_0 e^{X_t}$ for $t \geq 0$ and $G_0 > 0$ with drift μ and variance σ^2 . Then $\ln G_t = \ln G_0 + X_t$ is distributed Normally with mean $\ln G_0 + \mu t$ and variance $\sigma^2 t$.

$$\begin{aligned} E(G_t) &= E(G_0 e^{X_t}) \\ &= \int_{-\infty}^{\infty} G_0 e^x f_{X_t}(x) dx \\ &= G_0 \int_{-\infty}^{\infty} \exp(x) \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) dx \end{aligned}$$

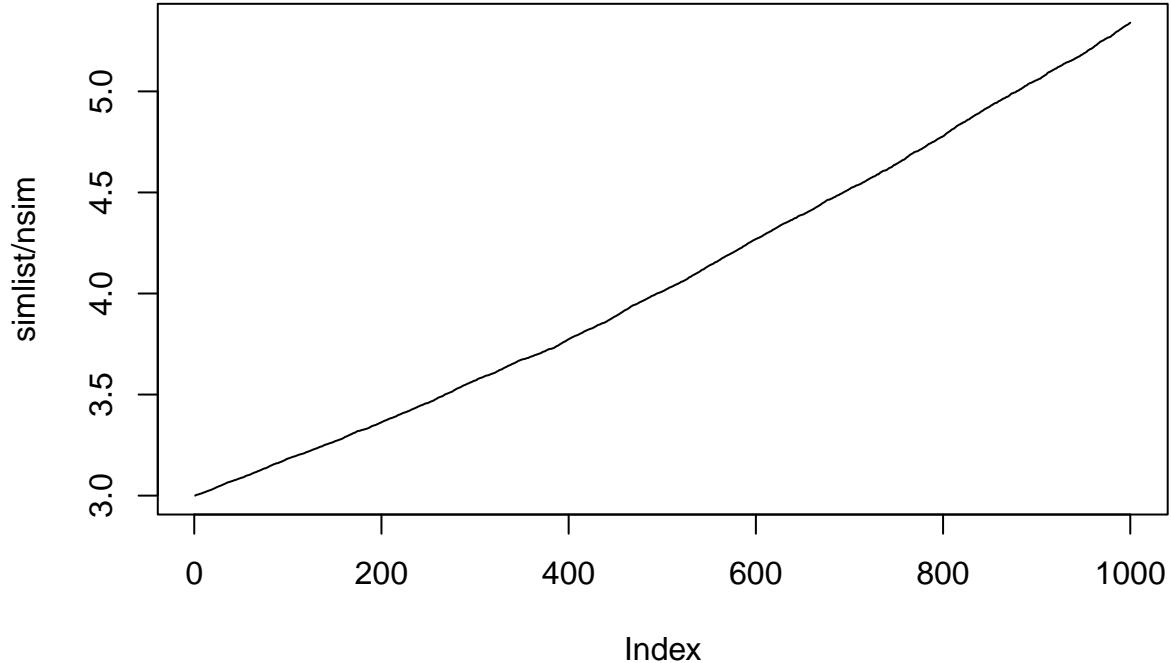
Simulating Geometric Brownian Motion

```
g0 <- 3
mu <- 0.4
sigma <- 0.6
nsim <- 10e3
n <- 1000
t <- seq(0, 1, length = n)
simlist <- numeric(n)
```

```

for (i in 1:n sim) {
  bm <- cumsum(c(0, rnorm(n-1, 0, 1))) / sqrt(n)
  simlist <- simlist + g0 * exp(mu * t + sigma * bm)
}
plot(simlist / nsim, type = "l")

```



Example 8.15

Assume XYZ stock sells for \$80 and follows a geometric Brownian motion with drift 0.10 and volatility 0.50. Find the probability that in 90 days the price of XYZ will rise to at least \$100.

$$\begin{aligned}
P(X_{0.25} \geq 100) &= P(80e^{0.1(0.25)+0.50B_{0.25}} \geq 100) \\
&= P(e^{0.025+0.50B_{0.25}} \geq 1.25) \\
&= P(0.025 + 0.50B_{0.25} \geq \ln 1.25) \\
&= P(B_{0.25} \geq (\ln 1.25 - 0.025)/0.5) = 0.214
\end{aligned}$$

Exercises

Exercise 8.1

Show that

$$f(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$$

satisfies the partial differential heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$$

Solution: Examining the first-order time partial differential:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \\
&= \frac{1}{\sqrt{2\pi t}} \frac{\partial}{\partial t} \exp\left(-\frac{x^2}{2t}\right) + \exp\left(-\frac{x^2}{2t}\right) \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi t}} \\
&= \frac{1}{\sqrt{2\pi t}} \frac{-x^2}{2} \frac{-1}{t^2} \exp\left(-\frac{x^2}{2t}\right) + \exp\left(-\frac{x^2}{2t}\right) \frac{1}{\sqrt{2\pi}} \frac{-1}{2} t^{-3/2} \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \left[\frac{x^2}{2t^2} - \frac{1}{2t} \right] \\
&= \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \left[\frac{x^2}{t^2} - \frac{1}{t} \right]
\end{aligned}$$

The right side, which includes a second-order in x term:

$$\begin{aligned}
\frac{1}{2} \frac{\partial^2 f}{\partial x^2} &= \frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \frac{\partial^2}{\partial x^2} \exp\left(-\frac{x^2}{2t}\right) \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \frac{\partial}{\partial x} \frac{-2x}{2t} \exp\left(-\frac{x^2}{2t}\right) \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{x^2}{2t}\right) \frac{\partial}{\partial x} \frac{-x}{t} - \frac{x}{t} \frac{\partial}{\partial x} \exp\left(-\frac{x^2}{2t}\right) \right] \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \left[-\frac{1}{t} \exp\left(-\frac{x^2}{2t}\right) + \frac{x^2}{t^2} \exp\left(-\frac{x^2}{2t}\right) \right] \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \left[\frac{x^2}{t^2} - \frac{1}{t} \right]
\end{aligned}$$

These two sides are equal, thus $f(x, t)$ satisfies the heat equation.

Exercise 8.2

For standard Brownian motion, find (a) $P(B_2 \leq 1)$

Solution: B_2 is distributed Normal with mean 0 and variance 2, so $P(B_2 \leq 1) = \Phi\left(\frac{1}{\sqrt{2}}\right) \approx 0.760$

(b) $E(B_4|B_1 = x)$

Solution: By independent increments, $B_4 - B_1$ and B_1 are independent random variables. By stationary increments, $B_4 - B_1$ is distributed the same as B_3 .

$$E(B_4|B_1 = x) = E(B_4 - B_1|B_1 = x) + E(B_1|B_1 = x) = E(B_3) + x = x$$

(c) $\text{Corr}(B_{t+s}, B_s)$

Solution:

$$\text{Corr}(B_{t+s}, B_s) = \frac{\text{Cov}(B_{t+s}, B_s)}{\sigma_{B_{t+s}} \sigma_{B_s}} = \frac{s}{\sqrt{t+s} \sqrt{s}} = \sqrt{\frac{s}{t+s}}$$

(d) $\text{Var}(B_4|B_1)$

Solution: Let X be the process from $t = 1$ to 4 . By stationary increments, $X = B_4 - B_1$ has the same distribution as B_3 and by independent increments, is uncorrelated with B_1 .

$$\text{Var}(B_4|B_1) = \text{Var}(B_1 + X|B_1) = 0 + \text{Var}(X) = 3$$

(e) $P(B_3 \leq 5|B_1 = 2)$

Solution: By the Markov property, the Wiener process restarts at $t = 1$. Thus, we can calculate the unconditional probability for the event $W_2 \leq 3$, where W is a standard Wiener process.

$$P(B_3 \leq 5|B_1 = 2) = P(W_2 \leq 3) = \Phi\left(\frac{3}{\sqrt{2}}\right)$$

Exercise 8.3

For standard Brownian motion started at $x = -3$, find (a) $P(X_1 + X_2 > -1)$

Solution: Consider the process X_t , where $t > 0$. This is distributed equal to a standard Brownian motion process but shifted down by 3, so $X_t = B_t - 3$. Then, we can rewrite X_2 as $B_2 - 3$ which is distributed $(B_2 - B_1) + B_1 - 3$.

$$P(X_1 + X_2 > -1) = P(B_1 - 3 + (B_2 - B_1) + B_1 - 3 > -1) = P(2B_1 + (B_2 - B_1) - 6 > -1) = P(2B_1 + (B_2 - B_1) > 5)$$

By stationary intervals, $(B_2 - B_1)$ is distributed Gaussian with variance 1. $2B_1 + (B_2 - B_1)$ is distributed $2X + Y$, where X and Y are both standard Normal random variables, so $2B_1 + (B_2 - B_1) \sim \mathcal{N}(0, 5)$. Therefore, $P(X_1 + X_2 > -1) = 1 - \Phi(\sqrt{5})$, which is approximately 0.013.

(b) The conditional density of X_2 given $X_1 = 0$

Solution: X_2 is distributed like $X_1 + B_1$ by stationary and independent increments, so we have:

$$X_2|X_1 = 0 \Rightarrow X_1 + B_1|X_1 = 0 \Rightarrow B_1$$

The conditional density of $X_2 | X_1 = 0$ is a standard Normal with variance 1. Intuitively, this makes sense. By independent increments, the process on the interval $t = 1$ to $t = 2$ is a new Brownian motion process restarted at the origin.

(c) $\text{Cov}(X_3, X_4)$

Solution:

$$\text{Cov}(X_3, X_4) = \text{Cov}(B_3 - 3, B_4 - 4) = \text{Cov}(B_3, B_4) = 3$$

(d) $E(X_4|X_1)$

Solution: By independence and stationary increments, X_4 is distributed $X_1 + B_3$.

$$E(X_4|X_1) = E(X_1|X_1) + E(B_3) = X_1 + 0 = X_1$$

Intuitively, the most recent information we have at $t = 4$ is the value of the process at $t = 1$. The process restarts at $t = 1$, so the conditional expectation is the given value X_1 . This is in line with the process's martingale property.

Exercise 8.4

In a race between Lisa and Cooper, let X_t denote the amount of time (in seconds) by which Lisa is ahead when $100t$ percent of the race has been completed. Assume that $(X_t)_{0 \leq t \leq 1}$ can be modeled by a Brownian motion with drift parameter 0 and variance parameter σ^2 . If Lisa is leading by $\sigma/2$ seconds after three-fourths of the race is complete, what is the probability that she is the winner?

Solution: The desired probability is $P(X_1 \geq 0 | X_{0.75} = \sigma/2)$. The race process can be rewritten as $X_1 = (X_1 - X_{0.75}) + X_{0.75}$. By stationary increments, $X_1 - X_{0.75} \sim \sigma B_{0.25}$ and by independent increments, is uncorrelated with the process $X_{0.75}$.

$$\begin{aligned} P(X_1 \geq 0 | X_{0.75} = \sigma/2) &= P((X_1 - X_{0.75}) + X_{0.75} > 0 | X_{0.75} = \sigma/2) \\ &= P(\sigma B_{0.25} + X_{0.75} > 0 | X_{0.75} = \sigma/2) \\ &= P(\sigma B_{0.25} + \sigma/2 > 0) \\ &= P(B_{0.25} > -1/2) \\ &= P(1/2 \cdot Z > -1/2) \\ &= 1 - \Phi(-1) \\ &= \Phi(1) \end{aligned}$$

This is approximately 0.841.

Exercise 8.5

Consider standard Brownian motion. Let $0 < s < t$.

- (a) Find the joint density of (B_s, B_t)

Solution: Because B_t is a standard Brownian motion process, B_s and B_t have a multivariate Normal distribution. $\vec{\mu} = (0, 0)^T$ and the covariance matrix is:

$$\text{Cov}(B_s, B_t) = \begin{bmatrix} s & s \\ s & t \end{bmatrix}$$

- (b) Show that the conditional distribution of B_s given $B_t = y$ is Normal, with mean and variance

$$E(B_s | B_t = y) = \frac{sy}{t} \quad \text{and} \quad \text{Var}(B_s | B_t = y) = \frac{s(t-s)}{t}$$

Solution: Because (B_s, B_t) is a bivariate Normal random variable, it follows that the conditional distribution $B_s | B_t = y$ is Normal. We can rewrite $B_t = B_s + (B_t - B_s)$, and setting $B_s = x$, we have:

$$\begin{aligned}
f_{B_s|B_t}(x|y) &= \frac{f_{B_t|B_s}(y|x)f_{B_s}(x)}{f_{B_t}(y)} \\
&= \frac{\sqrt{2\pi t}}{\sqrt{2\pi(t-s)} \cdot \sqrt{2\pi s}} \cdot \exp\left(-\frac{(y-x)^2}{2(t-s)} - \frac{x^2}{2s} + \frac{y^2}{2t}\right) \\
&= \frac{1}{\sqrt{2\pi s(t-s)/t}} \cdot \exp\left(-\frac{st(y^2 - 2xy + x^2) + t(t-s)x^2 - s(t-s)y^2}{2st(t-s)}\right) \\
&= \frac{1}{\sqrt{2\pi s(t-s)/t}} \cdot \exp\left(-\frac{sty^2 - 2stxy + stx^2 + t^2x^2 - stx^2 - sty^2 + s^2y^2}{2st(t-s)}\right) \\
&= \frac{1}{\sqrt{2\pi s(t-s)/t}} \cdot \exp\left(-\frac{t^2x^2 - 2stxy + s^2y^2}{2st(t-s)}\right) \\
&= \frac{1}{\sqrt{2\pi s(t-s)/t}} \cdot \exp\left(-\frac{(tx - sy)^2}{2st(t-s)}\right) \\
&= \frac{1}{\sqrt{2\pi s(t-s)/t}} \cdot \exp\left(-\frac{(x - \frac{s}{t}y)^2}{2s(t-s)/t}\right)
\end{aligned}$$

This corresponds to the density for a Normal random variable with mean $\frac{sy}{t}$ and standard deviation $\frac{s}{t}(t-s)$.

Exercise 8.6

For $s > 0$, show that the translation $(B_{t+s} - B_s)_{t \geq 0}$ is a standard Brownian motion.

Solution: To show that $X_t = (B_{t+s} - B_s)_{t \geq 0}$ is a standard Brownian motion, we must show the initial value $X_0 = 0$, the mean function is 0 for all t , the covariance between any 2 times s and t is $\min(s, t)$, and the function is continuous with probability 1.

Initial value: $X_0 = B_{0+s} - B_s = B_s - B_s = 0$.

Mean function:

$$E(X_t) = E(B_{t+s} - B_s) = E(B_{t+s}) - E(B_s) = 0 - 0 = 0$$

Covariance function: Let $0 < t < u$.

$$\text{Cov}(X_t, X_u) = \text{Cov}(B_{t+s} - B_s, B_{u+s} - B_s) = \text{Cov}(B_{t+s}, B_{u+s}) - \text{Cov}(B_{t+s}, B_s) - \text{Cov}(B_s, B_{u+s}) + \text{Cov}(B_s, B_s) = \min(t+s, u+s)$$

Continuity: The last condition is true because the difference of 2 functions that are almost surely continuous is also almost surely continuous.

As all of the conditions are met, X_t is a standard Brownian motion process.

Exercise 8.7

Show that the reflection $(-B_t)_{t \geq 0}$ is a standard Brownian motion.

Solution: Let $X_t = -B_t$ for $t \geq 0$.

Initial value: $X_0 = -B_0 = 0$.

Mean value: $E(X_t) = E(-B_t) = -E(B_t) = 0$.

Covariance function: Let $0 < s < t$. $\text{Cov}(X_s, X_t) = \text{Cov}(-B_s, -B_t) = \text{Cov}(B_s, B_t) = \min(s, t)$.

Continuity: Continuity is inherited by inspection.

As all of the conditions are met, X_t is a standard Brownian motion process.

Exercise 8.8

Find the covariance function for Brownian motion with drift.

Solution: Let $X_t = \mu t + \sigma B_t$. Let $0 < s < t$. The covariance function is:

$$\begin{aligned}\text{Cov}(X_s, X_t) &= \text{Cov}(\mu s + \sigma B_s, \mu t + \sigma B_t) \\ &= \text{Cov}(\sigma B_s, \sigma B_t) \\ &= \sigma^2 \text{Cov}(B_s, B_t) \\ &= \sigma^2 s\end{aligned}$$

Exercise 8.9

Let $W_t = B_{2t} - B_t$, for $t \geq 0$.

(a) Is $(W_t)_{t \geq 0}$ a Gaussian process?

Solution: W_t is a Gaussian process as B_{2t} and B_t are both Gaussian processes, and the difference of two Gaussian processes is Gaussian. We can see that at each time t , B_{2t} and B_t are distributed Normal, so W_t is Normal at each time t .

(b) Is $(W_t)_{t \geq 0}$ a Brownian motion process?

Solution: W_t is not a Brownian motion process. We look at the covariance function:

$$\begin{aligned}\text{Cov}(W_s, W_t) &= \text{Cov}(B_{2s} - B_s, B_{2t} - B_t) \\ &= \text{Cov}(B_{2s}, B_{2t}) - \text{Cov}(B_s, B_{2t}) - \text{Cov}(B_{2s}, B_t) + \text{Cov}(B_s, B_t) \\ &= 2 \min(s, t) - \min(s, 2t) - \min(2s, t) + \min(s, t) \\ &= 3 \min(s, t) - \min(s, 2t) - \min(2s, t)\end{aligned}$$

This expression cannot be further simplified, and does not collapse to $\min(s, t)$ unless $s = t$. This is not standard Brownian motion.

Exercise 8.10

Let $(B_t)_{t \geq 0}$ be a Brownian motion started in x . Let

$$X_t = B_t - t(B_1 - y), \text{ for } 0 \leq t \leq 1$$

The process $(X_t)_{t \geq 0}$ is a Brownian bridge with start in x and end in y . Find the mean and covariance functions.

Solution: The mean function is $E(X_t) = E(B_t - t(B_1 - y)) = E(B_t) - tE(B_1) + ty = x - tx + ty = (1-t)x + ty$. The mean varies linearly between x and y over the interval $0 \leq t \leq 1$.

The covariance function is:

$$\begin{aligned}\text{Cov}(X_s, X_t) &= \text{Cov}(B_s - s(B_1 - y), B_t - t(B_1 - y)) \\ &= \text{Cov}(B_s, B_t) + \text{Cov}(B_s, -t(B_1 - y)) + \text{Cov}(-s(B_1 - y), B_t) + \text{Cov}(-s(B_1 - y), -t(B_1 - y)) \\ &= \text{Cov}(B_s, B_t) + \text{Cov}(B_s, -tB_1) + \text{Cov}(-sB_1, B_t) + \text{Cov}(-sB_1, -tB_1) \\ &= \min(s, t) - ts - st + st \\ &= \min(s, t) - st\end{aligned}$$

When s, t are both small, then the covariance looks similar to standard Brownian motion. This is akin to the process just starting out, and the influence of the correction term is small. As s, t grow, then the covariance is reduced as the bridge term contributes more as time progresses.

Exercise 8.11

A standard Brownian motion crosses the t -axis at times $t = 2$ and $t = 5$. Find the probability that the process exceeds level $x = 1$ at time $t = 4$.

Solution: Due to the Markov property and by shifting the process, the event $\{B_4 > 1 | B_2 = 0, B_5 = 0\}$ is equivalent to $\{W_2 > 1\}$, where $W_t = B_t - \frac{t}{3}B_3$ is a Brownian bridge with ends tied down to 0 and $0 \leq t \leq 3$. The mean of this process is 0, and at each time t , the variance is:

$$\begin{aligned}\text{Var}(W_t) &= \text{Var}\left(B_t - \frac{t}{3}B_3\right) \\ &= \text{Var}(B_t) + \frac{t^2}{9}\text{Var}(B_3) - 2 \cdot \frac{t}{3}\text{Cov}(B_t, B_3) \\ &= t + \frac{3t^2}{9} - \frac{2t^2}{3} \\ &= t - \frac{t^2}{3} \\ &= t\left(1 - \frac{t}{3}\right)\end{aligned}$$

So at time $t = 2$, $W_2 \sim \mathcal{N}(0, 2(1 - \frac{2}{3}))$. The standard deviation is $\sqrt{2/3}$, so:

$$P(W_2 > 1) = 1 - \Phi\left(\frac{1}{\sqrt{2/3}}\right) = 1 - \Phi\left(\sqrt{\frac{3}{2}}\right) \approx 0.110$$

Exercise 8.12

Show that Brownian motion with drift has independent and stationary increments.

Solution: Let $X_t = \mu t + \sigma B_t$ be the drift process.

Stationary increments: We show that for all $s, t > 0$, $X_{t+s} - X_s$ has the same distribution as X_t .

$$\begin{aligned}X_{t+s} - X_s &= \mu(t+s) + \sigma B_{t+s} - [\mu s + \sigma B_s] \\ &= \mu t + \sigma(B_{t+s} - B_s) \\ &= \mu t + \sigma B_t\end{aligned}$$

The last equality is a property of standard Brownian motion. This is the same distribution as X_t , so the drift process exhibits stationary increments.

Independent increments: We show that for $0 \leq q < r \leq s < t$, $X_t - X_s$ and $X_r - X_q$ are independent random variables.

$$\begin{aligned}\text{Cov}(X_t - X_s, X_r - X_q) &= \text{Cov}(X_t, X_r) - \text{Cov}(X_s, X_r) - \text{Cov}(X_t, X_q) + \text{Cov}(X_s, X_q) \\ &= \sigma^2 [\min(t, r) - \min(s, r) - \min(t, q) + \min(s, q)] \\ &= \sigma^2 [r - r - q + q] \\ &= 0\end{aligned}$$

Each non-overlapping interval is uncorrelated, so the drift process exhibits independent increments.

Exercise 8.13

Let $(X_t)_{t \geq 0}$ denote a Brownian motion with drift μ and variance σ^2 . For $0 < s < t$, find $E(X_s X_t)$.

Solution: By definition, $\text{Cov}(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t)$, so $E(X_s X_t) = \text{Cov}(X_s, X_t) + E(X_s)E(X_t) = \sigma^2 \min(s, t) + (\mu t)(\mu s) = \sigma^2 s + \mu^2 s t = s(\sigma^2 s + \mu^2 t)$.

Exercise 8.14

A Brownian motion with drift parameter $\mu = -1$ and variance $\sigma^2 = 4$ starts at $x = 1.5$. Find the probability that the process is positive at $t = 3$.

Solution: Let $X_t = -t + 2B_t + 1.5$, where B_t is a standard Brownian motion started at the origin. This corresponds to the drift process in the problem statement.

$$P(X_3 > 0) = P(-3 + 2B_3 + 1.5 > 0) = P(2B_3 > 3 - 1.5) = P\left(B_3 > \frac{3}{4}\right)$$

Because $B_3 \sim \sqrt{3}Z$, this probability is $1 - \Phi\left(\frac{3}{4\sqrt{3}}\right) = \Phi\left(-\frac{\sqrt{3}}{4}\right) \approx 0.097$.

Exercise 8.15

See Example 8.11 on using Brownian motion to model the home team advantage in basketball. In Stern (1994), Table 8.3 is given based on the outcomes of 493 basketball games played in 1992.

Quarter	Variable	Mean	Standard Deviation
1	$X_{0.25}$	1.41	7.58
2	$X_{0.50} - X_{0.25}$	1.57	7.40
3	$X_{0.75} - X_{0.50}$	1.51	7.30
4	$X_1 - X_{0.75}$	0.22	6.99

Here, X_t is the difference between the home team's score and the visiting team's score after $t(100)$ percent of the game is played. The data show the mean and standard deviation of those differences at the end of each quarter. Why might the data support the use of a Brownian motion model? What aspects of the data give reason to doubt the Brownian motion model?

Solution: A process is Brownian motion iff there is 0 initial value, independent increments, Gaussian increments, and almost-surely continuous paths.

That the paths are almost surely continuous is trivial. It's also reasonable to assume that the initial value is 0, due to the rules of basketball. However, the fourth quarter mean differs from the other quarters, which suggests that this is not a Brownian motion process, even with drift.

Exercise 8.16

Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be independent, standard Brownian motions. Show that $Z_t = a(X_t - Y_t)$ defines a standard Brownian motion for some a . Find a .

Solution: Since the difference of a.s. continuous paths is also almost surely continuous, the process Z_t is a.s. continuous. The mean function is also 0 since the linear combination of Gaussian r.v.s is Gaussian. Letting $0 < s < t$, the covariance function is:

$$\begin{aligned} \text{Cov}(Z_s, Z_t) &= \text{Cov}(a(X_s - Y_s), a(X_t - Y_t)) \\ &= a^2[\text{Cov}(X_s, X_t) - \text{Cov}(X_s, Y_t) - \text{Cov}(Y_s, X_t) + \text{Cov}(Y_s, Y_t)] \\ &= a^2[\min(s, t) + \min(s, t)] \\ &= 2a^2 \min(s, t) \end{aligned}$$

If $a = \pm 1/\sqrt{2}$, then the covariance function is $\min(s, t)$.

Exercise 8.17

For $a > 0$, show that for standard Brownian motion the first hitting time T_a has the same distribution as $1/X^2$, where X is a normal random variable with mean 0 and variance $1/a^2$.

Solution: Let $T_a = 1/X^2$. The CDF is:

$$P(T_a \leq t) = P(1/X^2 \leq t) = P(X^2 \geq 1/t) = P(X \geq t^{-1/2}) + P(X \leq -t^{-1/2}) = 2P(X \leq -t^{-1/2})$$

Differentiating with respect to t :

$$\begin{aligned} P(T_a = t) &= \frac{d}{dt} 2P(X \leq -t^{-1/2}) \\ &= 2 \cdot 1/2t^{-3/2} \cdot f_X(-t^{1/2}) \\ &= \frac{1}{\sqrt{t^3}} \cdot \frac{1}{\sqrt{2\pi/a^2}} \exp\left(-\frac{(-t^{-1/2})^2}{2/a^2}\right) \\ &= \frac{|a|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \end{aligned}$$

This corresponds to the first hitting time distribution.

Exercise 8.18

Show that the first hitting time T_a has the same distribution as $a^2 T_1$.

Solution: $a^2 T_1$ is the time for a process to hit 1, scaled by a^2 . Assuming that the CDFs are equal:

$$P(T_a \leq t) = P(a^2 T_1 \leq t) = P(T_1 \leq t/a^2)$$

Differentiating with respect to t :

$$\begin{aligned} \frac{\partial}{\partial t} P(T_a \leq t) &= \frac{\partial}{\partial t} P(T_1 \leq t/a^2) \\ &= \frac{1}{a^2} f_{T_1}(t/a^2) \\ &= \frac{1}{a^2} \cdot \frac{1}{\sqrt{2\pi(t/a^2)^3}} \exp\left(-\frac{1^2}{2t/a^2}\right) \\ &= \frac{1}{a^2} \cdot \frac{|a^3|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \\ &= \frac{|a|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \end{aligned}$$

Exercise 8.19

Find the mean and variance of the maximum value of standard Brownian motion on $[0,t]$.

Solution: Let M_t be the maximum of the standard Brownian motion on $[0,t]$, and let $a > 0$. We can split the event $\{M_t > a\}$ into 2 possibilities: either B_t is greater than a at time t or B_t is less than a at time t . If at time t the Brownian motion exceeds a , then it's a certainty that $M_t > a$, so $\{B_t > a\}$ implies $\{M_t > a\}$, so:

$$\{M_t > a\} = \{M_t > a, B_t > a\} \cup \{M_t > a, B_t \leq a\} = \{B_t > a\} \cup \{M_t > a, B_t \leq a\}$$

In the second event, consider a time $0 < t^* \leq t$ where B_{t^*} hit a , then B_t finishes at t under the level a . By the reflection principle, the path of the Brownian motion on the interval $[t^*, t]$ that finishes below a corresponds

to a path that finishes at least a at time t , which gives $\{M_t > a, B_t \leq a\}$ as corresponding to the same event $\{\tilde{B}_t \geq a\} = B_t > a$.

$$\begin{aligned} P(M_t > a) &= P(B_t > a) + P(B_t < a) = 2P(B_t > a) \\ P(M_t > a) &= 2 \int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \\ P(M_t = a) &= \frac{\partial}{\partial a}[1 - P(M_t > a)] = -2 \frac{\partial}{\partial a} \int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= -2 \left[0 - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \right] \\ &= \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right), x > 0 \end{aligned}$$

This is the distribution for the absolute value of the Brownian motion path, $|B_t|$.

An alternative formulation draws on the already calculated hitting time distribution T_a . The event $\{M_t > a\}$ corresponds to the hitting time being prior to t , so this event is $\{T_a < t\}$:

$$P(M_t > a) = P(T_a < t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds$$

Let $a^2/s \Rightarrow x^2/t$. Then $s = (a^2 t)x^{-2} \Rightarrow ds = -2(a^2 t)x^{-3} dx$, and $x^2 = a^2 t/s$,

$$\begin{aligned} x^+ &= \sqrt{\lim_{s \rightarrow t} \frac{a^2 t}{s}} = \sqrt{a^2} = a \\ x^- &= \sqrt{\lim_{s \rightarrow 0} \frac{a^2 t}{s}} = \infty \end{aligned}$$

Substituting:

$$\begin{aligned} P(M_t > a) &= \int_{x=\infty}^{x=a} \frac{a}{\sqrt{2\pi}} (a^2 t x^{-2})^{-3/2} \exp\left(-\frac{x^2}{2t}\right) \cdot -2a^2 t x^{-3} dx \\ &= -2 \int_{x=\infty}^{x=a} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2t}\right) \cdot a^{-3} t^{-3/2} x^3 a^2 t x^{-3} dx \\ &= 2 \int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \end{aligned}$$

This is the same expression as the CDF from above.

$$\begin{aligned} E(M_t) &= E(|B_t|) \\ &= \int_0^\infty \sqrt{\frac{2}{\pi t}} x \exp\left(-\frac{x^2}{2t}\right) dx \\ &= \sqrt{\frac{2}{\pi t}} \int_0^\infty x \exp\left(-\frac{x^2}{2t}\right) dx \end{aligned}$$

Let $u = x^2/2, du = xdx$:

$$\begin{aligned} E(M_t) &= \sqrt{\frac{2}{\pi t}} \int_0^\infty \exp(-u/t) du \\ &= -t \sqrt{\frac{2}{\pi t}} \exp(-u/t)|_0^\infty \\ &= \sqrt{\frac{2t}{\pi}} \end{aligned}$$

To get the variance:

$$\begin{aligned}
\text{Var}(M_t) &= \text{Var}(|B_t|) \\
&= E(|B_t|^2) - E(|B_t|)^2 \\
&= E(B_t^2) - E(|B_t|)^2 \\
&= [\text{Var}(B_t) + E(B_t)^2] - E(|B_t|)^2 \\
&= t - \frac{2t}{\pi} \\
&= t \left(1 - \frac{2}{\pi}\right)
\end{aligned}$$

Exercise 8.20

Use the reflection principle to show

$$P(M_t \geq a, B_t \leq a - b) = P(B_t \geq a + b), \text{ for } a, b > 0.$$

Solution: Since $T_a : \inf\{t \geq 0 : B_t \geq a\}$ is a hitting time, we can apply the reflection principle.

Consider the process

$$\tilde{B}_t = \begin{cases} B_t & t < T_a \\ 2a - B_t & t \geq T_a \end{cases}$$

Consider the case that this process hits level a by time t and is at most $a - b$ at time t . Then, the event $\{M_t \geq a, B_t \leq a - b\}$ corresponds to the event $\{\tilde{B}_t \geq a + b\}$ (which is the same distance b from line a , reflected the other direction).

Exercise 8.21

From standard Brownian motion, let X_t be the process defined by

$$X_t = \begin{cases} B_t, & \text{if } t < T_a, \\ a, & \text{if } t \geq T_a \end{cases}$$

where T_a is the first hitting time of $a > 0$. The process $(X_t)_{t \geq 0}$ is called *Brownian motion absorbed at a* . The distribution of X_t has discrete and continuous parts.

(a) Show

$$P(X_t = a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

Solution:

$$P(X_t = a) = P(X_t = a | T_a > t)P(T_a > t) + P(X_t = a | T_a \leq t)P(T_a \leq t) = 0 \cdot P(T_a > t) + 1 \cdot P(T_a \leq t) = P(T_a \leq t) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

(b) For $x < a$, show

$$P(X_t \leq x) = P(B_t \leq x) - P(B_t \leq x - 2a) = \frac{1}{\sqrt{2\pi t}} \int_{x-2a}^x e^{-z^2/2t} dz$$

Solution: