Chapter 1

2025-09-28

Exercises

Exercise 1.1

Exercise 1.11

Consider the gambler's ruin process where at each wager, the gambler wins with probability p and loses with probability $q = 1 - p$. The gambler stops when reaching \$n or losing all their money. If the gambler starts with \$k, with $0 < k < n$, find the probability of eventual ruin. See example 1.10.

Solution: Let p_k be the probability of winning the overall game with current wallet \$k, and q_k be corresponding probability of losing. At current wallet \$k, the chance of winning restarts at the next step up or down, so we condition on the gambler winning or losing this round (LOTP).

$$p_k = p \cdot p_{k+1} + q \cdot p_{k-1} p \cdot p_k + q \cdot p_k = p \cdot p_{k+1} + q \cdot p_{k-1} p(p_{k+1} - p_k) = q(p_k - p_{k-1}) p_{k+1} - p_k = \frac{q}{p}(p_k - p_{k-1})$$

Telescoping and multiplying both sides by p/q:

$$p_1 - p_0 = \frac{p}{q}(p_2 - p_1) = \left(\frac{p}{q}\right)^2 (p_3 - p_2) = \cdots = \left(\frac{p}{q}\right)^{n-1} (p_n - p_{n-1})$$

At the lower extreme, $p_1 - p_0 = p_1$ since $p_0 = 0$. $p_1 = \frac{p}{q}(p_2 - p_1)$, so $p_1 + \frac{p}{q} p_1 = p_2$ or $p_2 = \frac{1}{q} p_1$. Then,

Exercise 1.12

In n rolls of a fair die, let X be the number of times 1 is rolled, and Y the number of times 2 is rolled. Find the conditional distribution of X given $Y = y$.

Solution: Since all the 2s have been counted, $X \mid Y$ can take on values $\{1, 3, 4, 5, 6\}$ with equal probability. There are $n - y$ remaining rolls, and the probability of getting a 1 is 1/5, so $X \mid Y = y \sim \text{Binom}(n - y, 1/5)$. Note that the support for $X \mid Y = y$ are the integers $0 \le y \le n$, as $X \mid Y = y$ is Binomial.

Exercise 1.13

Random variables X and Y have joint density function

$$f(x, y) = 3y, \quad \text{for } 0 < x < y < 1$$

(a) Find the conditional density of Y given $X = x$.

Solution: The marginal density:

$$f(x) = \int_Y f(x, y) dy = \int_x^1 3y dy = \left.\frac{3y^2}{2}\right|_x^1 = \frac{3 - 3x^2}{2}$$

The conditional density:

$$f(y \mid x) = \frac{f(x, y)}{f(x)} = \frac{3y}{(3 - 3x^2)/2} = \frac{2y}{1 - x^2} \text{ for } 0 < x < y < 1$$

(b) Find the conditional density of Y given $X = x$. Describe the conditional distribution.

Solution: The marginal distribution:

$$f(y) = \int_X f(x,y)dx = \int_0^y 3y dx = 3yx|_0^y = 3y^2$$

The conditional distribution:

$$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{3y}{3y^2} = \frac{1}{y} \text{ for } x < y < 1$$

Note that this conditional distribution is a function of y, and does not depend on x except for the bounds. Once Y has been fixed, then X can take on any value with uniform probability across the region $0 < x < y$.

Exercise 1.14

Random variables X and Y have joint density function

$$f(x,y) = 4e^{-2x}, \quad \text{for } 0 < y < x < \infty$$

(a) Find the conditional density of X given $Y = y$.

Solution: The marginal density $f_Y(y)$ is:

$$f_Y(y) = \int_X f(x,y)dx = \int_y^\infty 4e^{-2x}dx = -2e^{-2x}\Big|_y^\infty = 2e^{-2y}$$

The conditional density is:

$$f_{X|Y=y}(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)}$$

(b) Find the conditional density of Y given $X = x$. Describe the conditional distribution.

Solution: The marginal density:

$$f_X(x) = \int_Y f(x,y)dy = \int_0^x 4e^{-2x}dy = 4ye^{-2x}\Big|_0^x = 4xe^{-2x}$$

The conditional density:

$$f_{Y|X=x}(y \mid x) = \frac{f(x,y)}{f(x)} = \frac{4e^{-2x}}{4xe^{-2x}} = \frac{1}{x}$$

This is a uniform distribution. Given that X is a certain value, y can take any value between 0 and x, so this is a uniform distribution on the interval (0, x) which is captured in the original limits of the joint density function.

Exercise 1.15

Let X and Y be uniformly distributed on the disk of radius 1 centered at the origin. Find the conditional distribution of Y given $X = x$.

Solution: The joint distribution is:

$$f_{X,Y}(x,y) = \frac{1}{\pi} \text{ for } x^2 + y^2 \leq 1$$

From the definition of conditional probability:

$$f_{Y|X=x}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

The marginal distribution $f_X(x)$ can be found by integrating out the Y variable from the joint distribution

$$f_X(x) = \int_Y f_{X,Y}(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy = \frac{2\sqrt{1-x^2}}{\pi}$$

The conditional distribution is therefore:

$$f_{Y|X=x}(y \mid x) = \frac{2\sqrt{1-x^2}/\pi}{1/\pi} = 2\sqrt{1-x^2} \text{ for } -1 \leq x \leq 1$$

Exercise 1.16

A poker hand consists of five cards drawn from a standard 52-card deck. Find the expected number of aces in a poker hand given that the first card drawn is an ace.

Solution: Let X be the total number of aces drawn and A be the event that the first card is an ace. If A, then there are 3 aces to be drawn in the remaining 4 cards, so $X \mid A \sim \text{HGeom}(w = 3, b = 48, n = 4)$, with an additional ace at the beginning. Therefore,

$$E(X \mid A) = 1 + 4 \cdot \frac{3}{52} \approx 1.23$$

Exercise 1.17

Let X be a Poisson random variable with $\lambda = 3$. Find $E(X \mid X > 2)$.

Solution: The expression is:

$$E(X \mid X > 2) = \frac{\sum_{x=3}^{\infty} x \cdot P(X = x)}{P(X > 2)}$$

The numerator is:

$$\begin{aligned}
\sum_{x=3}^{\infty} x \cdot P(X = x) &= \underbrace{\sum_{x=0}^{\infty} x \cdot P(X = x)}_{=E(X)} - 0 \cdot P(X = 0) - 1 \cdot P(X = 1) - 2 \cdot P(X = 2) \\
&= 3 - P(X = 1) - 2P(X = 2) \\
&= 3 - \frac{3^1 e^{-3}}{1!} - 2 \cdot \frac{3^2 e^{-3}}{2!} \\
&= 3 - e^{-3}(3 + 9) \\
&= 3 - 12e^{-3}
\end{aligned}$$

The denominator is:

$$\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\
&= 1 - \frac{3^0 e^{-3}}{0!} - \frac{3^1 e^{-3}}{1!} - \frac{3^2 e^{-3}}{2!} \\
&= 1 - e^{-3}(1 + 3 + 9/2) \\
&= 1 - e^{-3}(17/2)
\end{aligned}$$

Combining both, we have:

$$E(X \mid X > 2) = \frac{3 - 12e^{-3}}{1 - 8.5e^{-3}}$$

Exercise 1.18

From the definition of conditional expectation given an event, show that

$$E(I_B \mid A) = P(B \mid A)$$

Solution:

$$E(I_B \mid A) = \frac{E(I_B I_A)}{P(A)} = \frac{E(I_{A\cap B})}{P(A)} = \frac{P(A\cap B)}{P(A)} = P(B \mid A)$$

Exercise 1.19 NOT DONE

Exercise 1.20

A fair coin is flipped repeatedly. (a) Find the expected number of flips needed to get three heads in a row.

Solution: These events partition the sample space: T, HT, HHT, and HHH. Let X be the flips to get 3 in a row.

$$\begin{aligned} E(X) &= E(X \mid T)P(T) + E(X \mid HT)P(HT) + E(X \mid HHT)P(HHT) + E(X \mid HHH)P(HHH) \\ &= E(X \mid T)\frac{1}{2} + E(X \mid HT)\frac{1}{4} + E(X \mid HHT)\frac{1}{8} + E(X \mid HHH)\frac{1}{8} \end{aligned}$$

If we get a tails, we restart. For simplicity, let $a = E(X)$. So $E(X \mid T) = 1 + E(X)$, $E(X \mid HT) = 2 + E(X)$, $E(X \mid HHT) = 3 + E(X)$, and $E(X \mid HHH) = 3$:

$$\begin{aligned} a &= \frac{1+a}{2} + \frac{2+a}{4} + \frac{3+a}{8} + \frac{3}{8} \\ a &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} + a\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\ \frac{a}{8} &= \frac{7}{4} \\ a &= 14 \end{aligned}$$

Exercise 1.21

Let T be a nonnegative, continuous random variable. Show

$$\int_0^\infty P(T > t)dt = \int_0^\infty \left(\int_t^\infty f_T(x)dx\right) dt$$

The inner integral runs from t to ∞, and the outer integral from 0 to ∞. Swapping the order of integration, the inner integral now runs from t to ∞, while the outer integral runs from 0 to x (note that the bounds of integration coincide at the line $x = t$ if we imagine x axis on the horizontal axis and t on the vertical axis).

$$\int_0^\infty \left(\int_t^\infty f_T(x)dx\right) dt = \int_0^\infty \left(\int_0^x f_T(x)dt\right) dx = \int_0^\infty f_T(x)\left(\int_0^x dt\right) dx = \int_0^\infty f_T(x)x dx = \int_0^\infty x f_T(x)dx = E(T)$$

Exercise 1.22

Let $E(Y \mid X)$ when (X, Y) is uniformly distributed on the following regions. (a) The rectangle $[a, b] \times [c, d]$.

Solution: Since this is a rectangle and (X, Y) is uniformly distributed, there is an equal likelihood of Y appearing between $[c, d]$ regardless of where X is, so $E(Y) = E(Y \mid X) = \frac{c+d}{2}$.

(b) The triangle with vertices (0,0),(1,0),(1,1).

Solution: The joint distribution is:

$$f_{X,Y}(x,y) = 2 \text{ for } 0 < y < x < 1$$

The marginal distribution of X is:

$$f_X(x) = \int_Y 2dy = \int_0^x 2dy = 2x$$

The conditional distribution of $Y \mid X$ is:

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$$

$$E(Y \mid X) = \int_Y y \cdot f_{Y|X}(y \mid x)dy = \int_0^x y \cdot \frac{1}{x}dy = \int_0^x \frac{y}{x}dy = \left.\frac{y^2}{2x}\right|_0^x = \frac{x^2}{2x} = \frac{x}{2}$$

(c) The disc of radius 1 centered at the origin.

Solution: The joint distribution is:

$$f_{X,Y}(x,y) = \frac{1}{\pi} \text{ for } x^2 + y^2 \le 1$$

The marginal distribution of X:

$$\begin{aligned} f_X(x) = \int_Y f_{X,Y}(x,y)dy &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \\ &= \left.\frac{y}{\pi}\right|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\ &= \frac{2\sqrt{1-x^2}}{\pi} \end{aligned}$$

Then the conditional distribution of $Y \mid X$ is:

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/\pi}{2\sqrt{1-x^2}/\pi} = \frac{1}{2\sqrt{1-x^2}}$$

The conditional expectation $E(Y \mid X)$:

$$E(Y \mid X) = \int_Y y \cdot f_{Y|X}(y \mid x)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y}{2\sqrt{1-x^2}}dy = \left.\frac{y^2}{2\sqrt{1-x^2}}\right|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1-x^2-(1-x^2)}{2\sqrt{1-x^2}} = 0$$

Exercise 1.23

Let $X_1, X_2, \ldots$ be an i.i.d sequence random variables with common mean μ. Let $S_n = X_1 + \cdots + X_n$ for $n \ge 1$.

(a) Find $E(S_M \mid S_n)$, for $m \le n$.

Solution: Since $S_m = X_1 + \cdots + X_m$ and $S_n = S_m + X_{m+1} + \cdots + X_n$:

$$\begin{aligned} E(S_m \mid S_n) &= E(S_n - X_n - \cdots - X_{m+1} \mid S_n) \\ &= E(S_n \mid S_n) - E(X_n \mid S_n) - \cdots - E(X_{m+1} \mid S_n) \\ &= S_n - E(X_n \mid S_n) - \cdots - E(X_{m+1} \mid S_n) \end{aligned}$$

By symmetry, $E(X_i \mid S_n) = S_n/n$, so $E(S_m \mid S_n) = S_n - S_n(n-m)/n = S_n(m/n)$.

(b) Find $E(S_m \mid S_n)$, for $mgen$.

Solution:

$$E(S_m \mid S_n) = E(S_n + X_{n+1} + \cdots + X_m \mid S_n) = S_n + E(X_{m+1} \mid S_n) + \cdots + E(X_m \mid S_n)$$

Since each X_i is independent of S_n for $i > n$, then $E(S_m \mid S_n) = S_n + (m - n)\mu$.

Exercise 1.24

Prove the law of total expectation $E(Y) = E(E(Y \mid X))$ for the continuous case.

Solution:

$$E(E(Y \mid X)) = \int_x E(Y \mid X = x) f_X(x) dx = \int_x \left(\int_y y f_{Y|X}(y \mid x) dy \right) f_X(x) dx$$

The conditional distribution $f_{Y|X} = \frac{f_{x,y}(x,y)}{f_X x}$.

$$\begin{aligned}
&\int_x \left(\int_y y f_{Y|X}(y \mid x) dy \right) f_X(x) dx \\
&= \int_x \left(\int_y y \frac{f_{X,Y}(x,y)}{f_X(x)} dy \right) f_X(x) dx \\
&= \int_x \left(\int_y y f_X(x) \frac{f_{X,Y}(x,y)}{f_X(x)} dy \right) dx \\
&= \int_x \int_y y f_{X,Y}(x,y) dy dx \\
&= \int_y y \left(\int_x f_{X,Y}(x,y) dx \right) dy \\
&\int_y y f_Y(y) dy \\
&= E(Y)
\end{aligned}$$

Exercise 1.25

Let X and Y be independent exponential random variables with respective parameters 1 and 2. Find $P(X/Y < 3)$ by conditioning.

Solution:

$$
\begin{aligned}
P(X/Y < 3) &= \int_0^\infty P(X/Y < 3 \mid Y = y) f_Y(y) dy \\
&= \int_0^\infty P(X/y < 3 \mid Y = y) f_Y(y) dy \\
&= \int_0^\infty P(X < 3y) P(Y = y) dy \\
&= \int_0^\infty (1 - e^{-3y}) 2e^{-2y} dy \\
&= \int_0^\infty 2e^{-2y} - 2e^{-5y} dy \\
&= -e^{-2y}\big|_0^\infty + \frac{2}{5}\, e^{-5y}\big|_0^\infty \\
&= 1 - \frac{2}{5} \\
&= \frac{3}{5}
\end{aligned}
$$

Exercise 1.26

The density of X is $f(x) = xe^{-x}$, for $x > 0$. Given $X = x$, Y is uniformly distributed on (0,x). Find $P(Y < 2)$ by conditioning on X.

Solution:

We have $f_{Y|X}(y \mid x) = 1/x$, and $F_{Y|X}(y \mid x) = \frac{y}{x}$ for $0 < y < x$, and 1 for $y \geq x$.

If $X < 2$, then Y will always be between 0 and 2, so $P(Y < 2 \mid X < 2) = 1$. If $X \geq 2$, then $P(Y < 2 \mid X < 2) = 2/x$.

$$
\begin{aligned}
P(Y < 2) &= \int_0^\infty P(Y < 2 \mid X = x) f_X(x) dx \\
&= \int_0^\infty P(Y < 2 \mid X = x) xe^{-x} dx \\
&= \underbrace{\int_0^2 1 \cdot xe^{-x} dx}_{P(0<X<2)} + \int_2^\infty y/x \cdot xe^{-x} dx \\
&= + \int_2^\infty 2e^{-x} dx \\
&= (1 - e^{-2}) - 2e^{-x} \mid_2^\infty \, dx \\
&= 1 - e^{-2} + 2e^{-2}
\end{aligned}
$$

```
exercise_1_26 <- function() {
  x <- rexp(1000, 1)
  y <- runif(1000, rep(0, 1000), x)
  mean(y < 2)
}
```

Exercise 1.27

A restaurant receives N customers per day, where N is a random variable with mean 200 and standard deviation 40. The amount spent by each customer is normally distributed with mean \$15 and standard deviation \$3. The amounts that customers spend are independent of each other and independent of N. Find the mean and standard deviation of the total amount spent at the restaurant per day.

Solution: Let X_i be the amount that the i customer spends, and $S = X_1 + \cdots + X_N$.

$$
\begin{aligned}
E(S) &= E(E(S \mid N)) = E(E(X_1 + \cdots + X_N \mid N)) = E(NX_i) = E(N)E(X_i) = 200 \cdot 15 = 3000 \\
Var(S) &= E(Var(S \mid N)) + Var(E(S \mid N)) \\
&= E(Var(X_1 + \cdots + X_N \mid N)) + Var(E(X_1 + \cdots + X_N \mid N)) \\
&= E(Var(X_1 \mid N) + \cdots + Var(X_N \mid N)) + Var(E(X_1 \mid N) + \cdots + E(X_N \mid N)) \\
&= E(N \cdot 3^2) + Var(N \cdot 15) \\
&= 200 \cdot 3^2 + 15^2 \cdot 1600
\end{aligned}
$$

So the standard deviation is $\sqrt{361,800} = 30\sqrt{201} \approx 601.50$.

```
nsim <- 1000
Ns <- rnbinom(nsim, size = 25/(1-0.125), prob = 0.125)
Ss <- sapply(Ns, function(n) sum(rnorm(n, mean = 15, sd = 3)))
mean(Ss) # should be 3000
```

```
## [1] 3016.168
```

```
sd(Ss) # should be around 601.5
```

```
## [1] 603.4623
```

Exercise 1.28

On any day, the number of accidents on the highway has a Poisson distributions with parameter Λ. The parameter Λ varies from day to day and is itself a random variable. Find the mean and variance of the number of accidents per day when Λ is uniformly distributed on (0,3).

Solution: Let the total number of accidents be T, thus $T \mid \Lambda \sim \text{Pois}(\Lambda)$.

$$
\begin{aligned}
E(T) &= E(E(T \mid \Lambda)) = E(\Lambda) = \frac{3}{2} \\
Var(T) &= E(Var(T \mid \Lambda)) + Var(E(T \mid \Lambda)) \\
&= E(\Lambda) + Var(\Lambda) \\
&= \frac{3}{2} + \frac{3^2}{12} = \frac{27}{12} \\
&= \frac{9}{4}
\end{aligned}
$$

Exercise 1.29

If X and Y are independent, does $Var(Y \mid X) = Var(Y)$?

Solution: Yes, since $Var(Y \mid X) = E(Y^2 \mid X) - E(Y \mid X)^2 = E(Y^2) - E(Y)^2 = Var(Y)$.

Exercise 1.30

Assume that $Y = g(X)$ is a function of X. Find simple expressions for (a) $E(Y \mid X)$.

Solution: $E(g(X) \mid X) = g(X)$

(b) $Var(Y \mid X)$.

Solution: $Var(Y \mid X) = E(Y^2 \mid X) - E(Y \mid X)^2 = E(g(X)^2 \mid X) - E(g(X) \mid X)^2 = g(X)^2 - g(X)^2 = 0$.

Exercise 1.31 NOT DONE

Consider a sequence of i.i.d. Bernoulli trials with success parameter p. Let X be the number of tirals needed until the first success occurs. Then, X has a geometric distribution with parameter p. Find the variance of X by conditioning on the first trial.

Solution: Let T_1 be the outcome of the first trial. If the first trial succeeds, then $X = 1$. Otherwise, if the trial fails, then we restart the game. The cases are:

$$X = \begin{cases} 0 & \text{with probability } p \\ 1 + X & \text{with probability } 1 - p \end{cases}$$

$$Var(X) = E(Var(X \mid T_1)) + Var(E(X \mid T_1)) = E()$$

Exercise 1.32

R: Simulate flipping three fair coins and counting the number of heads X.

(a) Use your simulation to estimate $P(X = 1)$ and $E(X)$.

Solution:

```
nsim <- 10000
heads_from_fair_coin <- rbinom(nsim, 3, 0.5)
mean(heads_from_fair_coin == 1) # P(X=1)
```

```
## [1] 0.3776
```

```
mean(heads_from_fair_coin) # E(X)
```

```
## [1] 1.4913
```

(b) Modify the above to allow for a biased coin where $P(\text{Heads}) = 3/4$.

Solution:

```
nsim <- 10000
p_biased_coin <- 0.75
heads_from_biased_coin <- rbinom(nsim, 3, p_biased_coin)
mean(heads_from_biased_coin == 1) # P(X=1)
```

```
## [1] 0.1394
```

```
mean(heads_from_biased_coin) # E(X)
```

```
## [1] 2.2598
```

Exercise 1.33

R: Cards are drawn from a standard deck, with replacement, until an ace appears. Simulate the mean and variance of the number of cards required.

Solution: