Chapter 2

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Exercises

Exercise 2.1

A Markov chain has transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix} \end{array}$$

with initial distribution $\alpha = (0.2, 0.3, 0.5)$. Find the following: (a) $P(X_7 = 3|X_6 = 2)$

Solution: By time homogeneity,

$$P(X_7 = 3|X_6 = 2) = P(X_1 = 3|X_0 = 2) = P_{23} = 0.6$$

(b) $P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3)$

Solution: By the Markov property and time homogeneity,

$$\begin{aligned} P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3) &= P(X_9 = 2|X_7 = 3) \\ &= P(X_2 = 2|X_0 = 3)(P^2)_{23} \\ &= 0.54 \end{aligned}$$

(c) $P(X_0 = 3|X_1 = 1)$

Solution: By Bayes' rule,

$$\begin{aligned} P(X_0 = 3|X_1 = 1) &= \frac{P(X_1 = 1|X_0 = 3)P(X_0 = 3)}{P(X_1 = 1)} \\ &= \frac{P_{31}\alpha_3}{(\alpha P)_1} \\ &= \frac{0.3 \cdot 0.5}{0.17} \\ &= \frac{15}{17} \approx 0.88 \end{aligned}$$

(d) $E(X_2)$

Solution:

$$E(X_2) = \sum_k kP(X_2 = k) = \sum_k k(\alpha P^2)_k = 2.363$$

Exercise 2.2

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$