MATH 245 PRACTICE EXAM #2

1. Suppose A, B, C and X are all invertible matrices, and $A(X+B) = CA$. Solve this equation for X. Show your work.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates points counterclockwise through an angle of $\pi/6$ radians about the origin, while at the same time moves points to three times their original distance from the origin. Find the standard matrix of T.

3. Suppose $\vec{v}$ is a vector in $\mathbb{R}^n$ such that $\vec{x} = \vec{v}$ gives a solution to the matrix equation $A\vec{x} = \vec{b}$.

   (a) Show that if $\vec{w}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{v} + \vec{w}$ is a solution to $A\vec{x} = \vec{b}$.

   (b) Show that if $\vec{u}$ is another solution to $A\vec{x} = \vec{b}$, then there is a vector $\vec{w}$ such that $A\vec{w} = \vec{0}$ and $\vec{u} = \vec{v} + \vec{w}$.

4. In a certain region about 4% of the city's population moves to the suburbs each year and the remaining 96% remain in the city. Moreover, 3% of the suburban population moves to the city, and 97% remain in the suburbs. In 2015 there were 800,000 residents in the city and 500,000 in the suburbs.

   (a) Set up the transition matrix T for this problem.

   (b) Use T to determine the city and suburban populations two years later, in 2017.

5. Prove that the set of 2×2 invertible matrices is not closed under either addition or scalar multiplication.

6. Prove that if A is invertible and $c \neq 0$, then cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

7. Prove that if A is idempotent (that is $A^2 = A$) then A^t is also idempotent.

8. Evaluate the following determinant $\begin{vmatrix} 1 & 2 & -1 & 0 \\ 1 & 4 & 2 & 1 \\ -1 & 2 & 6 & 6 \\ 2 & 2 & -4 & +2 \end{vmatrix}$.

9. Let $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$.

   (a) Determine A^{-1} using the method of Gauss-Jordan elimination.

   (b) Use A^{-1} to solve the linear system

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 3 \\ -x_1 + x_2 + 3x_3 &= 2 \\ x_1 - x_2 - 4x_3 &= 1 \end{aligned} \tag{1}$$

10. Mark each of the following statements as TRUE or FALSE.

    (a) If A and B are $m \times n$, then both AB^t and A^tB are defined.

    (b) If A and B are $n \times n$, then $(A+B)(A-B) = A^2 - B^2$.

    (c) If $AB = C$ and C has two columns, then A has two columns.

    (d) If two rows of a 3×3 matrix A are the same, then $|A| = 0$.

    (e) If A is 2×2 and $|A| = 3$, then $|2A^{-1}| = 4/3$.

11. Prove that if A and C are $n \times n$ matrices with C invertible, then $|C^{-1}AC| = |A|$.

12. An $n \times n$ matrix is called nilpotent if $A^2 = 0$. Prove that if A and B are nilpotent and $AB = BA$, then AB is nilpotent.

13. A square matrix A is called antisymmetric if $A = -A^t$.

    (a) Prove that for any square matrix B, the matrix $B - B^t$ is antisymmetric.

    (b) Prove that if an $n \times n$ matrix A is antisymmetric and n is odd, then $|A| = 0$.

14. Find x such that $2\begin{bmatrix} 2x & x \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$.

15. Determine whether the following transformations are linear. If it is linear, prove it. If it is not, show why not.

    (a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2 \\ 4y \end{bmatrix}$ of $\mathbb{R}^2 \to \mathbb{R}^2$

    (b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x-y \end{bmatrix}$ of $\mathbb{R}^2 \to \mathbb{R}^2$

Answers & Hints (not complete solutions)

1. $X = A^{-1} C A - B$

2. $\begin{bmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ \frac{3}{2} & 3\sqrt{3}/2 \end{bmatrix}$

3a. Show that $A(\vec{v} + \vec{w}) = \vec{b}$

b. Show $A(\vec{u} - \vec{v}) = \vec{0}$

4a. $T = \begin{array}{cc} \text{city} & \text{sub} \\ \left[\begin{matrix} .96 & .03 \\ .04 & .97 \end{matrix}\right. & \left.\right] \end{array} \begin{array}{l} \text{city} \\ \text{sub} \end{array}$

b. $T^2 X_0 = \begin{bmatrix} 767{,}190 \\ 532{,}810 \end{bmatrix} \begin{array}{l} \text{city} \\ \text{sub.} \end{array}$

5. Find 2×2 counterexamples

6. Show $cA(\frac{1}{c}A^{-1}) = I$ and $(\frac{1}{c}A^{-1})cA = I$

7. Use properties of transpose

8. -14

9. $A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$

Find (by hand)

9b. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -31 \\ -26 \\ -3 \end{bmatrix}$

10. a.) True
b.) False
c.) False
d.) True
e.) True

Should know how to justify these answers.

11. Use Thm 3.4 b & c

12. Show $(AB)^2 = 0$

13. a. Show $(B - B^t)^t = -(B - B^t)$
b.) use Thm 3.4 a & c

14. $x = 2$

15a. Not a lin. transf. Show a prop. fails.

b Yes. give proof.