Portfolio Allocation through Optimal Control

Alex Shaw, Johnson Merrell, Walker Hughes, Caleb Dame, Grant White

Portfolio allocation is a common problem in finance and optimization. These routines find vectors of weights that represent the portion of an investor's capital that should be invested in a set of assets. Most portfolio optimization routines seek to minimize portfolio variance given a specified expected return on investments and do not account for an investor's risk tolerance over time. By addressing this problem from an optimal control perspective with wealth as the state variable and portfolio weights as the control, we are able to minimize portfolio variance without bounding total returns from above. Our solution finds the optimal portfolio weights given an investor's initial wealth and risk tolerance over time.

1 Introduction

Portfolio optimization is a well-known problem in the field of mathematical finance and optimization. All of us at some point will tackle this problem, be it planning for retirement, buying a house or sending our children to college. This can be further complicated by the change of one's risk intolerance overtime as he/she nears retirement for example. So how do we allocate capital to maximize returns and minimize risk? And how do we accomplish this overtime as our risk intolerance changes? This project seeks to address the problem of portfolio optimization through optimal control: maximizing final wealth, while minimizing portfolio variance over time and controlling for time-variant risk tolerances.

2 Mathematical Model

Let $\vec{\alpha}(t)$, our control, represent our portfolio allocation at time t, where $\alpha_i(t)$ is the fraction of our wealth in asset i at time t. Let $w(t)$, our state, be our wealth at time t. Since we seek to maximize final wealth while minimizing portfolio variance, we create a cost functional $J[\vec{\alpha}]$

that penalizes variance throughout the time horizon and includes an endpoint cost to maximize final wealth (minimize negative wealth). Our optimal control problem is to minimize

$$J[\vec{\alpha}] = \int_0^{t_f} T(t)\vec{\alpha}^T(t)\Sigma\vec{\alpha}(t)dt - w(t_f)$$

subject to

$$w'(t) = w(t)\vec{r}^T\vec{\alpha}(t), \quad w(0) = w_0, \quad \vec{\alpha}(t) \geq 0, \quad \sum_{i=1}^{n} \alpha_i(t) = 1.$$

Note that $\vec{r}$ is a vector such that r_i is the expected return of asset i, and Σ is the covariance matrix of the assets. This gives the Hamiltonian H to be

$$H(t) = pw' - L = p(t)w(t)\vec{r}^T\vec{\alpha}(t) - T(t)\vec{\alpha}^T(t)\Sigma\vec{\alpha}(t).$$

The equality and inequality constraints modify the Lagrangian to be

$$\mathscr{L} = H + \lambda(t)(\mathbb{1}^T\vec{\alpha} - 1) - \vec{\mu}^T(t)\vec{\alpha}(t)$$

With $\lambda(t)$ and $\vec{\mu}(t)$ being functions of time, this makes the problem nearly impossible to solve. We instead relax the inequality constraints, allowing our portfolio to contain short positions. Likewise, we redefine our control vector $\vec{\alpha}$ in such a way that will force its elements to sum to one, but will not require constrained optimization techniques. These changes are reflected in our simplified optimal control problem below:

$$J[\vec{\alpha}] = \int_0^{t_f} (T(t)\vec{\alpha}^T\Sigma\vec{\alpha})dt - w(t_f)$$

$$w'(t) = w(t)\vec{r}^T\vec{\alpha}, \quad w(0) = w_0$$

$$\alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i.$$

3 Solution

To make the problem more tractable, we also assume all assets are independent, meaning Σ is a diagonal matrix.

Our Hamiltonian is as follows,

$$H(t) = pw(\vec{r}^T\vec{\alpha}) - T(t)\vec{\alpha}^T\Sigma\vec{\alpha}$$
$$= pw(\sum_{i=1}^{n-1}(r_i - r_n)\alpha_i + r_n) - T(t)(\sum_{i=1}^{n-1}\sigma_i^2\alpha_i^2 - 2\sigma_n^2(1 - \sum_{i=1}^{n-1}\alpha_i)^2)$$

with partial derivatives

$$\frac{\partial H}{\partial \alpha_i} = pw(r_i - r_n) - T(t)(2\sigma_i^2\alpha_i - 2\sigma_n^2(1 - \sum_{j=1}^{n-1}\alpha_j)).$$

As you can see this is starting to get messy very quickly. It suffices to say that after long and painful amounts of algebra we end up with the system of linear equations shown below that allow us to solve for $\vec{\alpha}$ in terms of p and w.

$$M = \begin{bmatrix} c_1 & 1 & \dots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & c_{n-1} \end{bmatrix}$$

In other words, M is a matrix of 1's with c_i along the diagonal where

$$c_i = \frac{\sigma_i^2 - \sigma_n^2}{\sigma_n^2}$$

and

$$\vec{b} = \left[\frac{pw(r_i - r_n)}{2T(t)\sigma_n^2} + 1\right]_{i=1}^{n-1}$$

These definitions allow us to solve the following linear equation for $\vec{\alpha}$.

$$\vec{\alpha} = M^{-1}\vec{b}$$

The final step is to append $\alpha_n = 1 - \sum_{i=1}^{n-1}\alpha_i$ to the resulting $\vec{\alpha}$, to bring us back into n dimensions. In practice, we have never run into a situation where M is not invertible.

Requiring all assets to have different variances is probably enough to guarantee invertibility. Further investigation would be necessary to confirm the previous statement.

Our evolution equations are much simpler and are given below.

$$p' = -\frac{\partial H}{\partial w} = -p\vec{r}^T\vec{\alpha}$$

$$p(t_f) = -\frac{D\phi}{Dw(t_f)} = 1$$

$$w' = \frac{\partial H}{\partial p} = w\vec{r}^T\vec{\alpha}$$

$$w(0) = w_0$$

4 Applications

The main feature of our model is its ability to adjust for time-variant personal risk preferences (through our function $T(t)$). We will investigate the effects of different risk intolerances in the subsections below. To solve the equations given in the previous section, we used SciPy's implementation for solving boundary value problems.

4.1 Modeling Risk Preferences

Generally, you are less tolerant to risk as you near the end of your investment period. This makes our choices of monotone increasing functions for $T(t)$ valid choices. Here we compare three different risk intolerance functions. It is worth mentioning that our functional is very sensitive to different T functions. The constants in the following T functions have been fine tuned to give meaningful results.

4.1.1 Linear Intolerance

Let $T(t)$ be given by

$$T(t) = \frac{1}{5}w_0(t+1).$$

A linear intolerance is the most natural starting point to explore the effect of changing intolerance over time. The resulting plots are shown in Section 4.2 below.

4.1.2 Quadratic Intolerance

Let $T(t)$ be given by

$$T(t) = \frac{3}{10}w_0(\frac{t^2}{10}+1).$$

A quadratic intolerance weighs ending portfolio variance a lot while ignoring portfolio variance during the rest of the time period. The resulting plots are shown in Section 4.2 below.

4.1.3 Scaled Sigmoid Intolerance

Let $T(t)$ be given by

$$T(t) = w_0(\frac{3}{1+e^{-(2t-10)}}+\frac{1}{2}).$$

A scaled sigmoid intolerance makes portfolio variance negligible during the first time periods and costly during the ending time periods. The resulting plots are shown in Section 4.2 below.

4.2 Plots

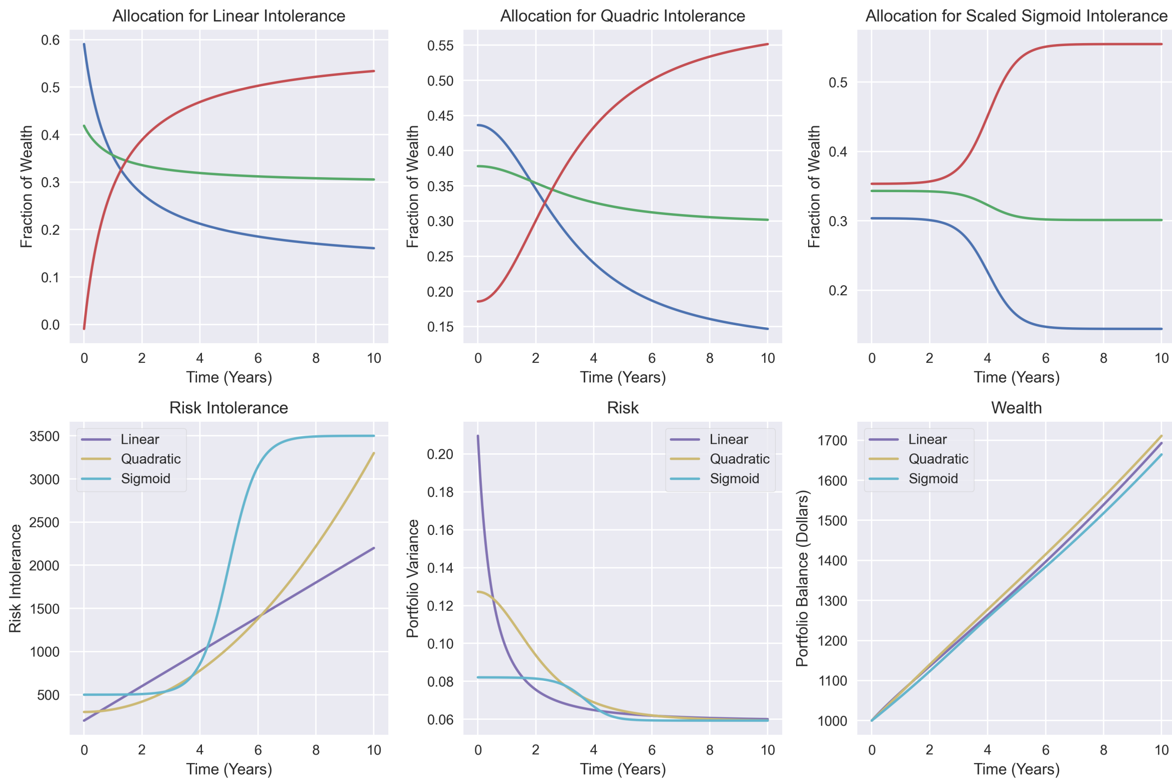

Figure 1. Allocation of wealth given various tolerance preferences. In the top row of plots, the red line corresponds to Asset 1 (3% expected return, 0.1 variance), the green line corresponds to Asset 2 (5% expected return, 0.2 variance), the blue line corresponds to Asset 3 (10% expected return, 0.5 variance).

5 Conclusion

Portfolio optimization is a difficult problem. While naive optimization techniques for this problem often employ restraining assumptions, our model allows us to minimize portfolio variance and maximize final wealth without intractable constraints.

For small values of t, the optimal portfolio allocation was highly dependent on risk intolerance used. However, with all three intolerance functions studied, portfolio allocation eventually seemed to converge to similar solutions at t_f. In all cases, final portfolio variance was similar. Likewise, final wealth was similar, with quadratic risk intolerance giving a marginally higher final wealth.

Future endeavors to improve our results could include a more rigorous treatment of covariances between assets and longer time horizons. We could also improve our results through a more robust use of inequality constraints (possibly through soft constraints in our cost functional). Finally, including a larger set of potential assets to invest in and comparing final total returns with those of other portfolio allocation routines would be an interesting point of comparison.