Bengt
06/23/24

A follow-up to my 05/19/24 handout – simplified and improved subdomain mappings

With use of the two functional equations (1) and (2) for the ${}_2F_1$-function as given in the previous handout, i.e., the mappings $z \to \frac{1}{z}$ and $z \to 1 - \frac{1}{z}$, we want to reduce calculations for an arbitrary z-value in the complex plane to one or two evaluations in a region in the unit disk that also keeps the longest distance possible from the singularity at $z = 1$. This task can be vastly simplified from the description in my previous handout. Consider Figure 1 below:

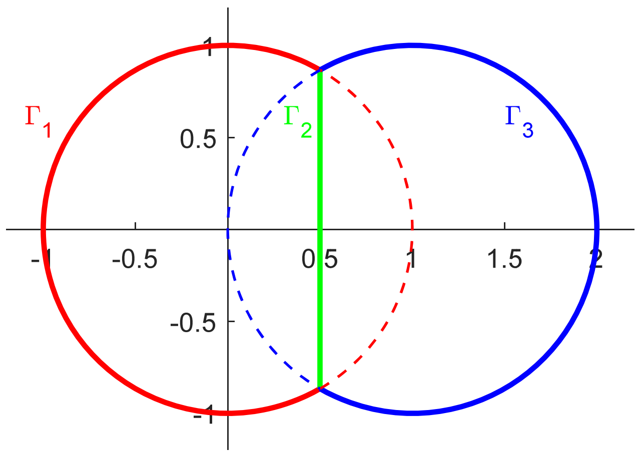

Figure 1. The unit circle (red) and the translation of it the distance one to the right (blue).

There are three curve segments shown as solid curves in the figure, denoted $\Gamma_1, \Gamma_2, \Gamma_3$. One finds by straightforward algebra that

$z \to 1 - \frac{1}{z}$ maps Γ_3 to Γ_2 and Γ_2 to Γ_1, i.e., the **in**terior of $\{\Gamma_2, \Gamma_3\}$ to the **in**terior of $\{\Gamma_1, \Gamma_2\}$,

$z \to \frac{1}{z}$ maps Γ_1 to Γ_1 and Γ_3 to Γ_2, i.e., the **ex**terior of $\{\Gamma_1, \Gamma_3\}$ to the **in**terior of $\{\Gamma_1, \Gamma_2\}$,

Therefore, we can map every point that does not already fall inside $\{\Gamma_1, \Gamma_2\}$ to a point that does. A numerical method for arbitrary complex z needs only to be fast and accurate within this region.

We can also note that quite a few numerical methods for evaluating the ${}_2F_1$-function exhibit particularly large errors around $z = e^{\pm \pi i/3}$. That is most certainly related to the fact that these points are the only ones that map to themselves for both of the mappings $z \to \frac{1}{z}$ and $z \to 1 - \frac{1}{z}$.