(1) Suppose we have a square matrix A and it's polar decomposition $A = SV$ where S is Hermitian positive semi-definite and V is a unitary matrix. Then, because S is Hermitian, it is diagonalizable as $S = UDU^*$ where U is unitary and D is a diagonal matrix with each entry as an eigenvalue of S. Furthermore, because S is positive semi-definite, each eigenvalue λ of S satisfies $\lambda \geq 0$ so each entry of D is non-negative. Then

$$A = SV = UDU^*V = UDW^*$$

where $W = V^*U$. Note, W is the product of two unitary matrices and so W is also unitary. So, from the polar form, we have the SVD of A as

$$A = UDW^*.$$

For a general $m \times n$ matrix A of rank $r \leq \min\{m, n\}$ has an SVD as

$$A = U\Sigma V^*$$

where U is $m \times r$ and V is $n \times r$ satisfying $U^*U = V^*V = I$, and Σ is an $r \times r$ diagonal matrix with diagonal entries $\sigma_1 \geq \ldots \geq \sigma_r > 0$.

(2) For each of the following statements, prove that it is true or give a counter example. In all questions $A = \mathbb{C}^{n\times n}$.

(a) If A is real and λ is an eigenvalue of A, the so is $-\lambda$.

False: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

In this case, A has eigenvalues of 1 and 2 but not -1 or -2.

(b) If A is real and λ is an eigenvalue of A, then so is $\bar{\lambda}$.

True: Suppose A is real and has an eigenvalue of λ. Then, λ satisfies $p(\lambda) = 0$ where $p(x)$ is the characteristic polynomial

$$p(x) = \det(A - xI).$$

But, because A is real, the coefficients of $p(x)$ are all real. Then from the properties of roots of polynomials, we know if λ is a root of $p(\lambda) = 0$, then so is $\bar{\lambda}$ (i.e. $p(\bar{\lambda}) = 0$) which implies $\bar{\lambda}$ is an eigenvalue of A.

(c) If λ is an eigenvalue of A and A is non-singular, then λ^{-1} is an eigenvalue of A^{-1}.

True: Suppose λ is an eigenvalue of A with eigenvector $\vec{x}$ and A is non-singular. Then

$$A\vec{x} = \lambda\vec{x} \implies \vec{x} = \lambda A^{-1}\vec{x} \implies \lambda^{-1}\vec{x} = A^{-1}\vec{x}$$

showing that λ^{-1} is an eigenvalue of A^{-1}.

(d) If A is Hermitian and λ is an eigenvalue of A, then $|\lambda|$ is a singular value of A.

True: Suppose A is Hermitian with λ as an eigenvalue. Then, because A is Hermitian, A is diagonalizable by a unitary matrix

$$A = UDU^*$$

where U is unitary and D is a diagonal matrix of the eigenvalues of A. Furthermore, because A is Hermitian, all of its eigenvalues are real and so we can write D as

$$D = |D| \operatorname{sign}(D) = \begin{pmatrix} |\lambda_1| & & & \\ & |\lambda_2| & & \\ & & \ddots & \\ & & & |\lambda_n| \end{pmatrix} \begin{pmatrix} \operatorname{sign}(\lambda_1) & & & \\ & \operatorname{sign}(\lambda_2) & & \\ & & \ddots & \\ & & & \operatorname{sign}(\lambda_n) \end{pmatrix}.$$

Then,

$$A = UDU^* = U\,|D|\underbrace{\operatorname{sign}(D)U^*}_{V^*} = U\,|D|\,V^*.$$

Then, because $\operatorname{sign}(D)$ is just a diagonal matrix of ± 1, $V^* = \operatorname{sign}(D)U^*$ is still a unitary matrix. So, because $|D|$ is a diagonal matrix of non-negative entries, $A = U\,|D|\,V^*$ is an SVD of A with singular values equal to $|\lambda|$ where λ are eigenvalues of A.

(3) A matrix $S \in \mathbb{C}^{n\times n}$ such that $S^* = -S$ is called skew-Hermitian. Show that

(a) eigenvalues of S are purely imaginary (or zero):
Suppose λ is an eigenvalue of S with corresponding eigenvector $\vec{x}$. Then

$$\begin{aligned}\lambda\|x\|^2 &= \lambda\langle\vec{x},\vec{x}\rangle = \langle\lambda\vec{x},\vec{x}\rangle = \langle A\vec{x},\vec{x}\rangle = \langle\vec{x},A^*\vec{x}\rangle \\ &= \langle\vec{x},-A\vec{x}\rangle = \langle\vec{x},-\lambda\vec{x}\rangle = -\bar{\lambda}\|\vec{x}\|^2\end{aligned}$$

showing that $\lambda = -\bar{\lambda}$ which implies that λ is purely imaginary or zero.

(b) matrix $I - S$ is non-singular:
Suppose S has eigenvalues $\{\lambda_i\}_{i=1}^n$. Then, $I - S$ has eigenvalues $\{1-\lambda_i\}_{i=1}^n$. But, from the previous part, we know each λ_i is purely imaginary or zero which implies $1 - \lambda_i \neq 0$ for $i = 1,\dots,n$. Thus each eigenvalue of $I - S$ is non-zero and so $I - S$ must be non-singular. The same argument can show that $(I + S)$ is also non-singular.

(c) matrix $Q = (I - S)^{-1}(I + S)$ is unitary:
From the previous part, we know $(I - S)$ and $(I + S)$ are both non-singular. So, by direct computation, we have

$$\begin{aligned}QQ^* &= (I-S)^{-1}(I+S)((I-S)^{-1}(I+S))^* \\ &= (I-S)^{-1}(I+S)(I+S^*)(I-S^*)^{-1} \\ &= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} \\ &= (I-S)^{-1}(I-S^2)(I+S)^{-1} \\ &= \underbrace{(I-S)^{-1}(I-S)}_{I}\underbrace{(I+S)(I+S)^{-1}}_{I} \\ &= I\end{aligned}$$

Showing that Q is unitary.

(4) Given $A \in \mathbb{C}^{n\times n}$, use Schur's decomposition to show that, for every $\varepsilon > 0$, there exists a diagonalizable matrix B such that $\|A - B\|_2 \leq \varepsilon$.

Suppose A has eigenvalues $\{\lambda_i\}_{i=1}^n$. Then, we have a Schur's decomposition of A as

$$A = UTU^*$$

where U is unitary and T is an upper triangular matrix with diagonal entries equal to the eigenvalues of A;

$$T = \begin{pmatrix} \lambda_1 & \times & \cdots & \times \\ 0 & \lambda_2 & \cdots & \times \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix}.$$

Now, let any $\varepsilon > 0$ be given. Then, define the matrix B as

$$B = U(T - D)U^*$$

where D is a diagonal matrix,

$$D = \operatorname{diag}(\delta_1, \delta_2, \dots, \delta_n)$$

such that δ_i satisfies

$$\max_{i=1,\dots,n} \{|\lambda_i - \delta_i|\} < \varepsilon$$

with each $\lambda_i - \delta_i$ being distinct. Now, note that the eigenvalues of B are $\{\lambda_i - \delta_i\}_{i=1}^n$ meaning each eigenvalue of B is distinct making B diagonalizable. Then

$$\begin{aligned}
\|A - B\|_2 &= \|UTU^* - U(T - D)U^*\|_2 \\
&= \|U(T - T + D)U^*\|_2 \\
&= \|UDU^*\|_2 \\
&\leq \|U\|_2\|D\|_2\|U^*\|_2 \\
&= \|D\|_2 \\
&= \sqrt{\max_{\lambda\in\sigma(D^*D)} \lambda} \\
&= \sqrt{\max_{i=1,\dots,n} |\lambda_i - \delta_i|^2} \\
&= \max_{i=1,\dots,n} |\lambda_i - \delta_i| \\
&< \varepsilon.
\end{aligned}$$

So, diagonalizable matrices are dense in $\mathbb{C}^{n\times n}$.