(1) Show that Jacobi's method for finding eigenvalues of a real symmetric matrix is ultimately quadratically convergent. Assume that all off-diagonal elements of the matrix A_k are $O(\varepsilon)$,where k enumerates Jacobi sweeps. Show that the all rotations of the next Jacobi sweep are of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 - O(\varepsilon^2) & \cdots & O(\varepsilon) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & O(\varepsilon) & \cdots & 1 - O(\varepsilon^2) & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Then demonstrate that this implies that, after the sweep, all off-diagonal elements of A_{k+1} are $O(\varepsilon^2)$. Assume that all eigenvalues are non-zero and distinct.

(2) Show that A is diagonalizable iff there is a positive definite self-adjoint matrix H such that $H^{-1}AH$ is normal.
Proof:

($\implies$)Suppose a matrix A is diagonalizable. Then we can write

$$A = PDP^{-1} \implies D = P^{-1}AP$$

where D is a diagonal and P is an invertible. Next, let's take the polar decomposition of P as $H = HU$ where H is positive definite self-adjoint and U is unitary. Note, H^{-1} is also positive definite-self-adjoint and $H = PU^*$. Then

$$H^{-1}AH = UP^{-1}APU^* = UDU^*.$$

Then

$$\begin{aligned}(H^*AH)^*(H^*AH) &= (UDU^*)^*(UDU^*)\\ &= (UD^*U^*)(UDU^*)\\ &= UD^*DU^*\\ &= UDD^*U^*\\ &= (UDU^*)(UD^*U^*)\\ &= (UDU^*)(UDU^*)^*\\ &= (H^{-1}AH)(H^{-1}AH)^*\end{aligned}$$

Showing that there exists a positive definite self-adjoint matrix H such that $H^{-1}AH$ is normal.

($\impliedby$) Now, suppose there exists a positive definite self-adjoint matrix H such that $H^{-1}AH$ is normal. Then, because $H^{-1}AH$ is normal, it is diagonalizable by a unitary matrix:

$$H^{-1}AH = UDU^*$$

where D is diagonal and U is unitary. Then, we have

$$D = U^*H^{-1}AHU = P^{-1}AP$$

where $P = HU$. So, A is similar to a diagonal matrix and is thus diagonalizable. □