Chapter 1

Problem 3

Find an equation relating the parameters c, m, n so that the function $u(x,t) = \sin(mt)\sin(nx)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ for $c > 0$.

First off, our derivatives are as follows,

$$\begin{aligned} u_{tt} &= -m^2 \sin(mt)\sin(nx) \\ u_{xx} &= -n^2 \sin(mt)\sin(nx). \end{aligned}$$

Then, plugging our derivatives into the wave equation yields

$$-m^2 \sin(mt)\sin(nx) = -c^2 n^2 \sin(mt)\sin(nx)$$

which implies

$$m^2 = c^2 n^2$$

for $c > 0$.

Problem 8

Consider the linear transport equation (1.8) with initial and boundary conditions (1.10).

$$\begin{cases} u_t + c u_x = 0 \\ u(x,0) = \phi(x), & x \geq 0 \\ u(0,t) = \psi(t), & t \geq 0. \end{cases}$$

(a) Suppose the data ϕ, ψ are differentiable functions. Show that the function $u : Q_q \to \mathbb{R}$ given by

$$u(x,t) = \begin{cases} \phi(x - ct), & \text{if } x \geq ct, \\ \psi(t - x/c), & \text{if } x \leq ct \end{cases} \tag{1}$$

satisfies the PDE away from the line $x = ct$ with $c > 0$, the boundary condition, and initial conditions.

When we are away from the line, boundary condition, and initial condition, we have

$$\begin{aligned} u_t &= \begin{cases} -c\phi'(x - ct), & x > ct > 0 \\ \psi'(t - x/c), & 0 < x < ct \end{cases} \\ u_x &= \begin{cases} \phi'(x - ct), & x > ct > 0 \\ -\frac{1}{c}\psi'(t - x/c), & 0 < x < ct \end{cases}. \end{aligned}$$

Using the derivatives above

$$\begin{aligned} u_t + c u_x &= \begin{cases} -c\phi'(x - ct) + c\phi'(x - ct), & x > ct > 0 \\ \psi'(t - x/c) - c\frac{1}{c}\psi'(t - x/c), & 0 < x < ct \end{cases} \\ &= \begin{cases} 0, & x > ct > 0 \\ 0, & 0 < x < ct \end{cases} \\ &= 0 \end{aligned}$$

for x, t not on the line, boundaries, or initial conditions. So, (1) defined above satisfies our PDE in the desired region.

(b) In solution (1), the line $x = ct$, which emerges from the origin $x = t = 0$, separates the quadrant Q_1 into two regions.

(i) Find conditions on the data ϕ, ψ so that the solution is continuous across the line $x = ct$.
When $x = ct$, we have $\phi(x - ct) = \phi(ct - ct) = \phi(0)$ and $\psi(t - x/c) = \psi(x/c - x/c) = \psi(0)$. Thus, for continuity when $x = ct$, we need

$$\phi(0) = \psi(0).$$

(ii) Find conditions on the data ϕ, ψ so that the solution is differentiable across the line $x = ct$.
For differentiablity of u across the line $x = ct$, we need differentiablity in both partial derivatives, u_t and u_x across the line. First, let's compute the derivatives on the line:

$$\begin{aligned}
\partial_x \phi(x - ct)\big|_{x=ct} &= \phi'(0) \\
\partial_x \psi(t - x/c)\big|_{x=ct} &= -\frac{1}{c}\psi'(0) \\
\partial_t \phi(x - ct)\big|_{x=ct} &= -c\phi'(0) \\
\partial_t \psi(x - ct)\big|_{x=ct} &= \psi'(0).
\end{aligned}$$

Equating our derivatives yields

$$\begin{aligned}
\phi'(0) &= -\frac{1}{c}\psi'(0) \\
-c\phi'(0) &= \psi'(0)
\end{aligned}$$

which reduces to

$$\phi'(0) = -\frac{1}{c}\psi'(0).$$

So, for u to be differentiable across the line $x = ct$, we need $\phi'(0) = -\frac{1}{c}\psi'(0)$.

Problem 9

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Verify that if $u(x, t)$ is differentiable and satisfies (1.12), that is, $u = f(x - ut)$, then $u(x, t)$ is a solution of the initial value problem

$$u_t + uu_x = 0, \ -\infty < x < \infty, \ t > 0, \ u(x, 0) = f(x), \ -\infty < x < \infty.$$

Suppose $u(x, t)$ is differentiable and $u = f(x - ut)$. Then, the initial condition easily verified as

$$u(x, 0) = f(x - u(x, 0) \cdot 0) = f(x).$$

Next, we can start verifying the PDE by computing the partial derivatives as follows

$$\begin{aligned}
u_t &= f'(x - ut)(-u_t t - u) \\
u_x &= f'(x - ut)(1 - u_x t).
\end{aligned}$$

Now, putting these derivative into our PDE yields

$$\begin{aligned} u_t + uu_x &= f'(x-ut)(-u_t t - u) + uf'(x-ut)(1-u_x t) \\ &= f'(x-ut)(-u_t t - u + u - uu_x t) \\ &= f'(x-ut)(-u_t t - uu_x t) \\ &= -tf'(x-ut)(u_t + uu_x). \end{aligned}$$

Clearly, at $t = 0$ our PDE equals zero and so u satisfies the PDE at $t = 0$, But, when $t > 0$, we have

$$u_t + uu_x = -tf'(x-ut)(u_t + uu_x)$$

which implies that either $1 = -tf'(x-ut)$ or $u_t + uu_x = 0$. The first equality implies that $f'(x-ut) = -1/t$ which doesn't hold because $f'(x-ut)$ is differentiable everywhere but $-1/t$ is not differentiable at $t = 0$. So, we must have the second equality, $u_t + uu_x = 0$ which shows that u solve the IVP!

Chapter 2

Problem 5

Problem 6

Additional Problems

Problem A1

Problem A2