1. Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix where the diagonal entries are given by a_j for $j = 1, \dots, n$, the lower diagonal entries are b_j for $j = 2, \dots, n$ and the upper diagonal entries are c_j for $j = 1, \dots, n-1$.

   (a) For $n = 3$, derive the LU factorization of the matrix A.

$$U = \begin{pmatrix} a_1 & c_1 & 0 \\ b_1 & a_2 & c_2 \\ 0 & b_2 & a_3 \end{pmatrix} \to \begin{pmatrix} a_1 & c_1 & 0 \\ 0 & a_2 - \frac{c_1 b_1}{a_1} & c_2 \\ 0 & b_2 & a_3 \end{pmatrix} \to \begin{pmatrix} a_1 & c_1 & 0 \\ 0 & a_2 - \frac{c_1 b_1}{a_1} & c_2 \\ 0 & 0 & a_3 - \frac{c_2 b_2}{a_2 - \frac{c_1 b_1}{a_1}} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ \frac{b_1}{a_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ \frac{b_1}{a_1} & 1 & 0 \\ 0 & \frac{b_2}{a_2 - \frac{c_1 b_1}{a_1}} & 1 \end{pmatrix}.$$

   So, our LU factorization in $n = 3$ is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{b_1}{a_1} & 1 & 0 \\ 0 & \frac{b_2}{a_2 - \frac{c_1 b_1}{a_1}} & 1 \end{pmatrix}$$

   and

$$U = \begin{pmatrix} a_1 & c_1 & 0 \\ 0 & a_2 - \frac{c_1 b_1}{a_1} & c_2 \\ 0 & 0 & a_3 - \frac{c_2 b_2}{a_2 - \frac{c_1 b_1}{a_1}} \end{pmatrix}.$$

   (b) What is the extension of the LU factorization for general n?

   Looking at the $n = 3$ case, we can see that the next entry in U and L can be turned into an iterative process. The iteration process is as follows.

   (1) Set U to be the zero $n \times n$ matrix and set L to be the $n \times n$ identity matrix.
   (2) Set $U_{11} = a_1$.
   (3) Set $k = 1$.
   (4) Set $U_{k+1,k+1} = a_k - \frac{c_k b_k}{U_{k,k}}$.
   (5) Set $U_{k,k+1} = c_k$.
   (6) Set $L_{k+1,k} = \frac{b_k}{U_{k,k}}$
   (7) Increase k by 1 and then repeat at step (4) until done.

   (c) What is the operation count when applying Gaussian Elimination to a tridiagonal system without pivoting.

   Looking at our operation count in part (b), we can see that step 1, 2, and 3, take 0 flops. Next, step 4 takes 3 flops. Because we are just doing Gaussian Elimination and we don't need to form LU, we can skip the rest of the steps except for the repeat step which occurs $n-1$ times. Thus, the total cost is given by

$$3(n-1) = 3n - 3 \text{ flops}.$$

2. Consider the linear system

$$\begin{aligned} 6x + 2y + 2z &= -2 \\ 2x + \frac{2}{3}y + \frac{1}{3}z &= 1 \\ x + 2y - z &= 0 \end{aligned}$$

(a) Verify that $(x, y, z) = (2.6, -3.8, -5)$ is the exact solution.
To verify the solution, let's first rewrite the LHS of the system and multiply by our vector to get

$$\begin{pmatrix} 6 & 2 & 2 \\ 2 & \frac{2}{3} & \frac{1}{3} \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2.6 \\ -3.8 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

which shows the exact solution is given by $(x, y, z) = (2.6, -3.8, -5)$.

(b) Let's create our augmented matrix and begin Gaussian elimination

$$\left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 2 & \frac{2}{3} & \frac{1}{3} & 1 \\ 1 & 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0.3333 & 0.3333 & -0.3333 \\ 2 & 0.6667 & 0.3333 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0.3333 & 0.3333 & -2 \\ 0 & 0.0001 & -0.3333 & 1.666 \\ 0 & 1.666 & -1.333 & 0.3333 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0.3333 & 0.3333 & -2 \\ 0 & 1 & -3333 & 16660 \\ 0 & 1.666 & -1.333 & 0.3333 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1111 & -5554 \\ 0 & 1 & -3333 & 16660 \\ 0 & 0 & 5551 & -27740 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1111 & -5554 \\ 0 & 1 & -3333 & 16660 \\ 0 & 0 & 1 & -4.997 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -4.997 \end{array}\right).$$

So, our solution in 4 digit arithmetic without pivoting is given by $(x, y, z) = (-3, 10, -4.997)$ which has an absolute error of 14.893

(c) Repeat part (b) with partial pivoting.

$$\left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 2 & \frac{2}{3} & \frac{1}{3} & 1 \\ 1 & 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 2 & 0.6667 & 0.3333 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 0 & 0.0001 & 0.3333 & 1 \\ 0 & 1.666 & -1.333 & 0.3333 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 0 & 1.666 & -1.333 & 0.3333 \\ 0 & 0.0001 & 0.3333 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 6 & 2 & 2 & -2 \\ 0 & 1.666 & -1.333 & 0.3333 \\ 0 & 0 & 0.3333 & 1 \end{array}\right)$$

which implies $z = 3.000$, $y = 2.6$, and $x = -2.644$ which has an absolute error of 11.5091.

(d) Gaussian elimination with partial pivoting was slightly more accurate in this case and kept us from losing so many significant digits by reducing divisions by relatively small numbers.

3. Consider the system $Ax = b$ where

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 \\ -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$