1) Find two-term asymptotic approximations to each of the roots of

(a) $\varepsilon x^3 + \varepsilon x^2 - x + 1 = 0$.
First, let's find the regularly perturbed solutions by assuming

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots .$$

Then, our equation becomes

$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 1 = 0.$$

So, equating our ordered terms, we have

$$\begin{array}{rccl} O(1): & -x_0 + 1 = 0 & \implies & x_0 = 1 \\ O(\varepsilon): & x_0^3 + x_0^2 - x_1 = 0 & \implies & x_1 = 2. \end{array}$$

So, our regularly perturbed root has a two-approximation of

$$\boxed{x = 1 + 2\varepsilon + O(\varepsilon^2).}$$

Now to get our singularly perturbed solutions, suppose

$$x = \frac{1}{\sqrt{\varepsilon}} y,$$

Then, our equation becomes

$$\frac{1}{\sqrt{\varepsilon}} y^3 + y^2 - \frac{1}{\sqrt{\varepsilon}} y + 1 = 0 \implies y^3 + \sqrt{\varepsilon} y^2 - y + \sqrt{\varepsilon} = 0$$

which is maximally balanced. Now, assume we can express y as

$$y = y_0 + \sqrt{\varepsilon} y_1 + \cdots .$$

Then, equating our ordered terms yields

$$\begin{array}{rccl} O(1): & y_0^3 - y_0 = 0 & \implies & y_0 = -1, 0, 1 \\ O(\sqrt{\varepsilon}): & 3y_0^2 y_1 + y_0^2 - y_1 + 1 = 0 & \implies & y_1 = \begin{cases} -1, & y_0 = \pm 1 \\ 1, & y_0 = 0 \end{cases}. \end{array}$$

We can ignore the case when $y_0 = 0$ because that will just lead to our regularly perturbed solution. So our singularly perturbed solutions are given by

$$\boxed{\begin{array}{l} x = \frac{1}{\sqrt{\varepsilon}} - 1 + O(\sqrt{\varepsilon}) \\ x = -\frac{1}{\sqrt{\varepsilon}} - 1 + O(\sqrt{\varepsilon}). \end{array}}$$

The order of accuracy can be found at the end of each expression in the Big-O notation. Furthermore, a convergence plot for this part can be found at the end of the problem.

(b) $2\varepsilon^3 x^5 - \varepsilon x^4 + x^3 - 3\varepsilon x^2 + 4x + 2\varepsilon = 0$.
Just like last time, let's find the regularly perturbed solutions by assuming

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 \cdots .$$

Then, plugging in our series for x into the polynomial equation and equating ordered terms yields

$$\begin{aligned} O(1):& \quad x_0^3 + 4x_0 = 0 &&\implies x_0 = -2i, 0, 2i \\ O(\varepsilon):& \quad -x_0^4 + 3x_0^2 x_1 - 3x_0^2 + 4x_1 + 2 = 0 &&\implies x_1 = \begin{cases} -\frac{1}{4}, & x_0 = \pm 2i \\ -\frac{1}{2}, & x_0 = 0 \end{cases}. \end{aligned}$$

We still need one more nonzero term for the case when $x_0 = 0$. So when $x_0 = 0$, we have

$$\begin{aligned} O(\varepsilon^2):& \quad 4x_2 = 0 &&\implies x_2 = 0 \\ O(\varepsilon^3):& \quad x_1^3 - 3x_1^2 + 4x_3 = 0 &&\implies x_3 = \tfrac{7}{32}. \end{aligned}$$

So, our three regularly perturbed solutions have the form

$$\boxed{\begin{aligned} x &= -2i - \tfrac{1}{4}\varepsilon + O(\varepsilon^2), \\ x &= -\tfrac{1}{2}\varepsilon + \tfrac{7}{32}\varepsilon^3 + O(\varepsilon^5), \\ x &= 2i - \tfrac{1}{4}\varepsilon + O(\varepsilon^2). \end{aligned}}$$

Now, let's look for singularly perturbed solutions by first considering

$$x = \frac{1}{\varepsilon^2} y.$$

In this case, our equation becomes

$$2y^5 - y^4 + \varepsilon y^3 - 3\varepsilon^4 y^2 + 4\varepsilon^4 y + 4\varepsilon^5 y + 2\varepsilon^8 = 0$$

which is maximally balanced by the first two terms. Just as before, let's assume

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots .$$

Now, plugging this in and equating our ordered terms yields

$$O(1): 2y_0^5 - y_0^4 = 0 \implies y_0 = 0, \frac{1}{2}.$$

Now, just taking $y_0 = \frac{1}{2}$, we have

$$O(\varepsilon): 10y_0^4 - 4y_0^3 y_1 + y_0^3 = 0 \implies y_1 = -1$$

which gives a solution of

$$\boxed{x = \frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon} + O(1).}$$

To find the other singularly perturbed solutions, assume

$$x = \frac{1}{\varepsilon} y.$$

Then, our equation becomes

$$2\varepsilon y^5 - y^4 + y^3 - 3\varepsilon^2 y^2 + 4\varepsilon^2 y + 2\varepsilon^4 = 0$$

which implies

$$O(1) : -y_0^4 + y_0^3 = 0 \implies y_0 = 0, 1.$$

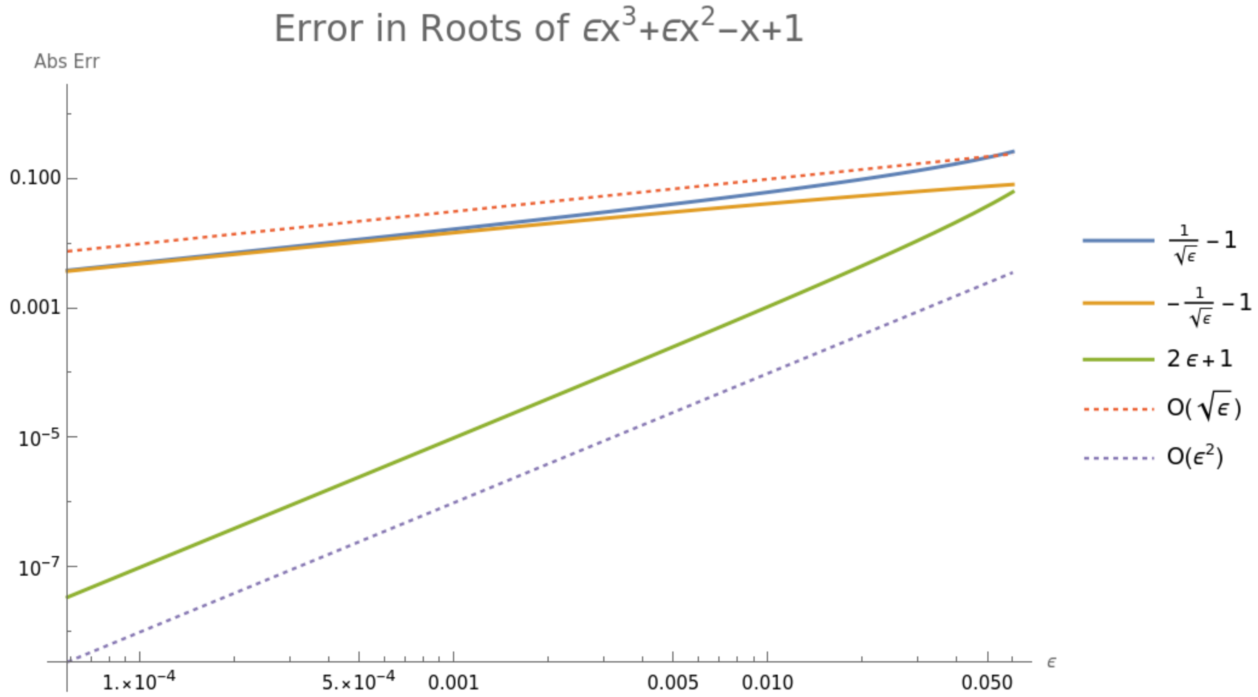
Then, taking $y_0 = 1$ yields

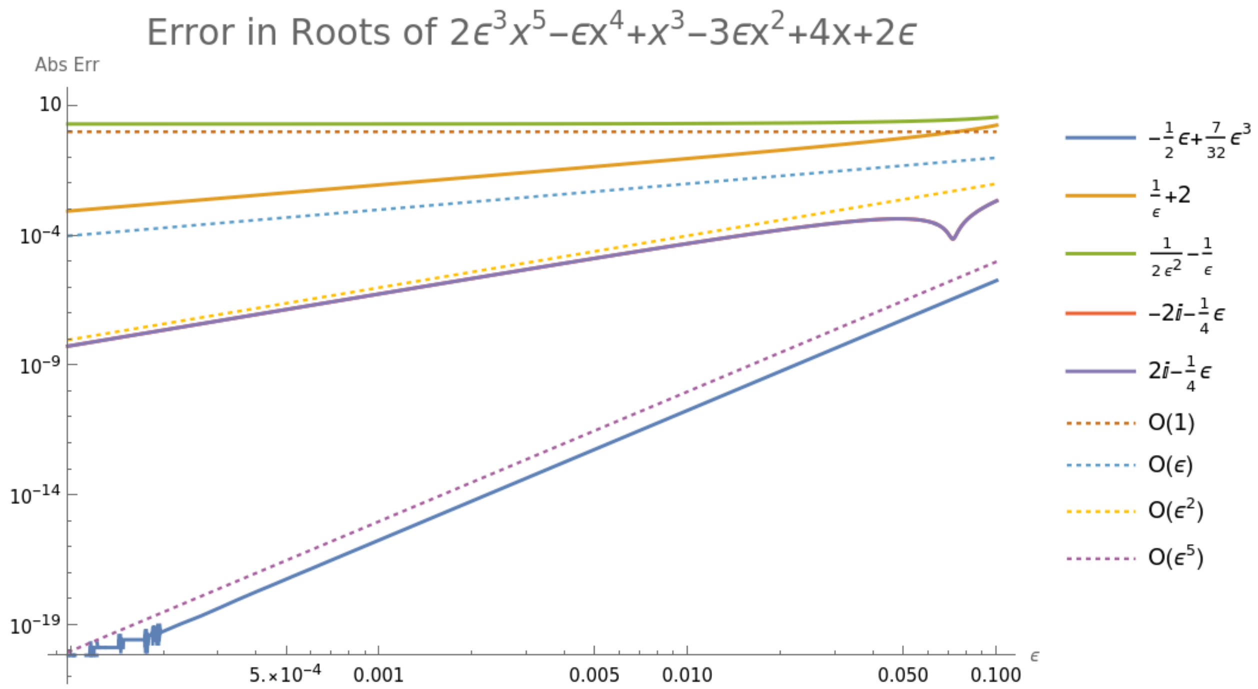
$$O(\varepsilon) : 2 - 4y_1 + 3y_1 = 0 \implies y_1 = 2.$$

So, our final singularly perturbed solution is given by

$$\boxed{x = \frac{1}{\varepsilon} + 2 + O(\varepsilon).}$$

Just like part (a), the order of accuracy for each approximation is in Big-O notation at the end of the estimate. To confirm the order of accuracies, the plot below shows the convergence of each root estimate.


Error in Roots of $\epsilon x^3+\epsilon x^2-x+1$
Abs Err
0.100
0.001
10^{-5}
10^{-7}
$1.\times10^{-4}$
$5.\times10^{-4}$
0.001
0.005
0.010
0.050
ϵ
$\frac{1}{\sqrt{\epsilon}}-1$
$-\frac{1}{\sqrt{\epsilon}}-1$
$2\epsilon+1$
$O(\sqrt{\epsilon})$
$O(\epsilon^2)$


Error in Roots of $2\epsilon^3x^5-\epsilon x^4+x^3-3\epsilon x^2+4x+2\epsilon$
Abs Err
10
10^{-4}
10^{-9}
10^{-14}
10^{-19}
$5.\times10^{-4}$
0.001
0.005
0.010
0.050
0.100
ϵ
$-\frac{1}{2}\epsilon+\frac{7}{32}\epsilon^3$
$\frac{1}{\epsilon}+2$
$\frac{1}{2\epsilon^2}-\frac{1}{\epsilon}$
$-2i-\frac{1}{4}\epsilon$
$2i-\frac{1}{4}\epsilon$
$O(1)$
$O(\epsilon)$
$O(\epsilon^2)$
$O(\epsilon^5)$

2) Find the first three terms of an asymptotic approximation to the solution $x(\varepsilon)$ of the transcendental equation

$$\frac{e^{-x^2}}{x} = \varepsilon, \quad \varepsilon \ll 1.$$

To find our approximation, we will use an iterative method. The iteration can be set up as follows

$$\begin{aligned}
&e^{-x^2} = \varepsilon x \\
\implies &-x^2 = \ln(\varepsilon) + \ln(x) \\
\implies &x^2 = \ln\left(\frac{1}{\varepsilon}\right) - \ln(x)
\end{aligned}$$

which gives us the iteration scheme

$$x_n^2 = \ln\frac{1}{\varepsilon} - \ln x_{n-1}$$

Now, suppose $x_0 = 1$. Then if we let $L_1 = \ln\frac{1}{\varepsilon}$, iterating yields

$$x_1^2 = \ln\frac{1}{\varepsilon} = L_1.$$

Next, let $L_2 = -\frac{1}{2}\ln\ln\frac{1}{\varepsilon} = -\frac{1}{2}\ln L_1$. Then iterating again

$$x_2^2 = \ln\frac{1}{\varepsilon} - \ln\sqrt{\ln\frac{1}{\varepsilon}} = \ln\frac{1}{\varepsilon} - \frac{1}{2}\ln\ln\frac{1}{\varepsilon} = L_1 + L_2.$$

Again,

$$\begin{aligned}
x_3^2 &= L_1 - \ln(L_1 + L_2) \\
&= L_1 - \frac{1}{2}\ln\left(L_1\left(1 + \frac{L_2}{L_1}\right)\right) \\
&= L_1 + L_2 - \frac{1}{2}\ln\left(1 + \frac{L_2}{L_1}\right) \\
&= L_1 + L_2 - \frac{1}{2}\frac{L_2}{L_1} + \frac{1}{4}\left(\frac{L_2}{L_1}\right)^2 + \cdots
\end{aligned}$$

So, truncating our series yields the asymptotic approximation

$$\boxed{x^2 \sim L_1 + L_2 - \frac{1}{2}\frac{L_2}{L_1} \text{ as } \varepsilon \to 0.}$$

3) Find the first order perturbations of the eigenvalues of the differential equation

$$\begin{cases} y'' + \lambda y + \varepsilon y^n = 0, & x \in (0, \pi) \\ y(0) = y(\pi) = 0 \end{cases}$$

for $n = 1, 2, 3$.

First, let's compute the eigenvalues and eigenfunctions for the unperturbed problem:

$$y_0'' + \lambda y_0 = 0 \implies y_0 = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Then, from the BCs,

$$y_0(0) = A = 0$$

and

$$y_0(\pi) = B\sin(\sqrt{\lambda}\pi) = 0 \implies \lambda_0 = k^2, \quad k = 1, 2, \ldots.$$

Next, from the Fredholm alternative, we can compute our first order eigenvalue term as

$$\lambda_1 = \frac{\langle (y_0)^n, w\rangle}{\langle y_0, w\rangle}$$

where

$$(\mathcal{L}^* - \lambda_0^*)w = -w'' - k^2 w = 0$$

which implies

$$w(x) = A\sin(kx).$$

Then,

$$\lambda_1 = \frac{\langle \sin^n(kx), A\sin(kx)\rangle}{\langle \sin(kx), A\sin(kx)\rangle} = \frac{\int_0^\pi A^n \sin^{n+1}(x)\mathrm{d}x}{\int_0^\pi A\sin^2(x)\mathrm{d}x} = \begin{cases} 1, & n = 1 \\ \frac{4}{3k\pi}A(1 - (-1)^k), & n = 2 \\ \frac{3A^2}{4}, & n = 3 \end{cases}.$$

So, putting everything together, we have the 1st order perturbation for λ as

$$\boxed{\lambda = k^2 + \varepsilon\left(\begin{cases} 1, & n = 1 \\ \frac{4}{3k\pi}A(1 - (-1)^k), & n = 2 \\ \frac{3A^2}{4}, & n = 3 \end{cases}\right) + O(\varepsilon^2).}$$

I know for odd k, the 1st order term goes to zero when $n = 2$ but I can not for the life of me figure out how to get a "λ_2" to keep two nonzero terms there.

4) Bessel's function of the first kind and 3/2 order is given by

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\cos x + \frac{\sin x}{x}\right).$$

Determine the two-term expansions for large roots of

(a) $J_{3/2}(x) = 0$.
First, note that

$$\begin{aligned} J_{3/2}(x) = 0 &\implies \sqrt{\frac{2}{\pi x}}\left(-\cos x + \frac{\sin x}{x}\right) = 0 \\ &\implies -\cos x + \frac{\sin x}{x} = 0 \\ &\implies \cot(x) = \frac{1}{x}. \end{aligned}$$

From our final expression, when $|x| \gg 0$, we would expect $\cot(x)$ to dominate the solution to the roots because $\frac{1}{x}$ decays. So we would expect large roots to look like the roots of $\cot(x)$ plus some corrector. The roots of $\cot(x)$ are given by

$$x_{\text{cot}} = (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

So, let's assume our solution is of the form

$$x = x_{\text{cot}} + \delta(k)$$

where $\delta(k)$ is our corrector. Then, rewriting our equation and expanding about x_{cot} yields

$$\begin{aligned} \cot(x) = \frac{1}{x} &\implies \sin(x) = x\cos(x) \\ &\implies (-1)^k + O(x^2) = \frac{1}{4}\pi(-1)^k(2k+1)(2\pi k - 2x + \pi) + O(x^2) \end{aligned}$$

Then, substituting $x = x_{\text{cot}} + \delta(k)$ in and truncating yields

$$\begin{aligned} (-1)^k &= -(-1)^k(2k+1)\frac{\pi}{2}\delta(k) \\ \implies \delta(k) &= -\frac{2}{2k\pi + \pi} + O\left(\frac{1}{k^2}\right). \end{aligned}$$

Putting everything together, we have an approximate solution of

$$\boxed{x = x_{\text{cot}} + \delta(k) = (2k+1)\frac{\pi}{2} - \frac{2}{2k\pi + \pi} + O\left(\frac{1}{k^4}\right).}$$

I know the problem asked for a comparison with the first 5 roots but I went a bit further to a 100 roots to see the trend continue; the convergence plot is at the end of the problem.

(b) $J'_{3/2}(x) = 0$.

Somewhat similar to before, we have

$$\begin{aligned} J'_{3/2}(x) = 0 &\implies \frac{(2x^2-3)\sin(x) + 3x\cos(x)}{\sqrt{2\pi}x^{5/2}} = 0 \\ &\implies \left(2x^2-3\right)\sin(x) + 3x\cos(x) = 0 \\ &\implies \tan(x) = \frac{3x}{3-2x^2}. \end{aligned}$$

From this expression, we can see that for large x, the $\tan(x)$ is the dominant term so we would expect the roots of $\tan(x)$ to dominate the solution. In this case, we have the roots of $\tan(x)$ as

$$x_{\tan} = k\pi, \quad k \in \mathbb{Z}.$$

Now, let's assume the roots of our equation have the form

$$x = x_{\tan} + \delta(k)$$

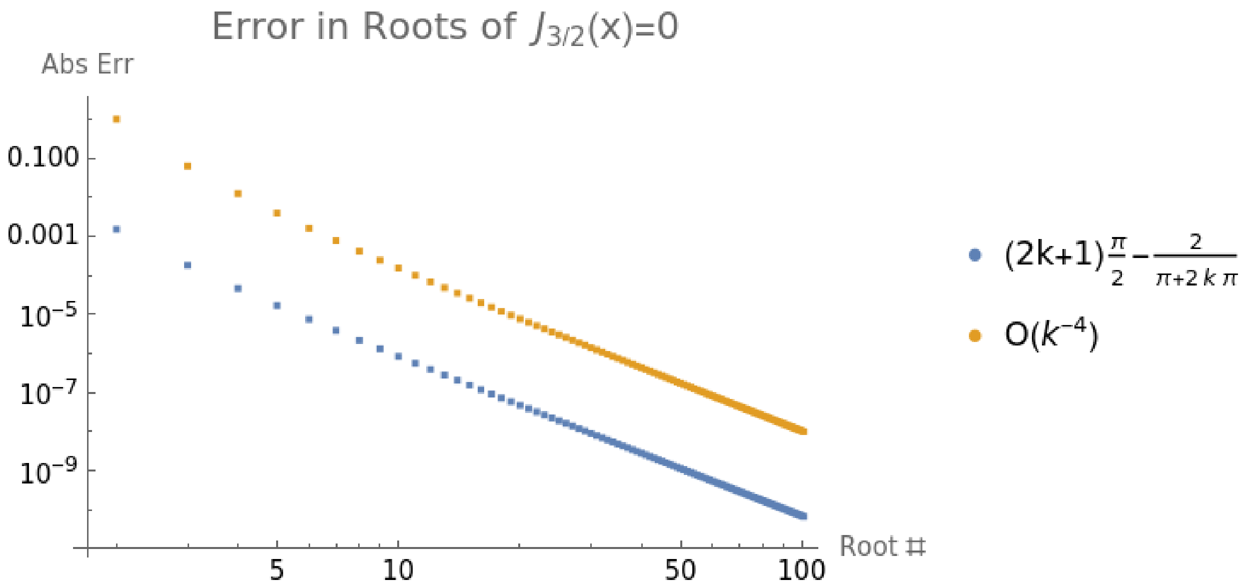
where $\delta(k)$ is a correction. Then rewriting our equation, expanding about $x_{\tan}$, and plugging in our assumption yields

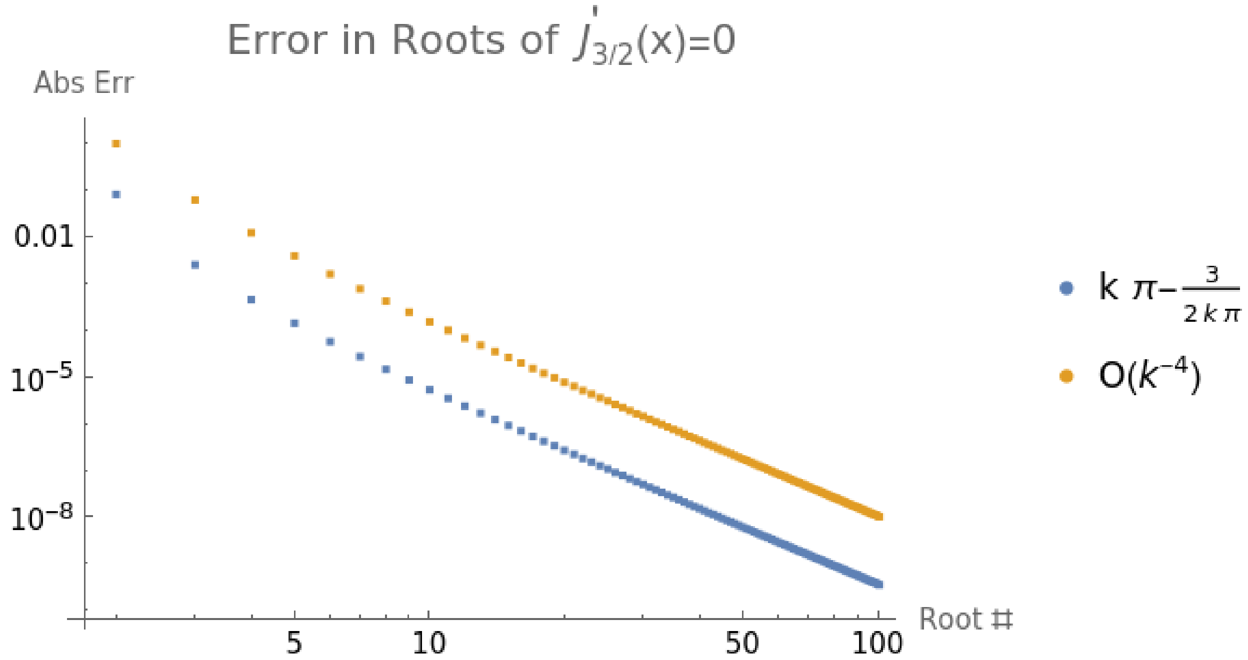
$$\begin{aligned} \tan(x) = \frac{3x}{3-2x^2} &\implies \left(3-2x^2\right)\sin(x) = 3x\cos(x) \\ &\implies (-1)^k\left(3-2\pi^2k^2\right)(x-\pi k) = 3(-1)^k x + O(x^2) \\ &\implies (-1)^k(3-2k^2\pi^2)\delta(k) = 3(-1)^k(k\pi + \delta(k)) \\ &\implies \delta(k) = -\frac{3}{2k\pi} + O\left(\frac{1}{k^4}\right). \end{aligned}$$

So, our large roots can be approximated using

$$\boxed{x = x_{\tan} + \delta(k) = k\pi - \frac{3}{2k\pi}.}$$

The convergence plot can be found on the next page.


Error in Roots of $J_{3/2}(x)=0$
Abs Err
0.100
0.001
10^{-5}
10^{-7}
10^{-9}
5
10
50
100
Root #
$(2k+1)\frac{\pi}{2}-\frac{2}{\pi+2k\pi}$
$O(k^{-4})$


Error in Roots of $J'_{3/2}(x)=0$
Abs Err
0.01
10^{-5}
10^{-8}
5
10
50
100
Root #
$k\pi-\frac{3}{2k\pi}$
$O(k^{-4})$

5) Determine the order of the following as $\varepsilon \to 0$:

(a) $\ln(\cot\varepsilon)$

$$\ln(\cot\varepsilon) = \ln\left(\frac{\cos\varepsilon}{\sin\varepsilon}\right) = \ln\left(\frac{1-\frac{1}{2}\varepsilon^2+\cdots}{\varepsilon-\frac{1}{6}\varepsilon^3+\cdots}\right) \sim \ln\left(\frac{1}{\varepsilon}\right) = O\left(\ln\left(\frac{1}{\varepsilon}\right)\right)$$

(b) $\sinh\frac{1}{\varepsilon}$

$$\sinh\frac{1}{\varepsilon} = \frac{1}{2}(e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}) \sim \frac{1}{2}e^{\frac{1}{\varepsilon}} = O(e^{\frac{1}{\varepsilon}})$$

(c) $\coth\frac{1}{\varepsilon}$

$$\coth\frac{1}{\varepsilon} = \frac{e^{\frac{1}{\varepsilon}} + e^{-\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}} \sim \frac{e^{\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}}} = 1 = O(1)$$

(d) $\frac{\varepsilon^{3/4}}{1-\cos\varepsilon}$

$$\frac{\varepsilon^{3/4}}{1-\cos\varepsilon} = \frac{\varepsilon^{3/4}}{1-1+\frac{1}{2}\varepsilon^2+\cdots} = \frac{\varepsilon^{3/4}}{\frac{1}{2}\varepsilon^2+\cdots} \sim 2\varepsilon^{3/4-2} = 2\varepsilon^{-5/4} = O(\varepsilon^{-5/4})$$

(e) $\ln\left(1+\ln\frac{1+2\varepsilon}{\varepsilon}\right)$

$$\ln\left(1+\ln\frac{1+2\varepsilon}{\varepsilon}\right) = \ln\left(1+\ln\left(\frac{1}{\varepsilon}+2\right)\right) \sim \ln\left(1+\ln\frac{1}{\varepsilon}\right) = O\left(\ln\left(\ln\frac{1}{\varepsilon}\right)\right)$$

6) Arrange the following in descending order for small ε

(a) Given $e^{-1/\varepsilon}, \ln\frac{1}{\varepsilon}, \varepsilon^{-0.01}, \cot\varepsilon, \sinh\frac{1}{\varepsilon}$, we can order them in descending order as

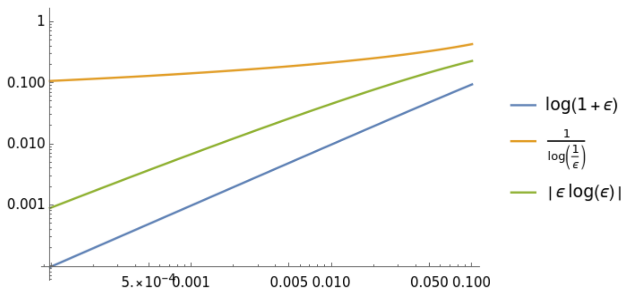
$$\sinh\frac{1}{\varepsilon} \gg \cot\varepsilon \gg \varepsilon^{-0.01} \gg \ln\frac{1}{\varepsilon} \gg e^{-1/\varepsilon}.$$

This ordering was pretty straight forward although $\varepsilon^{-0.01}$ and $\ln\frac{1}{\varepsilon}$ was a little tricky until I noticed the limit of the ratio between the two went to zero when $\ln\frac{1}{\varepsilon}$ was in the numerator.

(b) Given $\ln(1+\varepsilon) = O(\varepsilon), \cot\varepsilon = O\left(\frac{1}{\varepsilon}\right), \tanh\frac{1}{\varepsilon} = O(1), \frac{\sin\varepsilon}{\varepsilon^{3/4}} = O(\varepsilon^{1/4}), \varepsilon\ln\varepsilon, e^{-1/\varepsilon}, \sinh\frac{1}{\varepsilon} = O(e^{1/\varepsilon}), \frac{1}{\ln 1/\varepsilon}$, we can order them as

$$\sinh\frac{1}{\varepsilon} \gg \cot\varepsilon \gg \tanh\frac{1}{\varepsilon} \gg \frac{\sin\varepsilon}{\varepsilon^{3/4}} \gg \frac{1}{\ln 1/\varepsilon} \gg \varepsilon\ln\varepsilon \gg \ln(1+\varepsilon) \gg e^{-1/\varepsilon}$$

The logarithm terms were a little tricky to order, but plotting definitely helped.



(c) Given

$$\begin{aligned}
\ln(1+\varepsilon) &= O(\varepsilon), \\
\operatorname{sech}^{-1}\varepsilon = \ln\left(\sqrt{\frac{1}{\varepsilon}-1}\sqrt{\frac{1}{\varepsilon}+1}+\frac{1}{\varepsilon}\right) &= O\left(\ln\frac{1}{\varepsilon}\right), \\
\frac{1-\cos\varepsilon}{1+\cos\varepsilon} &= O(\varepsilon^2), \\
\sqrt{\varepsilon(1-\varepsilon)} &= O(\varepsilon^{1/2}), \\
e^{-\cosh(1/\varepsilon)} &= O(e^{-e^{1/\varepsilon}}), \\
\ln\left(1+\frac{\ln((1+2\varepsilon)/\varepsilon)}{1-2\varepsilon}\right) &= O\left(\ln\left(\ln\frac{1}{\varepsilon}\right)\right), \\
\ln\left(1+\frac{\ln(1+\varepsilon)}{\varepsilon(1-2\varepsilon)}\right) &= O(1), \\
\frac{\varepsilon^{1/2}}{1-\cos\varepsilon} &= O(\varepsilon^{-3/2})
\end{aligned}$$

which can be ordered as

$$\begin{aligned}
\frac{\varepsilon^{1/2}}{1-\cos\varepsilon} &\gg \operatorname{sech}^{-1}\varepsilon \\
&\gg \ln\left(1+\frac{\ln((1+2\varepsilon)/\varepsilon)}{1-2\varepsilon}\right) \\
&\gg \ln\left(1+\frac{\ln(1+\varepsilon)}{\varepsilon(1-2\varepsilon)}\right) \\
&\gg \sqrt{\varepsilon(1-\varepsilon)} \\
&\gg \ln(1+\varepsilon) \\
&\gg \frac{1-\cos\varepsilon}{1+\cos\varepsilon} \\
&\gg e^{-\cosh(1/\varepsilon)}
\end{aligned}$$

Again, the complex log terms were the trickiest to place in the list but after a little bit of playing I reduced them down to orders that I could make sense of.