(1) Prove that

(a) If all singular values of a matrix $A \in \mathbb{C}^{n\times n}$ are equal, then $A = \gamma U$,where U is unitary and γ is a constant.
Proof:

Suppose A has singular values all equal to $\gamma \geq 0$. Then A has the SVD

$$A = W\Sigma V^*$$

where W and V are unitary and Σ is a diagonal matrix of γ. Then

$$A = W\Sigma V^* = W\gamma I V^* = \gamma W V^* = \gamma U$$

where $U = WV^*$ is unitary because it is the product of two unitary matrices. $\square$

(b) If $A \in \mathbb{C}^{n\times b}$ is non-singular and λ is an eigenvalue of A, then $\|A^{-1}\|_2^{-1} \leq |\lambda| \leq \|A\|_2$.
Proof:

Suppose $A \in \mathbb{C}n \times n$ is non-singular with an eigenvalue λ. Then, by the properties of induced matrix-norms, we have

$$|\lambda| \leq \rho(A) \leq \|A\|_2$$

where $\rho(A)$ denotes the spectral radius of A. Now, because A is non-singular, A^{-1} exists and

$$\rho(A^{-1}) = \frac{1}{\min\limits_{i=1,\ldots,n} |\lambda_i|}$$

where λ_i denotes the ith eigenvalue of A. Then

$$\frac{1}{\|A^{-1}\|_2} \leq \frac{1}{\rho(A^{-1})} = \frac{1}{\frac{1}{\min\limits_{i=1,\ldots,n} |\lambda_i|}} = \min_{i=1,\ldots,n} |\lambda_i| \leq |\lambda|\,.$$

Putting everything together yields

$$\|A^{-1}\|_2^{-1} \leq |\lambda| \leq \|A\|_2.$$

$\square$

(2) Show that any square matrix $A \in \mathbb{C}^{n\times n}$ may be represented in the form $A = SU$, where S is a Hermitian non-negative definite matrix and U is a unitary matrix. Show that if A is invertible such representation is unique.
Proof:

Suppose we have a matrix $A \in \mathbb{C}^{n\times n}$. Then, A has the SVD

$$A = W\Sigma V^*$$

where W and V are unitary and Σ is a matrix of the singular values. Then

$$A = W\Sigma V^* = W\Sigma W^* W V^* = SU \tag{1}$$

where $S = W\Sigma W^*$ and $U = WV^*$. Note, because Σ is a diagonal matrix of non-negative entries and W is unitary, S must be positive semi-definite and Hermitian. Furthermore, U is unitary because it is the product of two unitary matrices. So, we have the desired decomposition of A.

Now, suppose A is non-singular. Then

$$A = \underbrace{(AA^*)^{\frac{1}{2}}}_{S}\underbrace{(AA^*)^{-\frac{1}{2}}A}_{U} = SU$$

Then, because AA^* is non-singular and Hermitian positive definite, $S = (A^*A)^{1/2}$ is Hermitian positive-definite and unique. Now let's show that $U = (AA^*)^{-\frac{1}{2}}A$ is unitary and unique. Using a spectral decomposition, we have

$$U = (AA^*)^{-\frac{1}{2}}A = PD^{-\frac{1}{2}}P^*A$$

where P is unitary and D is diagonal. So, because P is the unitary matrix that came from AA^* in the spectral decomposition, P is actually the left singular matrix of the SVD of A. Now, we can write the SVD of A as $A = PD^{\frac{1}{2}}W^*$ which gives us

$$U = PD^{-\frac{1}{2}}P^*A = PD^{-\frac{1}{2}}P^*PD^{\frac{1}{2}}W^* = PW^*.$$

This shows that U is the product of two unitary matrices and is thus unitary. From (1), we can write U as

$$U = S^{-1}A$$

but we know S and A are non-singular and so U must be uniquely determined here. Thus, the decomposition, $A = SU$, is unique when A is non-singular. □

(3) Consider the Discrete Fourier transform (DFT) matrix $F \in \mathbb{C}^{n\times n}$,

$$F = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}$$

where $\omega = e^{i\frac{2\pi}{n}}$ is the nth root of unity. Show that $F^*F = nI$.
Proof:

Note, because $\bar{\omega} = e^{-i\frac{2\pi}{n}}$, we have

$$F^* = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{pmatrix}.$$

Furthermore, we have the identity

$$1 + \omega + \cdots + \omega^{n-1} = \sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega} = \frac{1-e^{2\pi i}}{1-\omega} = 0. \tag{2}$$

Now, let's look at the ith row and jth column of F^*F. If $i = j$, then

$$[F^*F]_{ii} = \sum_{k=0}^{n-1} \omega^{-k(i-1)}\omega^{k(i-1)} = \sum_{k=0}^{n-1} 1 = n.$$

Then, if $i \neq j$, we have

$$[F^*F]_{ij} = \sum_{k=0}^{n-1} \omega^{-k(i-1)}\omega^{k(j-1)} = \sum_{k=0}^{n-1} \omega^{k(j-i)}$$

which is just a rearrangement of (2) because ω^k is n-periodic in k. So, we have

$$[F^*F]_{ij} = \sum_{k=0}^{n-1} \omega^{k(j-i)} = \sum_{k=0}^{n-1} \omega^k = 0.$$

This shows that the diagonal entries of F^*F are n and the off diagonal entries of F^*F are zero. So, we can factor out the diagonal to get

$$F^*F = nI$$

$\square$