1. Prove the following for $x \in \mathbb{C}^n$:

   (a) $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$.
   Proof:

   For the first inequality, we have

$$\begin{aligned}\|x\|_\infty &= \max_{1\leq i\leq n} |x_i| \\ &\leq \sum_{i=1}^{n} |x_i| \\ &= \|x\|_1.\end{aligned}$$

   For the second half of our inequality chain, we have

$$\begin{aligned}\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ &\leq \sum_{i=1}^{n} \left(\max_{1\leq j\leq n} |x_j|\right) \\ &= n \max_{1\leq j\leq n} |x_j| \\ &= n\|x\|_\infty\end{aligned}$$

   Thus, $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$. □

   (b) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$.
   Proof:

   For the first inequality, we have

$$\begin{aligned}\|x\|_\infty &= \max_{1\leq i\leq n} |x_i| \\ &= \sqrt{\left(\max_{1\leq i\leq n} |x_i|\right)^2} \\ &\leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \\ &= \|x\|_2.\end{aligned}$$

For the second half of our inequality chain, we have

$$\begin{aligned}
\|x\|_2 &= \sqrt{\sum_{i=1}^{n} |x_i|^2} \\
&\leq \sqrt{\sum_{i=1}^{n} \left(\max_{1\leq j\leq n} |x_j|\right)^2} \\
&= \sqrt{n\left(\max_{1\leq j\leq n} |x_j|\right)^2} \\
&= \sqrt{n} \max_{1\leq j\leq n} |x_j| \\
&= n\|x\|_\infty.
\end{aligned}$$

Thus, $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$. □

(c) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$.
Proof:

Let's work on the first half of the inequality:

$$\begin{aligned}
\|x\|_2 &= \sqrt{\sum_{i=1}^{n} |x_i|^2} \\
&\leq \sqrt{\sum_{i=1}^{n} |x_i|^2 + \sum_{i,j,i\neq j}^{n} |x_i|\,|x_j|} \\
&= \sqrt{\left(\sum_{i=1}^{n} |x_i|\right)\left(\sum_{j=1}^{n} |x_j|\right)} \\
&= \sqrt{\left(\sum_{i=1}^{n} |x_i|\right)^2} \\
&= \sum_{i=1}^{n} |x_i| \\
&= \|x\|_1.
\end{aligned}$$

Now, let's show the second half of the inequality

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\
&= \sum_{i=1}^{n} |1|\,|x_i| \\
&\leq \|\vec{1}\|_2\|x\|_2 \qquad \text{by Cauchy-Shwartz} \\
&= \left(\sum_{i=1}^{n} |1|^2\right)\|x\|_2 \\
&= \sqrt{n}\|x\|_2.
\end{aligned}$$

Thus, $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$. □

2. Let $A \in \mathbb{R}^{n\times m}$ be a non-zero matrix with rank r.

   (a) Write down the singular value decomposition of A. List the properties of the matrices you use in your decomposition.

   We can write the SVD of A as
$$A = U\Sigma V^T$$
   Where $U \in \mathbb{R}^{n\times n}$ is made up of the left singular vectors, $\Sigma \in \mathbb{R}^{n\times m}$ is a semi-diagonal matrix with positive singular values along the main diagonal, and $V \in \mathbb{R}^{m\times m}$ is made up of the right singular vectors.

   Some properties of U

   - it is orthonormal
   - the columns of U are made up of the normalized eigenvectors of AA^T (i.e. $U = [u_1|u_2|\dots|u_n]$).

   Some properties of V

   - it is orthonormal
   - the columns of V are made up of the normalized eigenvectors of A^TA (i.e. $V = [v_1|v_2|\dots|v_n]$).

   Finally, some properties of Σ

   - the diagonal entries of Σ are given by the singular values σ_i for $i = 1, 2, \dots, r$. The rest of the entries of Σ are zero.
   - by convention, we organize the singular values from the left diagonal to right diagonal as $\sigma_1 \ge \dots \ge \sigma_r \ge 0$.

   (b) Show that $\mathbb{R}^m$ has an orthonormal basis $v_1, \dots, v_m$, $\mathbb{R}^n$ has an orthonormal basis $u_1, \dots, u_n$ and there exists $\sigma_1 \ge \dots \ge \sigma_r \ge 0$ such that
$$Av_i = \begin{cases} \sigma_i u_i, & i = 1, \dots, r \\ 0, & i = r+1, \dots, m \end{cases},$$
$$A^T u_i = \begin{cases} \sigma_i v_i, & i = 1, \dots, r \\ 0, & i = r+1, \dots, m \end{cases}.$$

   First off, because we are guaranteed a SVD for all $A \in \mathbb{R}^{n\times m}$, we can get our decomposition given in part (a). Using our SVD from part (a), we know the columns of V form an orthonormal basis for $\mathbb{R}^m$ given by $v_1, \dots, v_m$, and the columns of U form an orthonormal basis for $\mathbb{R}^n$ given by $u_1, \dots, u_n$. So we can get the two orthonormal bases that we desired.

   Now, using the SVD in part (a), we have $\sigma_1 \ge \dots \ge \sigma_r \ge 0$ from the singular values matrix Σ. Putting all of this together,
$$\begin{aligned} Av_i = U\Sigma V^T v_i &= \begin{cases} \sigma_i u_i (v_i \cdot v_i), & i = 1, \dots, r \\ 0, & i = r+1, \dots, m \end{cases} \\ &= \begin{cases} \sigma_i u_i, & i = 1, \dots, r \\ 0, & i = r+1, \dots, m \end{cases} \end{aligned}$$

because $v_i \cdot v_j = 0$ if $i \neq j$ and $v_i \cdot v_j = 1$ if $i = j$ by the orthonormalness of V.
For the second equality, we have

$$A^T u_i = (U\Sigma V^T)^T u_i = V\Sigma^T U^T u_i = \begin{cases} \sigma_i v_i (u_i \cdot u_i), & i = 1, \dots, r \\ 0, & i = r+1, \dots, n \end{cases}$$
$$= \begin{cases} \sigma_i v_i, & i = 1, \dots, r \\ 0, & i = r+1, \dots, n \end{cases}$$

because $u_i \cdot u_j = 0$ if $i \neq j$ and $u_i \cdot u_j = 1$ if $i = j$ by the orthonormalness of U.

(c) Argue that

$$\text{Range}(A) = \text{span}\{u_1, \dots, u_r\} \tag{1}$$
$$\text{Null}(A) = \text{span}\{v_{r+1}, \dots, v_m\} \tag{2}$$
$$\text{Range}(A^T) = \text{span}\{v_1, \dots, v_r\} \tag{3}$$
$$\text{Null}(A^T) = \text{span}\{u_{r+1}, \dots, u_n\}. \tag{4}$$

- To understand (1), suppose we have any $x \in \mathbb{R}^m$. Then, because we only have r singular values and the rest of Σ is either rows of zeros or columns of zeros, we have

$$Ax = U\Sigma V^T x = \sum_{i=1}^{r} \sigma_i u_i v_i^T x = \sum_{i=1}^{r} (\sigma_i v_i^T x) u_i = \sum_{i=1}^{r} a_i u_i$$

where $a_i = \sigma_i v_i^T x$ for $i = 1, \dots, r$ are constants. Thus, Ax can be written as a linear combination of $\{u_1, \dots, u_r\}$ for all $x \in \mathbb{R}^m$ which implies

$$\text{Range}(A) = \text{span}(u_1, \dots, u_r).$$

- To argue (2), let's find the nullspace previous part:

$$Ax = \sum_{i=1}^{r} a_i u_i$$

where $a_i = \sigma_i v_i^T x$ for $i = 1, \dots r$. From this expression, we can see that nullspace is the set of all x such that $a_i = 0$ for $i = 1, \dots, r$. In other words, we need x to satisfy

$$\sigma_i v_i^T x = 0$$

for each $i = 1, \dots, r$. We know each $\{v_1, \dots, v_m\}$ is an orthogonal set and so the only x that make $a_i = 0$ are $x = b_{r+1} v_{r+1} + \cdots + b_m v_m$ which implies that

$$\text{Null}(A) = \text{span}(v_{r+1}, \dots, v_m).$$

- Similar to (1), we can show (3). Suppose we have any $x \in \mathbb{R}^n$. Then,

$$A^T x = V\Sigma^T U^T x = \sum_{i=1}^{r} \sigma_i v_i u_i^T x = \sum_{i=1}^{r} (\sigma_i u_i^T x) v_i = \sum_{i=1}^{r} a_i v_i$$

where $a_i = \sigma_i u_i^T x$ for $i = 1, \dots, r$ constants. Thus, $A^T x$ can be written as a linear combination of $\{v_1, \dots, v_r\}$ for all $x \in \mathbb{R}^n$ which implies

$$\text{Range}(A^T) = \text{span}(v_1, \dots, v_r).$$

- We can argue (4) by applying the argument for (2) to (3) with the roles of u_i and v_i switched.

(d) Now show that Range(A^T) is orthogonal to Null(A).
Suppose we have any $x \in$ Range(A^T) and any $y \in$ Null(A). Then, from part (c), we know that

$$x = a_1 v_1 + \cdot + a_r v_r$$

for some $a_i \in \mathbb{R}$ and that

$$y = b_{r+1} v_{r+1} + \cdot + b_m v_m$$

for some $b_i \in \mathbb{R}$. Then,

$$\begin{aligned}
x \cdot y &= \sum_{i=r+1}^{m} x \cdot (b_i v_i) \\
&= \sum_{i=r+1}^{m} \sum_{j=1}^{r} (a_j v_j) \cdot (b_i v_i) \\
&= \sum_{i=r+1}^{m} \sum_{j=1}^{r} a_j b_i (v_j \cdot v_i) \\
&= \sum_{i=r+1}^{m} \sum_{j=1}^{r} a_j b_i (0) \qquad \text{because } i \neq j \text{ and } v_j \cdot v_i = 0 \text{ for } i \neq j \\
&= 0.
\end{aligned}$$

Therefore, Range(A^T) is orthogonal to Null(A).

3. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and $u, v \in \mathbb{R}^n$.

(a) Prove the following matrix identity (Sherman-Morrison)

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

Proof:

To prove the Sherman-Morrison formula, we just need to show the RHS is the inverse of $A + uv^T$.

- First direction

$$\begin{aligned}
\left(A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u} \right) (A + uv^T) &= \\
&= I.
\end{aligned}$$

- The second direction can be shown as

$$\begin{aligned}
(A + uv^T) \left(A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u} \right) &= \\
&= I.
\end{aligned}$$

Therefore

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}$$

□