(1) A circulant matrix of size $(2n+1)\times(2n+1)$ has the form

$$C = \begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_{2n} \\ a_2 & a_0 & a_1 & \cdots & a_{2n-1} \\ a_{2n-1} & a_{2n} & a_0 & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{2n} & a_0 \end{pmatrix}.$$

Furthermore, let $S$ denote the matrix that shifts the index of a vector by 1. In this case, $S$ will be a $(2n+1)\times(2n+1)$ matrix of the form

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

(a) Show that any circulant matrix can be written as a polynomial of the $S$ matrix.

To create a polynomial that makes any circulant matrix, notice that each $a_0$ in $C$ is along the diagonal which can be formed by $a_0 I = a_0 S^0$. Next, notice that each $a_1$ in $C$ can be given by $a_1 S^1$. Continuing this trend, we can see that each $a_{2n}$ in $C$ can be given by $a_{2n}S^{2n}$. Finally, we can simply add up each term to get

$$C = a_0 + a_1 S + a_2 S^2 + \cdots + a_{2n}S^{2n}$$

which is a polynomial of $S$.

(b) Let $v^k$ denote the $k$th orthogonal Fourier basis vector where the $j$th entry of $v^k$ is given by

$$v_j^k = e^{\frac{2\pi ijk}{2n+1}}.$$

Prove that the vectors $v^k$ are all the eigenvectors of the circulant matrix. Furthermore, what are the eigenvalues?

To prove that the vectors $v^k$ are the eigenvectors of the circulant matrix, let's check that each of these “eigenvectors” is actually an eigenvector of $C$:

$$\begin{aligned}
Cv^k &= (a_0 + a_1 S + a_2 S^2 + \cdots + a_{2n} S^{2n}) v^k \\
&= a_0 v^k + a_1 S v^k + a_2 S^2 v^k + \cdots + a_{2n} S^{2n} v^k \\
&= \begin{pmatrix} a_0 v_0^k + a_1 v_1^k + a_2 v_2^k + \cdots + a_{2n} v_{2n}^k \\ a_0 v_1^k + a_1 v_2^k + a_2 v_3^k + \cdots + a_{2n} v_0^k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_0 v_{2n}^k + a_1 v_0^k + a_2 v_1^k + \cdots + a_{2n} v_{2n-1}^k \end{pmatrix} \\
&= \begin{pmatrix} a_0 e^{\frac{2\pi i 0k}{2n+1}} + a_1 e^{\frac{2\pi i 1k}{2n+1}} + a_2 e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i 2nk}{2n+1}} \\ a_0 e^{\frac{2\pi i 1k}{2n+1}} + a_1 e^{\frac{2\pi i 2k}{2n+1}} + a_2 e^{\frac{2\pi i 3k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i 0k}{2n+1}} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_0 e^{\frac{2\pi i 2nk}{2n+1}} + a_1 e^{\frac{2\pi i 0k}{2n+1}} + a_2 e^{\frac{2\pi i 1k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i (2n-1)k}{2n+1}} \end{pmatrix} \\
&= \left( a_0 + a_1 e^{\frac{2\pi i 1k}{2n+1}} + a_2 e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i 2nk}{2n+1}} \right) \begin{pmatrix} e^{\frac{2\pi i 0k}{2n+1}} \\ e^{\frac{2\pi i 1k}{2n+1}} \\ \vdots \\ e^{\frac{2\pi i 2nk}{2n+1}} \end{pmatrix} \\
&= \left( a_0 + a_1 e^{\frac{2\pi i 1k}{2n+1}} + a_2 e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i 2nk}{2n+1}} \right) v^k.
\end{aligned}$$

Thus, $v^k$ is an eigenvector of $C$ with eigenvalue

$$\lambda_k = a_0 + a_1 e^{\frac{2\pi i 1k}{2n+1}} + a_2 e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n} e^{\frac{2\pi i 2nk}{2n+1}}.$$

Finally, because the collection of $v^k$ forms a basis of dimension $2n+1$ and $C$ is a $(2n+1) \times (2n+1)$ matrix, the collection of $v^k$ is all of the eigenvectors of $C$.

(2) Let $0 \leq t_0 < t_1 < \cdots < t_{2n} < w\pi$ and consider the trigonometric polynomial interpolation problem: define

$$l_j(t) = \prod_{\substack{k=0 \\ k \neq j}}^{2n} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)}$$

for $j = 0, 1 \ldots, 2n$. It is easy to show that $l_j(t_i) = \delta_{ij}$ for each $j$.

Show that $l_j(t)$ is a trigonometric polynomial of degree less than or equal to $n$. Then the solution of the trigonometric interpolation problem is given by

$$p_n(t) = \sum_{j=0}^{2n} f(t_j) l_j(t).$$