Problems

1.

(i) Given $x_0 = -0.2, x_1 = 0$, and $x_2 = 0.2$ construct a second degree polynomial to approximate $f(x) = e^x$ via Newton's divided differences.
We want to derive a polynomial of the form

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

where $a_i = [x_0, \ldots, x_{i-1}]$ are the Newton Divided differences. For this problem, we have

$$\begin{aligned}
a_0 &= f[x_0] = e^{x_0} = e^{-0.2}, \\
a_1 &= f[x_0, x_1] = \frac{e^{x_1} - e^{x_0}}{x_1 - x_0} = \frac{1 - e^{-0.2}}{0.2},
\end{aligned}$$

and

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{e^{0.2}-1}{0.2} - \frac{1-e^{-0.2}}{0.2}}{0.4}$$

which makes our polynomial

$$\begin{aligned}
p(x) &= e^{-0.2} + \frac{1 - e^{-0.2}}{0.2}(x + 0.2) + \frac{\frac{e^{0.2}-1}{0.2} - \frac{1-e^{-0.2}}{0.2}}{0.4}(x + 0.2)(x) \\
&= 1 + 1.00668x + 0.501669x^2.
\end{aligned}$$

(ii) Derive an error bound for $p_2(x)$ when $x \in [-0.2, 0.2]$.
First, note that the third derivative of f is maximized over $[-0.2, 0.2]$ when $x = 0.2$. Then, we can obtain a bound on our error as

$$\begin{aligned}
E(t) &\leq \max_{t\in[-0.2,0.2]} E(t) \\
&= \max_{t\in[-0.2,0.2]} \frac{(t + 0.2)(t)(t - 0.2)}{6} e^{0.2} \\
&= \frac{(-\frac{\sqrt{3}}{15} + 0.2)(-\frac{\sqrt{3}}{15})(-\frac{\sqrt{3}}{15} - 0.2)}{6} e^{0.2} \\
&= 6.26824 \cdot 10^{-4}.
\end{aligned}$$

(iii) Compute the error $E(0.1) = f(0.1) - p_2(0.1)$. How does this compare with the error bound?
Our error is

$$E(0.1) = |1.10517 - 1.10568| = 5.136621 \cdot 10^{-4}$$

which is within our error bound! So our error bound holds $x = 0.1$.

2.

(i) Show there is a unique cubic polynomial $p(x)$ for which

$$p(x_0) = f(x_0) \qquad p(x_2) = f(x_2)$$
$$p'(x_1) = f'(x_1) \qquad p''(x_1) = f''(x_1)$$

where $f(x)$ is a given function and $x_0 \neq x_2$.
Suppose $p(x)$ and $q(x)$ are two cubic polynomials satisfying

$$p(x_0) = q(x_0) = f(x_0) \qquad p(x_2) = q(x_2) = f(x_2)$$
$$p'(x_1) = q(x_1) = f'(x_1) \qquad p''(x_1) = q''(x_1) = f''(x_1).$$

Now let $v(x) + p(x) - q(x)$. Then, by linearity, $v(x)$ is a cubic polynomial that satisfies

$$v(x_0) = 0 \qquad v(x_2) = 0$$
$$v'(x_1) = 0 \qquad v''(x_1) = 0.$$

Furthermore, because $v(x)$ is a cubic polynomial, there exists constants a_0, a_1, a_2, and a_3 such that

$$\begin{aligned} v(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ v'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \\ v''(x) &= 2a_2 + 6a_3 x. \end{aligned}$$

Then,

$$v''(x_1) = 2a_2 + 6a_3 x_1 = 0$$

which implies

$$a_2 = -3a_3.$$

Using this expression in our first derivative yields

$$v'(x_1) = a_1 - 6a_3 x_1^2 + 3a_3 x_1^2 = 0$$

which implies

$$a_1 = 3a_3 x_1^2.$$

Finally,

$$\begin{aligned} v(x) &= a_0 + 3a_3 x_1^2 x - 3a_3 x_1 x^2 + a_3 x^3 \\ &= a_0 + 3a_3 x_1^2 x - 3a_3 x_1 x^2 + a_3 x^3 - a_3 x_1^3 + a_3 x_1^3 \\ &= (a_0 + a_3 x_1^3) - a_3 (x - x_1)^3. \end{aligned}$$

Then, using our last constraints, we have the system

$$\begin{aligned} v(x_0) &= (a_0 + a_3 x_1^3) - a_3 (x_0 - x_1)^3 = 0 \\ v(x_2) &= (a_0 + a_3 x_1^3) - a_3 (x_2 - x_1)^3 = 0 \end{aligned}$$

which yields

$$a_3 (x_0 - x_1)^3 = a_3 (x_2 - x_1)^3.$$

Then, because $x_0 \neq x_2$, we must have $a_3 = 3$ which implies $a_0 = 0$. Therefore,

$$v(x) = 0$$

and so we must have

$$p(x) = q(x)$$

showing the uniqueness of our polynomial.

(ii) Derive a formula for $p(x)$.

We know $p(x)$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

for some constants a_0, a_1, a_2, and a_3. Then, using our second derivative information, we have

$$p''(x_1) = 2a_2 + 6a_3 x_1 = f''(x_1)$$

which implies

$$a_2 = \frac{1}{2} f''(x_1) - 3a_3 x_1.$$

Then, using our first derivative information, we have

$$p'(x_1) = a_1 + 2\left(\frac{1}{2} f''(x_1) - 3a_3 x_1\right) x_1 + 3a_3 x_1^2 = f'(x_1)$$

which implies

$$a_1 = f'(x_1) - f''(x_1)x_1 + 3a_3 x_1^2.$$

Next, we can use our function information to get the system

$$\begin{aligned} p(x_0) &= a_0 + f'(x_1)x_0 - f''(x_1)x_0 x_1 + 3a_3 x_0 x_1^2 + \frac{1}{2} f''(x_1)x_0^2 - 3a_3 x_0^2 x_1 + a_3 x_0^3 = f(x_0) \\ p(x_2) &= a_0 + f'(x_1)x_2 - f''(x_1)x_1 x_2 + 3a_3 x_1^2 x_2 + \frac{1}{2} f''(x_1)x_2^2 - 3a_3 x_1 x_2^2 + a_3 x_2^3 = f(x_2) \end{aligned}$$

which implies

$$\begin{aligned} a_3 &= \frac{f(x_2) - f(x_0) + f'(x_1)(x_0 - x_2) - f''(x_1)x_1(x_0 - x_2) + \frac{1}{2} f''(x_1)(x_0^2 - x_2^2)}{3x_1^2(x_2 - x_0) + 3x_1(x_0^2 - x_2^2) + x_2^3 - x_0^3} \\ a_0 &= f(x_0) - f'(x_1)x_0 + f''(x_1)x_0 x_1 + 3a_3 x_0 x_1^2 - \frac{1}{2} f''(x_1)x_0^2 + 3a_3 x_0^2 x_1 - a_3 x_0^3. \end{aligned}$$

Now, we construct our polynomial by first computing a_3, a_0, a_1, and a_2 in that order and then plugging them into our polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

(iii) Let $x_0 = -1, x_1 = 0$, and $x_2 = 1$. Assuming $f(x) \in C^4[-1, 1]$, show that for $x \in [-1, 1]$,

$$f(x) - p(x) = \frac{x^4 - 1}{4!} f^4(\eta_x)$$

for some $\eta_x \in [-1, 1]$.

3. Suppose we have m data points $\{(t_i, y_i)\}_{i=1}^{m}$, where the t-values all occur in some interval $[x_0, x_n]$. We subdivide the interval $[x_0, x_n]$ into n subintervals $\{[x_k, x_{k+1}]_{k=0}^{n-1}\}$ of equal length h and attempt to choose a spline function $s(x)$ with nodes at $\{x_k\}_{k=0}^{n}$ in such a way so that

$$\sum_{i=1}^{m} |y_i - s(t_i)|^2$$

is *minimized.*

   (i) *Sheeeeeeeeeeeeesh*

Code Used