Book Problems

1. Chapter 4, Problem 10: Let $f(x,t)$ be a continuous function, and let $\Delta(x,t)$ denote the domain of dependence of the point (x,t) for
$$u_{tt} = c^2 u_{xx}.$$
Show that $u(x,t) = \frac{1}{2c}\iint_{\Delta(y,\tau)} f(x,t)\mathrm{d}y\mathrm{d}\tau$ satisfies
$$u_{tt} = c^2 u_{xx} + f(x,t), u(x,0) = 0 = u_t(x,0).$$

First, let's rewrite u with more solid bounds:
$$u(x,t) = \frac{1}{2c}\int_0^t \int_{c(\tau-t)+x}^{-c(\tau-t)+x} f(y,\tau)\mathrm{d}y\mathrm{d}\tau.$$

Then, using Leibniz integral rule, our derivatives become
$$\begin{aligned}
u_{tt} &= \frac{1}{2c}\frac{\partial}{\partial t}\left(c\int_0^t f(c(t-\tau)+x,\tau) + f(-c(t-\tau)+x)\mathrm{d}\tau\right) \\
&= f(x,t) + \frac{1}{2c}c^2\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau \\
u_{xx} &= \frac{1}{2c}\frac{\partial}{\partial x}\left(\int_0^t f(c(t-\tau)+x,\tau) - f(-c(t-\tau)+x)\mathrm{d}\tau\right) \\
&= \frac{1}{2c}\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau.
\end{aligned}$$

Then, our PDE becomes
$$\begin{aligned}
u_{tt} - c^2u_{xx} - f(x,t) &= f(x,t) + \frac{1}{2c}c^2\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau \\
&\quad - c^2\frac{1}{2c}\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau - f(x,t) \\
&= f(x,t) + \frac{c}{2}\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau \\
&\quad - \frac{c}{2}\int_0^t f_x(c(t-\tau)+x,\tau) - f_x(-c(t-\tau)+x)\mathrm{d}\tau - f(x,t) \\
&= 0.
\end{aligned}$$

So u satisfies the PDE, now let's check the initial conditions
$$\begin{aligned}
u(x,0) &= \frac{1}{2c}\int_0^0 \int_{c(\tau-0)+x}^{-c(\tau-0)+x} f(y,\tau)\mathrm{d}y\mathrm{d}\tau = 0 \\
u_t(x,0) &= \frac{1}{2}\int_0^0 f(c(0-\tau)+x,\tau) + f(-c(0-\tau)+x)\mathrm{d}\tau = 0.
\end{aligned}$$

Thus, u solves the IVP.

2. Chapter 5, Problem 6: Solve the heat equation with initial condition $u(x,0) = H(x)e^{-x}$ where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x \end{cases}.$$

Using the fundamental solution, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} H(y) e^{-y} \mathrm{d}y \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y} \mathrm{d}y \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{x}^{-\infty} e^{-\frac{u^2}{4kt}} e^{u-x} \mathrm{d}u \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{x}^{-\infty} e^{-\frac{1}{4kt}(u^2-4ktu)} e^{-x} \mathrm{d}u \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{x}^{-\infty} e^{-\frac{1}{4kt}(u^2-4ktu+(2kt)^2-(2kt)^2)} e^{-x} \mathrm{d}u \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{x}^{-\infty} e^{-\frac{1}{4kt}(u-2kt)^2} e^{kt-x} \mathrm{d}u \\
&= e^{kt-x} \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x} e^{-\left(\frac{u}{2\sqrt{kt}}-\sqrt{kt}\right)^2} \mathrm{d}u \\
&= e^{kt-x} \frac{2\sqrt{kt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\frac{x}{2\sqrt{kt}}-\sqrt{kt}} e^{-v^2} \mathrm{d}v \\
&= e^{kt-x} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{kt}}-\sqrt{kt}} e^{-v^2} \mathrm{d}v \\
&= e^{kt-x} \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} - \sqrt{kt}\right)\right) \\
&= \frac{1}{2} e^{kt-x} \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} - \sqrt{kt}\right)\right).
\end{aligned}$$

Additional Problems

1. The integral form of Maxwell's equation can be written:

$$\text{Gauss' law:} \quad \int_{\partial\Omega} \mathbf{D} \cdot \mathbf{n} \mathrm{d}S = 0 \tag{1}$$

$$\text{no magnetic charge:} \quad \int_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} \mathrm{d}S = 0 \tag{2}$$

$$\text{Ampère's law:} \quad \int_{\partial\Sigma} \mathbf{H} \cdot \mathbf{n} \mathrm{d}l - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma} \mathbf{D} \cdot \mathbf{n} \mathrm{d}S = 0 \tag{3}$$

$$\text{Faraday's law:} \quad \int_{\partial\Sigma} \mathbf{E} \cdot \mathbf{n} \mathrm{d}l - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma} \mathbf{B} \cdot \mathbf{n} \mathrm{d}S = 0 \tag{4}$$

where $\Omega \subset \mathbb{R}^3$ is any three-dimensional volume and $\partial\Omega$ is its boundary, $\Sigma \subset \mathbb{R}^3$ is any two-dimensional surface and $\partial\Sigma$ is its boundary, and $\mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{E}$ are smooth vector fields mapping $\mathbb{R}^3 \times \mathbb{R}$ to $\mathbb{R}^3$.

(a) Express Maxwell's equation in local, differential form.

Using the divergence theorem and Stoke's theorem, we have

$$
\begin{aligned}
&(1) \int_{\partial\Omega} \mathbf{D}\cdot\mathbf{n}\mathrm{d}S = \int_{\Omega} \nabla\cdot\mathbf{D}\mathrm{d}V = 0 \implies \nabla\cdot\mathbf{D} = 0\\
&(2) \int_{\partial\Omega} \mathbf{B}\cdot\mathbf{n}\mathrm{d}S = \int_{\Omega} \nabla\cdot\mathbf{B}\mathrm{d}V = 0 \implies \nabla\cdot\mathbf{B} = 0\\
&(3) \int_{\partial\Sigma} \mathbf{H}\cdot\mathbf{n}\mathrm{d}l - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma}\mathbf{D}\cdot\mathbf{n}\mathrm{d}S = \int_{\Sigma}(\nabla\times\mathbf{H})\cdot\mathbf{n}\mathrm{d}S - \int_{\Sigma}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\cdot\mathbf{n}\mathrm{d}S\\
&= \int_{\Sigma}(\nabla\times\mathbf{H})\cdot\mathbf{n} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\cdot\mathbf{n}\mathrm{d}S = 0 \implies \nabla\times\mathbf{H} = \frac{\mathrm{d}\mathbf{D}}{\mathrm{d}t}\\
&(4) \int_{\partial\Sigma} \mathbf{E}\cdot\mathbf{n}\mathrm{d}l - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma}\mathbf{B}\cdot\mathbf{n}\mathrm{d}S = \int_{\Sigma}(\nabla\times\mathbf{E})\cdot\mathbf{n}\mathrm{d}S - \int_{\Sigma}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}\cdot\mathbf{n}\mathrm{d}S\\
&= \int_{\Sigma}(\nabla\times\mathbf{E})\cdot\mathbf{n} - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}\cdot\mathbf{n}\mathrm{d}S = 0 \implies \nabla\times\mathbf{E} = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}
\end{aligned}
$$

(b) Assuming the constitutive laws

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}$$

where $\varepsilon > 0$ and $\mu > 0$, derive the $(3+1)D$ (Vector) linear wave equation for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$. What is the electromagnetic wave speed?

Using the constitutive laws, we can rewrite Maxwell's local differential equations as

$$\varepsilon\nabla\cdot\mathbf{E} = 0 \tag{5}$$
$$\mu\nabla\cdot\mathbf{H} = 0 \tag{6}$$
$$\nabla\times\mathbf{H} = \varepsilon\mathbf{E}_t \tag{7}$$
$$\nabla\times\mathbf{E} = -\mu\mathbf{H}_t. \tag{8}$$

Now, we can start deriving the electromagnetic linear wave equations. From (7), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla\times\mathbf{H} = \varepsilon\frac{\partial}{\partial t}\mathbf{E}_t &\implies \nabla\times\mathbf{H}_t = \varepsilon\mathbf{E}_{tt}\\
\text{which from (8)} &\implies -\frac{1}{\mu}\nabla\times\nabla\times\mathbf{E} = \varepsilon\mathbf{E}_{tt}\\
&\implies -\frac{1}{\mu}(\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}) = \varepsilon\mathbf{E}_{tt}\\
&\implies \varepsilon\mathbf{E}_{tt} - \frac{1}{\mu}\nabla^2\mathbf{E} = -\frac{1}{\mu}\nabla(\nabla\cdot\mathbf{E})\\
\text{which by (5)} &\implies \mathbf{E}_{tt} - \frac{1}{\mu\varepsilon}\nabla^2\mathbf{E} = \mathbf{0}
\end{aligned}
$$

Now, by the same process, we can get the other wave equation. From (7)

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla \times \mathbf{E} = -\mu \mathbf{H}_{tt} &\implies \nabla \times \mathbf{E}_t = -\mu \mathbf{H}_{tt} \\
\text{which by (8)} &\implies \frac{1}{\varepsilon}\nabla \times \nabla \times \mathbf{H} = -\mu \mathbf{H}_{tt} \\
&\implies \frac{1}{\varepsilon}(\nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}) = -\mu \mathbf{H}_{tt} \\
&\implies \mu \mathbf{H}_{tt} - \frac{1}{\varepsilon}\nabla^2 \mathbf{H} = \frac{1}{\varepsilon}\nabla(\nabla \cdot \mathbf{H}) \\
\text{which by (6)} &\implies \mathbf{H}_{tt} - \frac{1}{\mu\varepsilon}\nabla^2 \mathbf{H} = \mathbf{0}.
\end{aligned}
$$

Thus, our electromagnetic wave equations are given by

$$
\begin{aligned}
\mathbf{E}_{tt} - \frac{1}{\mu\varepsilon}\nabla^2 \mathbf{E} = \mathbf{0} \\
\mathbf{H}_{tt} - \frac{1}{\mu\varepsilon}\nabla^2 \mathbf{H} = \mathbf{0}
\end{aligned}
$$

where the wave speed is given by $\sqrt{\frac{1}{\mu\varepsilon}}$. On a fun note, in a vacuum, this wave speed actually becomes the speed of light and so we can see that electromagnetic waves travel at the speed of light!

2. Assuming the following initial value problems have a solution, determine whether or not that solution is unique, justifying your answer. Assume u is twice continuously differentiable and its first and second derivatives are L^2 integrable.

   Note, because our functions are L^2 integrable up to the second derivative, boundary evaluations at $\pm\infty$ must go to zero.

   (a) Inhomogeneous wave equation with boundary:

$$
\begin{cases}
u_{tt} - c^2 u_{xx} + f(x,t), & c > 0, x > 0, t > 0 \\
u(x,0) = \phi(x), u_t(x,0) = \psi(x), & x > 0 \\
u(0,t) = h(t), & t > 0.
\end{cases}
$$

   To check the uniqueness of this problem, suppose we have to solutions to the IVBP, u_1 and u_2. Then by linearity, $v(x,t) = u_1 - u_2$ is a solution to the IVBP

$$
\begin{cases}
v_{tt} - c^2 v_{xx} = 0, & c > 0, x > 0, t > 0 \\
v(x,0) = 0, v_t(x,0) = 0, & x > 0 \\
u(0,t) = 0, & t > 0
\end{cases}.
$$

   From the boundary condition we must have

$$
u_t(0,t) = 0.
$$

Now, consider $v_t(v_{tt} - c^2 v_{xx}) = 0$. Integrating yields

$$\begin{aligned}
&\int_0^\infty v_t v_{tt} \mathrm{d}x - c^2 \int_0^\infty v_t v_{xx} \mathrm{d}x = 0 \\
\implies &\int_0^\infty \partial_t \left(\frac{1}{2} v_t^2\right) \mathrm{d}x - c^2 v_t v_x \big|_{x=0}^\infty + c^2 \int_0^\infty v_x v_{xt} \mathrm{d}x = 0 \\
\implies &\int_0^\infty \partial_t \left(\frac{1}{2} v_t^2\right) \mathrm{d}x + c^2 \int_0^\infty \partial_t \left(\frac{1}{2} v_x^2\right) \mathrm{d}x = c^2 v_t v_x \big|_{x=0}^\infty \\
\implies &\partial_t \underbrace{\int_0^\infty \frac{1}{2} v_t^2 + \frac{1}{2} c^2 v_x^2 \mathrm{d}x}_{E(t)} = c^2 v_t v_x \big|_{x=0}^\infty .
\end{aligned}$$

Then, because our initial data is compactly supported and bounded, and that $v_t(0, t) = 0$, we have

$$\partial_t E(t) = c^2 v_t v_x \big|_{x=0}^\infty = 0$$

which implies

$$E(t) = const.$$

But, we have

$$E(0) = \int_0^\infty \frac{1}{2} 0^2 + \frac{1}{2} c^2 0^2 \mathrm{d}x = 0 = const.$$

Thus, $E(t) = 0$ for all $t > 0$ which implies $v(x, t) = 0$ for all $x > 0$ and $t > 0$. Therefore,

$$u_1 = u_2$$

over our whole domain which implies that solutions to our original IBVP are unique.

(b) Damped wave equation:

$$\begin{cases} u_{tt} + \mu u_t - c^2 u_{xx} = 0, & \mu > 0, c > 0, x \in \mathbb{R}, t > 0 \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), & x \in \mathbb{R}. \end{cases}$$

Suppose u_1 and u_2 are two solutions to the IVP above. Then by linearity,$v = u_1 - u_2$ solves the IVP

$$\begin{cases} v_{tt} + \mu v_t - c^2 v_{xx} = 0, & \mu > 0, c > 0, x \in \mathbb{R}, t > 0 \\ v(x, 0) = 0, v_t(x, 0) = 0, & x \in \mathbb{R}. \end{cases}$$

Now, consider $v_t v_{tt} + \mu v_t v_t - c^2 v_t v_{xx} = 0$. Similar to part (a) integrating yields

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_t \left(\frac{1}{2} v_t^2\right) \mathrm{d}x + \int_{\mathbb{R}} \mu v_t^2 \mathrm{d}x - c^2 \int_{\mathbb{R}} v_t v_{xx} \mathrm{d}x = 0 \\
\implies &\partial_t \int_{\mathbb{R}} \left(\frac{1}{2} v_t^2\right) \mathrm{d}x + \int_{\mathbb{R}} \mu v_t^2 \mathrm{d}x - c^2 \ v_t v_x \big|_{x=-\infty}^\infty + \int_{\mathbb{R}} c^2 v_t v_{xt} \mathrm{d}x = 0 \\
\implies &\partial_t \int_{\mathbb{R}} \left(\frac{1}{2} v_t^2\right) \mathrm{d}x + \int_{\mathbb{R}} \mu v_t^2 \mathrm{d}x + \partial_t \int_{\mathbb{R}} \frac{1}{2} c^2 v_x^2 \mathrm{d}x = 0 \\
\implies &\partial_t \underbrace{\int_{\mathbb{R}} \frac{1}{2} v_t^2 + \frac{1}{2} c^2 v_x^2 \mathrm{d}x}_{E(t)} = -\int_{\mathbb{R}} \mu v_t^2 \mathrm{d}x \leq 0.
\end{aligned}$$

Then, $E(t) \geq 0$ for all $t > 0$ because the integrand of $E(t)$ is non-negative. Furthermore, $E(0) = \int_{\mathbb{R}} 0 \mathrm{d}x = 0$ and $\partial_t E(t) \leq 0$ which implies $E(t) \leq 0$ for all $t > 0$. Thus, putting everything together yields $0 \leq E(t) \leq 0$ which implies $E(t) = 0$ for all $t > 0$. Therefore $v(x,t) = 0$ for all $x \in \mathbb{R}, t > 0$ which implies $u_1 = u_2$ and so solutions to the original IVP must be unique.

(c) (3 + 1)D wave equation:

$$\begin{cases} u_{tt}(x,t) = \Delta u(x,t), & x \in \Omega \subset \mathbb{R}^3, t > 0 \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), & x \in Omega \\ \frac{\partial u}{\partial n}(x,t) = -a(x)\frac{\partial u}{\partial t}(x,t), & x \in \partial\Omega, t > 0 \end{cases}$$

with $a(x) \geq 0$ and $\Omega \subset \mathbb{R}^3$ is bounded.

Suppose u_1 and u_2 are solutions to the IVBP. Then by linearity, $v = u_1 - u_2$ solves

$$\begin{cases} v_{tt} - \Delta v = 0, & x \in \Omega, t > 0 \\ v(x,0) = 0, v_t(x,0) = 0, & x \in \Omega \\ \frac{\partial v}{\partial n}(x,t) = -a(x)\frac{\partial v}{\partial t}(x,t), & x \in \partial\Omega, t > 0. \end{cases}$$

Now, consider $v_t v_{tt} - v_t \Delta v = 0$. Then, taking the volume integral over Ω yields

$$\begin{aligned}
& \int_\Omega v_t v_{tt} - v_t \Delta v \mathrm{d}V = 0 \\
& \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V - \int_\Omega v_t \nabla \cdot \nabla v \mathrm{d}V \\
\text{div thm w/ IPT} & \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V - \int_{\partial\Omega} v_t \nabla v \cdot \mathbf{n} \mathrm{d}S + \int_\Omega (\nabla v_t) \cdot (\nabla v) \mathrm{d}V = 0 \\
& \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V - \int_{\partial\Omega} v_t \frac{\partial v}{\partial n} \mathrm{d}S + \int_\Omega (\nabla v_t) \cdot (\nabla v) \mathrm{d}V = 0 \\
& \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V - \int_{\partial\Omega} v_t(-a(x)v_t) \mathrm{d}S + \int_\Omega (\nabla v_t) \cdot (\nabla v) \mathrm{d}V = 0 \\
& \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V + \int_{\partial\Omega} a(x)v_t^2 \mathrm{d}S + \int_\Omega (\nabla v_t) \cdot (\nabla v) \mathrm{d}V = 0 \\
& \implies \int_\Omega \partial_t \left(\frac{1}{2}v_t^2\right) \mathrm{d}V + \int_{\partial\Omega} a(x)v_t^2 \mathrm{d}S + \int_\Omega \frac{1}{2}\partial_t(\nabla v \cdot \nabla v) \mathrm{d}V = 0 \\
& \implies \partial_t \underbrace{\int_\Omega \frac{1}{2}v_t^2 + \frac{1}{2}\nabla v \cdot \nabla v \mathrm{d}V}_{E(t)} = -\int_{\partial\Omega} a(x)v_t^2 \mathrm{d}S \leq 0.
\end{aligned}$$

Just like in part (b), $E(0) = 0$ and $E(t) \geq 0$ which when combined with the fact that $\partial_t E(t) \leq 0$ implies that $E(t) = 0$ for all $t > 0$. Therefore $v(x,t) = 0$ which implies $u_1 = u_2$ for all $x \in \Omega$, $t > 0$. So, solutions to the original IVBP must be unique.

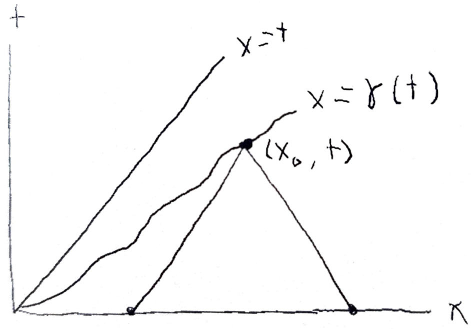
3. Consider the moving IBVP for the wave equation with unit wave speed

$$\begin{cases} u_{tt} - u_{xx} = 0, & x > \gamma(t), t > 0 \\ u(x,0) = f(x), & x > 0 \\ u_t(x,0) = g(x), & x > 0 \\ u(\gamma(t),t) = h(t), & t > 0. \end{cases}$$

Assume that $f \in C^2(\gamma(0), \infty), g \in C^1(\gamma(0), \infty), h \in C^2(0, \infty)$, and $\gamma C^1(0, \infty)$. Also assume $\gamma(0) = 0$.

(a) Is the IBVP with $\dot{\gamma}(t) > 1$ for some $t > 0$ well-posed?

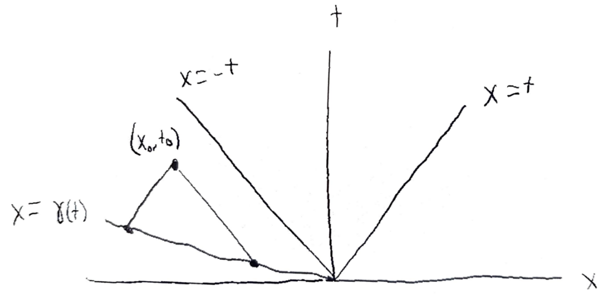
Below is a rough sketch of our solution space From the figure, we can see that when $x_0 > \gamma(t_0)$,



$u(x_0, t_0)$ will be given by d'Alembert's solution. Furthermore, if we go up to the boundary (i.e. $x_0 = \gamma(t_0)$), then $u(x_0, t_0)$ can also be given by d'Alembert be both left and right going characteristics touch at (x_0, t_0). But, we must also have $u(x_0, t_0) = u(\gamma(t0) = h(t)$ which will almost be equal to d'Alembert's solution. So we don't necessarily have existence along $x = \gamma(t)$ and so our IBVP is not well-posed for $\dot{(\gamma)} > 0$.

(b) Is the IBVP with $\dot{\gamma}(t) < -1$ for some $t > 0$ well-posed?

Below is a rough sketch of our solution space From the sketch, we can see that when $\gamma(t_0) <$

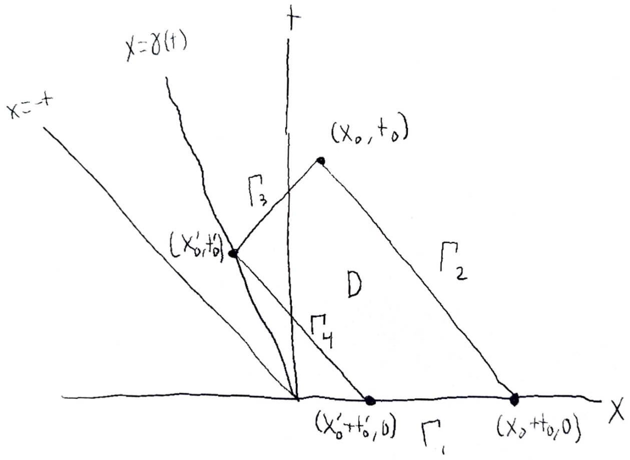


$x_0 < t_0$, both left and right moving characteristics originate from the line $x = \gamma(t)$ for $t > 0$. Furthermore, we know the general solution has the for $u(x_0, t_0) = F(x_0 - t_0) + G(x_0 + t_0)$ which means that we have two unknowns to find to obtain our explicit solution. But, because the solution is only given information from $x = \gamma(t)$, we only have the initial condition involving $h(t)$. We don't have derivative information or other constraints for $u(x_0, t_0)$ when $\gamma(t_0) < x_0 < t_0$ and so we can not uniquely determine u in this region. Therefore, the IBVP is ill-posed when $\dot{\gamma} < -1$.

(c) Solve the IBVP.

First, let's assume $\dot{\gamma(t)} \in (-1, 1)$ for all $t > 0$ so that we have a well-posed problem. Then, if $x_0 > t_0$, $u(x_0, t_0)$ only has a domain of dependence involving $f(x)$ and $g(x)$. Therefore, when $x_0 > t_0$, u is given by d'Alembert's solution

$$u(x_0, t_0) = \frac{1}{2}(f(x_0 - t_0) + f(x_0 - t_0)) + \frac{1}{2}\int_{x_0 - t_0}^{x_0 + t_0} g(y) \mathrm{d}y.$$



Now, if $\gamma(t_0) < x_0 < t$, our solution is given by the domain of dependence D shown in the figure above. We can rewrite the PDE in terms of a vector quantity as

$$u_{tt} - u_{xx} = \nabla \cdot \begin{pmatrix} -u_x \\ u_t \end{pmatrix} = 0$$

which implies

$$\int_D \nabla \cdot \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}A = 0.$$

Then, by the divergence theorem, we have

$$\int_\Gamma \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} = 0.$$

From the figure, we can break this line integral into integrating counter-clockwise over $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4. Before we continue, let (x_0', t_0') be the point that satisfies both

$$t_0' - t_0 = x_0' - x_0$$

and

$$x_0' = \gamma(t_0').$$

Then, our parameterizations of the different lines can be as follows

$$\begin{aligned}
\Gamma_1 :& (x(s), t(s)) = (s, 0) \\
\Gamma_2 :& (x(s), t(s)) = (s, -(s - x_0) + t_0) \\
\Gamma_3 :& (x(s), t(s)) = (s, s - x_0' + t_0') \\
\Gamma_4 :& (x(s), t(s)) = (s, -(s - x_0') + t_0')
\end{aligned}$$

Now let's break up our integral into manageable chunks

$$\int_\Gamma \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} = \int_{\Gamma_1} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} + \int_{\Gamma_2} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} + \int_{\Gamma_3} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} + \int_{\Gamma_4} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} = 0.$$

So for Γ_1, we have

$$\mathrm{d}\mathbf{S} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathrm{d}s \text{ with } x_0' + t_0' \le s \le x_0 + t_0$$

which yields

$$\begin{aligned}\int_{\Gamma_1}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\mathrm{d}\mathbf{S} &= \int_{x_0'+t_0'}^{x_0+t_0}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\cdot\begin{pmatrix}0\\-1\end{pmatrix}\mathrm{d}s\\ &= \int_{x_0'+t_0'}^{x_0+t_0} -u_t\mathrm{d}s\\ &= -\int_{x_0'+t_0'}^{x_0+t_0} g(s)\mathrm{d}s.\end{aligned}$$

Now, for Γ_2, we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = -u_x + u_t, \quad \mathrm{d}\mathbf{S} = \begin{pmatrix}1\\1\end{pmatrix}\mathrm{d}s, \text{ with } x_0 \le s \le x_0 + t_0$$

which yields

$$\begin{aligned}\int_{\Gamma_2}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\mathrm{d}\mathbf{S} &= \int_{x_0+t_0}^{x_0}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}\mathrm{d}s\\ &= \int_{x_0+t_0}^{x_0} -u_x + u_t\mathrm{d}s\\ &= \int_{x_0+t_0}^{x_0}\frac{\mathrm{d}u}{\mathrm{d}s}\mathrm{d}s\\ &= u(x_0,t_0) - u(x_0+t_0,0)\\ &= u(x_0,t_0) - f(x_0+t_0).\end{aligned}$$

Next, for Γ_3, we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = -u_x - u_t, \quad \mathrm{d}\mathbf{S} = \begin{pmatrix}-1\\1\end{pmatrix}\mathrm{d}s, \text{ with } x_0' \le s \le x_0$$

which yields

$$\begin{aligned}\int_{\Gamma_3}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\mathrm{d}\mathbf{S} &= \int_{x_0}^{x_0'}\begin{pmatrix}-u_x\\u_t\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix}\mathrm{d}s\\ &= \int_{x_0}^{x_0'} u_x + u_t\mathrm{d}s\\ &= -\int_{x_0}^{x_0'}\frac{\mathrm{d}u}{\mathrm{d}s}\mathrm{d}s\\ &= u(x_0,t_0) - u(x_0',t_0')\\ &= u(x_0,t_0) - h(t_0').\end{aligned}$$

Finally, for Γ_4, we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = u_x - u_t, \quad \mathrm{d}\mathbf{S} = \begin{pmatrix}-1\\-1\end{pmatrix}\mathrm{d}s, \text{ with } x_0' \le s \le x_0' + t_0'$$

which yields

$$\begin{aligned}\int_{\Gamma_4} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} &= \int_{x_0'}^{x_0'+t_0'} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \mathrm{d}s \\ &= \int_{x_0'}^{x_0'+t_0'} u_x - u_t \mathrm{d}s \\ &= \int_{x_0'}^{x_0'+t_0'} \frac{\mathrm{d}u}{\mathrm{d}s} \mathrm{d}s \\ &= u(x_0'+t_0', 0) - u(x_0', t_0') \\ &= f(x_0'+t_0') - h(t_0').\end{aligned}$$

Putting everything together, we have

$$\int_{\Gamma} \begin{pmatrix} -u_x \\ u_t \end{pmatrix} \mathrm{d}\mathbf{S} = -\int_{x_0'+t_0'}^{x_0+t_0} g(s)\mathrm{d}s + u(x_0,t_0) - f(x_0+t_0) + u(x_0,t_0) - h(t_0') + f(x_0'+t_0') - h(t_0') = 0$$

which implies

$$u(x_0,t_0) = \frac{1}{2}(f(x_0+t_0) - f(x_0'+t_0')) + h(t_0') + \frac{1}{2}\int_{x_0'+t_0'}^{x_0+t_0} g(s)\mathrm{d}s$$

for $\gamma(t_0) < x_0 < t$.

To briefly check our solution, let's assume $\gamma(t) = 0$. Then $(x_0', t_0') = (0, t_0 - x_0)$ and so our solution is given by

$$u(x_0,t_0) = \begin{cases} \frac{1}{2}(f(x_0-t_0) + f(x_0+t_0)) + \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} g(s)\mathrm{d}s, & x_0 > t_0 \\ \frac{1}{2}(f(x_0+t_0) - f(t_0-x_0)) + h(t_0-x_0) + \frac{1}{2}\int_{t_0-x_0}^{x_0+t_0} g(s)\mathrm{d}s, & 0 < x < t \end{cases}.$$

Which is exactly what we saw in class and in the book! We can see that the odd extension of f negatively interferes with the original left moving wave as expected.

4. Consider the Euler-Poisson-Darboux (EPD) equation

$$z_{\xi\eta} + \frac{N}{\xi+\eta}(z_\xi + z_\eta) = 0, z = z(\xi,\eta)$$

for N a natural number.

(a) Verify that

$$z(\xi,\eta) = k + \frac{\partial^{N-1}}{\partial\xi^{N-1}}\left(\frac{f(\xi)}{(\xi+\eta)^N}\right) + \frac{\partial^{N-1}}{\partial\eta^{N-1}}\left(\frac{g(\eta)}{(\xi+\eta)^N}\right)$$

for $k \in \mathbb{R}$ and f, g sufficiently smooth function, is the general purpose solution to the EPD equation. *Hint:* We have the identity

$$\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^N}\right) = (\xi+\eta)\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) + N\frac{\partial^{N-1}}{\partial\xi^{N-1}}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right).$$

To verify the solution, let's compute our need partials:

$$\begin{aligned}
z_\xi &= \frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^N}\right) + \frac{\partial^{N-1}}{\partial\eta^{N-1}}\frac{\partial}{\partial\xi}\left(\frac{g(\eta)}{(\xi+\eta)^N}\right)\\
&= (\xi+\eta)\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) + N\frac{\partial^{N-1}}{\partial\xi^{N-1}}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) - N\frac{\partial^{N-1}}{\partial\eta^{N-1}}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right)\\
z_\eta &= \frac{\partial^{N-1}}{\partial\xi^{N-1}}\frac{\partial}{\partial\eta}\left(\frac{f(\xi)}{(\xi+\eta)^N}\right) + \frac{\partial^N}{\partial\eta^N}\left(\frac{g(\eta)}{(\xi+\eta)^N}\right)\\
&= -N\frac{\partial^{N-1}}{\partial\xi^{N-1}}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) + (\xi+\eta)\frac{\partial^N}{\partial\eta^N}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right) + N\frac{\partial^{N-1}}{\partial\eta^{N-1}}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right)\\
z_{\xi\eta} &= \frac{\partial^N}{\partial\xi^N}\frac{\partial}{\partial\eta}\left(\frac{f(\xi)}{(\xi+\eta)^N}\right) + \frac{\partial^N}{\partial\eta^N}\frac{\partial}{\partial\xi}\left(\frac{g(\eta)}{(\xi+\eta)^N}\right)\\
&= -N\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) - N\frac{\partial^N}{\partial\eta^N}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right).
\end{aligned}$$

Then with two cancellations, we have

$$\begin{aligned}
\frac{N}{\xi+\eta}(z_\xi + z_\eta) &= \frac{N}{\xi+\eta}\left((\xi+\eta)\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) + (\xi+\eta)\frac{\partial^N}{\partial\eta^N}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right)\right)\\
&= N\frac{\partial^N}{\partial\xi^N}\left(\frac{f(\xi)}{(\xi+\eta)^{N+1}}\right) + N\frac{\partial^N}{\partial\eta^N}\left(\frac{g(\eta)}{(\xi+\eta)^{N+1}}\right).
\end{aligned}$$

From here, it is easy to see that the rest cancels to get

$$z_{\xi\eta} + \frac{N}{\xi+\eta}(z_\xi + z_\eta) = 0.$$

Therefore, $z(\xi,\eta)$ is a solution to the EPD equation.

(b) Show that if z satisfies the EPD equation, then $u(r,t) = z(r+t, r-t)$ satisfies the wave-type equation

$$u_{tt} - u_{rr} - \frac{2N}{r}u_r = 0.$$

With $u(r,t) = z(r+t, r-t)$, we must have $\xi = r+t$ and $\eta = r-t$ which implies

$$\xi + \eta = 2r \implies r = \frac{1}{2}(\xi + \eta).$$

Now, let's compute some needed partials

$$\begin{aligned} u_t &= z_\xi - z_\eta \\ u_{tt} &= z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta} \\ u_r &= z_\xi + z_\eta \\ u_{rr} &= z_{\xi\xi} + 2z_{\xi\eta} + z_{\eta\eta}. \end{aligned}$$

Next, using these partials, we have

$$\begin{aligned} u_{tt} - u_{rr} - \frac{2N}{r} &= z_{\xi\xi} - 2z_{\xi\eta} + z_{\eta\eta} - z_{\xi\xi} - 2z_{\xi\eta} - z_{\eta\eta} - \frac{2N}{r}(z_\xi + z_\eta) \\ &= -4z_{\xi\eta} - \frac{2N}{r}(z_\xi + z_\eta) \\ &= -4z_{\xi\eta} - \frac{2N}{\frac{1}{2}(\xi + \eta)}(z_\xi + z_\eta) \\ &= -4\left(z_{\xi\eta} - \frac{N}{\xi + \eta}(z_\xi + z_\eta)\right) \\ &= -4(0) \\ &= 0. \end{aligned}$$

Therefore, $u(r,t) = z(r+t, r-t)$ is a solution to the wave-type equation.

(c) The wave equation in n dimensions is

$$u_{tt} - \Delta u = 0$$

where $x \in \mathbb{R}^n$. Prove that radial solutions in the form of $u(x,t) = v(\|x\|, t)$, depending only on $r = \|x\|$, satisfy

$$v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0$$

implying that, when $N = \frac{1}{2}(n-1)$, n is an odd number, the EPD equation is equivalent to the radial wave equation.

Suppose $u(x,t) = v(\|x\|, t)$. Then the second time derivative is given by

$$u_{tt} = v_{tt}(\|x\|, t).$$

Furthermore, the Laplacian can be computed as

$$\begin{aligned}
\Delta u = \Delta v &= \sum_{i=1}^{n} \partial_{x_i x_i} v(\|x\|, t) \\
&= \sum_{i=1}^{n} \partial_{x_i} \left(v_r \frac{x_i}{\|x\|} \right) \\
&= \sum_{i=1}^{n} \left(v_{rr} \frac{x_i^2}{\|x\|^2} + v_r \frac{\|x\| - \frac{x_i^2}{\|x\|}}{\|x\|^2} \right) \\
&= v_{rr} \frac{1}{\|x\|^2} \underbrace{\sum_{i=1}^{n} x_i^2}_{\|x\|^2} + \frac{v_r}{\|x\|} \left(\underbrace{\sum_{i=1}^{n} (1)}_{n} - \frac{1}{\|x\|^2} \underbrace{\sum_{i=1}^{n} x_i^2}_{\|x\|^2} \right) \\
&= v_{rr} + \frac{v_r}{\|x\|}(n-1) \\
&= v_{rr} + \frac{n-1}{r} v_r.
\end{aligned}$$

Then, putting everything together, we have

$$0 = u_{tt} - \Delta u = v_{tt} - v_{rr} - \frac{n-1}{r} v_r.$$

Therefore v satisfies the radial wave equation.

(d) What is the general solution to the radial wave equation for an odd number of dimensions?
Let $x \in \mathbb{R}^n$ with n odd. Then by part (c) if $N = \frac{1}{2}(n-1)$, we have the solution to the radial wave equation in odd dimensions

$$v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0$$

which is given by

$$u(x,t) = v(\|x\|, t) = v(r,t) = z(r+t, r-t)$$

where from part (b) and (a),

$$u(x,t) = z(r+t, r-t) = k + \frac{\partial^{N-1}}{\partial \xi^{N-1}} \left(\frac{f(\xi)}{(\xi+\eta)^N} \right) \Bigg|_{\xi=r+t, \eta=r-t} + \frac{\partial^{N-1}}{\partial \eta^{N-1}} \left(\frac{g(\eta)}{(\xi+\eta)^N} \right) \Bigg|_{\xi=r+t, \eta=r-t}$$

with $r = \|x\|$.