# Chapter 1

## Problem 3

Find an equation relating the parameters $c, m, n$ so that the function $u(x, t) = \sin(mt)\sin(nx)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ for $c > 0$.

First off, our derivatives are as follows,

$$u_{tt} = -m^2 \sin(mt)\sin(nx)$$
$$u_{xx} = -n^2 \sin(mt)\sin(nx).$$

Then, plugging our derivatives into the wave equation yields

$$-m^2 \sin(mt)\sin(nx) = -c^2 n^2 \sin(mt)\sin(nx)$$

which implies

$$m^2 = c^2 n^2$$

for $c > 0$.

## Problem 8

Consider the linear transport equation (1.8) with initial and boundary conditions (1.10).

$$\begin{cases} u_t + cu_x = 0 \\ u(x, 0) = \phi(x), & x \geq 0 \\ u(0, t) = \psi(t), & t \geq 0. \end{cases}$$

**(a)** Suppose the data $\phi, \psi$ are differentiable functions. Show that the function $u : Q_q \to \mathbb{R}$ given by

$$u(x, t) = \begin{cases} \phi(x - ct), & \text{if } x \geq ct, \\ \psi(t - x/c), & \text{if } x \leq ct \end{cases} \tag{1}$$

satisfies the PDE away from the line $x = ct$ with $c > 0$, the boundary condition, and initial conditions.

When we are away from the line, boundary condition, and initial condition, we have

$$u_t = \begin{cases} -c\phi'(x - ct), & x > ct > 0 \\ \psi'(t - x/c), & 0 < x < ct \end{cases}$$

$$u_x = \begin{cases} \phi'(x - ct), & x > ct > 0 \\ -\frac{1}{c}\psi'(t - x/c), & 0 < x < ct \end{cases}.$$

Using these the derivatives above

$$u_t + cu_x = \begin{cases} -c\phi'(x - ct) + c\phi'(x - ct), & x > ct > 0 \\ \psi'(t - x/c) - c\frac{1}{c}\psi'(t - x/c), & 0 < x < ct \end{cases}$$

$$= \begin{cases} 0, & x > ct > 0 \\ 0, & 0 < x < ct \end{cases}$$

$$= 0$$

for $x, t$ not on the line, boundaries, or initial conditions. So, (1) defined above satisfies our PDE in the desired region.

(b) In solution (1), the line $x = ct$, which emerges from the origin $x = t = 0$, separates the quadrant $Q_1$ into two regions.

    **(i)** Find conditions on the data $\phi, \psi$ so that the solution is continuous across the line $x = ct$.
    When $x = ct$, we have $\phi(x - ct) = \phi(ct - ct) = \phi(0)$ and $\psi(t - x/c) = \psi(x/c - x/c) = \psi(0)$.
    Thus, for continuity when $x = ct$, we need

$$\phi(0) = \psi(0).$$

    **(ii)** Find conditions on the data $\phi, \psi$ so that the solution is differentiable across the line $x = ct$.
    For differentiablity of $u$ across the line $x = ct$, we need differentiablity in both partial derivatives, $u_t$ and $u_x$ across the line. First, let's compute the derivatives on the line:

$$\partial_x \phi(x - ct)|_{x=ct} = \phi'(0)$$
$$\partial_x \psi(t - x/c)|_{x=ct} = -\frac{1}{c}\psi'(0)$$
$$\partial_t \phi(x - ct)|_{x=ct} = -c\phi'(0)$$
$$\partial_t \psi(x - ct)|_{x=ct} = \psi'(0).$$

Equating our derivatives yields

$$\phi'(0) = -\frac{1}{c}\psi'(0)$$
$$-c\phi'(0) = \psi'(0)$$

which reduces to

$$\phi'(0) = -\frac{1}{c}\psi'(0).$$

So, for $u$ to be differentiable across the line $x = ct$, we need $\phi'(0) = -\frac{1}{c}\psi'(0)$.

## Problem 9

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Verify that if $u(x,t)$ is differentiable and satisfies (1.12), that is, $u = f(x - ut)$, then $u(x,t)$ is a solution of the initial value problem

$$u_t + uu_x = 0, \ -\infty < x < \infty, \ t > 0, \ u(x,0) = f(x), \ -\infty < x < \infty.$$

Suppose $u(x,t)$ is differentiable and $u = f(x - ut)$. Then, the initial condition easily verified as

$$u(x,0) = f(x - u(x,0) \cdot 0) = f(x).$$

Next, we can start verifying the PDE by computing the partial derivatives as follows

$$u_t = f'(x - ut)(-u_t t - u)$$
$$u_x = f'(x - ut)(1 - u_x t).$$

Now, putting these derivative into our PDE yields

$$\begin{aligned}
u_t + uu_x &= f'(x - ut)(-u_t t - u) + uf'(x - ut)(1 - u_x t) \\
&= f'(x - ut)(-u_t t - u + u - uu_x t) \\
&= f'(x - ut)(-u_t t - uu_x t) \\
&= -tf'(x - ut)(u_t + uu_x).
\end{aligned}$$

Clearly, at $t = 0$ our PDE equals zero and so $u$ satisfies the PDE at $t = 0$, But, when $t > 0$, we have

$$u_t + uu_x = -tf'(x - ut)(u_t + uu_x)$$

which implies that either $1 = -tf'(x - ut)$ or $u_t + uu_x = 0$. The first equality implies that $f'(x - ut) = -1/t$ which doesn't hold because $f'(x - ut)$ is differentiable everywhere but $-1/t$ is not differentiable at $t = 0$. So, we must have the second equality, $u_t + uu_x = 0$ which shows that $u$ solve the IVP!

# Chapter 2

## Problem 5

## Problem 6

# Additional Problems

## Problem A1

## Problem A2