1. Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   i. $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$
   Suppose
   $$x_{n+1} = g(x_n) = -16 + 6x_n + \frac{12}{x_n}.$$
   Then, Taylor expanding about x_* yields
   $$g(x_n) = g(x_*) + (6 - \frac{12}{\xi^2})(x_n - x_*) = x_* + (6 - \frac{12}{\xi^2})(x_n - x_*)$$
   for some ξ between x_n and x_*. Using this expansion, we can get
   $$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|6 - \frac{12}{\xi^2}\right|. \tag{1}$$
   Now, depending on our initial guess for x_n, the RHS of (1) can be greater than 1. For instance, suppose $x_n = 3$, then $\xi \in [2, 3]$ which implies
   $$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|6 - \frac{12}{\xi^2}\right| \geq 3.$$
   This shows that after an iteration near x_*, the error actually grows larger. But, if we choose $x_n = 1$, then
   $$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|6 - \frac{12}{\xi^2}\right| \geq \frac{2}{3}$$
   Which shows that the error might decrease near x_*. Putting these two points together shows that we are unsure if the iterate will converge or not.

   ii. $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{1/3}$
   Suppose
   $$g(x) = \frac{2}{3}x + \frac{1}{x^2}.$$
   Furthermore, suppose $x_0 \in [1, 2]$. Then $x_* \in [1, 2]$, $\min_{x\in[1,2]} g(x) = 3^{1/3}$, and $\max_{x\in[1,2]} g(x) = \frac{5}{3}$. Therefore $g([1, 2]) \subset [1, 2]$ So because $g(x)$ is continuously differentiable on $[1, 2]$, Theorem 2.6 of Atkinson tells us that with $x_0 \in [1, 2]$ and $x_{n+1} = g(x_n)$, that x_* is a unique solution to $x = g(x)$ in $[1, 2]$ and that
   $$\lim_{n\to\infty} x_n = x_*.$$
   In other words, for any x_0 in the neighborhood $[1, 2]$ of x_*, the fixed point iteration will converge.
   Now let's figure out the order of convergence. Taylor expanding $g(x_n)$ about $x_n = x_*$ yields
   $$x_{n+1} = g(x_n) = g(x_*) + \left(\frac{2}{3} - \frac{2}{x_*^3}\right)(x_n - x_*) + \frac{3}{\xi^4}(x_n - x_*)^2 = x_* + \frac{3}{\xi^4}(x_n - x_*)^2$$
   for ξ between x_n and x_*. Using the equality above, we can obtain
   $$\frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{3}{\xi^4}.$$

Now because $x_n \to x_*$ as $n \to \infty$ and ξ is between x_n and x_*, we must have $\xi \to x_*$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{3}{x_*^4}$$

which shows that $x_{n+1} = g(x_n)$ converges quadratically to x_* in the neighborhood $[1, 2]$.

iii. $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$

Just as in the previous part, let's first show that the iteration converges to x_* for x_0 sufficiently close to x_*. First, let

$$g(x) = \frac{12}{1 + x_n}, x_* = 3.$$

Then $x_{n+1} = g(x_n)$, $x_* \in [2, 4]$, $\min_{x\in[2,4]} g(x) = \frac{12}{5}$, and $\max_{x\in[2,4]} g(x) = 4$. Therefore, $g([2, 4]) \subset [2, 4]$ and so by Theorem 2.6 on Atkinson, x_* is a unique solution to $x = g(x)$ in $[2, 4]$ and any $x_0 \in [2, 4]$ with $x_{n+1} = g(x_n)$ will have

$$\lim_{n\to\infty} x_n = x_*.$$

In other words, for any $x_0 \in [2, 4]$ (i.e. sufficiently close to x_*), $x_{n+1} = g(x_n)$ will converge to x_*.

Now let's determine the order and rate of convergence. Taylor expanding $g(x_n)$ about $x_n = x_*$ yields

$$x_{n+1} = g(x_n) = g(x_*) - \frac{12}{(1 + \xi)^2}(x_n - x_*)$$

for some ξ between x_n and x_*. Then, rearranging yields

$$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|\frac{12}{(1 + \xi)^2}\right|.$$

Finally, because $\lim_{n\to\infty} x_n = x_*$, $\lim_{n\to\infty} \xi = x_*$ and so

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|\frac{12}{(1 + x_*)^2}\right| = \frac{3}{4}.$$

This limit shows us that $x_{n+1} = g(x_n)$ will converge linearly to x_* with a rate of $\frac{3}{4}$ provided $x_0 \in [2, 4]$ (i.e. x_0 is sufficiently close to x_*).

2. In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x, t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x, t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) 34$$

where T_s is the constant temperature during a cold period T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2) and

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \mathrm{d}s$$

Assume that $T_i = 20[\deg C]$, $T_s = -15[\deg C]$, $\alpha = 0.138 \cdot 10^{-6}$[meters^2 per second].

i. We want to determine how deep a water main should be buried so that it will only freeze after 60 days exposure at this constant surface temperature. Formulate the problem as a root finding problem $f(x) = 0$. What is f and what is f'? Plot the function f on $[0, \bar{x}]$, where $\bar{x}$ is chosen so that $f(\bar{x}) > 0$.

First, let's identify what we are are trying to solve. We want to x such that $T(x, 60[\text{days}]) = 0$. To find, this, we to solve for T which yields

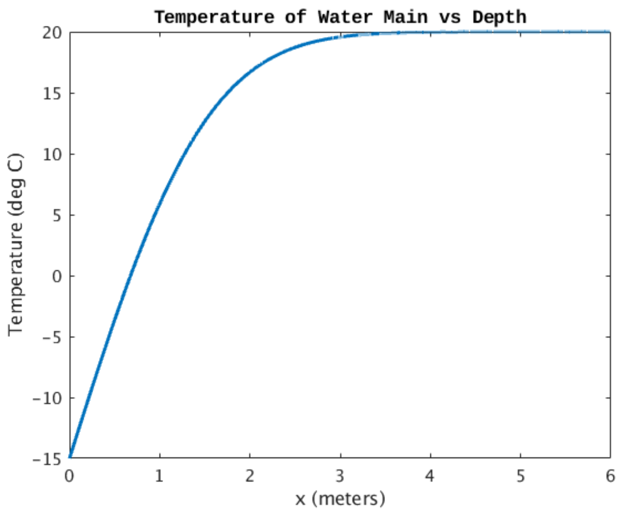
$$T(x,t) = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)(T_i - T_s) + T_s.$$

Setting this equation to zero reveals that our f is

$$f(x) = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)(T_i - T_s) + T_s$$

with $t = 60[\text{days}] = 5184000[\text{seconds}]$. Then,

$$f'(x) = \frac{T_i - T_s}{\sqrt{\pi\alpha t}} e^{-\frac{1}{4\alpha t}x^2}$$



ii. Compute an approximate depth using the Bisection Method with starting values $a_0 = 0[\text{meters}]$ and $b_0 = \bar{x}[\text{meters}]$.

Using $a_0 = 0$ and $b_0 = \bar{x} = 6$ with the bisection method converges to a root of $x = .67696185448190249$ after 46 iterations.

iii. Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01[\text{meters}]$.

Using $x_0 = 0.01$, Newton's method converges to a root of $x = 0.6769618544819366$ in 5 iterations. If instead we use $x_0 = \bar{x} = 6$, Newton fails after the first iteration due to a division by zero error. This faulty division is caused by our temperature function being flat away from the root and so when we used $\bar{x}$, our derivative shot to zero which caused the next iteration to fly to another very flat section causing the code to throw an error.

Overall, I would still prefer Newton's method here because it converges much more rapidly than the bisection method when it converges. Plus, in this case, when Newton wasn't going to work, it stopped only after a single iteration. Thus, it would still be faster to try another initial root guess for Newton than it would be to run the bisection method once.

(Code used is at the end of the document)

3. Consider applying Newton's method to a real cubic polynomial.

   i. In the case that the polynomial has three distinct real roots, $x = \alpha, x = \beta$, and $x = \gamma$, show that the starting guess $x_0 = \frac{1}{2}(\alpha + \beta)$ will yield the root γ in one step.

   Any cubic polynomial with roots $x \in \{\alpha, \beta, \gamma\}$ can be written as

   $$p(x) = a(x - \alpha)(x - \beta)(x - \gamma)$$

   where $a \in \mathbb{R}$ is a constant. Using this general cubic yields a newton iteration of

   $$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}.$$

   Then, using $x_0 = \frac{1}{2}(\alpha + \beta)$ yields an iteration of

   $$\begin{aligned} x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} &= \frac{1}{2}(\alpha + \beta) - \frac{-\frac{1}{8}a(\alpha - \beta)^2(\alpha + \beta - 2\gamma)}{-\frac{1}{4}a(\alpha - \beta)^2} \\ &= \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(2\gamma) \\ &= \gamma. \end{aligned}$$

   Thus, $x_1 = \gamma$ which shows Newton converges to $x = \gamma$ in one iteration when $x_0 = \frac{1}{2}(\alpha + \beta)$.

   ii. Give a heuristic argument showing that if two roots coincide, there is precisely one starting guess x_0 other than $x_0 = \beta$ for which Newton's method will fail, and that this one separates the basins of attraction for the distinct roots.

   Consider the cubic

   $$p(x) = a(x - \alpha)(x - \beta)^2$$

   with the constant $a \in \mathbb{R}$. Any plot of $p(x)$ will look similar to Figure 1. From the plot we can see that Newton fails whenever the derivative $p'(x) = 0$. One of these breaking points in the double root at β; the other breaking point is at the x_b between α and β for which $p'(x_b) = 0$. Now, Newton could potentially break at other points if after a certain iteration, we end up at x_b or β. But, because we have double root in this cubic, none of the tangent lines to $p(x)$ cross the x-axis at x_b or β. This lack of other points is due to the function bouncing off the x-axis at β, because any tangent line constructed on $p(x)$ will cross the x-axis towards α if $x < x_b$ and will cross the x-axis towards β if $x > x_b$. Thus, any starting x_0 will move away x_b. Even though iterations can move towards β, they will never get to β and thus, won't break after a certain iteration.

   From the discussion above and Figure 1, we can see that x_b forms the line between the basin of attraction for α and β because the tangents cross the x-axis closer to α or β depending on if the initial x_0 is to the left or right of x_b.

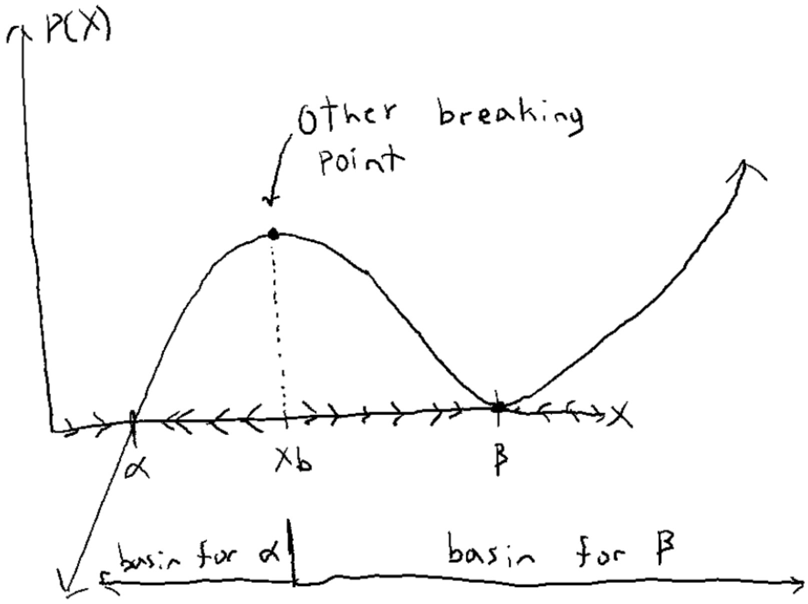


Figure 1: General plot of cubic polynomial with one double root

iii. Extend the argument in part ii. to the case when all three roots again are distinct. Explain why there are now infinitely many starting guesses x_0 for which the iteration will fail.

Consider the cubic polynomial:

$$p(x) = a(x-\alpha)(x-\beta)(x-\gamma)$$

where $a \in \mathbb{R}$ is a constant. All plots of $p(x)$ will look similar to Figure 2. From Figure 2, we can see that we still have two obvious breaking points x_{b1} and x_{b2} between α and β, and β and γ. However, now both breaking points are not at roots. In fact, one breaking point will correspond to $p(x) > 0$, and one will correspond to $p(x) < 0$. Looking at where the tangent lines of $p(x)$ cross the x-axis show that there is a tangent line that crosses at x_{b1} and x_{b2}. These tangents could send an initial point x_0 to one of the breaking points which would cause Newton to fail at the second iteration. Even worse, these new breaking points that map to the original breaking points have points that Newton will map to from somewhere else. We can keep repeating this process indefinitely to find an infinite sequence of points in a Newton iteration that will eventually work its way back to the first breaking points which will cause the method to fail. The light gray lines in Figure 2 show a potential path that Newton can take to eventually break after some amount of iterations.

This sequence of breaking points only exists because the local max and min are on the opposite sides of the x-axis and so the tangents will scan across the breaking points as x_n goes to the middle root from a local min or max.

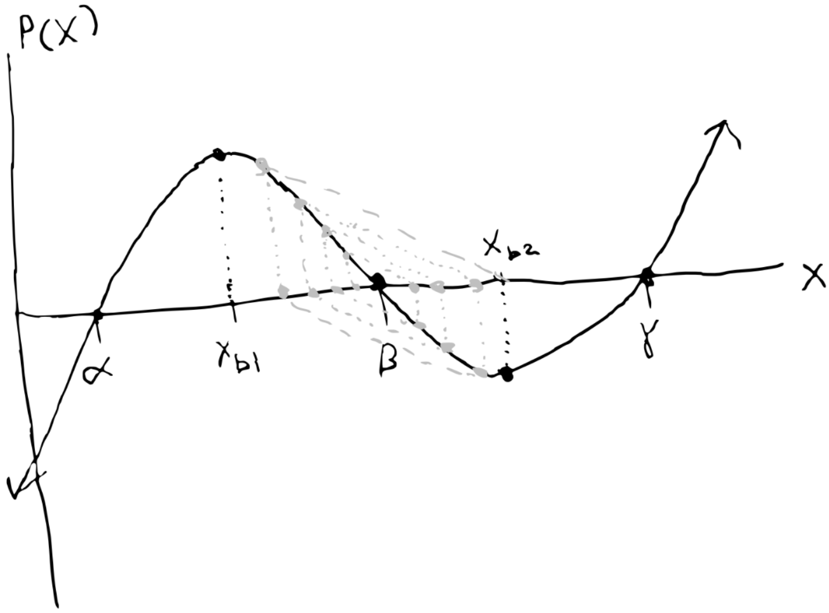


Figure 2: General plot of cubic polynomial with three distinct roots

4. The sequence x_k produced by Newton's method is quadratically convergent to x_* with $f(x_*) = 0$, $f'(x) \neq 0$ and $f''(x)$ continuous at x_*.

   Let $f(x) = (x - x_*)^p q(x)$ with p a positive integer with q twice continuously differentiable and $q(x_*) \neq 0$. Note: $f'(x_*) = 0$. In the following sub-problems, let $x_k, f_k = f(x_k), e_k = |x_* - x_k|$, etc.

   i. Prove that Newton's method converges linearly for $f(x)$.
   Using $f(x) = (x - x_*)^p q(x)$, we have the Newton iterate

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}.$$

   Then,

$$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \frac{|g(x_n) - x_*|}{|x_n - x_*|} = \frac{\left|x_n - x_* - \frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|}{|x_n - x_*|}$$
$$= \left|1 - \frac{q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|.$$

   Now, we can take the limit of both sides to get

$$\lim_{x->x^*} \frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|1 - \frac{q(x_*)}{pq(x_*)}\right| = \left|1 - \frac{1}{p}\right|$$

   which shows that Newton converges linearly.

   ii. Consider the modified Newton iteration defined by

$$x_{k+1} = x_k - p\frac{f_k}{f'_k}.$$

Prove that if x_k converges to x_*, then the rate of convergence is quadratic.
Firstly, let

$$g(x_n) = x_n - p\frac{f(x_n)}{f'(x_n)} = x_n - p\frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}.$$

Then,

$$\begin{aligned}
\frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{|g(x_n) - x_*|}{|x_n - x_*|^2} &= \frac{\left|x_n - x_* - p\frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|}{|x_n - x_*|^2} \\
&= \frac{\left|\frac{p(x_n - x_*)q(x_n) + (x_n - x_*)^2 q'(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)} - \frac{p(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|}{|x_n - x_*|^2} \\
&= \frac{\left|\frac{(x_n - x_*)^2 q'(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|}{|x_n - x_*|^2} \\
&= \left|\frac{q'(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|.
\end{aligned}$$

Now, assuming $x_n \to x_*$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \left|\frac{q'(x_*)}{pq(x_*)}\right| = C$$

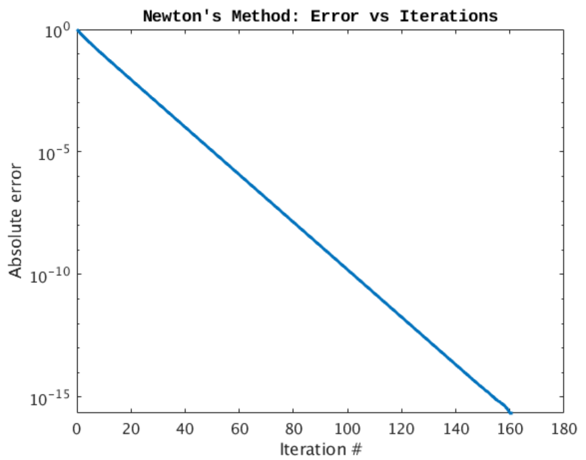
where $C > 0$ is a constant. Therefore, the modified Newton method converges quadratically when x_n converges to x_*.

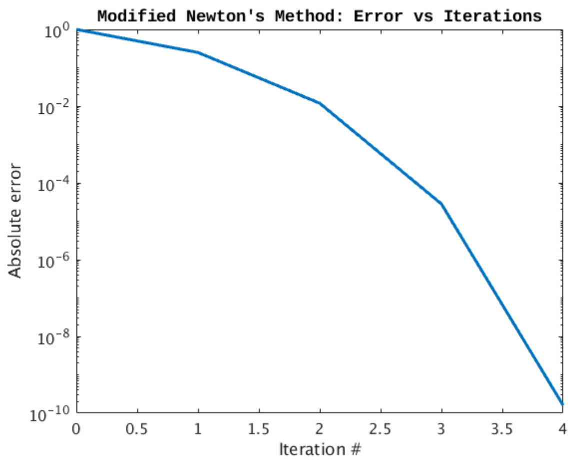
iii. Write MATLAB codes for both Newton and modified Newton methods. Apply these to the function

$$f(x) = (x - 1)^5 e^x$$

and compare the results. Use $x_0 = 0$ as a starting point.

Running my *Problem4.m* MATLAB code produces the figures:





The accompanying data can be seen in the tables below:
Convergence table for Newton's Method

```
-------x value---------------Absolute Error----
0                           1.000000000000000e+00
2.500000000000000e-01       7.500000000000000e-01
4.264705882352942e-01       5.735294117647058e-01
5.560386945475865e-01       4.439613054524135e-01
6.534832807807371e-01       3.465167192192629e-01
7.279472249514199e-01       2.720527750485801e-01
7.854886403066591e-01       2.145113596933409e-01
8.303140230035438e-01       1.696859769964562e-01
8.654434113183316e-01       1.345565886816684e-01
8.930989767635392e-01       1.069010232364608e-01
9.149462812517817e-01       8.505371874821832e-02
9.322513979063947e-01       6.774860209360534e-02
9.459872350785150e-01       5.401276492148499e-02
9.569077575858133e-01       4.309224241418674e-02
9.656011294461713e-01       3.439887055382873e-02
9.725285627331186e-01       2.747143726688139e-02
9.780532041545410e-01       2.194679584545900e-02
9.824619147378555e-01       1.753808526214451e-02
9.859818784751131e-01       1.401812152488691e-02
9.887933851886461e-01       1.120661481135388e-02
9.910397429641806e-01       8.960257035819374e-03
9.928350115850026e-01       7.164988414997353e-03
9.942700656972199e-01       5.729934302780126e-03
9.954173673503955e-01       4.582632649604457e-03
9.963347346718042e-01       3.665265328195777e-03
9.970683254984478e-01       2.931674501552184e-03
9.976550043890671e-01       2.344995610932932e-03
9.981242235746398e-01       1.875776425360187e-03
9.984995196540195e-01       1.500480345980493e-03
9.987997058079005e-01       1.200294192099549e-03
9.990398222884037e-01       9.601777115962884e-04
9.992318947154557e-01       7.681052845442959e-04
9.993855393754196e-01       6.144606245803708e-04
9.995084466046663e-01       4.915533953336881e-04
9.996067669496730e-01       3.932330503270132e-04
9.996854197455142e-01       3.145802544858300e-04
9.997483397550898e-01       2.516602449101901e-04
9.997986743375146e-01       2.013256624854298e-04
9.998389410913578e-01       1.610589086421532e-04
9.998711539107186e-01       1.288460892814314e-04
9.998969237926446e-01       1.030762073553904e-04
9.999175394591127e-01       8.246054088734134e-05
9.999340318392842e-01       6.596816071580136e-05
9.999472256455015e-01       5.277435449846379e-05
9.999577806278077e-01       4.221937219228700e-05
```

```
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9.999999346693744e-01    6.533062557600999e-08
9.999999477354997e-01    5.226450028317231e-08
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9.999999665507200e-01    3.344928001247638e-08
9.999999732405760e-01    2.675942400998110e-08
9.999999785924608e-01    2.140753918578042e-08
9.999999828739686e-01    1.712603137082880e-08
9.999999862991750e-01    1.370082503004966e-08
9.999999890393400e-01    1.096066004624419e-08
9.999999912314720e-01    8.768528014790888e-09
9.999999929851776e-01    7.014822389628250e-09
9.999999943881421e-01    5.611857867293679e-09
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9.999999964084110e-01    3.591589026186170e-09
9.999999971267288e-01    2.873271176540015e-09
9.999999977013830e-01    2.298616963436473e-09
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9.999999985288851e-01    1.471114896567371e-09
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9.999999999352996e-01    6.470035618377779e-11
9.999999999482397e-01    5.176026274256174e-11
9.999999999585918e-01    4.140821019404939e-11
9.999999999668734e-01    3.312661256416050e-11
9.999999999734988e-01    2.650124564240741e-11
9.999999999787990e-01    2.120104092284691e-11
9.999999999830391e-01    1.696087714719852e-11
9.999999999864313e-01    1.356870171775881e-11
9.999999999891450e-01    1.085498357866754e-11
9.999999999913161e-01    8.683942454013049e-12
9.999999999930529e-01    6.947109554289455e-12
9.999999999944423e-01    5.557665438971071e-12
9.999999999955539e-01    4.446110146716364e-12
9.999999999964431e-01    3.556932526294077e-12
9.999999999971545e-01    2.845501612114276e-12
9.999999999977236e-01    2.276401289691421e-12
9.999999999981789e-01    1.821098827292644e-12
9.999999999985432e-01    1.456834652913130e-12
9.999999999988345e-01    1.165512131251489e-12
9.999999999990676e-01    9.323652960802065e-13
9.999999999992542e-01    7.458478279431802e-13
9.999999999994034e-01    5.966338534335591e-13
9.999999999995227e-01    4.772848782863548e-13
9.999999999996182e-01    3.818056981685913e-13
9.999999999996946e-01    3.054223540743806e-13
9.999999999997556e-01    2.443600877199970e-13
9.999999999998045e-01    1.955102746364901e-13
9.999999999998436e-01    1.564304241696846e-13
9.999999999998749e-01    1.251221348752551e-13
9.999999999998999e-01    1.001421168211891e-13
9.999999999999198e-01    8.015810237793630e-14
9.999999999999358e-01    6.417089082333405e-14
9.999999999999487e-01    5.129230373768223e-14
9.999999999999589e-01    4.107825191113079e-14
9.999999999999671e-01    3.286260152890463e-14
9.999999999999737e-01    2.631228568361621e-14
9.999999999999789e-01    2.109423746787797e-14
```

```
9.999999999999831e-01     1.687538997430238e-14
9.999999999999865e-01     1.354472090042691e-14
9.999999999999891e-01     1.088018564132653e-14
9.999999999999913e-01     8.659739592076221e-15
9.999999999999931e-01     6.883382752675971e-15
9.999999999999944e-01     5.551115123125783e-15
9.999999999999956e-01     4.440892098500626e-15
9.999999999999964e-01     3.552713678800501e-15
9.999999999999971e-01     2.886579864025407e-15
9.999999999999977e-01     2.331468351712829e-15
9.999999999999981e-01     1.887379141862766e-15
9.999999999999984e-01     1.554312234475219e-15
9.999999999999988e-01     1.221245327087672e-15
9.999999999999990e-01     9.992007221626409e-16
9.999999999999992e-01     7.771561172376096e-16
9.999999999999993e-01     6.661338147750939e-16
9.999999999999994e-01     5.551115123125783e-16
9.999999999999996e-01     4.440892098500626e-16
9.999999999999997e-01     3.330669073875470e-16
9.999999999999998e-01     2.220446049250313e-16
9.999999999999998e-01     2.220446049250313e-16
```

Convergence table for Modified Newton's Method:

```
-------x value---------------Absolute Error----
0                         1.000000000000000e+00
1.250000000000000e+00     2.500000000000000e-01
1.011904761904762e+00     1.190476190476186e-02
1.000028277344192e+00     2.827734419175165e-05
1.000000000159921e+00     1.599207433145011e-10
1.000000000000000e+00                         0
```

From the error plots and tables, clearly the Modified Newton's method converges much faster (in only 5 iterations vs 161 iterations from Newton's method). But, to really understand what's happening, we can first turn to the tables. The Newton's method table shows an error that is decreasing at a nearly linear rate with the method gaining another power of ten of accuracy every ten or so iterations. Similarly, the plot of Newton's method shows a straight line when we logarithmically scale the y-axis. Both of these combined tell us that Newton's method is converging to the multiple root linearly.

On the other hand, the table of the Modified Newton's method shows the number of digits that we have correct doubles after each iteration (just like quadratic convergence in the theory describes). Furthermore, the error plot of the Modified Newton's method shows a decreasing, concave down curve as the number of iterations increases with a logarithmically scaled y-axis. This type of downward curve is characteristic of higher order methods which only support the theory more than the Modified Newton's method has quadratic convergence.

Problem 2 Code

```
%%
% Problem 2 driver
% Author: Caleb Jacobs
% Date last modified: 09-09-2021
format longE

%% Function parameters
Ti = 20;                    % Initial soil temperature [deg C]
Ts = -15;                   % Surrounding temperature [deg C]
alpha  = 0.138e-6;          % Thermal conductivety [meters^2/seconds]
t = 5184000;                % Time after cold snap [seconds]

%% Function Definitions
f   = @(x) erf(x / (2 * sqrt(alpha * t))) * (Ti - Ts) + Ts;
df  = @(x) ((Ti - Ts) / sqrt(pi * alpha * t)) * exp(-x.^2 / (4 * alpha * t))
   ;

%% Simulation parameters
x0 = 0.01;                  % Newton's method initial guess
a = 0;                      % Initial left bound of bisection interval
b = 6;                      % Initial right bound of bisection interval
tol  = 1e-13;               % Stopping tolerance
maxIts = 100;               % Maximum iterations allowed

%% Plot the temperature function
X = linspace(0, 6, 1000);
plot(X, f(X), 'LineWidth', 2)
xlabel('x (meters)')
ylabel('Temperature (deg C)')
title('Temperature of Water Main vs Depth')

%% Run each method!
newton(x0, f, df, maxIts, tol, 0.6769)

bisection(a, b, f, tol, maxIts, 0.6769)
```

Problem 4 Code

```matlab
%% Homework 2 Newton's method and Modified Newton
% Author: Caleb Jacobs
% Date last modified: 08-09-2021
format longE
close all

%% Function definitions
p  = 5;
f  = @(x) (x - 1).^p * exp(x);
df = @(x) p*(x - 1).^(p - 1) * exp(x) + (x - 1).^5 * exp(x);

%% Iteration settings
x0     = 0;         % Initial root guess
tol    = 1e-9;      % Accuracy goal
maxIts = 1000;      % Maximum allowed iterations
root   = 1;         % True root to our function

%% Run root finding functions
dat1 = newton(x0, f, df, maxIts, 1e-16, root);
dat2 = modNewton(x0, f, df, p, maxIts, tol, root);

%% Plot convergence data
figure()
semilogy(0:(length(dat1) - 1), dat1(:,2), 'LineWidth', 2)
title("Newton's Method: Error vs Iterations")
xlabel('Iteration #')
ylabel('Absolute error')

figure()
semilogy(0:(length(dat2) - 1), dat2(:,2), 'LineWidth', 2)
title("Modified Newton's Method: Error vs Iterations")
xlabel('Iteration #')
ylabel('Absolute error')

%% Function defintions
% Newton's method
% x0: initial root guess
% f:  function to find root of
% df: derivative of function
% maxIts: maximum iterations for convergence
% tol: convergence tolerance
% root: true root for error computation
% return: iteration data
function data = newton(x0, f, df, maxIts, tol, root)
    data = [x0 abs(x0 - root)];     % Initialize data set

    % Begin running iterations
    for i = 1:maxIts
```

```
        denom = df(x0);                 % Compute derivative of f

        if denom == 0                   % Check if derivative is flat
            fprintf('ERROR: Derivative is flat\n');
            return
        end

        x1 = x0 - f(x0) / denom;        % Compute next iteration

        data(i + 1, :) = [x1 abs(x1 - root)];    % Add data to array

        if abs(x1 - x0) <= tol          % Check stopping criteria
            return
        end

        x0 = x1;                        % Save current iteration
    end

    fprintf('ERROR: Method did not converge\n')
    return
end

% Modfied Newton's method
% x0: initial root guess
% f:  function to find root of
% df: derivative of function
% p: Multiplicity of root
% maxIts: maximum iterations for convergence
% tol: convergence tolerance
% root: true root for error computation
% return: iteration data
function data = modNewton(x0, f, df, p, maxIts, tol, root)
    data = [x0 abs(x0 - root)];         % Initialize data set

    % Begin running iterations
    for i = 1:maxIts
        denom = df(x0);                 % Compute derivative of f

        if denom == 0                   % Check if derivative is flat
            fprintf('ERROR: Derivative is flat\n');
            return
        end

        x1 = x0 - p * f(x0) / denom;        % Compute next iteration

        data(i + 1, :) = [x1 abs(x1 - root)];    % Add data to array

        if abs(x1 - x0) <= tol          % Check stopping criteria
            return
        end

```

```
            x0 = x1;                            % Save current iteration
        end

        fprintf('ERROR: Method did not converge\n')
        return
end
```

Bisection Code

```matlab
%%
% Bisection method for finding roots
% Author: Caleb Jacobs
% Date last modified: 09-09-2021
%
% a: left searching interval bound
% b: right searching interval bound
% f: function to compute the root of
% tol: stopping tolerance
% maxIts: maximum iterations allowed
% root: true root for error calculations
% return: error code
function ier = bisection(a, b, f, tol, maxIts, root)
    fprintf("Bisection Method\n")

    % Check if root is guaranteed in the interval
    if f(a) * f(b) > 0
        ier = 2;                        % Return 2 if solution not guaranteed
        return
    end

    % Begin iterations
    for i = 1:maxIts
        c = (b + a) / 2;                % Midpoint of interval

        fprintf('%3d: %.16e\n', i, abs(c - root) / abs(root))

        if f(c) == 0 || (b - c) < tol   % Check convergence criteria
            ier = 0;                    % Return 0 for success
            fprintf('Root = %.16e\n\n', c);
            return
        end

        if f(c) * f(a) >= 0             % Update searching interval
            a = c;
        else
            b = c;
        end
    end

    ier = 1;                            % Return 1 for no convergence
    return
end
```

Newton's Method Code

```
%%
% Newton's root finding method
% Author: Caleb Jacobs
% Date last modified: 07-09-2021
%
% x0: initial root guess
% f:  function to find root of
% df: derivative of function
% maxIts: maximum iterations for convergence
% tol: convergence tolerance
% root: true root for error computation
% return: error code
function ier = newton(x0, f, df, maxIts, tol, root)
    fprintf("Newton's Method\n")

    % Begin running iterations
    for i = 1:maxIts
        denom = df(x0);                 % Compute derivative of f

        if denom == 0                   % Check if derivative is flat
            ier = 2;
            return
        end

        x1 = x0 - f(x0) / denom;       % Compute next iteration

        fprintf('%3d: %.16e\n', i, abs(x1 - root) / abs(root))

        if abs(x1 - x0) <= tol          % Check stopping criteria
            ier = 0;                    % Return 0 for success
            fprintf('Root = %.16e\n\n', x1);
            return
        end

        x0 = x1;                        % Save current iteration
    end

    ier = 1;                            % Return 1 if method did not converge
    return
end
```