RBFs Over Near-Flat Surfaces

APPM 5480 Asymptotics

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April 25, 2022

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Outline

1. Introduction/Background
   - Radial Basis Functions
   - RBF Interpolants
   - Solving $A\boldsymbol{\lambda} = \boldsymbol{F}$
2. Enter Asymptotics!
   - A perturbation problem
   - Perturbed solution
   - Solution remarks
3. What's Next

Introduction/Background

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Radial Basis Functions

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Commonly used radial basis functions

Type of basis function	Radial function $\phi(r)$
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Multiquadric(MQ)	$\sqrt{1+(\varepsilon r)^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{1+(\varepsilon r)^2}$
Inverse Quadratic (IQ)	$1/(1+(\varepsilon r)^2)$
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$$s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \phi_\varepsilon(\|\mathbf{x} - \mathbf{x}_i\|)$$

where λ_i can be found by solving

$$\underbrace{\begin{pmatrix} \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_n\|) \\ \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_n\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_n\|) \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}}_{\boldsymbol{\lambda}} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}}_{\boldsymbol{F}}.$$

Introduction/Background

Solving $A\boldsymbol{\lambda} = \boldsymbol{F}$

Solving for our weights λ

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- Naively solving $A\boldsymbol{\lambda} = \boldsymbol{F}$ is called *RBF-Direct*. *Does not work when ε is small.*
- Solve this system stably using an indirect method for small ε.

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RBF Contour Padé (RBF-CP) (*precursor to RBF-RA*)

RBF Rational Approximations (RBF-RA) (circa 2017) [3]

RBF-RA issues

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RBF-RA works great when the data given $\{\mathbf{x}_i\}_{i=1}^{n}$ is

- on a completely flat surface
- on a curvy surface.

However, RBF-RA breaks down when the data is on a near-flat surface (i.e. surface with small curvature).

So, we would like understand/explore why RBF-RA breaks on near flat surfaces.

Enter Asymptotics!

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A perturbation problem

A perturbation problem

Suppose we have the pairs of points $\{(x_i, y_i)\}_{i=1}^{n}$ in the *unit disk*. Then, define our interpolation nodes as

$$\mathbf{x}_i = \left\langle x_i, y_i, \sqrt{\frac{1}{\kappa^2} - r_i^2} - \frac{1}{\kappa} \right\rangle, \quad r_i = \sqrt{x_i^2 + y_i^2}.$$

This is just data over a sphere with curvature κ with the top of the sphere at the origin:

$$x_i^2 + y_i^2 + (z_i + 1/\kappa)^2 = \frac{1}{\kappa^2}$$

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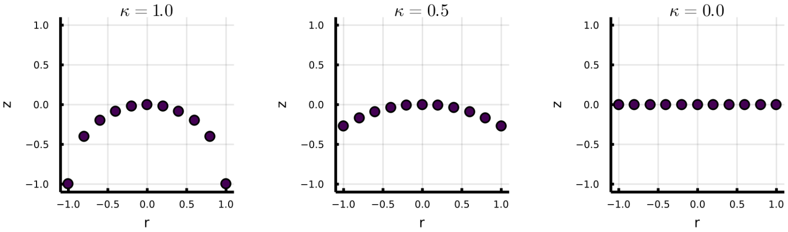

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Then, the difference between two nodes is given as

$$\|\mathbf{x}_i - \mathbf{x}_j\|_2^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + \left(\sqrt{\frac{1}{\kappa^2} - r_i^2} - \sqrt{\frac{1}{\kappa^2} - r_j^2} \right)^2$$

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Further, each entry of the collocation matrix A will be

$$A_{ij} = \phi_\varepsilon(\|\mathbf{x}_i - \mathbf{x}_j\|_2) = e^{-(\varepsilon\|\mathbf{x}_i - \mathbf{x}_j\|_2)^2} = e^{-\varepsilon^2\|\mathbf{x}_i - \mathbf{x}_j\|_2^2}$$

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&= e^{-\varepsilon^2((x_i-x_j)^2+(y_i-y_j)^2)}e^{-\varepsilon^2\left(\sqrt{\frac{1}{\kappa^2}-r_i^2}-\sqrt{\frac{1}{\kappa^2}-r_j^2}\right)^2} \\
&= e^{-\varepsilon^2((x_i-x_j)^2+(y_i-y_j)^2)}\left(1-\frac{1}{4}\varepsilon^2\kappa^2(r_i^2-r_j^2)^2+\cdots\right) \\
&= e^{-\varepsilon^2((x_i-x_j)^2+(y_i-y_j)^2)} - \kappa^2\frac{1}{4}\varepsilon^2(r_i^2-r_j^2)^2e^{-\varepsilon^2((x_i-x_j)^2+(y_i-y_j)^2)} + \mathcal{O}(\kappa^4).
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A = A_0 + \kappa^2 A_1 + \mathcal{O}(\kappa^4)
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A perturbation problem

With the expansion of A

$$A = A_0 + \kappa^2 A_1 + \cdots ,$$

our perturbed interpolant problem is then given by

$$A\boldsymbol{\lambda} = \boldsymbol{F} \implies (A_0 + \kappa^2 A_1 + \cdots)\boldsymbol{\lambda} = \boldsymbol{F}, \quad 0 < \kappa \ll 1.$$

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Note, A_0 is just the standard collocation matrix if our data was on a flat surface. We know

A_0 is nonsingular for $\varepsilon \in \mathbb{R}, \varepsilon \neq 0$,

A_0 is symmetric.

Enter Asymptotics!

Perturbed solution

Perturbed interpolation weights problem

$$(A_0 + \kappa^2 A_1 + \cdots)\boldsymbol{\lambda} = \boldsymbol{F}, \quad 0 < \kappa \ll 1$$

Assume $\boldsymbol{\lambda}$ has the regular expansion

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \kappa^2 \boldsymbol{\lambda}_1 + \cdots .$$

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Then at

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So in general,

$$\boldsymbol{\lambda}_n = -A_0^{-1}(A_1\boldsymbol{\lambda}_{n-1} + A_2\boldsymbol{\lambda}_{n-2} + \cdots + A_n\boldsymbol{\lambda}_0)$$

Enter Asymptotics!

Solution remarks

Remarks

Putting everything together, our interpolation weights can be written as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \kappa^2 \boldsymbol{\lambda}_1 + \cdots$$

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Remarks

- This shows that $\boldsymbol{\lambda}$ is asymptotically well behaved when $\kappa \to 0$. So ideally, RBF-RA ought to not break in this limit.
- We can use this asymptotic expansion to bypass the small curvature issue in RBF-RA.

Patched RBF-RA

The modified RBF-RA algorithm is highlighted below

(a) Compute κ estimate of data using desired algorithm
(b) Use RBF-RA to stably compute needed $\boldsymbol{\lambda}_i$
(c) Compute $\boldsymbol{\lambda}$ using $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \kappa^2 \boldsymbol{\lambda}_1 + \cdots$ up to desired term.

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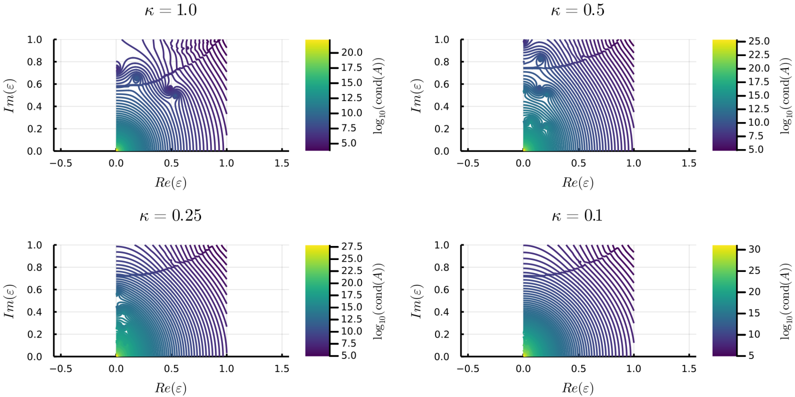
Patching an algorithm like this is not ideal. Instead, we would like to understand why the algorithm breaks and then fix the algorithm at the fundamental level.

What’s Next

A closer look at conditioning

The conditioning of the collocation matrix at various κ and ε in the complex plane can be seen below



A perturbed eigenvalue problem

Using the same perturbed collocation matrix as before, we can pose a perturbed eigenvalue problem as

$$(A_0 + \kappa^2 A_1 + \cdots)\mathbf{x} = \lambda \mathbf{x}.$$

Now, because A depends on arbitrary, real life data, it is reasonable to assume that A has all distinct eigenvalues. So, finding a perturbed eigenvalue/eigenvector solution to this is readily found using our formulas derived early on in our course or in *Hinch* [2]. In this case, the eigenvalues all have regular perturbations

$$\lambda(\varepsilon; \kappa) = \lambda_0(\varepsilon) + \kappa^2 \lambda_2(\varepsilon) + \cdots .$$

A perturbed root finding problem

The A matrix is normal and so

$$\mathrm{cond}(A) = \frac{|\lambda_{\mathsf{max}}|}{|\lambda_{\mathsf{min}}|}.$$

So, to understand when the condition number blows up, we want to know when $\lambda(\varepsilon;\kappa) = 0$ for any of the eigenvalues λ. So, an interesting perturbed root finding problem is find ε such that

$$\lambda_0(\varepsilon) + \kappa^2\lambda_2(\varepsilon) + \cdots = 0, \quad 0 < \kappa \ll 1.$$

Even for simple node-sets, $\lambda_i(\varepsilon)$ can be very nonlinear making this perturbed root finding problem quite difficult.

References

[1] B. Fornberg and N. Flyer, *A primer on radial basis functions with applications to the geosciences*, vol. 87 of CBMS-NSF regional conference series in applied mathematics, SIAM, 2015.

[2] E. J. Hinch, *Perturbation Methods*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1991.

[3] G. B. Wright and B. Fornberg, *Stable computations with flat radial basis functions using vector-valued rational approximations*, Journal of Computational Physics, 331 (2017), pp. 137–156.

Thank You

Any questions?