RBF Interpolants Over Near-Flat Surfaces

Background and Problem

Given a small parameter ε that we will call the shape parameter and data $\{\mathbf{x}_i, f_i\}_{i=1}^{n}$, we can write out a radial basis function (RBF) based interpolant of the data as

$$s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \phi_\varepsilon(\|\mathbf{x} - \mathbf{x}_i\|_2) \tag{1}$$

where λ_i are interpolant weights and ϕ_ε is a radial function (such as $\phi_\varepsilon(r) = e^{-(\varepsilon r)^2}$). In the most direct form, we can find λ_i by solving the system

$$\begin{pmatrix} \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_1 - \mathbf{x}_n\|) \\ \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_2 - \mathbf{x}_n\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_1\|) & \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_2\|) & \cdots & \phi_\varepsilon(\|\mathbf{x}_n - \mathbf{x}_n\|) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

In solving this system numerically, it is known that taking ε smaller and smaller tends to increase the accuracy of our interpolant. However, if ε becomes too small, the matrix above becomes severely ill-conditioned and the interpolant becomes unusable numerically even when it should be well behaved. So, to find λ_i even when ε is small, Bengt and some of his previous students have developed multiple methods that take ε to be complex and then they perform a contour integral about $\varepsilon = 0$ to recover the small ε interpolant stably. The most recent and robust method, so called RBF-RA, is based on rational approximations of our underlying function.

Now with small ε, we can apply RBF-RA to data that is on a surface with a large curvature κ or a completely flat surface (i.e. $\kappa = 0$) to get stable and desired results. However, applying RBF-RA to data on a surface that is nearly flat (i.e. $0 < \kappa \ll 1$) leads to erroneous results. Both Cécile and Bengt have numerically explored this discrepancy with a few visuals (a singularity appears to pop up at the origin) but have not yet come to understand it nor fix it. Intuition tells us that this singularity for $0 < \kappa \ll 1$ should be removable because both $\kappa = 0$ and $0 \ll \kappa$ lead to well behaved results.

Herein lies my project, I am seeking to understand how our interpolant behaves when we perturb κ near zero. The methods of Asymptotics appear to be a perfect candidate for developing this new understanding. However, before we can apply Asymptotics to this problem, we need to see where κ is embedded in our RBF interpolant. In this case, κ will appear in the data. For completely flat surfaces, the data all falls in a single plane. But, once we start perturbing κ, our data will start to deviate from this plane. The simplest example of a surface with curvature κ is given by a circle of radius $1/\kappa$

$$x^2 + y^2 = \frac{1}{\kappa^2} \implies y(x; \kappa) \tag{2}$$

Note that locally, any smooth curve can be approximated by this circle and that this circle can easily be extended to higher dimensions (i.e. hyperspheres). **So, our perturbation problem can be posed as asymptotically describe the interpolant $s(x; \kappa)$ given in (1) when we take our data to be $\mathbf{x}_i = (x_i, y_i)$ subject to (2) with $0 < \kappa \ll 1$.**

This problem is wide open and it doesn't appear anyone has been trying to tackle it in the field. Furthermore, if we can develop an understanding of how κ affects our interpolant, we could learn how to avoid numerical instability in the small κ regime and improve the RBF-RA method. As a personal impact of this project, Cécile, one of her grad students, and I have been using RBF-RA on general surfaces of which we are not guaranteed κ will be outside of the near-flat regime. So any new understanding of how κ affects RBF interpolants could greatly improve our results.