1. Derive a quadrature based on the cubic Hermite interpolating polynomial with data $f(a)$, $f(b)$, $f'(a)$, and $f'(b)$. Derive an upper bound on the error.

   Using the Hermite-Lagrange basis, we can construct our cubic Hermite polynomial as

$$p(x) = f(a)H_a(x) + f(b)H_b(x) + f'(a)K_a(x) + f'(b)K_b(x)$$

   where

$$\begin{aligned}
H_a(x) &= \left(1 - 2(x-a)\frac{1}{a-b}\right)\frac{(x-b)^2}{(a-b)^2} \\
H_b(x) &= \left(1 - 2(x-b)\frac{1}{b-a}\right)\frac{(x-a)^2}{(b-a)^2} \\
K_a(x) &= (x-a)\frac{(x-b)^2}{(a-b)^2} \\
K_b(x) &= (x-b)\frac{(x-a)^2}{(b-a)^2}.
\end{aligned}$$

   Now, integrating $p(x)$ over our interval, $[a, b]$, we obtain our quadrature as

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \int_a^b p(x)\,\mathrm{d}x \\
&= f(a)\int_a^b H_a(x)\,\mathrm{d}x + f(b)\int_a^b H_b(x)\,\mathrm{d}x + f'(a)\int_a^b K_a(x)\,\mathrm{d}x + f'(b)\int_a^b K_b(x)\,\mathrm{d}x \\
&= f(a)\frac{b-a}{2} + f(b)\frac{b-a}{2} + f'(a)\frac{(a-b)^2}{12} - f'(b)\frac{(a-b)^2}{12} \\
&= \boxed{(f(a) + f(b))\frac{b-a}{2} + (f'(a) - f'(b))\frac{(a-b)^2}{12}.}
\end{aligned}$$

   Now, assuming $f \in C^4[a, b]$, we can get an error bound for this quadrature by integrating the Hermite interpolant error as

$$\begin{aligned}
E = \int_a^b |f(x) - p(x)|\,\mathrm{d}x &= \int_a^b \left|\frac{f^{(4)}(\eta_x)}{4!}(x-a)^2(x-b)^2\right|\,\mathrm{d}x && \text{for some } \eta_x \in [a, b] \\
&\leq \frac{M}{24}\int_a^b (x-a)^2(x-b)^2\,\mathrm{d}x && \text{where } M = \max_{\eta\in[a,b]}\left|f^{(4)}(\eta)\right| \\
&= \frac{M}{24}\left(\frac{(b-a)^5}{30}\right) \\
&= \boxed{\frac{M(b-a)^5}{720}.}
\end{aligned}$$

2. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots .$$

Generalize the Richardson extrapolation process to obtain an estimate of I with an error on the order $\frac{1}{n^2\sqrt{n}}$. Assume that three values $I_n, I_{n/2}$, and $I_{n/4}$ have been computed.

From the error formula, we have the three equations

$$I = I_n + \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots \tag{1}$$

$$I = I_{n/2} + 2\sqrt{2}\frac{C_1}{n\sqrt{n}} + 4\frac{C_2}{n^2} + 4\sqrt{2}\frac{C_3}{n^2\sqrt{n}} + \cdots \tag{2}$$

$$I = I_{n/4} + 8\frac{C_1}{n\sqrt{n}} + 16\frac{C_2}{n^2} + 32\frac{C_3}{n^2\sqrt{n}} + \cdots . \tag{3}$$

Using these three equations, we want to eliminate the C_1 and C_2 error terms which we can do by reducing

$$\begin{pmatrix} 1 & 2\sqrt{2} & 8 \\ 1 & 4 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8\sqrt{2} \\ 0 & 1 & 2(\sqrt{2}+2) \end{pmatrix}$$

which tells us that

$$8\sqrt{2}(1) - 2(\sqrt{2}+2)(2) + (3)$$

will eliminate our desired error terms. So, we have the equation

$$(8\sqrt{2} - 2(\sqrt{2}+2) + 1)I = 8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4} + (16 - 8\sqrt{2})\frac{C_3}{n^2\sqrt{n}}$$

which implies

$$I = \frac{8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2}+2) + 1} + O\left(\frac{1}{n^2\sqrt{n}}\right).$$

So if we use the integration formula I' defined as

$$\boxed{I' = \frac{8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2}+2) + 1}}$$

we get our desired error

$$\boxed{I - I' = O\left(\frac{1}{n^2\sqrt{n}}\right)}.$$

3. Let $n \geq 0$.

(i) Give a formula for the Gauss quadrature points $x_j, j = 0, \ldots, n$, needed for the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$.
First, note that the Chebysheb polynomials are orthogonal under the given weight function. So, to find our nodes, x_j, we need to find the roots of the $n+1$ Chebyshev polynomial which is given by

$$T_{n+1}(x) = \cos((n+1)\arccos(x)).$$

To find the roots of T_{n+1}, we need

$$(n+1)\arccos(x) = \frac{\pi}{2} + j\pi$$

for any integer j. So, we must have

$$\boxed{x_j = \cos\left(\frac{\left(j+\frac{1}{2}\right)\pi}{n+1}\right).}$$

So we don't have overlapping x_j, restrict j to $j = 0, \ldots, n$.

(ii) Show that for positive integers n,

$$\sum_{j=0}^{n} \cos((2j+1)\theta) = \frac{\sin((2n+2)\theta)}{2\sin(\theta)},$$

unless θ is a multiple of π. What is the value of the sum when θ is a multiple of π?
To begin showing our sum, note that from a product-to-sum identity, we have

$$2\sin(\theta)\cos((2j+1)\theta) = \sin((2j+2)\theta) - \sin(2j\theta).$$

Using this identity, we have the telescoping sum

$$\begin{aligned}
\sum_{j=0}^{n} 2\sin(\theta)\cos((2j+1)\theta) =& + \sin(2\theta) - 0 \\
& + \sin(4\theta) - \sin(2\theta) \\
& + \sin(6\theta) - \sin(4\theta) \\
& + \cdots \\
& + \sin(2n\theta) - \sin((2n-2)\theta) \\
& + \sin((2n+2)\theta) - \sin(2n\theta) \\
=& \sin((2n+2)\theta)
\end{aligned}$$

which implies

$$\sum_{j=0}^{n} 2\sin(\theta)\cos((2j+1)\theta) = \sin((2n+2)\theta)$$

or solving for our desired sum,

$$\boxed{\sum_{j=0}^{n} \cos((2j+1)\theta) = \frac{\sin((2n+2)\theta)}{2\sin(\theta)}}$$

If θ is a multiple of π, then

$$\boxed{\sum_{j=0}^{n} \cos((2j+1)\theta) = \begin{cases} n+1, & \theta \text{ is an even multiple} \\ -n-1, & \theta \text{ is an odd multiple} \end{cases}.}$$

(iii) Suppose

$$T_n(x) = \cos(n \arccos(x)), \quad x \in [-1,1].$$

Then, for integers $k = 1, \ldots, n$, we have

$$\begin{aligned} \sum_{j=0}^{n} T_k(x_j) &= \sum_{j=0}^{n} \cos\left(k \arccos\left(\cos\left(\frac{\left(j+\frac{1}{2}\right)\pi}{n+1}\right)\right)\right) \\ &= \sum_{j=0}^{n} \cos\left(k\left(\frac{\left(j+\frac{1}{2}\right)\pi}{n+1}\right)\right) \\ &= \sum_{j=0}^{n} \cos\left((2j+1)\frac{k\pi}{2n+2}\right) \\ &= \frac{\sin\left((2n+2)\frac{k\pi}{2n+2}\right)}{2\sin\left(\frac{k\pi}{2n+2}\right)} \\ &= \frac{\sin(k\pi)}{2\sin\left(\frac{k\pi}{2n+2}\right)} \\ &= 0. \end{aligned}$$

However,

$$\begin{aligned} \int_{-1}^{1} \frac{T_k(x)}{\sqrt{1-x^2}} \, \mathrm{d}x &= \int_{-1}^{1} \frac{\cos(k \arccos(x))}{\sqrt{1-x^2}} \, \mathrm{d}x \\ &= \frac{\sin k\pi}{k} \\ &= 0. \end{aligned}$$

So

$$\boxed{\sum_{j=0}^{n} T_k(x_j) = \int_{-1}^{1} \frac{T_k(x)}{\sqrt{1-x^2}} \, \mathrm{d}x.}$$

Next, we have

$$\sum_{j=0}^{n} T_0(x_j) = \sum_{j=0}^{n} \cos(0) = \sum_{j=0}^{n} 1 = n+1.$$

Similarly

$$\frac{n+1}{\pi} \int_{-1}^{1} \frac{T_0(x)}{\sqrt{1-x^2}} \, \mathrm{d}x = \frac{n+1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \frac{n+1}{\pi}\pi = n+1.$$

So

$$\boxed{\sum_{j=0}^{n} T_0(x_j) = \frac{n+1}{\pi} \int_{-1}^{1} \frac{T_0(x)}{\sqrt{1-x^2}} \, \mathrm{d}x.}$$

(iv) Now, lets compute the quadrature weights with the weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

on the interval $(-1, 1)$.

$$\begin{aligned} W_k = \int_{-1}^{1} \frac{\phi(x)}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int_{-1}^{1} \frac{\sum_{s=0}^{n} C_s T_s(x)}{\sqrt{1-x^2}} \\ &= \frac{\pi}{n+1} C_0 \sum_{j=1}^{n} T_0(x_j) + \sum_{s=1}^{n}\sum_{j=0}^{n} C_s T_s(x_j) \\ &= \frac{\pi}{n+1}. \end{aligned}$$

4. The gamma function is defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad x > 0.$$

A simple program to approximate $\Gamma(x)$ using trapezoidal rule applied to a truncated line.

```
%%
% Quadrature for computing the gamma function
%
% Date last modified: 09-Dec-2021
% Author: Caleb Jacobs
format longe
gam(4, 1000) - gamma(4)


function val = gam(x, n)
    b = 20 * x;                        % Compute upper bound of
   integration

    t = linspace(0, b, n);             % Create evaluation nodes
    h = t(2) - t(1);                   % Find stepsize
    f = t .^ (x - 1) .* exp(-t);      % Compute function values at each
   node

    val = h * ((f(1) + f(n)) / 2 + sum(f(2 : n - 1)));
end
```

(a) My idea for choosing the interval to integrate over involved figuring out when the integrand was *small enough* compared to my desired tolerance. Because the integrand decays to zero as $t- > 0$ for all x, we are able to find some cutoff so that the tail end of the integral does not affect the overall numerical result to our desired tolerance.

(b) In playing with MATLAB, the quad routine in MATLAB used less function evaluations to obtain the same error.

(c) Using the given Gauss-Laguerre quadrature code to obtain our weights and nodes allowed the integral for Γ to converge much more rapidly and with less work!

5. Gaussian quadrature
$$\int_{-1}^{1} f(x)\ \mathrm{d}x = \sum_{k=0}^{n} w_k f(x_k)$$
with nodes including the endpoints ($x_0 = -1$ and $x_n = 1$) using Legendre polynomials are called *Gauss-Legendre-Lobatto* quadratures.

   (i) Show that if the interior nodes $x_1, \ldots, x_{n-1}$ in the quadrature are given by the roots of $p_n'(x)$ where $p_n(x)$ denotes the n-th degree Legendre polynomial, then the quadrature is exact for polynomials up to degree $2n-1$.

   Before we begin, note that $(x^2-1)p_n'(x) = \frac{x}{n}p_n(x) - \frac{1}{n}p_{n-1}(x)$. Then, for any $h \in \mathcal{P}_{2n-1}$ and some $q, r \in \mathcal{P}_{n-2}$, we have
$$h(x) = \left(\frac{x}{n}p_n(x) - \frac{1}{n}p_{n-1}(x)\right) q(x) + r(x).$$
   Now, define $\bar{q}(x) = xq(x)$; note that $\bar{q} \in \mathcal{P}_{n-1}$. Then
$$\begin{aligned}\int_{-1}^{1} h(x)\ \mathrm{d}x &= \int_{-1}^{1} \left(\frac{x}{n}p_n(x) - \frac{1}{n}p_{n-1}(x)\right) q(x) + r(x)\ \mathrm{d}x \\ &= \underbrace{\int_{-1}^{1} \frac{1}{n}p_n(x)\bar{q}(x) - \frac{1}{n}p_{n-1}(x)q(x)\ \mathrm{d}x}_{0 \because \text{ of orthogonality of } p_n \text{ and } p_{n-1}} + \int_{-1}^{1} r(x)\ \mathrm{d}x \\ &= \underbrace{\int_{-1}^{1} (x^2-1)p_n'(x)q(x)\ \mathrm{d}x}_{\text{still } 0} + \int_{-1}^{1} r(x)\ \mathrm{d}x\end{aligned}$$
   which implies that if we chose our boundary nodes $x_0 = -1, x_n = 1$ and every interior node to be a root of $p_n'(x)$, by construction, our quadrature will be exact for polynomials $h \in \mathcal{P}_{2n-1}$.

   (ii) Find the 4-point Gauss-Legendre-Lobatto quadrature for the integral $\int_{-1}^{1} f(x)\ \mathrm{d}x$.

   First, let's find the interior nodes. We know
$$p_3(x) = \frac{1}{2}(5x^3 - 3x) \implies p_3'(x) = \frac{1}{2}(15x^2 - 3)$$
   So, $p_3'(x) = 0$ when $x = \pm\frac{1}{\sqrt{5}}$. Thus, our nodes are $x_0 = -1, x_1 = -\frac{1}{\sqrt{5}}, x_2 = \frac{1}{\sqrt{5}}$, and $x_3 = 1$.

Next, we can compute our weights as

$$w_0 = \int_{-1}^{1} \frac{\left(x+\frac{1}{\sqrt{5}}\right)\left(x-\frac{1}{\sqrt{5}}\right)(x-1)}{\left(-1+\frac{1}{\sqrt{5}}\right)\left(-1-\frac{1}{\sqrt{5}}\right)(-1-1)} = \frac{1}{6}$$

$$w_1 = \int_{-1}^{1} \frac{(x+1)\left(x-\frac{1}{\sqrt{5}}\right)(x-1)}{\left(-\frac{1}{\sqrt{5}}+1\right)\left(-\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}-1\right)} = \frac{5}{6}$$

$$w_2 = \int_{-1}^{1} \frac{(x+1)\left(x+\frac{1}{\sqrt{5}}\right)(x-1)}{\left(\frac{1}{\sqrt{5}}+1\right)\left(\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}-1\right)} = \frac{5}{6}$$

$$w_3 = \int_{-1}^{1} \frac{(x+1)\left(x+\frac{1}{\sqrt{5}}\right)\left(x-\frac{1}{\sqrt{5}}\right)}{(1+1)\left(1+\frac{1}{\sqrt{5}}\right)\left(1-\frac{1}{\sqrt{5}}\right)} = \frac{1}{6}.$$

Therefore, our quadrature is

$$\boxed{\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \frac{1}{6}f(-1) + \frac{5}{6}f\left(-\frac{1}{\sqrt{5}}\right) + \frac{5}{6}f\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{6}f(1).}$$