1. Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   i. $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$

   ii. $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{1/3}$
   Suppose
   $$g(x) = \frac{2}{3}x + \frac{1}{x^2}.$$
   Furthermore, suppose $x_0 \in [1, 2]$. Then $x_* \in [1, 2]$, $\min_{x\in[1,2]} g(x) = 3^{1/3}$, and $\max_{x\in[1,2]} g(x) = \frac{5}{3}$. Therefore $g([1, 2]) \subset [1, 2]$ So because $g(x)$ is continuously differentiable on $[1, 2]$, Theorem 2.6 of Atkinson tells us that with $x_0 \in [1, 2]$ and $x_{n+1} = g(x_n)$, that x_* is a unique solution to $x = g(x)$ in $[1, 2]$ and that
   $$\lim_{n\to\infty} x_n = x_*.$$
   In other words, for any x_0 in the neighborhood $[1, 2]$ of x_*, the fixed point iteration will converge.
   Now let's figure out the order of convergence. Taylor expanding $g(x_n)$ about $x_n = x_*$ yields
   $$x_{n+1} = g(x_n) = g(x_*) + \left(\frac{2}{3} - \frac{2}{x_*^3}\right)(x_n - x_*) + \frac{3}{\xi^4}(x_n - x_*)^2 = x_* + \frac{3}{\xi^4}(x_n - x_*)^2$$
   for ξ between x_n and x_*. Using the equality above, we can obtain
   $$\frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{3}{\xi^4}.$$
   Now because $x_n \to x_*$ as $n \to \infty$ and ξ is between x_n and x_*, we must have $\xi \to x_*$ as $n \to \infty$. Thus
   $$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{3}{x_*^4}$$
   which shows that $x_{n+1} = g(x_n)$ converges quadratically to x_* in the neighborhood $[1, 2]$.

   iii. $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$
   Just as in the previous part, let's first show that the iteration converges to x_* for x_0 sufficiently close to x_*. First, let
   $$g(x) = \frac{12}{1 + x_n}, x_* = 3.$$
   Then $x_{n+1} = g(x_n)$, $x_* \in [2, 4]$, $\min_{x\in[2,4]} g(x) = \frac{12}{5}$, and $\max_{x\in[2,4]} g(x) = 4$. Therefore, $g([2, 4]) \subset [2, 4]$ and so by Theorem 2.6 on Atkinson, x_* is a unique solution to $x = g(x)$ in $[2, 4]$ and any $x_0 \in [2, 4]$ with $x_{n+1} = g(x_n)$ will have
   $$\lim_{n\to\infty} x_n = x_*.$$
   In other words, for any $x_0 \in [2, 4]$ (i.e. sufficiently close to x_*), $x_{n+1} = g(x_n)$ will converge to x_*.
   Now let's determine the order and rate of convergence. Taylor expanding $g(x_n)$ about $x_n = x_*$ yields
   $$x_{n+1} = g(x_n) = g(x_*) - \frac{12}{(1 + \xi)^2}(x_n - x_*)$$

for some ξ between x_n and x_*. Then, rearranging yields

$$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|\frac{12}{(1+\xi)^2}\right|.$$

Finally, because $\lim_{n\to\infty} x_n = x_*$, $\lim_{n\to\infty} \xi = x_*$ and so

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|\frac{12}{(1+x_*)^2}\right| = \frac{3}{4}.$$

This limit shows us that $x_{n+1} = g(x_n)$ will converge linearly to x_* with a rate of $\frac{3}{4}$ provided $x_0 \in [2, 4]$ (i.e. x_0 is sufficiently close to x_*).

2. In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x, t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where T_s is the constant temperature during a cold period T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2) and

$$\operatorname{erf}(t) = \frac{2}{\sqrt{2}} \int_0^t e^{-s^2} \mathrm{d}s$$

Assume that $T_i = 20[\deg C]$, $T_s = -15[\deg C]$, $\alpha = 0.138 \cdot 10^{-6}$[meters^2 per second].

3. Consider applying Newton's method to a real cubic polynomial.

   i. In the case that the polynomial has three distinct real roots, $x = \alpha, x = \beta$, and $x = \gamma$, show that the starting guess $x_0 = \frac{1}{2}(\alpha + \beta)$ will yield the root γ in one step.
   Any cubic polynomial with roots $x \in \{\alpha, \beta, \gamma\}$ can be written as

$$p(x) = a(x - \alpha)(x - \beta)(x - \gamma)$$

where $a \in \mathbb{R}$ is a constant. Using this general cubic yields a newton iteration of

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}.$$

Then, using $x_0 = \frac{1}{2}(\alpha + \beta)$ yieds an iteration of

$$\begin{aligned} x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} &= \frac{1}{2}(\alpha + \beta) - \frac{-\frac{1}{8}a(\alpha - \beta)^2(\alpha + \beta - 2\gamma)}{-\frac{1}{4}a(\alpha - \beta)^2} \\ &= \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(2\gamma) \\ &= \gamma. \end{aligned}$$

Thus, $x_1 = \gamma$ which shows Newton converges to $x = \gamma$ in one iteration when $x_0 = \frac{1}{2}(\alpha + \beta)$.

ii. Give a heuristic argument showing that if two roots coincide, there is precisely one starting guess x_0 other than $x_0 = \beta$ for which Newton's method will fail, and that this one separates the basins of attraction for the distinct roots.

Consider the cubic

$$p(x) = a(x-\alpha)(x-\beta)^2$$

with the constant $a \in \mathbb{R}$. Any plot of $p(x)$ will look similar to Figure 1. From the plot we can see that Newton fails whenever the derivative $p'(x) = 0$. One of these breaking points in the double root at β; the other breaking point is at the x_b between α and β for which $p'(x_b) = 0$. Now, Newton could potentially break at other points if after a certain iteration, we end up at x_b or β. But, because we have double root in this cubic, none of the tangent lines to $p(x)$ cross the x-axis at x_b or β. This lack of other points is due to the function bouncing off the x-axis at β, because any tangent line constructed on $p(x)$ will cross the x-axis towards α if $x < x_b$ and will cross the x-axis towards β if $x > x_b$. Thus, any starting x_0 will move away x_b. Even though iterations can move towards β, they will never get to β and thus, won't break after a certain iteration.

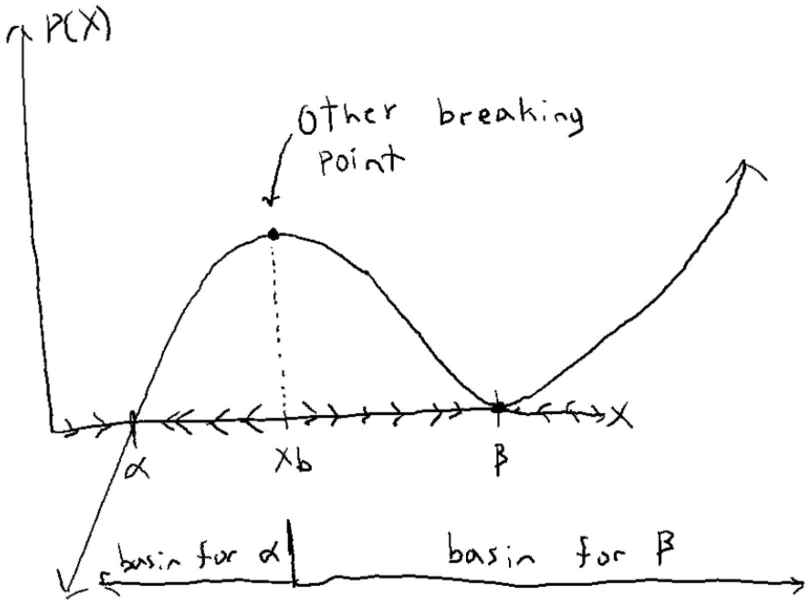

Figure 1: General plot of cubic polynomial with one double root

From the discussion above and Figure 1, we can see that x_b forms the line between the basin of attraction for α and β because the tangents cross the x-axis closer to α or β depending on if the initial x_0 is to the left or right of x_b.

iii. Extend the argument in part ii. to the case when all three roots again are distinct. Explain why there are now infinitely many starting guesses x_0 for which the iteration will fail.

Consider the cubic polynomial:

$$p(x) = a(x-\alpha)(x-\beta)(x-\gamma)$$

where $a \in \mathbb{R}$ is a constant. All plots of $p(x)$ will look similar to Figure 2. From Figure 2, we can see that we still have two obvious breaking points x_{b1} and x_{b2} between α and β, and β and γ. However, now both breaking points are not at roots. In fact, one breaking point will correspond to $p(x) > 0$, and one will correspond to $p(x) < 0$. Looking at where the tangent lines of $p(x)$ cross the x-axis show that there is a tangent line that crosses at x_{b1} and x_{b2}. These tangents could send an initial point x_0 to one of the breaking points which would cause Newton to fail at the second iteration. Even worse, these new breaking points that map to the original breaking points have points that Newton will map to from somewhere else. We can keep repeating this process indefinitely to find an infinite sequence of points in a Newton iteration that will eventually work its way back to the first breaking points which will cause the method to fail. The light gray lines in Figure 2 show a potential path that Newton can take to eventually break after some amount of iterations.

This sequence of breaking points only exists because the local max and min are on the opposite sides of the x-axis and so the tangents will scan across the breaking points as x_n goes to the middle root from a local min or max.

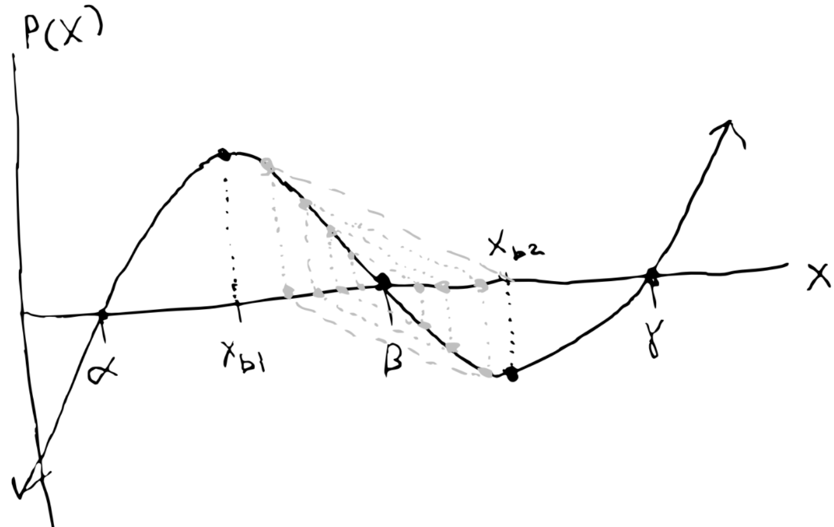

Figure 2: General plot of cubic polynomial with three distinct roots

4. The sequence x_k produced by Newton's method is quadratically convergent to x_* with $f(x_*) = 0$, $f'(x) \neq 0$ and $f''(x)$ continuous at x_*.

   Let $f(x) = (x - x_*)^p q(x)$ with p a positive integer with q twice continuously differentiable and $q(x_*) \neq 0$. Note: $f'(x_*) = 0$. In the following sub-problems, let $x_k, f_k = f(x_k), e_k = |x_* - x_k|$, etc.

   i. Prove that Newton's method converges linearly for $f(x)$.

      Using $f(x) = (x - x_*)^p q(x)$, we have the Newton iterate

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}.$$

      Then,

$$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \frac{|g(x_n) - x_*|}{|x_n - x_*|} = \frac{\left|x_n - x_* - \frac{(x_n - x_*)q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|}{|x_n - x_*|}$$
$$= \left|1 - \frac{q(x_n)}{pq(x_n) + (x_n - x_*)q'(x_n)}\right|.$$

      Now, we can take the limit of both sides to get

$$\lim_{x \to x^*} \frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \left|1 - \frac{q(x_*)}{pq(x_*)}\right| = \left|1 - \frac{1}{p}\right|$$

      which shows that Newton converges linearly.

   ii. Consider the modified Newton iteration defined by

$$x_{k+1} = x_k - p\frac{f_k}{f'_k}.$$

      Prove that if x_k converges to x_*, then the rate of convergence is quadratic.

   iii. Write MATLAB codes for both Newton and modified Newton methods. Apply these to the function

$$f(x) = (x - 1)^5 e^x$$

      and compare the results. Use $x_0 = 0$ as a starting point.