

- (1) A circulant matrix of size $(2n + 1) \times (2n + 1)$ has the form

$$C = \begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_{2n} \\ a_2 & a_0 & a_1 & \cdots & a_{2n-1} \\ a_{2n-1} & a_{2n} & a_0 & \cdots & a_{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{2n} & a_0 \end{pmatrix}.$$

Furthermore, let S denote the matrix that shifts the index of a vector by 1. In this case, S will be a $(2n + 1) \times (2n + 1)$ matrix of the form

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

- (a) Show that any circulant matrix can be written as a polynomial of the S matrix.

To create a polynomial that makes any circulant matrix, notice that each a_0 in C is along the diagonal which can be formed by $a_0 I = a_0 S^0$. Next, notice that each a_1 in C can be given by $a_1 S^1$. Continuing this trend, we can see that each a_{2n} in C can be given by $a_{2n} S^{2n}$. Finally, we can simply add up each term to get

$$C = a_0 + a_1 S + a_2 S^2 + \cdots + a_{2n} S^{2n}$$

which is a polynomial of S .

- (b) Let v^k denote the k th orthogonal Fourier basis vector where the j th entry of v^k is given by

$$v_j^k = e^{\frac{2\pi i j k}{2n+1}}.$$

Prove that the vectors v^k are all the eigenvectors of the circulant matrix. Furthermore, what are the eigenvalues?

To prove that the vectors v^k are the eigenvectors of the circulant matrix, let's check that

each of these “eigenvectors” is actually an eigenvector of C :

$$\begin{aligned}
 Cv^k &= (a_0 + a_1S + a_2S^2 + \cdots + a_{2n}S^{2n})v^k \\
 &= a_0v^k + a_1Sv^k + a_2S^2v^k + \cdots + a_{2n}S^{2n}v^k \\
 &= \begin{pmatrix} a_0v_0^k + a_1v_1^k + a_2v_2^k + \cdots + a_{2n}v_{2n}^k \\ a_0v_1^k + a_1v_2^k + a_2v_3^k + \cdots + a_{2n}v_0^k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_0v_{2n}^k + a_1v_0^k + a_2v_1^k + \cdots + a_{2n}v_{2n-1}^k \end{pmatrix} \\
 &= \begin{pmatrix} a_0e^{\frac{2\pi i 0k}{2n+1}} + a_1e^{\frac{2\pi i 1k}{2n+1}} + a_2e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i 2nk}{2n+1}} \\ a_0e^{\frac{2\pi i 1k}{2n+1}} + a_1e^{\frac{2\pi i 2k}{2n+1}} + a_2e^{\frac{2\pi i 3k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i 0k}{2n+1}} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_0e^{\frac{2\pi i 2nk}{2n+1}} + a_1e^{\frac{2\pi i 0k}{2n+1}} + a_2e^{\frac{2\pi i 1k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i (2n-1)k}{2n+1}} \end{pmatrix} \\
 &= \left(a_0 + a_1e^{\frac{2\pi i 1k}{2n+1}} + a_2e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i 2nk}{2n+1}} \right) \begin{pmatrix} e^{\frac{2\pi i 0k}{2n+1}} \\ e^{\frac{2\pi i 1k}{2n+1}} \\ \vdots \\ e^{\frac{2\pi i 2nk}{2n+1}} \end{pmatrix} \\
 &= \left(a_0 + a_1e^{\frac{2\pi i 1k}{2n+1}} + a_2e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i 2nk}{2n+1}} \right) v^k.
 \end{aligned}$$

Thus, v^k is an eigenvector of C with eigenvalue

$$\lambda_k = a_0 + a_1e^{\frac{2\pi i 1k}{2n+1}} + a_2e^{\frac{2\pi i 2k}{2n+1}} + \cdots + a_{2n}e^{\frac{2\pi i 2nk}{2n+1}}.$$

Finally, because the collection of v^k forms a basis of dimension $2n+1$ and C is a $(2n+1) \times (2n+1)$ matrix, the collection of v^k is all of the eigenvectors of C .

- (2) Let $0 \leq t_0 < t_1 < \dots < t_{2n} < w\pi$ and consider the trigonometric polynomial interpolation problem: define

$$l_j^{(n)}(t) = \prod_{\substack{k=0 \\ k \neq j}}^{2n} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)}$$

for $j = 0, 1, \dots, 2n$. It is easy to show that $l_j(t_i) = \delta_{ij}$ for each j .

Show that $l_j(t)$ is a trigonometric polynomial of degree less than or equal to n . Then the solution of the trigonometric interpolation problem is given by

$$p_n(t) = \sum_{j=0}^{2n} f(t_j) l_j(t).$$

To prove that the degree of our trig polynomial is less than or equal to n let's start with the base case when $n = 1$.

$$\begin{aligned} l_j^{(1)} &= \prod_{\substack{k=0 \\ k \neq j}}^2 \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)} \\ &= \frac{\sin\left(\frac{1}{2}(t - t_l)\right) \sin\left(\frac{1}{2}(t - t_s)\right)}{\prod_{\substack{k=0 \\ k \neq j}}^2 \sin\left(\frac{1}{2}(t_j - t_k)\right)} \\ &= \frac{\cos\left(\frac{1}{2}(t_s - t_l)\right) - \cos\left(t - \frac{1}{2}(t_s + t_l)\right)}{2 \prod_{\substack{k=0 \\ k \neq j}}^2 \sin\left(\frac{1}{2}(t_j - t_k)\right)} \\ &= \frac{\cos\left(\frac{1}{2}(t_s - t_l)\right) - \cos(t) \cos\left(\frac{1}{2}(t_s + t_l)\right) - \sin(t) \sin\left(\frac{1}{2}(t_s + t_l)\right)}{2 \prod_{\substack{k=0 \\ k \neq j}}^2 \sin\left(\frac{1}{2}(t_j - t_k)\right)} \\ &= \sum_{k=0}^1 (a_k \cos(kt) + b_k \sin(kt)) \end{aligned}$$

for some constants a_n and b_n . Therefore, $l_j^{(1)}$ is a trig polynomial of degree 1 or less.

Now, suppose $l_j^{(i-1)}$ is a trig polynomial of degree $i - 1$ or less. Then

$$\begin{aligned}
 l_j^{(i)} &= \prod_{\substack{k=0 \\ k \neq j}}^{2i} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)} \\
 &= \left(\prod_{\substack{k=0 \\ k \neq j}}^{2i-2} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)} \right) \left(\prod_{\substack{k=2i-1 \\ k \neq j}}^{2i} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)} \right) \\
 &= l_j^{(i-1)}(t) \left(\prod_{\substack{k=2i-1 \\ k \neq j}}^{2i} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)} \right) \\
 &= l_j^{(i-1)}(t) \frac{\sin\left(\frac{1}{2}(t - t_l)\right) \sin\left(\frac{1}{2}(t - t_s)\right)}{\prod_{\substack{k=2i-1 \\ k \neq j}}^{2i} \sin\left(\frac{1}{2}(t_j - t_k)\right)} \\
 &= l_j^{(i-1)}(t) \frac{\cos\left(\frac{1}{2}(t_s - t_l)\right) - \cos(t) \cos\left(\frac{1}{2}(t_s + t_l)\right) - \sin(t) \sin\left(\frac{1}{2}(t_s + t_l)\right)}{\prod_{\substack{k=2i-1 \\ k \neq j}}^{2i} \sin\left(\frac{1}{2}(t_j - t_k)\right)} \\
 &= l_j^{(i-1)}(t) (\alpha + \beta \cos(t) + \gamma \sin(t))
 \end{aligned}$$

for some constants α, β and γ . Then,

$$\begin{aligned}
 l_j^{(i)} &= l_j^{(i-1)}(t) (\alpha + \beta \cos(t) + \gamma \sin(t)) \\
 &= \sum_{k=0}^{i-1} (a_k \cos(kt) + b_k \sin(kt) (\alpha + \beta \cos(t) + \gamma \sin(t))) \\
 &= \sum_{k=0}^{i-1} A_k \cos((k-1)t) + B_k \cos((k+1)t) + C_k \sin((k-1)t) + D_k \sin((k+1)t) \\
 &= \sum_{k=0}^i A'_k \cos(kt) + B'_k \sin(kt)
 \end{aligned}$$

for some constants A'_k and B'_k . Therefore, $l_j^{(i)}$ has degree less than or equal to i . So, by induction l_j^n has degree less than or equal to n .