1.

2. Prove the following result:Let $f \in C^2[a,b]$ with $f''(x) > 0$ for $a \le x \le b$. If $q_i^*(x) = a_0 + a_1 x$ is the linear minimax approximation to $f(x)$ on $[a,b]$, then

$$a_1 = \frac{f(b) - f(a)}{b-a}, \quad a_0 = \frac{f(a) + f(c)}{2} - \frac{a+c}{2} \cdot \frac{f(b) - f(a)}{b-a}$$

where c is the unique solution of

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof:

From Cauchy's Equioscillation Theorem, we know there exists a unique polynomial of the form $p_1^*(x) = a_0 + a_1 x$ such that the error $E(x) = f(x) - p_1^*(x)$ satisfies

$$E(a) = \rho \tag{1}$$
$$E(c) = -\rho \tag{2}$$
$$E(b) = \rho \tag{3}$$
$$E'(c) = 0 \tag{4}$$

where ρ is the maximum error on $[a,b]$ and $c \in (a,b)$. Now using (4), we have

$$E'(c) = f'(c) - a_1 = 0 \implies a_1 = f'(c).$$

Then, ((1) - (3)) implies

$$f(a) - f(b) + a_1(b-a) = 0 \implies \boxed{a_1 = \frac{f(b) - f(a)}{b-a}}$$

which implies c is the solution to

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}.$$

Next, ((1) + (2)) implies

$$f(a) + f(c) - 2a_0 - a_1(a+c) = 0 \implies \boxed{a_0 = \frac{f(a) + f(c)}{2} - \frac{a+c}{2} \cdot \frac{f(b) - f(a)}{b-a}}.$$

Finally, the max error can be obtained from (1) as

$$\boxed{\rho = f(a) - \frac{f(a) + f(c)}{2} + \frac{f(b) - f(a)}{b-a}\left(\frac{a+c}{2} - a\right)}.$$

□