1. Let x_0, x_1 be two successive points from a secant method applied to solving $f(x) = 0$ with $f_0 = f(x_0), f_1 = f(x_1)$. Show that regardless of which point x_0 or x_1 is regarded as the most recent point, the new point derived from the secant step will be the same.

   Suppose x_1 is the most recent point. Then the next point x_2 produced by the secant method would be given by

$$\begin{aligned}
x_2 &= x_1 - f_1 \frac{x_1 - x_0}{f_1 - f_0} \\
&= \frac{x_1(f_1 - f_0) - f_1(x_1 - x_0)}{f_1 - f_0} \\
&= \frac{x_1 f_1 - x_1 f_0 - x_1 f_1 + x_0 f_1}{f_1 - f_0} \\
&= \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} \\
&= \frac{x_0 f_1 - x_1 f_0 + x_0 f_0 - x_0 f_0}{f_1 - f_0} \\
&= \frac{x_0(f_1 - f_0) + f_0(x_0 - x_1)}{f_1 - f_0} \\
&= x_0 - f_0 \frac{x_0 - x_1}{f_0 - f_1}
\end{aligned}$$

   which is the secant iteration if x_0 was the most recent point. Thus, regardless of which point is the most recent, the secant iteration will produce the same point x_2.

2. Determine whether the following sets of vectors are dependent or linearly independent:

   (a) $(1, 2, -1, 3), (3, -1, 1, 1), (1, 9, -5, 11)$.
   We can determine if this set of vectors in linearly independent by forming the matrix and row reducing as follows

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 9 \\ -1 & 1 & -5 \\ 3 & 1 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   This row reduction shows that our third vector can be written as a linear combination of the first two vectors and so our set is *linearly dependent.*

   (b) $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.
   Just as in part (a), we can test linear dependence by forming a matrix and row reducing as follows

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   The row reduction shows that each vector can not be written as a linear combination of the other two vectors and so our set is *linearly independent.*

3. Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ be linearly independent vectors in $\mathbb{R}^n$ and let A be a non-singular $n \times n$ matrix. Define $\vec{y}_i = A\vec{x}_i$ for $i = 1, 2, \ldots, k$. Show that $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k$ are linearly independent.
Proof:

Let A, x_i, and y_i be defined as above for $i = 1, 2, \ldots, k$. Then, because $\{x_i\}_{i=1}^k$ forms a linearly independent set, $x_i \neq 0$ for $i = 1, 2, \ldots, k$. Furthermore, because A is non-singular and $x_i \neq 0$,

$$y_i = Ax_i \neq 0$$

for each $i = 1, 2, \cdots, k$. With this information in mind, let's find constants a_i such that $a_1y_1 + a_2y_2 + \cdots + a_ky_k = 0$. Consider

$$\begin{aligned} a_1y_1 + a_2y_2 + \cdots + a_ky_k &= a_1Ax_1 + a_2Ax_2 + \cdots + a_kAx_k \\ &= A(a_1x_1 + a_2x_2 + \cdots + a_kx_k). \end{aligned} \tag{$*$}$$

Then, because A is non-singular, setting ($*$) equal to zero yields

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0. \tag{1}$$

Finally, because $\{x_i\}_{i=1}^k$ is linearly independent, the only solution to (1) is $a_i = 0$ for each $i = 1, 2, \ldots, k$. Thus, the only solution to $a_1y_1 + a_2y_2 + \cdots + a_ky_k = 0$ is when $a_i = 0$ for each $i = 1, 2, \ldots, k$. Therefore, $\{y_i\}_{i=1}^k$ forms a linearly independent set. □

4. Given the orthogonal vectors

$$\vec{u}_1 = (1, 2, -1) \quad \vec{u}_2 = (1, 1, 3)$$

produce a third vector $\vec{u}_3$ such that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Normalize the vectors to create an orthonormal basis.

For any number of dimensions, we could use the Gram-Schmidt process to generate orthogonal vectors to our given vectors but we are in $\mathbb{R}^3$ and so we can simply use the cross product to get a third orthogonal vector $\vec{u}_3$. We can produce $\vec{u}_3$ as

$$\vec{u}_3 = \vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 1 & 3 \end{vmatrix} = (6+1)\hat{i} - (3+1)\hat{j} + (1-2)\hat{k} = (7, -4, -1)$$

Thus, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{(1, 2, -1), (1, 1, 3), (7, -4, -1)\}$ is an orthogonal basis for $\mathbb{R}^3$. Going further, we can normalize each vector to get the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{66}} \begin{pmatrix} 7 \\ -4 \\ -1 \end{pmatrix} \right\}.$$

5. Prove that similar matrices have the same eigenvalues and that there is a one-to-one correspondence of the eigenvectors.
Proof:

Suppose we have similar square $n \times n$ matrices A and B. Then, by the definition of similar matrices, there exists an invertible $n \times n$ matrix P such that

$$A = P^{-1}BP. \tag{2}$$

Now, suppose A has an eigenvalue λ with corresponding eigenvector $\vec{\lambda}$. Then, $A\vec{\lambda} = \lambda\vec{\lambda}$. Furthermore,

$$P^{-1}BP\vec{\lambda} = A\vec{\lambda} = \lambda\vec{\lambda}.$$

Rearranging yields

$$B(P\vec{\lambda}) = \lambda(P\vec{\lambda}) \tag{3}$$

which implies $P\vec{\lambda}$ is an eigenvector of B with corresponding eigenvalue λ. Thus, A and B both have the same eigenvalue and because we picked any eigenvalue of A and A and B are the same size, A and B must have the same eigenvalues. Furthermore, from (3), we can form a one-to-one correspondence between the eigenvectors $\vec{\lambda}_A$ of A to the eigenvectors $\vec{\lambda}_B$ of B as

$$\vec{\lambda}_B = P\vec{\lambda}_A.$$

□

6. A matrix $A \in \mathbb{R}^{n\times n}$ is *positive definite* if and only if $\langle A\vec{x}, \vec{x}\rangle > 0$ for all $\vec{x} \in \mathbb{R}^n; x \neq 0$,

   Prove that if A is positive definite, then A is non-singular.
   Proof: Proof by contradiction:

   Suppose $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Then, $\langle A\vec{x}, \vec{x}\rangle > 0$ for all non-zero $x \in \mathbb{R}^n$.

   Now, suppose there exists some $\vec{x} \neq 0$ such that $A\vec{x} = 0$. Then

   $$\langle A\vec{x}, \vec{x}\rangle = \langle 0, \vec{x}\rangle = 0.$$

   But, because A is positive definite, we know if $\vec{x} \neq 0$, then $\langle A\vec{x}, \vec{x}\rangle > 0$ which is a contradiction. Therefore, $A\vec{x} = 0$ only if $\vec{x} = 0$ which shows that A is non-singular. □

7. Let M be any real $n \times n$ non-singular matrix and let $A = M^T M$. Prove that A is positive definite.
   Proof:

   Let M and A be defined as above and let any non-zero $\vec{x} \in \mathbb{R}^n$ be given. Then

   $$\begin{aligned} \vec{x}^T A\vec{x} &= \vec{x}^T M^T M\vec{x} \\ &= (M\vec{x})^T(M\vec{x}) \\ &= (M\vec{x}) \cdot (M\vec{x}). \end{aligned}$$

   Now, we have the dot product between two identical vectors. Furthermore, because M is non-singular and $\vec{x} \neq 0$, we must have $M\vec{x} \neq 0$. Finally, because the dot product is an inner product, the dot product between two identical non-zero vectors must be strictly positive. So, putting everything together, we must have

   $$\vec{x}^T A\vec{x} = (M\vec{x}) \cdot (M\vec{x}) > 0. \tag{4}$$

   Then, because the choice of non-zero $\vec{x}$ was arbitrary, (4) must hold for all non-zero $x \in \mathbb{R}^n$ which by definition means A is positive definite. □