Problems

1.

(i) Given $x_0 = -0.2, x_1 = 0$, and $x_2 = 0.2$ construct a second degree polynomial to approximate $f(x) = e^x$ via Newton's divided differences.

We want to derive a polynomial of the form

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

where $a_i = [x_0, \ldots, x_{i-1}]$ are the Newton Divided differences. For this problem, we have

$$\begin{aligned} a_0 &= f[x_0] = e^{x_0} = e^{-0.2}, \\ a_1 &= f[x_0, x_1] = \frac{e^{x_1} - e^{x_0}}{x_1 - x_0} = \frac{1 - e^{-0.2}}{0.2}, \end{aligned}$$

and

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{e^{0.2}-1}{0.2} - \frac{1-e^{-0.2}}{0.2}}{0.4}$$

which makes our polynomial

$$\begin{aligned} p(x) &= e^{-0.2} + \frac{1 - e^{-0.2}}{0.2}(x + 0.2) + \frac{\frac{e^{0.2}-1}{0.2} - \frac{1-e^{-0.2}}{0.2}}{0.4}(x + 0.2)(x) \\ &= 1 + 1.00668x + 0.501669x^2. \end{aligned}$$

(ii) Derive an error bound for $p_2(x)$ when $x \in [-0.2, 0.2]$.

First, note that the third derivative of f is maximized over $[-0.2, 0.2]$ when $x = 0.2$. Then, we can obtain a bound on our error as

$$\begin{aligned} E(t) &\leq \max_{t\in[-0.2,0.2]} E(t) \\ &= \max_{t\in[-0.2,0.2]} \frac{(t + 0.2)(t)(t - 0.2)}{6} e^{0.2} \\ &= \frac{(-\frac{\sqrt{3}}{15} + 0.2)(-\frac{\sqrt{3}}{15})(-\frac{\sqrt{3}}{15} - 0.2)}{6} e^{0.2} \\ &= 6.26824 \cdot 10^{-4}. \end{aligned}$$

(iii) Compute the error $E(0.1) = f(0.1) - p_2(0.1)$. How does this compare with the error bound?

Our error is

$$E(0.1) = |1.10517 - 1.10568| = 5.136621 \cdot 10^{-4}$$

which is within our error bound! So our error bound holds $x = 0.1$.

2.

(i) Show there is a unique cubic polynomial $p(x)$ for which

$$p(x_0) = f(x_0) \qquad p(x_2) = f(x_2)$$
$$p'(x_1) = f'(x_1) \qquad p''(x_1) = f''(x_1)$$

where $f(x)$ is a given function and $x_0 \neq x_2$.
Suppose $p(x)$ and $q(x)$ are two cubic polynomials satisfying

$$p(x_0) = q(x_0) = f(x_0) \qquad p(x_2) = q(x_2) = f(x_2)$$
$$p'(x_1) = q(x_1) = f'(x_1) \qquad p''(x_1) = q''(x_1) = f''(x_1).$$

Now let $v(x) + p(x) - q(x)$. Then, by linearity, $v(x)$ is a cubic polynomial that satisfies

$$v(x_0) = 0 \qquad v(x_2) = 0$$
$$v'(x_1) = 0 \qquad v''(x_1) = 0.$$

(ii) Derive a formula for $p(x)$.

(iii) Let $x_0 = -1, x_1 = 0$, and $x_2 = 1$. Assuming $f(x) \in C^4[-1, 1]$, show that for $x \in [-1, 1]$,

$$f(x) - p(x) = \frac{x^4 - 1}{4!} f^4(\eta_x)$$

for some $\eta_x \in [-1, 1]$.

3. Suppose we have m data points $\{(t_i, y_i)\}_{i=1}^m$, where the t-values all occur in some interval $[x_0, x_n]$. We subdivide the interval $[x_0, x_n]$ into n subintervals $\{[x_k, x_{k+1}]_{k=0}^{n-1}\}$ of equal length h and attempt to choose a spline function $s(x)$ with nodes at $\{x_k\}_{k=0}^n$ in such a way so that

$$\sum_{i=1}^m |y_i - s(t_i)|^2$$

is *minimized.*

(i) *Sheeeeeeeeeeeeesh*

Code Used