(1) Prove that

(a) If all singular values of a matrix $A \in \mathbb{C}^{n\times n}$ are equal, then $A = \gamma U$,where U is unitary and γ is a constant.
Proof:

Suppose A has singular values all equal to $\gamma \geq 0$. Then A has the SVD

$$A = W\Sigma V^*$$

where W and V are unitary and Σ is a diagonal matrix of γ. Then

$$A = W\Sigma V^* = W\gamma I V^* = \gamma W V^* = \gamma U$$

where $U = WV^*$ is unitary because it is the product of two unitary matrices. □

(b) If $A \in \mathbb{C}^{n\times b}$ is non-singular and λ is an eigenvalue of A, then $\|A^{-1}\|_2^{-1} \leq |\lambda| \leq \|A\|_2$.
Proof:

Suppose $A \in \mathbb{C}n \times n$ is non-singular with an eigenvalue λ. Then, by the properties of induced matrix-norms, we have

$$|\lambda| \leq \rho(A) \leq \|A\|_2$$

where $\rho(A)$ denotes the spectral radius of A. Now, because A is non-singular, A^{-1} exists and

$$\rho(A^{-1}) = \frac{1}{\min\limits_{i=1,\ldots,n} |\lambda_i|}$$

where λ_i denotes the ith eigenvalue of A. Then

$$\frac{1}{\|A^{-1}\|_2} \leq \frac{1}{\rho(A^{-1})} = \frac{1}{\frac{1}{\min\limits_{i=1,\ldots,n} |\lambda_i|}} = \min_{i=1,\ldots,n} |\lambda_i| \leq |\lambda|\,.$$

Putting everything together yields

$$\|A^{-1}\|_2^{-1} \leq |\lambda| \leq \|A\|_2.$$

□

(2) Show that any square matrix $A \in \mathbb{C}^{n\times n}$ may be represented in the form $A = SU$, where S is a Hermitian non-negative definite matrix and U is a unitary matrix. Show that if A is invertible such representation is unique.
Proof:

Suppose we have a matrix $A \in \mathbb{C}^{n\times n}$. Then, A has the SVD

$$A = W\Sigma V^*$$

where W and V are unitary and Σ is a matrix of the singular values. Then

$$A = W\Sigma V^* = W\Sigma W^* W V^* = SU$$

where $S = W\Sigma W^*$ and $U = WV^*$. Note, because Σ is a diagonal matrix of non-negative entries and W is unitary, S must be positive semi-definite and Hermitian. Furthermore,

U is unitary because it is the product of two unitary matrices. So, we have the desired decomposition of A.

Now, suppose A is non-singular. Then

$$A = \underbrace{(A^*A)^{\frac{1}{2}}}_{S}\underbrace{(A^*A)^{-\frac{1}{2}}A}_{U}$$

Then, because A^*A is non-singular and Hermitian positive definite, $S = (A^*A)^{1/2}$ is Hermitian positive-semidefinite and unique. $\square$