1. Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   i. $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$

   ii. $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{1/3}$

   iii. $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$

2. In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

   where T_s is the constant temperature during a cold period T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2) and

$$\operatorname{erf}(t) = \frac{2}{\sqrt{2}} \int_0^t e^{-s^2} \mathrm{d}s$$

   Assume that $T_i = 20[\deg C]$, $T_s = -15[\deg C]$, $\alpha = 0.138 \cdot 10^{-6}$[meters^2 per second].

3.

4. The sequence x_k produced by Newton's method is quadratically convergent to x_* with $f(x_*) = 0$, $f'(x) \neq 0$ and $f''(x)$ continuous at x_*.

   Let $f(x) = (x - x_*)^p q(x)$ with p a positive integer with q twice continously differentiable and $q(x_*) \neq 0$. Note: $f'(x_*) = 0$. In the following sub-problems, let $x_k, f_k = f(x_k), e_k = |x_* - x_k|$, etc.

   i. Prove that Newton's method converges linearly for $f(x)$.

   ii. Consider the modified Newton iteration defined by

$$x_{k+1} = x_k - p\frac{f_k}{f'_k}.$$

   Prove that if x_k converges to x_*, then the rate of convergence is quadratic.

   iii. Write MATLAB codes for both Newton and modified Newton methods. Apply these to the function

$$f(x) = (x-1)^5 e^x$$

   and compare the results. Use $x_0 = 0$ as a starting point.