

1. Derive a quadrature based on the cubic Hermite interpolating polynomial with data $f(a)$, $f(b)$, $f'(a)$, and $f'(b)$. Derive an upper bound on the error.

Using the Hermite-Lagrange basis, we can construct our cubic Hermite polynomial as

$$p(x) = f(a)H_a(x) + f(b)H_b(x) + f'(a)K_a(x) + f'(b)K_b(x)$$

where

$$\begin{aligned} H_a(x) &= \left(1 - 2(x-a)\frac{1}{a-b}\right) \frac{(x-b)^2}{(a-b)^2} \\ H_b(x) &= \left(1 - 2(x-b)\frac{1}{b-a}\right) \frac{(x-a)^2}{(b-a)^2} \\ K_a(x) &= (x-a)\frac{(x-b)^2}{(a-b)^2} \\ K_b(x) &= (x-b)\frac{(x-a)^2}{(b-a)^2}. \end{aligned}$$

Now, integrating $p(x)$ over our interval, $[a, b]$, we obtain our quadrature as

$$\begin{aligned} \int_a^b f(x) \, dx &\approx \int_a^b p(x) \, dx \\ &= f(a) \int_a^b H_a(x) \, dx + f(b) \int_a^b H_b(x) \, dx + f'(a) \int_a^b K_a(x) \, dx + f'(b) \int_a^b K_b(x) \, dx \\ &= f(a) \frac{b-a}{2} + f(b) \frac{b-a}{2} + f'(a) \frac{(a-b)^2}{12} - f'(b) \frac{(a-b)^2}{12} \\ &= \boxed{(f(a) + f(b)) \frac{b-a}{2} + (f'(a) - f'(b)) \frac{(a-b)^2}{12}}. \end{aligned}$$

Now, assuming $f \in C^4[a, b]$, we can get an error bound for this quadrature by integrating the Hermite interpolant error as

$$\begin{aligned} E &= \int_a^b |f(x) - p(x)| \, dx = \int_a^b \left| \frac{f^{(4)}(\eta_x)}{4!} (x-a)^2 (x-b)^2 \right| \, dx && \text{for some } \eta_x \in [a, b] \\ &\leq \frac{M}{24} \int_a^b (x-a)^2 (x-b)^2 \, dx && \text{where } M = \max_{\eta \in [a, b]} |f^{(4)}(\eta)| \\ &= \frac{M}{24} \left(\frac{(b-a)^5}{30} \right) \\ &= \boxed{\frac{M(b-a)^5}{720}}. \end{aligned}$$

2. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots.$$

Generalize the Richardson extrapolation process to obtain an estimate of I with an error on the order $\frac{1}{n^2\sqrt{n}}$. Assume that three values $I_n, I_{n/2}$, and $I_{n/4}$ have been computed.

From the error formula, we have the three equations

$$I = I_n + \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots \quad (1)$$

$$I = I_{n/2} + 2\sqrt{2}\frac{C_1}{n\sqrt{n}} + 4\frac{C_2}{n^2} + 4\sqrt{2}\frac{C_3}{n^2\sqrt{n}} + \cdots \quad (2)$$

$$I = I_{n/4} + 8\frac{C_1}{n\sqrt{n}} + 16\frac{C_2}{n^2} + 32\frac{C_3}{n^2\sqrt{n}} + \cdots \quad (3)$$

Using these three equations, we want to eliminate the C_1 and C_2 error terms which we can do by reducing

$$\begin{pmatrix} 1 & 2\sqrt{2} & 8 \\ 1 & 4 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8\sqrt{2} \\ 0 & 1 & 2(\sqrt{2} + 2) \end{pmatrix}$$

which tells us that

$$8\sqrt{2}(1) - 2(\sqrt{2} + 2)(2) + (3)$$

will eliminate our desired error terms. So, we have the equation

$$(8\sqrt{2} - 2(\sqrt{2} + 2) + 1)I = 8\sqrt{2}I_n - 2(\sqrt{2} + 2)I_{n/2} + I_{n/4} + (16 - 8\sqrt{2})\frac{C_3}{n^2\sqrt{n}}$$

which implies

$$I = \frac{8\sqrt{2}I_n - 2(\sqrt{2} + 2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2} + 2) + 1} + O\left(\frac{1}{n^2\sqrt{n}}\right).$$

So if we use the integration formula I' defined as

$$I' = \frac{8\sqrt{2}I_n - 2(\sqrt{2} + 2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2} + 2) + 1}$$

we get our desired error

$$I - I' = O\left(\frac{1}{n^2\sqrt{n}}\right).$$

3. Let $n \geq 0$.

- (i) Give a formula for the Gauss quadrature points $x_j, j = 0, \dots, n$, needed for the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$.

First, note that the Chebyshev polynomials are orthogonal under the given weight function. So, to find our nodes, x_j , we need to find the roots of the $n+1$ Chebyshev polynomial which is given by

$$T_{n+1}(x) = \cos((n+1) \arccos(x)).$$

To find the roots of T_{n+1} , we need

$$(n+1) \arccos(x) = \frac{\pi}{2} + j\pi$$

for any integer j . So, we must have

$$x_j = \cos\left(\frac{(j + \frac{1}{2})\pi}{n+1}\right).$$

So we don't have overlapping x_j , restrict j to $j = 0, \dots, n$.