1. Derive a quadrature based on the cubic Hermite interpolating polynomial with data $f(a)$, $f(b)$, $f'(a)$, and $f'(b)$. Derive an upper bound on the error.

   Using the Hermite-Lagrange basis, we can construct our cubic Hermite polynomial as

$$p(x) = f(a)H_a(x) + f(b)H_b(x) + f'(a)K_a(x) + f'(b)K_b(x)$$

   where

$$\begin{aligned}
H_a(x) &= \left(1 - 2(x-a)\frac{1}{a-b}\right)\frac{(x-b)^2}{(a-b)^2} \\
H_b(x) &= \left(1 - 2(x-b)\frac{1}{b-a}\right)\frac{(x-a)^2}{(b-a)^2} \\
K_a(x) &= (x-a)\frac{(x-b)^2}{(a-b)^2} \\
K_b(x) &= (x-b)\frac{(x-a)^2}{(b-a)^2}.
\end{aligned}$$

   Now, integrating $p(x)$ over our interval, $[a, b]$, we obtain our quadrature as

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \int_a^b p(x)\,\mathrm{d}x \\
&= f(a)\int_a^b H_a(x)\,\mathrm{d}x + f(b)\int_a^b H_b(x)\,\mathrm{d}x + f'(a)\int_a^b K_a(x)\,\mathrm{d}x + f'(b)\int_a^b K_b(x)\,\mathrm{d}x \\
&= f(a)\frac{b-a}{2} + f(b)\frac{b-a}{2} + f'(a)\frac{(a-b)^2}{12} - f'(b)\frac{(a-b)^2}{12} \\
&= \boxed{(f(a)+f(b))\frac{b-a}{2} + (f'(a)-f'(b))\frac{(a-b)^2}{12}.}
\end{aligned}$$

   Now, assuming $f \in C^4[a, b]$, we can get an error bound for this quadrature by integrating the Hermite interpolant error as

$$\begin{aligned}
E = \int_a^b |f(x) - p(x)|\,\mathrm{d}x &= \int_a^b \left|\frac{f^{(4)}(\eta_x)}{4!}(x-a)^2(x-b)^2\right|\,\mathrm{d}x && \text{for some } \eta_x \in [a,b] \\
&\leq \frac{M}{24}\int_a^b (x-a)^2(x-b)^2\,\mathrm{d}x && \text{where } M = \max_{\eta\in[a,b]}\left|f^{(4)}(\eta)\right| \\
&= \frac{M}{24}\left(\frac{(b-a)^5}{30}\right) \\
&= \boxed{\frac{M(b-a)^5}{720}.}
\end{aligned}$$

2. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots .$$

Generalize the Richardson extrapolation process to obtain an estimate of I with an error on the order $\frac{1}{n^2\sqrt{n}}$. Assume that three values $I_n, I_{n/2}$, and $I_{n/4}$ have been computed.

From the error formula, we have the three equations

$$I = I_n + \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \cdots \tag{1}$$

$$I = I_{n/2} + 2\sqrt{2}\frac{C_1}{n\sqrt{n}} + 4\frac{C_2}{n^2} + 4\sqrt{2}\frac{C_3}{n^2\sqrt{n}} + \cdots \tag{2}$$

$$I = I_{n/4} + 8\frac{C_1}{n\sqrt{n}} + 16\frac{C_2}{n^2} + 32\frac{C_3}{n^2\sqrt{n}} + \cdots . \tag{3}$$

Using these three equations, we want to eliminate the C_1 and C_2 error terms which we can do by reducing

$$\begin{pmatrix} 1 & 2\sqrt{2} & 8 \\ 1 & 4 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8\sqrt{2} \\ 0 & 1 & 2(\sqrt{2}+2) \end{pmatrix}$$

which tells us that

$$8\sqrt{2}(1) - 2(\sqrt{2}+2)(2) + (3)$$

will eliminate our desired error terms. So, we have the equation

$$(8\sqrt{2} - 2(\sqrt{2}+2) + 1)I = 8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4} + (16 - 8\sqrt{2})\frac{C_3}{n^2\sqrt{n}}$$

which implies

$$I = \frac{8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2}+2) + 1} + O\left(\frac{1}{n^2\sqrt{n}}\right).$$

So if we use the integration formula I' defined as

$$\boxed{I' = \frac{8\sqrt{2}I_n - 2(\sqrt{2}+2)I_{n/2} + I_{n/4}}{8\sqrt{2} - 2(\sqrt{2}+2) + 1}}$$

we get our desired error

$$\boxed{I - I' = O\left(\frac{1}{n^2\sqrt{n}}\right)}.$$

3. Let $n \geq 0$.

   (i) Give a formula for the Gauss quadrature points $x_j, j = 0, \dots, n$, needed for the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$.
   First, note that the Chebysheb polynomials are orthogonal under the given weight function. So, to find our nodes, x_j, we need to find the roots of the $n+1$ Chebyshev polynomial which is given by
   $$T_{n+1}(x) = \cos((n+1)\arccos(x)).$$
   To find the roots of T_{n+1}, we need
   $$(n+1)\arccos(x) = \frac{\pi}{2} + j\pi$$
   for any integer j. So, we must have
   $$\boxed{x_j = \cos\left(\frac{\left(j + \frac{1}{2}\right)\pi}{n+1}\right).}$$
   So we don't have overlapping x_j, restrict j to $j = 0, \dots, n$.

   (ii) Show that for positive integers n,
   $$\sum_{j=0}^{n} \cos((2j+1)\theta) = \frac{\sin((2n+2)\theta)}{2\sin(\theta)},$$
   unless θ is a multiple of π. What is the value of the sum when θ is a multiple of π?
   To begin showing our sum, note that from a product-to-sum identity, we have
   $$2\sin(\theta)\cos((2j+1)\theta) = \sin((2j+2)\theta) - \sin(2j\theta).$$
   Using this identity, we have the telescoping sum
   $$\begin{aligned}\sum_{j=0}^{n} 2\sin(\theta)\cos((2j+1)\theta) = &+ \sin(2\theta) - 0\\ &+ \sin(4\theta) - \sin(2\theta)\\ &+ \sin(6\theta) - \sin(4\theta)\\ &+ \cdots\\ &+ \sin(2n\theta) - \sin((2n-2)\theta)\\ &+ \sin((2n+2)\theta) - \sin(2n\theta)\\ = &\sin((2n+2)\theta)\end{aligned}$$
   which implies
   $$\sum_{j=0}^{n} 2\sin(\theta)\cos((2j+1)\theta) = \sin((2n+2)\theta)$$
   or solving for our desired sum,
   $$\sum_{j=0}^{n} \cos((2j+1)\theta) = \frac{\sin((2n+2)\theta)}{2\sin(\theta)}$$

   (iii)