Problem 5: Fourier Series

In[ ]:=
```
f[k_, t_] = 1/Sqrt[2 π] ⅇ^(ⅈ k t);
f[t_] = 2 t + 1;
```

$$f[k\_, t\_] = \frac{1}{\sqrt{2\pi}} e^{i k t};$$

$$f[t\_] = 2t + 1;$$

(a) Verify that each f_k has norm 1.

In[ ]:=
```
Table[{StringTemplate["k = ``"][k],
    Sqrt[Integrate[f[k, t] (f[k, t]*), {t, 0, 2 π}]]}, {k, -5, 5}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} k\ =\ -5 & 1 \\ k\ =\ -4 & 1 \\ k\ =\ -3 & 1 \\ k\ =\ -2 & 1 \\ k\ =\ -1 & 1 \\ k\ =\ 0 & 1 \\ k\ =\ 1 & 1 \\ k\ =\ 2 & 1 \\ k\ =\ 3 & 1 \\ k\ =\ 4 & 1 \\ k\ =\ 5 & 1 \end{pmatrix}$$

(b) Verify that f_k and f_j are orthogonal if k ≠ j

In[ ]:=
```
Assuming[k ≠ j && k ∈ ℤ && j ∈ ℤ,
  Integrate[f[k, t] (f[j, t]*), {t, 0, 2 π}]]
```

Out[ ]= 0

(c) Compute $\alpha_k = \langle f, f_k \rangle$ for k = 0, ±1, ±2, ±3, ±4, ±5

In[ ]:=
```
α[k_] := ∫_0^(2 π) f[t] (f[k, t]*) ⅆ t;
Table[{StringTemplate["α`` = "][k], α[k]},
  {k, -5, 5}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \alpha_{-5} & = & -\frac{2}{5} i \sqrt{2\pi} \\ \alpha_{-4} & = & -i \sqrt{\frac{\pi}{2}} \\ \alpha_{-3} & = & -\frac{2}{3} i \sqrt{2\pi} \\ \alpha_{-2} & = & -i \sqrt{2\pi} \\ \alpha_{-1} & = & -2 i \sqrt{2\pi} \\ \alpha_{0} & = & 2\sqrt{2}\,\pi^{3/2} + \sqrt{2\pi} \\ \alpha_{1} & = & 2 i \sqrt{2\pi} \\ \alpha_{2} & = & i \sqrt{2\pi} \\ \alpha_{3} & = & \frac{2}{3} i \sqrt{2\pi} \\ \alpha_{4} & = & i \sqrt{\frac{\pi}{2}} \\ \alpha_{5} & = & \frac{2}{5} i \sqrt{2\pi} \end{pmatrix}$$

(d) Compute the partial sums:

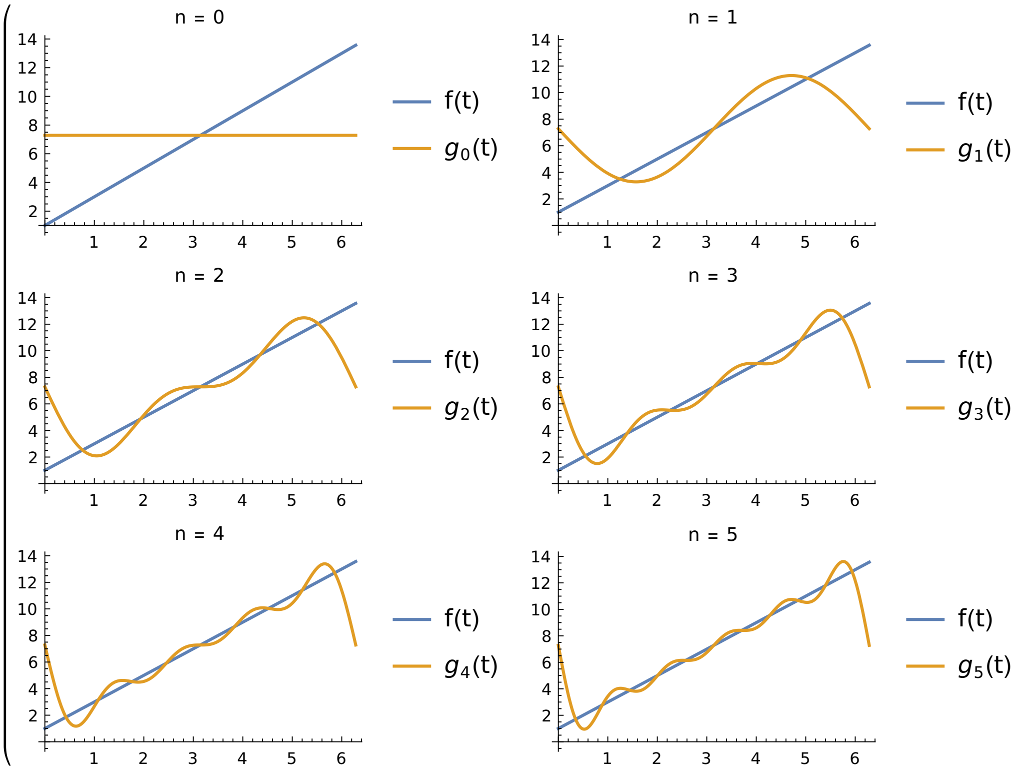
In[ ]:=
```
g[n_, t_] := Σ_(k=-n)^n α[k] × f[k, t];
partialSums = Table[{StringTemplate["g``(t) = "][n], g[n, t]},
    {n, 0, 5}] // FullSimplify // Expand;
partialSums // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} g_0(t) & = & 1 + 2\pi \\ g_1(t) & = & 1 + 2\pi - 4\,\mathrm{Sin}[t] \\ g_2(t) & = & 1 + 2\pi - 4\,\mathrm{Sin}[t] - 2\,\mathrm{Sin}[2t] \\ g_3(t) & = & 1 + 2\pi - 4\,\mathrm{Sin}[t] - 2\,\mathrm{Sin}[2t] - \frac{4}{3}\mathrm{Sin}[3t] \\ g_4(t) & = & 1 + 2\pi - 4\,\mathrm{Sin}[t] - 2\,\mathrm{Sin}[2t] - \frac{4}{3}\mathrm{Sin}[3t] - \mathrm{Sin}[4t] \\ g_5(t) & = & 1 + 2\pi - 4\,\mathrm{Sin}[t] - 2\,\mathrm{Sin}[2t] - \frac{4}{3}\mathrm{Sin}[3t] - \mathrm{Sin}[4t] - \frac{4}{5}\mathrm{Sin}[5t] \end{pmatrix}$$

In[ ]:=
```
ArrayReshape[Table[
    Plot[{f[t], partialSums[[i, 2]]}, {t, 0, 2 π},
     PlotLabel → StringTemplate["n = ``"][i - 1],
     PlotLegends → {"f(t)", StringTemplate["g``(t)"][i - 1]}],
   {i, 1, 6}], {3, 2}] // MatrixForm
```

Out[ ]//MatrixForm=



(e) What is $\| f \|_2^2$ and $\sum_{k=-5}^{5} | \alpha_k |^2$?

In[ ]:=
$$\int_0^{2\pi} f[t]\,(f[t]^*)\,dt \;//\; N$$
$$\sum_{k=-5}^{5} Abs[\alpha[k]]^2 \;//\; N$$

Out[ ]= 415.974

Out[ ]= 406.859

(f) For each value of n = 0,1,2,3,4,5, compute L_2-error.

```
Table[{StringTemplate["||f - g`` ||₂ = "][n],
   Sqrt[Integrate[(f[t] - partialSums[[n + 1, 2]]) ((f[t] - partialSums[[n + 1, 2]])*), {t, 0, 2 π}]]}, {n,
   0, 5}] // N // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \|f - g_0\|_2 = & 9.09304 \\ \|f - g_1\|_2 = & 5.69367 \\ \|f - g_2\|_2 = & 4.45551 \\ \|f - g_3\|_2 = & 3.7771 \\ \|f - g_4\|_2 = & 3.3354 \\ \|f - g_5\|_2 = & 3.01899 \end{pmatrix}$$

What is the “max norm” error $\| f - g_n \|_\infty = \max_{t \in [0,2\pi]} | f(t) - g_n(t) |$?

In[ ]:=
```
Table[{StringTemplate["n = ``"][n],
    Maximize[{Abs[f[t] - partialSums[[n + 1, 2]]],
      0 ≤ t ≤ 2 π}, t]},
  {n, 0, 5}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} n = 0 & \{2\pi, \{t \to 0\}\} \\ n = 1 & \{2\pi, \{t \to 0\}\} \\ n = 2 & \{2\pi, \{t \to 0\}\} \\ n = 3 & \{2\pi, \{t \to 0\}\} \\ n = 4 & \{2\pi, \{t \to 0\}\} \\ n = 5 & \{2\pi, \{t \to 0\}\} \end{pmatrix}$$