```
In[1]:= (*Elements of the Ambient Space*)
z = {z1, z2, z3};
R[{z1_, z2_, z3_}] =
  {z1,
   z2,
   z3}; (*R⃗*)

J[{z1_, z2_, z3_}] =
  Table[∂_z[[ii]] R[z][[i]],
    {i, 1, 3}, {ii, 1, 3}] // FullSimplify; (*J^i_i'*)

JJ[{z1_, z2_, z3_}] =
  Inverse[J[{z1, z2, z3}]] // FullSimplify; (*J^i'_i*)

Z[{z1_, z2_, z3_}] =
  Table[∂_z[[i]] R[z],
   {i, 1, 3}]; (*Z⃗_i*)

MZ[{z1_, z2_, z3_}] =
  (Table[Z[z][[i]].Z[z][[j]],
     {i, 1, 3}, {j, 1, 3}] // FullSimplify);(*Z_ij*)

MZZ[{z1_, z2_, z3_}] =
  Inverse[MZ[z]]; (*Z^ij*)

ZZ[{z1_, z2_, z3_}] =
  Table[Sum_{j=1}^{3} (MZZ[z][[i, j]] Z[z][[j]]),
   {i, 1, 3}]; (*Z⃗^i*)

e[i_, j_, k_] :=
  Signature[{i, j, k}]; (*e_ijk or e^ijk*)

V[{z1_, z2_, z3_}] =
  Det[MZ[z]] // FullSimplify; (*Det(Z..)*)

covε[i_, j_, k_] :=
  Sqrt[Det[MZ[z]]] Signature[{i, j, k}]; (*ε_ijk*)

contε[i_, j_, k_] :=
  Signature[{i, j, k}] / Sqrt[Det[MZ[z]]]; (*ε^ijk*)
```

```
Γ[{z1_, z2_, z3_}] =
   Table[ZZ[z][[k]].∂_{z[[j]]} (Z[z][[i]]),
    {k, 1, 3}, {i, 1, 3}, {j, 1, 3}];(*Γ^k_ij*)
```

In[14]:=

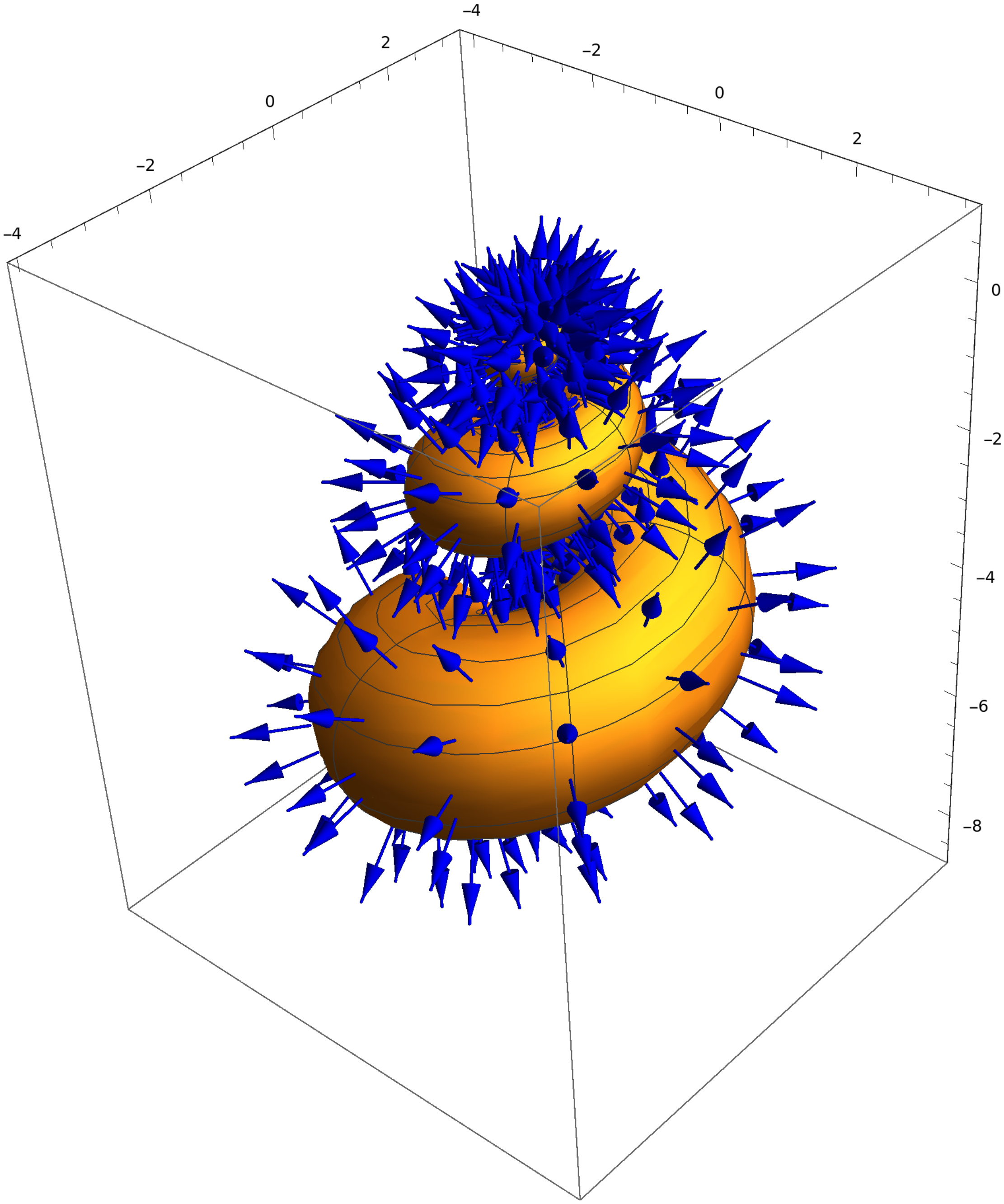
```
(*Elements of a Hypersurface*)
s = {s1, s2};
SR[{s1_, s2_}] =
  {2 (1 - ⅇ^(s1/(6 π))) (Cos[s1]) (Cos[s2/2])^2,
   -2 (1 - ⅇ^(s1/(6 π))) (Sin[s1]) (Cos[s2/2])^2,
   1 - ⅇ^(s1/(3 π)) - Sin[s2] + ⅇ^(s1/(6 π)) Sin[s2]}; (*R⃗(Z(S))*)

S[{s1_, s2_}] =
  Table[∂_{s[[α]]} R[SR[s]],
   {α, 1, 2}]; (*S⃗_α*)

ST[{s1_, s2_}] =
  Table[∂_{s[[α]]} R[SR[s]][[i]],
   {i, 1, 3}, {α, 1, 2}]; (*Z^i_α*)

MS[{s1_, s2_}] =
  Table[∑_{i=1}^{3} ∑_{j=1}^{3}
     (MZ[SR[s]][[i, j]] ST[s][[i, α]] ST[s][[j, β]]),
   {α, 1, 2}, {β, 1, 2}]; (*S_αβ*)

MSS[{s1_, s2_}] =
  Inverse[MS[s]];(*S^αβ*)

STT[{s1_, s2_}] =
  Table[∑_{α=1}^{2} ∑_{i=1}^{3}
     ST[s][[i, α]] MSS[s][[α, β]] MZ[SR[s]][[i, j]],
   {β, 1, 2}, {j, 1, 3}]; (*Z^α_i*)

SS[{s1_, s2_}] =
  Table[∑_{α=1}^{2} S[s][[α]] MSS[s][[α, β]], {β, 1, 2}]; (*S⃗^α*)

SΓ[{s1_, s2_}] =
  Table[SS[s][[α]].∂_{s[[γ]]} (S[s][[β]]),
   {α, 1, 2}, {β, 1, 2}, {γ, 1, 2}]; (*Γ^α_βγ*)
```

```
ee[α_, β_] :=
  Signature[{α, β}]; (*e_αβ or e^αβ*)

A[{s1_, s2_}] =
  Det[MS[s]]; (*Det(S..)*)

scovε[α_, β_] :=
  Sqrt[Det[MS[s]]] Signature[{α, β}]; (*ε_αβ*)

scontε[α_, β_] :=
  Signature[{α, β}]; / Sqrt[Det[MS[s]]]; ; (*ε^αβ*)

n[{s1_, s2_}] =
  -Table[Sum_{j=1}^{3} Sum_{k=1}^{3} Sum_{α=1}^{2} Sum_{β=1}^{2}
      (1/2 contε[i, j, k] scovε[α, β] STT[s][[α, j]] STT[s][[β, k]]),
    {i, 1, 3}]; (*N^i*)

P[{s1_, s2_}] =
  Table[Sum_{k=1}^{3} n[s][[i]] n[s][[k]] MZ[z][[j, k]],
   {i, 1, 3}, {j, 1, 3}]; (*P^i_j*)

T[{s1_, s2_}] =
  Table[Sum_{α=1}^{2} ST[s][[i, α]] STT[s][[α, j]],
   {i, 1, 3}, {j, 1, 3}];(*T^i_j*)
```

i. Finding $\vec{N}$

In[•]:=
```
norms = Table[Graphics3D[{Blue, Arrowheads[Large],
      Arrow[Tube[{SR[s], SR[s] + n[s]}, .02]]}],
    {s1, .01, 6 π, .5}, {s2, .01, 2 π, .5}];

Show[ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
  PlotRange → All, ImageSize → Large], norms]
```

Out[•]=



ii. Finding $\nabla_\alpha \nabla^\alpha F$

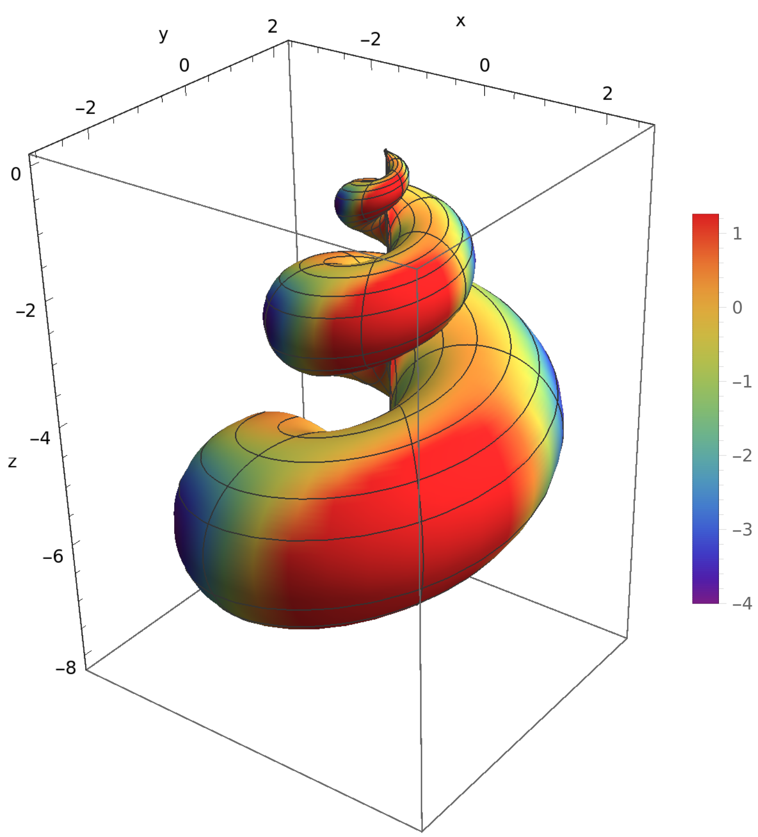
In[30]:=

```
F[{z1_, z2_, z3_}] = z1 z2;
soln2[{s1_, s2_}] =
```

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{1}{\sqrt{A[s]}} \partial_{s[[\alpha]]} \left(\sqrt{A[s]}\ MSS[s][[\alpha, \beta]]\ \partial_{s[[\beta]]} F[SR[s]] \right);$$

```
min = NMinValue[{soln2[s], 0 ≤ s1 ≤ 6 π && 0 ≤ s2 ≤ 2 π}, {s1, s2}];
max = NMaxValue[{soln2[s], 0 ≤ s1 ≤ 6 π && 0 ≤ s2 ≤ 2 π}, {s1, s2}];
ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
 ColorFunction →
  Function[{x, y, z, s1, s2},
   ColorData["Rainbow"][Rescale[soln2[s], {min, max}]]],
 ColorFunctionScaling → False,
 PlotLegends → BarLegend[{"Rainbow", {min, max}},
   ColorFunctionScaling → True],
 PlotRange → All, PlotPoints → 100, AxesLabel → {"x", "y", "z"}]
```

Out[•]=



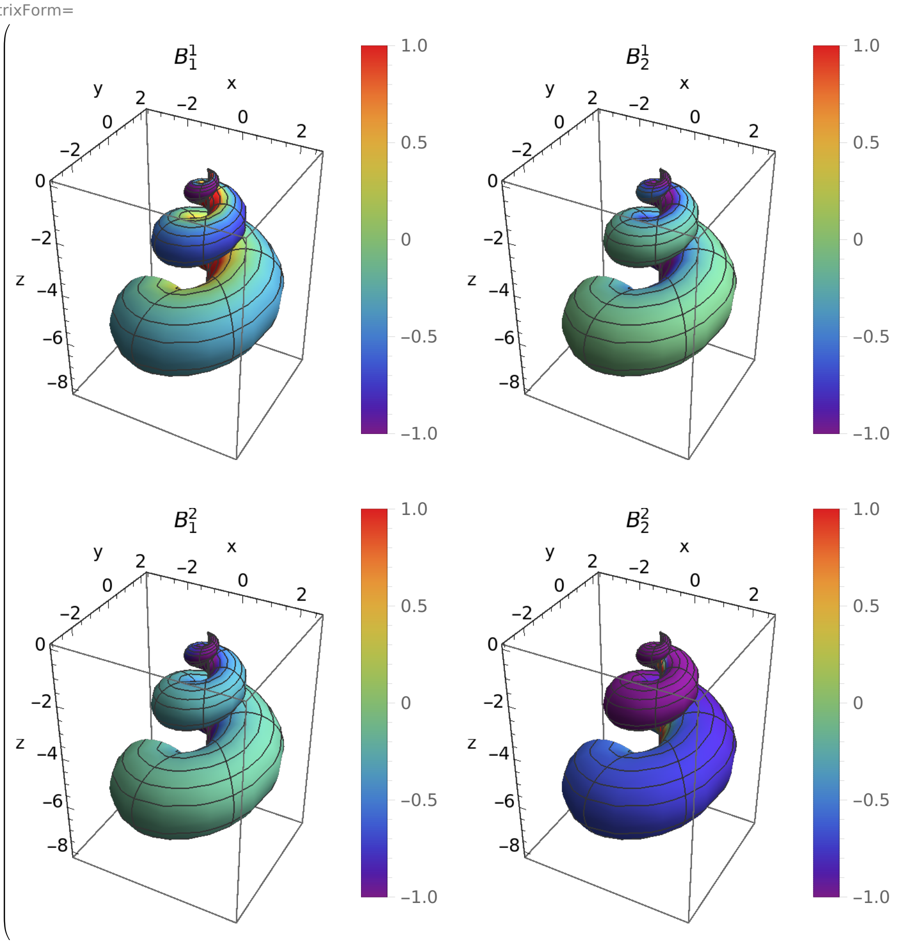
iii. Finding B_α^β

In[72]:=

```
B[{s1_, s2_}] = Table[∑_{i=1}^{3} ∑_{j=1}^{3} ∑_{γ=1}^{2} n[s]⟦j⟧ MZ[R[SR[s]]]⟦i, j⟧ ·
      (∂_{s⟦α⟧} (ST[s]⟦i, γ⟧) -
        ∑_{ω=1}^{2} SΓ[s]⟦ω, α, γ⟧ ST[s]⟦i, ω⟧ +
        ∑_{k=1}^{3} ∑_{m=1}^{3} ST[s]⟦k, α⟧ Γ[R[SR[s]]]⟦i, k, m⟧ ST[s]⟦m, γ⟧) ·
      MSS[s]⟦γ, β⟧, {β, 1, 2}, {α, 1, 2}];
```

```
Table[ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
   ColorFunction →
    Function[{x, y, z, s1, s2},
     ColorData["Rainbow"][Rescale[B[s]⟦β, α⟧, {-1, 1}]]],
   ColorFunctionScaling → False,
   PlotLegends → BarLegend[{"Rainbow", {min, max}},
     ColorFunctionScaling → True],
   PlotRange → All, PlotPoints → 10, AxesLabel → {"x", "y", "z"},
   PlotLabel → Subsuperscript[B, α, β]], {β, 1, 2}, {α, 1, 2}] // MatrixForm
```

Out[75]//MatrixForm=



iv. Finding $\nabla_\alpha N^i$

```
a = 1;
b = 1;
soln4[{s1_, s2_}] =
  Table[∂_{s⟦α⟧} n[s]⟦i⟧ + ∑_{k=1}^{3} ∑_{m=1}^{3} ST[s]⟦k, α⟧ Γ[R[SR[s]]]⟦i, k, m⟧ n[s]⟦m⟧,
   {α, 1, 2}, {i, 1, 3}];

soln4[{a, b}] // N // MatrixForm (*∇_α N^i*)

Table[-∑_{β=1}^{2} ST[{a, b}]⟦i, β⟧ B[{a, b}]⟦β, α⟧,
   {α, 1, 2}, {i, 1, 3}] // N // MatrixForm (*-Z^i_β B^β_α*)

(soln4[{a, b}] - Table[-∑_{β=1}^{2} ST[{a, b}]⟦i, β⟧ B[{a, b}]⟦β, α⟧,
     {α, 1, 2}, {i, 1, 3}]) // N // MatrixForm (*∇_α N^i+Z^i_β B^β_α≃0*)
```

Out[46]//MatrixForm=

$$\begin{pmatrix} 0.456152 & 0.118124 & 0.0113261 \\ 1.65198 & -0.0632867 & 0.355605 \end{pmatrix}$$

Out[47]//MatrixForm=

$$\begin{pmatrix} 0.456152 & 0.118124 & 0.0113261 \\ 1.65198 & -0.0632867 & 0.355605 \end{pmatrix}$$

Out[48]//MatrixForm=

$$\begin{pmatrix} -3.5059\cdot 10^{-11} & 1.17761\cdot 10^{-10} & 1.83831\cdot 10^{-10} \\ 1.14027\cdot 10^{-11} & -3.83023\cdot 10^{-11} & -5.97911\cdot 10^{-11} \end{pmatrix}$$

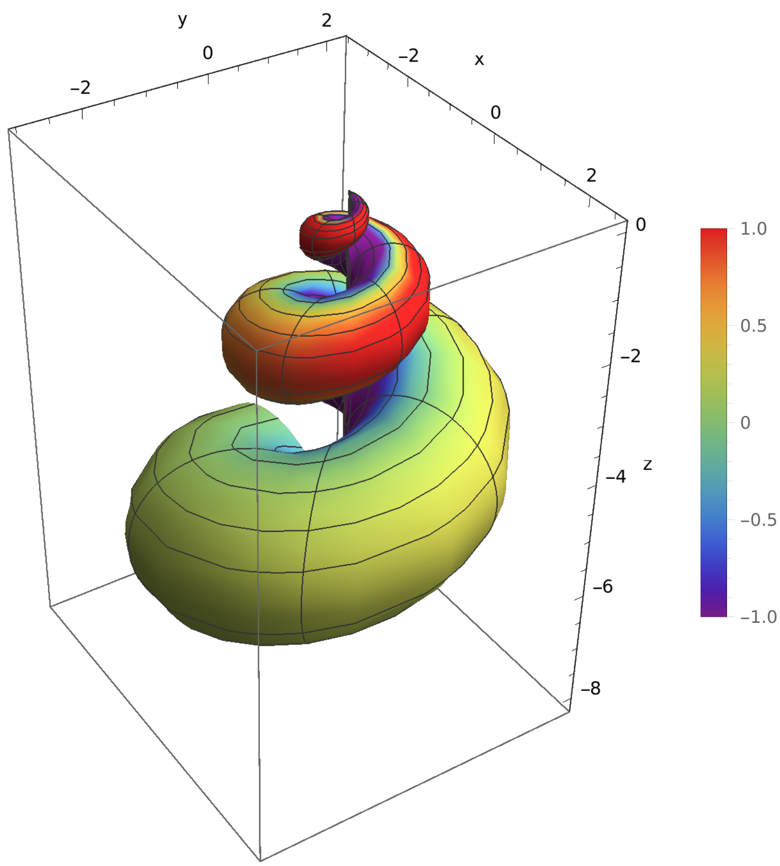
$\nabla_\alpha N^i$ can be seen as the derivative of the normal vector in the direction of the different bases of the surf ace in component form. So the calculation describes how the normal vectors change in the direction of the different bases on the surface!

v. Finding K

In[33]:=
```
K[{s1_, s2_}] = Det[B[s]];
```

```
min = -1;
max = 1;
ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
 ColorFunction →
  Function[{x, y, z, s1, s2},
   ColorData["Rainbow"][Rescale[K[s], {min, max}]]],
 ColorFunctionScaling → False,
 PlotLegends → BarLegend[{"Rainbow", {min, max}},
   ColorFunctionScaling → True],
 PlotRange → All, PlotPoints → 10, AxesLabel → {"x", "y", "z"}]
```

Out[ ]=

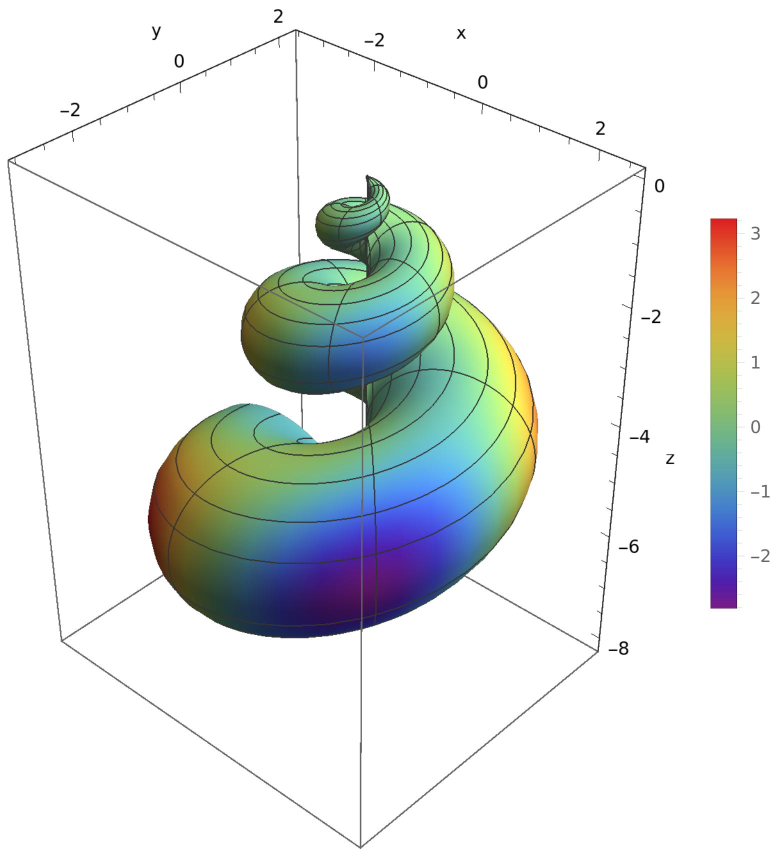


vi. Finding $\frac{\partial F}{\partial n}$

In[34]:=
```
F[{z1_, z2_, z3_}] = z1 z2;
delF[{z1_, z2_, z3_}] = Table[∂z⟦i⟧ F[z], {i, 1, 3}];
soln6[{s1_, s2_}] = ∑_{i=1}^{3} n[s]⟦i⟧ delF[R[SR[s]]]⟦i⟧ // FullSimplify;
```

```
min = -2.81;
max = NMaxValue[{soln6[s], 0 ≤ s1 ≤ 6 π && 0 ≤ s2 ≤ 2 π}, {s1, s2}];
ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
 ColorFunction →
  Function[{x, y, z, s1, s2},
   ColorData["Rainbow"][Rescale[soln6[s], {min, max}]]],
 ColorFunctionScaling → False,
 PlotLegends → BarLegend[{"Rainbow", {min, max}},
   ColorFunctionScaling → True],
 PlotRange → All, PlotPoints → 100, AxesLabel → {"x", "y", "z"}]
```

Out[•]=



vii. Finding $\frac{\partial^2 F}{\partial n^2}$

In[37]:=

```
F[{z1_, z2_, z3_}] = z1 z2;
del2F[{z1_, z2_, z3_}] =
  Table[∂z[[i]],z[[j]] F[z] - Sum[Γ[z][[m, i, j]] ∂z[[m]] F, {m, 1, 3}], {i, 1, 3}, {j, 1, 3}];
soln7[{s1_, s2_}] =
  Sum[Sum[n[s][[i]] n[s][[j]] del2F[R[SR[s]]][[i, j]], {j, 1, 3}], {i, 1, 3}] // FullSimplify;
```

```
min = NMinValue[{soln7[s], 0 ≤ s1 ≤ 6 π && 0 ≤ s2 ≤ 2 π}, {s1, s2}];
max = NMaxValue[{soln7[s], 0 ≤ s1 ≤ 6 π && 0 ≤ s2 ≤ 2 π}, {s1, s2}];
ParametricPlot3D[SR[s], {s1, 0, 6 π}, {s2, 0, 2 π},
 ColorFunction →
  Function[{x, y, z, s1, s2},
   ColorData["Rainbow"][Rescale[soln7[s], {min, max}]]],
 ColorFunctionScaling → False,
 PlotLegends → BarLegend[{"Rainbow", {min, max}},
   ColorFunctionScaling → True],
 PlotRange → All, PlotPoints → 100, AxesLabel → {"x", "y", "z"}]
```

Out[•]=