LOG(M) PROJECT: LARGE SCALE GEOMETRY OF INTEGERS DRAFT 0.1

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1. Introduction

The study of groups through their geometric properties has proven to be a fruitful area of research, providing deep insights into both algebraic and geometric structures [5, 3]. In this paper, we adopt a geometric group-theoretic approach to investigate the large-scale geometry of the integer group $\mathbb{Z}$ and the discrete Heisenberg group $H(4)(\mathbb{Z})$. Our main objective is to develop algorithms for computing the word metric in these groups with respect to specific generating sets. This work builds upon foundational concepts in abstract algebra [4] and is motivated by discussions with Professor Yanlong Hao on the geometric properties of integer groups [2].

2. Preliminaries

We begin by recalling essential definitions and concepts that underpin our study [4].

Definition 2.1 (Group)**.** A *group* $(G, *)$ is a set G equipped with a binary operation $* : G \times G \to G$ satisfying the following axioms:

1. **Associativity**: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
2. **Identity Element**: There exists an element $e \in G$ such that $e * a = a * e = a$ for all $a \in G$.
3. **Inverse Elements**: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Definition 2.2 (Subgroup and Generating Set)**.** A subset $H \subseteq G$ is called a *subgroup* of G if H is itself a group under the operation $*$. A subset $S \subseteq G$ is a *generating set* of G if every element $g \in G$ can be expressed as a finite product of elements of S and their inverses. In this case, we write $G = \langle S \rangle$.

Definition 2.3 (Word Metric)**.** Let $G = \langle S \rangle$ be a finitely generated group with generating set S. The *word length* of an element $g \in G$, denoted by $|g|$, is the minimal length n of a word $w = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}$, where $s_i \in S$, $\epsilon_i \in \{\pm 1\}$, such that $g = w$. The *word metric* $d_S : G \times G \to \mathbb{N}_0$ is defined by

$$d_S(g, h) = |g^{-1} h|.$$

The word metric turns the group G into a metric space, providing a framework for geometric analysis [3].

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3. Computing the Word Metric on $\mathbb{Z}$

We first address the problem of computing the word metric on the additive group of integers $\mathbb{Z}$ with respect to a finite generating set.

3.1. Algorithm Development.

Although $\mathbb{Z}$ is a cyclic group generated by a single integer, considering multiple generators introduces complexity in determining the minimal word length.

Lemma 3.1. *Let $X = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{N}$ be an ordered finite generating set for $\mathbb{Z}$, with $x_1 < x_2 < \cdots < x_k$. Define the constant*

$$M = \sum_{i=1}^{k-1} \left(\left\lfloor \frac{x_i + x_{i+1}}{2} \right\rfloor x_i \right) + x_k.$$

Then, for any $n \in \mathbb{Z}$ with $|n| > M$, the word length satisfies

$$|n| = |n - x_k| + 1.$$

Proof. Since $\gcd(X) = 1$ (by Bézout's identity [1]), the set X generates $\mathbb{Z}$. For $|n| > M$, subtracting the largest generator x_k reduces $|n|$ by one while keeping the result within the group generated by X. The specific form of M ensures that this recursive process eventually reaches an integer whose word length has been precomputed. □

Theorem 3.2 (Algorithm for Word Length in $\mathbb{Z}$)**.** *Given the ordered generating set $X = \{x_1, x_2, \ldots, x_k\}$ for $\mathbb{Z}$, the word length $|n|$ of any integer $n \in \mathbb{Z}$ can be computed using the following algorithm:*

1. *Verify that* $\gcd(X) = 1$ *to ensure X generates $\mathbb{Z}$* [1].
2. *Compute the constant M as defined in Lemma 3.1.*
3. *Precompute the word lengths $|n|$ for all integers with $|n| \leq M$.*
4. *For $|n| > M$, apply the recursive relation:*

$$|n| = |n - x_k| + 1.$$

Proof. The algorithm is justified by Lemma 3.1. By precomputing the word lengths for integers up to M, we ensure that the recursion terminates after a finite number of steps for any $n \in \mathbb{Z}$. □

This algorithm provides an efficient method for computing the word metric on $\mathbb{Z}$, which is fundamental in understanding its geometric structure.

4. Computing the Word Metric on the Heisenberg Group $H(4)(\mathbb{Z})$

We now extend our analysis to the discrete Heisenberg group $H(4)(\mathbb{Z})$, which is a central object of study in nilpotent group theory [5].

4.1. **Definition and Generators.**

Definition 4.1 (Heisenberg Group $H(4)(\mathbb{Z})$)**.** The group $H(4)(\mathbb{Z})$ consists of all 4×4 upper-triangular matrices of the form:

$$H = \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, y_1, y_2, z \in \mathbb{Z}$.

The group $H(4)(\mathbb{Z})$ can be generated by the set

$$S = \{X_1, X_2, Y_1, Y_2\},$$

where

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.2. **Challenges and Approach.** Computing the word metric in $H(4)(\mathbb{Z})$ presents significant challenges due to its non-abelian and higher-dimensional nature. To address this, we consider the group's commutator relations and exploit its nilpotent structure. Future work will focus on developing algorithms that generalize our methods from $\mathbb{Z}$ to $H(4)(\mathbb{Z})$, potentially drawing upon advanced concepts in group theory and geometry.

5. Conclusion

We have presented an algorithm for computing the word metric on the integer group $\mathbb{Z}$ with respect to an arbitrary finite generating set, utilizing fundamental principles from abstract algebra [4] and number theory [1]. Additionally, we have laid the groundwork for computing the word metric in the Heisenberg group $H(4)(\mathbb{Z})$, a key object in the study of nilpotent groups [5]. This research contributes to a deeper understanding of the interplay between algebraic and geometric properties in group theory.

References

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