1 Ivrii Problems

1. Show the identities

   (a)
   $$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2} = \frac{\pi^2}{(\sin \pi z)(\tan \pi z)}$$

   (b)
   $$\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2-n^2} = \frac{\pi}{\sin \pi z}$$

   (c)
   $$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2+a^2} = \frac{\pi}{a} \cdot \frac{\sinh 2\pi a}{\cosh 2\pi a - \cos 2\pi z}.$$

   Solution. First, note that each of the series in question are absolutely convergent (at least pointwise) by comparision with the series $\sum \frac{1}{n^2}$ wherever the terms in the series do not have a singularity. Thus we may rearrange these series and still be certain that they converge pointwise away from singularities of their terms.

   (a) Separating even and odd terms we get,

   $$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2} &= \sum_{k=-\infty}^{\infty} \frac{1}{(z-2k)^2} - \sum_{k=-\infty}^{\infty} \frac{1}{(z-2k+1)^2} \\ &= \frac{1}{4}\left(\frac{\pi^2}{\sin^2 \frac{\pi z}{2}} - \frac{\pi^2}{\sin^2 \frac{\pi(z-1)}{2}} \right) \\ &= \frac{\pi^2(\cos^2 \frac{\pi z}{2} - \sin^2 \frac{\pi z}{2})}{4 \sin^2 \frac{\pi z}{2} \cos^2 \frac{\pi z}{2}} \\ &= \frac{\pi^2 \cos \pi z}{\sin^2 \pi z}\end{aligned}$$

2. Prove the identity by taking the logarithmic derivative of both sides

   $$\pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \sinh \pi x.$$

3. Let $\mathcal{F}$ be the family of holomorphic functions in the unit disk satisfying $|f^{(n)} \leq n!$ for all $n \geq 0$. Show that $\mathcal{F}$ is a normal family.

4. Suppose f is an entire function. Show that f has an $n-th$ root if and only if all zeros of f have multiplicity divisible by n.

5. Suppose $f(z)$ is a holomorphic function defined on the unit disk with $|f(z)| \leq M$. Let the zeros of f be $a_1, a_2, \ldots, a_n$ counted with their multiplicities. Show that

$$|f(z)| \leq M \left| \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a_k} z} \right|.$$

In particular, $|f(0)| \leq M \prod_{k=1}^{n} |a_k|$ and if $f(0) = 0$ then $|f(z)| \leq M|z|$.

6. (Blashke Product) Suppose $f(z)$ is a holomorphic function defined on the unit disk with a zero at 0 of order s and the other zeros $\{a_k\}$ satisfying $\sum_k (1 - |a_k|) < \infty$ (or equivalently $\sum_k \log |a_k| > -\infty$). It then admits a nice factoriztion $f = BG$ where B is a product of

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a_k} z}$$

and $G(z)$ is a holomorphic function without zeros. Show that $B(z)$ is holomorphic.

7. Show that a bounded holomorphic function admits a Blashke product, ie that the sum $\sum_k \log |a_k|$ converges (Hint: use the first corollary of Problem 5).

8. (Jensen's formula) Suppose $f(z)$ is a holomorphic function in the unit disk without zeros. As $\log |f(z)|$ is harmonic, we have $\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta$ for $0 < r < 1$. But what if $f(z)$ has zeros in the unit disk? Suppose that $f(0) \neq 0$ and denote its zeros by $\{a_k\}$. In this case,

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \sum_{|a_k| < r} \log(\frac{r}{|a_k|}).$$

Here $\log |f(0)|$ is not equal to, but is actually less than the mean value. Such functions are called subharmonic. Prove the above formula (Hint: first consider r for which no $\{a_k\}$ lie on $|z| = r$, also show that RHS is continuous).

9. (Canonical Product) Look back at the proof of Weierstrass' theorem and observe the following: Suppose $f(z)$ is an entire function with zeros $\{a_k\}$ satisfying $\sum_k \frac{1}{|a_k|^{m+1}} \leq \infty$ for some integer m. Then it is possible to choose all $m_k = m$.

10. Prove that

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Hint: The Weierstrass theorem implies that

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

To find $g(z)$, take the logarithmic derivative and use Example 2 from Section 1.2.

11. (Wedderburn's lemma) Suppose f and g are entire functions without common zeros. Show that there exists entire functions a, b such that $af + bg = 1$.

    Find the residues at the poles of the Γ function (Hint: use the functional equation).

12. (Bohr-Mollerup theorem) Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a logarithmically convex function satisfying $f(x+1) = f(x)$ and $f(1) = 1$. Then necessarily, $f(x) = \Gamma(x)$ for all $x > 0$. Hint: Show that for all natural $n \geq 2$ and positive x

    $$(n-1)^x (n-1)! \leq f(x+n) \leq n^x (n-1)!$$

    and
    $$\frac{n^x n!}{x(x+1)\cdots(x+n)} \leq f(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \cdot \frac{x+n}{n}.$$

13. Show that the automorphisms of the upper half-plane which preserve i are given by

    $$w(z) = \frac{z + \tan\theta/2}{1 - z\tan\theta/2}.$$

    Write the formula for $\theta = \pi$.

14. Find the automorphism group of $\mathbb{C} \setminus \{0\}$.

15. Show that two annuli are biholomorphic if and only if the ration of their radii are the same (complete the proofs below).

    (a) Use the Previous problem.

    (b) Use the uniqueness of solution to the Dirichlet problem:

16. Show that two tori $\mathbb{C}/\Gamma_1$ (with Γ_1 generated by e_1, e_2) and $\mathbb{C}/\Gamma_2$ (with Γ_2 generated by f_1, f_2) are biholomorphic if and only if there is a fractional linear transformation with integer coefficients and determinant 1 which takes (e_1, e_2) to (f_1, f_2).

17. (Schwarz-Christoffel formula) Show that the mapping $F : \mathbb{H}^+ \to \Omega$ (where Ω is a polygon) given by the formula

    $$F(w) = C \int_0^w \prod_{k=1}^{n-1} (w - w_k)^{-\beta_k} dw + C$$

    is conformal for some distinct real numers w_k and $\sum \beta_k = 2$.

18. Show that $F(w) = \int_0^w (1 - w^n)^{-2/n} dw$ maps the unit disk onto the interior of a regular polygon with n sides.

19. Find the image of the unit disk under the mapping $F(z) = \frac{1}{z} \prod_{k=1}^n (z - a_k)^{\lambda_k}$ where λ_k are positive with $\sum_{k=1}^n \lambda_k = 2$.

20. Prove the addition theorem:

$$\mathfrak{p}(z_1+z_2) = -\mathfrak{p}(z_1) - \mathfrak{p}(z_2) + \frac{1}{4}\left(\frac{\mathfrak{p}'(z_1)-\mathfrak{p}'(z_2)}{\mathfrak{p}(z_1)-\mathfrak{p}(z_2)}\right)^2$$

21. Another form of the addition theorem: if $u+v+w=0$ in $\mathbb{C}/\Gamma$ then

$$\begin{vmatrix} 1 & 1 & 1 \\ \mathfrak{p}(u) & \mathfrak{p}(v) & \mathfrak{p}(w) \\ \mathfrak{p}'(u) & \mathfrak{p}'(v) & \mathfrak{p}'(w) \end{vmatrix} = 0$$

22. Suppose that $f(z)$ is an even doubly periodic function (let the group of periods be Γ). There exists points $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n \in \mathbb{C}/\Gamma$ such that

$$f(z) = c\prod_{k=1}^{n} \frac{\mathfrak{p}(z)-\mathfrak{p}(a_k)}{\mathfrak{p}(z)-\mathfrak{p}(b_k)}$$

23. Show that a doubly periodic function is a rational function of the $\mathfrak{p}$ and $\mathfrak{p}'$ (Hint: use the previous problem).

24. Show that while the function $f(z) = \sum_n z^{2^n}$ is holomorphic in the unit disk, it does not extend holomorphically to any larger open set (Hint: $f(z^2) = f(z) - z$).

25. Find the radius of convergence of $f(z) = \sum_{k\geq 1} \frac{z^{2k}}{k+1}$. Find a maximal domain of existence (a maximal open set in $\mathbb{C}$ to which $f(z)$ may be analytically continued).

26. The set of solutions (z, w) of $w^2 - 2wz + 1 = 0$ can be completed to a compact Riemann surface over $\mathbb{C}P^1$. Find the residues of the differential form $\frac{dz}{\sqrt{z^2-1}}$ points at infinity.

27. Prove
$$\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}.$$

28. Prove Picard's theorem for meromorphic functions: if a meromorphic function defined on the entire complex plane omits three values, it is necessarily constant.

    Solution. If f is meromorphic and omits a, b, and c then $\frac{1}{f(z)-a}$ is a holomorphic function which omits $\frac{1}{b-a}$ and $\frac{1}{c-a}$. Thus by Picard's Little theorem f is constant.

29. Show that a non-constant holomorphic function defined on $\mathbb{C} \setminus \{0\}$ omits at most 1 value.

    Solution. If f has an essential singularity at 0 or ∞ then an application of Picard's Great theorem completes the proof. If not then f is a meromorphic function on the Riemann sphere and is thus rational (see Section 2). Then the fundamental theorem of algebra completes the proof.

30. Show that the function $f(z) = ze^z$ attains every value from $\mathbb{C}$.

    Solution. Note first that f has an essential singularity at ∞. That is, $f(1/z) = \frac{e^{1/z}}{z}$ has an essential singularity at 0. Thus by Picard's theorem on the neighbourhood of infinity $|z| > 1$, f omits at most one value. This value must be 0 and since f acheives 0 at $z = 0$, f attains every value.

31. Suppose f, g are meromorphic functions such that $f^3 + g^3 = 1$. Show that actually f and g are constant functions. Is result still true if 3 is replaced by a larger positive integer?

    Solution. Note that the function $\frac{f^3}{g^3}+1 = \frac{1}{g^3}$ is nonvanishing. Thus $\frac{f}{g}$ omits the 3 distinct roots of -1. Thus by Picard's theorem for meromorphic functions $\frac{f}{g} = c$. But then $(c^3 + 1)g^3 = 1$ so g and therefore f is constant.

32. Suppose f, g are entire functions satisfying $e^f + e^g = 1$. Show that f, g are actually constant functions.

    Solution. If $e^f + e^g = 1$ then the entire function $e^f = 1 - e^g$ omits 0 and 1, therefore by Picard's Little theorem it must be constant. Then by differentiating we find that f' must be zero so f is constant.

2 Other Problems

1. If f is an entire function without an essential singularity at ∞ then f is a polynomial.

   Solution. Since f either has a pole or a removeable singularity at ∞, $\lim_{z\to\infty} \frac{f(z)}{z^k} = 0$ for some integer k. But then by Cauchy's estimate

   $$|f^{(n)}(0)| \leq \frac{n!Cr^k}{r^n}$$

   for large enough radius r. But then if $n > k$, $f^{(n)}(0) = 0$ so f is a polynomial.

2. If f is a meromorphic function with a nonessential singularity at ∞ then f is a rational function.

   Solution. Since f has a nonessential singularity at ∞, this singularity must be isolated so there is a neighbourhood of ∞ which has no poles. This imples that f only has finitely many poles say $z_1, \ldots, z_n$ counted with multiplicity. Then $(z - z_1) \cdots (z - z_n) f(z)$ is a holomorphic function with no essential singularity at ∞ so applying the previous problem completes the proof.