Paper Review

High-dimensional Covariate Balancing Propensity Score by Ning, Peng, and Imai

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Motivation

Observation Study

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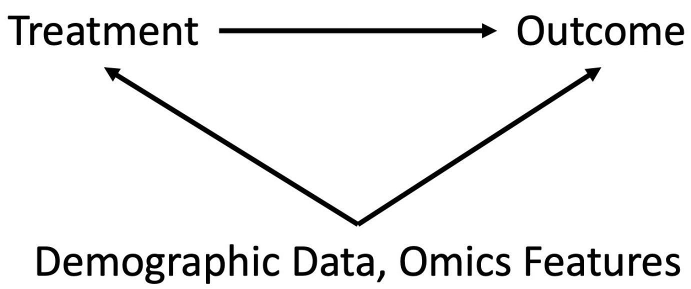

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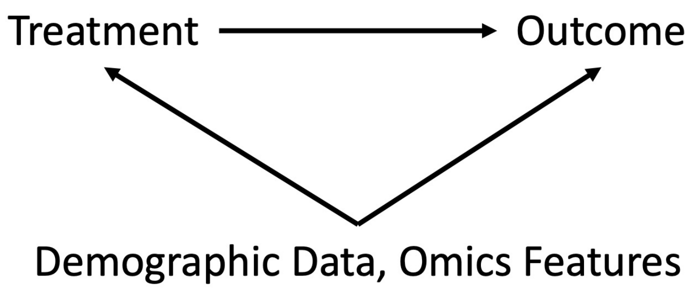

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> How to estimate average treatment effect?
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> Propensity score? Outcome model?

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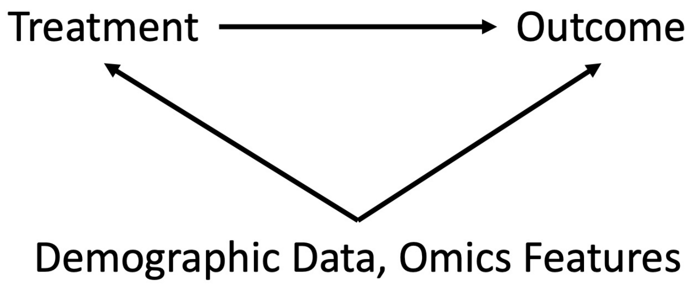

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High-dimensional regression

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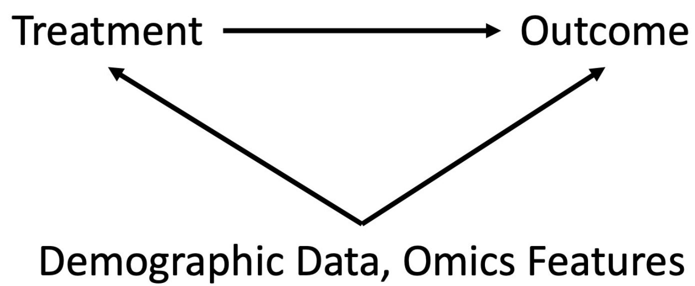

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High-dimensional regression

See Bradic (2019): Rate/sparsity double robust

Background

Covariate Balancing Propensity Score (CBPS)

See Imai (2014): Estimate propensity score so that covariate balance is optimized

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Propensity Score

Conditional probability of treatment assignment

$$\pi(\mathbf{X}_i) = \Pr(T_i = 1|\mathbf{X}_i)$$

Balancing property (without assumption)

$$T_i \perp\!\!\!\perp \mathbf{X}_i|\pi(\mathbf{X}_i)$$

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Use of Propensity Score for Causal Inference

- Matching
- Subclassification
- Weighting (Horvitz-Thompson)
- Double-robust estimators

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Propensity Score Tautology

- Propensity score is unknown
- Diagnostics: covariate balance checking
- Misspecification is possible especially for non-binary treatments
- Skewed covariates are common in applied settings
- Propensity score methods can be sensitive to misspecification

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Covariate balancing condition (C.1)

$$\mathrm{E}[\frac{T_i\tilde{\mathbf{X}}_i}{\pi_\beta(\mathbf{X}_i)} - \frac{(1-T_i)\tilde{\mathbf{X}}_i}{1-\pi_\beta(\mathbf{X}_i)}] = 0,\ \tilde{\mathbf{X}}_i = f(\mathbf{X}_i) \text{ any vector-valued measurable function}$$

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Score condition (C.2)

$$\mathrm{E}[\frac{T_i\pi'_\beta(\mathbf{X}_i)}{\pi_\beta(\mathbf{X}_i)} - \frac{(1-T_i)\pi'_\beta(\mathbf{X}_i)}{1-\pi_\beta(\mathbf{X}_i)}] = 0,\ (f(\mathbf{X}_i) = \pi'(\mathbf{X}_i))$$

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Covariate Balancing Propensity Score (CBPS)

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- Over-identified CBPS: combine them with score conditions

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MLE

$$\hat{\beta}_{MLE} = \text{argmax}_{\beta \in \Theta} \sum_{i=1}^{N} T_i \log(\pi_\beta(\mathbf{X}_i)) + (1 - T_i)\log(1 - \pi_\beta(\mathbf{X}_i))$$

$$\frac{1}{N}\sum_{i=1}^{N} s_\beta(T_i, \mathbf{X}_i) = 0, \;\; s_\beta(T_i, \mathbf{X}_i) = \frac{T_i \pi'_\beta(\mathbf{X}_i)}{\pi_\beta(\mathbf{X}_i)} - \frac{(1 - T_i)\pi'_\beta(\mathbf{X}_i)}{1 - \pi_\beta(\mathbf{X}_i)}$$

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GMM

$$\hat{\boldsymbol{\beta}}_{GMM} = \operatorname{argmin} \bar{g}_{\beta}(\mathbf{T}, \mathbf{X})' \boldsymbol{\Sigma}_{\beta}(\mathbf{T}, \mathbf{X})^{-1} g_{\beta}(\mathbf{T}, \mathbf{X}), \text{ where } g_{\beta}(\mathbf{T}, \mathbf{X}) = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\begin{bmatrix} \text{score condition} \\ \text{balancing condition} \end{bmatrix}}_{g_{\beta}(T_i, \mathbf{X}_i)}$$

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The balancing condition: $\frac{1}{N}\sum_{i=1}^{N} w_\beta(T_i,\mathbf{X}_i)\tilde{\mathbf{X}}_i$

- $w_\beta(T_i,\mathbf{X}_i) = \frac{T_i-\pi_\beta(\mathbf{X}_i)}{\pi_\beta(\mathbf{X}_i)[1-\pi_\beta(\mathbf{X}_i)]}$, the sample analogue of C.1
- $w_\beta(T_i,\mathbf{X}_i) = \frac{N}{N_1}\frac{T_i-\pi_\beta(\mathbf{X}_i)}{1-\pi_\beta(\mathbf{X}_i)}$, the sample analogue of C.2

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Pros

- Single model determines the treatment assignment mechanism and the covariate balancing weights
- Optimize the covariate balance while modeling the treatment assignment
- Estimate propensity score without consulting the outcome data
- Improve WLS and DR estimator (reduce s.e.)
- **Robust to model misspecification**

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Empirical Study

The CBPS method can be applied to HT, IPW, WLS, and DR estimator of ATE

High-dimensional CBPS

HD-CBPS Methodology

HD-CBPS Methodology

Set-up

- Data: $T_i, Y_i, \mathbf{X}_i$, $\dim(\mathbf{X}_i) = d$, $d \gg N$
- Potential outcomes: $Y_i(1), Y_i(0)$
- SUTVA: $Y_i = T_i Y_i(1) + (1 - T_i) Y_i(0)$
- Positivity: $0 < \Pr(T_i = 1 | \mathbf{X}_i) < 1$
- Ignorability: $\{Y_i(0), Y_i(1) \perp\!\!\!\perp T_i | \mathbf{X}_i\}$
- ATE: $\mu^* = \mathrm{E}[Y_i(1) - Y_i(0)]$
- Focus on: $\mu_1^* = \mathrm{E}[Y_i(1)]$

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Two correctly specified high-dimensional working models

- $\Pr(T_i = 1|\mathbf{X}_i) = \pi(\mathbf{X}_i' \boldsymbol{\beta}^* + u_i)$
  - where u_i is the approximation error
- $\mathrm{E}[Y_i(1)|\mathbf{X}_i] = K_1(\mathbf{X}_i) + r_i$, $K_1(\mathbf{X}_i) = \boldsymbol{\alpha}^{*\prime} \mathbf{X}_i$
  - where r_i is the approximation error

HD-CBPS Methodology

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Recall

Covariate balancing properties

Let $\hat{\pi} = \hat{\pi}(\mathbf{X}'\hat{\boldsymbol{\beta}})$ be an estimator of the propensity score

$$\sum_{i=1}^{N}(\frac{T_i}{\hat{\pi}} - 1)\mathbf{X}_i = \mathbf{0}, \text{ strong condition}$$

e.g., balancing the mean or every component of $\mathbf{X}_i$ (Imai, 2014; Fan et al., 2016)

$$\sum_{i=1}^{N}(\frac{T_i}{\hat{\pi}} - 1)\boldsymbol{\alpha}^{*\prime}\mathbf{X}_i = 0, \text{ weak condition}$$

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Challenge

In high-dimensional setting, $\hat{\boldsymbol{\beta}}$ that satisfies strong condition is not unique ?

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Main idea

Update estimated propensity score by balancing a **subset** of covariates that are predictive to the outcome

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Step 1 (Generalized Quasilikelihood)

$$Q_N(\boldsymbol{\beta}) = \frac{1}{N}\sum_{i=1}^{N}\int_0^{\boldsymbol{\beta}'\mathbf{X}_i}[\frac{T_i}{\pi(u)} - 1]w_1(u)du$$

$$\hat{\boldsymbol{\beta}} = \text{argmin}_{\beta}\{-Q_N(\boldsymbol{\beta}) + \lambda_{\beta}||\boldsymbol{\beta}||_1\}$$

Remark: $\frac{\partial Q_N(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \frac{1}{N}\sum_{i=1}^{N}[\frac{T_i}{\pi(\boldsymbol{\beta}'\mathbf{X}_i)} - 1]w_1(\boldsymbol{\beta}'\mathbf{X}_i)\mathbf{X}_i$

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Step 2

$$L_N(\boldsymbol{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}T_i w_2(\hat{\boldsymbol{\beta}}'\mathbf{X}_i)(Y_i - \boldsymbol{\alpha}'\mathbf{X}_i)^2$$

$$\tilde{\boldsymbol{\alpha}} = \text{argmin}_\alpha\{L_N(\boldsymbol{\alpha}) + \lambda_\alpha||\boldsymbol{\alpha}||_1\}$$

HD-CBPS Methodology

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Step 3 (Update! Remove bias)

Active set: $\tilde{S} = \{j : \tilde{\alpha}_j \neq 0\}$; and $\mathbf{X}_{\tilde{S}}$

$$g_N(\boldsymbol{\gamma}) = \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{T_i}{\pi(\boldsymbol{\gamma}'\mathbf{X}_{i\tilde{S}} + \hat{\boldsymbol{\beta}}'_{\tilde{S}^c}\mathbf{X}_{i\tilde{S}^c})} - 1\right\}\mathbf{X}_{i\tilde{S}}$$

$$\tilde{\gamma} = \text{argmin}_{\gamma\in\mathbb{R}^{|\tilde{S}|}} ||g_N(\boldsymbol{\gamma})||_2^2$$

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Step 4

Let $\tilde{\boldsymbol{\beta}} = (\tilde{\boldsymbol{\gamma}}', \hat{\boldsymbol{\beta}}'_{\tilde{S}^c})'$, and $\tilde{\pi}_i = \pi(\tilde{\boldsymbol{\beta}}'\mathbf{X}_i)$

- Remark: $\sum_{i=1}^{N}\left(\frac{T_i}{\tilde{\pi}_i} - 1\right)\mathbf{X}_i'\boldsymbol{\alpha}^* \approx \sum_{i=1}^{N}\left(\frac{T_i}{\tilde{\pi}_i} - 1\right)\mathbf{X}_i'\tilde{\boldsymbol{\alpha}} = \sum_{i=1}^{N}\left(\frac{T_i}{\tilde{\pi}_i} - 1\right)\mathbf{X}'_{i\tilde{S}}\tilde{\boldsymbol{\alpha}}_{\tilde{S}} = 0$

Horvitz–Thompson estimator:

$$\hat{\mu}_1 = \frac{1}{N}\sum_{i=1}^{N}\frac{T_iY_i}{\tilde{\pi}_i}$$

Theoretical Properties

Assumptions

A.1 (Unconfoundedness) $\{Y_i(0), Y_i(1) \perp\!\!\!\perp T_i | \mathbf{X}_i\}$.

A.2 (Overlap) $\pi_i^* \geq c_0$, for some constant c_0.

A.3 (Sub-Guassian condition) both $\epsilon_1 = Y(1) - \boldsymbol{\alpha}^{*\prime}\mathbf{X}$ and X_j satisfy $||\epsilon_1||_{\psi_2} \leq c_\epsilon$, $||X_j||_{\psi_2} \leq c_X$.

A.4 (Sparsity) $\frac{(s_1 \vee s_2)\log(d \vee N)}{\sqrt{N}} = o(1)$, $s_1 = ||\boldsymbol{\beta}^*||_1$, $s_2 = ||\boldsymbol{\alpha}^*||_1$; $\sum_{i=1}^N u_i^2 = O(s_1)$, $\sum_{i=1}^N r_i^2 = O(s_2)$, and $\sum_{i=1}^N u_i r_i = o(\sqrt{N})$.

A.5 (Eigenvalue condition) $\boldsymbol{\Sigma} = \mathrm{E}(\mathbf{X}\mathbf{X}')$, then $C \leq \lambda_{min}(\Sigma_{SS}) \leq \lambda_{max}(\Sigma_{SS}) \leq 1/C$ for some constant C and any $S \subset \{1, 2, \ldots, d\}$ with $|S| \leq (s_1 \vee s_2)\log(N)$.

A.6 (Propensity score and weight functions)

- i) PS model $\pi(u)$ is bounded away from 0, and $\pi'(u)$ is locally Lipschitz
- ii) weight function $w_1(u)$ is bounded away from 0 and $w_1'(u)$ is locally Lipschitz
- iii) weight function $w_2(u)$ is bounded away from 0 and $w_2'(u)$ is continuous.

Theoretical Properties

Under correct model specification

Theorem 1 (Asymp. Normality & Semi-Effi)

When $\Pr(T_i = 1|\mathbf{X}_i) = \pi(\mathbf{X}_i'\boldsymbol{\beta}^* + u_i)$ and $\mathrm{E}[Y_i(1)|\mathbf{X}_i] = K_1(\mathbf{X}_i) + r_i$ are correctly specified, and A.1-A.6 hold, $\hat{\mu}_1$ with ANY weight functions satisfies

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$$\hat{\mu}_1 - \mu_1^* = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{T_i}{\pi_i^*}\{Y_i(1) - \boldsymbol{\alpha}^{*\prime}\mathbf{X}_i\} + \boldsymbol{\alpha}^{*\prime}\mathbf{X}_i - \mu_1^*\right] + O_p\left\{\frac{(s_1 \vee s_2)\log(d \vee N)}{N}\right\}$$

as $d, N \to \infty$.

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as $d, N \to \infty$.

Let V be the semiparametric asymptotic variance bound, i.e.

$$V = \mathrm{E}\Big[\frac{1}{\pi^*}\mathrm{E}(\epsilon_1^2|\mathbf{X}) + (\boldsymbol{\alpha}^{*\prime}\mathbf{X} - \mu_1^*)^2\Big].$$

Assume $\mathrm{E}(\epsilon_1^2|\mathbf{X}) \geq c$ for some $c > 0$ and $\mathrm{E}(\boldsymbol{\alpha}^{*\prime}\mathbf{X})^4 = O(s_2^2)$. Then

$$\sqrt{N}(\hat{\mu}_1 - \mu_1^*)/V^{1/2} \xrightarrow{d} N(0, 1).$$

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True variance

$$V = \mathrm{E}\Big[\frac{1}{\pi^*}\mathrm{E}(\epsilon_1^2|\mathbf{X}) + (\boldsymbol{\alpha}^{*\prime}\mathbf{X} - \mu_1^*)^2\Big]$$

Estimated variance ?

$$\hat{V} = \frac{1}{N}\sum_{i=1}^{N}\Big\{\frac{T_i}{\tilde{\pi}_i^2}(Y_i - \tilde{\boldsymbol{\alpha}}'\mathbf{X}_i)^2 + (\tilde{\boldsymbol{\alpha}}'\mathbf{X}_i - \hat{\mu}_1)^2\Big\}$$

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Honest CI

When all assumptions in Theorem 1 hold, then

$$|\hat{V} - V| = O_p\Big[(s_1 \vee s_2)\Big\{\frac{\log(d \vee N)}{N}\Big\}^{1/2}\Big].$$

Given $0 < \eta \leq 1$, $\mathcal{I} = \big(\hat{\mu}_1 - z_{1-\eta/2}(\hat{V}/N)^{1/2}, \hat{\mu}_1 + z_{1-\eta/2}(\hat{V}/N)^{1/2}\big)$

Theoretical Properties

Under misspecified PS model (Robustness)

$\pi^* = \Pr(T = 1|\mathbf{X})$ does not belong to the assumed parametric class $\{\pi(\mathbf{X}'\boldsymbol{\beta})\}$

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New step 1

$$\boldsymbol{\beta}^o = \text{argmax E}\Big[\int_0^{\mathbf{X}_i'\boldsymbol{\beta}} \{\frac{T_i}{\pi(u)} - 1\} w_1(u) du\Big],\ \boldsymbol{\beta}^o \text{ is sparse}$$

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Remark

- When PS model is correct, $\boldsymbol{\beta}^o$ reduces to $\boldsymbol{\beta}^*$
- The estimand, $\boldsymbol{\beta}^o$ implicitly depends on the choice of $w_1(u)$
- By choosing $w_1(u) = 1$, $\boldsymbol{\beta}^o$ satisfies

$$\mathrm{E}\big[\{\frac{T_i}{\pi(\mathbf{X}_i'\boldsymbol{\beta}^o)} - 1\}\mathbf{X}_i\big] = 0$$

- $\boldsymbol{\beta}^o$: an estimand by solving the strong covariate balancing equation
- $\pi^o = \pi(\mathbf{X}_i'\boldsymbol{\beta}^o)$

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$$I_1 = \frac{1}{N}\sum_{i=1}^{N}\Big(\frac{T_i}{\tilde{\pi}_i} - \frac{T_i}{\pi_i^o}\Big)\{Y_i(1) - \mathbf{X}_i'\boldsymbol{\alpha}^*\}$$

$$I_2 = \frac{1}{N}\sum_{i=1}^{N}\Big(\frac{T_i}{\tilde{\pi}_i} - 1\Big)\mathbf{X}_i'\boldsymbol{\alpha}^*$$

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Proposition 1 (Only outcome model is correctly specified)

......,

$$\hat{\mu}_1 - \mu_1^* = \frac{1}{N}\sum_{i=1}^{N}\Big[\frac{T_i}{\pi_i^o}\{Y_i(1) - \boldsymbol{\alpha}^{*\prime}\mathbf{X}_i\} + \boldsymbol{\alpha}^{*\prime}\mathbf{X}_i - \mu_1^*\Big] + O_p\Big\{\frac{(s_1 \vee s_2)\log(d \vee N)}{N}\Big\}$$

$$V_{mis-ps} = \mathrm{E}\Big[\frac{\pi^*}{(\pi^o)^2}\mathrm{E}(\epsilon_1^2|\mathbf{X}) + (\boldsymbol{\alpha}^{*\prime}\mathbf{X} - \mu_1^*)^2\Big],\ \sqrt{N}(\hat{\mu}_1 - \mu_1^*)/V_{mis-ps}^{1/2} \xrightarrow{d} N(0,1)$$

Remark: under misspecified PS model, $\hat{V}$ is still consistent for V_{mis-ps}, see the proof of Corollary 1.

Theoretical Properties

Under misspecified outcome model (Robustness)

$\mathrm{E}[Y_i(1)|\mathbf{X}_i]$ is non-linear in $\mathbf{X}_i$

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$$\boldsymbol{\alpha}^o = \operatorname{argmin} \mathrm{E}\{T_i w_2(\mathbf{X}_i'\boldsymbol{\beta})(Y_i - \mathbf{X}_i'\boldsymbol{\alpha})^2\}$$

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Proposition 2 (Only PS model is correctly specified)

......,

$$\hat{\mu}_1 - \mu_1^* = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{T_i}{\pi_i^*}\{Y_i(1) - \boldsymbol{\alpha}^{o\prime}\mathbf{X}_i\} + \boldsymbol{\alpha}^{o\prime}\mathbf{X}_i - \mu_1^*\right] + O_p\left\{\frac{(s_1 \vee s_2)\log(d \vee N)}{N}\right\}$$

$$V_{mis-o} = \mathrm{E}\left[\frac{1}{\pi^*}\mathrm{E}(\epsilon^{o2}|\mathbf{X}) + (\boldsymbol{\alpha}^{o\prime}\mathbf{X} - \mu_1^*)^2\right],\ \sqrt{N}(\hat{\mu}_1 - \mu_1^*)/V_{mis-o}^{1/2} \xrightarrow{d} N(0,1)$$

Remark 1: $w_2(u)$ can be $\frac{\pi'(u)}{\pi^2(u)}$

Remark 2: under misspecified outcome model, $\hat{V}$ is still consistent for V_{mis-ps}

Theoretical Properties

Double Robust

> Proposition 1 and Proposition 2 together imply $\hat{\mu}_1$ is root-n consistent and asymptotically normal provided either the PS-model or O-model is correctly specified. And $\hat{V}$ is always consistent, the same confidence interval is valid so long as one of the two models is correctly specified.
>
> Recommendation: $w_1(u) = 1$ and $w_2(u) = \pi'(u)/\pi^2(u)$.

Remark: this method can also be used for generalized linear models, along with great theoretical properties. See chapter 4.

Simulation Study

Simulation

Simulation

Set-up

$$\mathbf{X}_i \sim N(0, \Sigma),\ \Sigma_{jk} = \rho^{|j-k|},\ \rho = .2,\ j, k \in \{1, 2, \ldots, d\}$$

$$\pi(\mathbf{X}_i) = 1 - \frac{1}{1 + \exp(-X_{i1} + \frac{1}{2}X_{i2} - \frac{1}{4}X_{i3} - \frac{1}{10}X_{i4} - \frac{1}{10}X_{i5} + \frac{1}{10}X_{i6})}$$

$$Y_i(1) = 2 + .137(X_{i5} + X_{i6} + X_{i7} + X_{i8}) + \epsilon_{1i}$$

$$Y_i(0) = 1 + .291(X_{i5} + X_{i6} + X_{i7} + X_{i8} + X_{i9} + X_{i10}) + \epsilon_{0i}$$

$$\begin{aligned} X_{mis} =& \{\exp(X_1/2),\ X_2/\{1 + \exp(X_1)\} + 10,\ (X_1X_3/25 + 0.6)^3,\ (X_2 + X_4 + 20)^2, \\ & X_6,\ \exp(X_6 + X_7),\ X_9^2,\ X_7^3 - 20,\ X_9, \ldots,\ X_d\} \end{aligned}$$

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Candidate methods

- HD-CBPS: the proposed method
- PB: approximate residual balancing
- AIPW: regularized augmented inverse probability weighting estimator
- D-SELECT: double selection

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Candidate methods

- HD-CBPS: the proposed method
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Evaluation metrics

- Std.err: standard error
- RMSE: standardized root-mean-squared-error
- Coverage: coverage probability of 95% confidence intervals

Simulation

Both models correct

Simulation

Both models correct

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias       -0.0026 -0.0017 -0.0498  -0.0910
## Std err     0.0936  0.1074  0.0926   0.0979
## RMSE        0.0936  0.0174  0.1052   0.1337
## Coverage    0.9650  0.9300  0.9150   0.8900
## CI length   0.3867  0.4231  0.3775   0.4294
```

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## CI length    0.3867  0.4231  0.3775   0.4294
```

N=500, d=2000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0595 -0.0580 -0.1200  -0.0397
## Std err      0.1061  0.1155  0.1011   0.1279
## RMSE         0.1216  0.1292  0.1569   0.1279
## Coverage     0.9100  0.9100  0.8550   0.9450
## CI length    0.3862  0.4359  0.3731   0.5034
```

Simulation

Propensity score model is misspecified

Simulation

Propensity score model is misspecified

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias       -0.0120 -0.0303 -0.1078  -0.0782
## Std err     0.0984  0.1153  0.0963   0.1034
## RMSE        0.0991  0.1193  0.1446   0.1296
## Coverage    0.9650  0.9450  0.8150   0.9050
## CI length   0.3864  0.4431  0.3732   0.4227
```

Simulation

Propensity score model is misspecified

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```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0120 -0.0303 -0.1078  -0.0782
## Std err      0.0984  0.1153  0.0963   0.1034
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## Coverage     0.9650  0.9450  0.8150   0.9050
## CI length    0.3864  0.4431  0.3732   0.4227
```

N=500, d=2000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0446 -0.0685 -0.1234  -0.0357
## Std err      0.0924  0.1041  0.0921   0.1214
## RMSE         0.1025  0.1246  0.1540   0.1265
## Coverage     0.9300  0.9100  0.7400   0.9400
## CI length    0.3839  0.4382  0.3702   0.5023
```

Simulation

Outcome model is misspecified

Simulation

Outcome model is misspecified

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0034 -0.0321 -0.0562  -0.0991
## Std err      0.0917  0.0982  0.0914   0.1023
## RMSE         0.0917  0.1033  0.1072   0.1424
## Coverage     0.9600  0.9600  0.9050   0.8450
## CI length    0.3874  0.4292  0.3815   0.4327
```

Simulation

Outcome model is misspecified

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0034 -0.0321 -0.0562  -0.0991
## Std err      0.0917  0.0982  0.0914   0.1023
## RMSE         0.0917  0.1033  0.1072   0.1424
## Coverage     0.9600  0.9600  0.9050   0.8450
## CI length    0.3874  0.4292  0.3815   0.4327
```

N=500, d=2000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias        -0.0317 -0.0572 -0.1215  -0.0443
## Std err      0.0944  0.0992  0.0921   0.1026
## RMSE         0.0995  0.1145  0.1525   0.1118
## Coverage     0.9500  0.9550  0.7700   0.9450
## CI length    0.3890  0.4403  0.3728   0.4261
```

Simulation

Both are misspecified

Simulation

Both are misspecified

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias       -0.0547 -0.1201 -0.1873  -0.1005
## Std err     0.1106  0.1038  0.0903   0.0950
## RMSE        0.1234  0.1588  0.2079   0.1383
## Coverage    0.8900  0.8150  0.7750   0.8750
## CI length   0.3994  0.4586  0.3790   0.4333
```

Simulation

Both are misspecified

N=500, d=1000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias       -0.0547 -0.1201 -0.1873  -0.1005
## Std err     0.1106  0.1038  0.0903   0.0950
## RMSE        0.1234  0.1588  0.2079   0.1383
## Coverage    0.8900  0.8150  0.7750   0.8750
## CI length   0.3994  0.4586  0.3790   0.4333
```

N=500, d=2000

```
##             HD-CBPS      RB    AIPW D-SELECT
## Bias       -0.0243 -0.0599 -0.1393  -0.0518
## Std err     0.0969  0.1060  0.0921   0.0965
## RMSE        0.0999  0.1218  0.1670   0.1095
## Coverage    0.9400  0.9400  0.7200   0.9500
## CI length   0.3948  0.4545  0.3781   0.4334
```

Summary

Summary

HD-CBPS

- Method: straightforward
- Appealling theoretical results
- Good empirical performance
    - CBPS tends to outperform the AIPW in low-dimensional settings
    - Conclusion appears to hold in high-dimensional settings as well

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HD-CBPS

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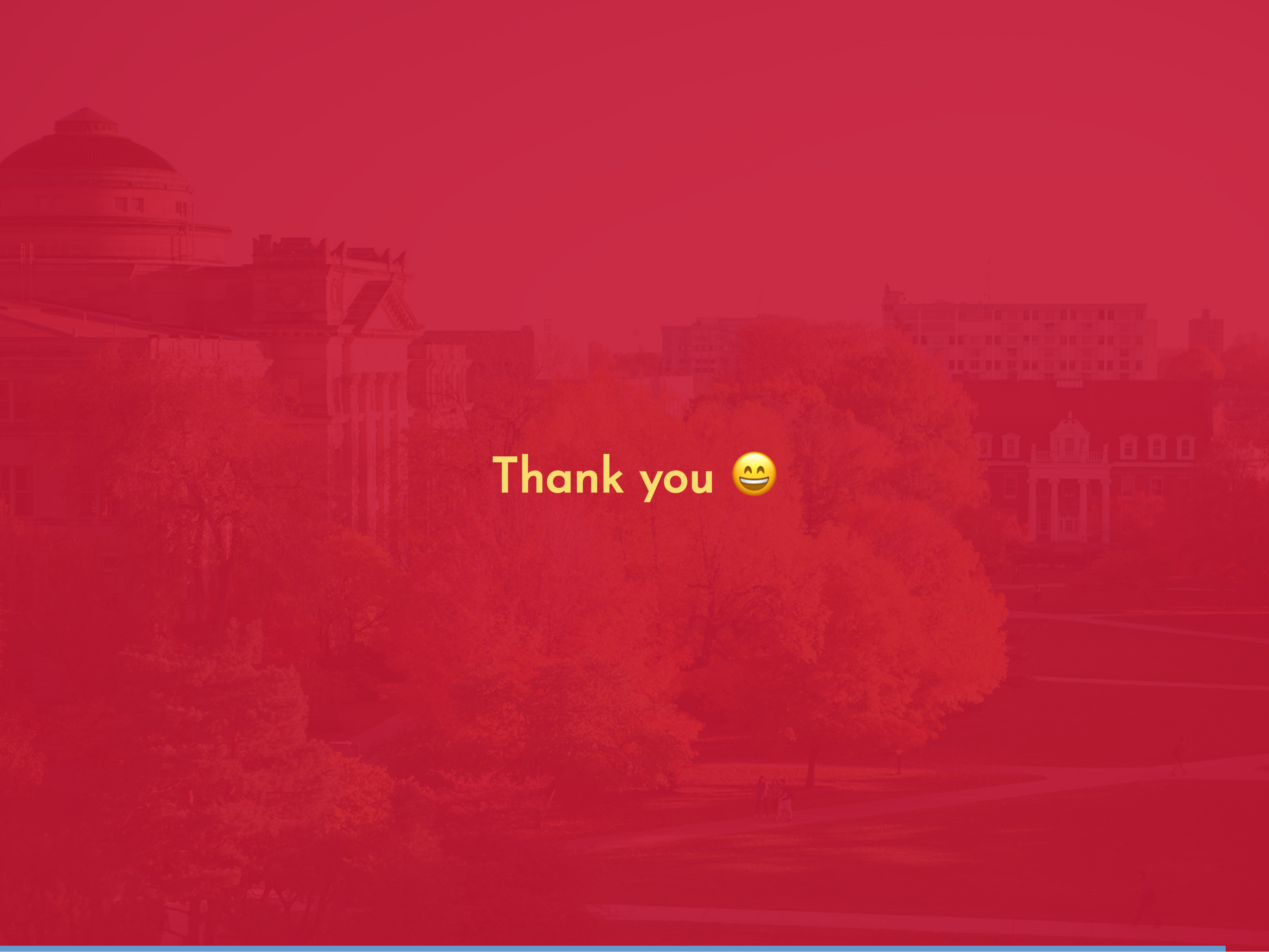
  - Conclusion appears to hold in high-dimensional settings as well

Potential

- Applied on different high-dimensional dataset
- Inference based on cross-validated estimators
  - Sample-splitting?
- Dense theory
  - Relax some conditions?

Questions


Thank you 😄

Reference

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Ning, Y., Sida, P., & Imai, K. (2020). Robust estimation of causal effects via a high-dimensional covariate balancing propensity score. Biometrika, 107(3), 533-554.