Non-Monotone Missingness: The Minimal Variance of the Linear Estimate

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Introduction

The goal of this project is to get an optimal (or near optimal) estimator of a general estimator when the data has a non-monotone missing structure. So far we have reduced the problem to estimating the following:

$$\hat{\theta} = \frac{\delta_{11}}{\pi_{11}} g(Z) + \lambda_0(\delta)\alpha_0(X) + \lambda_1(\delta)\alpha_1(X, Y_1) + \lambda_2(\delta)\alpha_2(X, Y_2). \tag{1}$$

For notation we have variables $Z = (X, Y_1, Y_2)$ that are observed in the segment noted in Table 1.

Table 1: This table matches the segment with its associated observed variables.

Segment	Variables
A_{00}	X
A_{10}	X, Y_1
A_{01}	X, Y_2
A_{11}	X, Y_1, Y_2

For each segment $A_{d_1 d_2}$ the probability of an observation being in a specific segment is known to be $\pi_{d_1 d_2}$, and the associated random variable that indicates whether observation i is in $A_{d_1 d_2}$ is $\delta_{d_1 d_2 i}$. For brevity, it will sometimes be convienent to drop the subscript i. To get an optimal estimator, we need to choose values of λ and α to minimize the variance of $\hat{\theta}$.

Simplifications

To simplify this model consider the case in which each λ *and* α are linear functions. This means that we have the following: $\lambda_0 = \lambda_0^{(0)} + \lambda_1^{(0)}\delta_{00} + \lambda_2^{(0)}\delta_{10} + \lambda_3^{(0)}\delta_{01} + \lambda_4^{(0)}\delta_{11}$, $\lambda_1 = \lambda_0^{(1)} + \lambda_2^{(1)}\delta_{10} + \lambda_4^{(1)}\delta_{11}$, and $\lambda_2 = \lambda_0^{(2)} + \lambda_3^{(2)}\delta_{01} + \lambda_4^{(2)}\delta_{11}$. Also, $\alpha_0 = \alpha_0^{(0)} + \alpha_1^{(0)}x$, $\alpha_1 = \alpha_0^{(1)} + \alpha_1^{(1)}x + \alpha_2^{(1)}y_1$, and $\alpha_2 = \alpha_0^{(2)} + \alpha_1^{(1)}x + \alpha_3^{(1)}y_2$. This means that we can express the estimator in Equation 1 as the following matrix equation:

$$\hat{\theta} = n^{-1}\mathbf{1}_n' \left(\frac{\delta_{11}}{\pi_{11}} g(Z) + (\boldsymbol{\alpha}\boldsymbol{\lambda}'\boldsymbol{\delta})'\mathbf{Z} \right).$$

where

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0^{(0)} & \alpha_0^{(1)} & \alpha_0^{(2)} \\ \alpha_1^{(0)} & \alpha_1^{(1)} & \alpha_1^{(2)} \\ 0 & \alpha_2^{(1)} & 0 \\ 0 & 0 & \alpha_3^{(2)} \end{bmatrix}, \boldsymbol{\lambda} = \begin{bmatrix} \lambda_0^{(0)} & \lambda_0^{(1)} & \lambda_0^{(2)} \\ \lambda_1^{(0)} & 0 & 0 \\ \lambda_2^{(0)} & \lambda_2^{(1)} & 0 \\ \lambda_3^{(0)} & 0 & \lambda_3^{(2)} \\ \lambda_4^{(0)} & \lambda_4^{(1)} & \lambda_4^{(2)} \end{bmatrix}, \boldsymbol{\delta} = \begin{bmatrix} 1 \\ \delta_{00} \\ \delta_{10} \\ \delta_{01} \\ \delta_{11} \end{bmatrix} \mathbf{Z}_i = \begin{bmatrix} 1 \\ X_i \\ Y_{1i} \\ Y_{2i} \end{bmatrix}$$

This is a linear estimator and the goal is to find the optimal $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ that,

$$\text{minimize } \text{Var}(\hat{\theta}) \text{ such that } \boldsymbol{\lambda}' E[\boldsymbol{\delta}] = 0.$$

Simpliying the Variance

We know that

$$\begin{aligned} \text{Var}\left(\frac{\delta_{11}}{\pi_{11}} g(Z)) + (\boldsymbol{\alpha}\boldsymbol{\lambda}'\boldsymbol{\delta})'\mathbf{Z} \right) &= \text{Var}\left(\frac{\delta_{11}}{\pi_{11}} g(Z) \right) + \text{Var}((\boldsymbol{\alpha}\boldsymbol{\lambda}'\boldsymbol{\delta})'\mathbf{Z}) + 2\text{Cov}\left(\frac{\delta_{11}}{\pi_{11}} g(Z), (\boldsymbol{\alpha}\boldsymbol{\lambda}'\boldsymbol{\delta})'\mathbf{Z} \right) \\ &:= V_1 + V_2 + V_3 \end{aligned}$$

Hence, we have for $\Pi := E[\delta]$,

$$
\begin{aligned}
V_1 &= \operatorname{Var}\left(\frac{\delta_{11}}{\pi_{11}} g(Z)\right) \\
&= \operatorname{Var}\left(E\left[\frac{\delta_{11}}{\pi_{11}} g(Z) \mid Z\right]\right) + E\left[\operatorname{Var}\left(\frac{\delta_{11}}{\pi_{11}} g(Z) \mid Z\right)\right] \\
&= E[g^2(Z)] - E[g(Z)]^2 + \frac{1}{\pi_{11}} E[g^2(Z)] - E[g^2(Z)] \\
&= \frac{1}{\pi_{11}} E[g^2(Z)] - E[g(Z)]^2
\end{aligned}
$$

$$
\begin{aligned}
V_2 &= \operatorname{Var}((\alpha\lambda'\delta)'Z) \\
&= \operatorname{Var}(E[(\alpha\lambda'\delta)'Z \mid Z]) + E[\operatorname{Var}((\alpha\lambda'\delta)'Z \mid Z)] \\
&= \Pi'\lambda\alpha' \operatorname{Var}(Z)\alpha\lambda'\Pi + E[Z'\alpha\lambda' \operatorname{Var}(\delta)\lambda\alpha' Z] \\
&= \Pi'\lambda\alpha'(E[ZZ'] - E[Z]E[Z'])\alpha\lambda'\Pi + E[Z'\alpha\lambda'(E[\delta\delta'] - \Pi\Pi')\lambda\alpha' Z] \\
&= \Pi'\lambda\alpha' E[ZZ']\alpha\lambda'\Pi - \Pi'\lambda\alpha' E[Z]E[Z']\alpha\lambda'\Pi + E[Z'\alpha\lambda' E[\delta\delta']\lambda\alpha' Z] - E[Z'\alpha\lambda'\Pi\Pi'\lambda\alpha' Z] \\
&= \Pi'\lambda\alpha' E[ZZ']\alpha\lambda'\Pi - \Pi'\lambda\alpha' E[Z]E[Z']\alpha\lambda'\Pi + E[Z'\alpha\lambda' E[\delta\delta']\lambda\alpha' Z] - \Pi'\lambda\alpha' E[ZZ']\alpha\lambda'\Pi \\
&= \Pi'\lambda\alpha' E[Z]E[Z']\alpha\lambda'\Pi + E[Z'\alpha\lambda' E[\delta\delta']\lambda\alpha' Z] \\
&= E[Z'\alpha\lambda' E[\delta\delta']\lambda\alpha' Z]
\end{aligned}
$$

The third to last equality holds because a 1×1 matrix is always symmetric and the last equality holds because $\Pi'\lambda = (\lambda'\Pi)' = 0'$. We can expand this expression but we get the horribly ugly result:

$X^2\left(\alpha_1^{(0)}\right)^2\left(\lambda_0^{(0)}\right)^2 + 2X^2\left(\alpha_1^{(0)}\right)^2\lambda_0^{(0)}\lambda_1^{(0)}\pi_{00} + 2X^2\left(\alpha_1^{(0)}\right)^2\lambda_0^{(0)}\lambda_2^{(0)}\pi_{10} + 2X^2\left(\alpha_1^{(0)}\right)^2\lambda_0^{(0)}\lambda_3^{(0)}\pi_{01} +$
$2X^2\left(\alpha_1^{(0)}\right)^2\lambda_0^{(0)}\lambda_4^{(0)}\pi_{11} + X^2\left(\alpha_1^{(0)}\right)^2\left(\lambda_1^{(0)}\right)^2\pi_{00} + X^2\left(\alpha_1^{(0)}\right)^2\left(\lambda_2^{(0)}\right)^2\pi_{10} + X^2\left(\alpha_1^{(0)}\right)^2\left(\lambda_3^{(0)}\right)^2\pi_{01} +$
$X^2\left(\alpha_1^{(0)}\right)^2\left(\lambda_4^{(0)}\right)^2\pi_{11} + 2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10} + 2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_0^{(1)} + 2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11} +$
$2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00} + 4X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10} + 2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01} + 2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11} +$
$2X^2\alpha_1^{(0)}\alpha_1^{(1)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_0^{(2)} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11} +$
$2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11} +$
$2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01} + 2X^2\alpha_1^{(0)}\alpha_1^{(2)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11} + 3X^2\left(\alpha_1^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2\pi_{10} + X^2\left(\alpha_1^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2 +$
$2X^2\left(\alpha_1^{(1)}\right)^2\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11} + X^2\left(\alpha_1^{(1)}\right)^2\left(\lambda_4^{(1)}\right)^2\pi_{11} + 2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10} + 2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_0^{(2)} +$
$2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01} + 2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11} + 2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11} + 2X^2\alpha_1^{(1)}\alpha_1^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11} +$
$X^2\left(\alpha_1^{(2)}\right)^2\left(\lambda_0^{(2)}\right)^2 + 2X^2\left(\alpha_1^{(2)}\right)^2\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01} + 2X^2\left(\alpha_1^{(2)}\right)^2\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11} + X^2\left(\alpha_1^{(2)}\right)^2\left(\lambda_3^{(2)}\right)^2\pi_{01} +$
$X^2\left(\alpha_1^{(2)}\right)^2\left(\lambda_4^{(2)}\right)^2\pi_{11} + 2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10} + 2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_0^{(1)} + 2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11} +$

$$2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00} + 4XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10} + 2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01} +$$
$$2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11} + 2XY_1\alpha_1^{(0)}\alpha_2^{(1)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11} + 6XY_1\alpha_1^{(1)}\alpha_2^{(1)}\left(\lambda_0^{(1)}\right)^2\pi_{10} +$$
$$2XY_1\alpha_1^{(1)}\alpha_2^{(1)}\left(\lambda_0^{(1)}\right)^2+4XY_1\alpha_1^{(1)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11}+2XY_1\alpha_1^{(1)}\alpha_2^{(1)}\left(\lambda_4^{(1)}\right)^2\pi_{11}+2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10}+$$
$$2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_0^{(2)}+2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+$$
$$2XY_1\alpha_1^{(2)}\alpha_2^{(1)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_0^{(2)}+2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01}+2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11}+$$
$$2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00} + 2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10} + 2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01} +$$
$$2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11} + 2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01} + 2XY_2\alpha_1^{(0)}\alpha_3^{(2)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11} +$$
$$2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10}+2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}+2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+$$
$$2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+2XY_2\alpha_1^{(1)}\alpha_3^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+2XY_2\alpha_1^{(2)}\alpha_3^{(2)}\left(\lambda_0^{(2)}\right)^2+4XY_2\alpha_1^{(2)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01}+$$
$$4XY_2\alpha_1^{(2)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11} + 2XY_2\alpha_1^{(2)}\alpha_3^{(2)}\left(\lambda_3^{(2)}\right)^2\pi_{01} + 2XY_2\alpha_1^{(2)}\alpha_3^{(2)}\left(\lambda_4^{(2)}\right)^2\pi_{11} +$$
$$2X\alpha_0^{(0)}\alpha_1^{(0)}\left(\lambda_0^{(0)}\right)^2 + 4X\alpha_0^{(0)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_1^{(0)}\pi_{00} + 4X\alpha_0^{(0)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_2^{(0)}\pi_{10} + 4X\alpha_0^{(0)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_3^{(0)}\pi_{01} +$$
$$4X\alpha_0^{(0)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_4^{(0)}\pi_{11}+2X\alpha_0^{(0)}\alpha_1^{(0)}\left(\lambda_1^{(0)}\right)^2\pi_{00}+2X\alpha_0^{(0)}\alpha_1^{(0)}\left(\lambda_2^{(0)}\right)^2\pi_{10}+2X\alpha_0^{(0)}\alpha_1^{(0)}\left(\lambda_3^{(0)}\right)^2\pi_{01}+$$
$$2X\alpha_0^{(0)}\alpha_1^{(0)}\left(\lambda_4^{(0)}\right)^2\pi_{11} + 2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10} + 2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_0^{(1)} + 2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11} +$$
$$2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00} + 4X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10} + 2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01} + 2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11} +$$
$$2X\alpha_0^{(0)}\alpha_1^{(1)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_0^{(2)} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11} +$$
$$2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11} +$$
$$2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01} + 2X\alpha_0^{(0)}\alpha_1^{(2)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10} + 2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_0^{(1)} +$$
$$2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00} + 4X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10} + 2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01} +$$
$$2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(0)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11} + 6X\alpha_0^{(1)}\alpha_1^{(1)}\left(\lambda_0^{(1)}\right)^2\pi_{10} + 2X\alpha_0^{(1)}\alpha_1^{(1)}\left(\lambda_0^{(1)}\right)^2 +$$
$$4X\alpha_0^{(1)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(1)}\left(\lambda_4^{(1)}\right)^2\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10} + 2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_0^{(2)} +$$
$$2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01} + 2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11} + 2X\alpha_0^{(1)}\alpha_1^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11} +$$
$$2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_0^{(2)} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00} +$$
$$2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01} +$$
$$2X\alpha_0^{(2)}\alpha_1^{(0)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10} + 2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_0^{(2)} + 2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01} +$$
$$2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(1)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11} + 2X\alpha_0^{(2)}\alpha_1^{(2)}\left(\lambda_0^{(2)}\right)^2 +$$
$$4X\alpha_0^{(2)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01}+4X\alpha_0^{(2)}\alpha_1^{(2)}\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11}+2X\alpha_0^{(2)}\alpha_1^{(2)}\left(\lambda_3^{(2)}\right)^2\pi_{01}+2X\alpha_0^{(2)}\alpha_1^{(2)}\left(\lambda_4^{(2)}\right)^2\pi_{11}+$$
$$3Y_1^2\left(\alpha_2^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2\pi_{10} + Y_1^2\left(\alpha_2^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2 + 2Y_1^2\left(\alpha_2^{(1)}\right)^2\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11} + Y_1^2\left(\alpha_2^{(1)}\right)^2\left(\lambda_4^{(1)}\right)^2\pi_{11} +$$
$$2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10}+2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}+2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+$$
$$2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+2Y_1Y_2\alpha_2^{(1)}\alpha_3^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10}+2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}+$$
$$2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11}+2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00}+4Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10}+2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01}+$$
$$2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11} + 2Y_1\alpha_0^{(0)}\alpha_2^{(1)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11} + 6Y_1\alpha_0^{(1)}\alpha_2^{(1)}\left(\lambda_0^{(1)}\right)^2\pi_{10} + 2Y_1\alpha_0^{(1)}\alpha_2^{(1)}\left(\lambda_0^{(1)}\right)^2 +$$
$$4Y_1\alpha_0^{(1)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11} + 2Y_1\alpha_0^{(1)}\alpha_2^{(1)}\left(\lambda_4^{(1)}\right)^2\pi_{11} + 2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10} + 2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_0^{(2)} +$$

$2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+2Y_1\alpha_0^{(2)}\alpha_2^{(1)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+$
$Y_2^2\left(\alpha_3^{(2)}\right)^2\left(\lambda_0^{(2)}\right)^2+2Y_2^2\left(\alpha_3^{(2)}\right)^2\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01}+2Y_2^2\left(\alpha_3^{(2)}\right)^2\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11}+Y_2^2\left(\alpha_3^{(2)}\right)^2\left(\lambda_3^{(2)}\right)^2\pi_{01}+$
$Y_2^2\left(\alpha_3^{(2)}\right)^2\left(\lambda_4^{(2)}\right)^2\pi_{11}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_0^{(2)}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11}+$
$2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11}+$
$2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01}+2Y_2\alpha_0^{(0)}\alpha_3^{(2)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11}+2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10}+2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}+$
$2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+2Y_2\alpha_0^{(1)}\alpha_3^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+$
$2Y_2\alpha_0^{(2)}\alpha_3^{(2)}\left(\lambda_0^{(2)}\right)^2+4Y_2\alpha_0^{(2)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01}+4Y_2\alpha_0^{(2)}\alpha_3^{(2)}\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11}+2Y_2\alpha_0^{(2)}\alpha_3^{(2)}\left(\lambda_3^{(2)}\right)^2\pi_{01}+$
$2Y_2\alpha_0^{(2)}\alpha_3^{(2)}\left(\lambda_4^{(2)}\right)^2\pi_{11}+\left(\alpha_0^{(0)}\right)^2\left(\lambda_0^{(0)}\right)^2+2\left(\alpha_0^{(0)}\right)^2\lambda_0^{(0)}\lambda_1^{(0)}\pi_{00}+2\left(\alpha_0^{(0)}\right)^2\lambda_0^{(0)}\lambda_2^{(0)}\pi_{10}+$
$2\left(\alpha_0^{(0)}\right)^2\lambda_0^{(0)}\lambda_3^{(0)}\pi_{01}+2\left(\alpha_0^{(0)}\right)^2\lambda_0^{(0)}\lambda_4^{(0)}\pi_{11}+\left(\alpha_0^{(0)}\right)^2\left(\lambda_1^{(0)}\right)^2\pi_{00}+\left(\alpha_0^{(0)}\right)^2\left(\lambda_2^{(0)}\right)^2\pi_{10}+$
$\left(\alpha_0^{(0)}\right)^2\left(\lambda_3^{(0)}\right)^2\pi_{01}+\left(\alpha_0^{(0)}\right)^2\left(\lambda_4^{(0)}\right)^2\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}\pi_{10}+2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(0)}\lambda_0^{(1)}+$
$2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(0)}\lambda_4^{(1)}\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(1)}\lambda_1^{(0)}\pi_{00}+4\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(1)}\lambda_2^{(0)}\pi_{10}+2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(1)}\lambda_3^{(0)}\pi_{01}+$
$2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_0^{(1)}\lambda_4^{(0)}\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(1)}\lambda_4^{(0)}\lambda_4^{(1)}\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(0)}\lambda_0^{(2)}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(0)}\lambda_3^{(2)}\pi_{01}+$
$2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(0)}\lambda_4^{(2)}\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(2)}\lambda_1^{(0)}\pi_{00}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(2)}\lambda_2^{(0)}\pi_{10}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(2)}\lambda_3^{(0)}\pi_{01}+$
$2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_0^{(2)}\lambda_4^{(0)}\pi_{11}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_3^{(0)}\lambda_3^{(2)}\pi_{01}+2\alpha_0^{(0)}\alpha_0^{(2)}\lambda_4^{(0)}\lambda_4^{(2)}\pi_{11}+3\left(\alpha_0^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2\pi_{10}+$
$\left(\alpha_0^{(1)}\right)^2\left(\lambda_0^{(1)}\right)^2+2\left(\alpha_0^{(1)}\right)^2\lambda_0^{(1)}\lambda_4^{(1)}\pi_{11}+\left(\alpha_0^{(1)}\right)^2\left(\lambda_4^{(1)}\right)^2\pi_{11}+2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}\pi_{10}+$
$2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_0^{(1)}\lambda_0^{(2)}+2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_0^{(1)}\lambda_3^{(2)}\pi_{01}+2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_0^{(1)}\lambda_4^{(2)}\pi_{11}+2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_0^{(2)}\lambda_4^{(1)}\pi_{11}+$
$2\alpha_0^{(1)}\alpha_0^{(2)}\lambda_4^{(1)}\lambda_4^{(2)}\pi_{11}+\left(\alpha_0^{(2)}\right)^2\left(\lambda_0^{(2)}\right)^2+2\left(\alpha_0^{(2)}\right)^2\lambda_0^{(2)}\lambda_3^{(2)}\pi_{01}+2\left(\alpha_0^{(2)}\right)^2\lambda_0^{(2)}\lambda_4^{(2)}\pi_{11}+$
$\left(\alpha_0^{(2)}\right)^2\left(\lambda_3^{(2)}\right)^2\pi_{01}+\left(\alpha_0^{(2)}\right)^2\left(\lambda_4^{(2)}\right)^2\pi_{11}$

Finally, to understand V_3 we can solve for the third covariance term.

$$
\begin{aligned}
&\operatorname{Cov}\left(\frac{\delta_{11}}{\pi_{11}}g(Z),(\alpha\lambda'\delta)'Z\right)\\
&\quad=E\left[\frac{\delta_{11}}{\pi_{11}}g(Z)(\alpha\lambda'\delta)'Z\right]-E\left[\frac{\delta_{11}}{\pi_{11}}g(Z)\right]E[(\alpha\lambda'\delta)'Z]\\
&\quad=E\left[g(Z)\frac{\delta_{11}}{\pi_{11}}\delta'\lambda\alpha'Z\right]-E[g(Z)](\alpha\lambda'\Pi)'E[Z]\\
&\quad=E[g(Z)\begin{bmatrix}1&0&0&0&1\end{bmatrix}\lambda\alpha'Z]\\
&\quad=(\lambda_0^{(0)}+\lambda_4^{(0)})(\alpha_0^{(0)}+\alpha_1^{(0)}E[g(Z)x])+(\lambda_0^{(1)}+\lambda_4^{(1)})(\alpha_0^{(1)}+\alpha_1^{(1)}E[g(Z)x]+\alpha_2^{(1)}E[g(Z)y_1])\\
&\qquad\qquad+(\lambda_0^{(2)}+\lambda_4^{(2)})(\alpha_0^{(2)}+\alpha_1^{(2)}E[g(Z)x]+\alpha_3^{(2)}E[g(Z)y_2])
\end{aligned}
$$

Since we have an understanding of $\operatorname{Var}(\widehat{\theta})$, we can find the minimum by differentiating with respect to each coefficient in λ and α.

This is still a work in progress. I need to run a simulation and test it out. I also believe that the terms $\alpha_0^{(0)}$, $\alpha_0^{(1)}$, and $\alpha_0^{(2)}$ hinder identifiability. But I will work on it.