

Introduction

This document reports the optimality of the proposed non-monotone estimator. It turns out that the proposed estimator is not optimal. We show a better estimator and present a simulation study.

Setup

- Let $U = \{1, \dots, N\}$ be a finite population.
- Consider the random variables $(X_i, Y_{1i}, Y_{2i}, \pi_i) \stackrel{\text{ind}}{\sim} F$ for some unknown F for all $i \in U$.
- We assume that x_i is observed throughout the finite population.
- Let $I_i \stackrel{\text{ind}}{\sim} \pi_i$ for $i \in U$ be the first phase sample inclusion indicator.
- If $i \in \{1, \dots, N\}$ is in the first phase sample ($I_i = 1$) then π_i is observed.
- The second phase sampling indicators are δ_{00} , δ_{01} , δ_{10} , and δ_{11} . These mutual exclusive variables encode the following: if both Y_1 and Y_2 are observed then $\delta_{11} = 1$; if Y_1 is observed and Y_2 is missing then $\delta_{10} = 1$; if Y_1 is missing and Y_2 is observed then $\delta_{01} = 1$; and if both Y_1 and Y_2 are missing then $\delta_{00} = 1$. (For notational purposes, it is sometimes convenient to write $\delta_1 = \delta_{11} + \delta_{10}$ and $\delta_2 = \delta_{11} + \delta_{01}$. A similar notation will be used for π . We have $\pi_{1+} = \pi_{11} + \pi_{10}$ and $\pi_{2+} = \pi_{11} + \pi_{01}$.)
- All of the sampling indicators were drawn via *Poisson* sampling. So the sample size of each category A_{ij} is random. However, this does ensure independence between observations.
- The goal is to estimate $\theta = E[g(X, Y_1, Y_2)]$ in the population.
- The proposed estimator is

$$\begin{aligned}
\hat{\theta}_{\text{eff}} = & n^{-1} \sum_{i=1}^n E[g_i | X_i] \\
& + n^{-1} \sum_{i=1}^n \frac{\delta_{1i}}{\pi_{1+}(X_i)} (E[g_i | X_i, Y_{1i}] - E[g_i | X_i]) \\
& + n^{-1} \sum_{i=1}^n \frac{\delta_{2i}}{\pi_{2+}(X_i)} (E[g_i | X_i, Y_{2i}] - E[g_i | X_i]) \\
& + n^{-1} \sum_{i=1}^n \frac{\delta_{1i}\delta_{2i}}{\pi_{11}(X_i)} (g_i - E[g_i | X_i, Y_{1i}] - E[g_i | X_i, Y_{2i}] + E[g_i | X_i]) \quad (1)
\end{aligned}$$

- The goal is to show that this estimator is optimal within the class of estimators:

$$\begin{aligned}
\hat{\theta}_{\text{eff}} &= n^{-1} \sum_{i=1}^n E[g_i \mid X_i] \\
&+ n^{-1} \sum_{i=1}^n \frac{\delta_{1i}}{\pi_{1+}(X_i)} (b(X_i, Y_{1i}) - E[g_i \mid X_i]) \\
&+ n^{-1} \sum_{i=1}^n \frac{\delta_{2i}}{\pi_{2+}(X_i)} (a(X_i, Y_{2i}) - E[g_i \mid X_i]) \\
&+ n^{-1} \sum_{i=1}^n \frac{\delta_{1i}\delta_{2i}}{\pi_{11}(X_i)} (g_i - b_i - a_i + E[g_i \mid X_i]). \tag{2}
\end{aligned}$$

Results

The first part recaps what Dr. Fuller and I discussed previously.

- Let $g = E[Y_2]$, and suppose that we know the distribution of X and the covariance structure of $[X, Y_1, Y_2, \pi]$. Then instead of modeling the relationship between X and Y_k , we can use difference estimators: $Z_1 \equiv Y_1 - b_1 \tilde{X}$ and $Z_2 \equiv Y_2 - b_2 \tilde{X}$, where $\tilde{X} = X - E[X]$. To minimize the variance of Z_k we can choose the optimal value of b_k , which is

$$b_k = \frac{\text{Cov}(Y_k, X)}{\text{Var}(X)}.$$

Since these covariance values are known, b_k is known. This means that we now have the following table:

	Z_1	Z_2
A_{11}	✓	✓
A_{10}	✓	
A_{01}		✓
A_{00}		

Because we assumed that the distribution of X is known, the section A_{00} contains no additional information about Y_1 or Y_2 . Let $\mu_k = E[Y_k]$.

- We now consider a normal model:

$$\begin{pmatrix} x_i \\ e_{1i} \\ e_{2i} \end{pmatrix} \stackrel{\text{ind}}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{12} & \sigma_{22} \end{bmatrix} \right)$$

and define $y_{1i} = \theta_1 + x_i + e_{1i}$ and $y_{2i} = \theta_2 + x_i + e_{2i}$. Then $b_1 = b_2 = 1$. We define $\bar{z}_k^{(ij)}$ as the mean of y_k in segment A_{ij} . This means that we have means $\bar{z}_1^{(11)}$, $\bar{z}_2^{(11)}$, $\bar{z}_1^{(10)}$, and $\bar{z}_2^{(01)}$. Let $W = [\bar{z}_1^{(11)}, \bar{z}_2^{(11)}, \bar{z}_1^{(10)}, \bar{z}_2^{(01)}]'$, then for $n_{ij} = |A_{ij}|$, we have

$$Z - M\mu \sim N(\vec{0}, V)$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} \frac{\sigma_{11}}{n_{11}} & \frac{\sigma_{12}}{n_{11}} & 0 & 0 \\ \frac{\sigma_{12}}{n_{11}} & \frac{\sigma_{22}}{n_{11}} & 0 & 0 \\ 0 & 0 & \frac{\sigma_{11}}{n_{10}} & 0 \\ 0 & 0 & 0 & \frac{\sigma_{22}}{n_{01}} \end{bmatrix}.$$

Thus, the BLUE for $\mu = [\mu_1, \mu_2]'$ is

$$\hat{\mu} = (M'V^{-1}M)^{-1}M'V^{-1}W. \quad (3)$$

- Since this is the BLUE, we would expect it to be at least as good as the proposed estimator. And if the proposed estimator is optimal it should be equivalent to the BLUE in the case that X , Y_1 and Y_2 are normal. So I ran a simulation study to test this. For $i = \{1, \dots, n = 1000\}$,

$$\begin{pmatrix} x_i \\ e_{1i} \\ e_{2i} \end{pmatrix} \stackrel{\text{ind}}{\sim} N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \sigma_{12} \\ 0 & \sigma_{12} & 1 \end{bmatrix} \right) \text{ and } y_{1i} = \theta_1 + x_i + e_{1i}, y_{2i} = \theta_2 + x_i + e_{2i}.$$

Each observation i was then assigned a segment A_{11} , A_{10} , A_{01} or A_{00} independently with each draw having probability $p_{11} = 0.4$, $p_{10} = 0.2$, $p_{01} = 0.2$ and $p_{00} = 0.2$ respectively. This means that $\pi_{11} = 0.4$ and $\pi_{1+} = 0.6 = \pi_{2+}$. We let $g = E[Y_2]$ and we test the following estimators:

- Oracle: This computes the average value of Y_2 in all observations (even when Y_2 is not supposed to be observed).

$$\hat{\theta} = n^{-1} \sum_{i=1}^n y_{2i}.$$

- OracleX: This computes the average value of $Y_2 - X$ in all observations (even when Y_2 is not supposed to be observed.)

$$\hat{\theta} = n^{-1} \sum_{i=1}^n (y_{2i} - x_i).$$

- CC: This computes the average value of Y_2 in all observations in which Y_2 is observed.

$$\hat{\theta} = \frac{\sum_{i=1}^n \delta_{2i} y_{2i}}{\sum_{i=1}^n \delta_{2i}}.$$

- Proposed: This is the proposed estimator from Equation 1.
- WLS: This is the weighted linear estimator from Equation 3. Note, since $g = E[Y_2]$ this only contains the second element from Equation 3.

The results are shown in the table below. This simulation was run with the number of observations $n = 1000$ and the Monte Carlo sample size of $B = 3000$.

Table 1: This table shows the estimators of $\theta = E[Y_2]$. The true value of θ is 5 and the true value of $\sigma_{12} = 0.5$. The bias column shows the average bias of the estimator and the actual value of $\theta = 5$ across the $B = 3000$ simulations. The SD column shows the average standard deviation across the $B = 3000$ simulations for each algorithm. The Tstat column displays the t-statistic of a t-test comparing the estimator to the actual value. The value of this column is computed via $\frac{\bar{\hat{\theta}} - \theta}{\sqrt{\text{Var}\hat{\theta}/B}}$. The Pval column displays the p-value of the t-statistic.

Algorithm	Bias	SD	Tstat	Pval
Oracle	0.001	0.044	1.546	0.061
OracleX	0.000	0.032	0.479	0.316
CC	0.002	0.058	1.549	0.061
WLS	0.001	0.040	1.207	0.114
Prop	0.002	0.051	1.918	0.028

Computing the Variance by Hand

Table demonstrates that the WLS estimator outperforms the Oracle estimator. I was initially confused about why this is the case, but realized that I could approximate the standard error of both estimators reasonably well. It turns out that I was able to get reasonable estimates for the variance of all of the proposed estimators including $\hat{\theta}_{prop}$. In this part of the report, I detail my computations and show how the simulated results are accurate.

$$\begin{aligned}
\text{Var}(\hat{\theta}_{Oracle}) &= \text{Var}\left(n^{-1} \sum_{i=1}^n y_{2i}\right) \\
&= n^{-1} \text{Var}(y_2) \\
&= n^{-1}(1 + \sigma_{22}) && \text{because } \text{Var}(x) = 1.
\end{aligned} \tag{4}$$

$$\begin{aligned}
\text{Var}(\hat{\theta}_{OracleX}) &= \text{Var}\left(n^{-1} \sum_{i=1}^n y_{2i} - x_i\right) \\
&= n^{-1} \text{Var}(e_2) \\
&= n^{-1} \sigma_{22}.
\end{aligned} \tag{5}$$

Using a Taylor expansion,

$$\begin{aligned}
\text{Var}(\hat{\theta}_{cc}) &= \text{Var}\left(\theta_2 + \pi_{2+}^{-1} \left(n^{-1} \sum_{i=1}^n \delta_{2i} y_{2i} - \pi_{2+} \theta_2\right) - \frac{\theta_2 \pi_{2+}}{\pi_{2+}^2} \left(n^{-1} \sum_{i=1}^n \delta_{2i} - \pi_{2+}\right) + o_p(1)\right) \\
&\quad \text{(I can add more steps later if necessary.)}
\end{aligned} \tag{6}$$

$$= \frac{1 + \sigma_{22}}{n\pi_{2+}} + o_p(1). \tag{7}$$

To compute the variance of $\hat{\theta}_{WLS}$, we have,

$$\begin{aligned}
\text{Var}(\hat{\theta}_{WLS}) &= \text{Var}((M'V^{-1}V)^{-1}M'V^{-1}Z) \\
&= E[\text{Var}((M'V^{-1}V)^{-1}M'V^{-1}Z \mid n_1)] \\
&= E[(M'V^{-1}M)]
\end{aligned} \tag{8}$$

$$\tag{9}$$

By Slutsky's Theorem, $\text{Var}(\hat{\theta}_{WLS}) \xrightarrow{P} (M'E[V]^{-1}M)^{-1}$.

Finally, we compute the variance for the proposed estimator. First, we estimate the variance for a similar estimator $\tilde{\theta}_{prop}$. This estimator is the same as the proposed estimator except that we assume that the estimates from $E[Y_{2i} \mid X, Y_1]$ and $E[Y_{2i} \mid X]$ are population coefficients and independent of the rest of the sample. This could occur if there is a separate sample in which we can estimate these expectations. Since the population coefficients of the regression $Y = \beta_0 + \beta_1 X$ are

$$\beta_0 = E[Y] - E[X]\text{Var}(X)^{-1}\text{Cov}(X'Y) \text{ and } \beta_1 = \text{Var}(X)^{-1}\text{Cov}(X, Y),$$

we have that,

$$E[Y_2 \mid X] = E[Y_2] - \beta_1 E[X] + \beta_1 x = \theta_2 - x$$

and

$$\begin{aligned}
E[Y_2 | X, Y_1] &= \beta_0 + [X, Y_1][\beta_1 \beta_2]' \\
&= E[Y_2] - E[[X, Y_1]]\text{Var}([X, Y_1])^{-1}\text{Cov}([X, Y_1], Y_2) + [X, Y_1]\text{Var}([X, Y_1])^{-1}\text{Cov}([X, Y_1], Y_2) \\
&= \theta_2 + [X, Y_1]\sigma_{11}^{-1} \begin{bmatrix} 1 + \sigma_{11} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 + \sigma_{12} \end{bmatrix} \\
&= \theta_2 + x \left(\frac{\sigma_{11} - \sigma_{12}}{\sigma_{11}} \right) + y_1 \left(\frac{\sigma_{12}}{\sigma_{11}} \right) \\
&= \theta_2 + x(1 - \sigma_{12}) + y_1(\sigma_{12}).
\end{aligned}$$

This means that

$$E[Y_2 | X, Y_1] - E[Y_2 | X] = \sigma_{12}(y_1 - x) = \sigma_{12}e_1.$$

Hence, whereas for $\theta = E[Y_2]$,

$$\hat{\theta}_{prop} = n^{-1} \sum_{i=1}^n \left\{ E[Y_2 | X] + \left(\frac{\delta_{1i}}{\pi_{1+}} - \frac{\delta_{11i}}{\pi_{11}} \right) (E[Y_2 | X, Y_1] - E[Y_2 | X]) + \frac{\delta_{2i}}{\pi_{2+}} (y_{2i} - E[Y_2 | X]) \right\}$$

for the similar estimator we have

$$\tilde{\theta}_{prop} = n^{-1} \sum_{i=1}^n \left(\theta_2 + x_i + \left(\frac{\delta_{1i}}{\pi_{1+}} - \frac{\delta_{11i}}{\pi_{11}} \right) \sigma_{12}e_{1i} + \frac{\delta_{2i}}{\pi_{2+}} (e_{2i}) \right)$$

This means that

$$\begin{aligned}
\text{Var}(\tilde{\theta}_{prop}) &= n^{-1} \text{Var} \left(x + \left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \sigma_{12}e_1 + \frac{\delta_2}{\pi_{2+}}(e_2) \right) \\
&= n^{-1} \left(\text{Var}(x) + \text{Var} \left(\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \sigma_{12}e_1 \right) + \text{Var} \left(\frac{\delta_2}{\pi_{2+}}(e_2) \right) \right. \\
&\quad \left. + 2\text{Cov} \left(\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \sigma_{12}e_1, \frac{\delta_2}{\pi_{2+}}(e_2) \right) \right) \\
&\equiv A + B + C + D.
\end{aligned}$$

$$A = \text{Var}(x) = 1.$$

$$\begin{aligned}
B &= \sigma_{12}^2 \sigma_{11} \text{Var} \left(\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \right) \\
&= \sigma_{12}^2 \sigma_{11} (\text{Var}(\delta_1/\pi_{1+}) + \text{Var}(\delta_{11}/\pi_{11}) - 2\text{Cov}(\delta_1/\pi_{1+}, \delta_{11}/\pi_{11})) \\
&= \sigma_{12}^2 \sigma_{11} (\pi_{1+}^{-1} + \pi_{11}^{-1} - 2 - 2(\pi_{11} - \pi_{1+}\pi_{11})/(\pi_{1+}\pi_{11})) \\
&= \sigma_{12}^2 \sigma_{11} (\pi_{11}^{-1} - \pi_{1+}^{-1}).
\end{aligned}$$

$$\begin{aligned}
C &= \text{Var}(\delta_2 e_2 / \pi_{2+}) \\
&= \pi_{2+}^{-2} (\text{Var}(\delta_2) \text{Var}(e_2) + \text{Var}(\delta_2) E[e_2]^2 + \text{Var}(e_2) E[\delta_2]^2) \\
&= \pi_{2+}^{-2} (\pi_{2+}(1 - \pi_{2+}) \sigma_{22} + \sigma_{22} \pi_{2+}^2) \\
&= \sigma_{22} / \pi_{2+}.
\end{aligned}$$

$$\begin{aligned}
D &= 2\text{Cov} \left(\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \sigma_{12} e_1, \frac{\delta_2}{\pi_{2+}}(e_2) \right) \\
&= 2E \left[\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \sigma_{12} e_1 \frac{\delta_2}{\pi_{2+}}(e_2) \right] \\
&= 2E \left[\left(\frac{\delta_1}{\pi_{1+}} - \frac{\delta_{11}}{\pi_{11}} \right) \frac{\delta_2}{\pi_{2+}} \right] \sigma_{12}^2 \\
&= 2E \left[\frac{\delta_{11}}{\pi_{1+}\pi_{2+}} - \frac{\delta_{11}}{\pi_{2+}} \right] \sigma_{12}^2 \\
&= 2 \frac{\pi_{11}}{\pi_{2+}} (\pi_{1+}^{-1} - 1) \sigma_{12}^2.
\end{aligned}$$

Hence,

$$\text{Var}(\tilde{\theta}_{prop}) = n^{-1} (1 + \sigma_{12}^2 \sigma_{11} (\pi_{11}^{-1} - \pi_{1+}^{-1}) + \sigma_{22} \pi_{2+}^{-1} 2\sigma_{12}^2 (\pi_{1+}^{-1} - 1) \pi_{11} / \pi_{2+}). \quad (10)$$

Since $\hat{\beta} - \beta = O(n)$ from Theorem 2.2.1 of [1], $\text{Var}(\hat{\theta}_{prop}) \rightarrow \text{Var}(\tilde{\theta}_{prop})$. (I think that I also need to show that the dependency between the estimated coefficients and the observed data is bounded and that this bound goes to zero as the sample size increases.)

These estimated variances are almost exactly in line with the estimated standard deviations from the simulation study. With values of $n = 1000$, $\sigma_{11} = 1$, $\sigma_{22} = 1$, $\sigma_{12} = 0.5$, $\theta_2 = 5$, $\pi_{11} = 0.4$, $\pi_{10} = 0.2$, $\pi_{01} = 0.2$, and $\pi_{00} = 0.2$ the standard deviations from the previous calculations and from the Monte Carlo simulation are listed in Table 2:

References

- [1] Wayne A Fuller. *Sampling statistics*. John Wiley & Sons, 2009.

Table 2: This table compares the calculated and estimated standard errors for each of the algorithms listed. The calculated values are computed as the square root of the variance derived in the previous section. We use Equations 4, 5, 6 8, and 10 to calculate the variance of Oracle, OracleX, CC, WLS, and Prop respectively and then take the square root of the answer. For the estimated standard error, we take the standard deviation of the B estimators for each algorithm.

Algorithm	Calculated Standard Error	Estimated Standard Error
Oracle	0.045	0.044
OracleX	0.032	0.032
CC	0.058	0.058
WLS	0.040	0.040
Prop	0.056	0.051