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Enumeration of Mapping Patterns

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Pólya's enumeration theorem is generalized in the following way. We have sets R and D, and a group G acting (by means of representations) on R and D simultaneously. This induces an equivalence relation in the set of all mappings (or of all one-to-one mappings) of R into D. The number of equivalence classes is determined for both cases. The example of types of mappings of a set into itself is treated in detail.

1. INTRODUCTION

Let G be a finite group. Let χ be a representation of G by means of permutations of a finite set D, and let ζ be a representation of G by means of permutations of a finite set R. The set of all mappings of D into R is denoted by R^D. The set of all one-to-one mappings of D into R is denoted by R_D. (So $R_D \subseteq R^D$.)

In R^D we consider the following equivalence relation $E_{\chi,\zeta}$. If $f_1 \in R^D$, $f_2 \in R^D$ we say that f_1 and f_2 are equivalent if there exists a $\gamma \in G$ such that $\zeta(\gamma) f_1 = f_2 \chi(\gamma)$.

The equivalence classes will be called (χ, ζ)-*patterns.*

If a (χ, ζ)-pattern contains an f that maps D one-to-one into R, then all functions in that pattern are one-to-one. In that case the pattern will be called an *injective* (χ, ζ)-pattern.

We shall be interested in the *number* of (χ, ζ)-patterns and in the *number* of injective (χ, ζ)-patterns.

These questions form a common generalization of some questions that have been treated previously [1]:

(i) If K, H are groups of permutations of D and R, respectively, and if $G = K \times H$ is the direct product of K and H, then we can take χ, ζ as the projections onto K and H:

$$\chi(\kappa, \eta) = \kappa, \qquad \zeta(\kappa, \eta) = \eta \qquad (\kappa \in K, \eta \in H).$$

With this choice of χ and ζ, two mappings f_1, f_2 are equivalent if $\kappa \in K$,

$\eta \in H$ exist such that $\eta f_1 = f_2 \kappa$. We shall return to this case in Example 1 of Section 3.

(ii) Let G be a group of permutations of the finite set D. Two permutations f_1, f_2 of D are called equivalent if $\gamma \in G$ exists such that $f_2 = \gamma f_1 \gamma^{-1}$. The number of equivalence classes was evaluated in [1, Theorem 4]. These classes are injective patterns in the following special situation: we take $R = D$ and we define χ and ζ by $\chi(\gamma) = \zeta(\gamma) = \gamma$ for all $\gamma \in G$.

NOTATION. If S is a finite set then $| S |$ is the number of its elements. If κ is a permutation of the finite set D, and if j is a positive integer, then $c(\kappa, j)$ denotes the number of cycles with length j. Hence $\sum_{j=1}^{\infty} c(\kappa, j) = | D |$. If, moreover, $x_1, x_2, \ldots$ are variables, we write

$$Z(\kappa; x_1, x_2, \ldots) := x_1^{c(\kappa,1)} x_2^{c(\kappa,2)} \cdots .$$

If χ and ζ are representations of a group G by means of permutations of finite sets D and R, respectively, and if we have two sets of variables: $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, then we define

$$\begin{aligned} &T(\chi, \zeta; x_1, x_2, \ldots; y_1, y_2, \ldots): \\ &\qquad = | G |^{-1} \sum_{\gamma \in G} Z(\chi(\gamma); x_1, x_2 \ldots.) \, Z(\zeta(\gamma); y_1, y_2, \ldots). \end{aligned}$$

This notation is taken from [3], where the expression T (with an arbitrary number of representations instead of two) was used for counting "vector mapping patterns."

For the sake of completeness we give the definition of the cycle index of a permutation group G:

$$P_G(x_1, x_2, \ldots) := | G |^{-1} \sum_{\gamma \in G} Z(\gamma; x_1, x_2, \ldots).$$

2. ENUMERATION OF (χ, ζ)-PATTERNS

THEOREM 1. *The number of (χ, ζ)-patterns is the value of*

$$T\left(\chi, \zeta; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, e^{z_1 + z_2 + \cdots}, e^{2(z_2 + z_4 + \cdots)}, e^{3(z_3 + z_6 + \cdots)}, \ldots\right)$$

at the point $z_1 = z_2 = \cdots = 0$.

(It has to be agreed that in $T(\chi, \zeta; x_1, x_2, \ldots; y_1, y_2, \ldots)$ all x_i's are

written in front *of all y_j's, in order to comply with the convention to write differential operators in front of the things they act on.*)

Proof. Let Q be an equivalence class (under the equivalence $E_{\chi,\zeta}$). By Burnside's lemma (see [2, Section 5.3]) we have

$$1 = |G|^{-1} \sum_{\gamma \in G} |S_{\gamma,Q}|,$$

where

$$S_{\gamma,Q} := \{f \mid f \in Q, \zeta(\gamma) f = f\chi(\gamma)\}.$$

If we sum with respect to Q we get for the number of (χ, ζ)-patterns

$$|G|^{-1} \sum_{\gamma \in G} |\{f \mid f \in R^D, \zeta(\gamma) f = f\chi(\gamma)\}| \tag{1}$$

If g is a permutation of D, and h a permutation of R then we agree

$$N(g, h) := |\{f \mid f \in R^D, hf = fg\}|. \tag{2}$$

We can write (1) as

$$|G|^{-1} \sum_{\gamma \in G} N(\chi(\gamma), \zeta(\gamma)). \tag{3}$$

The value of $N(g, h)$ is easily evaluated (see [2, Section 5.12]). If we abbreviate $b_j := c(\chi(\gamma), j)$, $c_j := c(\zeta(\gamma), j)$, we obtain

$$\begin{aligned} N(\chi(\gamma), \zeta(\gamma)) &= \prod_{i=1}^{\infty} \Big(\sum_{j \mid i} (jc_j)^{b_i}\Big) \\ &= c_1^{b_1}(c_1 + 2c_2)^{b_2} (c_1 + 3c_3)^{b_3} (c_1 + 2c_2 + 4c_4)^{b_4} \cdots \end{aligned}$$

(powers with exponent 0 have to be interpreted as 1). We can also write this as

$$\begin{aligned} &\left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \left(\frac{\partial}{\partial z_3}\right)^{b_3} \cdots \exp\Big(\sum_{j=1}^{\infty} jc_j(z_j + z_{2j} + z_{3j} + \cdots)\Big) \\ &= Z\left(\chi(\gamma), \zeta(\gamma); \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots; e^{z_1+z_2+\cdots}, e^{2(z_2+z_4+\cdots)}, e^{3(z_3+z_6+\cdots)}, \ldots\right) \end{aligned}$$

(evaluated at $z_1 = z_2 = \cdots = 0$).

Now summing for γ, and dividing by $|G|$, we infer that (3) equals the expression occurring in the theorem.

THEOREM 2. *The number of injective (χ, ζ)-patterns is the value of*

$$T\left(\chi, \zeta; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots; 1 + z_1, 1 + 2z_2, 1 + 3z_3, \ldots\right)$$

evaluated at $z_1 = z_2 = \cdots = 0$ (as in Theorem 1, the differential operators are to be written in front).

Proof. The only difference from the proof of Theorem 1 is that instead of (2) we have to deal with

$$M(g, h) := |\{f \mid f \in R_D, hf = fg\}|,$$

for which we obtain (cf. [2, Section 5.10])

$$\begin{aligned} M(\chi(\gamma), \zeta(\gamma)) &= \prod_{j=1}^{\infty} (j^{b_j} c_j(c_j - 1) \cdots (c_j - b_j + 1)) \\ &= \left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \left(\frac{\partial}{\partial z_3}\right)^{b_3} \cdots \\ &\qquad (1 + z_1)^{c_1} (1 + 2z_2)^{c_2} (1 + 3z_3)^{c_3} \cdots . \end{aligned}$$

3. EXAMPLES

1. If we take the situation indicated in the introduction under (i), we have

$$T(\chi, \zeta; x_1, x_2, \ldots; y_1, y_2, \ldots) = P_K(x_1, x_2, \ldots) P_H(y_1, y_2, \ldots),$$

where P_K and P_H are the cycle indices. Now Theorems 1 and 2 turn into known theorems, viz., Theorems 5.4 and 5.2 of [2]. (These theorems are special cases of a more general theorem (Theorem 1 in [1]).) That general theorem can, of course, also be generalized to the case of (χ, ζ)-patterns.

2. If we take the situation indicated in the introduction under (ii), we have, in terms of the cycle index P_G,

$$T(\chi, \zeta; x_1, x_2, \ldots; y_1, y_2, \ldots) = P_G(x_1 y_1, x_2 y_2, \ldots).$$

It follows that the number of patterns equals

$$\sum_{\gamma \in G} \prod_{j=1}^{\infty} j^{c(\gamma, j)} (c(\gamma, j))!$$

(see the proof of Theorem 2, with $b_j = c_j = c(\gamma, j)$). This result was obtained in [1, Theorem 4], where it was also expressed as

$$\int_0^\infty \cdots \int_0^\infty \exp(-(x_1 + \cdots + x_m))\, P_G(x_1, 2x_2, 3x_3, \ldots)\, dx_1 \cdots dx_m$$
$$= \left\{ P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots\right) \times (1 - z_1)^{-1}(1 - 2z_2)^{-1}(1 - 3z_3)^{-1} \cdots \right\}_{z_1 = z_2 = \ldots = 0}$$

with $m := |D|$.

3. Let us consider a finite set D and a group G of permutations of D. Two mappings f_1, f_2 of D into itself will be called equivalent if $\gamma \in G$ exists such that $\gamma f_1 \gamma^{-1} = f_2$. We ask for the number of equivalence classes.

The situation arises from the one considered in Theorem 1 if we take $R = D$, and, as in the previous example, χ and ζ such that $\chi(\gamma) = \zeta(\gamma) = \gamma$ for all $\gamma \in G$. For the number of classes we obtain

$$|G|^{-1} \sum_{\gamma \in G} \prod_{i=1}^{\infty} \sum_{j|i} (jc_j)^{c_i},$$

where $c_j = c(\gamma, j)$. In particular, if we take for G the symmetric group (the group of all permutations of D), we shall refer to the equivalence classes as *mapping types*, and we get as the number S_K of mapping types on a set D with $|D| = k$:

$$S_k = \sum_{(n)} \prod_{i=1}^{\infty} \left(\sum_{j|i} (jn_j)\right)^{n_i} i^{-n_i}/n_i!,$$

where the first summation is over all sequences $n_1, n_2, \ldots$ with

$$n_1 + 2n_2 + 3n_3 + \cdots = k.$$

We give the first values here:

$S_1 = 1,\quad S_2 = 3,\quad S_3 = 7,\quad S_4 = 19,\quad S_5 = 47,\quad S_6 = 130,$
$S_7 = 343,\quad S_8 = 951,\quad S_9 = 2615,\quad S_{10} = 7318,$
$S_{11} = 20491,\quad S_{12} = 57903,\quad S_{13} = 163898,$
$S_{14} = 466199,\quad S_{15} = 1328993.$

We briefly refer to an entirely different method for counting the mapping types. A mapping of D into itself can be considered as a directed graph with vertex set D, possibly with loops. A mapping is called *primitive* if its graph is connected, and we shall use the same adjective for mapping types consisting of primitive mappings. Let p_k denote the number of primitive mapping types. Then we can derive

$$\sum_{k=0}^{\infty} S_k x^k = \prod_{j=1}^{\infty} (1 - x^j)^{-p_j}$$

(with $S_0 = 1$), by remarking that a mapping type on a set with k elements corresponds to a non-negative integer-valued function on the set of all primitive mapping types (the values of the function indicate how often the respective primitive mapping types occur in the given mapping type).

It remains to show how to compute p_k. A primitive mapping type can be described as a cycle (possibly of length 1) with oriented trees growing on the respective points of the cycle (the orientation in the trees is toward the cycle).

Let t_k denote the number of topologic rooted trees with k points, and $T(x) = \sum_1^{\infty} t_k x^k$. Thus

$$T(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + \cdots,$$

a function whose coefficients can be determined by means of Cayley's functional equation (see [4])

$$T(x) = x \prod_{j=1}^{\infty} (1 - x^j)^{-t_j}.$$

Now Pólya's fundamental theorem gives us the number of primitive mapping types whose cycle has length m. This number is the number of patterns of mapping a cycle of m elements into the set of all topologic trees. The counting series is

$$P_{Z_m}(T(x), T(x^2), T(x^3), \ldots),$$

where Z_m stands for the cyclic group of order m:

$$P_{Z_m} = m^{-1} \sum_{d \mid m} \varphi(d)(x_d)^{m/d}$$

(φ denotes Euler's totient).

Finally we obtain

$$\sum_{k=1}^{\infty} p_k x^k = \sum_{m=1}^{\infty} P_{Z_m}(T(x), T(x^2), \ldots).$$

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