Numerical Stability

1. Example that is the basis of stability analysis: $\dot{x} = \alpha x$

$$f = \alpha x, \quad \alpha = \alpha_r + i\alpha_i, \text{ Let } x_0 = A \text{ be i.c. :}$$

Exact: $f = Ae^{\alpha t}$

Numerical $x_{n+1} = (1+\alpha\Delta t)\, x_n$

$$x_0 = A$$
$$x_1 = (1+\alpha\Delta t)x_0 = (1+\alpha\Delta t)A$$
$$x_2 = (1+\alpha\Delta t)x_1 = (1+\alpha\Delta t)^2 A$$
$$x_n = (1+\alpha\Delta t)^n A$$

depends only on $\alpha\,\Delta t$

Stable example: $\alpha_i = 0,\ \alpha_r\Delta t = -0.1,\ A = 1$

$$x_1 = 0.90 \text{ exact } 0.904$$
$$x_2 = 0.81 \text{ exact } 0.819$$

etc.

Error over 1 interval $Error = Ae^{\alpha\Delta t} - A(1+\alpha\Delta t) \approx \tfrac{1}{2} A(\alpha\Delta t)^2$

First order accurate: error per time step $\tfrac{1}{2}(\alpha\Delta t)^2 = 0.005$

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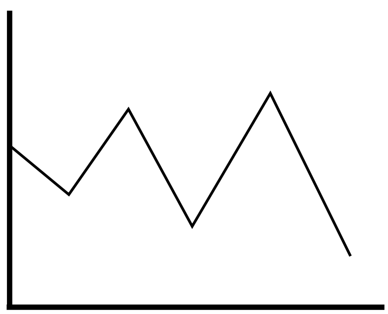
Unstable example: $\alpha_i = 0$, $\alpha_r\Delta t = -2.1$, $A = 1$



$x_1 = -1.1$ exact 0.12
$x_2 = 1.21$ exact 0.015
$x_3 = -1.331$ ...
etc.

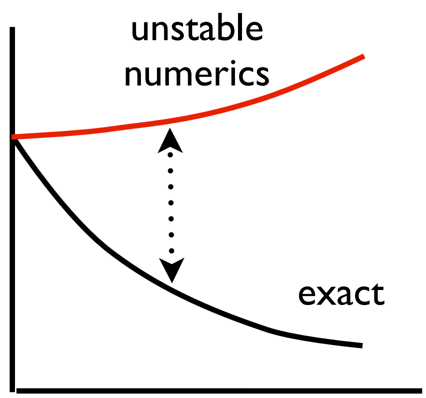
error *amplifies*

Exact amplification $\left|\dfrac{e^{\alpha(t+\Delta t)}}{e^{\alpha t}}\right| = e^{\alpha_r\Delta t} = e^{-2.1} < 1$

Stable method should also give amplification < 1

2. Introduction to stability analysis

Stability is a question about the behavior of the **error** --- not of the **solution**



Let $\alpha=\alpha_r+i\alpha_i$ where $i=\sqrt{-1}$

Magnitude of complex number

$|\alpha|^2 = \alpha_r^2 + \alpha_i^2 = (\alpha_r + i\alpha_i)(\alpha_r - i\alpha_i) = \alpha \times \alpha^*$

Hence $|e^\alpha|^2 = e^\alpha e^{\alpha^*} = e^{2\alpha_r}$

With $\alpha_r < 0$ the correct solution is damped. Numerics can cause spurious growth (e.g. Δt too large)

Computational
$$\left|\frac{x_{n+1}}{x_n}\right|^2 = |1+\alpha\Delta t|^2 = (1+\alpha_r\Delta t)^2 + (\alpha_i\Delta t)^2$$

Want this to be < 1
$$\Rightarrow \Delta t < \frac{-2\alpha_r}{\alpha_r^2+\alpha_i^2}$$

N.B.: $\alpha_r < 0$

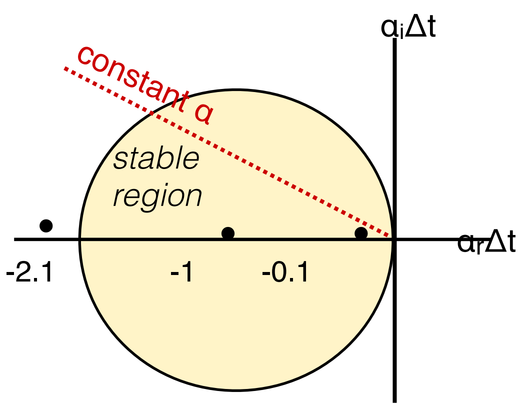
$\alpha_r = 0$ is unconditionally unstable; need damping

Generally, explicit methods have time-step restriction

$\alpha_i=0$ is stable if $0 > \alpha_r \Delta t > -2$; time step restriction $\Delta t < -2/\alpha_r$

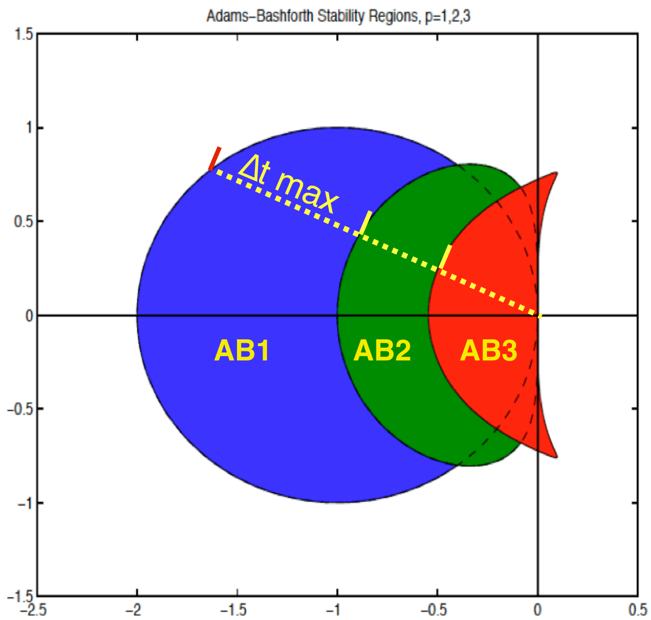
Geometric view: Stability criterion: $|1+\alpha\Delta t| < 1$

The equation of a circle is $(x+x_0)^2+ (y+y_0)^2 = R^2$; $(1+\alpha_r \Delta t)^2+ (\alpha_i \Delta t)^2 = 1$ is unit circle centered at (-1,0)

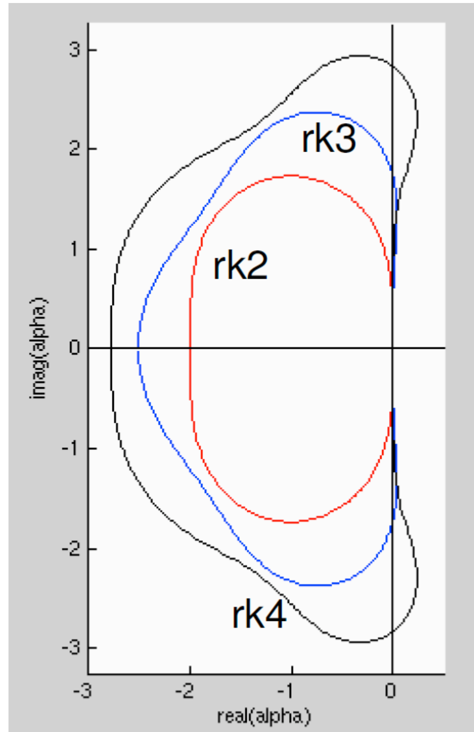


Sometimes called 'unstable for convection' because imaginary axis is in unstable region

3. Stability of RK and AB methods



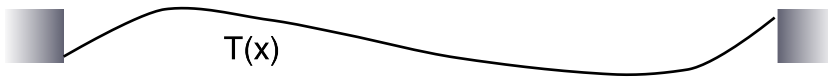
Higher order cross imaginary axis, but need smaller time-step



Cross imaginary axis without reduced time-step

Boundary value problems

solve d.e. given T(0), T(1); say T_0 and T_1. (Lead-in to implicit methods)



E.g., $d^2T/dx^2 - T = 0$: second order, need 2 data values.

Aside: *Exact solution*

$$T = A\cosh x + B\sinh x ;$$

NB: cosh(0)=1, sinh(0)=0 and cosh'(0)=0, sinh'(0)=1

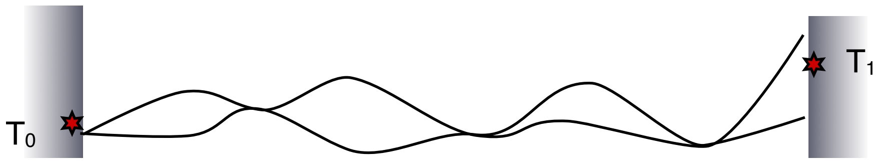
$$T = T_0\cosh(x) + [T_1 - T_0\cosh(1)]\sinh(x)/\sinh(1)$$

How do to this numerically?

Aside: For linear equations can solve with $T(0)=1$, $T'(0)=0$ ($\rightarrow \tilde{T}_1(x)$), then with $T(0)=0$, $T'(0)=1$ ($\rightarrow \tilde{T}_2(x)$) and take linear combination:

$$T = T_0\,\tilde{T}_1(x) + (T_1 - T_0\,\tilde{T}_1(1))\tilde{T}_2(x)/\tilde{T}_2(1)$$

Not OK for non-linear equations: solution is not sum of two independent solutions



Skip shooting methods

1. Shooting method -- OK for non-linear equations. Uses marching method for o.d.e.'s

```
T = T0
TP = G
DO x=dx,1,dx (or i=1,N)
  CALL RK(T,TP,x)
ENDDO
```

$T = T_1$? i.e., $||T - T_1||/T_1 < \varepsilon$? If no, then new quess; if yes, done.

How to guess: bi-section or Newton's method (root finding routine) form is $T(N \mid G) - T_1$ is of form $f(x)=0$, find x where x=G.

a. Bi-section: carry two estimates f(G1)<0 f(G2)>0 (or T(1;G1) and T(1;G2)).

DO WHILE $|G_1 - G_2| > \varepsilon$

Let $\tilde{G} = (G_1 + G_2)/2$

IF $T(1;\tilde{G}) < 0$ $\quad G_1 = \tilde{G}$, $G_2 = G_2$

IF $T(1;\tilde{G}) > 0$ $\quad G_1 = G_1$, $G_2 = \tilde{G}$

ENDDO

Print solution T(x)

iteration

G1 Ĝ G2 sign f(Ĝ)

+

G1 Ĝ G2

-

G1 Ĝ G2

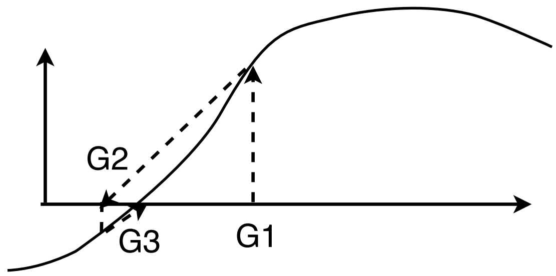
+

G1 G2

Skip shooting methods

Slow but robust. (Guess and improve; bounding box).

b. Newton's method: (Gradient based) Let $f(G) = T(1;G)-T_1$



Near $f(G)=0$, $f = f(G_n) + f'(G_n)\delta G$

$$G^{n+1} = G^n - f(G^n)/f'(G^n)$$

or $$\boxed{G^{n+1} = G^n - 2\Delta G f(G^n)/\left(f(G+\Delta G) - f(G-\Delta G)\right)}$$

$$f(\tilde{G}) \approx f(G) + f'(G)(\tilde{G} - G)$$

Integrate o.d.e. with $T(0)=T_0$ and $T'(0) = G$: let $F=T-T1$. Then integrate with $G+\Delta G$; then with $G-\Delta G$

```
          dG = 1.e-3   !  delta guess
          G=1.d0       !  initial guess
Newton:   DO WHILE(abs(err)>1.e-6)
          DO ig=1,3
             T(1)=0.d0
             T(2)=G+(ig-2)*dG
             t = 0.d0
             DO i=1,N
               call rk2(2,RHS,t,U,t+dt)        !  See HWK1
             ENDDO
             F(ig) = 1-T(1)                    !  save T at x=1
          ENDDO
          err = F(2)*2.*dG/(F(3)-F(1))
          G = G-err
          ENDDO Newton
```

R-K2 call: find T(x) by solving with $T(0)=0$, $T'(0)=G$