Hyperbolic Equations

Physics: Convection and wave propagation; sound in compressible flow.

$\partial_t q = -\nabla \cdot \mathbf{F}_q$. Now consider convective (dominated) flux $\mathbf{u}q$.

A. Examples:

Linearized compressible potential flow

$(1-M^2)\, \partial_x{}^2\phi + \partial_y{}^2\phi = 0$ M>1 is hyperbolic: recall asymptotes=radiation

c.f. Mach waves

Shallow water waves (long waves; non-dispersive): $k = 2\pi/\lambda$; $\omega = 2\pi/T$

$\partial_t{}^2 h - gH\partial_x{}^2 h = 0$ (or $\nabla^2 h$) ; $a = \sqrt{gH} = |\omega/k|$; $\omega = \pm k\sqrt{gH}$

Pictures: expansions and shocks; .gif animations. Will encounter in 1-D model equation and Euler equations. Physics <-> numerics.

B. Hyperbolic conservation laws: *now sound, later 1-D Euler*

1. Mass conservation control volume sketch, mass in-out = rate of change

$\partial_t\rho + \nabla\cdot(\mathbf{u}\rho) = 0$; or $\partial_t\rho + (\mathbf{u}\cdot\nabla)\rho + \rho\,\nabla\cdot\mathbf{u} = 0$; or $D_t\rho + \rho\,\nabla\cdot\mathbf{u} = 0$

conservation form *convective form*

2. Momentum conservation: F/vol = m a/vol = $\rho D_t\mathbf{u}$. F = pressure force: $(A P_{left} - A P_{right})/\mathrm{Vol} = -\Delta P/\Delta x \rightarrow -\nabla P$. Inviscid momentum equation

$$\rho D_t\mathbf{u} = -\nabla P \quad \text{(convective form)}$$

Conservation form for momentum:

$\rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla P$

$\mathbf{u}(\partial_t\rho + \nabla\cdot(\mathbf{u}\rho) = 0)$

$\rightarrow \partial_t\rho\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}\rho) = -\boldsymbol{\nabla}P$; $\partial_t\rho u_i + \partial_j(u_j u_i\rho) = -\partial_i P$

Note $\boldsymbol{\nabla}P = \boldsymbol{\nabla}\cdot(\mathbf{I}\,P)$: $\partial_t\rho\mathbf{u} + \nabla\cdot(\mathbf{u}\mathbf{u}\rho + \mathbf{I}\,P) = 0$: Hyperbolic conservation law

$\partial_t\rho\mathbf{u} + \nabla\cdot\mathbf{F}_\mathbf{u} = 0$, $\mathbf{F}_\mathbf{u} = \mathbf{u}\mathbf{u}\rho + \mathbf{I}\,P$ is flux function (tensor)

$\partial_t\rho + \nabla\cdot\mathbf{F}_\rho = 0$, $\mathbf{F}_\rho = \mathbf{u}\rho$ will encounter later

3. Total enthalpy (or thermodynamic variable. Euler equations later). For now: isentropic <-> sound speed given

$$a^2 = \partial P/\partial\rho\,|_s$$

C. Compressible flow contains sound waves. Sound waves are implicit in equations, affect numerics They are small disturbances.

Linearize about uniform flow $\mathbf{U} = (U,0,0)$, $\bar{\rho}$ where U and $\bar{\rho}$ are constant. $\mathbf{u} = \mathbf{U} + \mathbf{u}'(\mathbf{x},t)$. $\rho = \bar{\rho} + \rho'$; small perturbations $|\mathbf{u}| \ll U$ etc. .

$\mathbf{u} \cdot \nabla \mathbf{u} = (\mathbf{U} + \mathbf{u}') \cdot \nabla (\mathbf{U} + \mathbf{u}') = (\mathbf{U} + \mathbf{u}') \cdot \nabla \mathbf{u}' \approx U\partial_x \mathbf{u}'$

$$\boxed{\begin{aligned} \bar{\rho}\, \partial_t \mathbf{u}' + \bar{\rho} U \partial_x \mathbf{u}' &= -\nabla P' \\ \partial_t \rho' + U\partial_x \rho' + \bar{\rho}(\partial_x u' + \partial_y v') &= 0 \end{aligned}}$$

2 equations in 3 unknowns. Additional piece of info: $a^2 = \partial P/\partial \rho$ measures compressibility of a barotropic fluid: will show that *a* is sound speed; in an isentropic gas $a^2 = \gamma RT$. Use $dP' = a^2 d\rho'$ in first equation.

$$\nabla \cdot \{ \bar{\rho}\, D_t \mathbf{u}' = -a^2 \nabla \rho' \} \rightarrow \bar{\rho}\, D_t\, \nabla \cdot \mathbf{u}' + a^2 \nabla^2 \rho' = 0$$

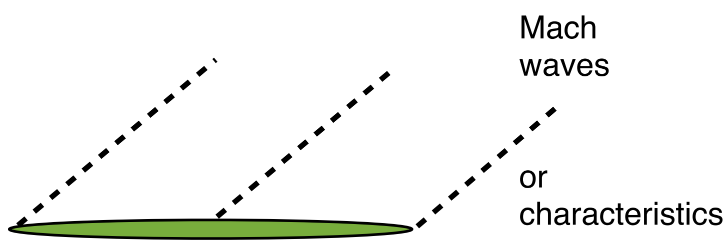
$$D_t \{ D_t \rho' + \bar{\rho} \nabla \cdot \mathbf{u} = 0 \} \rightarrow D^2_t \rho' + \bar{\rho}\, D_t\, \nabla \cdot \mathbf{u} = 0$$

Give

$$D_t^2 \rho' - a^2 \nabla^2 \rho' = 0$$

This is the (linear) equation of sound propagation on a mean flow; with $D_t = \partial_t + U\partial_x$

D. Case A. Steady (drop primes)



$$U^2 \partial_x^2 \rho - a^2 (\partial_x^2 \rho + \partial_y^2 \rho) = 0$$

or $$(M^2 - 1)\partial_x^2 \rho - \partial_y^2 \rho = 0$$

where M=U/a is the Mach number. Slender body equation. Hyperbolic in super-sonic case.

Solution: try $\rho = f(x-\alpha y)$: $\eta = x-\alpha y$; $\partial_x \rho = f'(\eta)$; $\partial_y \rho = -\alpha f'(\eta)$ etc. Substitute and find that $\alpha = \sqrt{M^2 - 1}$.

ρ is constant along Mach lines: $d\rho = dx\, \partial_x \rho + dy\, \partial_y \rho \rightarrow$

$$dy/dx = -\partial_x \rho / \partial_y \rho = 1/\alpha = 1/\sqrt{M^2 - 1}$$

Mach angle: $\tan\theta = 1/\sqrt{M^2 - 1}$ or $\sin\theta = 1/M$

E. Case B. Unsteady, U=0

$\partial^2_t\rho - a^2\nabla^2\rho = 0$. Linear, acoustic wave equation. In 1-D

$$\boxed{\partial^2_t\rho - a^2\partial^2_x\rho = 0}$$

Hyperbolic because of - sign. Wave propagation.

1. General solution is

   fcn(x-αt) → $\alpha^2 = a^2$ → $\alpha = \pm a$ (previous example ignored - sign; Mach waves on lower side)

   so $\rho = f(x-at) + g(x+at)$, verify by substitution.

2. sum of right and left moving waves <----- thunder ----->

   $f(x-at) = \text{const} \rightarrow xd/dt=a$; $g(x+at) = \text{const} \rightarrow xd/dt=-a$

Right and left moving waves -- undistorted (numerical objective)

$\eta = x\text{-}at = \text{const}$ $\xi = x+at = \text{const}$

General solution: $\rho(x,t) = g(x+at) + f(x-at)$. f and g determined by initial conditions.
E.g. $\rho(x,0) = 2\sin(x)$, $\partial_t\rho(x,0) = 0$.

$$\begin{aligned} g'(x) - f'(x) &= 0 \\ g(x) + f(x) &= 2\sin(x) \\ \Rightarrow\ g'(x) + f'(x) &= 2\cos(x) \end{aligned}$$

$$g(x) = \sin(x)$$
$$f(x) = \sin(x)$$
$$\rho(x,t) = \sin(x+at) + \sin(x-at) = 2\sin(x)\cos(at)$$

Start with sine wave, amplitude 2, split into left and right waves: in this case superposition is a standing wave

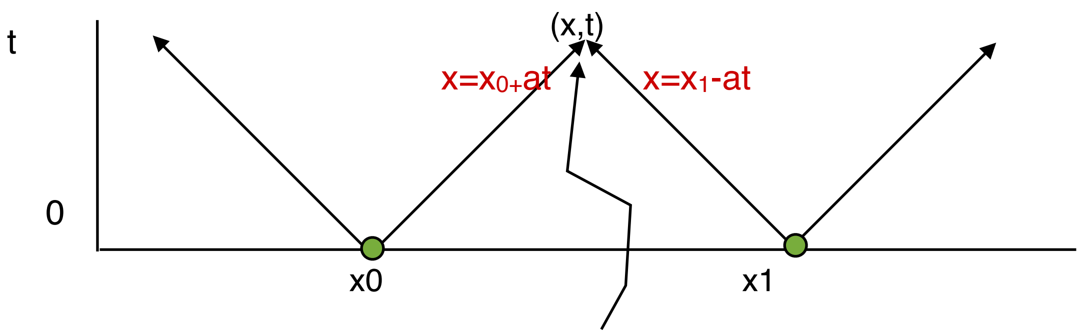
Characteristics and domain of dependence: another perspective on i.v.p.

A. Method of characteristics. *Reversed time; like particle motion*

Recall general solution to wave equation $\partial^2_t \rho - a^2\partial^2_x \rho = 0$
left and right propagating waves

$$\rho(x,t) = f(x-at) + g(x+at)$$

where f and g are determined by initial conditions



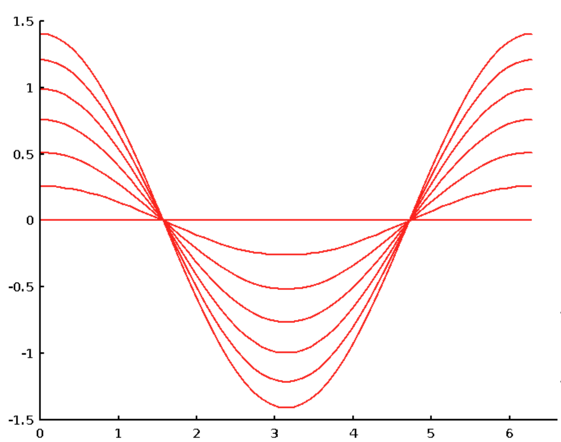
$$\rho(x,t) = f(x_0)+g(x_1) = f(x-at)+g(x+at)$$

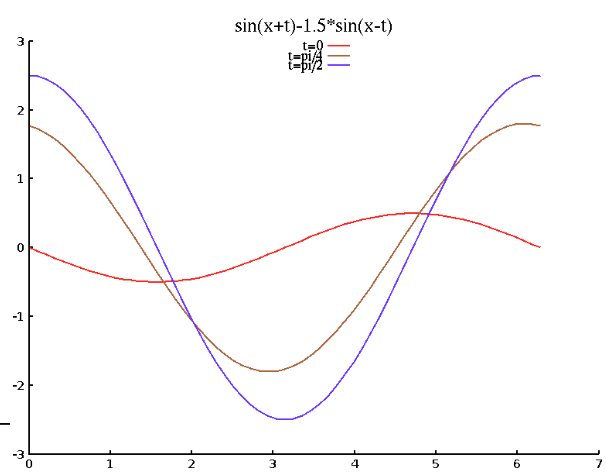
Information is carried on characteristics: $f(x_0)$, $g(x_1)$ are set by i.c. then carried. Follow characteristic back to find source. Note: contribution from x_0 is the right-moving wave, etc. Need 2 initial conditions.

Example: $f(x_0) = \sin(x_0)$; $g(x_1) = -\sin(x_1)$ at t=0.
Characteristics through x,t originated at x_0 = x-at and x_1 = x+at so

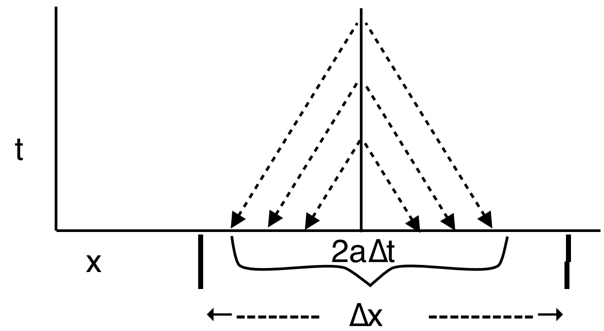
$$\rho(x,t) = \sin(x-at)-\sin(x+at) = 2\cos(x)\sin(at)$$

Right figure is g=sin(x) ; f = -1.5*sin(x)





B. Domain of dependence



Solution is determined by rays that reach x: whole domain must be included in numerical stencil. For simple waves zone is just the left or right wedge. Cone or sphere in 2, 3-D.

Points inside domain of dependence influence solution at time t. Points outside zone are irrelevant. Whole zone has influence between t and t+Δt:

$\rho(x,t+\Delta t) = \rho(x,t) + \int_0^{\Delta t} \dot{\rho}\, dt$.

For a time interval Δt, $\Delta x = a\Delta t$, so zone of dependence is the interval $x \pm a\Delta t$, or $|\Delta x| \le a\Delta t$ $\rightarrow$ if $|\Delta x| > a\Delta t$ physical domain $\in$ computational domain

Computational domain must include physical domain:

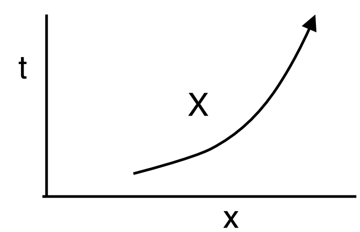
$a\Delta t/\Delta x \equiv$ CFL number < 1: rule of thumb. CFL = Courant-Fredrichs-Lewy

C. Characteristics can be curved: along characteristic ρ=const -> $\partial_x\rho dX + \partial_t\rho dt=0$ -> $dX/dt = -\partial_t\rho/\partial_x\rho$ E.g. if

$$\partial_t\rho + 1/x\, \partial_x\rho = 0$$

then $dX/dt = 1/X$; $X^2 = x^2_0 + 2t$

Given $\rho(t=0) = f(x_0)$ the solution is $\rho(x,t) = f(\sqrt{x^2-2t})$



Simple Waves:

Note that

$$\partial^2_t\rho - a^2\, \partial^2_x\rho = [\, (\partial_t - a\, \partial_x)\, (\partial_t + a\, \partial_x)\,]\rho$$

is a decomposition into left and right wave operators.

Consider right moving wave f(x-at). This solves reduced equation

$$\boxed{\partial_t f + a\, \partial_x f = 0}$$