(*Review finite-diff. and metrics*)

Second derivatives: Evaluate $\partial^2\psi/\partial^2 x$ as $\partial_x F$ with $F = \partial_x \psi$

Outside, then in, but iterated central would give 5 *i*-points.

Wrong:

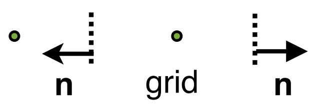
$$\delta_\xi(F) = (F_{i+1} - F_{i-1})/2 \text{ with } F = \delta_\xi\psi \rightarrow [(\psi_{i+2} - \psi_i) - (\psi_i - \psi_{i-2})]/4.$$

Instead use:

$$(F_{i+1/2} - F_{i-1/2})/2 \rightarrow [(\psi_{i+1} - \psi_i) - (\psi_i - \psi_{i-1})]/4.$$

$F_{i+1/2}$ = value on cell face F_{i+1} is data at cell center.

Or recall formulating equations in conservation form: Flux in - Flux out + source = rate of change inside c.v. Recall 1-D diffusion



$\partial_t T = \partial_x(\kappa\partial_x T) = -\partial_x F$ where $F = -\kappa\partial_x T$

$i-1/2 \leftarrow \Delta x \rightarrow i+1/2$ Flux in is in -**n** direction.

$\partial_t \int T dx = -(F\cdot\mathbf{n})_{i-1/2} - (F\cdot\mathbf{n})_{i+1/2} = F_{i-1/2} - F_{i+1/2}$ is exact conservation.

Finite difference method for curvilinear grids

We have a method for chain rule derivatives:

Rtxy(*,1,1) = ∂ξ/∂x ; Rtxy(*,1,2) = ∂ξ/∂y
Rtxy(*,2,1) = ∂η/∂x ; Rtxy(*,2,2) = ∂η/∂y

e.g., grid, x(i,j), y(i,j) → Rtxy(*,1,1) = δy/δj / (δx/δi · δy/δj - δx/δj · δy/δi)

∂ψ/∂x = ∂ξ/∂x ∂ψ/∂ξ + ∂η/∂x ∂ψ/∂η = Rtxy(i,j,1,1) [ψ(i+1,j) - ψ(i-1,j)]/2
+ Rtxy(i,j,2,1) [ψ(i,j+1) - ψ(i,j-1)]/2

Use this to discretize Poisson equation by centered finite differences.
Second derivatives make it `messy'; chain rule leads to cross derivatives $\partial^2\psi/\partial\xi\partial\eta$
(explain how this -> 9 point stencil)

Gauss-Seidel in Δ-form

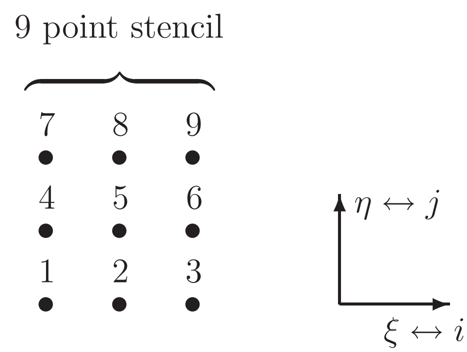
`Loop over` i, j

$$R = \omega - \left[\frac{\delta^2}{\delta^2 x} + \frac{\delta^2}{\delta^2 y}\right]\psi \qquad (1)$$

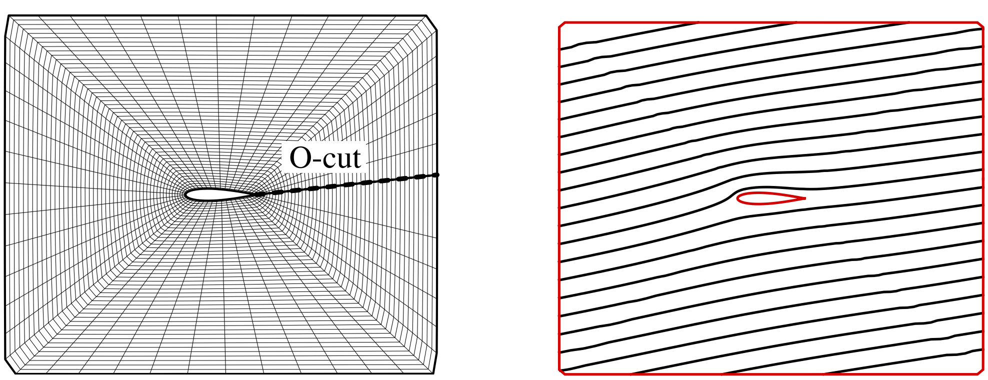
$$\Delta\psi = R(i,j)/A_5 \qquad (2)$$

$$\psi^{n+1}(i,j) = \psi^n(i,j) + \lambda\Delta\psi \qquad (3)$$

`endloop`



R is evaluated with the current values of ψ — a combination of n and $n+1$.



So R can be evaluated in aggregate. A5 is needed too.

Basic idea: Let $F = \partial_x\psi$ and $G = \partial_y\psi$. Then $\nabla^2\psi = \partial_x F + \partial_y G$.
(Recall the heat equation and gradient diffusion.) Term-by-term evaluation

A) $\partial_x F_{i,j} = (\partial\xi/\partial x)_{i,j}\, \partial_\xi F_{\,i,j} + (\partial\eta/\partial x)_{i,j}\, \partial_\eta F_{\,i,j}$

Use centered formulas; consider first term. Note that ξ and η directions stencils differ.

B) $\delta_\xi F_{i,j} = \delta_\xi\, [\partial_x\psi\,]_{i,j} = \delta_\xi\, [\partial\xi/\partial x\; \partial\psi/\partial\xi + \partial\eta/\partial x\; \partial\psi/\partial\eta\,]_{i,j}$

$$= (\partial\xi/\partial x)_{i+1/2,j}\;\; \partial_\xi\psi_{\,i+1/2,j} + \tfrac{1}{2}(\partial\eta/\partial x\, \partial_\eta\psi)_{i+1,j}$$

$$- (\partial\xi/\partial x)_{i-1/2,j}\;\; \partial_\xi\psi_{\,i-1/2,j} - \tfrac{1}{2}(\partial\eta/\partial x\, \partial_\eta\psi)_{i-1,j}$$

Where $2(\partial\xi/\partial x)_{i+1/2,j} = (\partial\xi/\partial x)_{\,i+1,j} + (\partial\xi/\partial x)_{\,i,j}$ *etc.*

For the derivatives in computational space:

1) $\delta_\xi\psi_{\,i+1/2,j} = \psi_{\,i+1,j} - \psi_{\,i,j}$

2) $\tfrac{1}{2}(\partial\eta/\partial x\; \delta_\eta\psi)_{\,i+1,j} = \tfrac{1}{4}\, \partial\eta/\partial x_{\,i+1,j}\, [\, \psi_{\,i+1,j+1} - \psi_{\,i+1,j-1}\,]$

Note that metric terms are at grid points.

Repeat with *i*-index shifted down to obtain second line of B:

3) $\partial_\xi\psi_{\,i-1/2,j} = \psi_{\,i,j} - \psi_{\,i-1,j}$

4) $\tfrac{1}{2}(\partial\eta/\partial x\; \partial_\eta\psi)_{\,i-1,j} = \tfrac{1}{4}\, \partial\eta/\partial x_{\,i-1,j}\, [\, \psi_{\,i-1,j+1} - \psi_{\,i-1,j-1}\,]$

Stencil contains $\psi_{\,i+1,j+1}$, $\psi_{\,i+1,j-1}$, $\psi_{\,i-1,j+1}$, $\psi_{\,i-1,j-1}$ in addition to the 5-point stencil; so now it is a 9-point stencil.

This can be coded as: evaluate 1,2,3,4 ; use them to evaluate B, then evaluate A.

Same for $\partial_y G_{i,j}$.

Gauss-Seidel requires diagonal element (A_5). Can work it out, or apply the above with

$\psi_{1\text{-}4} = 0 = \psi_{6\text{-}9}$, $\psi_5 = 1$: which can be described as $A_5 = \Sigma_{i=1-5}\; A_i\, \delta_{i5}$

General geometry: From

$$\frac{\partial F}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial F}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial F}{\partial \eta} \text{ and } \frac{\partial G}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial G}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial G}{\partial \eta}$$

with $F = \partial_x \psi$ and $G = \partial_y \psi$,

and $\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \xi}{\partial x}\dfrac{\partial \psi}{\partial \xi} + \dfrac{\partial \eta}{\partial x}\dfrac{\partial \psi}{\partial \eta}$

$$\begin{aligned} R &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\overbrace{\left[\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta}\right]}^{\partial_x \psi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left[\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta}\right] \\ &+ \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}\left[\frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial \psi}{\partial \eta}\right] + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\left[\frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial \psi}{\partial \eta}\right] \end{aligned}$$

Discretize with central differences, using *Δξ=1* etc. Look at first two terms

1) Diagonal term

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left[\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi}\right] = \frac{\partial \xi}{\partial x}\left\{\left.\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi}\right|_{i+1/2} - \left.\frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi}\right|_{i-1/2}\right\}$$

$$= \frac{\partial \xi}{\partial x}(i,j)\left\{\frac{\partial \xi}{\partial x}(i+1/2,j)\Big(\psi(i+1,j) - \psi(i,j)\Big) - \frac{\partial \xi}{\partial x}(i-1/2,j)\Big(\psi(i,j) - \psi(i-1,j)\Big)\right\}$$

Using $Rxy_{11} = \dfrac{\partial \xi}{\partial x}$

$$= Rxy_{11}(i,j)\Big[Rxy_{11}(i+1/2,j)\Big(\psi(i+1,j) - \psi(i,j)\Big) - Rxy_{11}(i-1/2,j)\Big(\psi(i,j) - \psi(i-1,j)\Big)\Big]$$

2) cross term

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left[\frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta}\right] = \frac{\partial \xi}{\partial x}\left\{\frac{1}{2}\left(\left.\frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta}\right|_{i+1} - \left.\frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta}\right|_{i-1}\right)\right\}$$

$$= \frac{\partial \xi}{\partial x}(i,j)\,\frac{1}{4}\left(\frac{\partial \eta}{\partial x}(i+1,j)(\psi(i+1,j+1) - \psi(i+1,j-1)) - \frac{\partial \eta}{\partial x}(i-1,j)(\psi(i-1,j+1) - \psi(i-1,j-1))\right)$$

If $R^{xy}_{21} = \dfrac{\partial \eta}{\partial x}(i,j)$

$$= R^{xy}_{11}(i,j)\left[R^{xy}_{21}(i+1,j)\Big(\psi(i+1,j+1) - \psi(i+1,j-1)\Big) - R^{xy}_{21}(i-1,j)\Big(\psi(i-1,j+1) - \psi(i-1,j-1)\Big)\right.$$

Note: ψ(i+1,j+1) etc. fill in a 9-point stencil

The residual can be evaluated in this form, without writing the coefficients A $_{[1\text{-}9]}$; but A_5 is needed also.

Summing the coefficients of ψ(i,j) in the discretization of R gives the diagonal element

$$\begin{aligned} A_5 = \quad & - \quad \frac{\partial \xi}{\partial x}(i,j)\left[\frac{\partial \xi}{\partial x}(i-{}^1\!/\!_2\,,j) + \frac{\partial \xi}{\partial x}(i+{}^1\!/\!_2\,,j)\right] - \frac{\partial \eta}{\partial x}(i,j)\left[\frac{\partial \eta}{\partial x}(i,j-{}^1\!/\!_2\,) + \frac{\partial \eta}{\partial x}(i,j+{}^1\!/\!_2\,)\right] \\ & - \quad \frac{\partial \xi}{\partial y}(i,j)\left[\frac{\partial \xi}{\partial y}(i-{}^1\!/\!_2\,,j) + \frac{\partial \xi}{\partial y}(i+{}^1\!/\!_2\,,j)\right] - \frac{\partial \eta}{\partial y}(i,j)\left[\frac{\partial \eta}{\partial y}(i,j+{}^1\!/\!_2\,) + \frac{\partial \eta}{\partial y}(i,j-{}^1\!/\!_2\,)\right] \end{aligned}$$

In these formulas ∂η/∂x(i+1/2,j)= ½ [∂η/∂x(i,j) + ∂η/∂x(i+1,j)], etc.

Recall what is stored in the `metric tensor'

$^{T}Rxy(*,1,1) = \partial\xi/\partial x$; $^{T}Rxy(*,1,2) = \partial\xi/\partial y$; $^{T}Rxy(*,2,1) = \partial\eta/\partial x$; $^{T}Rxy(*,2,2) = \partial\eta/\partial y$

!-- 1,1 --------------------------------------

$\partial\xi/\partial x_{1/2} \cdot \partial\psi/\partial\xi_{1/2}$

```
xy1p =  (rtxy(i,j,1,1)+rtxy(i+1,j,1,1))/2  *(ψij(i+1,j)-ψij(i,j))
```

$\partial\xi/\partial x_{-1/2} \cdot \partial\psi/\partial\xi_{-1/2}$

```
xy1m = (rtxy(i,j,1,1)+rtxy(i-1,j,1,1))/2  *(ψij(i,j)-ψij(i-1,j))
```

! *First term of Residual* $\omega - \partial^2\Psi/\partial x^2$

! $\partial\xi/\partial x \{ \quad \partial/\partial\xi [\partial\xi/\partial x \cdot \partial\psi/\partial\xi]$

```
R   = ω -  rtxy(i,j,1,1)*(xy1p-xy1m  &
```

! $\partial/\partial\xi [\partial\eta/\partial x \cdot \partial\psi/\partial\eta] \}$

```
        + .25*rtxy(i+1,j,2,1)*(ψij(i+1,j+1)-ψij(i+1,j-1)) &
        - .25*rtxy(i-1,j,2,1)*(ψij(i-1,j+1)-ψij(i-1,j-1)) )
```

! *coef of ij from red terms*

```
A5   = - rtxy(i,j,1,1)*(rtxy(i,j,1,1)+rtxy(i+1,j,1,1)/2+rtxy(i-1,j,1,1)/2)
```

!-- 1,2 -------------------------------------

```
xy2p = (rtxy(i,j,1,2)+rtxy(i+1,j,1,2))/2*(ψij(i+1,j)-ψij(i,j))
xy2m = (rtxy(i,j,1,2)+rtxy(i-1,j,1,2))/2*(ψij(i,j)-ψij(i-1,j))
R    = R  - rtxy(i,j,1,2)*(xy2p-xy2m &
        + .25*rtxy(i+1,j,2,2)*(ψij(i+1,j+1)-ψij(i+1,j-1)) &
        - .25*rtxy(i-1,j,2,2)*(ψij(i-1,j+1)-ψij(i-1,j-1)) )
A5  =  A5 - rtxy(i,j,2,1)*(rtxy(i,j,2,1)+rtxy(i,j+1,2,1)/2+rtxy(i,j-1,2,1)/2)
```

!-- 2,1 -------------------------------------

```
xy3p = (rtxy(i,j,2,1)+rtxy(i,j+1,2,1))/2*(ψij(i,j+1)-ψij(i,j))
xy3m = (rtxy(i,j,2,1)+rtxy(i,j-1,2,1))/2*(ψij(i,j)-ψij(i,j-1))
R    = R  - rtxy(i,j,2,1)*(xy3p-xy3m &
        + .25*rtxy(i,j+1,1,1)*(ψij(i+1,j+1)-ψij(i-1,j+1)) &
        - .25*rtxy(i,j-1,1,1)*(ψij(i+1,j-1)-ψij(i-1,j-1)) )
A5  =  A5 - rtxy(i,j,1,2)*(rtxy(i,j,1,2)+rtxy(i+1,j,1,2)/2+rtxy(i-1,j,1,2)/2)
```

!-- 2,2 -------------------------------------

```
xy4p = (rtxy(i,j,2,2)+rtxy(i,j+1,2,2))/2*(ψij(i,j+1)-ψij(i,j))
xy4m = (rtxy(i,j,2,2)+rtxy(i,j-1,2,2))/2*(ψij(i,j)-ψij(i,j-1))
R    = R  - rtxy(i,j,2,2)*(xy4p-xy4m &
        + .25*rtxy(i,j+1,1,2)*(ψij(i+1,j+1)-ψij(i-1,j+1)) &
```

```
        - .25*rtxy(i,j-1,1,2)*(ψij(i+1,j-1)-ψij(i-1,j-1)) )
A5  =  A5 - rtxy(i,j,2,2)*(rtxy(i,j,2,2)+rtxy(i,j+1,2,2)/2+rtxy(i,j-1,2,2)/2)
```