AerE 546 **Homework 8**

Due on last day of class

Exercise 1, *Linear convection*

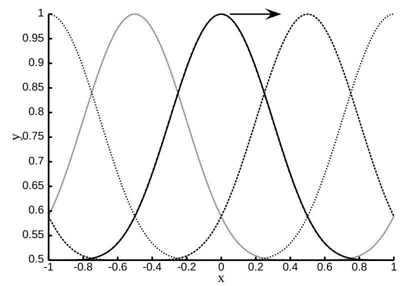
Convect a pulse,

$$u_0(x) = 0.5 + 0.5(1 - x^2)^6; \quad -1 \le x \le 1 \tag{1}$$

That is, solve

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2}$$

with initial condition (1). Apply a periodic boundary condition $u(-1) = u(1)$; or, in computational space $u(J) = u(1)$; $u(J+1) = u(2)$; etc. if $1 \le j \le J$.



The exact solution is $u = u_0(x - at)$ where u_0 is the function given above (and the argument should be adjusted for periodicity).

Use Euler explicit in time, with a linear combination of upwind and downwind differences in space:

$$\Delta x\frac{\partial u}{\partial x} = p_1\delta_- u + (1 - p_1)\delta_+ u$$

where

$$\delta_- u = u_j - u_{j-1}, \quad \delta_+ u = u_{j+1} - u_j.$$

For the following, plot $u(x)$ at times $a\,t = 0.0,\ 0.5,\ 1.0,\ 1.5,\ 2.0$ all in a single figure, as in the example above. Let $\Delta x = 0.01$. The c.f.l. number is $c \equiv a\Delta t/\Delta x$, so $a\Delta t = c\Delta x$.

Give a text answer to each question indicating the question (*e.g.*, 4a: The calculation blows up.)

1. Let $p_1 = 0.5$. Plot solution with $c = 0.3$. a: What happens? b: Why?
2. Set $p_1 = 1$ and $c = 0.5$. Plot your computed $u(x, t)$ at the specified times. Also plot the exact solution at the final time (2.0) as a dashed curve. a: Why do they differ?
3. Set $p_1 = 0.55$. Plot a solution with $c = 0.1$.
   a) Run your code for a few small values of c (but do not give me plots). For how large a value c will the solution remain stable?
   b) Derive the theoretical stability condition on c for a given p_1.

Exercise 2, *Burger's equation*

Burger's equation is a simple, scalar model for compressible flow. It contains non-linear convection. The equation can be written either as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0; \quad \text{or} \quad \frac{\partial u}{\partial t} + {}^1/_2\,\frac{\partial u^2}{\partial x} = 0$$

The first is convection form, the second is conservation form. Consider the domain $-1 \leq x \leq 1$ with the periodic condition $u(-1) = u(1)$. Let the initial condition be

$$u(x, 0) = 0.5 + 0.5[1 - x^2]^6$$

Solve Burger's equation numerically with 201 grid points in x. Use (maximum) c.f.l. numbers of 0.2 and 1.0. Integrate to a time of $t = 1.2$, or until the solution fails. Provide a single plot containing profiles of $u(x)$ at time intervals of 0.2. Use the following discretizations:

- RK2 in time and for $\delta_x(\cdot)$
  - a) central
  - b) first order upwind
- Lax-Wendroff
- MacCormick's scheme

Provide in 8 plots:

RK2 with central c.f.l.= 0.2
RK2 with central c.f.l.= 1.0
RK2 with upwind c.f.l.= 0.2
RK2 with upwind c.f.l.= 1.0
L-W. with c.f.l.= 0.2
L-W. with c.f.l.= 1.0
MacM. with c.f.l.= 0.2
MacM. with c.f.l.= 1.0

Exercise 3, *Top hat*

Solve Burger's equation by Euler explicit plus first order upwind in (a) convection form and (b) conservation form. Also solve with (c) Lax-Wendroff in conservation form. Now the initial condition is the 'top hat'

$$u(x, 0) = \begin{cases} 0 & |x| > 1/3 \\ 1 & |x| \leq 1/3 \end{cases}$$

Set the c.f.l. number to 0.2. Use periodic conditions as previously. Integrate to a time of $t = 1.8$. Provide a single plot containing profiles of $u(x)$ at time intervals of 0.3.

What should the velocity of the shock be? Which methods give it accurately? Which methods produce spurious behavior? Think about the pros and cons of each method.

What I am looking for: 3 Plots and a text statement whether the method is accurate.
E.E. c.f.l.=0.2, upwind convection form
E.E. c.f.l.=0.2, upwind conservation form
L.W. c.f.l.=0.2