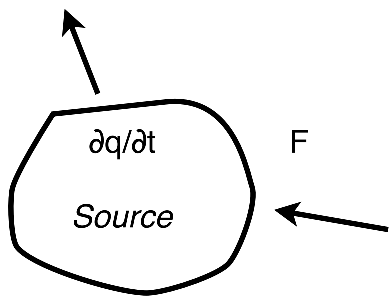
Finite volume method

Natural for unstructured meshed. Also, working with fluxes leads into hyperbolic equations.

A. Recall origin of p.d.e.'s



$$\partial_t \int_{CV} q dV = -\int_{CS} \boldsymbol{F} \cdot \boldsymbol{n} dS + \int_{CV} Sources\, dV$$
$$= -\int_{CV} \nabla \cdot \boldsymbol{F} dV + \int_{CV} Sources\, dV$$
$$\Rightarrow \partial_t q = -\nabla \cdot \boldsymbol{F} + Sources$$

Don't take second step: apply discretization to the integral balance. Control volume is polyline (polyhedron in 3-D).

Divergence theorem (Gauss') is used in f.v. method. Rationale:

$f = \int df/dx\, dx$ but now $f \rightarrow \int \mathbf{F} \cdot d\mathbf{S}$ and $df/dx\, dx \rightarrow$

$\int \boldsymbol{\nabla} \cdot \mathbf{F}\, d\mathbf{V} = \int \boldsymbol{\nabla} \cdot \mathbf{F}\, d\mathbf{S}\, dx_n = \int d\mathbf{F}/dx_n\, dx_n\, d\mathbf{S} = \int \mathbf{n} \cdot \mathbf{F} dS$

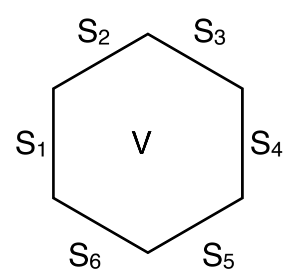
Divergence theorem <-> fundamental theorem of calculus:
Sometimes useful to start with p.d.e. (see below)

1. Control volumes are defined by mesh:

$$\partial_t \int_{CV} q dV = -\int_{CS} \boldsymbol{F} \cdot \boldsymbol{n} dS$$

(without sources). The integrals are

$$\boxed{\partial_t \int q\, dV = -\Sigma \int F \cdot \mathbf{n}\, dS}$$



This is exact. In 2-D, V = area and S = length: $\partial_t \int q\, dA = -\Sigma \int F \cdot \mathbf{n}\, d\ell$

The integrals must be discretized.

What is F? Convective $\mathbf{u}T$; diffusive $-\kappa\nabla T$ --- How does one compute the gradient?

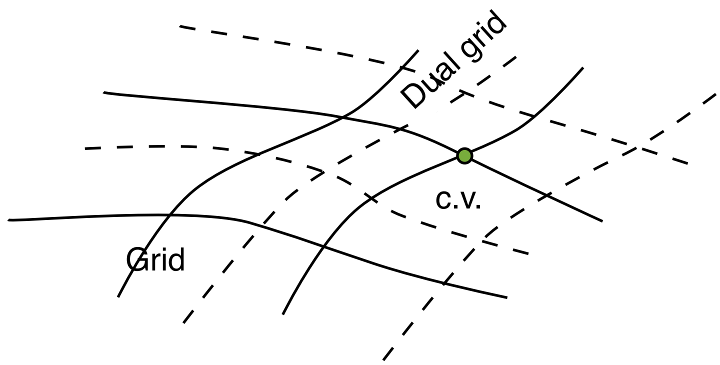
2. Sometimes have to think of starting with diff eq. E.g., down gradient heat flux:

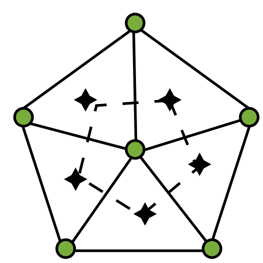
   $\int -\nabla T\, dV = -\int \nabla \cdot (\mathbf{I}\, T)\, dV = -\int T\, (\mathbf{n} \cdot \mathbf{I})\, dS = -\int T\, \mathbf{n} dS$ $\quad$ **n** = outward normal

   Temperature gradient is computed from face temperatures: $\int -\nabla T\, dV = -\Sigma \int \mathbf{n}\, T\, dS$

B. Control volumes
1. Order of accuracy determined by `reconstruction'.
2. Natural approach for unstructured grids (vs. finite difference)
3. Can derive by integrating diff. eqs. over control volume = mesh cell
4. Centers and vertices define dual grids

Data can be stored on grid (vertex) or on dual (cell center)

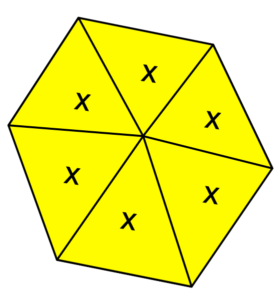


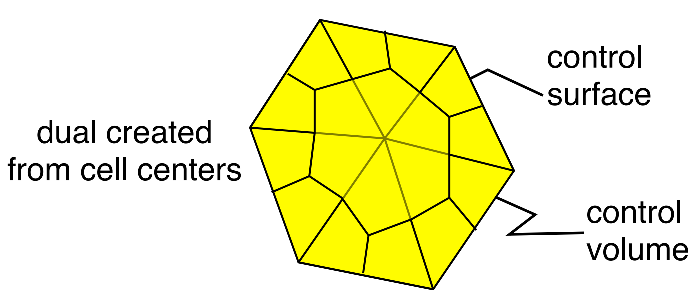


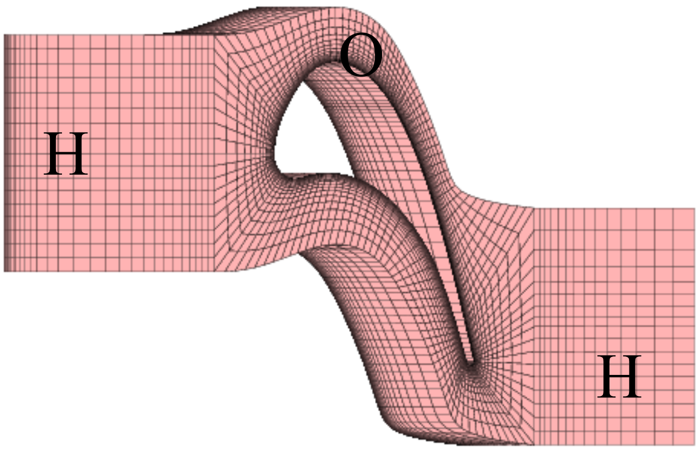
Mesh = set of control volumes. $\partial_t \int q\, dV = -\Sigma \int F \cdot \mathbf{n}\, dS$

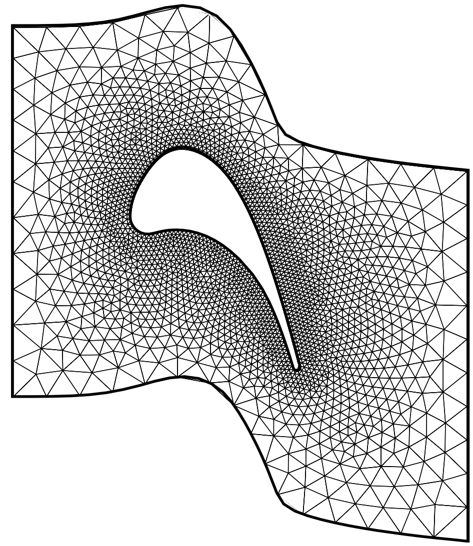
Note flux out of one cell = flux into neighbor.

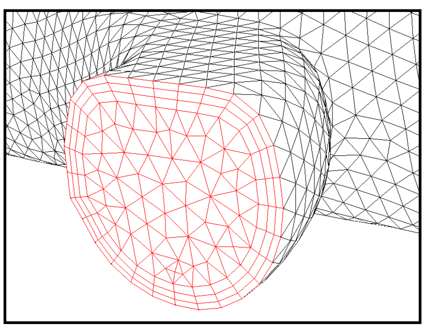
X's can be computational nodes of triangular c.v.s. Or vertices can be nodes and dual defines c.v.'s



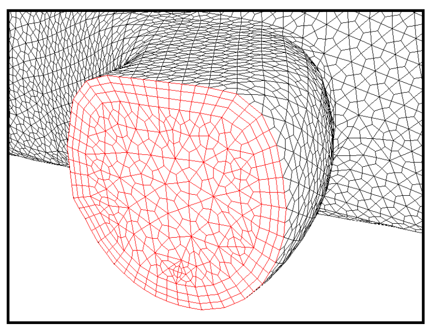




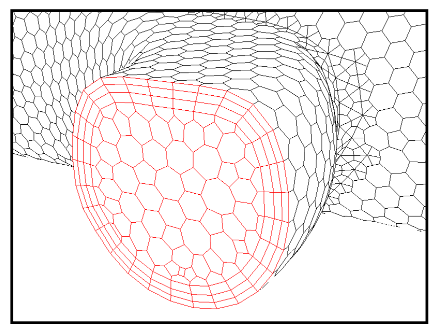




Tets and Prisms



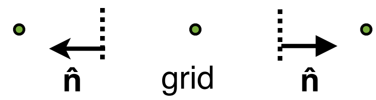
Dual mesh



Arbitrary polygonal mesh

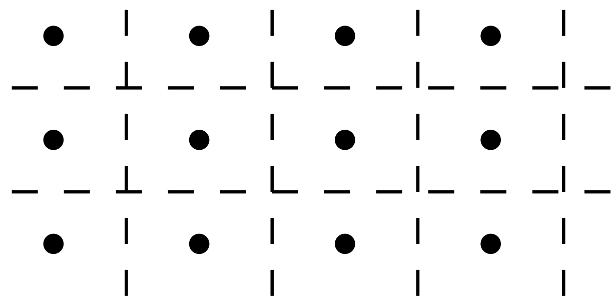
C. Finite Volume approximation

Reminder: $\partial_t T = \partial_x (\kappa \partial_x T) = -\partial_x F$ where $F = -\kappa \partial_x T$



$\partial_t \int T\,dx = F_{i-1/2} - F_{i+1/2}$ is exact conservation. Approximation: $\partial_t \int T\,dx \sim \partial_t T_i \Delta x$.

1. Flux balance applied to cell centers; volumes defined by mesh, think of Cartesian case



2. q = quantity per unit volume

   $\int_{cv} \partial_t q\, d\mathbf{V} = -\int_{cs} F \cdot \tilde{\mathbf{n}}\, dS + \int_{cv} \text{sources}\, d\mathbf{V}$: Flux = convection $\mathbf{u}q$ or diffusion $-\kappa \nabla q$

   Poisson: $\partial_t q = 0$ and $F = \nabla\psi$, source = ω: $\int_{cs} \tilde{\mathbf{n}} \cdot \nabla\psi\, dS = \int_{cv} \omega\, d\mathbf{V}$

   but ∇ complicates finite volume, analogous to Laplacian in finite diff.

3. Approximate the integrals to second order (midpoint rule)

   $$\int_{cell} \partial_t q\, d\mathbf{V} \approx \partial_t q_{\text{cell center}} \Delta\mathbf{V}$$

   $$\int_{cs} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \Sigma_{faces} \int \mathbf{F} \cdot \hat{\mathbf{n}}_i\, dS_i \approx \Sigma_i \, (\mathbf{F} \cdot \hat{\mathbf{n}}_i)_{\text{face center}} \Delta S_i$$

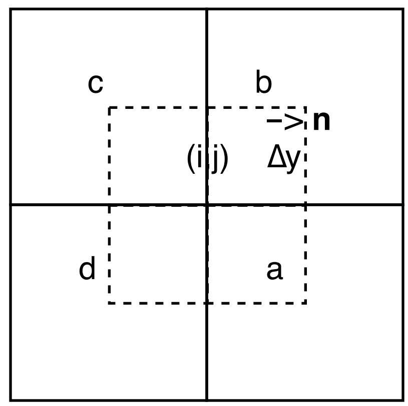
   Higher order: use polynomial 'reconstruction' from nodal values. Discrete conservation equation

   $$\boxed{\partial_t q_{cc} = \Sigma_i \, [(F \cdot \hat{\mathbf{n}})|_i \Delta S_i]_{fc} / \Delta\mathbf{V}}$$

E.g. continuity $q=\rho=$ constant:

$$0 = \Sigma_i \mathbf{F} \cdot \hat{\mathbf{n}}\, \Delta S_i \;/\Delta \mathbf{V}$$

With $\mathbf{F}=\rho\mathbf{u}$ and rectangular control volume:



$n=(1,0)$, $\Delta l_{a\text{-}b}=\Delta y$

$n=(0,-1)$, $\Delta l_{d\text{-}a}=x_a - x_d=\Delta x$

$$0 = [-\Delta x\, v_{i,j-1/2} + \Delta y\, u_{i+1/2,j} + \Delta x\, v_{i,j+1/2} - \Delta y\, u_{i-1/2,j}] \;/\; \Delta x \Delta y$$

$0 = (u_{i+1/2,j} - u_{i-1/2,j})/\Delta x + (v_{i,j+1/2} - v_{i,j-1/2})/\Delta y$ as expected -- $\nabla \cdot \mathbf{u}$.

With $u_{i+1/2} = (u_{i+1} + u_i)/2$

$0 = (u_{i+1,j} - u_{i-1,j})/ 2\Delta x + (v_{i,j+1} - v_{i,j-1}) / 2\Delta y$

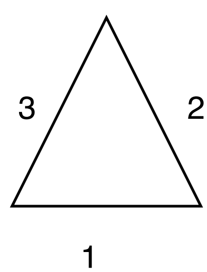
Equilateral triangle control volume . $\hat{\mathbf{n}}_2 = (\pm\sin \pi/6, \cos \pi/6) = (\sqrt{3},1)/2$

$\hat{\mathbf{n}}_1\, \Delta S= (0,-1)\, \ell$

$\hat{\mathbf{n}}_2\, \Delta S= (\sqrt{3},1)/2\; \ell$

$\hat{\mathbf{n}}_3\, \Delta S= (-\sqrt{3},1)/2\; \ell$

$V= \sqrt{3}\; \ell^2/4$



$$\frac{(-v_1 + \sqrt{3}/2\, u_2 + v_2/2 - \sqrt{3}/2\, u_3 + v_3/2)\, \ell}{\sqrt{3}\; \ell^2/4} = \frac{[v_2+v_3-2v_1]}{\sqrt{3}\,\ell/2} + \frac{2(u_2 - u_3)}{\ell}$$