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AER E 546 Fluid Mechanics and Heat Transfer I
Homework 3

1. A slab of metal is initially at uniform temperature. One end is suddenly raised to a high temperature, while the other end is kept cool. Compute the penetration of heat into the slab as a function of time. In dimensional form the temperature diffusion equation, initial and boundary values are

$$\frac{\partial T^*}{\partial t_*} = \kappa \frac{\partial^2 T^*}{\partial {x_*}^2} \qquad T^*(x,0) = 0 \qquad T^*(0,t) = 0, T^*(L,t) = T_w.$$

   Non-dimensionalize temperature by T_w and length by L and time by L^2/κ. Integrate by Euler Explicit, up to a non-dimensional time of 0.3. Use $N_x = 121$ grid points in x. Let $\Delta t = \alpha \Delta x^2$. Try a value of $\alpha > 0.5$. What happens? Why? How small must α be to obtain an accurate solution? Provide a single figure with line plots of the solution at time intervals of 0.04. Note that the computational time-step will be smaller than 0.04. The bulk heat transfer coefficient is defined as

$$h_T = \frac{Q}{T(1) - T(0)}$$

   where $T = \left.\frac{\partial T}{\partial x}\right|_{x=1}$ is the heat flux into the slab. Plot h_T as a function of time for $t > 0.01$.

2. A slab is heated by shining a laser on it. The laser is shut off and the heat diffuses throughout the slab. Its ends are insulated. This is modeled as the non-dimensional problem: solve

$$\frac{\partial T}{\partial x} = \frac{\partial *}{\partial \kappa \frac{\partial T}{\partial x}} x$$

   in the interval $0 \leq x \leq 1$, with the initial condition

$$T(x,0) = \frac{e^{-(x-0.5)^2/\sigma^2}}{\sigma\sqrt{\pi}}, \quad \sigma = 0.1$$

   The slab length is normalized to unity and σ characterizes the region heated by the laser. Consider a material with variable diffusivity. Let

$$\kappa = 0.1 + 2.0e^{-5x}$$

   The no-flux boundary condition

$$\frac{\partial T}{\partial x}(0,t) = 0 = \frac{\partial T}{\partial x}(1,t)$$

   is applied at the insulated ends. Use second order Runge-Kutta. Solve with about $N_x = 250$ grid points in x. Choose a small time-step to obtain an accurate solution. Integrate up to a non-dimensional time of 0.05. Provide a single plot containing the initial condition and curves showing the solution $T(x)$ at time intervals of 0.01. Plot $\int_0^1 T(x)\,\mathrm{d}x$ versus time. What should the value of the integral be?

3. Now consider the case where one end of the slab is insulated and the other is held at constant temperature:

$$\frac{\partial T}{\partial x}(0) = 0 \quad T(1) = 1$$

   solve the constant diffusivity, diffusion equation as in the first problem, but use Crank-Nicholson.

(a) Set $\Delta t = \alpha \Delta x^2$. Try a couple of relatively large value of α and see whether your calculation converges, or blows up (should it?).

(b) Provide a single figure with plots of $T(x)$ at intervalue of 0.04 up to $t = 0.4$. Explain why your solution makes sense. The bulk heat transfer coefficient is defined as

$$h_T = \frac{Q}{T(1) - T(0)}$$

where $T = \left.\frac{\partial T}{\partial x}\right|_{x=1}$ is the heat flux into the slab. Plot h_T as a function of time for $t > 0.01$.

4. (a) Is the scheme

$$U_i^{n+1} = U_i^n - \frac{C}{2}(U_{i+1}^n - U_{i-1}^n)$$

stable, conditionally stable or absolutely unstable? C is a constant.

(b) Is the scheme

$$U_i^{n+1} = U_i^n - \frac{C}{2}\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right)$$

stable, conditionally stable or absolutely unstable? C is a constant.

(c) Which method would be called implicit?

The scheme in (b) would be called implicit as the value of U_i^{n+1} depends on the values U_{i+1}^{n+1} and U_{i-1}^{n+1}. The solution for a point at time t^{n+1} depends on the points next to it at the same time. This means that a system of equations must be solved in order to update the solution.