Summary of explicit methods:

Euler	upwind = central + $\varepsilon \cdot$ diffusion, first order accurate
Runge-Kutta	recall RK is `stable for convection', time-stepping adds dissipation: variations (low storage, higher damping) are used in CFD.
Lax-Wendroff	minimum dissipation (ε=C), 2nd order accurate in space and time
MacCormick	two-step method, similar to L-W

Generally CFL < 1, or CFL_{stab} → time-step restriction $\Delta t <$ min[$CFL_{stab}\, \Delta x / a$

NOTE: distinction is *stencil*, algorithm uses RHS * Δt

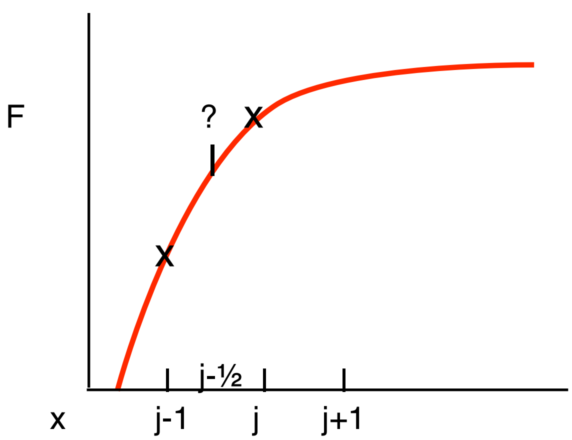
Finite volume thinking: flux interpolation, vs. finite-difference. $\partial_t u = -\partial_x F$.

A. Finite volume-like semi-discretization

$$\int \partial_t u \, dx \approx \partial_t u_j \, \Delta x = -[\, F_{j+1/2} - F_{j-1/2} \,]$$

• ¦ • ¦ •
j-1 j-½ j j+ ½ j+1

Recall u_j is cell average: $\partial u/\partial t \; V_j = -F_{j+1/2} A_{j+1/2} + F_{j-1/2} A_{j-1/2}$. The focus of hyperbolic numerics is the RHS -- e.g., flux interpolation, limiters, splitting, etc.

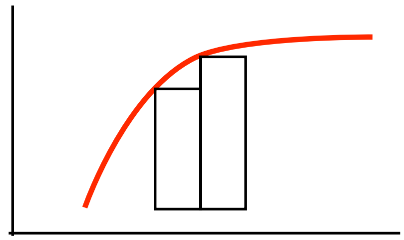


What are the cell face fluxes? F= true flux; $\tilde{F}$ = numerical flux

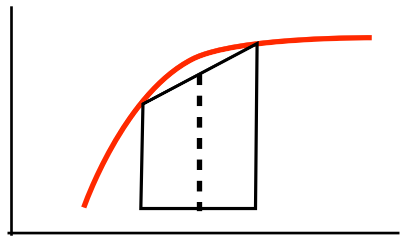
Numerical ≠ True, e.g. upwinding

B. Estimate of $\tilde{F}_{j-1/2}$: Interpolation methods: F = a +b s + c s^2 ....

Piecewise constant



Piecewise linear



$$1^{st} \text{ order upwind:} \quad \tilde{F}_{j-1/2} = \begin{cases} F_{j-1} + O(\Delta x), \; u > 0 \\ F_{j+1} + O(\Delta x), \; u < 0 \end{cases}$$

$$2^{nd} \text{ order central:} \quad \tilde{F}_{j-1/2} = \frac{1}{2}(F_j + F_{j-1}) + O(\Delta x)^2$$

$$\text{QUICK:} \quad \tilde{F}_{j-1/2} = \begin{cases} \frac{1}{8}(3F_j + 6F_{j-1} - F_{j-2}), \; u > 0 \\ \frac{1}{8}(3F_{j-1} + 6F_j - F_{j+1}), \; u < 0 \end{cases} \qquad + O(\Delta x)^3$$

QUICK is quadratic: upwind biased. Lagrange interpolation at j+1/2:

$$F = F_{j-1}\frac{(x_{j+1}-x)(x_j-x)}{2\Delta x^2} - F_j\frac{(x_{j+1}-x)(x_{j-1}-x)}{\Delta x^2} + F_{j+1}\frac{(x_j-x)(x_{j-1}-x)}{2\Delta x^2}$$

$$F_{j+1/2} = -\frac{F_{j-1}}{8} + \frac{3F_j}{4} + \frac{3F_{j+1}}{8}$$

Or fit a quadratic, F = a +b s + c s^2

s=0,-1,-2: a= F_j ; $F_{j-1} = F_j$ - b+c ; $F_{j-2} = F_j$ - 2b+ 4c → $F_{j-1/2} = F_j$ - b/2+c/4

$F_{j-1/2} = F_j$ (1/2)(3/2)/2 + F_{j-1} 1/2(3/2) - F_{j-2} 1/2(1/2)/2 = 3/8F_j + 3/4F_{j-1} - 1/8F_{j-2}

C. Plug approx. flux into discrete form, but just for understanding. In practice code is written in terms of fluxes across cell faces.

Into $\partial_t u_j + (\tilde{F}_{j+1/2} - \tilde{F}_{j-1/2})/\Delta x = 0$ substitute:

Piecewise const ⇔ 1st order, upwind:

$$\tilde{F}_{j+1/2} = F_j;\ \tilde{F}_{j-1/2} = F_{j-1}$$

$$\partial_t u_j + (F_j - F_{j-1})/\Delta x = 0$$

Linear ⇔ 2nd order, central

$$\tilde{F}_{j+1/2} = (F_j + F_{j+1})/2;\ \tilde{F}_{j-1/2} = (F_j + F_{j-1})/2$$

$$\partial_t u_j + (F_{j+1} - F_{j-1})/2\Delta x = 0$$

QUICK (popular in incompressible flow)

$$\partial u_j/\partial t + [(3F_{j+1} + 6F_j - F_{j-1}) - (3F_j + 6F_{j-1} - F_{j-2})]/8\Delta x = 0$$

$$\partial u_j/\partial t + [(3F_{j+1} + 3F_j - 7F_{j-1} + F_{j-2})]/8\Delta x = 0$$

NB: This is not the same as 3rd order finite diff: $(2F_{j+1} + 3F_j - 6F_{j-1} + F_{j-2})/6\Delta x$

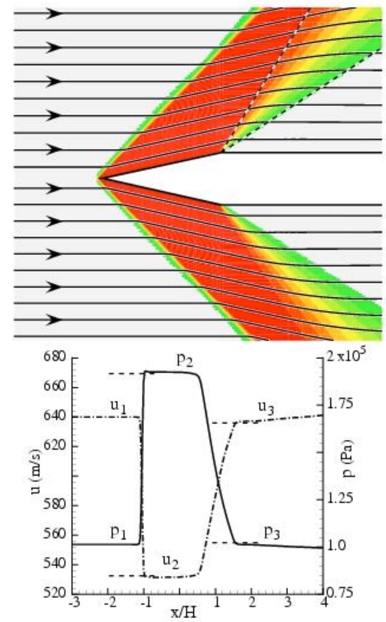
QUICK = 3/4 $\delta F/\delta x|_{\text{3rd order upwind}}$ + 1/4 $\delta F/\delta x|_{\text{2rd order central}}$

so it is 2nd order. QUICK is dissipative due to upwind bias --- 4th order numerical diffusion. Dissipative property provides stability; not as bad as 1st order upwind.

Burger's equation

A model for non-linear wave propagation : Burger's paper was on turbulence.

Has closed form solutions: can develop shocks and expansions



A. Recall conservation form $\partial_t U + \partial_x F(U) = 0$

Let $F = \frac{1}{2}U^2$. Inviscid Burger's equation is

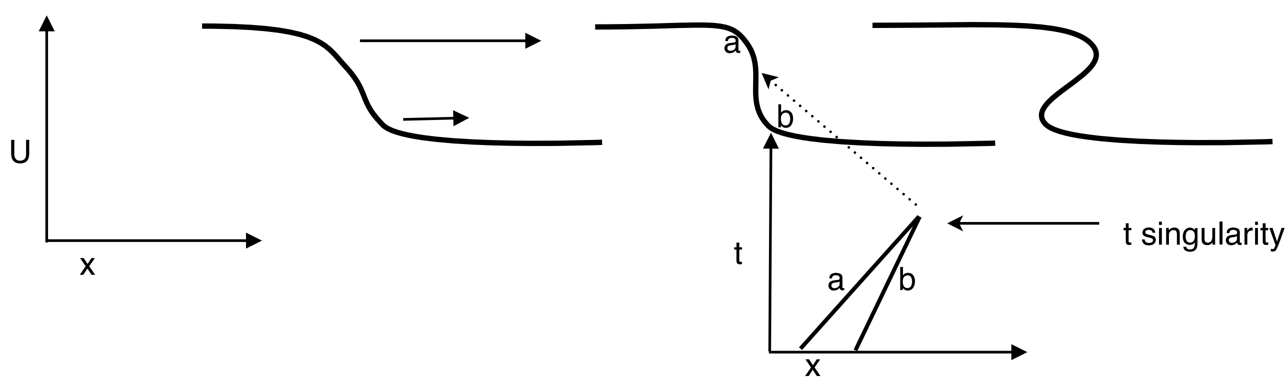
Conservation form: $\partial_t U + \partial_x \frac{1}{2}U^2 = 0$

Convective form: $\partial_t U + U\partial_x U = 0$ $[\partial_t U + U\partial_x U = 1/Re\, \partial^2_x U]$

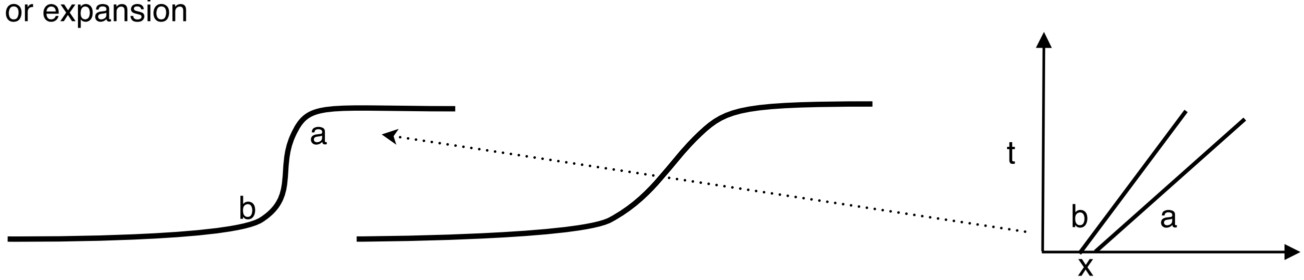
Two forms have different numerical properties (HWK).

From convective form, implicit solution is $U = f(x-Ut)$ --- solve for $U(x,t)$. May have multivalued solution.

B. Model for shock -- with any finite viscosity, solution cannot be double valued



or expansion



C. Time to shock (double value):

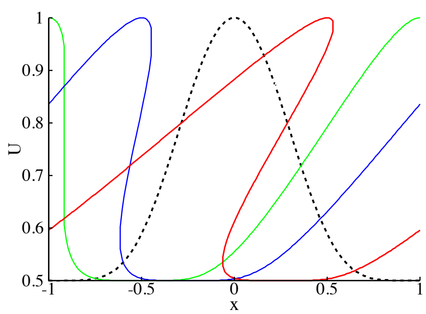
$x_a+U(x_a)t_{sing} = x_b+U(x_b)t_{sing}$

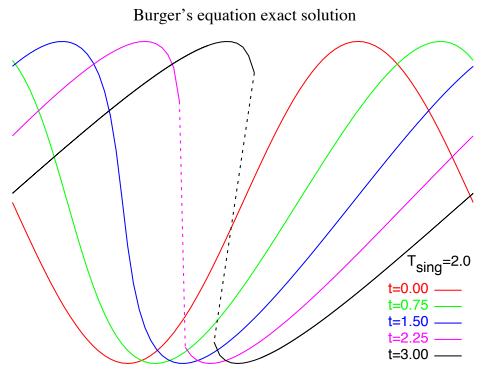
$t_{sing} = -\Delta x/\Delta U|_{min}$ => (N.B. $dU/dx<0$ is required for shock to occur)

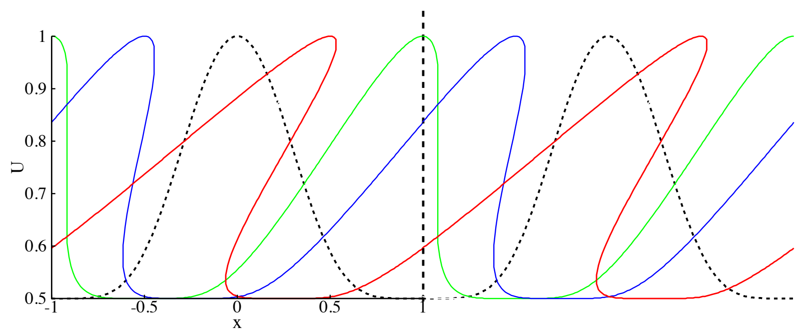
$$t_{sing} = 1/\max_x(-dU/dx)$$

E.g. if initial condition is $U=1+½\sin(\pi x)$ then

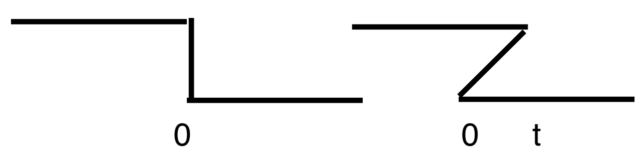
$t_{sing} = -2/\pi \min[\cos(\pi x)] = 2/\pi$







D. Exact (weak or generalized solution)



$U = 0 \qquad x > 0$

$U = 1 \qquad x < t$

$U = x/t \qquad 0 < x < t$

$U = x/t$ solves Burger's equation: $\partial (x/t)/\partial t + (x/t)\, \partial(x/t)/\partial x = 0$

E. Exact (implicit) solution to i.v.p. $U(x) = G(x)$ at $t=0$.

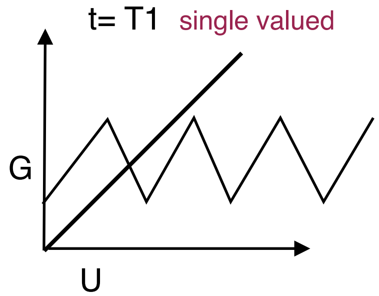
$U(x,t) = G(x - U(x,t)\, t)$. Verify $\partial_t G(\eta) = G'\, \partial_t \eta$

$\partial_t U = \partial_t G = -(U + t\partial_t U)\, G'$; $\partial_x U = (1 - t\partial_x U)\, G'$

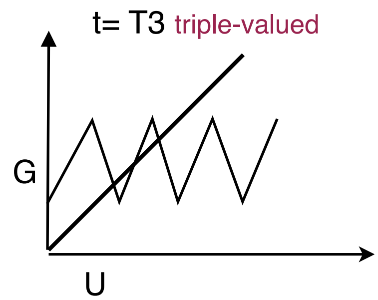
$\partial_t U + U\partial_x U = (-U - t\partial_t U + U - tU\partial_x U)\, G' = -t\,(\partial_t U + U\partial_x U)\, G' \rightarrow (1+tG')\,(\partial_t U + U\partial_x U)=0$

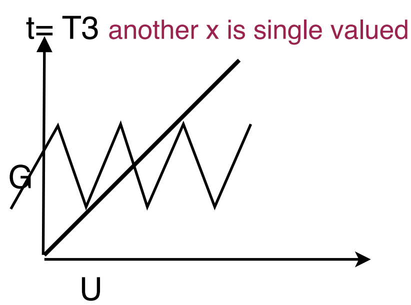
If $(1+tG') \neq 0$ then $\partial_t U + U\partial_x U = 0$ $\therefore$

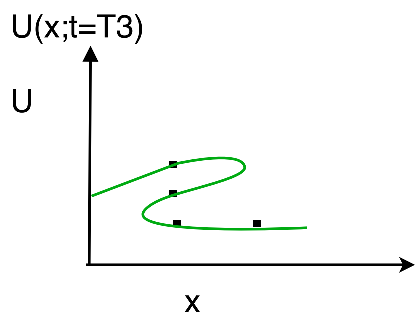
F. U (x,y) = G(x - Ut) is implicit: for each x,t solve by Newton's method -- but might be multiple solutions. Graphical method. As a function of U, G(x - Ut) oscillates more rapidly as t increases. Consider a particular *x*, t (e.g. 0.25,0.1 ; G=sin(.25-.01U))

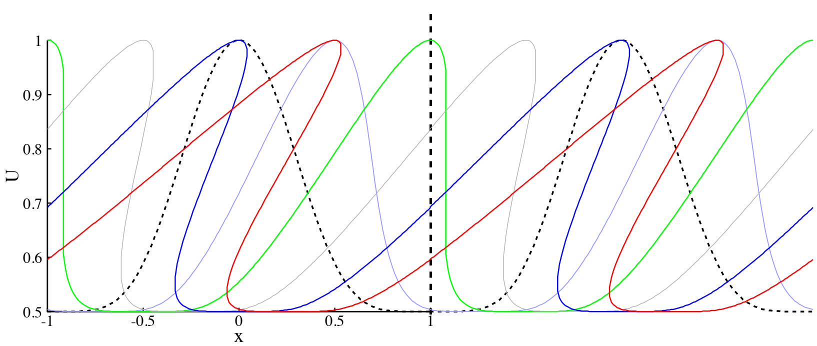


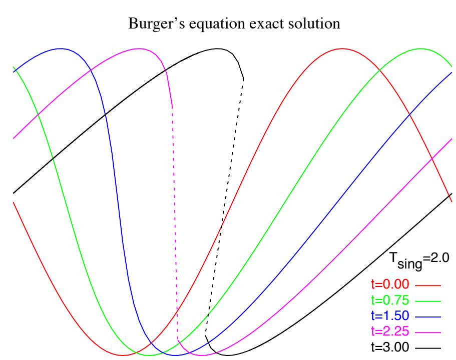
T2 double valued



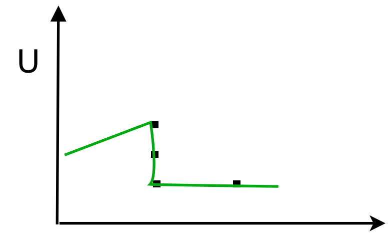








G. But numerical solution is stored as U(j), cannot be three values. With any viscosity (numerical or physical) solution will stay single valued and form a shock.
This is called `weak' solution: Burger's equation everywhere except at discontinuity.



H. Jump formula (c.f. Rankine-Hugoniot) follows from conservation form:

$$\int^{\delta}_{-\delta} \partial_t U\, dx = \int^{\delta}_{-\delta} -\partial_x F\, dx$$

Wave is propagating at speed a_s so locally $U(x-a_s t)$ and $\partial_t U = -a_s \partial_x U$

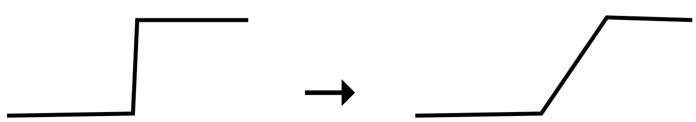
$$\int^{\delta}_{-\delta} -a_s \partial_x U dx = \int^{\delta}_{-\delta} -\partial_x F dx \;\Rightarrow\; a_s [U(\delta) - U(-\delta)] = F(\delta) - F(-\delta)$$

as $\delta \to 0$. Thus $a_s = \Delta F/\Delta U$. E.g. for Burger flux

$$a_s = ½(U^2(\delta) - U^2(-\delta))/[U(\delta) - U(-\delta)] = ½(U(\delta) + U(-\delta))$$

is the average of the velocity before and after the shock: $U_s = ½(U_a + U_b)$

H. Expansion: If U increases with x solution spreads

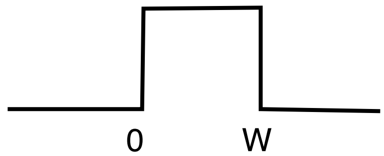


U=0 x<0, U= x/t, 0<x<t, U=1 x>t

Check $\partial U/\partial t = -x\dot{L}/L^2 = -x/L^2$. $U\partial U/\partial x = x/L^2$ (Recall previous soln for shock)

I. Solution for square wave. Initial condition

$U=0 \quad x < 0$
$U=U_t \quad x > W$ (initial width)

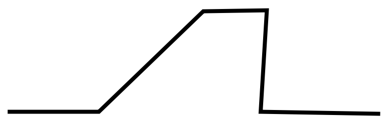


Shock speed ½ U_t : jump condition

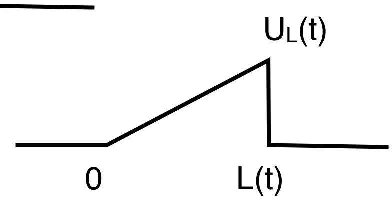
$c = \Delta F/\Delta U = ½(U_+^2 - U_-^2)/ (U_+ - U_-) = ½ (U_+ + U_-)$

Solution is

$U=0 \quad x < 0$
$U= U_t x/t \quad U_t t > x > 0$ expansion
$U= U_t \quad U_t t < x < W+ ½ U_t t$
$U=0 \quad x > W+ ½ U_t t$



when $U_t t < 2W$. Substitute and check.



When $t > 2W/U_t$ expansion catches shock. Note that

$\int^{\infty}_{-\infty} \partial_t U dx = \int^{\infty}_{-\infty} -\partial_x F\, dx = 0$ so $\int^{\infty}_{-\infty} U dx$ = constant = $U_t W$ from initial condition.

Thus

$½U_L(t)L(t)= U_t W$

and shock speed gives $d_t L(t) = ½ U_L(t)$. Then $[\, t_0=2W/U_t \,;\, L_0=2W]$

$L(t)d_t L= U_t W \;\to\; L^2 = (2W)^2 + 2U_t W(t-t_0)$

$L = \sqrt{(2W)^2 +2U_t W (t-t_0)}$

$U_L = U_t W / [\, (2W)^2 + 2U_t W (t-t_0) \,]^{1/2}$

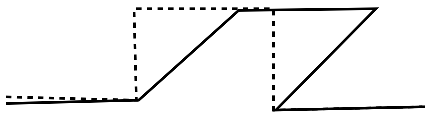
for large t,

$L = \sqrt{2U_t W t}$

$U_L = \sqrt{2U_t W / t}$

Reiterate: Burger as model of non-linearity: steepening; shock capturing
Solution for square wave. Show animation of supersonic wedge, exact soln

Exact, multi-valued



Weak solution. Recall

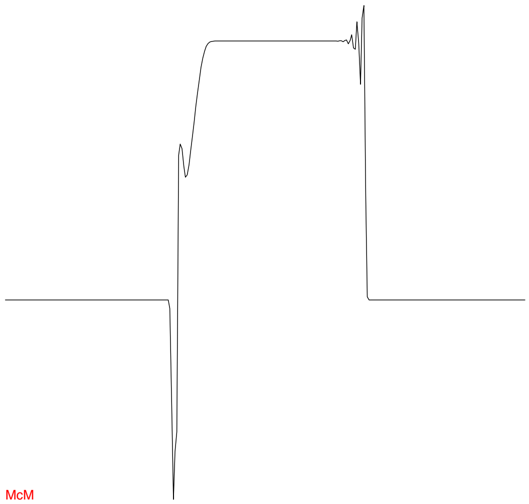
$$a_s = [F(\delta) - F(-\delta)]/ /[U(\delta) - U(-\delta)]$$

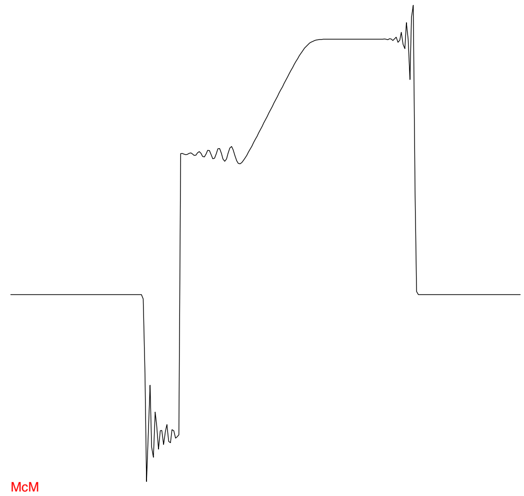
$$a_s = ½[U^2(\delta) - U^2(-\delta)] /[U(\delta) - U(-\delta)] = ½(U(\delta) + U(-\delta))$$

Pseudo-code, Burger's equation conservation form

```
  READ CFL, N
   dx = 2 /(N-1);   x = [-1:1:dx]
   dt = CFL*dx /Umax

  Initial Condition: U(x) = 0.5*(1.0-(x +0.2)²)⁶ +.5

!****************  Integrate in time  ******************

time: DO t = dt,T,dt
  IF(McC)
     DO j=1,N           ! downwind   S= U*
        jp = j+1   ;  IF(jp == N+1) jp = 2
        S(j) =  U(j)-dt*(U(jp)²-U(j)²)/(2 dx)
    ENDDO
    DO j=1,N            ! upwind
       jm = j-1   ;  IF(jm == 0  ) jm  = N-1
       U(j) =  .5*(U(j)+S(j))-.5*dt*(S(j)²-S(jm)²)/(2 dx)
  ELSEIF(Euler)
    S(j) =  U(j)
    DO j=1,N            ! upwind
       jm = j-1   ;  IF(jm == 0) jm  = N-1
       U(j) =  U(j)- dt*(S(j)²-S(jm)²)/(2 dx)
    ENDDO
  ENDIF
        OUTPUT at desired times
ENDDO time
```

Show animation: spurious for top-hat for MacCormick; ok for EE_up





NB: Convection form

```
U(j) =  U(j)- dt*U(j)(U(j)-U(jm))/ dx
```

Before shock `U(j)-U(jm)=0.` After shock `U(j)=0.` So shock cannot move.