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AER E 546 Fluid Mechanics and Heat Transfer I
Homework 1

#1

- (a) How many 'data' points are needed to obtain a third order accurate polynomial approximation? Derive a finite difference formula for $\partial T / \partial x$ that is third order accurate in Δx . Use only the minimum number of points.

Four data points are needed to obtain a third order accurate polynomial approximation, as the Taylor series for a function, f with four coefficients is of the following form.

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{6}f'''(x_0)(x - x_0)^3 + O((x - x_0)^4)$$

Note that this polynomial has errors of order $(x - x_0)^4$, so the approximation is third order accurate.

In order to derive a finite difference formula for $\frac{\partial T}{\partial x}$ that is third order accurate I will first find a third order accurate polynomial approximation using four points. The four points I will use will be equally spaces with spacing Δx and they will be labeled x_{-2}, x_{-1}, x_0, x_1 with function values f_{-2}, f_{-1}, f_0, f_1 respectively. The polynomial approximation will solve the following equations for a, b, c , and d .

$$\begin{aligned} f_{-2} &= a + b(-2\Delta x) + c(-2\Delta x)^2 + d(-2\Delta x)^3 \\ f_{-1} &= a + b(-\Delta x) + c(-\Delta x)^2 + d(-\Delta x)^3 \\ f_0 &= a \\ f_1 &= a + b\Delta x + c(\Delta x)^2 + d(\Delta x)^3 \end{aligned}$$

Clearly $a = f_0$. The 2nd and 4th equations can be added to solve for c . Summing these equations gives

$$\begin{aligned} f_{-1} + f_1 &= 2f_0 + 2c(\Delta x)^2 \\ c &= \frac{f_{-1} + f_1 - 2f_0}{2(\Delta x)^2} \end{aligned}$$

Summing the first equation and -2 times the second equation gives

$$\begin{aligned} f_{-2} - 2f_{-1} &= -f_0 + 2c(\Delta x)^2 + -6d(\Delta x)^3 \\ f_{-2} - 2f_{-1} &= -f_0 + f_{-1} + f_1 - 2f_0 + -6d(\Delta x)^3 \\ f_{-2} - 3f_{-1} + 3f_0 - 1f_1 &= -6d(\Delta x)^3 \\ d &= \frac{f_{-2} - 3f_{-1} + 3f_0 - 1f_1}{-6(\Delta x)^3} \end{aligned}$$

Plugging all these values into the final equation allows for b to be found.

$$\begin{aligned} f_1 &= f_0 + b\Delta x + \frac{f_{-1} + f_1 - 2f_0}{2} + \frac{-f_{-2} + 3f_{-1} - 3f_0 + 1f_1}{6} \\ \frac{6f_1}{6} &= \frac{6f_0}{6} + b\Delta x + \frac{3f_{-1} + 3f_1 - 6f_0}{6} + \frac{-f_{-2} + 3f_{-1} - 3f_0 + 1f_1}{6} \\ \frac{f_{-2} - 6f_{-1} + 3f_0 + 2f_1}{6} &= b\Delta x \\ b &= \frac{f_{-2} - 6f_{-1} + 3f_0 + 2f_1}{6\Delta x} \end{aligned}$$

Now that we have a polynomial approximation of

$$p(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3$$

where the values of a , b , c , and d were computed above. We can now compute the first derivative of this approximation, which is

$$p'(x) = b + 2c(x - x_0) + 3d(x - x_0)^2.$$

The first derivative at x_0 is thus b , or $p'(x_0) = b$. Therefore a third order approximation of the first derivative at the point x_0 is

$$b = \frac{f_{-2} - 6f_{-1} + 3f_0 + 2f_1}{6\Delta x}$$

- (b) Derive the second order accurate centered difference formula for $\frac{\partial^2 T}{\partial x^2}$.

First I will derive the second order polynomial approximation for three points centered around x_0 . I will label the points x_{-1} , x_0 , and x_1 with function values f_{-1} , f_0 , and f_1 respectively. The third order accurate polynomial approximation will be of the form

$$a + b(x - x_0) + c(x - x_0)^2$$

Note that the second derivative of this approximation is always c , so the formula for c will also be the centered finite difference for the second derivative of second order. Finding this approximation amounts to solving the following three equations.

$$\begin{aligned} f_{-1} &= a - b\Delta x + c(\Delta x)^2 \\ f_0 &= a \\ f_1 &= a + b\Delta x + c(\Delta x)^2 \end{aligned}$$

Clearly $a = f_0$. The first and third equations can be summed to find c .

$$\begin{aligned} f_{-1} + f_1 &= 2f_0 + 2c(\Delta x)^2 \\ c &= \frac{f_{-1} - f_0 + f_1}{2(\Delta x)^2} \end{aligned}$$

Thus we don't even need to solve for b because the second order central finite difference for the second derivative is

$$\frac{f_{-1} - f_0 + f_1}{2(\Delta x)^2}$$

#2

- (a) The equation for a damped oscillator is

$$\ddot{Y} + \sigma \dot{Y} + \omega^2 Y = 0.$$

Let the non-dimensional frequency be $\omega = 1$. Consider the two damping rates $\sigma = 0.0$ and $\sigma = 0.5$. Solve this by RK2, out to $t = 32$, with the initial conditions $Y(0) = 1$ and $\dot{Y}(0) = 0$. The time-step can be $\Delta t = 32/N$, where N is the number of integration points. Plot solutions with $N = 21, 101, 301$. What is the analytical solution? Compare your numerical solutions to the exact result.

First I will compute the analytical solution to this differential equation. This can be done by finding the characteristic polynomial of the equation, which is

$$r^2 + \sigma r + 1 = 0.$$

Using the quadratic formula, we see that the roots of this polynomial are $r = -\frac{\sigma}{2} \pm \frac{\sqrt{\sigma^2 - 4}}{2}$. When $\sigma = 0.0$, the roots are $r = \pm i$. In the case of complex roots the general solution will be

$$Y(t) = c_1 \cos(t) + c_2 \sin(t).$$

Using the initial conditions we see that the exact solution is

$$Y(t) = \cos(t).$$

When $\sigma = 0.5$ the roots are $r = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$. In this case the general solution is

$$Y(t) = e^{-\frac{1}{4}t} \left(c_1 \cos\left(\frac{\sqrt{15}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{15}}{4}t\right) \right)$$

and the exact solution with boundary conditions is

$$Y(t) = e^{-\frac{1}{4}t} \cos\left(\frac{\sqrt{15}}{4}t\right).$$

Now in order to solve this equation numerically with RK2, we first need to transform this second order differential equation into a system of first order differential equations. To do this let $Z = \dot{Y}$, then the system becomes

$$\begin{aligned}\dot{Y} &= Z \\ \dot{Z} &= -\sigma Z - \omega^2 Y\end{aligned}$$

This is in the form $\dot{x} = RHS(x)$ where

$$\begin{aligned}x &= [Y, Z]^T \\ RHS(x) &= [\sigma x_2, -\sigma x_2 - \omega^2 x_1]^T.\end{aligned}$$

(b) The equation for a nonlinear spring (without damping) is

$$\ddot{Y} + Y - BY^3 = 0.$$

Solve by RK2 out to $t = 32$ with the initial conditions $Y(0) = 1$ and $\dot{Y}(0) = 0$. Plot $Y(t)$ for $B = 0.2, 0.6, 0.9, 0.999$. Chose N large enough to get an accurate solution; that will depend on the value of B .

#3 Repeat the linear spring computation (ex. 2.a) with AB2. What does the solution for $\sigma = 0.0$ tell you about the stability of AB2?