Classification of p.d.e.'s

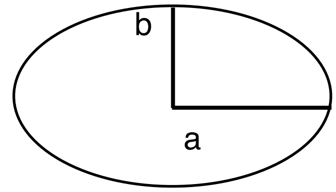
A. Recall conic sections

Ellipse

$x^2/a^2 + y^2/b^2 = 1$

linear term just shifts origin

or $(x-x_0)^2/a^2 + y^2/b^2 = x^2/a^2 + y^2/b^2 - 2x\, x_0/a^2 = \text{const}$



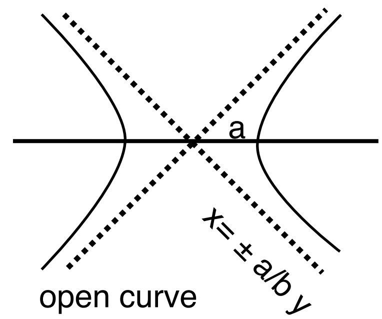
closed curve

Closed curve. The type is determined by the quadratic terms.
c.f.: n-dimensions, all positive signs

Hyperbola

$x^2/a^2 - y^2/b^2 = 1$

minus sign makes it an hyperbola. *Open* curve



open curve

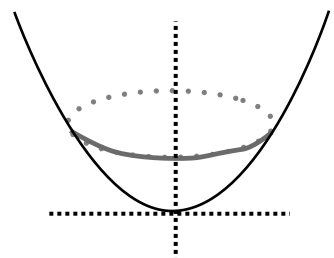
Parabola

$x^2/a^2 - y/b = 0$ or $x^2/a^2 - (y-y_0)/b = 0$

Linear in y. In between ellipse and hyperbola?
0 is between + and -, but more like the former.
C.f. y=time, closed in space, spreading in time

(Quadratic form $ax^2 + cy^2 + bxy$. Ellipse if $b^2 < 4ac$

Hyperbola if $b^2 > 4ac$)



Ellipse spreading (*xy-t*)

B. Differential operators $x \rightarrow \partial/\partial x$ (covariant, contravariant)

$x \longleftrightarrow \partial/\partial x \qquad y \longleftrightarrow \partial/\partial y$

$a \longleftrightarrow 1/a \qquad b \longleftrightarrow 1/b$

operate on some function, $\phi(x,y)$

Elliptic: $\underbrace{a^2\partial^2\phi/\partial x^2 + b^2\partial^2\phi/\partial y^2}_{\text{second order terms, all + signs}} + c\partial\phi/\partial x \ldots$ first derivatives don't alter type -- like a shift of origin.

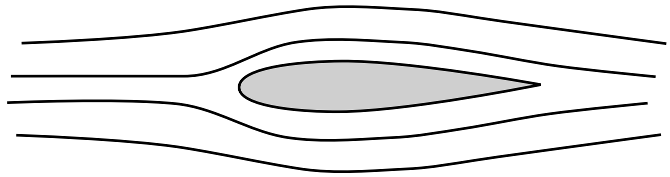
Hyperbolic: $\underbrace{a^2\partial^2\phi/\partial x^2 - b^2\partial^2\phi/\partial y^2}_{\text{one or more - sign}} + c\partial\phi/\partial x \ldots$

Parabolic: $\underbrace{a^2\partial^2\phi/\partial x^2 - b\partial\phi/\partial y}_{\text{first deriv. in y}}$ commonly in time: $a^2\partial^2\phi/\partial x^2 - b\partial\phi/\partial t$

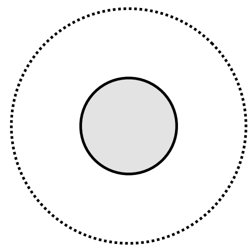
C. Physics → suitable numerics. I.e., where do these types of p.d.e. arise?

1. Elliptic: action at a distance, pressure field in incompressible flow, gravity, electrostatics.. Kinematics

$$\partial_x^2\psi + \partial_y^2\psi = \omega$$

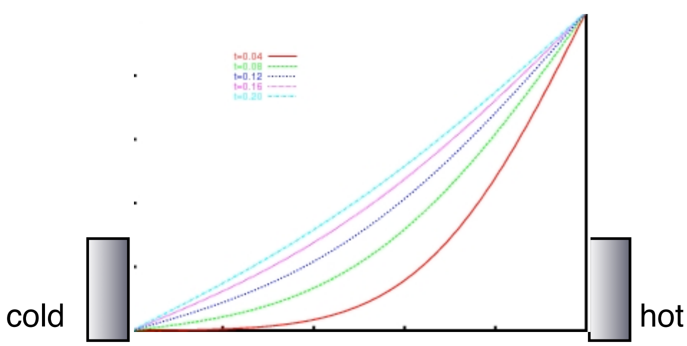


$$\nabla^2\psi = \omega$$

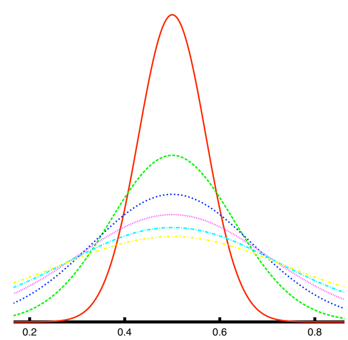


Not just 1/r, but that is the idea. Non-local

2. Parabolic: diffusion. Time-dependent heat transfer. Suggests marching in time direction (or downstream for boundary layers).



$$\delta \sim \sqrt{4\kappa t}$$

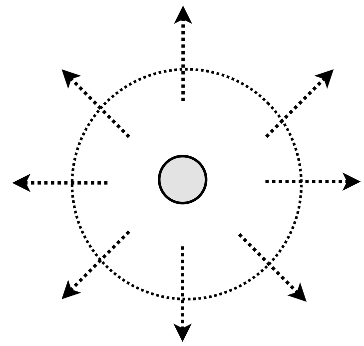


$$\int_{-x}^{x} T dx = \kappa \partial_x T|_{-x}^{x} \to 0 \; x \to \infty$$

unbounded hot spot

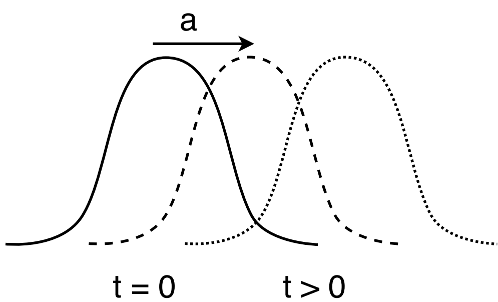
total heat is conserved

3. Hyperbolic: wave propagation
   Sound, E&M. capillary waves on pond



$$\partial_t^2 \psi - a^2 \partial_y^2 \psi = 0$$

$$\psi(x, t=0) = \psi_0(x) \;\; \rightarrow \;\; \psi = \psi_0(y \pm at)$$



left and right moving: depends on other i.c
give derivation (η=x ± at).

General classification: $ax^2 + bxy + cy^2$,

$$(x, y) \cdot \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{x} \cdot {}^T\boldsymbol{U} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \boldsymbol{U} \cdot {}^T\boldsymbol{x} = \boldsymbol{x'} \cdot \boldsymbol{\Lambda} \cdot {}^T\boldsymbol{x'}$$

Diagonalized form: $\lambda_1 \xi_1^2 + \lambda_2 \xi_2^2$ Type depends on sign of eigenvalues.
Elliptic if eigenvalues have same sign (>0 w/o.l.g) ; hyperbolic if one positive, one negative.

$\lambda = [\, a+c \pm \sqrt{b^2 + (a-c)^2}\,]/2$; both are same sign if $(a+c)^2 > b^2 + (a-c)^2$ or $\boxed{b^2 < 4ac}$

b=4ac is degenerate, $\lambda_2=0$ (parabolic)

E.g., $ax^2 + 2bxy$ (+ex) = 5 is hyperbola (c=0) $\Rightarrow$ $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial x\partial y = 5$ is hyperbolic.

$$\eta = x + y,\ \xi = x - y$$

So is $\partial^2\phi/\partial x\partial y=0$:

$$\begin{aligned} \partial_x\phi &= (\partial\eta/\partial x)\partial_\eta\phi + (\partial\xi/\partial x)\partial_\xi\phi = \partial_\eta\phi + \partial_\xi\phi \\ \partial_y(\partial_x\phi) &= (\partial\eta/\partial y)\partial_\eta(\partial_x\phi) + (\partial\xi/\partial y)\partial_\xi(\partial_x\phi) \\ &= \partial_\eta(\partial_\eta\phi + \partial_\xi\phi) - \partial_\xi(\partial_\eta\phi + \partial_\xi\phi) \\ &= \partial_\eta^2\phi - \partial_\xi^2\phi \end{aligned}$$

$\partial^2_t \phi - c^2\partial^2_x \phi = 0$	$\phi = \sin(x \pm ct)$	hyperbolic: c.f. wave
$\partial^2_y \phi + c^2\partial^2_x \phi = 0$	$\phi = \sin(x)e^{\pm cy}$	elliptic: c.f. + and - charge layer
$\partial_t \phi - c^2\partial^2_x \phi = 0$	$\phi = \sin(x)\, e^{-ct}$	parabolic: c.f. hot and cold wall

Parabolic adds damping: recall