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MATH 517 Finite Difference Methods
Homework 1

1. Consider a nonuniform grid $x_1 < x_2 < x_3 < x_4$. Derive a finite difference approximation of $u''(x_2)$ that is as accurate as possible for smooth functions $u(x)$, based on the four values $U_1 = u(x_1)$, $U_2 = u(x_2)$, $U_3 = u(x_3)$, and $U_4 = u(x_4)$. Give an expression for the dominant term in the error.

   First let $h_1 = x_2 - x_1$, $h_2 = x_3 - x_2$ and $h_3 = x_4 - x_3$. In order to approximate $u''(x_2)$, we will use a linear combination of $U_1, \dots, U_4$, that is we will find coefficients $\omega_1, \dots, \omega_4$ such that $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + E$, where the error, E, is as small as possible.

   U_1, U_3, and U_4 can be expressed as Taylor expansions about U_2 as follows

$$U_1 = u(x_1) = u(x_2) + u'(x_2)(-h_1) + \frac{1}{2}u''(x_2)(-h_1)^2 + \frac{1}{6}u'''(x_2)(-h_1)^3 + \frac{1}{24}u^{(4)}(c_1)(-h_1)^4$$
$$U_3 = u(x_1) = u(x_2) + u'(x_2)(h_2) + \frac{1}{2}u''(x_2)(h_2)^2 + \frac{1}{6}u'''(x_2)(h_2)^3 + \frac{1}{24}u^{(4)}(c_2)(h_2)^4$$
$$U_4 = u(x_1) = u(x_2) + u'(x_2)(h_2 + h_3) + \frac{1}{2}u''(x_2)(h_2 + h_3)^2 + \frac{1}{6}u'''(x_2)(h_2 + h_3)^3 + \frac{1}{24}u^{(4)}(c_3)(h_2 + h_3)^4$$

   where $c_1 \in [x_1, x_2]$, $c_2 \in [x_2, x_3]$, and $c_3 \in [x_2, x_4]$.

   Substituting these Taylor expansions into the linear combination and gathering the function and derivative values of u at x_2 results in

$$\begin{aligned}
\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 &= (\omega_1 + \omega_2 + \omega_3 + \omega_4)u(x_2) \\
&+ (-h_1\omega_1 + h_2\omega_3 + (h_2 + h_3)\omega_4)u'(x_2) \\
&+ \frac{1}{2}\Big(h_1^2\omega_1 + h_2^2\omega_3 + (h_2 + h_3)^2\omega_4\Big)u''(x_2) \\
&+ \frac{1}{6}\Big(-h_1^3\omega_1 + h_2^3\omega_3 + (h_2 + h_3)^3\omega_4\Big)u'''(x_2) \\
&+ \frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)^4\omega_4 u^{(4)}(c_3)\Big)
\end{aligned}$$

   Since there are four coefficients to set in the linear combination we can specify up to 4 conditions on these coefficients to get the best possible approximation of $u''(x_2)$. These equations are as follows

$$\begin{aligned}
\omega_1 + \omega_2 + \omega_3 + \omega_4 &= 0 \\
-h_1\omega_1 + h_2\omega_3 + (h_2 + h_3)\omega_4 &= 0 \\
h_1^2\omega_1 + h_2^2\omega_3 + (h_2 + h_3)^2\omega_4 &= 2 \\
-h_1^3\omega_1 + h_2^3\omega_3 + (h_2 + h_3)^3\omega_4 &= 0
\end{aligned}$$

   If these equations are satisfied, then

$$\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + \frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)^4\omega_4 u^{(4)}(c_3)\Big)$$

   where the approximation is $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4$ and the error is $\frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)$

Using Mathematica, this system of equations can be solved, to find that the coefficients are

$$\begin{aligned}
\omega_1 &= \frac{2(2h_2+h_3)}{h_1(h_1+h_2)(h_1+h_2+h_3)} \\
\omega_2 &= \frac{2h_1-4h_2-2h_3}{h_1h_2^2+h_1h_2h_3} \\
\omega_3 &= \frac{2(-h_1+h_2+h_3)}{h_2(h_1+h_2)h_3} \\
\omega_4 &= \frac{2(h_1-h_2)}{h_3(h_2+h_3)(h_1+h_2+h_3)}
\end{aligned}$$

Since u is a smooth function, the error can be simplified using the Intermediate Value Theorem, by noting that

$$\begin{aligned}
\frac{h_1^4\omega_1 u^{(4)}(c_1)+h_2^4\omega_3 u^{(4)}(c_2)}{h_1^4\omega_1+h_2^4\omega_3} &= u^{(4)}(\rho) \\
h_1^4\omega_1 u^{(4)}(c_1)+h_2^4\omega_3 u^{(4)}(c_2) &= \left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)
\end{aligned}$$

for some $\rho\in[x_1,x_3]$. Thus the error becomes

$$\frac{1}{24}\left(\left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3)\right).$$

The Intermediate Value Theorem can be used again to see that

$$\begin{aligned}
\frac{(h_1^4\omega_1+h_2^4\omega_3)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3)}{h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4} &= u^{(4)}(\mu) \\
\left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3) &= \left(h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4\right)u^{(4)}(\mu)
\end{aligned}$$

for $\mu\in[x_1,x_4]$.

The error can thus be written as

$$E=\frac{1}{24}\left(h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4\right)u^{(4)}(\mu).$$

Substituting in for ω_1, ω_3, and ω_4 and simplifying results in

$$E=-\frac{1}{12}(h_2(h_2+h_3)-h_1(2h_2+h_3))u^{(4)}(\mu).$$

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4. Consider the following 2-pt BVP:

$$\begin{gathered}
u''+u=f(x), \quad \text{on} \quad 0\le x\le 10 \\
u'(0)-u(0)=0, \quad u'(10)+u(10)=0
\end{gathered}$$

Construct a second order accurate finite-difference method for this BVP. Write your method as a linear system in the form $A\mathbf{u}=\mathbf{f}$.

First the interval $[0,10]$ needs to be discretized. Let $x_0=0$, $x_{N+1}=10$, and let $x_i=ih$ for $1\le i\le N$, where $h=\frac{10}{N+1}$. Thus the interval $[0,10]$ is described as a grid of $N+2$ equally spaced points with grid spacing h. Thus the solution to this BVP will be approximated on these grid points. Let U_i be the approximate value of $u(x_i)$. Thus an approximate solution to this BVP will be the values of U_i for $0 \leq i \leq N+1$.

Based on the boundary conditions be can find expressions for U_0 and U_{N+1}. The first boundary condition states that

$$u'(0) - u(0) = 0$$

We can approximate $u'(0)$ with a second order finite difference.

$$u'(0) \approx \frac{-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0}{h}$$

Thus the boundary condition can be rewritten in terms of the discretization as follows

$$-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0 - hU_0 = 0U_0 \qquad = (4U_1 - U_2)\frac{1}{3+2h}$$

The second boundary condition can be similarly manipulated.

$$\begin{aligned} u'(10) + u(10) &= 0 \\ u'(10) &\approx \frac{\frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1}}{h} \\ \frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1} + hU_{N+1} &= 0 \\ U_{N+1} &= (4U_N - U_{N-1})\frac{1}{3+2h} \end{aligned}$$

Now that expressions for U_0 and U_{N+1} have been found, we can find N equations for the remaining N unkowns, U_i for $1 \leq i \leq N$. In order that the finite-difference method is second order accurate, I will use the second order central difference to approximate the second derivative. This finite difference is

$$u''(x_i) \approx \frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1})$$

for $1 \leq i \leq N$.

This finite difference can then be used in the differential equation to create N equations as follows

$$\begin{aligned} \frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1}) + U_i &= f(x_i) \\ \frac{1}{h^2}\Big(U_{i-1} + (-2 + h^2)U_i + U_{i+1}\Big) &= f(x_i) \end{aligned}$$

for $2 \leq i \leq N-1$. For $i = 1$ and $i = N$, we need to substitute the expression for U_0 and U_{N+1} respectively. This results in

$$\begin{aligned} f(x_1) &= \frac{1}{h^2}\left((4U_1 - U_2)\frac{1}{3+2h} - 2U_1 + U_2\right) + U_1 \\ f(x_1) &= \frac{1}{h^2}\left(\left(\frac{4}{3+2h} - 2 + h^2\right)U_1 + \left(1 - \frac{1}{3+2h}\right)U_2\right) \\ f(x_N) &= \frac{1}{h^2}\left(U_{N-1} - 2U_N + (4U_N - U_{N-1})\frac{1}{3+2h}\right) + U_N \\ f(x_N) &= \frac{1}{h^2}\left(\left(1 - \frac{1}{3+2h}\right)U_{N-1} + \left(\frac{4}{3+2h} - 2 + h^2\right)U_N\right) \end{aligned}$$

These N equations can be expressed as the matrix equation

$$A\mathbf{u} = \mathbf{f}$$

where

$$\begin{aligned}
\mathbf{u} &= [U_1, U_2, \cdots, U_N]^T \\
\mathbf{f} &= [f(x_1), f(x_2), \cdots, f(x_N)]^T \\
A &= \frac{1}{h^2}\begin{bmatrix} \frac{4}{3+2h} - 2 + h^2 & 1 - \frac{1}{3+2h} & & & \\ 1 & -2 + h^2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 + h^2 & 1 \\ & & & 1 - \frac{1}{3+2h} & \frac{4}{3+2h} - 2 + h^2 \end{bmatrix}
\end{aligned}$$

A is a tridiagonal matrix, so all other entries are zero. Therefore to approximate the solution solve the system $A\mathbf{u} = \mathbf{f}$ and then plug in values for the expressions of U_0 and U_{N+1}.

5. Construct the exact solution to the BVP with $f(x) = -e^x$.

   This is a nonhomogenous ODE, so the solution to this BVP is found by summing the solutions to the homogenous and nonhomogenous equations. First I will begin by solving the homogenous version of the BVP, which is

$$u'' + u = 0$$

   It is known that the general solution to this homogenous ODE is of the form

$$\begin{aligned}
u(x) &= a\sin(x) + b\cos(x) \\
u''(x) &= -a\sin(x) - b\cos(x)
\end{aligned}$$

   which clearly satisfies the homogenous ODE.

   A solution to the nonhomogenous ODE can be found as well. In this case the ODE is

$$u'' + u = -e^x$$

   I will guess that the solution is of the form

$$\begin{aligned}
u(x) &= c_1e^x + c_2e^{-x} \\
u''(x) &= c_1e^x + c_2e^{-x}.
\end{aligned}$$

   Substituting this into the ODE results in

$$2c_1e^x + 2c_2e^{-x} = -e^x.$$

   Therefore

$$\begin{aligned}
c_1 &= -\frac{1}{2} \\
c_2 &= 0
\end{aligned}$$

   Thus the nonhomogenous solution is

$$u(x) = -\frac{1}{2}e^x$$

Therefore the overall solution to this BVP is

$$u(x) = -\frac{1}{2}e^x + a\sin(x) + b\cos(x)$$

Finally we must find a and b such that $u(x)$ satisfies the boundary conditions.

$$\begin{aligned}
u'(x) &= -\frac{1}{2}e^x + a\cos(x) - b\sin(x) \\
u(0) &= -\frac{1}{2} + b \\
u'(0) &= -\frac{1}{2} + a \\
0 &= u'(0) - u(0) \\
&= -\frac{1}{2} + a + \frac{1}{2} - b \\
a &= b \\
u(10) &= -\frac{1}{2}e^{10} + a\sin(10) + b\cos(10) \\
u'(10) &= -\frac{1}{2}e^{10} + a\cos(10) - b\sin(10) \\
0 &= u'(10) + u(10) \\
&= -e^{10} + (\cos(10) + \sin(10))a + (\cos(10) - \sin(10))b
\end{aligned}$$

Substituting in a for b.

$$\begin{aligned}
0 &= -e^{10} + 2\cos(10)a \\
a &= \frac{e^{10}}{2\cos(10)} \\
b &= \frac{e^{10}}{2\cos(10)}
\end{aligned}$$

Therefore the exact solution to the BVP is

$$u(x) = -\frac{1}{2}e^x + \frac{e^{10}}{2\cos(10)}(\sin(x) + \cos(x))$$

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