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MATH 517 Finite Difference Methods
Homework 2

1. Consider the 2-pt boundary value problem:

$$-u'' = f(x) \text{ on } 0 < x < L$$
$$u(0) = \alpha, \quad u'(L) = \sigma.$$

Discretize this problem using the $O(h^2)$ central finite differences and a ghost point near $x = L$ to handle the Neumann boundary condition. Write out the resulting linear system.

To discretize this problem let $x_i = ih$ where $h = \frac{L}{N+1}$ and N is the number of points in the discretization. Thus the solution to this BVP will be approximated on these grid points. Let U_i be the approximate value of $u(x_i)$. Thus an approximate solution to this BVP will be the values of U_i for $0 \le i \le N+1$. From the boundary conditions, it can be noted that $U_0 = \alpha$. The other boundary condition can be handled by introducing a ghost point U_{N+2} with the following two conditions

$$\frac{1}{h^2}(-U_N + 2U_{N+1} - U_{N+2}) = f(x_{N+1}) = f(L)\frac{1}{2h}(U_{N+2} - U_N) \qquad\qquad = \sigma.$$

In other words the central finite difference for the second derivative must hold on U_N, 2_{N+1}, and U_{N+2}, and the central difference for the first derivative must equal σ when centered on U_{N+1}. These two equations can be combined so that the ghost point U_{N+2} no longer appears in our finite difference method.

$$\frac{1}{h^2}(-2U_N + 2U_{N+1}) = f(x_{N+1}) + \frac{2\sigma}{h}$$

Now instead of a system of N equations for $i = 1, 2, \dots, N$, we have a system of $N+1$ equations for $i = 1, 2, \dots, N+1$. For $i = 2, \dots, N$, the central finite difference can be applied directly to the PDE to result in the following $N-1$ equations.

$$\frac{1}{h^2}(-U_{i-1} + 2U_i - U_{i+1}) = f(x_i)$$

For $i = 1$, we simply replace U_0 with α and simplify to

$$\frac{1}{h^2}(2U_1 - U_2) = f(x_1) + \frac{\alpha}{h^2}$$

For $i = N+1$, we use the equation we found before after removing the ghost point.

$$\frac{1}{h^2}(-2U_N + 2U_{N+1}) = f(x_{N+1}) + \frac{2\sigma}{h}$$

Then this system of equations cen be written as a matrix equation as follows

$$A\mathbf{U} = \mathbf{f}$$

where

$$\mathbf{U} = [U_1, U_2, \cdots, U_{N+1}]^T$$
$$\mathbf{f} = \left[f(x_1) + \frac{\alpha}{h^2}, f(x_2), \cdots, f(x_N), f(x_{N+1}) + \frac{2\sigma}{h}\right]^T$$
$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}$$

2. Find the Green's function that satisfies:

$$-G'' = \delta(x-\xi), \quad G(0;\xi) = 0, \quad G'(L;\xi) = 0.$$

3. Use the result from Problem 2 to write out the exact solution to the boundary value problem with general $f(x)$, α, and σ.

4. Use the results from Problems 2 and 3 to find the exact inverse to the finite difference matrix found in Problem 1.

5. Use the result in Problem 4 to prove that the finite difference method in Problem 1 is L_∞-stable.

6. Consider the uniform mesh $x_j = jh$ and let

$$U_j = u(x_j) \quad \text{and} \quad W_j \approx u'(x_j).$$

In standard finite differences, we typically find linear combinations of U_j to define the approximation W_i to $u'(x_j)$:

$$W_i = \sum_j \beta_j U_j$$

In compact finite differences we are allowed to generalize this to

$$\sum_j \alpha_j W_j = \sum_j \beta_j U_j$$

Find the compact finite difference with the optimal local truncation error that has the following form:

$$\alpha W_{j-1} + W_j + \alpha W_{j+1} = \beta\left(\frac{U_{j+1} - U_{j-1}}{2h}\right).$$

We can find the local truncation error by inserting the exact solution into the finite difference equation and using Taylor series.

$$\begin{aligned}
\tau_j &= -\alpha u'(x_{j-1}) - u'(x_j) - \alpha u'(x_{j+1}) + \beta\left(\frac{u(x_{j+1}) - u(x_{j-1})}{2h}\right) \\
u(x_{j-1}) &= u(x_j) - hu'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) - \frac{h^5}{120}u^{(5)}(x_j) + O(h^6) \\
u(x_{j+1}) &= u(x_j) + hu'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) + \frac{h^5}{120}u^{(5)}(x_j) + O(h^6) \\
u'(x_{j-1}) &= u'(x_j) - hu''(x_j) + \frac{h^2}{2}u'''(x_j) - \frac{h^3}{6}u^{(4)}(x_j) + \frac{h^4}{24}u^{(5)}(x_j) - \frac{h^5}{120}u^{(6)}(x_j) + O(h^6) \\
u'(x_{j+1}) &= u'(x_j) + hu''(x_j) + \frac{h^2}{2}u'''(x_j) + \frac{h^3}{6}u^{(4)}(x_j) + \frac{h^4}{24}u^{(5)}(x_j) + \frac{h^5}{120}u^{(6)}(x_j) + O(h^6)
\end{aligned}$$

Substituting in these Taylor series and simplifying results in

$$\tau_j = (-1 - 2\alpha + \beta)u'(x_j) + \frac{h^2}{6}(-6\alpha + \beta)u^{(3)}(x_j) + \frac{h^4}{120}(-10\alpha + \beta)u^{(5)}(x_j) + O(h^6)$$

To make the local truncation error as small as possible, we must choose α and β such that the following two equations are satisfied.

$$\begin{aligned}
0 &= -1 - 2\alpha + \beta \\
0 &= -6\alpha + \beta
\end{aligned}$$

If both of these equations are satisfied then this finite difference will be a fourth order approximation. These two equations can be solved as follows.

$$\begin{aligned}
\beta &= 6\alpha \\
0 &= -1 - 2\alpha + 6\alpha \\
1 &= 4\alpha \\
\alpha &= \frac{1}{4} \\
\beta &= \frac{3}{2}
\end{aligned}$$

If α and β are set to these values, then the local truncation error becomes

$$\tau_j = \frac{h^4}{120}(-10\alpha + \beta)u^{(5)}(x_j) + O(h^6)$$

.

Thus the compact finite difference operator will be

$$\frac{1}{4}W_{j-1} + W_j + \frac{1}{4}W_{j+1} = 3\Big(\frac{U_{j+1} - U_{j-1}}{4h}\Big).$$

which is 4th order accurate, because the local truncation error is $\tau = O(h^4)$.

7. Consider Poisson's equation in 2D:

$$\begin{aligned}
-u_{xx} - u_{yy} = f(x, y) \text{ in } \Omega = [0, 1] \times [0, 1], \\
u = g(x, y) \text{ on } \partial\Omega
\end{aligned}$$

Discretize this equation using the 5-point Laplacian on a uniform mesh $\Delta x = \Delta y = h$. Use the standard natural row-wise ordering.

8. Write a MATLAB code that constructs the sparse coefficient matrix A and the appropriate right hand side vector $\mathbf{F}$.

9. Using your code, do a numerical convergence study for the following right-hand side forcing and exact solution:

$$f(x, y) = -1.25e^{x+.5y} \quad \text{and} \quad u(x, y) = e^{x+.5y}$$