Assigned: Thursday March 3, 2016
Due: Thursday March 10, 2016

MATH 517: Homework 4
Spring 2016

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Numerical Methods for ODE IVPs

1. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

   (a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$;

   (b) $U^{n+1} = U^n$;

   (c) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$;

   (d) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

2. (a) Determine the general solution to the linear difference equation $2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$.

   Hint: One root of the characteristic polynomial is at $\zeta = 1$.

   (b) Determine the solution to this difference equation with the starting values $U^0 = 11$, $U^1 = 5$, and $U^2 = 1$. What is U^{10}?

   (c) Consider the LMM

   $$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1})).$$

   For what values of β_0 and β_1 is local truncation error $\mathcal{O}(k^2)$?

   (d) Suppose you use the values of β_0 and β_1 just determined in this LMM. Is this a convergent method?

3. Consider the so-called θ-method for $u'(t) = f(u(t))$,

   $$U^{n+1} = U^n + k\left[(1-\theta)f(U^n) + \theta f(U^{n+1})\right],$$

   where θ is a fixed parameter. Note that $\theta = 0,\ 1/2,\ 1$ all give familiar methods.

   (a) Show that this method is A-stable for $\theta \geq 1/2$.

   (b) Plot the stability region $\mathcal{S}$ for $\theta = 0,\ 1/4,\ 1/2,\ 3/4,\ 1$ and comment on how the stability region will look for other values of θ.

4. Consider the following Runge-Kutta method for the ODE $u' = f(u)$:

$$\begin{aligned} Y_1 &= f\left(U^n + \frac{1}{4}kY_1 + \left(\frac{1}{4} - \gamma\right) kY_2\right) \\ Y_2 &= f\left(U^n + \left(\frac{1}{4} + \gamma\right) kY_1 + \frac{1}{4}kY_2\right) \\ U^{n+1} &= U^n + \frac{1}{2}k\left(Y_1 + Y_2\right), \end{aligned}$$

where $\gamma = \sqrt{3}/6$.

(a) Compute the local truncation error for this method. **HINT:** it suffices to consider the function $f(u) = \lambda u$.

(b) Prove or disprove the following statement: this method is A-stable.

(c) Prove or disprove the following statement: this method is L-stable.