MATH 517 Finite Differences Homework 4

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1. Consider the Leapfrog method

$$U^{n+1} = U^{n\text{âĹŠ}1} + 2kf(U^n)$$

applied to the test problem $u\text{âĂš} = \lambda u$. The method is zero-stable and second order accurate, and hence convergent. If $\lambda < 0$ then the true solution is exponentially decaying.

On the other hand, for $\lambda < 0$ and $k > 0$ the point $z = k\lambda$ is never in the region of absolute stability of this method, and hence the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

Suppose we take $U^0 = \eta$, use Forward Euler to generate U^1, and then use the midpoint method for $n = 2, 3, \cdots$. Work out the exact solution U^n by solving the linear difference equation and explain how the apparent paradox described above is resolved

We know that $U^0 = \eta$ and $U^1 = U^0 + k\lambda U^0 = (1 + k\lambda)\eta$ by the Euler method. We also know that the solution is governed by the difference equation

$$U^{n+1} - 2k\lambda U^n - U^{n-1} = 0$$

We know that solutions to this difference equation are solutions of the following equation

$$\zeta^2 - 2k\lambda\zeta - 1 = 0$$

Using the quadractic formula solutions to this equation can be found

$$\begin{aligned}\zeta &= \frac{2k\lambda \pm \sqrt{4(k\lambda)^2 - 4(1)(-1)}}{2} \\ &= k\lambda \pm \sqrt{(k\lambda)^2 + 1}\end{aligned}$$

This implies that the general solution to the linear difference equation is of the form

$$U^n = c_1(k\lambda + \sqrt{(k\lambda)^2 + 1})^n + c_2(k\lambda - \sqrt{(k\lambda)^2 + 1})^n$$

Using the initial conditions that $U^0 = \eta$ and $U^1 = (1 + k\lambda)\eta$, it is possible to solve for the constants c_1 and c_2.

$$\begin{aligned}\eta &= c_1 + c_2 \\ c_2 &= \eta - c_1 \\ (1 + k\lambda)\eta &= c_1(k\lambda + \sqrt{(k\lambda)^2 + 1}) + c_2(k\lambda - \sqrt{(k\lambda)^2 + 1}) \\ (1 + k\lambda)\eta &= c_1(k\lambda + \sqrt{(k\lambda)^2 + 1}) + (\eta - c_1)(k\lambda - \sqrt{(k\lambda)^2 + 1}) \\ (1 + k\lambda)\eta &= 2c_1\sqrt{(k\lambda)^2 + 1} + \eta(k\lambda - \sqrt{(k\lambda)^2 + 1}) \\ c_1 &= \frac{\eta(1 + \sqrt{(k\lambda)^2 + 1})}{2\sqrt{(k\lambda)^2 + 1}} \\ c_1 &= \frac{\eta(\sqrt{(k\lambda)^2 + 1} + (k\lambda)^2 + 1)}{2(k\lambda)^2 + 2} \\ c_2 &= \eta - \frac{\eta(\sqrt{(k\lambda)^2 + 1} + (k\lambda)^2 + 1)}{2(k\lambda)^2 + 2} \\ c_2 &= \eta\frac{(k\lambda)^2 + 1 - \sqrt{(k\lambda)^2 + 1}}{2(k\lambda)^2 + 2}\end{aligned}$$

2. (a) Find the general solution of the linear difference equation

$$U^{n+3} + 2U^{n+2} - 4U^{n+1} - 8U^n = 0$$

The general solution of the linear difference equation can be found by finding the roots of the following polynomial.

$$\zeta^3 + 2\zeta^2 - 4\zeta - 8$$

This polynomial factors to

$$(\zeta - 2)(\zeta + 2)^2$$

Thus the general solution to this linear difference equation is

$$U^n = c_1 2^n + c_2(-2)^n + c_3 n(-2)^n$$

(b) Determine the particular solution with initial data $U_0 = 4$, $U_1 = -2$, and $U_2 = 8$. The particular solution can be found as follows.

$$\begin{aligned}
U^0 &= 4 = c_1 + c_2 \\
c_1 &= 4 - c_2 \\
U^1 &= -2 = 2c_1 - 2c_2 - 2c_3 \\
-2 &= 2(4 - c_2) - 2c_2 - 2c_3 \\
-10 + 4c_2 &= -2c_3 \\
c_3 &= 5 - 2c_2 U^2 & &= 8 = 4c_1 + 4c_2 + 8c_3 \\
8 &= 4(4 - c_2) + 4c_2 + 8(5 - 2c_2) \\
8 &= 16 - 4c_2 + 4c_2 + 40 - 16c_2 \\
-48 &= -16c_2 \\
c_2 &= 3 \\
c_1 &= 1 \\
c_3 &= -1
\end{aligned}$$

Thus the particular solution is $U^n = 2^n + (3 - n)(-2)^n$.

(c) Consider the iteration:

$$\begin{bmatrix} U^{n+1} \\ U^{n+2} \\ U^{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} U^n \\ U^{n+1} \\ U^{n+2} \end{bmatrix}$$

The matrix appearing here is the companion matrix for the difference equation. If this matrix is called A, then we can determine U^n from the starting values if we know A^n, the nth power of A. If $A = R\Lambda R^{-1}$ is the Jordan Canonical form for the matrix, then $A^n = R\Lambda^n R^{-1}$. Determine the eigenvalues and Jordan canonical form for this matrix and show how this is related to the general solution found in (a). The eigenvalues of A can be found by solving the following equation

$$\det(A - \lambda I) = 0 - \lambda^3 - 2\lambda^2 + 4\lambda + 8 \qquad = 0$$

This equation has the same solutions as the root of the polynomial in part (a). Therefore the eigenvalues are $\lambda = 2, -2, -2$. The Jordan Cononical form of A is

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

The eigenvalues correspond directly roots of the characteristic polynomial of the lienar difference equation.

3. Write a MATLAB script to plot the region of absolute stability of the 4-stage Runge-Kutta method. (See example 5.13 on Page 126)

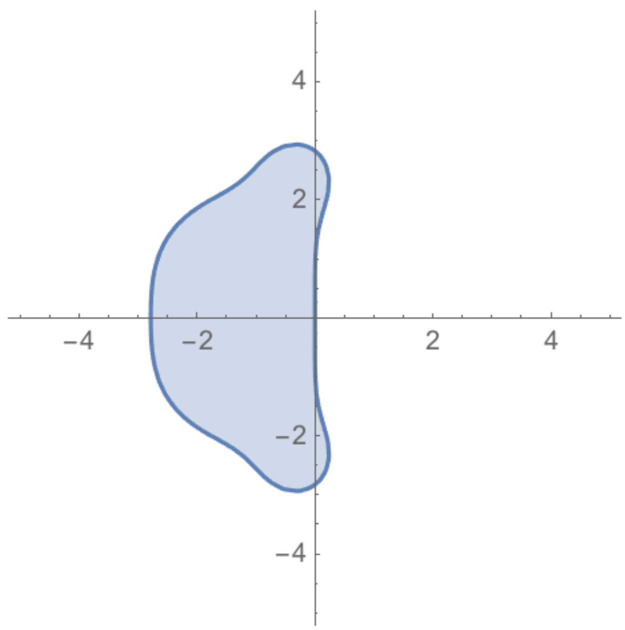
   I chose to use Mathematica instead to plot the region of absolute stability with the following script.

```
Y1 = Un;
Y2 = Un + 1/2 k L Y1;
Y3 = Un +1/2 k L (Y2);
Y4 = Un + k L(Y3);
Un1 = Un + 1/6 k L(Y1 + 2(Y2) + 2(Y3) +Y4)//FullSimplify//Expand

Un+k Un L+1/2 k^2 Un L^2+1/6 k^3 Un L^3+1/24 k^4 Un L^4

(1 + k L + 1/2 (k L)^2 + 1/6 (k L)^3 + 1/24 (k L)^4)Un

p[z_]:=1 + z + 1/2 z^2 + 1/6 z^3 + 1/24 z^4
RegionPlot[Abs[p[x + I y]]<=1,{x,-5,5},{y,-5,5},Axes->True,Frame->False]
```



4. (a) We know that the stages of a Runge-Kutta method can be expressed as

$$Y_i = f(t_n + c_i h, U^n + k\sum_{j=1}^{s}(a_{ij}Y_j))$$

$$Y_i = \lambda(U^n + k\sum_{j=1}^{s}(a_{ij}Y_j))$$

Note that the summation is equivalent to a matrix product, that is $\sum_{j=1}^{s}(a_{ij}Y_j) = (A\mathbf{Y})_i$, therefore

$$Y_i = \lambda(U^n + k(A\mathbf{Y})_i)$$

If every entry is structured this way the equivalent vector equation is

$$\mathbf{Y} = \lambda U^n \mathbf{e} + k\lambda A \mathbf{Y}$$

This vector equation can be solved for $\mathbf{Y}$

$$\begin{aligned}
\mathbf{Y} - k\lambda A \mathbf{Y} &= \lambda U^n \mathbf{e} \\
(I - k\lambda A)\mathbf{Y} &= \lambda U^n \mathbf{e} \\
\mathbf{Y} &= (I - k\lambda A)^{-1} \lambda U^n \mathbf{e}
\end{aligned}$$

Letting $z = k\lambda$

$$\mathbf{Y} = \lambda U^n (I - zA)^{-1} \mathbf{e}$$

(b) The actual timestep to calculate U^{n+1} has the following formula.

$$U^{n+1} = U^n + k \sum_{i=1}^{s} (b_i Y_i)$$

Since $\sum_{i=1}^{s} (b_i Y_i) = \mathbf{b}^T \mathbf{Y}$

$$U^{n+1} = U^n + k\mathbf{b}^T \mathbf{Y}$$

Using part (a)

$$\begin{aligned}
U^{n+1} &= U^n + k\mathbf{b}^T \lambda U^n (I - zA)^{-1} \mathbf{e} \\
U^{n+1} &= U^n + U^n z \mathbf{b}^T (I - zA)^{-1} \mathbf{e} \\
U^{n+1} &= U^n (1 + z\mathbf{b}^T (I - zA)^{-1} \mathbf{e})
\end{aligned}$$

Therefore

$$R(z) = 1 + z\mathbf{b}^T (I - zA)^{-1} \mathbf{e}$$

(c) The determinant of $A - \lambda I$ for this explicit case is

$$\det A - \lambda I = -\lambda^s.$$

Now the Cayley Hamilton theorem states that

$$-A^s = 0I.$$

Hence

$$A^s = 0I.$$

(d) Taylor expanding $(I - zA)^{-1}$ shows this.

(e) Using parts (b) and (d) we can write $R(z)$ for any explicit Runge-Kutta method as

$$\begin{aligned}
R(z) &= 1 + z\mathbf{b}^T (I - zA)^{-1} \mathbf{e} \\
R(z) &= 1 + z\mathbf{b}^T \sum_{i=0}^{s-1} \left(z^i A^i \right) \mathbf{e}
\end{aligned}$$

Let $a_i = \mathbf{b}^T A^i \mathbf{e} \in \mathbb{R}$, then

$$R(z) = 1 + \sum_{i=0}^{s-1} \left(a_i z^{i+1} \right)$$

This is a polynomial of a most degree s as the last term in the sum with have a power of s.