Due: Thursday Feb. 4, 2016

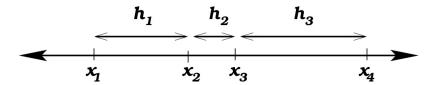
## MATH 517: Homework 1 Spring 2016

**NOTE:** For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the **subplot** command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

## Part 1: Non-Uniform Grid

Consider the non-uniform grid:



1. Derive a finite difference approximation to  $u''(x_2)$  that is accurate as possible for smooth functions u(x), based on the four values  $U_1 = u(x_1), \ldots, U_4 = u(x_4)$ . Give an expression for the dominant term in the error.

In the next two questions, you will try to determine an order of accuracy for your method:

- 2. To get a better feel for how the error behaves as the grid gets finer, take 500 values of H, where H spans 2 or 3 orders of magnitude, and for each value of H, randomly generate three numbers,  $h_1$ ,  $h_2$ , and  $h_3$ , where each  $h_i \in [0, H]$ . For each value H, compute your approximation to  $u''(x_2)$  using the randomly generated  $h_i \in [0, H]$ . Plot the error against H on a log-log plot to get a scatter plot of the behavior as  $H \to 0$ . **NOTE:** in MATLAB the command x = H\*rand(1) will produce a single random number in the range [0,H].
- 3. Estimate the order of accuracy by doing a least squares fit of the form

$$\log(E(H)) = K + p\log(H)$$

to determine K and p based on the 500 data points. Recall that this can be done by solving the following linear system in the least squares sense:

$$\begin{bmatrix} 1 & \log(H_1) \\ 1 & \log(H_2) \\ \vdots & \vdots \\ 1 & \log(H_{500}) \end{bmatrix} \begin{bmatrix} K \\ p \end{bmatrix} = \begin{bmatrix} \log(E(H_1)) \\ \log(E(H_2)) \\ \vdots \\ \log(E(H_{500})) \end{bmatrix}.$$

**NOTE:** "In the least-squares sense" means that one should solve the rectangular system Ax = b, by solving the (square) normal equation:  $A^T Ax = A^T b$ .

## Part 2: Mixed Conditions

Consider the following 2-point BVP:

$$u'' + u = f(x)$$
, on  $0 \le x \le 10$   
 $u'(0) - u(0) = 0$ ,  $u'(10) + u(10) = 0$ .

- 4. Construct a second-order accurate finite-difference method for this BVP. Write your method as a linear system of the form  $A\vec{u} = \vec{f}$ .
- 5. Construct the exact solution to this BVP with  $f(x) = -e^x$ .
- 6. Verify that your method is second order accurate by solving the BVP with  $f(x) = -e^x$  at four different grid spacings h.

**HINT 1:** Use the **spdiags** command in MATLAB to create the sparse matrix A – this will save storage and allow MATLAB to use a fast solver. Modify the following commands, which generate a tri-diagonal matrix with a [1,-2,1] structure, to model your specific BVP:

**HINT 2:** Use the command  $u = A \setminus f$  to invert your system – this will use a fast solver if you created your matrix with spdiags.

## Part 3: Periodic Boundary Conditions

Consider the following 2-point BVP:

$$-u'' + u = f(x), \quad \text{on} \quad 0 \le x \le 1$$
 
$$u(0) = u(1) \quad u'(0) = u'(1) \quad \text{(these are periodic BCs)}.$$

- 7. Construct a fourth-order accurate finite-difference method for this BVP based on the fourth-order central finite difference.
  - Write your method as a linear system of the form  $A\vec{u} = \vec{f}$ .
- 8. Construct the exact solution to this BVP with  $f(x) = \sin(4\pi x)$ .
- 9. Verify that your method is fourth-order accurate by solving the BVP with  $f(x) = \sin(4\pi x)$  at four different grid spacings h.
- 10. Prove that your method is consistent with truncation error  $\|\vec{\tau}\| = O(h^4)$  and  $L_2$ -stable, thereby proving that your method converges at  $O(h^4)$  in the  $L_2$ -norm.

**HINT:** what is the smallest eigenvalue of the discrete version of -u''? How does +u modify this?