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MATH 517 Finite Difference Methods
Homework 1

1. Consider a nonuniform grid $x_1 < x_2 < x_3 < x_4$. Derive a finite difference approximation of $u''(x_2)$ that is as accurate as possible for smooth functions $u(x)$, based on the four values $U_1 = u(x_1)$, $U_2 = u(x_2)$, $U_3 = u(x_3)$, and $U_4 = u(x_4)$. Give an expression for the dominant term in the error.

   First let $h_1 = x_2 - x_1$, $h_2 = x_3 - x_2$ and $h_3 = x_4 - x_3$. In order to approximate $u''(x_2)$, we will use a linear combination of $U_1, \dots, U_4$, that is we will find coefficients $\omega_1, \dots, \omega_4$ such that $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + E$, where the error, E, is as small as possible.

   U_1, U_3, and U_4 can be expressed as Taylor expansions about U_2 as follows

$$
\begin{aligned}
U_1 &= u(x_1) = u(x_2) + u'(x_2)(-h_1) + \frac{1}{2}u''(x_2)(-h_1)^2 + \frac{1}{6}u'''(x_2)(-h_1)^3 + \frac{1}{24}u^{(4)}(c_1)(-h_1)^4 \\
U_3 &= u(x_1) = u(x_2) + u'(x_2)(h_2) + \frac{1}{2}u''(x_2)(h_2)^2 + \frac{1}{6}u'''(x_2)(h_2)^3 + \frac{1}{24}u^{(4)}(c_2)(h_2)^4 \\
U_4 &= u(x_1) = u(x_2) + u'(x_2)(h_2+h_3) + \frac{1}{2}u''(x_2)(h_2+h_3)^2 + \frac{1}{6}u'''(x_2)(h_2+h_3)^3 + \frac{1}{24}u^{(4)}(c_3)(h_2+h_3)^4
\end{aligned}
$$

   where $c_1 \in [x_1, x_2]$, $c_2 \in [x_2, x_3]$, and $c_3 \in [x_2, x_4]$.

   Substituting these Taylor expansions into the linear combination and gathering the function and derivative values of u at x_2 results in

$$
\begin{aligned}
\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = {} & (\omega_1 + \omega_2 + \omega_3 + \omega_4)u(x_2) \\
& + (-h_1\omega_1 + h_2\omega_3 + (h_2+h_3)\omega_4)u'(x_2) \\
& + \frac{1}{2}\left(h_1^2\omega_1 + h_2^2\omega_3 + (h_2+h_3)^2\omega_4\right)u''(x_2) \\
& + \frac{1}{6}\left(-h_1^3\omega_1 + h_2^3\omega_3 + (h_2+h_3)^3\omega_4\right)u'''(x_2) \\
& + \frac{1}{24}\left(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)^4\omega_4 u^{(4)}(c_3)\right)
\end{aligned}
$$

   Since there are four coefficients to set in the linear combination we can specify up to 4 conditions on these coefficients to get the best possible approximation of $u''(x_2)$. These equations are as follows

$$
\begin{aligned}
\omega_1 + \omega_2 + \omega_3 + \omega_4 &= 0 \\
-h_1\omega_1 + h_2\omega_3 + (h_2+h_3)\omega_4 &= 0 \\
h_1^2\omega_1 + h_2^2\omega_3 + (h_2+h_3)^2\omega_4 &= 2 \\
-h_1^3\omega_1 + h_2^3\omega_3 + (h_2+h_3)^3\omega_4 &= 0
\end{aligned}
$$

   If these equations are satisfied, then

$$
\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + \frac{1}{24}\left(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)^4\omega_4 u^{(4)}(c_3)\right)
$$

   where the approximation is $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4$ and the error is $\frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)$

Using Mathematica, this system of equations can be solved, to find that the coefficients are

$$\begin{aligned}
\omega_1 &= \frac{2(2h_2 + h_3)}{h_1(h_1 + h_2)(h_1 + h_2 + h + 3)} \\
\omega_2 &= \frac{2h_1 - 4h_2 - 2h_3}{h_1 h_2^2 + h_1 h_2 h_3} \\
\omega_3 &= \frac{2(-h_1 + h_2 + h_3)}{h_2(h_1 + h_2)h_3} \\
\omega_4 &= \frac{2(h_1 - h_2)}{h_3(h_2 + h_3)(h_1 + h_2 + h_3)}
\end{aligned}$$

Since u is a smooth function, the error can be simplified using the Intermediate Value Theorem, by noting that

$$\begin{aligned}
\frac{h_1^4 \omega_1 u^{(4)}(c_1) + h_2^4 \omega_3 u^{(4)}(c_2)}{h_1^4 \omega_1 + h_2^4 \omega_3} &= u^{(4)}(\rho) \\
h_1^4 \omega_1 u^{(4)}(c_1) + h_2^4 \omega_3 u^{(4)}(c_2) &= \left(h_1^4 \omega_1 + h_2^4 \omega_3\right) u^{(4)}(\rho)
\end{aligned}$$

for some $\rho \in [x_1, x_3]$. Thus the error becomes

$$\frac{1}{24}\left(\left(h_1^4 \omega_1 + h_2^4 \omega_3\right) u^{(4)}(\rho) + (h_2 + h_3)^4 \omega_4 u^{(4)}(c_3)\right)$$

The Intermediate Value Theorem can be used again to see that

$$\begin{aligned}
\frac{(h_1^4 \omega_1 + h_2^4 \omega_3) u^{(4)}(\rho) + (h_2 + h_3)^4 \omega_4 u^{(4)}(c_3)}{h_1^4 \omega_1 + h_2^4 \omega_3 + (h_2 + h_3)^4 \omega_4} &= u^{(4)}(\mu) \\
\left(h_1^4 \omega_1 + h_2^4 \omega_3\right) u^{(4)}(\rho) + (h_2 + h_3)^4 \omega_4 u^{(4)}(c_3) &= \left(h_1^4 \omega_1 + h_2^4 \omega_3 + (h_2 + h_3)^4 \omega_4\right) u^{(4)}(\mu)
\end{aligned}$$

2.