MATH 517: HOMEWORK 6
SPRING 2016

**NOTE:** For each homework assignment observe the following guidelines:

- Include a cover page.

- Always clearly label all plots (title, $x$-label, $y$-label, and legend).

- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

1. Consider the following method for solving the heat equation $u_{,t} = u_{,x,x}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}).$$

   (a) Determine the order of accuracy of this method (in both space and time).

   (b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of $\alpha$ (if any) will this method be Lax-Richtmyer stable and hence convergent?

   **Hint:** Consider the MOL interpretation and the stability region of the time-discretization being used.

   (c) Is this a useful method?

2. Consider the one-dimensional heat equation:

$$
\begin{aligned}
\textbf{PDE}: \quad & u_{,t} = \kappa u_{,x,x} \quad \text{for} \quad (t,x) \in [0,T] \times [0,1], \\
\textbf{BCs}: \quad & u(t,0) = g_0, \quad u(t,1) = g_1, \\
\textbf{ICs}: \quad & u(0,x) = f(x).
\end{aligned}
$$

   (a) The m-file `heat_CN.m` solves the heat equation $u_{,t} = \kappa u_{,x,x}$ using the Crank-Nicolson method. Run this code, and by changing the number of grid points, confirm that it is second-order accurate. Create a table of errors for various $h$ values and $k = 4h$ at $T = 1$.

   (b) Modify `heat_CN.m` to produce a new m-file `heat_trbdf2.m` that implements the TR-BDF2 method on the same problem. Test it to confirm that it is also second order accurate. Explain how you determined the proper boundary conditions in each stage of this Runge-Kutta method.

3. Again, consider the 1D heat equation.

   (a) Modify `heat_CN.m` to solve the heat equation for $-1 \leq x \leq 1$ with step function initial data
$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

With appropriate Dirichlet boundary conditions, the exact solution is

$$u(t,x) = \frac{1}{2} \operatorname{erfc}\left(x/\sqrt{4\kappa t}\right),$$

where erfc is the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2}\, dz.$$

   i. Test this routine $m = 39$ and $k = 4h$. Note that there is an initial rapid transient decay of the high wave numbers that is not captured well with this size time step.

   ii. How small do you need to take the time step to get reasonable results? For a suitably small time step, explain why you get much better results by using $m = 38$ than $m = 39$. What is the observed order of accuracy as $k \to 0$ when $k = \alpha h$ with $\alpha$ suitably small and $m$ even?

(b) Modify `heat_trbdf2.m` (see Problem 2) to solve the heat equation for $-1 \le x \le 1$ with step function initial data as above. Test this routine using $k = 4h$ and estimate the order of accuracy as $k \to 0$ with $m$ even. Why does the TR-BDF2 method work better than Crank-Nicolson?

4. Consider the following scheme for solving the 1D heat equation:

$$U_j^{n+1} = U_j^{n-1} + \frac{2k}{h^2}\left(U_{j+1}^n - \left(U_j^{n+1} + U_j^{n-1}\right) + U_{j-1}^n\right).$$

(a) Determine the local truncation error of this scheme.

(b) Using von Neumann stability analysis, determine conditions on the time step $k$ for which this method is stable.