MATH 517: HOMEWORK 5
SPRING 2016

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Numerical Methods for ODE IVPs

1. Consider the Leapfrog method

$$U^{n+1} = U^{n-1} + 2kf(U^n)$$

applied to the test problem $u' = \lambda u$. The method is zero-stable and second order accurate, and hence convergent. If $\lambda < 0$ then the true solution is exponentially decaying.

On the other hand, for $\lambda < 0$ and $k > 0$ the point $z = k\lambda$ is never in the region of absolute stability of this method, and hence the numerical solution should be growing exponentially for any nonzero time step. (And yet it converges to a function that is exponentially decaying.)

Suppose we take $U^0 = \eta$, use Forward Euler to generate U^1, and then use the midpoint method for $n = 2,\ 3,\ \ldots$. Work out the exact solution U^n by solving the linear difference equation and explain how the apparent paradox described above is resolved.

2. (a) Find the general solution of the linear difference equation:

$$U^{n+3} + 2U^{n+2} - 4U^{n+1} - 8U^n = 0.$$

(b) Determine the particular solution with initial data $U_0 = 4$, $U_1 = -2$, $U_2 = 8$.

(c) Consider the iteration:

$$\begin{bmatrix} U^{n+1} \\ U^{n+2} \\ U^{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 4 & -2 \end{bmatrix} \begin{bmatrix} U^{n} \\ U^{n+1} \\ U^{n+2} \end{bmatrix}$$

The matrix appearing here is the companion matrix for the difference equation. If this matrix is called A, then we can determine U^n from the starting values if we know A^n, the n^{th} power of A. If $A = R\Lambda R^{-1}$ is the Jordan Canonical form for the matrix, then $A^n = R\Lambda^n R^{-1}$. Determine the eigenvalues and Jordan Canonical form for this matrix and show how this is related to the general solution found in (a).

3. Write a MATLAB script to plot the region of absolute stability of the 4-stage Runge-Kutta method (see Example 5.13 on Page 126).

4. A general Runge-Kutta method has a Butcher tableau of the following form:

$$\begin{array}{c|c} \vec{c} & \mathcal{A} \\ \hline & \vec{b}^T \end{array}$$

where $\mathcal{A}$ is an $s \times s$ matrix, $\vec{b} = [b_1, b_2, \ldots, b_s]$, $\vec{c} = [c_1, c_2, \ldots, c_s]$.

(a) Show that this method when applied to $u' = \lambda u$ can be written as

$$\vec{Y} = \lambda U^n \left(\mathbb{I} - z\mathcal{A}\right)^{-1} \vec{e},$$

where $z = k\lambda$, $\vec{Y} = [Y_1, Y_2, \ldots, Y_s]$, $\mathbb{I}$ is the $s \times s$ identity matrix, and $\vec{e} = [1, 1, 1, \ldots, 1]$ is a vector of length s.

(b) Use this result to show that

$$R(z) = 1 + z\vec{b}^T \left(\mathbb{I} - z\mathcal{A}\right)^{-1} \vec{e}.$$

(c) For the remainder of this problem we concentrate on the explicit case where $\mathcal{A}$ is a strictly lower triangular matrix:

$$\mathcal{A} = \begin{bmatrix} 0 & & & & \\ a_{21} & 0 & & & \\ a_{31} & a_{32} & 0 & & \\ & \ddots & \ddots & \ddots & \\ a_{s1} & a_{s2} & \cdots & a_{s\,s-1} & 0 \end{bmatrix}.$$

Show that taking s powers of the matrix $\mathcal{A}$ results in the zero matrix:

$$\mathcal{A}^s = 0\mathbb{I}.$$

HINT: Prove this result using the Cayley-Hamilton Theorem, which states that if you compute the characteristic polynomial of a matrix $\mathcal{A}$:

$$p(\lambda) = \det(\mathcal{A} - \lambda\mathbb{I}),$$

then you can replace all the λ's in the characteristic polynomial by the matrix $\mathcal{A}$ and then

$$p(\mathcal{A}) = 0\mathbb{I}.$$

(d) Use the result from Part (c) to show prove that

$$\left(\mathbb{I} - z\mathcal{A}\right)^{-1} = \mathbb{I} + z\mathcal{A} + z^2\mathcal{A}^2 + z^3\mathcal{A}^3 + \cdots + z^{s-1}\mathcal{A}^{s-1}.$$

HINT: Taylor expand $f(z) = \left(\mathbb{I} - z\mathcal{A}\right)^{-1}$ about $z = 0$.

(e) Use the result from Parts (c) and (d) to prove that for any s-stage explicit Runge-Kutta method $R(z)$ is a polynomial of degree of at most s.