Assigned: Thursday Feb. 4, 2016
Due: Thursday Feb. 18, 2016

MATH 517: Homework 2
Spring 2016

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Part 1: Green's function

Consider the 2-point boundary value problem:

$$-u'' = f(x) \quad \text{on} \quad 0 < x < L$$
$$u(0) = \alpha, \quad u'(L) = \sigma.$$

1. Discretize this problem using $\mathcal{O}\left(h^2\right)$ central finite differences and a ghost point near $x = L$ to handle the Neumann boundary condition. Write out the resulting linear system.
2. Find the Green's function that satisfies:
$$-G'' = \delta\left(x - \xi\right), \quad G(0;\xi) = 0, \quad G'(L;\xi) = 0.$$
3. Use the result from Problem 2 to write out the exact solution to the boundary value problem with general $f(x)$, α, and σ.
4. Use the results from Problems 2 and 3 to find the exact inverse of the finite difference matrix found in Problem 1.
5. Use the result in Problem 4 to prove that the finite difference method from Problem 1 is L_∞-stable.

Part 2: Compact finite differences

Consider the uniform mesh $x_j = jh$ and let

$$U_j = u(x_j) \qquad \text{and} \qquad W_j \approx u'(x_j).$$

In standard finite differences, we typically find linear combinations of U_j to define the approximation W_i to $u'(x_i)$:

$$W_i = \sum_j \beta_j \, U_j.$$

In *compact* finite differences we are allowed to generalize this to

$$\sum_j \alpha_j W_j = \sum_j \beta_j \, U_j.$$

6. Find the compact finite difference operator with the optimal local truncation error that has the following form:

$$\alpha W_{j-1} + W_j + \alpha W_{j+1} = \beta \left(\frac{U_{j+1} - U_{j-1}}{2h} \right).$$

Part 3: 5-point Laplacian

Consider Poisson's equation in 2D:

$$\begin{aligned} -u_{,x,x} - u_{,y,y} &= f(x,y) \;\text{ in } \Omega = [0,1] \times [0,1], \\ u &= g(x,y) \;\text{ on } \partial\Omega. \end{aligned}$$

7. Discretize this equation using the 5-point Laplacian on a uniform mesh $\Delta x = \Delta y = h$. Use the standard natural row-wise ordering.

8. Write a MATLAB code that constructs the sparse coefficient matrix A and the appropriate right-hand side vector $\vec{F}$.

9. Using your code, do a numerical convergence study for the following right-hand side forcing and exact solution:

$$f(x,y) = -1.25 e^{x+0.5y} \quad \text{and} \quad u(x,y) = e^{x+0.5y}.$$

   Just use the built-in backslash operator in MATLAB to solve the linear system (in this case the backslash operator will use a sparse LU decomposition + forward and backward substitution).