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MATH 517 Finite Difference Methods
Homework 1

1. Consider a nonuniform grid $x_1 < x_2 < x_3 < x_4$. Derive a finite difference approximation of $u''(x_2)$ that is as accurate as possible for smooth functions $u(x)$, based on the four values $U_1 = u(x_1)$, $U_2 = u(x_2)$, $U_3 = u(x_3)$, and $U_4 = u(x_4)$. Give an expression for the dominant term in the error.

   First let $h_1 = x_2 - x_1$, $h_2 = x_3 - x_2$ and $h_3 = x_4 - x_3$. In order to approximate $u''(x_2)$, we will use a linear combination of $U_1, \dots, U_4$, that is we will find coefficients $\omega_1, \dots, \omega_4$ such that $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + E$, where the error, E, is as small as possible.

   U_1, U_3, and U_4 can be expressed as Taylor expansions about U_2 as follows

$$U_1 = u(x_1) = u(x_2) + u'(x_2)(-h_1) + \frac{1}{2}u''(x_2)(-h_1)^2 + \frac{1}{6}u'''(x_2)(-h_1)^3 + \frac{1}{24}u^{(4)}(c_1)(-h_1)^4$$
$$U_3 = u(x_1) = u(x_2) + u'(x_2)(h_2) + \frac{1}{2}u''(x_2)(h_2)^2 + \frac{1}{6}u'''(x_2)(h_2)^3 + \frac{1}{24}u^{(4)}(c_2)(h_2)^4$$
$$U_4 = u(x_1) = u(x_2) + u'(x_2)(h_2+h_3) + \frac{1}{2}u''(x_2)(h_2+h_3)^2 + \frac{1}{6}u'''(x_2)(h_2+h_3)^3 + \frac{1}{24}u^{(4)}(c_3)(h_2+h_3)^4$$

   where $c_1 \in [x_1, x_2]$, $c_2 \in [x_2, x_3]$, and $c_3 \in [x_2, x_4]$.

   Substituting these Taylor expansions into the linear combination and gathering the function and derivative values of u at x_2 results in

$$\begin{aligned}
\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 &= (\omega_1 + \omega_2 + \omega_3 + \omega_4)u(x_2) \\
&+ (-h_1\omega_1 + h_2\omega_3 + (h_2+h_3)\omega_4)u'(x_2) \\
&+ \frac{1}{2}\Big(h_1^2\omega_1 + h_2^2\omega_3 + (h_2+h_3)^2\omega_4\Big)u''(x_2) \\
&+ \frac{1}{6}\Big(-h_1^3\omega_1 + h_2^3\omega_3 + (h_2+h_3)^3\omega_4\Big)u'''(x_2) \\
&+ \frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)^4\omega_4 u^{(4)}(c_3)\Big)
\end{aligned}$$

   Since there are four coefficients to set in the linear combination we can specify up to 4 conditions on these coefficients to get the best possible approximation of $u''(x_2)$. These equations are as follows

$$\begin{aligned}
\omega_1 + \omega_2 + \omega_3 + \omega_4 &= 0 \\
-h_1\omega_1 + h_2\omega_3 + (h_2+h_3)\omega_4 &= 0 \\
h_1^2\omega_1 + h_2^2\omega_3 + (h_2+h_3)^2\omega_4 &= 2 \\
-h_1^3\omega_1 + h_2^3\omega_3 + (h_2+h_3)^3\omega_4 &= 0
\end{aligned}$$

   If these equations are satisfied, then

$$\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + \frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)^4\omega_4 u^{(4)}(c_3)\Big)$$

   where the approximation is $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4$ and the error is $\frac{1}{24}\Big(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2+h_3)^4\omega_4 u^{(4)}(c_3)\Big)$

Using Mathematica, this system of equations can be solved, to find that the coefficients are

$$
\begin{aligned}
\omega_1 &= \frac{2(2h_2+h_3)}{h_1(h_1+h_2)(h_1+h_2+h_3)}\\
\omega_2 &= \frac{2h_1-4h_2-2h_3}{h_1h_2^2+h_1h_2h_3}\\
\omega_3 &= \frac{2(-h_1+h_2+h_3)}{h_2(h_1+h_2)h_3}\\
\omega_4 &= \frac{2(h_1-h_2)}{h_3(h_2+h_3)(h_1+h_2+h_3)}
\end{aligned}
$$

Since u is a smooth function, the error can be simplified using the Intermediate Value Theorem, by noting that

$$
\begin{aligned}
\frac{h_1^4\omega_1 u^{(4)}(c_1)+h_2^4\omega_3 u^{(4)}(c_2)}{h_1^4\omega_1+h_2^4\omega_3} &= u^{(4)}(\rho)\\
h_1^4\omega_1 u^{(4)}(c_1)+h_2^4\omega_3 u^{(4)}(c_2) &= \left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)
\end{aligned}
$$

for some $\rho \in [x_1, x_3]$. Thus the error becomes

$$\frac{1}{24}\left(\left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3)\right).$$

The Intermediate Value Theorem can be used again to see that

$$
\begin{aligned}
\frac{(h_1^4\omega_1+h_2^4\omega_3)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3)}{h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4} &= u^{(4)}(\mu)\\
\left(h_1^4\omega_1+h_2^4\omega_3\right)u^{(4)}(\rho)+(h_2+h_3)^4\omega_4 u^{(4)}(c_3) &= \left(h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4\right)u^{(4)}(\mu)
\end{aligned}
$$

for $\mu \in [x_1, x_4]$.

The error can thus be written as

$$E = \frac{1}{24}\left(h_1^4\omega_1+h_2^4\omega_3+(h_2+h_3)^4\omega_4\right)u^{(4)}(\mu).$$

Substituting in for ω_1, ω_3, and ω_4 and simplifying results in

$$E = -\frac{1}{12}(h_2(h_2+h_3)-h_1(2h_2+h_3))u^{(4)}(\mu).$$

2.

3. The script for problems 2 and 3 is shown below with output shown after that.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Create test function
u = @(x) exp(x);
% Create exact second derivative
u2d = @(x) exp(x);

% randomly generate h
sz = [500,1];
```

```
H = exp(linspace(-7,2,500)');
h1 = H.*rand(sz);
h2 = H.*rand(sz);
h3 = H.*rand(sz);

% create x values from h values
x1 = -1*h1;
x2 = 0*H;
x3 = h2;
x4 = h2 + h3;

% compute weights
w1 = (2*(2*h2 + h3))./(h1.*(h1 + h2).*(h1 + h2 + h3));
w2 = (2*h1 - 4*h2 - 2*h3)./(h1.*h2.^2 + h1.*h2.*h3);
w3 = (2*(-1*h1 + h2 + h3))./(h2.*(h1 + h2).*h3);
w4 = (2*(h1 - h2))./(h3.*(h2 + h3).*(h1 + h2 + h3));

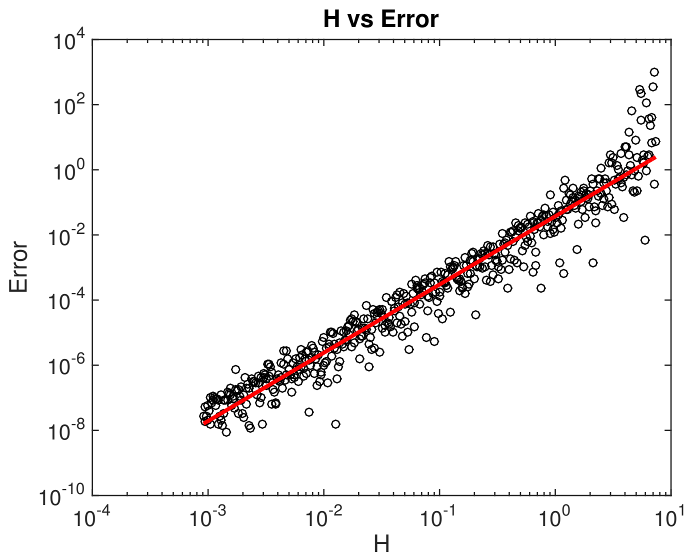
% approximate second derivative
u2da = w1.*u(x1) + w2.*u(x2) + w3.*u(x3) + w4.*u(x4);

% compute error
E = abs(u2d(x2) - u2da);
loglog(H, E, 'ko');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% do a least squares fit on the error in terms of H
A = [ones(sz), log(H)];
b = log(E);

% least squares fit Ax = b
lsf = (A'*A)\(A'*b);
lsf2 = A\b;
K = lsf(1);
p = lsf(2);
hold on
x = min(H):.01:max(H);
y = exp(K)*x.^p;
loglog(x,y,'r', 'LineWidth', 3);
set(gca, 'FontSize', 16);
ylabel('Error');
xlabel('H');
title('H vs Error')

p
K
```




```
>> H01_23

p =

    2.0956


K =

   -3.2838
```

4. Consider the following 2-pt BVP:

$$
\begin{aligned}
u'' + u = f(x), &\quad \text{on} \quad 0 \le x \le 10 \\
u'(0) - u(0) = 0, &\quad u'(10) + u(10) = 0
\end{aligned}
$$

Construct a second order accurate finite-difference method for this BVP. Write your method as a linear system in the form $A\mathbf{u} = \mathbf{f}$.

First the interval $[0, 10]$ needs to be discretized. Let $x_0 = 0$, $x_{N+1} = 10$, and let $x_i = ih$ for $1 \le i \le N$, where $h = \frac{10}{N+1}$. Thus the interval $[0, 10]$ is described as a grid of $N + 2$ equally spaced points with grid spacing h. Thus the solution to this BVP will be approximated on these grid points. Let U_i be the approximate value of $u(x_i)$. Thus an approximate solution to this BVP will be the values of U_i for $0 \le i \le N + 1$.

Based on the boundary conditions be can find expressions for U_0 and U_{N+1}. The first boundary condition states that

$$u'(0) - u(0) = 0$$

We can approximate $u'(0)$ with a second order finite difference.

$$u'(0) \approx \frac{-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0}{h}$$

Thus the boundary condition can be rewritten in terms of the discretization as follows

$$-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0 - hU_0 = 0$$
$$U_0 = (4U_1 - U_2)\frac{1}{3+2h}$$

The second boundary condition can be similarly manipulated.

$$u'(10) + u(10) = 0$$
$$u'(10) \approx \frac{\frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1}}{h}$$
$$\frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1} + hU_{N+1} = 0$$
$$U_{N+1} = (4U_N - U_{N-1})\frac{1}{3+2h}$$

Now that expressions for U_0 and U_{N+1} have been found, we can find N equations for the remaining N unkowns, U_i for $1 \leq i \leq N$. In order that the finite-difference method is second order accurate, I will use the second order central difference to approximate the second derivative. This finite difference is

$$u''(x_i) \approx \frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1})$$

for $1 \leq i \leq N$.

This finite difference can then be used in the differential equation to create N equations as follows

$$\frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1}) + U_i = f(x_i)$$
$$\frac{1}{h^2}\Big(U_{i-1} + (-2+h^2)U_i + U_{i+1}\Big) = f(x_i)$$

for $2 \leq i \leq N-1$. For $i = 1$ and $i = N$, we need to substitute the expression for U_0 and U_{N+1} respectively. This results in

$$f(x_1) = \frac{1}{h^2}\left((4U_1 - U_2)\frac{1}{3+2h} - 2U_1 + U_2\right) + U_1$$
$$f(x_1) = \frac{1}{h^2}\left(\left(\frac{4}{3+2h} - 2 + h^2\right)U_1 + \left(1 - \frac{1}{3+2h}\right)U_2\right)$$
$$f(x_N) = \frac{1}{h^2}\left(U_{N-1} - 2U_N + (4U_N - U_{N-1})\frac{1}{3+2h}\right) + U_N$$
$$f(x_N) = \frac{1}{h^2}\left(\left(1 - \frac{1}{3+2h}\right)U_{N-1} + \left(\frac{4}{3+2h} - 2 + h^2\right)U_N\right)$$

These N equations can be expressed as the matrix equation

$$A\mathbf{u} = \mathbf{f}$$

where

$$\mathbf{u} = [U_1, U_2, \cdots, U_N]^T$$
$$\mathbf{f} = [f(x_1), f(x_2), \cdots, f(x_N)]^T$$
$$A = \frac{1}{h^2}\begin{bmatrix} \frac{4}{3+2h} - 2 + h^2 & 1 - \frac{1}{3+2h} & & & \\ 1 & -2+h^2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2+h^2 & 1 \\ & & & 1 - \frac{1}{3+2h} & \frac{4}{3+2h} - 2 + h^2 \end{bmatrix}$$

A is a tridiagonal matrix, so all other entries are zero. Therefore to approximate the solution solve the system $A\mathbf{u} = \mathbf{f}$ and then plug in values for the expressions of U_0 and U_{N+1}.

5. Construct the exact solution to the BVP with $f(x) = -e^x$.

   This is a nonhomogenous ODE, so the solution to this BVP is found by summing the solutions to the homogenous and nonhomogenous equations. First I will begin by solving the homogenous version of the BVP, which is

$$u'' + u = 0$$

   It is known that the general solution to this homogenous ODE is of the form

$$\begin{aligned} u(x) &= a\sin(x) + b\cos(x) \\ u''(x) &= -a\sin(x) - b\cos(x) \end{aligned}$$

   which clearly satisfies the homogenous ODE.

   A solution to the nonhomogenous ODE can be found as well. In this case the ODE is

$$u'' + u = -e^x$$

   I will guess that the solution is of the form

$$\begin{aligned} u(x) &= c_1 e^x + c_2 e^{-x} \\ u''(x) &= c_1 e^x + c_2 e^{-x}. \end{aligned}$$

   Substituting this into the ODE results in

$$2c_1 e^x + 2c_2 e^{-x} = -e^x.$$

   Therefore

$$\begin{aligned} c_1 &= -\frac{1}{2} \\ c_2 &= 0 \end{aligned}$$

   Thus the nonhomogenous solution is

$$u(x) = -\frac{1}{2}e^x$$

   Therefore the overall solution to this BVP is

$$u(x) = -\frac{1}{2}e^x + a\sin(x) + b\cos(x)$$

Finally we must find a and b such that $u(x)$ satisfies the boundary conditions.

$$\begin{aligned}
u'(x) &= -\frac{1}{2}e^x + a\cos(x) - b\sin(x) \\
u(0) &= -\frac{1}{2} + b \\
u'(0) &= -\frac{1}{2} + a \\
0 &= u'(0) - u(0) \\
&= -\frac{1}{2} + a + \frac{1}{2} - b \\
a &= b \\
u(10) &= -\frac{1}{2}e^{10} + a\sin(10) + b\cos(10) \\
u'(10) &= -\frac{1}{2}e^{10} + a\cos(10) - b\sin(10) \\
0 &= u'(10) + u(10) \\
&= -e^{10} + (\cos(10) + \sin(10))a + (\cos(10) - \sin(10))b
\end{aligned}$$

Substituting in a for b.

$$\begin{aligned}
0 &= -e^{10} + 2\cos(10)a \\
a &= \frac{e^{10}}{2\cos(10)} \\
b &= \frac{e^{10}}{2\cos(10)}
\end{aligned}$$

Therefore the exact solution to the BVP is

$$u(x) = -\frac{1}{2}e^x + \frac{e^{10}}{2\cos(10)}(\sin(x) + \cos(x))$$

6. Verify that your method is second order accurate by solving the BVP with $f(x) = -e^x$ for four different grid spacings h.

   The script for solving this BVP is shown below.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 6
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% solving BVP u'' + u = f(x) for f(x) = -e^x
% with BCs u'(0) - u(0) = 0 and u'(L) + u(L) = 0, where L = 10

L = 10;
f = @(x) -exp(x);

% exact solution
u = @(x) (-1/2)*exp(x) + (exp(L)/(2*cos(L)))*(sin(x) + cos(x));

% create vector for storing errors at respective h values
E = [];
% N is number of interior point
```

```
for N=[10, 20, 40, 80, 160, 320, 640, 1280, 2560, 5120]
    % find spacing
    h = L/(N+1);

    % create x grid and function values on grid
    x = 0:h:L;
    fx = f(x(2:N+1))';

    % create matrix A that solves the finite difference problem
    e = ones(N, 1);
    mid = (-2 + h^2)*e;
    mid(1) = mid(1) + 4/(3 + 2*h);
    mid(N) = mid(N) + 4/(3 + 2*h);
    low = e;
    low(end - 1) = 1 - 1/(3 + 2*h);
    up = e;
    up(2) = 1 - 1/(3 + 2*h);
    A = 1/h^2 * spdiags([low, mid, up], -1:1, N, N);
    U = A\fx;

    % add on first and last nodes, U_0 and U_{N+1}
    U = [U(1)/(1 + h); U; U(end)/(1 + h)];

    % find exact solution
    ux = u(x)';

    % record error
    E = [E; h, norm(U - ux, inf)];

    plot(x,ux, x,U);
    pause;
end

% show ratio of decrease in error
hRatios = E(1:end-1,1)./E(2:end,1);
errorRatios = E(1:end-1,2)./E(2:end,2);
order = log(errorRatios)./log(hRatios);
table(hRatios, errorRatios, order)
```

This outputs the following, which shows that as h decreases by a factor of 2 the error decreases by a factor of 4 which shows that the method is of order 2.

```
>> H01_6

ans =

    hRatios    errorRatios    order
    _______    ___________    ______

    1.9091     3.6953         2.0214
    1.9524     3.5994         1.9143
    1.9756     3.7616         1.9458
    1.9877     3.8869         1.9763
    1.9938     3.9406         1.9873
    1.9969     3.9697         1.9935
    1.9984     3.9849         1.9968
    1.9992     3.9924         1.9984
```

```
1.9996      3.9963          1.9992
```

7. Consider the following 2-point BVP:

$$
\begin{aligned}
-u'' + u = f(x), \quad &\text{on} \quad 0 \le x \le 1 \\
u(0) = u(1) \quad &u'(0) = u'(1)
\end{aligned}
$$

Construct a fourth-order accurate finite-difference method for this BVP based on the fourth-order central finite difference. Write your method as a linear system of the form $A\mathbf{u} = \mathbf{f}$.

First the interval $[0, 1]$ needs to be discretized. Let $x_0 = 0$, $x_{N+1} = 1$, and let $x_i = ih$ for $1 \le i \le N$, where $h = \frac{1}{N+1}$. Thus the interval $[0, 1]$ is described as a grid of $N + 2$ equally spaced points with grid spacing h. Thus the solution to this BVP will be approximated on these grid points. Let U_i be the approximate value of $u(x_i)$. Thus an approximate solution to this BVP will be the values of U_i for $0 \le i \le N + 1$.

The boundary conditions are periodic, this implies that we can use the following equalities

$$
\begin{aligned}
U_0 &= U_{N+1} \\
U_{-1} &= U_N \\
U_{-2} &= U_{N-1} \\
U_{N+2} &= U_1
\end{aligned}
$$

The periodic boundary conditions reduces the number of unknowns from $N + 2$ to $N + 1$, and it allows for the finite difference to be used at all points $i = 0, 1, \dots, N$.

In order that this method is fourth-order accurate, I will use the fourth-order central difference to approximate the second derivative, which is

$$
u''(x_i) \approx \frac{-U_{i-2} + 16U_{i-1} - 30U_i + 16U_{i+1} - U_{i+2}}{12h^2}
$$

Therefore the finite difference equation for this differential equations is

$$
\frac{U_{i-2} - 16U_{i-1} + (30 + 12h^2)U_i - 16U_{i+1} + U_{i+2}}{12h^2} = f(x_i)
$$

for $i = 0, 1, \dots, N$.

This is equivalent to the following matrix equation

$$
A\mathbf{u} = \mathbf{f}
$$

where

$$
\begin{aligned}
\mathbf{u} &= [U_0, U_1, \cdots, U_N]^T \\
\mathbf{f} &= [f(x_0), f(x_1), \cdots, f(x_N)]^T
\end{aligned}
$$

$$
A = \frac{1}{12h^2}\begin{bmatrix}
30 + 12h^2 & -16 & 1 & & \cdots & 1 & -16 \\
-16 & 30 + 12h^2 & -16 & 1 & & \cdots & 1 \\
1 & -16 & 30 + 12h^2 & -16 & 1 & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 & & 1 & -16 & 30 + 12h^2 & -16 & 1 \\
1 & \cdots & & 1 & -16 & 30 + 12h^2 & -16 \\
-16 & 1 & \cdots & & 1 & -16 & 30 + 12h^2
\end{bmatrix}
$$

8. Construct an exact solution to the BVP for $f(x) = \sin(4\pi x)$.

As we have seen the exact solution to this BVP will be the sum of the homogenous solution, u_h, and nonhomogenous solution, u_n, to this BVP. The homogenous version of this BVP is

$$-u_h'' + u_h = 0.$$

The solution to this ODE is of the form $u_h(x) = a\sin(x) + b\cos(x)$.

The nonhomogenous solution can be found as follows. First I will guess that the solution is of the form $u_n(x) = c_1 \sin(4\pi x) + c_2 \cos(4\pi x)$.

The constants in this function can be found by plugging this function into the differential equation.

$$\begin{aligned} u_n''(x) &= -16\pi^2 c_1 \sin(4\pi x) - 16\pi^2 c_2 \cos(4\pi x) \\ \sin(4\pi x) &= -u_n''(x) + u_n(x) \\ &= \left(16\pi^2 + 1\right) c_1 \sin(4\pi x) + \left(16\pi^2 + 1\right) c_2 \cos(4\pi x) \end{aligned}$$

This results in the following two equations

$$\begin{aligned} 1 &= \left(16\pi^2 + 1\right) c_1 \\ c_1 &= \frac{1}{16\pi^2 + 1} \\ 0 &= \left(16\pi^2 + 1\right) c_2 \\ c_2 &= 0 \end{aligned}$$

Thus the nonhomogenous solution to this BVP is $u_n(x) = \frac{1}{16\pi^2+1}\sin(4\pi x)$.

Therefore the full solution is of the form $u(x) = a\sin(x) + b\cos(x) + \frac{1}{16\pi^2+1}\sin(4\pi x)$. Now we must select a and b to satisfy the periodic boundary conditions.

$$\begin{aligned} u(x) &= a\sin(x) + b\cos(x) + \frac{1}{16\pi^2 + 1}\sin(4\pi x) \\ u(0) &= b \\ u(1) &= a\sin(1) + b\cos(1) \\ u(0) &= u(1) \\ b &= a\sin(1) + b\cos(1) \\ a &= \left(\frac{1 - \cos(1)}{\sin(1)}\right) b \\ u'(x) &= a\cos(x) - b\sin(x) + \frac{4\pi}{16\pi^2 + 1}\cos(4\pi x) \\ u'(0) &= a + \frac{4\pi}{16\pi^2 + 1} \\ u'(1) &= a\cos(1) - b\sin(1) + \frac{4\pi}{16\pi^2 + 1} \\ u'(0) &= u'(1) \\ a + \frac{4\pi}{16\pi^2 + 1} &= a\cos(1) - b\sin(1) + \frac{4\pi}{16\pi^2 + 1} \\ a &= a\cos(1) - b\sin(1) \\ 0 &= (\cos(1) - 1)a - b\sin(1) \end{aligned}$$

Substituting for a

$$
\begin{aligned}
0 &= (\cos(1) - 1)\left(\frac{1-\cos(1)}{\sin(1)}\right)b - \sin(1)b \\
0 &= \left(\frac{-\cos(1)^2 + 2\cos(1) - 1 - \sin(1)^2}{\sin(1)}\right)b \\
0 &= \left(\frac{2\cos(1) - 2}{\sin(1)}\right)b
\end{aligned}
$$

Since $\frac{2\cos(1)-2}{\sin(1)} \neq 0$, then

$$b = 0$$

Therefore

$$a = 0$$

Thus the exact solution to the BVP is $u(x) = \frac{1}{16\pi^2+1}\sin(4\pi x)$.

9. Verify that your method is fourth-order accurate by solving the BVP with $f(x) = \sin(4\pi x)$ at four different grid spacings h.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 9
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% solving BVP -u'' + u = f(x) for f(x) = sin(4pi x) on 0 \le x \le 1
% with periodic BCs u(0) = u(1) and u'(0) = u'(1)

L = 1;
f = @(x) sin(4*pi*x);

% exact solution
u = @(x) 1/(16*pi^2 + 1)*sin(4*pi*x);

% create vector for storing errors at respective h values
E = [];
% N is number of interior point
for N=[10, 20, 40, 80, 160, 320, 640, 1280]
    % find spacing
    h = L/(N+1);

    % create x grid and function values on grid
    % x_0 to x_{N+1}
    x = 0:h:L;
    % leave out x_{N+1}
    fx = f(x(1:end-1))';

    % create matrix A that solves the finite difference problem
    e = ones(N+1, 1);
    A = 1/(12*h^2) * spdiags([-16*e, e, e, -16*e, (30 + 12*h^2)*e, -16*e, e, e, ...
        -16*e], [-N, -N+1, -2:2, N-1, N], N+1, N+1);
    U = A\fx;

    % add last node at x_{N+1}
    % U_{N+1} = U_{0}
    U = [U; U(1)];
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    % compute exact solution
    ux = u(x)';

    % record error
    E = [E; h, norm(U - ux, inf)];

    % plot approximate and exact solution
    plot(x,ux, x,U);
    %pause;
end

% show ratio of decrease in error
hRatios = E(1:end-1,1)./E(2:end,1);
errorRatios = E(1:end-1,2)./E(2:end,2);
order = log(errorRatios)./log(hRatios);
table(hRatios, errorRatios, order)
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>> H01_9

ans =

    hRatios    errorRatios    order
    _______    ___________    ______

    1.9091     12.309         3.8822
    1.9524      14.18         3.9636
    1.9756     15.132         3.9902
    1.9877     15.581         3.9975
    1.9938     15.794         3.9993
    1.9969     15.898         3.9998
    1.9984       14.9         3.9017
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10. Prove that your method is consistent with truncation error $\|\tau\| = O(h^4)$ and L_2-stable, thereby proving that your method converges at $O(h^4)$ in the L_2-norm.

First I will prove that my method is consistent with fourth order accuracy. A method is considered consistent and fourth order accurate if the norm of the truncation error approaches zero at the same rate as h^4. The truncation error is defined as the difference between the

In this case the truncation error is

$$\tau_i = \frac{1}{12h^2}\Big(u(x_{i-2}) - 16u(x_{i-1}) + \Big(30 + 12h^2\Big)u(x_i) - 16u(x_{i+1}) + u(x_{i+2})\Big) - f(x_i)$$

This can be simplified using the following Taylor series

$$u(x_{i-2}) = u(x_i) - 2hu'(x_i) + 2h^2u''(x_i) - \frac{4}{3}h^3u'''(x_i) + \frac{2}{3}h^4u^{(4)}(x_i) - \frac{4}{15}h^5u^{(5)}(x_i) + \frac{4}{45}h^6u^{(6)}(c_{-2})$$

$$-16u(x_{i-1}) = -16u(x_i) + 16hu'(x_i) - 8h^2u''(x_i) + \frac{8}{3}h^3u'''(x_i) - \frac{2}{3}h^4u^{(4)}(x_i) + \frac{2}{15}h^5u^{(5)}(x_i) - \frac{1}{45}h^6u^{(6)}(c_{-1})$$

$$-16u(x_{i+1}) = -16u(x_i) - 16hu'(x_i) - 8h^2u''(x_i) - \frac{8}{3}h^3u'''(x_i) - \frac{2}{3}h^4u^{(4)}(x_i) - \frac{2}{15}h^5u^{(5)}(x_i) - \frac{1}{45}h^6u^{(6)}(c_1)$$

$$u(x_{i+2}) = u(x_i) + 2hu'(x_i) + 2h^2u''(x_i) + \frac{4}{3}h^3u'''(x_i) + \frac{2}{3}h^4u^{(4)}(x_i) + \frac{4}{15}h^5u^{(5)}(x_i) + \frac{4}{45}h^6u^{(6)}(c_2)$$

Adding these four Taylor series results in

$$u(x_{i-2}) - 16u(x_{i-1}) - 16u(x_{i+1}) + u(x_{i+2}) = -30u(x_i) - 12h^2u''(x_i) + \frac{1}{45}h^6\Big(4u^{(6)}(c_{-2}) - u^{(6)}(c_{-1}) - u^{(6)}(c_1) + 4u^{(6)}(c_2)\Big)$$

The Intermediate Value Theorem can be used several times to simplify the final term

$$u(x_{i-2}) - 16u(x_{i-1}) - 16u(x_{i+1}) + u(x_{i+2}) = -30u(x_i) - 12h^2u''(x_i) + \frac{6}{45}h^6u^{(6)}(\mu)$$

Plugging this expression back into the truncation error formula results in

$$\begin{aligned}\tau_i &= \frac{1}{12h^2}\Big(\Big(30 + 12h^2\Big)u(x_i) - 30u(x_i) - 12h^2u''(x_i) + \frac{6}{45}u^{(6)}(\mu)\Big) - f(x_i)\\ &= u(x_i) - u''(x_i) + \frac{1}{90}h^4u^{(6)}(\mu) - f(x_i)\end{aligned}$$

Since u is a solution to the differential equation, $f(x_i) = -u''(x_i) + u(x_i)$

$$\tau_i = \frac{1}{90}h^4u^{(6)}(\mu)$$

Therefore

$$\tau_i = O(h^4)$$

This shows that our method is consistent and is fourth order accurate.

In order to show that our method is L_2 stable, we must show that $\|A^{-1}\|_2 \leq C$.

Since A is symmetric we know that $\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\min\{\lambda_i\}}$, where λ_i are the eigenvalues of A. To find the eigenvalues of A, we must first find the eigenfunctions

of the original BVP, that is we must find functions v that satisfy

$$\begin{aligned}-v'' + v = \lambda v, \quad \text{on} \quad 0 \leq x \leq 1\\ v(0) = 0 \quad v(1) = 0\end{aligned}$$

This is equivalent to the ODE

$$-v'' = (\lambda - 1)v$$

The general solution to this ODE is

$$v(x) = a\sin\Big(\sqrt{\lambda - 1}x\Big) + b\cos\Big(\sqrt{\lambda - 1}x\Big)$$

Based on the boundary condition $v(0) = 0$

$$\begin{aligned}0 &= a\sin(0) + b\cos(0)\\ b &= 0\end{aligned}$$

and by the boundary condition $v(1) = 0$

$$0 = a\sin\Big(\sqrt{\lambda - 1}\Big)$$

To find nontrivial solutions implies that

$$\begin{aligned}\sqrt{\lambda - 1} &= n\pi \\ \lambda &= n^2\pi^2 + 1\end{aligned}$$

Therefore the eigenvalues of this continuous problem are $n^2\pi^2 + 1$ for $n = 1, 2, \ldots$, and the eigenfunctions are $v^p = \sin(p\pi x)$.

Therefore for the continuous problem I will guess that the eigenvectors have the same form as the eigenvalues. Thus I will quess that the eigenvectors, $\mathbf{v}^p$ of A has entries $v_j^p = \sin(p\pi x_j) = \sin(p\pi jh)$, that is the eigenvectors are the eigenfunctions sampled on the mesh. To verify this we must compute the entries of $A\mathbf{v}^p$. The jth entry of $A\mathbf{v}^p$ can be found as follows

$$\begin{aligned}(A\mathbf{v}^p)_j &= \frac{1}{12h^2}\left(v_{j-2}^p - 16v_{j-1}^p + \left(30 + 12h^2\right)v_j^p - 16v_{j+1}^p + v_{j+2}^p\right) \\ &= \frac{1}{12h^2}(\sin(p\pi(j-2)h) - 16\sin(p\pi(j-1)h) \\ &+ \left(30 + 12h^2\right)\sin(p\pi jh) - 16\sin(p\pi(j+1)h) + \sin(p\pi(j+2)h))\end{aligned}$$

The following trigonometric identies can be used to simplify this expression

$$\begin{aligned}\sin(p\pi(j-2)h) &= \sin(p\pi jh)\cos(2p\pi h) - \cos(p\pi jh)\sin(2p\pi h) \\ -16\sin(p\pi(j-1)h) &= -16(\sin(p\pi jh)\cos(p\pi h) - \cos(p\pi jh)\sin(p\pi h)) \\ -16\sin(p\pi(j+1)h) &= -16(\sin(p\pi jh)\cos(p\pi h) + \cos(p\pi jh)\sin(p\pi h)) \\ \sin(p\pi(j+2)h) &= \sin(p\pi jh)\cos(2p\pi h) + \cos(p\pi jh)\sin(2p\pi h)\end{aligned}$$

Substituting these identities in and simplifying results in

$$\begin{aligned}(A\mathbf{v}^p)_j &= \frac{1}{12h^2}\left(\left(30 + 12h^2\right)\sin(p\pi jh) + 2\sin(p\pi jh)\cos(2p\pi h) - 32\sin(p\pi jh)\cos(p\pi h)\right) \\ &= \frac{1}{12h^2}\left(30 + 12h^2 + 2\cos(2p\pi h) - 32\cos(p\pi h)\right)\sin(p\pi jh) \\ &= \frac{1}{12h^2}\left(30 + 12h^2 + 2\cos(2p\pi h) - 32\cos(p\pi h)\right)v_j^p\end{aligned}$$

So this vector is indeed an eigenvector with eigenvalue

$$\lambda^p = \frac{1}{12h^2}\left(30 + 12h^2 + 2\cos(2p\pi h) - 32\cos(p\pi h)\right)$$

Doing a Taylor series expansion on this eigenvalue results in

$$\lambda^p \approx (1 + p^2\pi^2) - \frac{1}{90}p^6\pi^6h^4 + O(h^6)\min\{\lambda^p\} \qquad = \lambda^1 \approx (1 + \pi^2) - \frac{1}{90}\pi^6h^4 + O(h^6)$$

Coming back to our initial problem, we now know that

$$\begin{aligned}\left\|A^{-1}\right\|_2 &= \rho(A^{-1}) \\ &= \frac{1}{\lambda^1} \\ &\leq \frac{1}{1 + pi^2}\end{aligned}$$

Therefore the $\left\|A^{-1}\right\|_2$ is bounded and out method is L_2-stable.