Assigned: Thursday Jan. 14, 2016
Due: Thursday Feb. 4, 2016

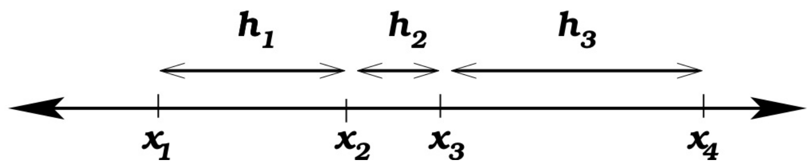
MATH 517: Homework 1
Spring 2016

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Part 1: Non-Uniform Grid

Consider the non-uniform grid:



1. Derive a finite difference approximation to $u''(x_2)$ that is accurate as possible for smooth functions $u(x)$, based on the four values $U_1 = u(x_1)$, $\ldots$, $U_4 = u(x_4)$. Give an expression for the dominant term in the error.

In the next two questions, you will try to determine an order of accuracy for your method:

2. To get a better feel for how the error behaves as the grid gets finer, take 500 values of H, where H spans 2 or 3 orders of magnitude, and for each value of H, randomly generate three numbers, h_1, h_2, and h_3, where each $h_i \in [0, H]$. For each value H, compute your approximation to $u''(x_2)$ using the randomly generated $h_i \in [0, H]$. Plot the error against H on a log-log plot to get a scatter plot of the behavior as $H \to 0$. **NOTE:** in MATLAB the command `x = H*rand(1)` will produce a single random number in the range `[0,H]`.

3. Estimate the order of accuracy by doing a least squares fit of the form

$$\log(E(H)) = K + p\log(H)$$

to determine K and p based on the 500 data points. Recall that this can be done by solving the following linear system in the least squares sense:

$$\begin{bmatrix} 1 & \log(H_1) \\ 1 & \log(H_2) \\ \vdots & \vdots \\ 1 & \log(H_{500}) \end{bmatrix} \begin{bmatrix} K \\ p \end{bmatrix} = \begin{bmatrix} \log(E(H_1)) \\ \log(E(H_2)) \\ \vdots \\ \log(E(H_{500})) \end{bmatrix}.$$

NOTE: "In the least-squares sense" means that one should solve the rectangular system $Ax = b$, by solving the (square) normal equation: $A^T Ax = A^T b$.

Part 2: Mixed Conditions

Consider the following 2-point BVP:

$$u'' + u = f(x), \quad \text{on} \quad 0 \leq x \leq 10$$
$$u'(0) - u(0) = 0, \quad u'(10) + u(10) = 0.$$

4. Construct a second-order accurate finite-difference method for this BVP. Write your method as a linear system of the form $A\vec{u} = \vec{f}$.

5. Construct the exact solution to this BVP with $f(x) = -e^x$.

6. Verify that your method is second order accurate by solving the BVP with $f(x) = -e^x$ at four different grid spacings h.

   HINT 1: Use the `spdiags` command in MATLAB to create the sparse matrix A – this will save storage and allow MATLAB to use a fast solver. Modify the following commands, which generate a tri-diagonal matrix with a [1,-2,1] structure, to model your specific BVP:

```
e = ones(n,1);
A = spdiags([e -2*e e], -1:1, n, n);
```

   HINT 2: Use the command `u = A\f` to invert your system – this will use a fast solver if you created your matrix with `spdiags`.

Part 3: Periodic Boundary Conditions

Consider the following 2-point BVP:

$$-u'' + u = f(x), \quad \text{on} \quad 0 \leq x \leq 1$$
$$u(0) = u(1) \quad u'(0) = u'(1) \quad \text{(these are periodic BCs)}.$$

7. Construct a fourth-order accurate finite-difference method for this BVP based on the fourth-order central finite difference.

   Write your method as a linear system of the form $A\vec{u} = \vec{f}$.

8. Construct the exact solution to this BVP with $f(x) = \sin(4\pi x)$.

9. Verify that your method is fourth-order accurate by solving the BVP with $f(x) = \sin(4\pi x)$ at four different grid spacings h.

10. Prove that your method is consistent with truncation error $\|\vec{\tau}\| = O(h^4)$ and L_2-stable, thereby proving that your method converges at $O(h^4)$ in the L_2-norm.

    HINT: what is the smallest eigenvalue of the discrete version of $-u''$? How does $+u$ modify this?