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MATH 517 Finite Difference Methods
Homework 2

1. Consider the 2-pt boundary value problem:

$$-u'' = f(x) \text{ on } 0 < x < L$$
$$u(0) = \alpha, \quad u'(L) = \sigma.$$

Discretize this problem using the $O(h^2)$ central finite differences and a ghost point near $x = L$ to handle the Neumann boundary condition. Write out the resulting linear system.

To discretize this problem let $x_i = ih$ where $h = \frac{L}{N+1}$ and N is the number of points in the discretization.

2. Find the Green's function that satisfies:

$$-G'' = \delta(x - \xi), \quad G(0; \xi) = 0, \quad G'(L; \xi) = 0.$$

3. Use the result from Problem 2 to write out the exact solution to the boundary value problem with general $f(x)$, α, and σ.

4. Use the results from Problems 2 and 3 to find the exact inverse to the finite difference matrix found in Problem 1.

5. Use the result in Problem 4 to prove that the finite difference method in Problem 1 is L_∞-stable.

6. Consider the uniform mesh $x_j = jh$ and let

$$U_j = u(x_j) \quad \text{and} \quad W_j \approx u'(x_j).$$

In standard finite differences, we typically find linear combinations of U_j to define the approximation W_i to $u'(x_j)$:

$$W_i = \sum_j \beta_j U_j$$

In compact finite differences we are allowed to generalize this to

$$\sum_j \alpha_j W_j = \sum_j \beta_j U_j$$

Find the compact finite difference with the optimal local truncation error that has the following form:

$$\alpha W_{j-1} + W_j + \alpha W_{j+1} = \beta\left(\frac{U_{j+1} - U_{j-1}}{2h}\right).$$

We can find the local truncation error by inserting the exact solution into the finite difference equation and using Taylor series.

$$\tau_j = -\alpha u'(x_{j-1}) - u'(x_j) - \alpha u'(x_{j+1}) + \beta\left(\frac{u(x_{j+1}) - u(x_{j-1})}{2h}\right)$$

$$u(x_{j-1}) = u(x_j) - hu'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) - \frac{h^5}{120}u^{(5)}(x_j) + O(h^6)$$

$$u(x_{j+1}) = u(x_j) + hu'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) + \frac{h^5}{120}u^{(5)}(x_j) + O(h^6)$$

$$u'(x_{j-1}) = u'(x_j) - hu''(x_j) + \frac{h^2}{2}u'''(x_j) - \frac{h^3}{6}u^{(4)}(x_j) + \frac{h^4}{24}u^{(5)}(x_j) - \frac{h^5}{120}u^{(6)}(x_j) + O(h^6)$$

$$u'(x_{j+1}) = u'(x_j) + hu''(x_j) + \frac{h^2}{2}u'''(x_j) + \frac{h^3}{6}u^{(4)}(x_j) + \frac{h^4}{24}u^{(5)}(x_j) + \frac{h^5}{120}u^{(6)}(x_j) + O(h^6)$$

7. Consider Poisson's equation in 2D:

$$\begin{aligned} -u_{xx} - u_{yy} = f(x,y) \text{ in } \Omega = [0,1] \times [0,1], \\ u = g(x,y) \text{ on } \partial\Omega \end{aligned}$$

   Discretize this equation using the 5-point Laplacian on a uniform mesh $\Delta x = \Delta y = h$. Use the standard natural row-wise ordering.

8. Write a MATLAB code that constructs the sparse coefficient matrix A and the appropriate right hand side vector $\mathbf{F}$.

9. Using your code, do a numerical convergence study for the following right-hand side forcing and exact solution:

$$f(x,y) = -1.25e^{x+.5y} \quad \text{and} \quad u(x,y) = e^{x+.5y}$$