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## MATH 517 Finite Difference Methods

### Homework 1

1. Consider a nonuniform grid  $x_1 < x_2 < x_3 < x_4$ . Derive a finite difference approximation of  $u''(x_2)$  that is as accurate as possible for smooth functions  $u(x)$ , based on the four values  $U_1 = u(x_1)$ ,  $U_2 = u(x_2)$ ,  $U_3 = u(x_3)$ , and  $U_4 = u(x_4)$ . Give an expression for the dominant term in the error.

First let  $h_1 = x_2 - x_1$ ,  $h_2 = x_3 - x_2$  and  $h_3 = x_4 - x_3$ . In order to approximate  $u''(x_2)$ , we will use a linear combination of  $U_1, \dots, U_4$ , that is we will find coefficients  $\omega_1, \dots, \omega_4$  such that  $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + E$ , where the error,  $E$ , is as small as possible.

$U_1, U_3$ , and  $U_4$  can be expressed as Taylor expansions about  $U_2$  as follows

$$\begin{aligned} U_1 &= u(x_1) = u(x_2) + u'(x_2)(-h_1) + \frac{1}{2}u''(x_2)(-h_1)^2 + \frac{1}{6}u'''(x_2)(-h_1)^3 + \frac{1}{24}u^{(4)}(c_1)(-h_1)^4 \\ U_3 &= u(x_3) = u(x_2) + u'(x_2)(h_2) + \frac{1}{2}u''(x_2)(h_2)^2 + \frac{1}{6}u'''(x_2)(h_2)^3 + \frac{1}{24}u^{(4)}(c_2)(h_2)^4 \\ U_4 &= u(x_4) = u(x_2) + u'(x_2)(h_2 + h_3) + \frac{1}{2}u''(x_2)(h_2 + h_3)^2 + \frac{1}{6}u'''(x_2)(h_2 + h_3)^3 + \frac{1}{24}u^{(4)}(c_3)(h_2 + h_3)^4 \end{aligned}$$

where  $c_1 \in [x_1, x_2]$ ,  $c_2 \in [x_2, x_3]$ , and  $c_3 \in [x_2, x_4]$ .

Substituting these Taylor expansions into the linear combination and gathering the function and derivative values of  $u$  at  $x_2$  results in

$$\begin{aligned} \omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 &= (\omega_1 + \omega_2 + \omega_3 + \omega_4)u(x_2) \\ &\quad + (-h_1\omega_1 + h_2\omega_3 + (h_2 + h_3)\omega_4)u'(x_2) \\ &\quad + \frac{1}{2}(h_1^2\omega_1 + h_2^2\omega_3 + (h_2 + h_3)^2\omega_4)u''(x_2) \\ &\quad + \frac{1}{6}(-h_1^3\omega_1 + h_2^3\omega_3 + (h_2 + h_3)^3\omega_4)u'''(x_2) \\ &\quad + \frac{1}{24}(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)^4\omega_4 u^{(4)}(c_3)) \end{aligned}$$

Since there are four coefficients to set in the linear combination we can specify up to 4 conditions on these coefficients to get the best possible approximation of  $u''(x_2)$ . These equations are as follows

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 + \omega_4 &= 0 \\ -h_1\omega_1 + h_2\omega_3 + (h_2 + h_3)\omega_4 &= 0 \\ h_1^2\omega_1 + h_2^2\omega_3 + (h_2 + h_3)^2\omega_4 &= 2 \\ -h_1^3\omega_1 + h_2^3\omega_3 + (h_2 + h_3)^3\omega_4 &= 0 \end{aligned}$$

If these equations are satisfied, then

$$\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4 = u''(x_2) + \frac{1}{24}(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)^4\omega_4 u^{(4)}(c_3))$$

where the approximation is  $\omega_1 U_1 + \omega_2 U_2 + \omega_3 U_3 + \omega_4 U_4$  and the error is  $\frac{1}{24}(h_1^4\omega_1 u^{(4)}(c_1) + h_2^4\omega_3 u^{(4)}(c_2) + (h_2 + h_3)^4\omega_4 u^{(4)}(c_3))$

Using Mathematica, this system of equations can be solved, to find that the coefficients are

$$\begin{aligned}\omega_1 &= \frac{2(2h_2 + h_3)}{h_1(h_1 + h_2)(h_1 + h_2 + h_3)} \\ \omega_2 &= \frac{2h_1 - 4h_2 - 2h_3}{h_1h_2^2 + h_1h_2h_3} \\ \omega_3 &= \frac{2(-h_1 + h_2 + h_3)}{h_2(h_1 + h_2)h_3} \\ \omega_4 &= \frac{2(h_1 - h_2)}{h_3(h_2 + h_3)(h_1 + h_2 + h_3)}\end{aligned}$$

Since  $u$  is a smooth function, the error can be simplified using the Intermediate Value Theorem, by noting that

$$\begin{aligned}\frac{h_1^4\omega_1u^{(4)}(c_1) + h_2^4\omega_3u^{(4)}(c_2)}{h_1^4\omega_1 + h_2^4\omega_3} &= u^{(4)}(\rho) \\ h_1^4\omega_1u^{(4)}(c_1) + h_2^4\omega_3u^{(4)}(c_2) &= (h_1^4\omega_1 + h_2^4\omega_3)u^{(4)}(\rho)\end{aligned}$$

for some  $\rho \in [x_1, x_3]$ . Thus the error becomes

$$\frac{1}{24} \left( (h_1^4\omega_1 + h_2^4\omega_3)u^{(4)}(\rho) + (h_2 + h_3)^4\omega_4u^{(4)}(c_3) \right).$$

The Intermediate Value Theorem can be used again to see that

$$\begin{aligned}\frac{(h_1^4\omega_1 + h_2^4\omega_3)u^{(4)}(\rho) + (h_2 + h_3)^4\omega_4u^{(4)}(c_3)}{h_1^4\omega_1 + h_2^4\omega_3 + (h_2 + h_3)^4\omega_4} &= u^{(4)}(\mu) \\ (h_1^4\omega_1 + h_2^4\omega_3)u^{(4)}(\rho) + (h_2 + h_3)^4\omega_4u^{(4)}(c_3) &= (h_1^4\omega_1 + h_2^4\omega_3 + (h_2 + h_3)^4\omega_4)u^{(4)}(\mu)\end{aligned}$$

for  $\mu \in [x_1, x_4]$ .

The error can thus be written as

$$E = \frac{1}{24} (h_1^4\omega_1 + h_2^4\omega_3 + (h_2 + h_3)^4\omega_4)u^{(4)}(\mu).$$

Substituting in for  $\omega_1$ ,  $\omega_3$ , and  $\omega_4$  and simplifying results in

$$E = -\frac{1}{12}(h_2(h_2 + h_3) - h_1(2h_2 + h_3))u^{(4)}(\mu).$$

2.

3. The script for problems 2 and 3 is shown below with output shown after that.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 2
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Create test function
u = @(x) exp(x);
% Create exact second derivative
u2d = @(x) exp(x);

% randomly generate h
sz = [500,11];
```

```

H = exp(linspace(-7,2,500)');
h1 = H.*rand(sz);
h2 = H.*rand(sz);
h3 = H.*rand(sz);

% create x values from h values
x1 = -1*h1;
x2 = 0*H;
x3 = h2;
x4 = h2 + h3;

% compute weights
w1 = (2*(2*h2 + h3))./(h1.*(h1 + h2).*(h1 + h2 + h3));
w2 = (2*h1 - 4*h2 - 2*h3)./(h1.*h2.^2 + h1.*h2.*h3);
w3 = (2*(-1*h1 + h2 + h3))./(h2.*(h1 + h2).*h3);
w4 = (2*(h1 - h2))./(h3.*(h2 + h3).*(h1 + h2 + h3));

% approximate second derivative
u2da = w1.*u(x1) + w2.*u(x2) + w3.*u(x3) + w4.*u(x4);

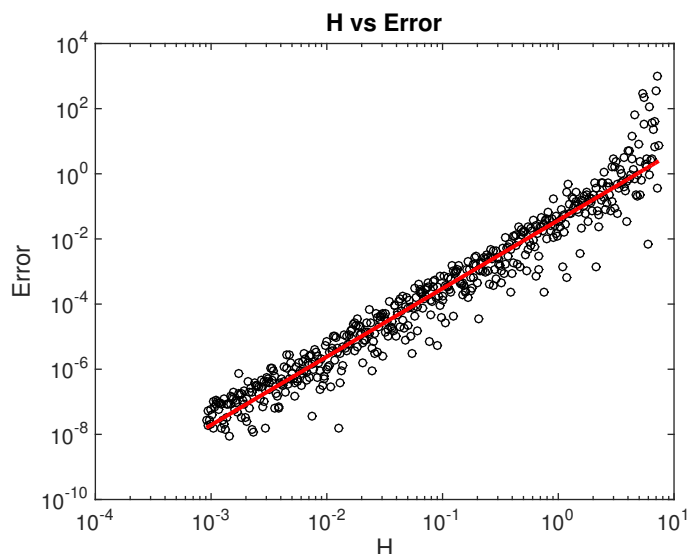
% compute error
E = abs(u2d(x2) - u2da);
loglog(H, E, 'ko');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% do a least squares fit on the error in terms of H
A = [ones(sz), log(H)];
b = log(E);

% least squares fit Ax = b
lsf = (A'*A)\(A'*b);
lsf2 = A\b;
K = lsf(1);
p = lsf(2);
hold on
x = min(H):.01:max(H);
y = exp(K)*x.^p;
loglog(x,y,'r', 'LineWidth', 3);
set(gca, 'FontSize', 16);
ylabel('Error');
xlabel('H');
title('H vs Error')

p
K

```



```
>> H01_23
```

```
p =
```

```
2.0956
```

```
K =
```

```
-3.2838
```

4. Consider the following 2-pt BVP:

$$\begin{aligned} u'' + u &= f(x), \quad \text{on } 0 \leq x \leq 10 \\ u'(0) - u(0) &= 0, \quad u'(10) + u(10) = 0 \end{aligned}$$

Construct a second order accurate finite-difference method for this BVP. Write your method as a linear system in the form  $A\mathbf{u} = \mathbf{f}$ .

First the interval  $[0, 10]$  needs to be discretized. Let  $x_0 = 0$ ,  $x_{N+1} = 10$ , and let  $x_i = ih$  for  $1 \leq i \leq N$ , where  $h = \frac{10}{N+1}$ . Thus the interval  $[0, 10]$  is described as a grid of  $N + 2$  equally spaced points with grid spacing  $h$ . Thus the solution to this BVP will be approximated on these grid points. Let  $U_i$  be the approximate value of  $u(x_i)$ . Thus an approximate solution to this BVP will be the values of  $U_i$  for  $0 \leq i \leq N + 1$ .

Based on the boundary conditions we can find expressions for  $U_0$  and  $U_{N+1}$ . The first boundary condition states that

$$u'(0) - u(0) = 0$$

We can approximate  $u'(0)$  with a second order finite difference.

$$u'(0) \approx \frac{-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0}{h}$$

Thus the boundary condition can be rewritten in terms of the discretization as follows

$$-\frac{1}{2}U_2 + 2U_1 - \frac{3}{2}U_0 - hU_0 = 0$$

$$U_0 = (4U_1 - U_2)\frac{1}{3+2h}$$

The second boundary condition can be similarly manipulated.

$$u'(10) + u(10) = 0$$

$$u'(10) \approx \frac{\frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1}}{h}$$

$$\frac{3}{2}U_{N+1} - 2U_N + \frac{1}{2}U_{N-1} + hU_{N+1} = 0$$

$$U_{N+1} = (4U_N - U_{N-1})\frac{1}{3+2h}$$

Now that expressions for  $U_0$  and  $U_{N+1}$  have been found, we can find  $N$  equations for the remaining  $N$  unknowns,  $U_i$  for  $1 \leq i \leq N$ . In order that the finite-difference method is second order accurate, I will use the second order central difference to approximate the second derivative. This finite difference is

$$u''(x_i) \approx \frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1})$$

for  $1 \leq i \leq N$ .

This finite difference can then be used in the differential equation to create  $N$  equations as follows

$$\frac{1}{h^2}(U_{i-1} - 2U_i + U_{i+1}) + U_i = f(x_i)$$

$$\frac{1}{h^2}(U_{i-1} + (-2 + h^2)U_i + U_{i+1}) = f(x_i)$$

for  $2 \leq i \leq N-1$ . For  $i = 1$  and  $i = N$ , we need to substitute the expression for  $U_0$  and  $U_{N+1}$  respectively. This results in

$$f(x_1) = \frac{1}{h^2}\left((4U_1 - U_2)\frac{1}{3+2h} - 2U_1 + U_2\right) + U_1$$

$$f(x_1) = \frac{1}{h^2}\left(\left(\frac{4}{3+2h} - 2 + h^2\right)U_1 + \left(1 - \frac{1}{3+2h}\right)U_2\right)$$

$$f(x_N) = \frac{1}{h^2}\left(U_{N-1} - 2U_N + (4U_N - U_{N-1})\frac{1}{3+2h}\right) + U_N$$

$$f(x_N) = \frac{1}{h^2}\left(\left(1 - \frac{1}{3+2h}\right)U_{N-1} + \left(\frac{4}{3+2h} - 2 + h^2\right)U_N\right)$$

These  $N$  equations can be expressed as the matrix equation

$$A\mathbf{u} = \mathbf{f}$$

where

$$\mathbf{u} = [U_1, U_2, \dots, U_N]^T$$

$$\mathbf{f} = [f(x_1), f(x_2), \dots, f(x_N)]^T$$

$$A = \frac{1}{h^2} \begin{bmatrix} \frac{4}{3+2h} - 2 + h^2 & 1 - \frac{1}{3+2h} & & & \\ 1 & -2 + h^2 & & & \\ & & \ddots & \ddots & \\ & & & 1 & -2 + h^2 & 1 \\ & & & 1 - \frac{1}{3+2h} & \frac{4}{3+2h} - 2 + h^2 \end{bmatrix}$$

$A$  is a tridiagonal matrix, so all other entries are zero. Therefore to approximate the solution solve the system  $A\mathbf{u} = \mathbf{f}$  and then plug in values for the expressions of  $U_0$  and  $U_{N+1}$ .

5. Construct the exact solution to the BVP with  $f(x) = -e^x$ .

This is a nonhomogenous ODE, so the solution to this BVP is found by summing the solutions to the homogenous and nonhomogenous equations. First I will begin by solving the homogenous version of the BVP, which is

$$u'' + u = 0$$

It is known that the general solution to this homogenous ODE is of the form

$$\begin{aligned} u(x) &= a \sin(x) + b \cos(x) \\ u''(x) &= -a \sin(x) - b \cos(x) \end{aligned}$$

which clearly satisfies the homogenous ODE.

A solution to the nonhomogenous ODE can be found as well. In this case the ODE is

$$u'' + u = -e^x$$

I will guess that the solution is of the form

$$\begin{aligned} u(x) &= c_1 e^x + c_2 e^{-x} \\ u''(x) &= c_1 e^x + c_2 e^{-x}. \end{aligned}$$

Substituting this into the ODE results in

$$2c_1 e^x + 2c_2 e^{-x} = -e^x.$$

Therefore

$$\begin{aligned} c_1 &= -\frac{1}{2} \\ c_2 &= 0 \end{aligned}$$

Thus the nonhomogenous solution is

$$u(x) = -\frac{1}{2}e^x$$

Therefore the overall solution to this BVP is

$$u(x) = -\frac{1}{2}e^x + a \sin(x) + b \cos(x)$$

Finally we must find  $a$  and  $b$  such that  $u(x)$  satisfies the boundary conditions.

$$\begin{aligned}
 u'(x) &= -\frac{1}{2}e^x + a \cos(x) - b \sin(x) \\
 u(0) &= -\frac{1}{2} + b \\
 u'(0) &= -\frac{1}{2} + a \\
 0 &= u'(0) - u(0) \\
 &= -\frac{1}{2} + a + \frac{1}{2} - b \\
 a &= b \\
 u(10) &= -\frac{1}{2}e^{10} + a \sin(10) + b \cos(10) \\
 u'(10) &= -\frac{1}{2}e^{10} + a \cos(10) - b \sin(10) \\
 0 &= u'(10) + u(10) \\
 &= -e^{10} + (\cos(10) + \sin(10))a + (\cos(10) - \sin(10))b
 \end{aligned}$$

Substituting in  $a$  for  $b$ .

$$\begin{aligned}
 0 &= -e^{10} + 2 \cos(10)a \\
 a &= \frac{e^{10}}{2 \cos(10)} \\
 b &= \frac{e^{10}}{2 \cos(10)}
 \end{aligned}$$

Therefore the exact solution to the BVP is

$$u(x) = -\frac{1}{2}e^x + \frac{e^{10}}{2 \cos(10)}(\sin(x) + \cos(x))$$

6. Verify that your method is second order accurate by solving the BVP with  $f(x) = -e^x$  for four different grid spacings  $h$ .

The script for solving this BVP is shown below.

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 6
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% solving BVP u'' + u = f(x) for f(x) = -e^x
% with BCs u'(0) - u(0) = 0 and u'(L) + u(L) = 0, where L = 10

L = 10;
f = @(x) -exp(x);

% exact solution
u = @(x) (-1/2)*exp(x) + (exp(L)/(2*cos(L)))*(sin(x) + cos(x));

% create vector for storing errors at respective h values
E = [];
% N is number of interior point

```

```

for N=[10, 20, 40, 80, 160, 320, 640, 1280, 2560, 5120]
    % find spacing
    h = L/(N+1);

    % create x grid and function values on grid
    x = 0:h:L;
    fx = f(x(2:N+1))';

    % create matrix A that solves the finite difference problem
    e = ones(N, 1);
    mid = (-2 + h^2)*e;
    mid(1) = mid(1) + 4/(3 + 2*h);
    mid(N) = mid(N) + 4/(3 + 2*h);
    low = e;
    low(end - 1) = 1 - 1/(3 + 2*h);
    up = e;
    up(2) = 1 - 1/(3 + 2*h);
    A = 1/h^2 * spdiags([low, mid, up], -1:1, N, N);
    U = A\fx;

    % add on first and last nodes, U_0 and U_{N+1}
    U = [U(1)/(1 + h); U; U(end)/(1 + h)];

    % find exact solution
    ux = u(x)';

    % record error
    E = [E; h, norm(U - ux, inf)];

    plot(x,ux, x,U);
    pause;
end

% show ratio of decrease in error
hRatios = E(1:end-1,1)./E(2:end,1);
errorRatios = E(1:end-1,2)./E(2:end,2);
order = log(errorRatios)./log(hRatios);
table(hRatios, errorRatios, order)

```

This outputs the following, which shows that as  $h$  decreases by a factor of 2 the error decreases by a factor of 4 which shows that the method is of order 2.

```
>> H01_6
```

```
ans =
```

hRatios	errorRatios	order
-----	-----	-----
1.9091	3.6953	2.0214
1.9524	3.5994	1.9143
1.9756	3.7616	1.9458
1.9877	3.8869	1.9763
1.9938	3.9406	1.9873
1.9969	3.9697	1.9935
1.9984	3.9849	1.9968
1.9992	3.9924	1.9984



7. Consider the following 2-point BVP:

$$\begin{aligned} -u'' + u &= f(x), \quad \text{on } 0 \leq x \leq 1 \\ u(0) &= u(1) \quad u'(0) = u'(1) \end{aligned}$$

Construct a fourth-order accurate finite-difference method for this BVP based on the fourth-order central finite difference. Write your method as a linear system of the form  $\mathbf{A}\mathbf{u} = \mathbf{f}$ .

First the interval  $[0, 1]$  needs to be discretized. Let  $x_0 = 0$ ,  $x_{N+1} = 1$ , and let  $x_i = ih$  for  $1 \leq i \leq N$ , where  $h = \frac{1}{N+1}$ . Thus the interval  $[0, 1]$  is described as a grid of  $N + 2$  equally spaced points with grid spacing  $h$ . Thus the solution to this BVP will be approximated on these grid points. Let  $U_i$  be the approximate value of  $u(x_i)$ . Thus an approximate solution to this BVP will be the values of  $U_i$  for  $0 \leq i \leq N + 1$ .

The boundary conditions are periodic, this implies that we can use the following equalities

$$\begin{aligned} U_0 &= U_{N+1} \\ U_{-1} &= U_N \\ U_{-2} &= U_{N-1} \\ U_{N+2} &= U_1 \end{aligned}$$

The periodic boundary conditions reduces the number of unknowns from  $N + 2$  to  $N + 1$ , and it allows for the finite difference to be used at all points  $i = 0, 1, \dots, N$ .

In order that this method is fourth-order accurate, I will use the fourth-order central difference to approximate the second derivative, which is

$$u''(x_i) \approx \frac{-U_{i-2} + 16U_{i-1} - 30U_i + 16U_{i+1} - U_{i+2}}{12h^2}$$

Therefore the finite difference equation for this differential equations is

$$\frac{U_{i-2} - 16U_{i-1} + (30 + 12h^2)U_i - 16U_{i+1} + U_{i+2}}{12h^2} = f(x_i)$$

for  $i = 0, 1, \dots, N$ .

This is equivalent to the following matrix equation

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$

where

$$\begin{aligned} \mathbf{u} &= [U_0, U_1, \dots, U_N]^T \\ \mathbf{f} &= [f(x_0), f(x_1), \dots, f(x_N)]^T \\ A &= \frac{1}{12h^2} \begin{bmatrix} 30 + 12h^2 & -16 & 1 & & \cdots & 1 & -16 \\ -16 & 30 + 12h^2 & -16 & 1 & & \cdots & 1 \\ 1 & -16 & 30 + 12h^2 & -16 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -16 & 30 + 12h^2 & -16 & 1 \\ -16 & 1 & \cdots & 1 & -16 & 30 + 12h^2 & -16 \end{bmatrix} \end{aligned}$$

8. Construct an exact solution to the BVP for  $f(x) = \sin(4\pi x)$ .

As we have seen the exact solution to this BVP will be the sum of the homogenous solution,  $u_h$ , and nonhomogenous solution,  $u_n$ , to this BVP. The homogenous version of this BVP is

$$-u_h'' + u_h = 0.$$

The solution to this ODE is of the form  $u_h(x) = a \sin(x) + b \cos(x)$ .

The nonhomogenous solution can be found as follows. First I will guess that the solution is of the form  $u_n(x) = c_1 \sin(4\pi x) + c_2 \cos(4\pi x)$ .

The constants in this function can be found by plugging this function into the differential equation.

$$\begin{aligned} u_n''(x) &= -16\pi^2 c_1 \sin(4\pi x) - 16\pi^2 c_2 \cos(4\pi x) \\ \sin(4\pi x) &= -u_n''(x) + u_n(x) \\ &= (16\pi^2 + 1)c_1 \sin(4\pi x) + (16\pi^2 + 1)c_2 \cos(4\pi x) \end{aligned}$$

This results in the following two equations

$$\begin{aligned} 1 &= (16\pi^2 + 1)c_1 \\ c_1 &= \frac{1}{16\pi^2 + 1} \\ 0 &= (16\pi^2 + 1)c_2 \\ c_2 &= 0 \end{aligned}$$

Thus the nonhomogenous solution to this BVP is  $u_n(x) = \frac{1}{16\pi^2 + 1} \sin(4\pi x)$ .

Therefore the full solution is of the form  $u(x) = a \sin(x) + b \cos(x) + \frac{1}{16\pi^2 + 1} \sin(4\pi x)$ . Now we must select  $a$  and  $b$  to satisfy the periodic boundary conditions.

$$\begin{aligned} u(x) &= a \sin(x) + b \cos(x) + \frac{1}{16\pi^2 + 1} \sin(4\pi x) \\ u(0) &= b \\ u(1) &= a \sin(1) + b \cos(1) \\ u(0) &= u(1) \\ b &= a \sin(1) + b \cos(1) \\ a &= \left( \frac{1 - \cos(1)}{\sin(1)} \right) b \\ u'(x) &= a \cos(x) - b \sin(x) + \frac{4\pi}{16\pi^2 + 1} \cos(4\pi x) \\ u'(0) &= a + \frac{4\pi}{16\pi^2 + 1} \\ u'(1) &= a \cos(1) - b \sin(1) + \frac{4\pi}{16\pi^2 + 1} \\ u'(0) &= u'(1) \\ a + \frac{4\pi}{16\pi^2 + 1} &= a \cos(1) - b \sin(1) + \frac{4\pi}{16\pi^2 + 1} \\ a &= a \cos(1) - b \sin(1) \\ 0 &= (\cos(1) - 1)a - b \sin(1) \end{aligned}$$

Substituting for  $a$

$$\begin{aligned} 0 &= (\cos(1) - 1) \left( \frac{1 - \cos(1)}{\sin(1)} \right) b - \sin(1)b \\ 0 &= \left( \frac{-\cos(1)^2 + 2\cos(1) - 1 - \sin(1)^2}{\sin(1)} \right) b \\ 0 &= \left( \frac{2\cos(1) - 2}{\sin(1)} \right) b \end{aligned}$$

Since  $\frac{2\cos(1)-2}{\sin(1)} \neq 0$ , then

$$b = 0$$

Therefore

$$a = 0$$

Thus the exact solution to the BVP is  $u(x) = \frac{1}{16\pi^2+1} \sin(4\pi x)$ .

9. Verify that your method is fourth-order accurate by solving the BVP with  $f(x) = \sin(4\pi x)$  at four different grid spacings  $h$ .

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Problem 9
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% solving BVP -u'' + u = f(x) for f(x) = sin(4pi x) on 0 \le x \le 1
% with periodic BCs u(0) = u(1) and u'(0) = u'(1)

L = 1;
f = @(x) sin(4*pi*x);

% exact solution
u = @(x) 1/(16*pi^2 + 1)*sin(4*pi*x);

% create vector for storing errors at respective h values
E = [];
% N is number of interior point
for N=[10, 20, 40, 80, 160, 320, 640, 1280]
    % find spacing
    h = L/(N+1);

    % create x grid and function values on grid
    % x_0 to x_{N+1}
    x = 0:h:L;
    % leave out x_{N+1}
    fx = f(x(1:end-1))';

    % create matrix A that solves the finite difference problem
    e = ones(N+1, 1);
    A = 1/(12*h^2) * spdiags([-16*e, e, e, -16*e, (30 + 12*h^2)*e, -16*e, e, e, ...
        -16*e], [-N, -N+1, -2:2, N-1, N], N+1, N+1);
    U = A\fx;

    % add last node at x_{N+1}
    % U_{N+1} = U_{0}
    U = [U; U(1)];

```

```

% compute exact solution
ux = u(x)';

% record error
E = [E; h, norm(U - ux, inf)];

% plot approximate and exact solution
plot(x, ux, x, U);
%pause;
end

% show ratio of decrease in error
hRatios = E(1:end-1,1)./E(2:end,1);
errorRatios = E(1:end-1,2)./E(2:end,2);
order = log(errorRatios)./log(hRatios);
table(hRatios, errorRatios, order)

```

>> H01\_9

ans =

hRatios	errorRatios	order
-----	-----	-----
1.9091	12.309	3.8822
1.9524	14.18	3.9636
1.9756	15.132	3.9902
1.9877	15.581	3.9975
1.9938	15.794	3.9993
1.9969	15.898	3.9998
1.9984	14.9	3.9017

10. Prove that your method is consistent with truncation error  $\|\tau\| = O(h^4)$  and  $L_2$ -stable, thereby proving that your method converges at  $O(h^4)$  in the  $L_2$ -norm.

First I will prove that my method is consistent with fourth order accuracy. A method is considered consistent and fourth order accurate if the norm of the truncation error approaches zero at the same rate as  $h^4$ . The truncation error is defined as the difference between the

In this case the truncation error is

$$\tau_i = \frac{1}{12h^2} \left( u(x_{i-2}) - 16u(x_{i-1}) + (30 + 12h^2)u(x_i) - 16u(x_{i+1}) + u(x_{i+2}) \right) - f(x_i)$$

This can be simplified using the following Taylor series

$$\begin{aligned}
u(x_{i-2}) &= u(x_i) - 2hu'(x_i) + 2h^2u''(x_i) - \frac{4}{3}h^3u'''(x_i) + \frac{2}{3}h^4u^{(4)}(x_i) - \frac{4}{15}h^5u^{(5)}(x_i) + \frac{4}{45}h^6u^{(6)}(c_{-2}) \\
-16u(x_{i-1}) &= -16u(x_i) + 16hu'(x_i) - 8h^2u''(x_i) + \frac{8}{3}h^3u'''(x_i) - \frac{2}{3}h^4u^{(4)}(x_i) + \frac{2}{15}h^5u^{(5)}(x_i) - \frac{1}{45}h^6u^{(6)}(c_{-1}) \\
-16u(x_{i+1}) &= -16u(x_i) - 16hu'(x_i) - 8h^2u''(x_i) - \frac{8}{3}h^3u'''(x_i) - \frac{2}{3}h^4u^{(4)}(x_i) - \frac{2}{15}h^5u^{(5)}(x_i) - \frac{1}{45}h^6u^{(6)}(c_1) \\
u(x_{i+2}) &= u(x_i) + 2hu'(x_i) + 2h^2u''(x_i) + \frac{4}{3}h^3u'''(x_i) + \frac{2}{3}h^4u^{(4)}(x_i) + \frac{4}{15}h^5u^{(5)}(x_i) + \frac{4}{45}h^6u^{(6)}(c_2)
\end{aligned}$$

Adding these four Taylor series results in

$$u(x_{i-2}) - 16u(x_{i-1}) - 16u(x_{i+1}) + u(x_{i+2}) = -30u(x_i) - 12h^2u''(x_i) + \frac{1}{45}h^6(4u^{(6)}(c_{-2}) - u^{(6)}(c_{-1}) - u^{(6)}(c_1) + 4u^{(6)}(c_2))$$

The Intermediate Value Theorem can be used several times to simplify the final term

$$u(x_{i-2}) - 16u(x_{i-1}) - 16u(x_{i+1}) + u(x_{i+2}) = -30u(x_i) - 12h^2u''(x_i) + \frac{6}{45}h^6u^{(6)}(\mu)$$

Plugging this expression back into the truncation error formula results in

$$\begin{aligned}\tau_i &= \frac{1}{12h^2} \left( (30 + 12h^2)u(x_i) - 30u(x_i) - 12h^2u''(x_i) + \frac{6}{45}u^{(6)}(\mu) \right) - f(x_i) \\ &= u(x_i) - u''(x_i) + \frac{1}{90}h^4u^{(6)}(\mu) - f(x_i)\end{aligned}$$

Since  $u$  is a solution to the differential equation,  $f(x_i) = -u''(x_i) + u(x_i)$

$$\tau_i = \frac{1}{90}h^4u^{(6)}(\mu)$$

Therefore

$$\tau_i = O(h^4)$$

This shows that our method is consistent and is fourth order accurate.

In order to show that our method is  $L_2$  stable, we must show that  $\|A^{-1}\|_2 \leq C$ .

Since  $A$  is symmetric we know that  $\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\min\{\lambda_i\}}$ , where  $\lambda_i$  are the eigenvalues of  $A$ . To find the eigenvalues of  $A$ , we must first find the eigenfunctions

of the original BVP, that is we must find functions  $v$  that satisfy

$$\begin{aligned}-v'' + v &= \lambda v, \quad \text{on } 0 \leq x \leq 1 \\ v(0) &= 0 \quad v(1) = 0\end{aligned}$$