MATH 517 Finite Differences Homework 6

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1. Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right)$$

(a) Determine the order of accuracy of this method (in both space and time).

In order to find the order of accuracy of this method, the local truncation error for this method must be found. The local truncation error for this method is given as

$$\tau = \frac{1}{2k}(u(t+k,x) - u(t-k,x)) - \frac{1}{h^2}(u(t,x-h) - 2u(t,x) + u(t,x+h))$$

I will consider the time and space terms seperately at first

$$u(t+k,x) = u(t,x) + ku_t(t,x) + \frac{1}{2}k^2 u_{tt}(t,x) + \frac{1}{6}k^3 u_{ttt}(t,x) + O(k^4)$$

$$u(t-k,x) = u(t,x) - ku_t(t,x) + \frac{1}{2}k^2 u_{tt}(t,x) - \frac{1}{6}k^3 u_{ttt}(t,x) + O(k^4)$$

$$\frac{1}{2k}(u(t+k,x) - u(t-k,x)) = \frac{1}{2k}\left(2ku_t(t,x) + \frac{1}{3}k^3 u_{ttt}(t,x) + O(k^4)\right)$$

$$\frac{1}{2k}(u(t+k,x) - u(t-k,x)) = u_t(t,x) + \frac{1}{6}k^2 u_{ttt}(t,x) + O(k^3)$$

$$u(t,x-h) = u(t,x) - hu_x(t,x) + \frac{1}{2}h^2 u_{xx}(t,x) - \frac{1}{6}h^3 u_{xxx}(t,x) + \frac{1}{24}h^4 u_{xxxx}(t,x) + O(h^5)$$

$$u(t,x+h) = u(t,x) + hu_x(t,x) + \frac{1}{2}h^2 u_{xx}(t,x) + \frac{1}{6}h^3 u_{xxx}(t,x) + \frac{1}{24}h^4 u_{xxxx}(t,x) + O(h^5)$$

$$\frac{1}{h^2}(u(t,x-h) - 2u(t,x) + u(t,x+h)) = \frac{1}{h^2}\left(h^2 u_{xx}(t,x) + \frac{1}{12}h^4 u_{xxxx} + O(h^6)\right)$$

$$\frac{1}{h^2}(u(t,x-h) - 2u(t,x) + u(t,x+h)) = u_{xx}(t,x) + \frac{1}{12}h^2 u_{xxxx} + O(h^4)$$

Note that in the spacial terms the 5th order error canceled out.
Using these two expressions the full truncation error is

$$\tau = u_t(t,x) + \frac{1}{6}k^2 u_{ttt}(t,x) + O(k^3) - u_{xx}(t,x) - \frac{1}{12}h^2 u_{xxxx} + O(h^4)$$

The heat equation states that $u_t(t,x) = u_{xx}(t,x)$

$$\tau = \frac{1}{6}k^2 u_{ttt}(t,x) + O(k^3) - u_{xx}(t,x) - \frac{1}{12}h^2 u_{xxxx} + O(h^4)$$

$$\tau = O(k^2 + h^2)$$

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α will this method be Lax-Richtmyer stable and hence convergent?

(c) Is this method useful?

2. Consider the one-dimensional heat equation:

$$PDE : u_t = \kappa u_{xx} \text{ for } (t,x) \in [0,T] \times [0,1]$$
$$BCs : u(t,0) = g_0, \ u(t,1) = g_1$$
$$ICs : u(0,x) = f(x)$$

(a) The m-file heat_CN.m solves the heat equation $u_t = \kappa u_{xx}$ using the Crank-Nicolson method. Run this code and by changing the number of grid points, confirm that it is second order accurate. Create a table of errors for varius h values, and $k = 4h$ at $T = 1$.

(b)

3.

4. Consider the following scheme for solving the 1D heat equation:

$$U_j^{n+1} = U_j^{n-1} + \frac{2k}{h^2}\left(U_{j+1}^n - \left(U_j^{n+1} + U_j^{n-1}\right) + U_{j-1}^n\right)$$

(a) Determine the local truncation error of this scheme.

The local truncation error for this scheme is given by

$$\tau = \frac{1}{2k}(u(t+k,x) - u(t-k,x)) - \frac{1}{h^2}(u(t,x+h) - (u(t+k,x) + u(t-k,x)) + u(t,x-h))$$

We have shown previously in problem 1(a) that

$$\frac{1}{2k}(u(t+k,x) - u(t-k,x)) = u_t(t,x) + \frac{1}{6}k^2 u_{ttt}(t,x) + O(k^3).$$

We can now consider the second term

$$u(t,x-h) = u(t,x) - h u_x(t,x) + \frac{1}{2}h^2 u_{xx}(t,x) - \frac{1}{6}h^3 u_{xxx}(t,x) + \frac{1}{24}h^4 u_{xxxx}(t,x) + O(h^5)$$

$$u(t,x+h) = u(t,x) + h u_x(t,x) + \frac{1}{2}h^2 u_{xx}(t,x) + \frac{1}{6}h^3 u_{xxx}(t,x) + \frac{1}{24}h^4 u_{xxxx}(t,x) + O(h^5)$$

$$u(t+k,x) = u(t,x) + k u_t(t,x) + \frac{1}{2}k^2 u_{tt}(t,x) + \frac{1}{6}k^3 u_{ttt}(t,x) + O(k^4)$$

$$u(t-k,x) = u(t,x) - k u_t(t,x) + \frac{1}{2}k^2 u_{tt}(t,x) - \frac{1}{6}k^3 u_{ttt}(t,x) + O(k^4)$$

$$\frac{1}{h^2}(u(t,x+h) - (u(t+k,x) + u(t-k,x)) + u(t,x-h))$$

$$= \frac{1}{h^2}\left(h^2 u_{xx} - k^2 u_{tt} + \frac{1}{12}h^4 u_{xxxx} + O(h^6) + O(k^4)\right)$$

$$= u_{xx} + \left(\frac{1}{12}h^2 - \frac{k^2}{h^2}\right)u_{xxxx} + O(h^4) + O(\frac{k^4}{h^2})$$

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