MATH 517 Finite Differences Homework 4

Caleb Logemann

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1. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

   (a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

   First from the book we see that the first two terms of the truncation error for any Linear Multistep method is

$$\tau = \frac{1}{k}\sum_{j=0}^{r}(\alpha_j)u(t_n) + \sum_{j=0}^{r}(j\alpha_j - \beta_j)u'(t_n) + O(k)$$

   Therefore we see that for a method to be consistent, that is at least of order k, the following two equalities must be met.

$$\sum_{j=0}^{r}(\alpha_j) = 0$$

$$\sum_{j=0}^{r}(j\alpha_j) = \sum_{j=0}^{r}(\beta_j)$$

   In these equtions α_j is the coefficient of U^{n+j} and β_j is the coefficient of $kf(U^{n+j})$ and r is the number of steps. Also for a Linear Multistep method to be zero-stable, the root condition must be met, which states that the magnitude of the roots of the characteristic polynomial are less than or equal to one or strictly less than one if the root is repeated. These two conditions will be used to test all the following methods.

   The previous method can be rewritten as

$$U^{n+2} - \frac{1}{2}U^{n+1} - \frac{1}{2}U^n = 2kf(U^{n+1})$$

   For this method

$$\sum_{j=0}^{r}(\alpha_j) = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$\sum_{j=0}^{r}(j\alpha_j) = 0 \times -\frac{1}{2} + 1 \times -\frac{1}{2} + 2 \times 1 = \frac{3}{2}\sum_{j=0}^{r}(\beta_j) \qquad = 2$$

   The second consistency condition is not satisfied, so this method is inconsistent.

   The characteristic polynomial for this method is

$$\rho(\zeta) = \zeta^3 - \frac{1}{2}\zeta - \frac{1}{2}$$

$$\rho(\zeta) = (\zeta - 1)\left(\zeta + \frac{1}{2}\right)$$

   The roots of this polynomial are $\zeta_1 = 1$ and $\zeta_2 = -\frac{1}{2}$. Neither of these roots are repeated and they both satisify $|\zeta_j| \leq 1$, therefore this method is zero-stable.

   Since this method is zero-stable but not consistent, this method is not convergent.

   (b) $U^{n+1} = U^n$

This method is equivalent to $U^{n+1} - U^n = 0$.

$$\sum_{j=0}^{r} (\alpha_j) = 1 - 1 = 0$$

$$\sum_{j=0}^{r} (j\alpha_j) = 0(-1) + 1(1) = 1$$

$$\sum_{j=0}^{r} (\beta_j) = 0$$

This method does not satisfy the consistency conditions.

The characteristic polynomial for this method is

$$\rho(\zeta) = \zeta - 1.$$

The root of this polynomial is $\zeta = 1$, this does satisfy the root condition, so the method is zero-stable.

This method is zero-stable, but not consistent and therefore not convergent.

(c) $U^{n+4} = U^n + \frac{4}{3}k\big(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1})\big)$

This method is equivalently $U^{n+4} - U^n = \frac{4}{3}k\big(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1})\big)$.

$$\sum_{j=0}^{r} (\alpha_j) = 1 - 1 = 0$$

$$\sum_{j=0}^{r} (j\alpha_j) = 0(-1) + 4(1) = 4$$

$$\sum_{j=0}^{r} (\beta_j) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$$

This method does satisfy both consistency conditions, and therefore is consistent.

The characteristic polynomial for this method is

$$\rho(\zeta) = \zeta^4 - 1.$$

The roots of this polynomial are $\zeta = 1, -1, i, -i$. None of these are repeated and all satisfy $|\zeta| \leq 1$, therefore the root condition is satisfied and this method is zero-stable.

Since this method is both zero-stable and consistent, this method is convergent.

(d) $U^{n+3} + U^{n+2} - U^{n+1} - U^n = 2k\big(f(U^{n+2}) + f(U^{n+1})\big)$

$$\sum_{j=0}^{r} (\alpha_j) = 1 + 1 - 1 - 1 = 0$$

$$\sum_{j=0}^{r} (j\alpha_j) = 0(-1) + 1(-1) + 2(1) + 3(1) = 4$$

$$\sum_{j=0}^{r} (\beta_j) = 2 + 2 = 4$$

This method does satisfy both consistency conditions, and therefore is consistent.

The characteristic polynomial for this method is

$$\begin{aligned}
\rho(\zeta) &= \zeta^3 + \zeta^2 - \zeta - 1 \\
\rho(\zeta) &= (\zeta - 1)(\zeta^2 + 2\zeta + 1) \\
\rho(\zeta) &= (\zeta - 1)(\zeta + 1)^2
\end{aligned}$$

The roots of this polynomial are $\zeta = 1, -1, -1$ The root $\zeta = -1$ has multiplicity 2 and does not satisfy $|\zeta| < 1$, therefore the root condition is not satisfied and the method is not zero-stable.

2. (a) Determine the general solution to the linear difference equation $2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$.

We know that the general solution to the linear difference equation is a linear combination of the roots of the characteristic polynomial. The characteristic polynomial for this linear difference equation is

$$\begin{aligned}
\rho(\zeta) &= 2\zeta^3 - 5\zeta^2 + 4\zeta - 1 \\
\rho(\zeta) &= (\zeta - 1)(2\zeta^2 - 3\zeta + 1) \\
\rho(\zeta) &= (\zeta - 1)^2(2\zeta - 1)
\end{aligned}$$

The roots of this equation are $\zeta = 1, 1, \frac{1}{2}$. Since we have repeated roots the general solution to this difference equation is $U^n = c_1 + c_2 n + c_3 \frac{1}{2^n}$.

(b) Determine the solution to this difference equation with the starting values $U^0 = 11$, $U^1 = 5$, and $U^2 = 1$. What is the value of U^{10}?

The 3 initial values create 3 equations in terms of c_1, c_2, c_3.

$$\begin{aligned}
11 &= c_1 + c_3 \\
5 &= c_1 + c_2 + \frac{1}{2}c_3 \\
1 &= c_1 + 2c_2 + \frac{1}{4}c_3
\end{aligned}$$

These equations can be solved as follows.

$$\begin{aligned}
6 &= -c_2 + \frac{1}{2}c_3 \\
-4 &= c_2 - \frac{1}{4}c_3 \\
2 &= \frac{1}{4}c_3 \\
8 &= c_3 \\
c_1 &= 3 \\
c_2 &= 5 - 3 - 4 = -2
\end{aligned}$$

The solution is therefore

$$U^n = 3 - 2n + \frac{8}{2^n}$$

The value of U^{10} can then be found as follows

$$\begin{aligned}
U^{10} &= 3 - 2 \times 10 + \frac{8}{2^{10}} \\
&= -17 + \frac{8}{2^{10}} \\
&= -\frac{2175}{128}
\end{aligned}$$

(c) Consider the LMM

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1}))$$

For what values of β_0 and β_1 is the local truncation error $O(k^2)$.
In order for this method to be 2nd order the following two equations must be satistified.

$$\sum_{j=0}^{r} (j\alpha_j) = \sum_{j=0}^{r} (\beta_j)$$
$$\sum_{j=0}^{r} \left(j^2\alpha_j\right) = \sum_{j=0}^{r} (j\beta_j)$$

For this problem these two equations are equivalent to

$$4 + 2(-5) + 3(2) = 0 = \beta_0 + \beta_1$$
$$4 + 4(-5) + 9(2) = 2 = \beta_1$$
$$\beta_0 = -2$$

Thus for this method to be second order $\beta_0 = -2$ and $\beta_1 = 2$.

(d) Suppose you use the values of β_0 and β_1 just determined in this LMM. Is this a convergent method?
No this method is not convergent, because the roots of the characteristic polynomial as shown in part (a) are $\zeta = 1, 1, \frac{1}{2}$. The root $\zeta = 1$ has multiplicity 2 and does not satisfy $|\zeta| < 1$. Therefore the root condition is not satisfied and the method is not zero-stable. This implies that the method is not convergent.

3. Consider the so-called θ-method for $u'(t) = f(u(t))$,

$$U^{n+1} = U^n + k\Big[(1-\theta)f(U^n) + \theta f(U^{n+1})\Big],$$

where θ is a fixed parameter.

(a) Show that the method is A-stable for $\theta \geq \frac{1}{2}$.
A method is A-stable if the region of absolute stability contains the entire left-half complex plane. The stability region for this method can be found as follows. The stability polynomial for this method is

$$\pi(\zeta; z) = (-1 - z(1-\theta)) + (1 - z\theta)\zeta$$

The root of this polynomial is

$$\zeta = \frac{-1 - z + z\theta}{1 - z\theta}$$
$$\zeta = -1 + \frac{z}{z\theta - 1}$$

In order for the root condition to be met

$$\left|-1+\frac{z}{z\theta-1}\right| \leq 1$$

$$\left(-1+\frac{z}{z\theta-1}\right)\left(-1+\frac{z^*}{z^*\theta-1}\right) \leq 1$$

$$1-\frac{z}{z\theta-1}-\frac{z^*}{z^*\theta-1}+\frac{zz^*}{(z\theta-1)(z^*\theta-1)} \leq 1$$

$$\frac{-z(z^*\theta-1)-z^*(z\theta-1)+zz^*}{(z\theta-1)(z^*\theta-1)} \leq 0$$

$$\frac{(1-2\theta)zz^*+z^*+z}{zz^*\theta^2-\theta(z^*+z)+1} \leq 0$$

$$\frac{(1-2\theta)zz^*+2\,\mathrm{Re}(z)}{zz^*\theta^2-2\theta\,\mathrm{Re}(z)+1} \leq 0$$

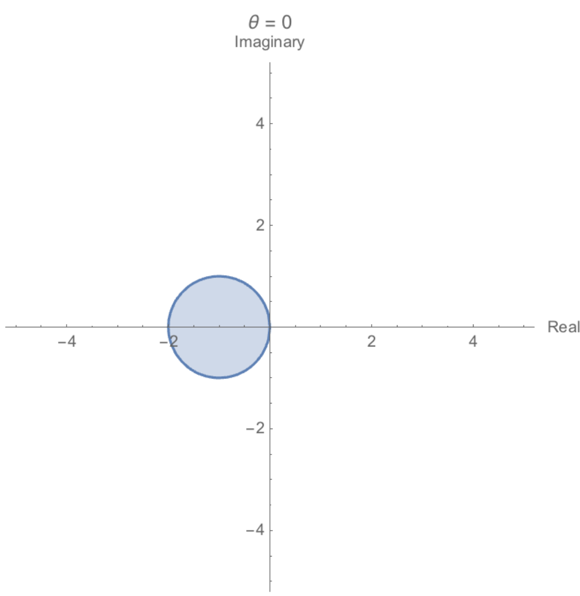
If $\mathrm{Re}(z) \leq 0$, then $zz^*\theta^2 - 2\theta\,\mathrm{Re}(z)+1 > 0$ so

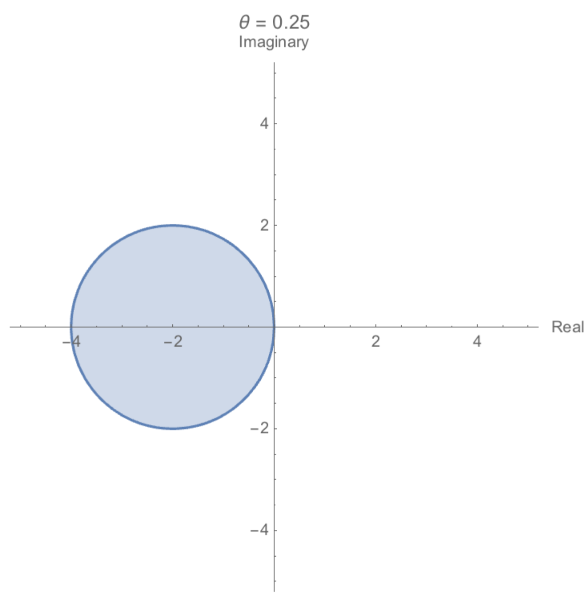
$$(1-2\theta)zz^* + 2\,\mathrm{Re}(z) \leq 0$$

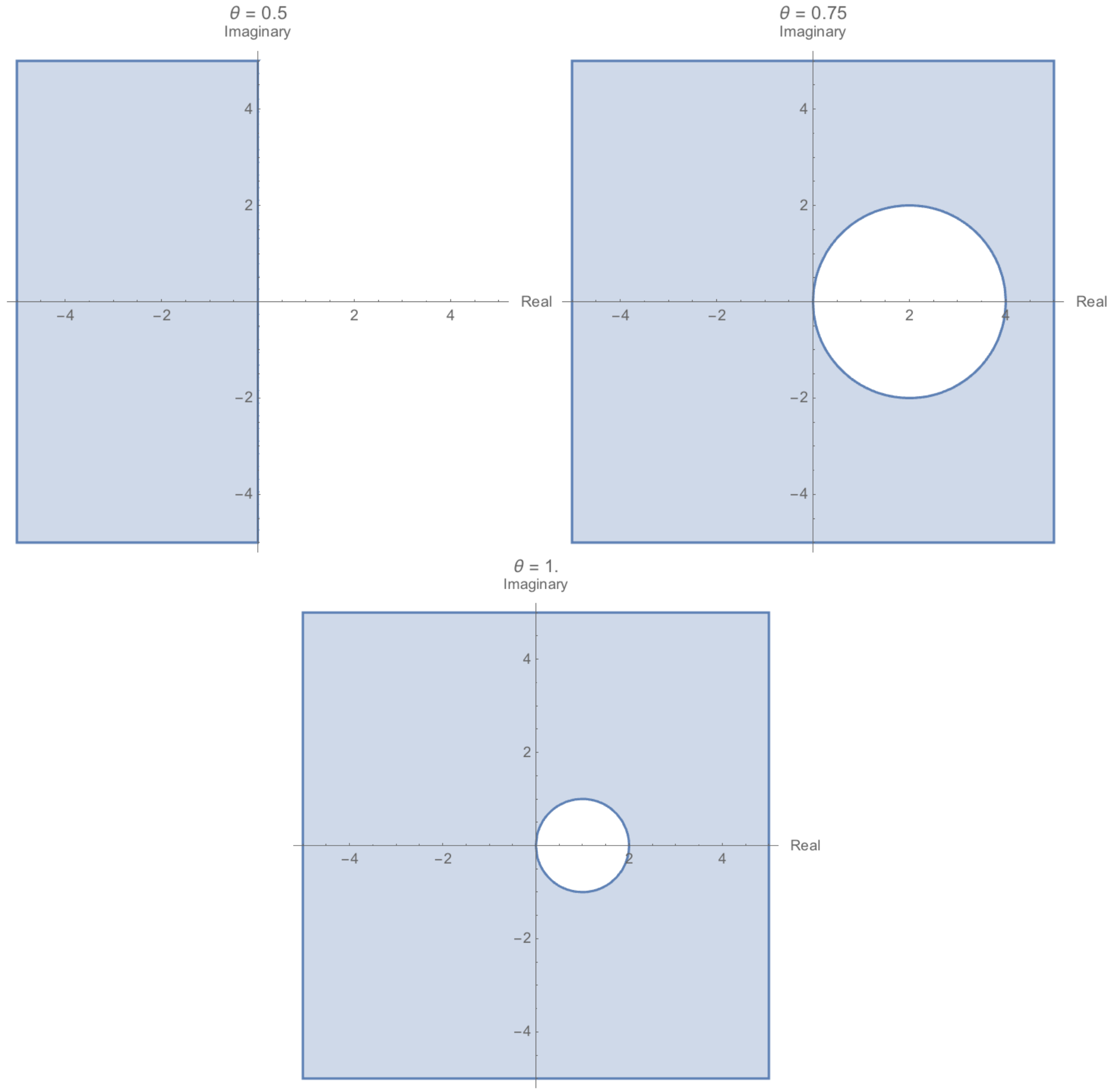
This equality is met if $\theta \geq \frac{1}{2}$, since $zz^* > 0$ and $\mathrm{Re}(z) < 0$. Therefore for $\theta \geq \frac{1}{2}$ the root condition is met for all z such that $\mathrm{Re}(z) \leq 0$. This means that the region of stability includes the left-half plane for $\theta \geq \frac{1}{2}$. Thus this method is A-stable for $\theta \geq \frac{1}{2}$.

(b) Plot the stability region, S, for $\theta = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and comment how the stability region will look for other values of θ.

Plots of the stability regions are shown below. For values of $\theta < .5$, the region of stability is a circle in the left half plane whose radius is increasing as θ approaches .5. At $\theta = .5$ the region of stability is the left-half plane. For $\theta > .5$ the region of stability is everything except a circle in the right-half plane. This circle's radius is decreasing as θ approaches 1.







4. (a) On the attached mathematica printout it is shown that this method for $f(u) = \lambda u$, that this method is equivalent to

$$U^{n+1} = \left(\frac{12 + 6k\lambda + (k\lambda)^2}{12 - 6k\lambda + (k\lambda)^2}\right)U^n.$$

This is equivalent to

$$U^{n+1} = e^{k\lambda}U^n - \frac{(k\lambda)^5}{720} + O(k^6).$$

Thus the one-step error is order k^5, and the local truncation error is order k^4.

(b) We showed in part (a) that the stability function was $R(z) = \frac{12+6z+z^2}{12-6z+z^2}$. In order for this method to be A-stable $|R(z)| \leq 1$ for all z such that $\text{Re}(z) \leq 1$.

From the picture on the mathematica printout we can see that this condition is satisfied, so this method is A-stable.

(c) This method is not L-stable because the limit of $R(z)$ as $z \to \infty$ along the real axis is 1. In other words

$$\lim_{z\to\infty}(R(z)) = 1.$$

Thus this method is not L-stable.