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MATH 520 Methods of Applied Math II
Homework 5

Section 12.4

#6 Let T denote the right shift operator on ℓ^2.

(a) Show that $\sigma_p(T) = \varnothing$.

Proof. First I will let $S_+ = T$, so as to better represent the right shift operator. In order to show that $\sigma_p(S_+) = \varnothing$, we must show that S_+ has no eigenvalues. First I will show that 0 is not an eigenvalue, if 0 was an eigenvalue then $S_+x = 0x = 0$ would have a nonzero solution $x \in \ell^2$. However the equation $S_+x = 0$ guarantees that

$$(S_+x)_k = x_{k-1} = 0$$

for $k \geq 1$, which implies that $x = 0$. This shows that the only solution to $S_+x = 0$ is $x = 0$, so 0 is not an eigenvalue of S_+, e.g. $0 \notin \sigma_p(S_+)$.
Next I will show that no nonzero value can be an eigenvalue. Assume to the contrary that $\lambda \neq 0 \in \sigma_p(T)$, that is λ is an eigenvalue of T. This implies that $S_+x = \lambda x$ has a nonzero solution $x \in \ell^2$. Both S_+x and λx are sequences and for the sequences to be equal each term in the sequences must be equal. Therefore I will compare the terms of these sequences. Note that by definition $(S_+x)_k = x_{k-1}$ for $k \geq 1$ and $(S_+x)_0 = 0$. Using this definition, in $S_+x = \lambda x$ two conditions arise first for $k = 0$

$$\lambda x_0 = (\lambda x)_0 = (S_+x)_0 = 0$$

and for $k \geq 1$

$$\lambda x_k = (\lambda x)_k = (S_+x)_k = x_{k-1}$$

The first condition shows that $x_0 = 0$, and the second that $x_k = \frac{x_{k-1}}{\lambda}$. However these two statements together inductively show that $x_k = 0$ for $k \geq 0$. Thus $x = 0$ is the only solution to the equation $S_+x = \lambda x$, and $\lambda \notin \sigma_p(S_+)$. This shows that no complex number can be an eigenvalue of S_+ and so $\sigma_p(S_+) = \varnothing$. □

(b) Show that $\sigma_c(T) = \{\lambda : |\lambda| = 1\}$.

(c) Show that $\sigma_r(T) = \{\lambda : |\lambda| < 1\}$.

Proof. Again I will let $S_+ = T$ and I will prove (b) and (c) simultaneaously. From part (a) it is clear that for any λ, the operator $\lambda I - S_+$ is one-to-one. If $\lambda I - S_+$ was not one-to-one then $\lambda \in \sigma_p(S_+)$, however we have already shown that $\sigma_p(S_+)$ is empty. Thus for any $\lambda \in \mathbb{C}$, λ must be in the resolvant set, the continuous spectrum, or the residual spectrum. From Example 12.5 it is known that for $|\lambda| > 1$, then $\lambda \in \rho(S_+)$. This is shown by noting that for bounded operators $\lambda \in \sigma(S_+)$ implies that $|\lambda| \leq \|S_+\| = 1$. Let $\lambda \in \sigma(S_+)$, then $|\lambda| \leq 1$. Note that $I, S_+ \in \mathcal{B}(\ell^2)$, so

$$(\lambda I - S_+)^* = \overline{\lambda} I^* - S_+^* = \overline{\lambda} I - S_-.$$

Also since $\lambda I - S_+$ is densely defined linear operator

$$R(\lambda I - S_+)^\perp = N((\lambda I - S_+)^*) = N\left(\overline{\lambda} I - S_-\right)$$

Since $R(\lambda I - S_+)$ determines whether λ is in the continuous spectrum or the residual spectrum, I will inspect $N\left(\overline{\lambda}I - S_-\right)$. Let $x \in N\left(\overline{\lambda}I - S_-\right)$, then

$$\begin{aligned} \left(\overline{\lambda}I - S_-\right)x &= 0 \\ \overline{\lambda}x - S_-x &= 0 \\ S_-x &= \overline{\lambda}x \\ (S_-x)_n &= \overline{\lambda}x_n \\ x_{n+1} &= \overline{\lambda}x_n \end{aligned}$$

Inducting on this formula, we find that a explicit formula for x_n

$$x_n = \overline{\lambda}^n x_0$$

This sequence $x_n = \overline{\lambda}^n x_0$ for an arbitrary x_0 is a potential element in $N\left(\overline{\lambda}I - S_-\right)$, yet it remains to be seen if $\{x_n\} \in \ell^2$. In order to see if $\{x_n\}$ is in ℓ^2 consider $\sum_{n=0}^{\infty}\left(|x_n|^2\right)$. If this sum is convergent then $x = \{x_n\}$ is in ℓ^2 and if the sum is not convergent then x is not in ℓ^2.

$$\begin{aligned} \sum_{n=0}^{\infty}\left(|x_n|^2\right) &= \sum_{n=0}^{\infty}\left(\left|\overline{\lambda}^n x_0\right|^2\right) \\ &= \sum_{n=0}^{\infty}\left(\left|\overline{\lambda}\right|^{2n}|x_0|^2\right) \\ &= |x_0|^2\sum_{n=0}^{\infty}\left(\left|\overline{\lambda}\right|^{2n}\right) \end{aligned}$$

Since we already know that $|\lambda| \leq 1$, there are two possible cases, either $|\lambda| = 1$ or $|\lambda| < 1$. If $|\lambda| < 1$, then

$$\sum_{n=0}^{\infty}\left(|x_n|^2\right) = |x_0|^2\sum_{n=0}^{\infty}\left(\left|\overline{\lambda}\right|^{2n}\right) = \frac{|x_0|^2}{1 - \left|\overline{\lambda}\right|^2}$$

as this is a geometric series. In this case, the sum converges for any x_0. Therefore for any x_0 the sequence $x_n = \overline{\lambda}^n x_0$ is in $N\left(\overline{\lambda}I - S_-\right)$. This shows that $N\left(\overline{\lambda}I - S_-\right) \neq \{0\}$. If we examine the relationship with the range again we see that

$$\begin{aligned} R(\lambda I - S_+)^{\perp} &= N\left(\overline{\lambda}I - S_-\right) \\ \left(R(\lambda I - S_+)^{\perp}\right)^{\perp} &= \left(N\left(\overline{\lambda}I - S_-\right)\right)^{\perp} \\ \overline{R(\lambda I - S_+)} &\neq (\{0\})^{\perp} \\ \overline{R(\lambda I - S_+)} &\neq \ell^2 \end{aligned}$$

Thus if $|\lambda| < 1$, then $R(\lambda I - S_+)$ is not dense and $\lambda \in \sigma_r(S_+)$. This shows that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_r(S_+)$.
If one the other hand $|\lambda| = 1$, then

$$\sum_{n=0}^{\infty}\left(|x_n|^2\right) = |x_0|^2\sum_{n=0}^{\infty}\left(\left|\overline{\lambda}\right|^{2n}\right) = |x_0|^2\sum_{n=0}^{\infty}(1)$$

which only converges if $x_0 = 0$. Thus if $|\lambda| = 1$, then $N\left(\overline{\lambda}I - S_-\right) = \{0\}$. Using the relationship with the range of $\lambda I - S_+$, we see that

$$
\begin{aligned}
R(\lambda I - S_+)^\perp &= N\left(\overline{\lambda}I - S_-\right) \\
\left(R(\lambda I - S_+)^\perp\right)^\perp &= \left(N\left(\overline{\lambda}I - S_-\right)\right)^\perp \\
\overline{R(\lambda I - S_+)} &= (\{0\})^\perp \\
\overline{R(\lambda I - S_+)} &= \ell^2
\end{aligned}
$$

This shows that the range of $\lambda I - S_+$ is dense in ℓ^2 when $|\lambda| = 1$.
Note that this shows that if $\lambda \in \sigma_r(S_+)$ then $|\lambda| \ngeq 1$, so $|\lambda| < 1$ and $\sigma_r(S_+) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Thus $\sigma_r(S_+) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ in fact.
Now if $|\lambda| = 1$, then $R(\lambda I - S_+)$ is dense, so either $\lambda \in \sigma_c(S_+)$ or $\lambda \in \rho(S_+)$. Suppose $\lambda \in \rho(S_+)$, then because $\rho(S_+)$ must be an open set λ must be an interior point of $\rho(S_+)$. Let B be a ball of radius ϵ around λ. Since $|\lambda| = 1$, there must be some $\mu \in B$ such that $1 - \epsilon < |mu| < 1$. However we have previously seen that if $|\mu| < 1$, then $\mu \in \sigma_r(S_+)$. This contradicts the fact that λ is an interior point of $\rho(S_+)$, because any ball around λ will contain points in the residual spectrum. Therefore $\lambda \notin \rho(S_+)$ and $\lambda \in \sigma_c(S_+)$. Thus $\{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \sigma_c(S_+)$. But since $\sigma_r(S_+)$, $\sigma_c(S_+)$, and $\rho(S_+)$ are disjoint this shows that $\{\lambda \in \mathbb{C} : |\lambda| = 1\} = \sigma_c(S_+)$. This is because $|\lambda| < 1$ implies $\lambda \in \sigma_r(S_+)$ and $|\lambda| > 1$ implies that $\lambda \in \rho(S_+)$.
In conclusion this shows for part (b) that

$$\{\lambda \in \mathbb{C} : |\lambda| = 1\} = \sigma_c(S_+)$$

and for part (c) that

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} = \sigma_c(S_+).$$

$\square$

#7 If $\lambda \neq \pm 1, \pm i$ show that λ is in the resolvant set of the Fourier Transform $\mathcal{F}$. (Suggestion: Assuming that a solution of $\mathcal{F}u - \lambda u = f$ exists, derive an explicit formula for it by justifying and using the identity

$$\mathcal{F}^4 u = \lambda^4 u + \lambda^3 f + \lambda^2 \mathcal{F} f + \mathcal{F}^3 f$$

together with the fact that $\mathcal{F}^4 = I$.)

Proof. Let $\lambda \in \mathbb{C}$ such that $\lambda \neq \pm 1, \pm i$. We have previously shown that this implies that $\lambda \notin \sigma_p(\mathcal{F})$. Therefore $\lambda \in \rho(\mathcal{F}) \cup \sigma_c(\mathcal{F}) \cup \sigma_r(\mathcal{F})$. This implies that for some $f \in L^2 \cap L^1$, there exists a solution $u \in L^2 \cap L^2$ such that $\mathcal{F}u - \lambda u = f$. This is equivalent to $\mathcal{F}u = f + \lambda u$. If we take the Fourier transform of each side several times the equality is perserved.

$$\begin{aligned}
\mathcal{F}u &= \lambda u + f \\
\mathcal{F}^2 u &= \lambda \mathcal{F} u + \mathcal{F} f \\
\mathcal{F}^2 u &= \lambda(\lambda u + f) + \mathcal{F} f \\
\mathcal{F}^2 u &= \lambda^2 u + \lambda f + \mathcal{F} f \\
\mathcal{F}^3 u &= \lambda^2 \mathcal{F} u + \lambda \mathcal{F} f + \mathcal{F}^2 f \\
\mathcal{F}^3 u &= \lambda^2(\lambda u + f) + \lambda \mathcal{F} f + \mathcal{F}^2 f \\
\mathcal{F}^3 u &= \lambda^3 u + \lambda^2 f + \lambda \mathcal{F} f + \mathcal{F}^2 f \\
\mathcal{F}^4 u &= \lambda^3 \mathcal{F} u + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f \\
\mathcal{F}^4 u &= \lambda^3(\lambda u + f) + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f \\
\mathcal{F}^4 u &= \lambda^4 u + \lambda^3 f + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f
\end{aligned}$$

Since $\mathcal{F}^4 = I$

$$\begin{aligned}
u &= \lambda^4 u + \lambda^3 f + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f \\
u - \lambda^4 u &= \lambda^3 f + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f \\
u &= \frac{1}{1 - \lambda^4}\left(\lambda^3 f + \lambda^2 \mathcal{F} f + \lambda \mathcal{F}^2 f + \mathcal{F}^3 f\right)
\end{aligned}$$

Since $f \in L^2 \cap \text{Ł}^1$ and $\lambda \neq \pm 1, \pm i$ this formula for u is well-defined. It remains to be seen that $u \in L^2 \cap L^2$. However $\mathcal{F} f \in L^2 \cap L^1$ so u is the sum of functions in $L^2 \cap L^1$ so $u \in L^2 \cap L^1$. Therefore we have an explicit formula for u given any f, this shows that $R(\lambda I - \mathcal{F}) = L^2 \cap L^1$, so $\lambda \in \rho(\mathcal{F})$. $\square$

#8 Let $\mathbf{H} = L^2(0,1)$, $T_1 u = T_2 u = T_3 u = u'$ on the domains

$$\begin{aligned} D(T_1) &= H^1(0,1) \\ D(T_2) &= \left\{u \in H^1(0,1) : u(0) = 0\right\} \\ D(T_3) &= \left\{u \in H^1(0,1) : u(0) = u(1) = 0\right\} \end{aligned}$$

(i) Show that $\sigma(T_1) = \sigma_p(T_1) = \mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$ and consider the equation $T_1 u = \lambda u$. Let $u(x) = e^{\lambda x}$, and note that $u \in H^1(0,1)$, because $u \in L^2(0,1)$ and $u'(x) = \lambda e^{\lambda x} \in L^2(0,1)$. This is true because

$$\int_0^1 \left|e^{\lambda x}\right|^2 \, \mathrm{d}x < \infty$$

and

$$\int_0^1 \left|\lambda e^{\lambda x}\right|^2 \, \mathrm{d}x < \infty.$$

Since $u' = \lambda e^{\lambda x}$ it is clear that $u' = \lambda u$. Therefore u is a nonzero solution to $T_1 u = \lambda u$, and therefore $\lambda \in \sigma_p(T_1)$. This shows that $\sigma_p(T_1) = \mathbb{C}$. □

(ii) Show that $\sigma(T_2) = \varnothing$.

Proof. Let $\lambda \in \mathbb{C}$, and suppose that $\lambda \in \sigma(T_2)$. As in part (i), if there is a solution to $T_2 u = \lambda u$, then $u(x) = Ae^{\lambda x}$ where $A \neq 0$ is some scalar. However any u of this form is not in $D(T_2)$ because even though $u \in H^1(0,1)$, we have $u(0) = A \neq 0$. Therefore $\lambda \notin \sigma_p(T_2)$. This also shows that $\lambda I - T_2$ is a one-to-one function for any λ, because if $\lambda I - T_2$ was not one-to-one then $\lambda \in \sigma_p(T_2)$. Consider the equation $(T_2 - \lambda I)u = f$ for some $f \in L^2(0,1)$. This is a differential equation that can be solved using an integrating factor.

$$\begin{aligned} (T_2 - \lambda I)u &= f \\ T_2 u - \lambda u &= f \\ u' - \lambda u &= f \end{aligned}$$

Let $\mu = e^{-\lambda x}$ and multiply both sides of the equation

$$\begin{aligned} e^{-\lambda x} u' - \lambda e^{-\lambda x} u &= e^{-\lambda x} f \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda x} u\right) &= e^{-\lambda x} f \\ e^{-\lambda x} u &= \int_0^x e^{-\lambda y} f \, \mathrm{d}y \\ u &= e^{\lambda x} \int_0^x e^{-\lambda y} f \, \mathrm{d}y \end{aligned}$$

Note that this is one possible solution to this equation. The important quality of this solution is that $u \in D(T_2)$. To see this note that

$$u(0) = e^{\lambda x} \int_0^0 e^{-\lambda y} f \, \mathrm{d}y = 0$$

Also $u \in L^2(0,1)$ as it is the product of functions in $L^2(0,1)$. The derivative of u also exists by the product rule,

$$u'(x) = \lambda e^{\lambda x} \int_0^x e^{-\lambda x} f \, \mathrm{d}x + e^{\lambda x}\Big(e^{-\lambda x} f(x) - f(0)\Big).$$

This derivative is in $L^2(0,1)$ as well. Therefore $u \in H^1(0,1)$ and $u \in D(T_2)$. This shows that for any f there is a solution $u \in D(T_2)$ or in other words $R(\lambda I - T_2) = L^2(0,1)$ for all λ. Thus $\lambda \in \rho(T_2)$, and not in $\sigma_r(T_2)$ or $\sigma_c(T_2)$. This shows that the spectrum is empty, i.e. $\sigma(T_2) = \varnothing$. $\square$

(iii) Show that $\sigma(T_3) = \sigma_r(T_3) = \mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$, Again as in part (i) if $T_3 u = \lambda u$ is going to have a solution, then it must be in the form $u(x) = Ae^{\lambda x}$. However if $u \in D(T_3)$, then $u(0) = u(1) = 0$ and this implies that $A = 0$ which makes $u(x) = 0$. Thus there is no nonzero solution to $T_3 u = \lambda u$. Thus for any $\lambda \in \mathbb{C}$, the operator $T_3 - \lambda I$ is one-to-one.

As is part (ii) we can say that the equation $T_3 u - \lambda u = f$ has a solution if and only if $\frac{\mathrm{d}}{\mathrm{d}=}\Big(e^{-\lambda x} u\Big) e^{-\lambda x} f$. Let $G(x)$ be any antiderivative of $e^{-\lambda x} f$, then $u = e^{\lambda x} G(x)$. So in order for u to be in $D(T_3)$ this implies that $G(0) = G(1) = 0$. This can only happen if

$$\int_0^1 f(x) e^{-\lambda x} \, \mathrm{d}x = 0$$

So there is a solution to $(\lambda I - T_3)u = f$ when f has this property. The set of $f \in L^2(0,1)$, which have this property are not a dense subset of $L^2(0,1)$, so this implies that $\lambda \in \sigma_r(T_3)$. Since this was true for any λ, this means that

$$\sigma_r(T_3) = \mathbb{C}$$

$\square$

#10 Let $Tu(x) = \int_0^x K(x,y)u(y)\,\mathrm{d}y$ be a Volterra intergral operator on $L^2(0,1)$ with a bounded kernel, $|K(x,y)| \le M$. Show that $\sigma(T) = \{0\}$. (There are several ways to show that T has no nonzero eigenvalues. Here is one approach: Define the equivalent norm on $L^2(0,1)$

$$\|u\|_\theta^2 = \int_0^1 |u(x)|^2 e^{-2\theta x}\,\mathrm{d}x$$

and show that the supremum of $\frac{\|Tu\|_\theta}{\|u\|_\theta}$ can be made arbitrarily small by choosing θ sufficiently large.

Proof. First I will show that $\|u\|_\theta$ is indeed a norm. Let $u = 0$ then

$$\begin{aligned}
\|0\|_\theta^2 &= \int_0^1 |0|^2 e^{-2\theta x}\,\mathrm{d}x \\
&= \int_0^1 0\,\mathrm{d}x = 0
\end{aligned}$$

Now suppose $\|u\|_\theta^2 = 0$,

$$\begin{aligned}
\|u\|_\theta^2 &= \int_0^1 |u|^2 e^{-2\theta x}\,\mathrm{d}x \\
0 &= \int_0^1 |u|^2 e^{-2\theta x}\,\mathrm{d}x.
\end{aligned}$$

Since this is an integral of a nonnegative function it can only be zero if the integrand is zero, this implies that

$$u = 0.$$

This shows that $\|u\|_\theta = 0$ if and only if $u = 0$.

Now suppose $u \in L^2(0,1)$ and λ is some scalar, then

$$\begin{aligned}
\|\lambda u\|_\theta &= \sqrt{\int_0^1 |\lambda u|^2 e^{-2\theta x}\,\mathrm{d}x} \\
&= \sqrt{|\lambda|^2 \int_0^1 |u|^2 e^{-2\theta x}\,\mathrm{d}x} \\
&= |\lambda| \sqrt{\int_0^1 |u|^2 e^{-2\theta x}\,\mathrm{d}x} \\
&= |\lambda| \|u\|_\theta
\end{aligned}$$

This shows that $\|\lambda u\|_\theta = |\lambda| \|u\|_\theta$.

Let $u, v \in L^2(0,1)$, then

$$\begin{aligned}
\|u+v\|_\theta &= \sqrt{\int_0^1 |u(x)+v(x)|^2 e^{-2\theta x} \, \mathrm{d}x} \\
&= \sqrt{\int_0^1 |u(x)e^{-\theta x} + v(x)e^{-\theta x}|^2 \, \mathrm{d}x} \\
&= \left\| u(x)e^{-\theta x} + v(x)e^{-\theta x} \right\|_{L^2(0,1)} \\
&\leq \left\| u(x)e^{-\theta x} \right\|_{L^2(0,1)} + \left\| v(x)e^{-\theta x} \right\|_{L^2} \\
&= \sqrt{\int_0^1 |u(x)e^{-\theta x}|^2 \, \mathrm{d}x} + \sqrt{\int_0^1 |u(x)e^{-\theta x}|^2 \, \mathrm{d}x} \\
&= \sqrt{\int_0^1 |u(x)|^2 e^{-2\theta x} \, \mathrm{d}x} + \sqrt{\int_0^1 |u(x)|^2 e^{-2\theta x} \, \mathrm{d}x} \\
&= \|u(x)\|_\theta + \|u(x)\|_\theta
\end{aligned}$$

Thus this function also satisfies the triangle inequality and therefore it is a norm.

Now that we have established that $\|\cdot\|_\theta$ is a norm, I will show that this norm is equivalent to the L^2 norm. First note that $e^{-2\theta} \leq e^{-2\theta x} \leq 1$ when $x \in [0,1]$. Using this it is clear that

$$e^{-2\theta} \int_0^1 |u(x)|^2 \, \mathrm{d}x \leq \int_0^1 |u(x)|^2 e^{-2\theta x} \, \mathrm{d}x \leq \int_0^1 |u(x)|^2 \, \mathrm{d}x$$

or equivalently

$$e^{-2\theta} \|u\|_{L^2} \leq \|u\|_\theta \leq \|u\|_{L^2}.$$

for any $u \in L^2(0,1)$. This shows that the norm $\|\cdot\|_\theta$ is equivalent to $\|\cdot\|_{L^2}$.

Next I will show that $\lim \left(\frac{\|Tu\|_\theta}{\|u\|_\theta} \right)$ arbitrarily small, by making θ large enough. Let $\epsilon > 0$ be fixed.

$$\begin{aligned}
\|Tu\|_\theta^2 &= \int_0^1 |Tu(x)|^2 e^{-2\theta x} \, \mathrm{d}x \\
&= \int_0^1 \left| \int_0^x K(x,y)u(y) \, \mathrm{d}y \right|^2 e^{-2\theta x} \, \mathrm{d}x
\end{aligned}$$

Using Holder's Inequality

$$\leq \int_0^1 e^{-2\theta x} \, \mathrm{d}x \int_0^1 \int_0^x |K(x,y)u(y)|^2 \, \mathrm{d}y \, \mathrm{d}x$$

Using Fubini's Theorem

$$= \int_0^1 e^{-2\theta x} \, \mathrm{d}x \int_0^1 |u(y)|^2 \int_y^1 |K(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

Since $K(x,y)$ is bounded on $[0,1] \times [0,1]$, there exists M such that

$$\begin{aligned}
&\leq \int_0^1 e^{-2\theta x} \, \mathrm{d}x \int_0^1 M|u(y)|^2 \, \mathrm{d}y \\
&= \frac{M}{-2\theta e^{-2\theta}} \|u\|_\theta^2
\end{aligned}$$

This implies that

$$\frac{\|Tu\|_\theta}{\|u\|_\theta} \le \sqrt{\frac{M}{-2\theta e^{-2\theta}}}$$

Since θ can be any number there exists θ such that

$$\frac{\|Tu\|_\theta}{\|u\|_\theta} \le \epsilon$$

Now that we have shown that

$$\|T\|_\theta = \lim\left(\frac{\|Tu\|_\theta}{\|u\|_\theta}\right) < \epsilon$$

for any $\epsilon > 0$, we can show that $\sigma(T) = \{0\}$. Theorem 12.1 states that $|\lambda| \le \|T\|$ for any $\lambda \in \sigma(T)$, because $T \in \mathcal{B}(L^2(0,1))$. However since $\|T\|_\theta$ is equivalent to $\|T\|$, this also means that $|\lambda| \le \|T\|_\theta$ for any θ. Now since $\|T\|_\theta$ can be arbitrarily small this implies that $\lambda = 0$ is the only possible element of $\sigma(T)$. Also by Theorem 12.3 we know that $\sigma(T)$ is nonempty because $T \in \mathcal{B}(L^2(0,1))$. This shows that $\sigma(T) = \{0\}$. $\square$

#11 If T is a symmetric operator, show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \mathbb{R}$$

(It is almost the same as showing that $\sigma(T) \subset \mathbb{R}$ for a self-adjoint operator.)

Proof. Let $\lambda = \xi + i\eta$ with $\eta \neq 0$. Now consider for $u \in D(T)$,

$$\begin{aligned}
\|(\lambda I - T)u\|^2 &= \|\lambda u - Tu\|^2 \\
&= \langle \lambda u - Tu, \lambda u - Tu \rangle \\
&= \langle \xi u + i\eta u - Tu, \xi u + i\eta u - Tu \rangle \\
&= \langle \xi u - Tu, \xi u - Tu \rangle + \langle i\eta u, \xi u - Tu \rangle + \langle \xi u - Tu, i\eta u \rangle + \langle i\eta u, i\eta u \rangle \\
&= \|\xi u - Tu\|^2 + \langle i\eta u, \xi u - Tu \rangle + \langle \xi u - Tu, i\eta u \rangle + \|i\eta u\|^2 \\
&= \|\xi u - Tu\|^2 + \langle i\eta u, \xi u - Tu \rangle + \langle \xi u - Tu, i\eta u \rangle + |\eta|^2\|u\|^2
\end{aligned}$$

Now note that

$$\begin{aligned}
\langle i\eta u, \xi u - Tu \rangle + \langle \xi u - Tu, i\eta u \rangle &= i\eta\langle u, \xi u - Tu \rangle - i\eta\langle \xi u - Tu, u \rangle \\
&= i\eta(\langle u, \xi u - Tu \rangle - \langle \xi u - Tu, u \rangle) \\
&= i\eta(\langle u, \xi u - Tu \rangle - (\langle \xi u, u \rangle - \langle Tu, u \rangle)) \\
&= i\eta(\langle u, \xi u - Tu \rangle - (\xi\langle u, u \rangle - \langle Tu, u \rangle))
\end{aligned}$$

As T is symmetric

$$= i\eta(\langle u, \xi u - Tu \rangle - (\xi\langle u, u \rangle - \langle u, Tu \rangle))$$

As ξ is real

$$\begin{aligned}
&= i\eta(\langle u, \xi u - Tu \rangle - (\langle u, \xi u \rangle - \langle u, Tu \rangle)) \\
&= i\eta(\langle u, \xi u - Tu \rangle - \langle u, \xi u - Tu \rangle) \\
&= i\eta(0) = 0
\end{aligned}$$

Therefore

$$\|(\lambda I - T)u\|^2 = \|\xi u - Tu\|^2 + |\eta|^2\|u\|^2$$

Now as $\|\xi u - Tu\|^2 \geq 0$, this implies that

$$\|(\lambda I - T)u\|^2 \geq |\eta|^2\|u\|^2$$

or

$$\|(\lambda I - T)u\| \geq |\eta|\|u\|.$$

Since $|\eta| > 0$ this implies that $\lambda I - T$ is one-to-one because if $\|u\| > 0$ then $\|(\lambda I - T)u\| > 0$ which shows that $N(\lambda I - T) = \{0\}$. Since $\lambda I - T$ is one-to-one for any complex λ this shows that $\lambda \notin \sigma_p(T)$. Therefore $\sigma_p(T) \subset \mathbb{R}$.

Since $\lambda \notin \sigma_p(T)$ this implies that $\lambda \in \sigma_c(T)$. Since $\lambda \in \sigma_c(T)$, then $\overline{R(\lambda I - T)} = H$.

I now claim that $R(\lambda I - T)$ must be closed. To see this note that $(\lambda I - T)^{-1}$ is well-defined because $\lambda I - T$ is one-to-one. Also $(\lambda I - T)^{-1}$ is bounded. If $u \in D(\lambda I - T)$, then there exists

$v \in R(\lambda I - T) = D((\lambda I - T)^{-1})$ such that $u = (\lambda I - T)^{-1}v$. Now

$$\begin{aligned}
\|(\lambda I - T)u\| &\geq |\eta|\|u\| \\
\left\|(\lambda I - T)(\lambda I - T)^{-1}u\right\| &\geq |\eta|\left\|(\lambda I - T)^{-1}u\right\| \\
\|u\| &\geq |\eta|\left\|(\lambda I - T)^{-1}u\right\| \\
\frac{\left\|(\lambda I - T)^{-1}u\right\|}{\|u\|} &\leq \frac{1}{|\eta|} \\
\left\|(\lambda I - T)^{-1}\right\| &\leq \frac{1}{|\eta|}
\end{aligned}$$

Thus this shows that $(\lambda I - T)^{-1}$ is bounded and continuous. Now to see that $R(\lambda I - T)$ is closed let $v_n \in R(\lambda I - T)$, such that $v_n \to v \in H$. Since $v_n \in R(\lambda I - T)$, there exists $u_n \in D(\lambda I - T)$ such that $(\lambda I - T)u_n = v_n$. It can be shown that $\{u_n\}$ converges. To see this note that $\|v_n - v_m\| \to 0$, this is equivalent to

$$\begin{aligned}
\|(\lambda I - T)u_n - (\lambda I - T)u_m\| &= \|(\lambda I - T)(u_n - u_m)\| \\
&\geq |eta|\|u_n - u_m\|
\end{aligned}$$

Thus $\|u_n - u_m\| \to 0$ as well. This shows that $\{u_n\}$ is Cauchy and since this is a Hilbert space the sequence is convergent. Therefore $u_n \to u$. Consider now

$$\begin{aligned}
u &= \lim_{n \to \infty}(u_n) \\
&= \lim_{n \to \infty}\left((\lambda I - T)^{-1}(\lambda I - T)u_n\right)
\end{aligned}$$

Since $(\lambda I - T)$ is continuous

$$\begin{aligned}
&= (\lambda I - T)^{-1}\lim_{n \to \infty}((\lambda I - T)u_n) \\
&= (\lambda I - T)^{-1}\lim_{n \to \infty}(v_n) \\
&= (\lambda I - T)^{-1}v
\end{aligned}$$

This shows that $u \in R((\lambda I - T)^{-1}) = D(\lambda I - T)$ and therefore $(\lambda I - T)u = v$. Therefore $v \in R(\lambda I - T)$, so $R(\lambda I - T)$ is closed. In conclusion since $R(\lambda I - T)$ is dense and closed, this implies that $R(\lambda I - T) = H$. This contradicts that $\lambda \in \sigma_c(T)$, so this shows that if $\lambda \in \mathbb{C}$, then $\lambda \notin \sigma_c(T)$. Therefore $\sigma_c(T) \subset \mathbb{R}$. $\square$