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MATH 520 Methods of Applied Math II
Homework 12

Section 16.8

#20 Show that the Fréchet derivative, if it exists, must be unique.

Proof. Let X, Y be Banach spaces and let $F : D(F) \subset X \rightarrow Y$. Now suppose that $A_1, A_2 \in B(X, Y)$ exist such that $A_1 \neq A_2$ and they both are the Fréchet derivative of F at some $x_0 \in D(F)$. This means that

$$\lim_{x \rightarrow x_0} \frac{\|F(x) - F(x_0) - A_1(x - x_0)\|}{\|x - x_0\|} = 0$$

and

$$\lim_{x \rightarrow x_0} \frac{\|F(x) - F(x_0) - A_2(x - x_0)\|}{\|x - x_0\|} = 0$$

□

#21 If $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$F(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

show that F is Gâteaux differentiable but not Fréchet differentiable at the origin.

The Gâteaux derivative of F at the origin can be computed as follows.

$$\begin{aligned} DF(0, 0)(u, v) &= \left. \frac{d}{dt} (F(0 + tu, 0 + tv)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left(\frac{tu(tv)^2}{(tu)^2 + (tv)^4} \right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left(\frac{tuv^2}{u^2 + t^2v^4} \right) \right|_{t=0} \\ &= \left. \frac{(u^2 + t^2v^4)uv^2 + tuv^2(2v^4t)}{(u^2 + t^2v^4)^2} \right|_{t=0} \\ &= \frac{u^3v^2}{u^4} \\ &= \frac{v^2}{u} \end{aligned}$$

This shows that the Gâteaux derivative of F is $A(u, v) = \frac{v^2}{u}$

Now we will consider the Fréchet derivative of F at $(0, 0)$. If the Fréchet derivative exists, then $A \in B(X, Y)$ will exist such that

$$\lim_{x \rightarrow x_0} \frac{\|F(x) - F(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0.$$

This can be simplified by noting that $x_0 = (0, 0)$ and using the definition of F .

$$\lim_{(u,v) \rightarrow (0,0)} \frac{\left| \frac{uv^2}{u^2+v^4} - A(u, v) \right|}{\sqrt{u^2 + v^2}} = 0.$$

#27 Let X, Y be Banach spaces, $F : D(F) \subset X \rightarrow Y$, and let $x, x_0 \in D(F)$ be such that $tx + (1-t)x_0 \in D(F)$ for $t \in [0, 1]$. If

$$M := \sup_{0 \leq t \leq 1} \|DF(tx + (1-t)x_0)\|$$

show that

$$\|F(x) - F(x_0)\| \leq M\|x - x_0\|$$

(Suggestion: justify and use a suitable version of the fundamental theorem of calculus.)

Section 17.5

#2 Let λ_1 be the smallest Dirichlet eigenvalue for $-\Delta$ in Ω , assume that $c \in C(\overline{\Omega})$ and $c(x) > -\lambda_1$ in $\overline{\Omega}$. If $f \in L^2(\Omega)$ prove the existence of a solution of

$$-\Delta u + c(x)u = f \quad x \in \Omega \quad u = 0 \quad \forall x \in \partial\Omega$$

Proof.

□

#3 Let $\lambda > 0$ and define

$$A[u, v] = \int_{\Omega} a_{jk}(x) u_{x_k}(x) v_{x_j}(x) \, dx + \lambda \int_{\Omega} uv \, dx$$

for all $u, v \in H^1(\Omega)$. Assume the ellipticity property (17.1.3) and that $a_{jk} \in L^\infty(\Omega)$. If $f \in L^2(\Omega)$ show that there exists a unique solution of

$$u \in H^1(\Omega) \quad A[u, v] = \int_{\Omega} f v \, dx \quad \forall v \in H^1(\Omega).$$

Justify that u may be regarded as the weak solution of

$$-(a_{jk} u_{x_k})_{x_j} + \lambda u = f(x) \quad x \in \Omega \quad a_{jk} u_{x_k} n_j = 0 \quad x \in \partial\Omega$$

The above boundary condition is said to be of conormal type.

#6 Let f and g be in $L^2(0, 1)$. Use the Lax-Milgram Theorem to prove there is a unique weak solution $\{u, v\} \in H_0^1(0, 1)$ to

$$\begin{aligned} -u'' + u + v' &= f \\ -v'' + v + u' &= g, \end{aligned}$$

where $u(0) = v(0) = 0$ and $u(1) = v(1) = 0$. (Hint: Start by defining the bilinear form

$$A[(u, v), (\phi, \psi)] = \int_0^1 u' \phi' + u \phi + v' \phi + v' \psi' + v \psi + u' \psi \, dx$$

on $H_0^1(0, 1) \times H_0^1(0, 1)$.