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MATH 520 Methods of Applied Math II
Homework 7

Section 13.6

#7 Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of nonzero real numbers satisfying

$$\sum_{j=1}^{\infty} \left(\lambda_j^2\right) < \infty$$

Construct a summetric Hilber Schmidt kernel K such that the corresonding integral operator has eigenvalues λ_j, $j = 1, 2, \dots$ and for which 0 is an eigenvalue of infinite multiplicity. (Suggestion: look for such a K in the form $K(x, y) = \sum_{j=1}^{\infty} \left(\lambda_j u_j(x) \overline{u_j(y)}\right)$ where $\{u_j\}$ are orthonormal but not complete in $L^2(\Omega)$.)

#12 Compute the singular value decomposition of the Volterra operator

$$Tu(x) = \int_0^x u(s)\,\mathrm{d}s$$

in $L^2 * = (0,1)$ and use it to find $\|T\|$. Is T normal? (Suggestion: The equation $T^*Tu = \lambda u$ is equivalent to an ODE eigenvalue problem which you can solve explicitly.)

Section 14.5

#1 Let $Lu = (x-2)u'' + (1-x)u' + u$ on $(0,1)$.

(a) Find the Green's function for

$$Lu = f \qquad u'(0) = 0 \quad u(1) = 0$$

(Hint First show that $x-1$, e^x are linearly independent solutions of $Lu = 0$.

(b) Find the adjoint operator and boundary conditions.

#2 Let

$$Tu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

on the domain

$$D(T) = \left\{u \in H^2(1,2) : u(1) = u(2) = 0\right\}$$

(a) Show that $N(T) = \{0\}$.

(b) Find the Green's function for the boundary value problem $Tu = f$.

(c) State and prove a result about the continuous dependence of the solution u on f in part (b).

#4 Prove the validity of (14.1.22). (Suggestions: start by writing $u(x)$ in the form

$$u(x) = \phi_2(x) \int_a^x C_2(y) f(y) \, \mathrm{d}y + \phi_1(x) \int_x^b C_1(y) f(y) \, \mathrm{d}y$$

and note that some of the terms that arise in the expression for u'(x) will cancel.)