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MATH 520 Methods of Applied Math II
Homework 9

Section 14.5

#5 Let $Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$ with $a_2' = a$, so that L is formally self adjoint. If $B_1 u = C_1 u(a) + C_2 u'(a)$, $B_2 u = C_3 u(b) + C_4 u'(b)$, show that $\{B_1^*, B_2^*\} = \{B_1, B_2\}$.

Proof. Consider the operators $B_1^* \psi = C_1 \psi(a) + C_2 \psi'(a) = B_1 \psi$ and $B_2^* \psi = C_3 \psi(b) + C_4 \psi'(b) = B_2 \psi$. In order to show that $\{B_1^*, B_2^*\}$ is adjoint to $\{B_1, B_2\}$ it must be shown that

$$J(\phi, \psi)|_a^b = 0$$

whenever $B_1 \phi = B_2 \phi = B_1^* \psi = B_2^* \psi = 0$.

Therefore let ϕ and ψ be chosen such that

$$B_1 \phi = B_2 \phi = B_1^* \psi = B_2^* \psi = 0.$$

First consider $B_1 \phi = B_1^* \psi = 0$. These equations can be rewritten as

$$C_1 \phi(a) = -C_2 \phi'(a)$$
$$C_1 \psi(a) = -C_2 \psi'(a).$$

Multiplying these equations gives

$$-C_1 C_2 \phi(a) \psi'(a) = -C_1 C_2 \psi(a) \phi'(a)$$

or

$$\phi(a)\psi'(a) - \phi'(a)\psi(a) = 0$$

Note that this statement is still true if one of C_1 or C_2 is zero. If one of them is zero, then either $\phi(a) = \psi(a) = 0$ or $\phi'(a) = \psi'(a) = 0$, which still gives

$$\phi(a)\psi'(a) - \phi'(a)\psi(a) = 0 - 0 = 0$$

Similary with $B_2 \phi = B_2^* \psi = 0$, it is possible to rewrite these equations as

$$C_3 \phi(b) = -C_4 \phi'(b)$$
$$C_3 \psi(b) = -C_4 \psi'(b).$$

Multiplying these equations gives

$$-C_3 C_4 \phi(b) \psi'(b) = -C_3 C_4 \psi(b) \phi'(b)$$

or

$$\phi(b)\psi'(b) - \phi'(b)\psi(b) = 0$$

Again if C_3 or C_4 is zero, then $\phi(b) = \psi(b) = 0$ or $\phi'(b) = \psi'(a) = 0$ which implies

$$\phi(b)\psi'(b) - \phi'(b)\psi(b) = 0$$

Now consider the boundary functional $J(\phi, \psi)$. The boundary function $J(\phi, \psi)$ can be fully expressed as

$$J(\phi, \psi) = a_2(\phi'\psi - \phi\psi') + (a_1 - a_2')\phi\psi.$$

Since the differential operator is formally self-adjoint, $a_1 = a_2'$, so this is equivalent to

$$J(\phi, \psi) = a_2(\phi'\psi - \phi\psi')$$

Now the condition $J(\phi, \psi)\big|_a^b = 0$ is equivalent to

$$a_2(\phi'(b)\psi(b) - \phi(b)\psi'(b)) - a_2(\phi'(a)\psi(a) - \phi(a)\psi'(a)) = 0.$$

Since we have already shown that $\phi'(b)\psi(b) - \phi(b)\psi'(b) = 0$ and $\phi'(a)\psi(a) - \phi(a)\psi'(a) = 0$ when the boundary operators are met. This condition is cleary met, when $B_1\phi = B_2\phi = B_1^*\psi = B_2^*\psi = 0$.

Therefore $\{B_1^*, B_2^*\}$ is adjoint to $\{B_1, B_2\}$ and since $B_1 = B_1^*$ and $B_2 = B_2^*$ this implies that $\{B_1^*, B_2^*\} = \{B_1, B_2\}$. In other words $\{B_1, B_2\}$ is adjoint to itself. The purpose of this exercise is to show that a formally self-adjoint differential operator with boundary conditions of the form found in B_1 and B_2 forms a self-adjoint linear operator. $\square$

#8 When we rewrite $a_2(x)u'' + a_1(x)u' + a_0(x)u = \lambda u$ as

$$-(p(x)u')' + q(x)u = \lambda\rho(x)u$$

the latter is often referred to as the *Liouville normal form.* Consider the eigenvalue problem

$$x^2u'' + xu' + u = \lambda u \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

(a) Find the Liouville normal form.

In order to find the Liouville normal form, the function $a_2(x)$ must be strictly less than zero, so I will first rewrite this eigenvalue problem as

$$-x^2u'' - xu' - u = -\lambda u \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

The functions $p(x)$, $\rho(x)$, and $q(x)$ can be found as follows.

$$\begin{aligned}
p(x) &= \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\,\mathrm{d}s\right)\\
&= \exp\left(\int_a^x \frac{-s}{-s^2}\,\mathrm{d}s\right)\\
&= \exp\left(\int_a^x \frac{1}{s}\,\mathrm{d}s\right)\\
&= \exp(\ln(s)|_{s=a}^x)\\
&= e^{\ln(x)-\ln(a)}\\
&= e^{\ln\left(\frac{x}{a}\right)}\\
&= \frac{x}{a}\\
\rho(x) &= -\frac{p(x)}{a_2(x)}\\
&= -\frac{x/a}{-x^2}\\
&= \frac{1}{ax}\\
q(x) &= a_0(x)\rho(x)\\
&= (-1)\frac{1}{ax}\\
&= -\frac{1}{ax}
\end{aligned}$$

Therefore the Liouville normal form of this eigenvalue problem is

$$-\left(\frac{x}{a}\phi'\right)' - \frac{1}{ax}\phi = -\lambda\frac{1}{ax}\phi$$

or

$$\left(\frac{x}{a}\phi'\right)' + \frac{1}{ax}\phi = \lambda\frac{1}{ax}\phi$$

(b) What is the orthogonality relationship satisfied by the eigenfunctions?

The eigenfunctions of this linear operator satisfy an orthogonality relationship with respect to the weight ρ. In mathematical terms,

$$\int_a^b \phi_n(x)\phi_m(x)\rho(x)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

or

$$\int_a^b \frac{\phi_n(x)\phi_m(x)}{ax}\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

(c) Find the eigenvalues and eigenfunctions. (You may find the original form of the equation easier to work with than the Liouville normal form when computing the eigenvalues and eigenfunctions.)

#10 Consider the Sturm-Liouville problem

$$u'' + \lambda u = 0 \qquad 0 < x < 1$$
$$u(0) - u'(0) = u(1) = 0$$

(a) Multiply the equation by u and integrate by parts to show that any eigenvalue is positive.

First I will note a few useful facts, first since $u(0)-u'(0) = 0$, this implies that $u(0) = u'(0)$. Also if u is nontrivial this guarantees that $\int_0^1 u^2(x)\,\mathrm{d}x > 0$. Finally if u is a nontrivial solution then $u'(x) \neq 0$ as $u(1) = 0$ makes any constant function is zero. This shows that $\int_0^1 (u'(x))^2\,\mathrm{d}x > 0$ as well.

Multiplying by u gives the following equation

$$uu'' + \lambda u^2 = 0.$$

Integrating both sides over $[0, 1]$ gives

$$\int_0^1 u(x)u''(x)\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x$$

This can be simplified using integration by parts

$$\begin{aligned}
\int_0^1 u(x)u''(x)\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0 \\
u(x)u'(x)\big|_{x=0}^1 - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0 \\
u(1)u'(1) - u(0)u'(0) - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0
\end{aligned}$$

Since $u(1) = 0$ and $u(0) = u'(0)$

$$-u^2(0) - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x = 0.$$

Since $\int_0^1 u^2(x)\,\mathrm{d}x > 0$

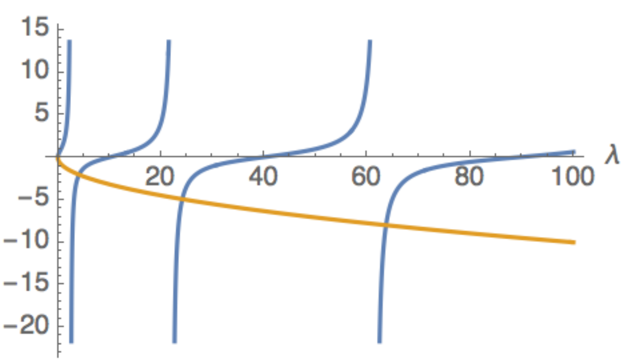
$$\lambda = \frac{u^2(0) + \int_0^1 (u'(x))^2\,\mathrm{d}x}{\int_0^1 u^2(x)\,\mathrm{d}x} > 0.$$

(b) Show that the eigenvalues are the positive solutions of $\tan\left(\sqrt{\lambda}\right) = -\sqrt{\lambda}$.

(c) Show graphically that such roots exist, and form an infinite sequence λ_k such that $(k-1/2)\pi < \sqrt{\lambda_k} < k\pi$ and

$$\lim_{k\to\infty}\left(\sqrt{\lambda_k} - (k - 1/2)\pi\right) = 0$$

First this graph shows that solutions to the equation $\tan\left(\sqrt{\lambda}\right) = -\sqrt{\lambda}$ exist.



Next consider the sequence $\lambda_k = \left((k - 1/2)\pi + \frac{1}{k}\right)^2$ meets these criterion. Note that $\sqrt{\lambda_k} = (k - 1/2)\pi + \frac{1}{k}$. Clearly $\sqrt{\lambda_k} > (k - 1/2)\pi$ and $\sqrt{\lambda_k} < k\pi$ as $\frac{1}{k} < \frac{\pi}{2}$.
Also

$$\begin{aligned}
\lim_{k \to \infty}\left(\sqrt{\lambda_k} - (k - 1/2)\pi\right) &= \lim_{k \to \infty}\left((k - 1/2)\pi + \frac{1}{k} - (k - 1/2)\pi\right) \\
&= \lim_{k \to \infty}\left(\frac{1}{k}\right) = 0
\end{aligned}$$

#14 If $\{\psi_n\}_{n=1}^{\infty}$ are Dirichlet eigenfunctions of the Laplacian making up an orthonormal basis of $L^2(\Omega)$, let $\zeta_n = \psi_n/\sqrt{\lambda_n}$ (λ_n the corresponding eigenvalue).

(a) Show that $\{\zeta_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$.

Proof. First I will show that ζ_n is an orthonormal set. In order to do this I will use that fact that

$$\int_{\Omega} \nabla u \nabla v \, \mathrm{d}x = \lambda \int_{\Omega} uv \, \mathrm{d}x$$

for any $v \in H_0^1(\Omega)$ where λ and u are a Dirichlet eigenvalue and eigenvector pair for the Laplacian on Ω.
Consider $\langle \zeta_n, \zeta_m \rangle_{H_0^1(\Omega)}$

$$\begin{aligned}
\langle \zeta_n, \zeta_m \rangle_{H_0^1(\Omega)} &= \int_{\Omega} \nabla \zeta_n \cdot \nabla \zeta_m \, \mathrm{d}x \\
&= \frac{1}{\sqrt{\lambda_n \lambda_m}} \int_{\Omega} \nabla \psi_n \cdot \nabla \psi_m \, \mathrm{d}x
\end{aligned}$$

Since $\psi_m \in H_0^1(\Omega)$ and ψ_n an eigenfunction

$$\begin{aligned}
&= \frac{\lambda_n}{\sqrt{\lambda_n \lambda_m}} \int_{\Omega} \psi_n \psi_m \, \mathrm{d}x \\
&= \frac{\lambda_n}{\sqrt{\lambda_n \lambda_m}} \langle \psi_n, \psi_m \rangle_{L^2(\Omega)}.
\end{aligned}$$

Since $\{\psi\}$ already forms an orthonormal basis of L^2, so

$$= \frac{\lambda_n}{\sqrt{\lambda_n \lambda_m}} \delta_{nm}$$

where δ_{nm} is the Kronecker delta. This shows that if $n = m$, then $\langle \zeta_n, \zeta_m \rangle_{H_0^1(\Omega)} = 1$ and if $n \neq m$, then $\langle \zeta_n, \zeta_m \rangle_{H_0^1(\Omega)} = 0$. This shows that $\{\zeta_n\}$ is an orthonormal set in $H_0^1(\Omega)$.
Now I will show that $\{\zeta_n\}$ is a basis of $H_0^1(\Omega)$. In order to show that this is a basis, I will show that this set is complete in $H_0^1(\Omega)$ or equivalently that the zero function is the only function orthogonal to the entire set. Therefore let $u \in H_0^1(\Omega)$, such that

$$\langle \zeta_n, u \rangle_{H_0^1(\Omega)} = 0$$

for all $n \in \mathbb{N}$.

$$\begin{aligned}
0 &= \langle \zeta_n, u \rangle_{H_0^1(\Omega)} \\
&= \int_{\Omega} \nabla \zeta_n \cdot \nabla u \, \mathrm{d}x \\
&= \frac{1}{\sqrt{\lambda_n}} \int_{\Omega} \nabla \psi_n \cdot \nabla u \, \mathrm{d}x
\end{aligned}$$

Since $u \in H_0^1(\Omega)$ and ψ_n is a Dirichlet eigenfunction.

$$\begin{aligned}
&= \frac{\lambda_n}{\sqrt{\lambda_n}} \int_{\Omega} \psi_n u \, \mathrm{d}x \\
&= \frac{\lambda_n}{\sqrt{\lambda_n}} \langle \psi_n, u \rangle_{L^2(\Omega)}
\end{aligned}$$

This shows that $\langle \psi_n, u \rangle_{L^2(\Omega)} = 0$ for all n. Since $\{\psi_n\}$ is already a basis of L^2 this means that $u = 0$. Therefore $\{\zeta_n\}$ is complete in $H_0^1(\Omega)$ and thus is a basis of $H_0^1(\Omega)$. $\square$

(b) Show that $f \in H_0^1(\Omega)$ if and only if $\sum_{n=1}^{\infty} \left(\lambda_n |\langle f, \psi_n \rangle|^2\right) < \infty$

Proof. First I will do some manipulation on the given sum.

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\lambda_n |\langle f, \psi_n \rangle|^2\right) &= \sum_{n=1}^{\infty} \left(\lambda_n \left|\int_\Omega f \psi_n \,\mathrm{d}x\right|^2\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left|\int_\Omega f \lambda_n \psi_n \,\mathrm{d}x\right|^2\right)
\end{aligned}$$

Since ψ_n and λ_n are an eigenfunction and eigenvalue pair for the Laplacian

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left|\int_\Omega f \Delta \psi_n \,\mathrm{d}x\right|^2\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left|\int_\Omega \nabla f \nabla \psi_n \,\mathrm{d}x\right|^2\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left|\langle f, \psi_n \rangle_{H_0^1(\Omega)}\right|^2\right) \\
&= \sum_{n=1}^{\infty} \left(\left|\left\langle f, \psi_n/\sqrt{\lambda_n} \right\rangle_{H_0^1(\Omega)}\right|^2\right) \\
&= \sum_{n=1}^{\infty} \left(\left|\langle f, \zeta_n \rangle_{H_0^1(\Omega)}\right|^2\right)
\end{aligned}$$

Now if $f \in H_0^1(\Omega)$ then

$$\sum_{n=1}^{\infty} \left(\left|\langle f, \zeta_n \rangle_{H_0^1(\Omega)}\right|^2\right) < \infty$$

and equivalently

$$\sum_{n=1}^{\infty} \left(\lambda_n |\langle f, \psi_n \rangle|^2\right) < \infty.$$

If on the other hand

$$\sum_{n=1}^{\infty} \left(\lambda_n |\langle f, \psi_n \rangle|^2\right) < \infty.$$

then

$$\sum_{n=1}^{\infty} \left(\left|\langle f, \zeta_n \rangle_{H_0^1(\Omega)}\right|^2\right) < \infty$$

which implies that $f \in H_0^1(\Omega)$. □

#15 If $\Omega < \mathbb{R}^n$ is a bounded open set with smooth enough boundary, find a solution of the wave equation problem

$$\begin{aligned} u_{tt} - \Delta u = 0 \qquad x \in \Omega \quad t > 0 \\ u(x, t) = 0 \qquad x \in \partial\Omega \quad t > 0 \\ u(x, 0) = f(x) \quad u_t(x, 0) = g(x) \qquad x \in \Omega \end{aligned}$$

in the form

$$u(x, t) = \sum_{n=1}^{\infty} (c_n(t)\psi_n(x))$$

where $\{\psi_n\}_{n=1}^{\infty}$ are the Dirichlet eigenfunctions of $-\Delta$ in Ω.

#16 Derive formally that

$$G(x, y) = \sum_{n=1}^{\infty} \left(\frac{\psi_n(x)\psi_n(y)}{\lambda_n} \right)$$

where λ_n, ψ_n are the Dirichlet eigenvalues and normalized eigenfunctions for the domain Ω, and $G(x, y)$ is the corresponding Green's function in (14.4.96). (Suggestion: if $-\Delta u = f$, expand both u and f in the ψ_n basis.)