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MATH 520 Methods of Applied Math II
Homework 6

Section 13.6

#1 Show that if $S \in \mathcal{B}\mathbf{H}$ and T is compact, the TS and ST are compact.

Proof. Let $E \subset \mathbf{H}$ be bounded. Since $S \in \mathcal{B}(\mathbf{H})$, this implies that the set $S(E)$ is also bounded. Now since T is compact the set $T(S(E))$ will be precompact. This shows that $TS(E) = T(S(E))$ is precompact and that therefore the operator TS is compact.

Also since T is compact this implies that $T(E)$ is precompact or that $\overline{T(E)}$ is compact. Let $\{u_n\} \in T(E)$, then there exists a subsequence $\left\{u_{n_j}\right\}$ such that $u_{n_j} \to u \in \overline{T(E)}$. Consider $\left\{S(u_{n_j})\right\} \in S(T(E))$ and note that since $S \in \mathcal{B}(\mathbf{H})$ there exists $v \in \overline{S(T(E))}$ such that $S(u_{n_j}) \to v$. This shows that $S(T(E))$ is also precompact, and therefore ST is a compact operator. $\square$

#2 If $T \in \mathcal{B}(\mathbf{H})$ and T^*T is compact, show that T must be compact. Use this to show that if T is compact then T^* must also be compact.

Proof. Let $T \in \mathcal{B}(\mathbf{H})$ such that T^*T is compact and let $u_n \xrightarrow{w} u$ in $\mathbf{H}$ Since T^*T is compact, $T^*Tu_n \to u$. To show that T compact we wish to show $Tu_n \to Tu$.

$$\begin{aligned}
\lim_{n\to\infty}\left(\|Tu_n - Tu\|^2\right) &= \lim_{n\to\infty}\left(|\langle Tu_n - Tu, Tu_n - Tu\rangle|\right) \\
&= \lim_{n\to\infty}\left(|\langle Tu_n - Tu, Tu_n - Tu\rangle|\right) \\
&= \lim_{n\to\infty}\left(|\langle T^*Tu_n - T^*Tu, u_n - u\rangle|\right) \\
&= \lim_{n\to\infty}\left(|\langle T^*Tu_n, u_n\rangle - \langle T^*Tu_n, u\rangle - \langle T^*Tu, u_n\rangle - \langle T^*Tu, u\rangle|\right)
\end{aligned}$$

Since $u_n \xrightarrow{w} u$ and $T^*Tu_n \to T^*Tu$

$$\begin{aligned}
&= |\langle T^*Tu, u\rangle - \langle T^*Tu, u\rangle - \langle T^*Tu, u\rangle - \langle T^*Tu, u\rangle| \\
&= 0
\end{aligned}$$

This shows that $Tu_n \to Tu$, so T is a compact operator.

Now let T be a compact operator, then by problem 1 TT^* is compact as $T^* \in \mathcal{B}(\mathbf{H})$. Now since $T^{**} = T$, this implies $(T^*)^*T^*$ is compact as well. Therefore by first part of this problem T^* is compact. ☐

#4 It $T \in \mathcal{B}(\mathbf{H})$ is compact and $\mathbf{H}$ is of infinite dimension, show that $0 \in \sigma(T)$.

Proof. Let $T \in \mathcal{B}(\mathbf{H})$ is compact and let $\mathbf{H}$ be of infinite dimension.

If $0 \in \sigma(T)$, then T is not one-to-one and onto. Therefore assume to contrary the that T is one-to-one and onto. Since $\mathbf{H}$ is infinite dimensional there exists an orthonormal basis $U = \{u_n\}_{n=1}^{\infty}$. Note that no subsequence, $\{u_{n_k}\}$ can be convergent because all the elements u_n are mutually orthogonal. Consider the set $T(U) = \{Tu_n\}$, since T is compact and U is bounded this set is precompact. Therefore there exists some subsequence Tu_{n_k} that converges to $v \in \overline{T(U)}$. Since $T \in \mathcal{B}(\mathbf{H})$ and is one-to-one and onto, T^{-1} is defined for all $x \in \mathbf{H}$ and is continuous. Thus we can consider the sequence $T^{-1}Tu_{n_k}$. Since T^{-1} is continuous, $T^{-1}Tu_{n_k} \to T^{-1}v$. However this is identical to $u_{n_k} \to T^{-1}v$ which is a contradiction as no subsequence u_{n_k} is convergent. Therefore T is not one-to-one and onto, which implies that $0 \in \sigma(T)$. $\square$

#13 The concept of a Hilbert-Schmidt operator can be defined abstractly as follows. If **H** is a separable Hilber space, we say that $T \in \mathcal{B}(\mathbf{H})$ is Hilbert-Schmidt if

$$\sum_{n=1}^{\infty} \left(\|Tu_n\|^2 \right) < \infty$$

for some orthonormal basis $\{u_n\}_{n=1}^{\infty}$ of **H**.

(a) Show that if T is Hilbert-Schmidt then the sum must be finite for any orthonormal basis of **H**.

Proof. First note that given any element $x \in \mathbf{H}$ and any orthonormal basis $\{u_n\}_{n=1}^{\infty}$, x can be represented as its projection onto the basis, that is

$$x = \sum_{n=1}^{\infty} \left(\langle x, u_n \rangle u_n \right)$$

This relationship can be used to rewrite $\|x\|^2$.

$$\begin{aligned}
\|x\|^2 &= \langle x, x \rangle \\
&= \left\langle x, \sum_{n=1}^{\infty} \left(\langle x, u_n \rangle u_n \right) \right\rangle \\
&= \sum_{n=1}^{\infty} \left(\langle x, \langle x, u_n \rangle u_n \rangle \right) \\
&= \sum_{n=1}^{\infty} \left(\overline{\langle x, u_n \rangle} \langle x, u_n \rangle \right) \\
&= \sum_{n=1}^{\infty} \left(|\langle x, u_n \rangle|^2 \right)
\end{aligned}$$

Therefore $\|x\|^2 = \sum_{n=1}^{\infty} \left(|\langle x, u_n \rangle|^2 \right)$ for any $x \in \mathbf{H}$ and any orthonormal basis $\{u_n\}_{n=1}^{\infty}$.

Now I will show $\sum_{n=1}^{\infty} \left(\|Tv_n\|^2 \right)$ is finite for any orthonormal basis $\{v_n\}_{n=1}^{\infty}$ when T is a Hilbert-Schmidt operator. Let $\{v_n\}_{n=1}^{\infty}$ be an orthonormal basis in **H**, and let T be a Hilbert-Schmidt operator, then there exists another orthonormal basis $\{u_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \left(\|Tu_n\|^2 \right) < \infty$$

Now since $\|Tv_n\|^2 = \sum_{m=1}^{\infty} \left(|\langle Tv_n, u_n \rangle|^2 \right)$,

$$\sum_{n=1}^{\infty} \left(\|Tv_n\|^2 \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle Tv_n, u_m \rangle|^2 \right) \right)$$

Since $T \in \mathcal{B}\mathbf{H}$, T^* exists

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle v_n, T^* u_m \rangle|^2 \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle T^* u_m, v_n \rangle|^2 \right) \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T^* u_m, v_n \rangle|^2 \right) \right)
\end{aligned}$$

Since $\{v_n\}$ is an orthonormal basis, $\sum_{n=1}^{\infty} \left(|\langle T^* u_m, v_n \rangle|^2 \right) = \|T^* u_m\|^2$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \left(\|T^* u_m\|^2 \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T^* u_m, u_n \rangle|^2 \right) \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle u_m, T u_n \rangle|^2 \right) \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T u_n, u_m \rangle|^2 \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle T u_n, u_m \rangle|^2 \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\|T u_n\|^2 \right) \\
&< \infty
\end{aligned}
$$

This shows that $\sum_{n=1}^{\infty} \left(\|T v_n\|^2 \right) < \infty$ for any orthonormal basis. $\square$

(b) Show that a Hilbert-Schmidt operator is compact.

Proof. Let T be a Hilbert-Schmidt operator. Since $\mathcal{K}(\mathbf{H})$ is a closed subspace of $\mathcal{B}(\mathbf{H})$, if there exists some sequence of operators $T_N \in \mathcal{K}(\mathbf{H})$ such that $T_N \to T$, then $T \in \mathcal{K}(\mathbf{H})$ because $\mathcal{K}(\mathbf{H})$ is closed. To this end, I will let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$ and I will define

$$T_N x = \sum_{n=1}^{N} \left(\langle x, u_n \rangle T u_n \right)$$

First I will show that $T_N \in \mathcal{K}(\mathbf{H})$ for any N. Note that $T_N \in \mathcal{B}(\mathbf{H})$,

$$
\begin{aligned}
\|T_N x\| &= \left\| \sum_{n=1}^{N} \left(\langle x, u_n \rangle T u_n \right) \right\| \\
&\leq \sum_{n=1}^{N} \left(|\langle x, u_n \rangle| \|T u_n\| \right)
\end{aligned}
$$

By Cauchy-Schwarze

$$
\begin{aligned}
&\leq \sum_{n=1}^{N} \left(\|x\| \|u_n\| \|T u_n\| \right) \\
&= \sum_{n=1}^{N} \left(\|x\| \|T u_n\| \right)
\end{aligned}
$$

Since $T \in \mathcal{B}(\mathbf{H})$

$$\begin{aligned}
&\leq \sum_{n=1}^{N} \left(\|x\|\|T\|\|u_n\|\right) \\
&= \sum_{n=1}^{N} \left(\|x\|\|T\|\right) \\
&= N\|T\|\|x\|
\end{aligned}$$

This shows that $\|T_N\| \leq N\|T\|$ and therefore that $T_N \in \mathcal{B}(\mathbf{H})$. Next note that $T_N x \in \operatorname{span}(\{Tu_n : 1 \leq n \leq N\})$. This shows that $R(T_N) \subset \operatorname{span}(\{Tu_n : 1 \leq n \leq N\})$ and that $\dim(R(T_N)) < N$. Since T_N is both bounded and of finite rank, this implies that $T_N \in \mathcal{K}(\mathbf{H})$.
Secondly I will show that $T_N \to T$, and I will use the fact that

$$Tx = T\left(\sum_{n=1}^{\infty} \left(\langle x, u_n\rangle u_n\right)\right) = \sum_{n=1}^{\infty} \left(\langle x, u_n\rangle Tu_n\right)$$

$$\begin{aligned}
\|T_N - T\|^2 &= \sup_{\|x\|=1} \left(\|T_N x - Tx\|^2\right) \\
&= \sup_{\|x\|=1} \left(\left\|\sum_{n=1}^{N} \left(\langle x, u_n\rangle Tu_n\right) - \sum_{n=1}^{\infty} \left(\langle x, u_n\rangle Tu_n\right)\right\|^2\right) \\
&= \sup_{\|x\|=1} \left(\left\|\sum_{n=N+1}^{\infty} \left(\langle x, u_n\rangle Tu_n\right)\right\|^2\right) \\
&\leq \sup_{\|x\|=1} \left(\sum_{n=N+1}^{\infty} \left(|\langle x, u_n\rangle|^2 \|Tu_n\|^2\right)\right) \\
&\leq \sup_{\|x\|=1} \left(\sum_{n=N+1}^{\infty} \left(\|x\|^2\|u_n\|^2\|Tu_n\|^2\right)\right) \\
&= \sum_{n=N+1}^{\infty} \left(\|Tu_n\|^2\right)
\end{aligned}$$

This last sum goes to 0 at $N \to \infty$ because $\sum_{n=1}^{\infty} \left(\|Tu_n\|^2\right) < \infty$.
This shows that $T_N \in \mathcal{K}(\mathbf{H})$ and that $T_N \to T$, now since $\mathcal{K}(\mathbf{H})$ is closed this implies that $T \in \mathcal{K}(\mathbf{H})$. $\square$