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MATH 520 Methods of Applied Math II
Homework 12

Section 16.8

#20 Show that the Fréchet derivative, if it exists, must be unique.

Proof. Let X, Y be Banach spaces and let $F : D(F) \subset X \to Y$. Now suppose that $A_1, A_2 \in B(X, Y)$ exist such that $A_1 \neq A_2$ and they both are the Fréchet derivative of F for all $x_0 \in D(F)$. This means that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_1(x - x_0)\|}{\|x - x_0\|} = 0$$

and

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_2(x - x_0)\|}{\|x - x_0\|} = 0$$

This can be written equivalently as

$$\lim_{z \to 0} \frac{\|F(z) - A_i(z)\|}{\|z\|} = 0$$

Let $\epsilon > 0$ be given, and then I will show that $\|A_1 - A_2\| < \epsilon$. Since A_1 and A_2 are derivatives of F, there exists δ_1 and δ_2 such that

$$\frac{\|F(z) - A_i(z)\|}{\|z\|} < \epsilon/2$$

for $\|z\| \leq \delta_i$ for $i = 1, 2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now consider the following for $\|z\| \leq \delta$.

$$\begin{aligned}
\|A_1(z) - A_2(z)\| &= \|A_1(z) - F(z) + F(z) - A_2(z)\| \\
&\leq \|A_1(z) - F(z)\| + \|F(z) - A_2(z)\| \\
&\leq \epsilon/2\|z\| + \epsilon/2\|z\| \\
&= \epsilon\|z\|
\end{aligned}$$

Finally consider $\|A_1 - A_2\|$.

$$\begin{aligned}
\|A_1 - A_2\| &= \sup_{z \in X}\left(\frac{\|A_1(z) - A_2(z)\|}{\|z\|}\right) \\
&= \sup_{\|z\| = \delta}\left(\frac{\|A_1(z) - A_2(z)\|}{\delta}\right) \\
&\leq \sup_{\|z\| = \delta}\left(\frac{\epsilon\|z\|}{\delta}\right) \\
&= \frac{\epsilon\delta}{\delta} \\
&= \epsilon
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this shows that $\|A_1 - A_2\| = 0$ or that $A_1 = A_2$, thus there can only be one Fréchet derivative of an operator. $\square$

#21 If $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

show that F is Gâteaux differentiable but not Fréchet differentiable at the origin.

The Gâteaux derivative of F at the origin can be computed as follows.

$$\begin{aligned} DF(0, 0)(u, v) &= \frac{\mathrm{d}}{\mathrm{d}t}(F(0 + tu, 0 + tv))\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tu(tv)^2}{(tu)^2 + (tv)^4}\right)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tuv^2}{u^2 + t^2v^4}\right)\bigg|_{t=0} \\ &= \frac{(u^2 + t^2v^4)uv^2 + tuv^2(2v^4t)}{(u^2 + t^2v^4)^2}\bigg|_{t=0} \\ &= \frac{u^3v^2}{u^4} \\ &= \frac{v^2}{u} \end{aligned}$$

This shows that the Gâteaux derivative of F is $A(u, v) = \frac{v^2}{u}$

Now we will consider the Fréchet derivative of F at $(0, 0)$. If the Fréchet derivative exists, then $A \in B(X, Y)$ will exist such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0.$$

This can be simplified by noting that $x_0 = (0, 0)$ and using the definition of F.

$$\lim_{(u,v)\to(0,0)} \frac{\left|\frac{uv^2}{u^2+v^4} - A(u, v)\right|}{\sqrt{u^2 + v^2}} = 0.$$

If the Fréchet derivative exists, then it will coincide with the Gâteaux derivative, therefore it must be that $A(u, v) = \frac{v^2}{u}$. The limit is now

$$\lim_{(u,v)\to(0,0)} \frac{\left|\frac{uv^2}{u^2+v^4} - \frac{v^2}{u}\right|}{\sqrt{u^2 + v^2}} = 0.$$

Also if the previous limit is going to exist, it must exist along any path to $(0, 0)$. Therefore I will consider the path along $u = v^2$, this gives

$$\begin{aligned} \lim_{(u,v)\to(0,0)} \frac{\left|\frac{uv^2}{u^2+v^4} - \frac{v^2}{u}\right|}{\sqrt{u^2 + v^2}} &= \lim_{(u,v)\to(0,0)} \frac{\left|\frac{v^4}{v^4+v^4} - \frac{v^2}{v^2}\right|}{\sqrt{v^4 + v^2}} \\ &= \lim_{(u,v)\to(0,0)} \frac{\frac{1}{2}}{\sqrt{v^4 + v^2}} \to \infty \end{aligned}$$

This shows that along the path $u = v^2$ the limit actually goes to ∞ as $(u, v) \to (0, 0)$, thus the Fréchet derivative does not exist.

#27 Let X, Y be Banach spaces, $F : D(F) \subset X \to Y$, and let $x, x_0 \in D(F)$ be such that $tx + (1-t)x_0 \in D(F)$ for $t \in [0, 1]$. If
$$M := \sup_{0 \le t \le 1} \|DF(tx + (1-t)x_0)\|$$
show that
$$\|F(x) - F(x_0)\| \le M\|x - x_0\|$$
(Suggestion: justify and use a suitable version of the fundamental theorem of calculus.)

Section 17.5

#2 Let λ_1 be the smallest Dirichlet eigenvalue for $-\Delta$ in Ω, assume that $c \in C(\overline{\Omega})$ and $c(x) > -\lambda_1$ in $\overline{\Omega}$. If $f \in L^2(\Omega)$ prove the existence of a solution of

$$-\Delta u + c(x)u = f \quad x \in \Omega \qquad u = 0 \quad \forall x \in \partial\Omega$$

Proof. Even though it isn't stated I will operate in the space $H_0^1(\Omega)$ as this is the natural Hilbert space for the Dirichlet Laplacian. First I will rewrite this PDE in weak form as

$$\int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in H_0^1(\Omega)$. Next I will define the following function

$$A[u,v] = \int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x.$$

To see that this function is bilinear, note that

$$\begin{aligned}
A[u_1 + u_2, v] &= \int_\Omega \nabla(u_1 + u_2)\nabla v + c(x)(u_1 + u_2)v \,\mathrm{d}x \\
&= \int_\Omega \nabla u_1 \nabla v + \nabla u_2 \nabla v + c(x)u_1 v + c(x)u_2 v \,\mathrm{d}x \\
&= \int_\Omega \nabla u_1 \nabla v + c(x)u_1 v \,\mathrm{d}x + \int_\Omega \nabla u_2 \nabla v + c(x)u_2 v \,\mathrm{d}x \\
&= A[u_1, v] + A[u_2, v]
\end{aligned}$$

and that

$$\begin{aligned}
A[u, v_1 + v_2] &= \int_\Omega \nabla u \nabla(v_1 + v_2) + c(x)u(v_1 + v_2) \,\mathrm{d}x \\
&= \int_\Omega \nabla u \nabla v_1 + \nabla u \nabla v_2 + c(x)uv_1 + c(x)uv_2 \,\mathrm{d}x \\
&= \int_\Omega \nabla u \nabla v_1 + c(x)uv_1 \,\mathrm{d}x + \int_\Omega \nabla u \nabla v_2 + c(x)uv_2 \,\mathrm{d}x \\
&= A[u, v_1] + A[u, v_2].
\end{aligned}$$

This shows that A is bilinear.

Next I will show that A is bounded. Note that since c is continuous on a closed set, so this implies that c achieves its maximum on $\overline{\Omega}$. Let $M = \max_{x \in \overline{\Omega}}\{c(x)\}$.

$$\begin{aligned}
A[u,v] &= \int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x \\
&= \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_\Omega c(x)uv \,\mathrm{d}x \\
&\leq \int_\Omega \nabla u \nabla v \,\mathrm{d}x + M \int_\Omega uv \,\mathrm{d}x
\end{aligned}$$

By Holder's inequality

$$\leq \|u\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)} + M\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

By Poincaré's Inequality

$$
\begin{aligned}
&\leq \|u\|_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)} + MC^2\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} \\
&= (1 + MC^2)\|u\|_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)}
\end{aligned}
$$

This shows that A is bounded.

Finally I will show that A is coercive. To see this note that c is continuous on a closed set, so this implies that c achieves its minimum on $\overline{\Omega}$. Let $m = \min_{x \in \overline{\Omega}}\{c(x)\}$. Also since $c(x) > -\lambda_1$ this implies that $m > -\lambda_1$. Since m is strictly greater than $-\lambda_1$ a value $-\epsilon$ can be chosen such that $m > -\epsilon > -\lambda_1$. This value can be chosen in such a way that $\epsilon > 0$. If $m \leq 0$, then $-\epsilon$ is any number between m and $-\lambda$ and $-\epsilon < 0$. If $m > 0$, then $-\epsilon$ can be chosen such that $0 > -\epsilon > -\lambda$ as $-\lambda < 0$.

$$
\begin{aligned}
A[u,u] &= \int_\Omega |\nabla u|^2 + c(x)u^2 \,\mathrm{d}x \\
&= \|u\|_{H_0^1(\Omega)}^2 + \int_\Omega c(x)u^2 \,\mathrm{d}x
\end{aligned}
$$

Since $c(x) > m > -\epsilon$

$$
\begin{aligned}
&\geq \|u\|_{H_0^1(\Omega)}^2 - \epsilon \int_\Omega u^2 \,\mathrm{d}x \\
&= \|u\|_{H_0^1(\Omega)}^2 - \epsilon\|u\|_{L^2(\Omega)}^2
\end{aligned}
$$

Since we are subtracting a positive number, subtracting a larger positive number will decrease the total. In this case the Poincaré inequality can be used. It is known that the smallest constant that satisfies the Poincaré inequality is $1/\lambda_1$.

$$
\begin{aligned}
&\geq \|u\|_{H_0^1(\Omega)}^2 - \frac{\epsilon}{\lambda_1}\|u\|_{H_0^1(\Omega)}^2 \\
&\geq \left(1 - \frac{\epsilon}{\lambda_1}\right)\|u\|_{H_0^1(\Omega)}^2
\end{aligned}
$$

Since $-\epsilon > -\lambda_1$, this implies that $\epsilon < \lambda_1$ and that $1 - \frac{\epsilon}{\lambda_1} > 0$. Thus A is coercive.

Now Lax-Milgram's Theorem states that there exists a unique solution to

$$
\int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x
$$

for all $v \in H_0^1(\Omega)$. This also implies that there is a weak solution to

$$
-\Delta u + c(x)u = f \quad x \in \Omega \qquad u = 0 \quad \forall x \in \partial\Omega
$$

$\square$

#3 Let $\lambda > 0$ and define

$$A[u,v] = \int_\Omega a_{jk}(x)u_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv\,\mathrm{d}x$$

for all $u, v \in H^1(\Omega)$. Assume the ellipticity property (17.1.3) and that $a_{jk} \in L^\infty(\Omega)$. If $f \in L^2(\Omega)$ show that there exists a unique solution of

$$u \in H^1(\Omega) \qquad A[u,v] = \int_\Omega fv\,\mathrm{d}x \quad \forall v \in H^1(\Omega).$$

Justify that u may be regarded as the weak solution of

$$-(a_{jk}u_{x_k})_{x_j} + \lambda u = f(x) \quad x \in \Omega \qquad a_{jk}u_{x_k}n_j = 0 \quad x \in \partial\Omega$$

The above boundary condition is said to be of conormal type.

Proof. Lax-Milgram's Theorem can be used to show that there exists a unique solution to

$$u \in H^1(\Omega) \qquad A[u,v] = \int_\Omega fv\,\mathrm{d}x \quad \forall v \in H^1(\Omega).$$

First I will show that A is bilinear.

$$\begin{aligned}
A[u_1+u_2,v] &= \int_\Omega a_{jk}(x)(u_1+u_2)_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega (u_1+u_2)v\,\mathrm{d}x \\
&= \int_\Omega a_{jk}(x)(u_1)_{x_k}(x)v_{x_j}(x) + a_{jk}(x)(u_2)_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega u_1v + u_2v\,\mathrm{d}x \\
&= \int_\Omega a_{jk}(x)(u_1)_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \int_\Omega a_{jk}(x)(u_2)_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega u_1v\,\mathrm{d}x + \lambda\int_\Omega u_2v\,\mathrm{d}x \\
&= A[u_1,v] + A[u_2,v] \\
A[u,v_1+v_2] &= \int_\Omega a_{jk}(x)u_{x_k}(x)(v_1+v_2)_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega u(v_1+v_2)\,\mathrm{d}x \\
&= \int_\Omega a_{jk}(x)u_{x_k}(x)(v_1)_{x_j}(x) + a_{jk}(x)u_{x_k}(x)(v_2)_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv_1 + uv_2\,\mathrm{d}x \\
&= \int_\Omega a_{jk}(x)u_{x_k}(x)(v_1)_{x_j}(x)\,\mathrm{d}x + \int_\Omega a_{jk}(x)u_{x_k}(x)(v_2)_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv_1\,\mathrm{d}x + \lambda\int_\Omega uv_2\,\mathrm{d}x \\
&= A[u,v_1] + A[u,v_2]
\end{aligned}$$

This shows that A is bilinear.

Next I will show that A is bounded. Let $M = \max_{j,k}\left\{\|a_{jk}\|_{L^\infty(\Omega)}\right\}$.

$$\begin{aligned}
A[u,v] &= \int_\Omega a_{jk}(x)u_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv\,\mathrm{d}x \\
&\le M\int_\Omega u_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv\,\mathrm{d}x \\
&\le M\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + \lambda\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}
\end{aligned}$$

By Poincaré's Inequality

$$\begin{aligned}
&\le M\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + \lambda C^2\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} \\
&= (M+\lambda C^2)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}
\end{aligned}$$

Since $\|u\|_{H^1(\Omega)} = \sqrt{\|u\|_{L^2(\Omega)}^2 + \|u\|_{H_0^1(\Omega)}^2}$, this implies that $\|u\|_{H_0^1(\Omega)} \le \|u\|_{H^1(\Omega)}$, therefore

$$\le (M+\lambda C^2)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

This shows that A is bounded.

Lastly I will show that A is coercive.

$$\begin{aligned} A[u,u] &= \int_\Omega a_{jk}(x) u_{x_k}(x) u_{x_j}(x)\,\mathrm{d}x + \lambda \int_\Omega u^2\,\mathrm{d}x \\ &= \int_\Omega a_{jk}(x) (\nabla u)_k (\nabla u)_j\,\mathrm{d}x + \lambda \int_\Omega u^2\,\mathrm{d}x \end{aligned}$$

By the ellipticity condition

$$\begin{aligned} &\geq \theta \int_\Omega |\nabla u|^2\,\mathrm{d}x + \lambda \int_\Omega u^2\,\mathrm{d}x \\ &= \theta \|u\|_{H_0^1(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 \end{aligned}$$

Let $\gamma = \min\{\theta, \lambda\} > 0$, then

$$\begin{aligned} &\geq \gamma\Big(\|u\|_{H_0^1(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\Big) \\ &= \gamma \|u\|_{H^1(\Omega)}^2 \end{aligned}$$

This shows that A is coercive.

Now Lax-Milgram's Theorem states that there exists a unique $u \in H^1(\Omega)$ such that

$$A[u,v] = \int_\Omega f v\,\mathrm{d}x \quad \forall v \in H^1(\Omega).$$

$\square$

#6 Let f and g be in $L^2(0,1)$. Use the Lax-Milgram Theorem to prove there is a unique weak solution $\{u,v\} \in H_0^1(0,1)$ to

$$\begin{aligned} -u'' + u + v' &= f \\ -v'' + v + u' &= g, \end{aligned}$$

where $u(0) = v(0) = 0$ and $u(1) = v(1) = 0$. (Hint: Start by defining the bilinear form

$$A[(u,v),(\phi,\psi)] = \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x$$

on $H_0^1(0,1) \times H_0^1(0,1)$.

Proof. First I will rewrite this system of PDEs in weak form.

$$\int_0^1 -u''\phi + u\phi + v'\phi - v''\psi + v\psi + u'\psi \,\mathrm{d}x = \int_0^1 f\phi + g\psi \,\mathrm{d}x$$

for all $\phi, \psi \in H_0^1(0,1)$. Integrating by parts were necessary gives

$$\int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x = \int_0^1 f\phi + g\psi \,\mathrm{d}x.$$

Now I will define the following bilinear function

$$A[(u,v),(\phi,\psi)] = \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x.$$

This function is bilinear because differentiation and integration are both linear operations. To verify this note that

$$\begin{aligned}
&A[(u_1 + u_2, v_1 + v_2),(\phi,\psi)] \\
&= \int_0^1 (u_1+u_2)'\phi' + (u_1+u_2)\phi + (v_1+v_2)'\phi + (v_1+v_2)'\psi' + (v_1+v_2)\psi + (u_1+u_2)'\psi \,\mathrm{d}x \\
&= \int_0^1 u_1'\phi + u_2'\phi' + u_1\phi + u_2\phi + v_1'\phi + v_2'\phi + v_1'\psi + v_2'\psi' + v_1\psi + v_2\psi + u_1'\psi + u_2'\psi \,\mathrm{d}x \\
&= \int_0^1 u_1'\phi + u_1\phi + v_1'\phi + v_1'\psi + v_1\psi + u_1'\psi \,\mathrm{d}x + \int_0^1 u_2'\phi' + u_2\phi + v_2'\phi + v_2'\psi' + v_2\psi + u_2'\psi \,\mathrm{d}x \\
&= A[(u_1,v_1),(\phi,\psi)] + A[(u_2,v_2),(\phi,\psi)]
\end{aligned}$$

and the same can be shown for the second argument.

Next I will show that A is bounded.

$$\begin{aligned}
&A[(u,v),(\phi,\psi)] \\
&= \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x \\
&\le \|u\|_{H_0^1}\|\phi\|_{H_0^1} + \|u\|_{L^2}\|\phi\|_{L^2} + \|v\|_{H_0^1}\|\phi\|_{L^2} + \|v\|_{H_0^1}\|\psi\|_{H_0^1} + \|v\|_{L^2}\|\psi\|_{L^2} + \|u\|_{H_0^1}\|\psi\|_{L^2}
\end{aligned}$$

Using Poincaré's Inequality many times

$$\begin{aligned}
&\le \|u\|_{H_0^1}\|\phi\|_{H_0^1} + C^2\|u\|_{H_0^1}\|\phi\|_{H_0^1} + C\|v\|_{H_0^1}\|\phi\|_{H_0^1} + \|v\|_{H_0^1}\|\psi\|_{H_0^1} + C^2\|v\|_{H_0^1}\|\psi\|_{H_0^1} + C\|u\|_{H_0^1}\|\psi\|_{H_0^1} \\
&= \|u\|_{H_0^1}\Big((1+C^2)\|\phi\|_{H_0^1} + C\|\psi\|_{H_0^1}\Big) + \|v\|_{H_0^1}\Big(C\|\phi\|_{H_0^1} + (1+C^2)\|\psi\|_{H_0^1}\Big)
\end{aligned}$$

Let $M = \max\{1 + C^2, C\}$

$$
\begin{aligned}
&\leq M\|u\|_{H_0^1}\left(\|\phi\|_{H_0^1} + \|\psi\|_{H_0^1}\right) + M\|v\|_{H_0^1}\left(\|\phi\|_{H_0^1} + \|\psi\|_{H_0^1}\right) \\
&\leq M\left(\|u\|_{H_0^1} + \|v\|_{H_0^1}\right)\left(\|\phi\|_{H_0^1} + \|\psi\|_{H_0^1}\right)
\end{aligned}
$$

Using Cauchy-Schwarze it is possible to show that $|x| + |y| \leq \sqrt{2}\sqrt{x^2 + y^2}$. Using this fact results in

$$
\begin{aligned}
&\leq 2M\sqrt{\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2}\sqrt{\|\phi\|_{H_0^1}^2 + \|v\|_{H_0^1}^2} \\
&= 2M\|(u, v)\|_{H_0^1 \times H_0^1}\|(\phi, \psi)\|_{H_0^1 \times H_0^1}
\end{aligned}
$$

Thus A is bounded.

Lastly I will show that A is coercive. Let $u, v \in H_0^1(0, 1)$, then

$$
\begin{aligned}
A[(u, v), (u, v)] &= \int_0^1 (u')^2 + u^2 + uv' + (v')^2 + v^2 + u'v \,\mathrm{d}x \\
&= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x + \int_0^1 uv' \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x + \int_0^1 u'v \,\mathrm{d}x
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
&= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x - \int_0^1 u'v \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x + \int_0^1 u'v \,\mathrm{d}x \\
&= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x \\
&= \|u\|_{H_0^1}^2 + \|u\|_{L^2}^2 + \|v\|_{H_0^1}^2 + \|v\|_{L^2}^2 \\
&\geq \|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2 \\
&= \|(u, v)\|_{H_0^1 \times H_0^1}
\end{aligned}
$$

Thus A is coercive.

Now the Lax-Milgram Theorem allows us to conclude that there exists a unique weak solution to

$$
\begin{aligned}
-u'' + u + v' &= f \\
-v'' + v + u' &= g,
\end{aligned}
$$

where $u(0) = v(0) = 0$ and $u(1) = v(1) = 0$. $\square$