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MATH 520 Methods of Applied Math II
Homework 2

Section 10.9

#10 Let S_+ and S_- be the left and right shift operators on ℓ^2. Show that $S_- = S_+^*$ and $S_+ = S_-^*$.

Proof. Both S_+ and S_- are in $\mathcal{B}(\ell^2)$ therefore they both have unique adjoints. Consider $x, y \in \ell^2$, then

$$\begin{aligned}
\langle S_+x, y\rangle &= \sum_{n=1}^{\infty} \left((S_+x)_n \cdot \overline{y_n}\right) \\
&= \sum_{n=2}^{\infty} \left(x_{n-1} \cdot \overline{y_n}\right) \\
&= \overline{\sum_{n=2}^{\infty} \left(\overline{x_{n-1}} y_n\right)}
\end{aligned}$$

$\square$

#11 Let T be the Volterra integral operator $Tu = \int_0^x u(y)\,\mathrm{d}y$ considered as an operator on $L^2(0,1)$. Find T^* and $N(T^*)$.

#12 Suppose $T \in \mathcal{B}(\mathbf{H})$ is self-adjoint and there exists a constant $c > 0$ such that $\|Tu\| \geq c\|u\|$ for all $u \in \mathbf{H}$. Show that there exists a solution of $Tu = f$ for all $f \in \mathbf{H}$. Show by example that the conclusion may be false if the assumption of self-adjointedness is removed.

Proof. This conclusion may be false if that operator is not self-adjoint. Consider the operator S_+ on ℓ^2. We have already shown that $S_+^* = S_-$ so S_+ is not self-adjoint. However $\|S_+x\| = \|x\|$ for all $x \in \ell^2$, so with $c = 1$ S_+ satisfies $\|S_+x\| \geq c\|x\|$ for all $x \in \ell^2$. However $R(S_+) = \{x \in \ell^2 : x_1 = 0\}$, so $S_+u = x$ will not have a solution if $x_1 \neq 0$. □

#13 Let M be the multiplication operator $Mu(x) = xu(x)$ in $L^2(0,1)$. Show that $R(M)$ is dense but not closed.

#15 An operator $T \in \mathcal{B}(\mathbf{H})$ is said to be normal if it commutes with its adjoint, i.e. $TT^* = T^*T$.

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