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MATH 520 Methods of Applied Math II
Homework 4

Section 11.4

#6 Let $\mathbf{H} = L^2(0,1)$ and $T_1 u = T_2 u = iu'$ with domains

$$D(T_1) = \left\{u \in H^1(0,1) : u(0) = u(1)\right\} D(T_2) \qquad = \left\{u \in H^1(0,1) : u(0) = u(1) = 0\right\}$$

Show that T_1 is self-adjoint, and that T_2 is closed and symmetric but not self-adjoint. What is T_2^*?

Proof. First I will show that T_1 is self-adjoint. Let $v \in D(T_1)$, then (v, iv') is an admissable pair for T_1^*. To see this note that

$$\begin{aligned}
\langle T_1 u, v\rangle &= \int_0^1 iu'(x)\overline{v(x)}\,\mathrm{d}x \\
&= iu(x)\overline{v(x)}\Big|_{x=0}^1 - \int_0^1 iu(x)\overline{v'(x)}\,\mathrm{d}x \\
&= iu(x)\overline{v(x)}\Big|_{x=0}^1 + \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x \\
&= iu(1)\overline{v(1)} - iu(0)\overline{v(0)} + \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x
\end{aligned}$$

Since $u(0) = u(1)$ and $v(0) = v(1)$.

$$\begin{aligned}
&= \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x \\
&= \langle u, T_1 v\rangle
\end{aligned}$$

This shows that $D(T_1) \subset D(T_1^*)$ and that $T_1 u = T_1^* u = iu'$ for $u \in D(T)$.

Now let $v \in D(T_1^*)$, if we can show that $v \in D(T)$ and $T_1^* v = iv'$, then we know that $T_1 = T_1^*$. Since $v \in D(T_1^*)$ then there exists $g \in L^2(0,1)$ such that (v, g) is an admissable pair for T_1^* and $\int_0^1 g(x)\,\mathrm{d}x = 0$ Now by definition $T_1^* v = g$. Also since (v, g) is an admissable pair,

$$\langle Tu, v\rangle = \langle u, g\rangle$$

for all $u \in D(T)$. Next I will define the following function

$$G(x) = \int_0^x g(y)\,\mathrm{d}y + \alpha$$

where

$$\alpha = i\int_0^1 v(s)\,\mathrm{d}s - \int_0^1\int_0^s g(y)\,\mathrm{d}y\,\mathrm{d}s.$$

Since $v, g \in L^2(0,1)$, this function is well-defined. Note that by the Fundamental Theorem of Calculus for L^2 functions, $G'(x) = g(x)$. Also note that since $\int_0^1 g(x)\,\mathrm{d}x = 0$,

$$G(0) = \int_0^0 g(y)\,\mathrm{d}y + \alpha = \alpha = \int_0^1 g(y)\,\mathrm{d}y + \alpha = G(1)$$

Now reconsider the inner product $\langle u, g\rangle$.

$$\begin{aligned}
\langle u, g\rangle &= \int_0^1 u(x)\overline{g(x)}\,\mathrm{d}x \\
&= \int_0^1 u(x)\overline{G'(x)}\,\mathrm{d}x \\
&= u(x)G(x)|_{x=0}^1 - \int_0^1 u'(x)\overline{G(x)}\,\mathrm{d}x \\
&= (u(1)G(1) - u(0)G(0)) - \int_0^1 u'(x)\overline{G(x)}\,\mathrm{d}x
\end{aligned}$$

Since $u \in D(T)$, $u(0) = u(1)$ and as shown before $G(0) = G(1)$, the first term is zero.

$$\langle u, g\rangle = -\int_0^1 u'(x)\overline{G(x)}\,\mathrm{d}x \qquad = -\langle u', G\rangle$$

Now using the definition of admissible pair it is possible to see that

$$\langle Tu, v\rangle = \langle u, g\rangle = -\langle u', G\rangle$$

for all $u \in D(T)$. Equivalently this is

$$\begin{aligned}
\int_0^1 iu'(x)\overline{v(x)}\,\mathrm{d}x &= -\int_0^1 u'(x)\overline{G(x)}\,\mathrm{d}x \\
\int_0^1 u'(x)\overline{G(x) - iv(x)}\,\mathrm{d}x &= 0
\end{aligned}$$

Since this is true for any $u \in D(T)$ this is true in particular for

$$u(x) = \int_0^x G(x) - iv(x)\,\mathrm{d}x$$

In order to verify that $u \in D(T)$

$\square$

#7 If T is symmetric with $R(T) = \mathbf{H}$ show that T is self-adjoint.

Proof. $\square$

#16 We say that a linear operator on a Hilbert space **H** is bounded below if there exists a constant $c_0 > 0$ such that

$$\langle Tu, u \rangle \geq -c_0 \|u\|^2 \quad \forall u \in D(T)$$

Show that Theorem 11.6 remains valid if the condition that T be positive is replaced by the assumption that T is bounded below.

Proof. $\square$

Section 12.4

#3 Recall that the resolvent operator of T is defined to be $R_\lambda = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$.

(a) Prove the resolvant identity (12.1.3).

Proof. □

(b) Deduce from this that R_λ and R_μ commute.

Proof. Let $\lambda, \mu \in \mathbb{C}$. If $\lambda = \mu$, then $R_\lambda = R_\mu$ so

$$R_\lambda R_\mu = R_\lambda^2 = R_\mu R_\lambda.$$

In this case R_λ and R_μ commute trivially. Now let $\lambda \neq \mu$, in this case the resolvant identity states that

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\lambda - \mu}.$$

Now consider the following

$$\begin{aligned} R_\lambda R_\mu &= \frac{R_\lambda - R_\mu}{\lambda - \mu} \\ &= \frac{R_\mu - R_\lambda}{\mu - \lambda} \\ &= R_\mu R_\lambda \end{aligned}$$

This shows that R_λ and R_μ commute. □

(c) Show also that T and R_λ commute for $\lambda \in \rho(T)$.

Proof. □

#4 Show that $\lambda \to R_\lambda$ is continuously differentiable, regarded as a mapping from $\rho(T) \subset \mathbb{C}$ into $\mathcal{B}(\mathbf{H})$, with

$$\frac{\mathrm{d}R_\lambda}{\mathrm{d}\lambda} = -R_\lambda^2$$

Proof. $\square$