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MATH 520 Methods of Applied Math II
Homework 1

Section 10.9

#3 Prove Proposition 10.1. Proposition 10.1 states that if T is bounded on its domain then it has a unique norm preserving extension to $\overline{D(T)}$. That is to say there exists a unique linear operator $S : \overline{D(T)} \subset X \to Y$ such that $Sx = Tx$ for $x \in D(T)$ and $\|S\| = \|T\|$.

Proof. Let $S : \overline{D(T)} \subset X \to Y$ be defined as follows.

$$Sx = \lim_{n\to\infty}(Tx_n)$$

where the sequence $\{x_n\}_{n=1}^{\infty}$ is any sequence in $D(T)$ that converges to x. Note that for any $x \in \overline{D(T)}$, x is a limit point of $D(T)$ so the sequence $\{x_n\}$ exists. Also since T is bounded it is also continuous, so the limit always exists.

Next I will show that S is linear. Consider $x_1, x_2 \in \overline{D(T)}$ and $c_1, c_2 \in \mathbb{C}$. Then there exists sequences in $D(T)$, $\{a_n\}_{n=1}^{\infty}$ that converges to x_1 and $\{b_n\}_{n=1}^{\infty}$ that converges to x_2. Now note that the sequence $c_1a_n + c_2b_n{}_{n=1}^{\infty}$ converges to $c_1x_1 + c_2x_2$ by the linearity of limits. Therefore

$$S(c_1x_1 + c_2x_2) = \lim_{n\to\infty}(T(c_1a_n + c_2b_n))$$

Because T is linear

$$S(c_1x_1 + c_2x_2) = \lim_{n\to\infty}(c_1T(a_n) + c_2T(b_n))$$

By the linearity of limits

$$S(c_1x_1 + c_2x_2) = c_1\lim_{n\to\infty}(T(a_n)) + c_2\lim_{n\to\infty}(T(b_n))$$
$$S(c_1x_1 + c_2x_2) = c_1S(x_1) + c_2S(x_2)$$

This shows that S is a linear operator.

Next I will show that $Sx = Tx$ for $x \in D(T)$. Let $\{x_n\}_{n=1}^{\infty}$ converge to x in $D(T)$, then because T is continuous, $\lim_{n\to\infty}(T(x_n)) = T(x)$. Therefore $Sx = Tx$.

Lastly I will show that $\|S\| = \|T\|$. Consider the following.

$$\|S\| = \lim_{x\in\overline{D(T)}}\frac{\|Sx\|_Y}{\|x\|_X}$$

□

#6 Show that a linear operator $T : \mathbb{C}^N \to \mathbb{C}^M$ is always bounded for any choice of norms on $\mathbb{C}^N$ and $\mathbb{C}^M$.

Proof. Let $T : \mathbb{C}^N \to \mathbb{C}^M$ be a linear operator. It is known that any linear operator from $\mathbb{C}^N \to \mathbb{C}^M$ can be expressed as a matrix multiplication, that is there exists matrix $A \in \mathbb{C}^{m \times n}$ such that $Tx = Ax$ for every $x \in \mathbb{C}^N$. It is well known that for finite dimensional vector spaces any two norms are equivalently. More precisely let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a finite dimensional vector space, then there exists constants C_1 and C_2 such that

$$0 < C_1 \leq \frac{\|x\|_1}{\|x\|_2} \leq C_2 < \infty$$

for any nonzero x in the vector space. Since both N and M are finite, norms on $\mathbb{C}^N$ are equivalent and norms on $\mathbb{C}^M$ are equivalent. Therefore I will let $\|\cdot\|_N$ represent any norm on $\mathbb{C}^N$ and $\|\cdot\|_M$ represent any norm on $\mathbb{C}^M$. $\square$

#7 If $T, T^{-1} \in \mathcal{B}(\mathbf{H})$ show that $(T^*)^{-1} \in \mathcal{B}(\mathbf{H})$ and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let $\mathbf{H}$ be a Hilbert Space and let $T \in \mathcal{B}(\mathbf{H})$ with $\square$

#14 If $T \in \mathcal{B}(\mathbf{H})$ show that T^* restricted to $R(T)$ is one-to-one.

Proof. Let $\mathbf{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathbf{H})$. Consider T^* restricted to $R(T)$. In other words let S be a linear function such that $D(S) = R(T)$ and $Sx = T^*x$ for every $x \in D(S)$. Let $x, y \in D(S) = R(T)$ such that $Sx = Sy$. Since S is linear this is equivalent to $S(x - y) = 0$, which implies that $x - y \in N(S)$. Note that $N(S) \subseteq N(T^*)$, so $x - y \in N(T^*)$. Also note that $R(T)$ is a subspace of $\mathbf{H}$ so that $x - y \in R(T)$. It has already been stated that $R(T) \subset N(T^*)^\perp$, which implies that $x - y \in N(T^*)^\perp$. Since $x - y \in N(T^*)^\perp$ and $x - y \in N(T^*)$, $x - y$ must be orthogonal to itself.

$$\begin{aligned}
\langle x - y, x - y \rangle &= 0 \\
\|x - y\|^2 &= 0 \\
\|x - y\| &= 0 \\
x - y &= 0 \\
x &= y
\end{aligned}$$

This follows since the only element of $\mathbf{H}$ with norm 0 is the zero element. This shows that when $Sx = Sy$ then $x = y$. Therefore S is one to one or T^* is one to one when restricted to the range of T. $\square$