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MATH 520 Methods of Applied Math II
Homework 6

Section 13.6

#1 Show that if $S \in \mathcal{B}\mathbf{H}$ and T is compact, the TS and ST are compact.

Proof. $\square$

#2 If $T \in \mathcal{B}(\mathbf{H})$ and T^*T is compact, show that T must be compact. Use this to show that if T is compact then T^* must also be compact.

Proof. $\square$

#4 It $T \in \mathcal{B}(\mathbf{H})$ is compact and $\mathbf{H}$ is of infinite dimension, show that $0 \in \sigma(T)$.

Proof. $\square$

#13 The concept of a Hilbert-Schmidt operator can be defined abstractly as follows. If $\mathbf{H}$ is a separable Hilber space, we say that $T \in \mathcal{B}(\mathbf{H})$ is Hilbert-Schmidt if

$$\sum_{n=1}^{\infty} \left(\|Tu_n\|^2\right) < \infty$$

for some orthonormal basis $\{u_n\}_{n=1}^{\infty}$ of $\mathbf{H}$.

(a) Show that if T is Hilbert-Schmidt then the sum must be finite for any orthonormal basis of $\mathbf{H}$.

Proof. First note that given any element $x \in \mathbf{H}$ and any orthonormal basis $\{u_n\}_{n=1}^{\infty}$, x can be represented as its projection onto the basis, that is

$$x = \sum_{n=1}^{\infty} \left(\langle x, u_n\rangle u_n\right)$$

This relationship can be used to rewrite $\|x\|^2$.

$$\begin{aligned}
\|x\|^2 &= \langle x, x\rangle \\
&= \left\langle x, \sum_{n=1}^{\infty} \left(\langle x, u_n\rangle u_n\right)\right\rangle \\
&= \sum_{n=1}^{\infty} \left(\langle x, \langle x, u_n\rangle u_n\rangle\right) \\
&= \sum_{n=1}^{\infty} \left(\overline{\langle x, u_n\rangle}\langle x, u_n\rangle\right) \\
&= \sum_{n=1}^{\infty} \left(|\langle x, u_n\rangle|^2\right)
\end{aligned}$$

Therefore $\|x\|^2 = \sum_{n=1}^{\infty} \left(|\langle x, u_n\rangle|^2\right)$ for any $x \in \mathbf{H}$ and any orthonormal basis $\{u_n\}_{n=1}^{\infty}$.
Now I will show $\sum_{n=1}^{\infty} \left(\|Tv_n\|^2\right)$ is finite for any orthonormal basis $\{v_n\}_{n=1}^{\infty}$ when T is a Hilbert-Schmidt operator. Let $\{v_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathbf{H}$, and let T be a Hilbert-Schmidt operator, then there exists another orthonormal basis $\{u_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \left(\|Tu_n\|^2\right) < \infty$$

Now since $\|Tv_n\|^2 = \sum_{m=1}^{\infty} \left(|\langle Tv_n, u_n\rangle|^2\right)$,

$$\sum_{n=1}^{\infty} \left(\|Tv_n\|^2\right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle Tv_n, u_m\rangle|^2\right)\right)$$

Since $T \in \mathcal{B}\mathbf{H}$, T^* exists

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle v_n, T^* u_m\rangle|^2\right)\right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle T^* u_m, v_n\rangle|^2\right)\right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T^* u_m, v_n\rangle|^2\right)\right)
\end{aligned}$$

Since $\{v_n\}$ is an orthonormal basis, $\sum_{n=1}^{\infty} \left(|\langle T^* u_m, v_n \rangle|^2 \right) = \|T^* u_m\|^2$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \left(\|T^* u_m\|^2 \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T^* u_m, u_n \rangle|^2 \right) \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle u_m, T u_n \rangle|^2 \right) \right) \\
&= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(|\langle T u_n, u_m \rangle|^2 \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(|\langle T u_n, u_m \rangle|^2 \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\|T u_n\|^2 \right) \\
&< \infty
\end{aligned}
$$

This shows that $\sum_{n=1}^{\infty} \left(\|T v_n\|^2 \right) < \infty$ for any orthonormal basis. □

(b) Show that a Hilbert-Schmidt operator is compact.

Proof. Let T be a Hilbert-Schmidt operator. Since $\mathcal{K}(\mathbf{H})$ is a closed subspace of $\mathcal{B}(\mathbf{H})$, if there exists some sequence of operators $T_N \in \mathcal{K}(\mathbf{H})$ such that $T_N \to T$, then $T \in \mathcal{K}(\mathbf{H})$ because $\mathcal{K}(\mathbf{H})$ is closed. To this end, I will let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$ and I will define

$$
T_N x = \sum_{n=1}^{N} \left(\langle x, u_n \rangle T u_n \right)
$$

First I will show that $T_N \in \mathcal{K}(\mathbf{H})$ for any N. Note that $T_N \in \mathcal{B}(\mathbf{H})$,

$$
\begin{aligned}
\|T_N x\| &= \left\| \sum_{n=1}^{N} \left(\langle x, u_n \rangle T u_n \right) \right\| \\
&\le \sum_{n=1}^{N} \left(|\langle x, u_n \rangle| \|T u_n\| \right)
\end{aligned}
$$

By Cauchy-Schwarze

$$
\begin{aligned}
&\le \sum_{n=1}^{N} \left(\|x\| \|u_n\| \|T u_n\| \right) \\
&= \sum_{n=1}^{N} \left(\|x\| \|T u_n\| \right)
\end{aligned}
$$

Since $T \in \mathcal{B}(\mathbf{H})$

$$
\begin{aligned}
&\leq \sum_{n=1}^{N} \left(\|x\| \|T\| \|u_n\|\right) \\
&= \sum_{n=1}^{N} \left(\|x\| \|T\|\right) \\
&= N\|T\| \|x\|
\end{aligned}
$$

This shows that $\|T_N\| \leq N\|T\|$ and therefore that $T_N \in \mathcal{B}(\mathbf{H})$. Next note that $T_N x \in \operatorname{span}(\{Tu_n : 1 \leq n \leq N\})$. This shows that $R(T_N) \subset \operatorname{span}(\{Tu_n : 1 \leq n \leq N\})$ and that $\dim(R(T_N)) < N$. Since T_N is both bounded and of finite rank, this implies that $T_N \in \mathcal{K}(\mathbf{H})$.
Secondly I will show that $T_N \to T$, and I will use the fact that

$$
Tx = T\left(\sum_{n=1}^{\infty} \left(\langle x, u_n \rangle u_n\right)\right) = \sum_{n=1}^{\infty} \left(\langle x, u_n \rangle T u_n\right)
$$

$$
\begin{aligned}
\|T_N - T\| &= \sup_{\|x\|=1} \left(\|T_N x - Tx\|\right) \\
&= \sup_{\|x\|=1} \left(\left\|\sum_{n=1}^{N} \left(\langle x, u_n \rangle T u_n\right) - \sum_{n=1}^{\infty} \left(\langle x, u_n \rangle T u_n\right)\right\|\right) \\
&= \sup_{\|x\|=1} \left(\left\|\sum_{n=N+1}^{\infty} \left(\langle x, u_n \rangle T u_n\right)\right\|\right) \\
&\leq \sup_{\|x\|=1} \left(\sum_{n=N+1}^{\infty} \left(|\langle x, u_n \rangle| \|T u_n\|\right)\right) \\
&\leq \sup_{\|x\|=1} \left(\sum_{n=N+1}^{\infty} \left(\|x\| \|u_n\| \|T u_n\|\right)\right) \\
&= \sum_{n=N+1}^{\infty} \left(\|T u_n\|\right)
\end{aligned}
$$

$\square$