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MATH 520 Methods of Applied Math II
Homework 7

Section 13.6

#6 Let

$$Tu(x) = \frac{1}{x}\int_0^x u(y)\,\mathrm{d}y \qquad u \in L^2(0,1)$$

Show that $(0,2) \subset \sigma_p(T)$ and that T is not compact.

Proof. Let $\lambda \in (0,2)$ and define $\alpha = \frac{1}{\lambda} - 1$. Since $\lambda \in (0,2)$ this is well-defined and this implies that $\alpha \in (-1/2, \infty)$. Now let $u(x) = x^\alpha$, and first note that $u \in L^2(0,1)$ for any □

#9 Let T be the integral operator with kernel $K(x,y) = e^{-|x-y|}$ on $L^2(-1,1)$. Find all the eigenvalues and eigenfunctions of T. Suggestion: $Tu = \lambda u$ is equivalent to an ODE problem. Don't forget about boundary conditions. the eigenvalues may need to be characterized in terms of the roots of a certain nonlinear functions.

#10 We say that $T \in \mathcal{B}(\mathbf{H})$ is a positive operator if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathbf{H}$. If T is a positive self-adjoint compact operator show that T has a square root, more precisely there exists a compact self-adjoint operator S such that $S^2 = T$. (Suggestion: If $T = \sum_{n=1}^{\infty}(\lambda_n P_n)$ try $S = \sum_{n=1}^{\infty}(\sqrt{\lambda_n} P_n)$. In a similar manner, one can define other fractional powers of T.)

Proof. □

#14 If $Q \in \mathcal{B}(\mathbf{H})$ is a Fredholm operator of index zero, show that there exists a one-to-one operator $S \in \mathcal{B}(\mathbf{H})$ and $T \in \mathcal{K}(\mathbf{H})$ such that $Q = S + T$. (Hint: Define $T = AP$ where P is the orthogonal projecion onto $N(Q)$ and $A : N(Q) \to N(Q^*)$ is one-to-one and onto.)