Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 3

Section 11.4

#2 Verify that $\mathbf{H} \times \mathbf{H}$ is a Hilbert space with the inner product given by (11.1.2), and prove Proposition 11.1.

Proof. First note that the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

satisfies all of the axioms of an inner product.

Next I will show that $\mathbf{H} \times \mathbf{H}$ is complete under this inner product. Let $x_n = (u_n, v_n)$ be a Cauchy sequence in $\mathbf{H} \times \mathbf{H}$. Let $\epsilon > 0$ be given then there exists $N \in \mathbb{N}$ such that $\|(u_n, v_n) - (u_m, v_m)\| < \epsilon$ for all $n, m > N$. This implies that

$$\begin{aligned}
\|(u_n, v_n) - (u_m, v_m)\|^2 &\leq \epsilon^2 \\
\|(u_n - u_m, v_n - v_m)\|^2 &\leq \epsilon^2 \\
\langle (u_n - u_m, v_n - v_m), (u_n - u_m, v_n - v_m) \rangle &\leq \epsilon^2 \\
\langle u_n - u_m, u_n - u_m \rangle + \langle v_n - v_m, v_n - v_m \rangle &\leq \epsilon^2 \\
\|u_n - u_m\|^2 + \|v_n - v_m\|^2 &\leq \epsilon^2
\end{aligned}$$

Since the left had is the sum of nonnegative numbers each term must be less than ϵ^2

$$\|u_n - u_m\| \leq \epsilon \|v_n - v_m\| \qquad \leq \epsilon$$

This shows that the sequences $u_n, v_n \in \mathbf{H}$ are both Cauchy. Since $\mathbf{H}$ is complete this implies that these sequences converge. Let $u = \lim_{n\to\infty}(u_n)$ and $v = \lim_{n\to\infty}(v_n)$. Now consider

$$\begin{aligned}
\|(u_n, v_n) - (u, v)\|^2 &= \|(u_n - u, v_n - v\|^2 \\
&= \langle (u_n - u, v_n - v), (u_n - u, v_n - v) \rangle \\
&= \langle u_n - u, u_n - u \rangle + \langle v_n - v, v_n - v \rangle \\
&= \|u_n - u\|^2 + \|v_n - v\|^2
\end{aligned}$$

Now let $n \to \infty$

$$\begin{aligned}
\lim_{n\to\infty} \left(\|(u_n, v_n) - (u, v)\|^2 \right) &= \lim_{n\to\infty} \left(\|u_n - u\|^2 + \|v_n - v\|^2 \right) \\
&= \lim_{n\to\infty} \left(\|u_n - u\|^2 \right) + \lim_{n\to\infty} \left(\|v_n - v\|^2 \right)
\end{aligned}$$

Since $u_n \to u$ and $v_n \to v$

$$= 0 + 0 = 0$$

Thus $(u_n, v_n) \to (u, v)$ as $n \to \infty$. This shows that any Cauchy sequence in $\mathbf{H} \times \mathbf{H}$ converges. Thus $\mathbf{H} \times \mathbf{H}$ is complete and is also a Hilbert space.

Finally I will prove Proposition 11.1. Let $T : D(T) \subset \mathbf{H} \to \mathbf{H}$ be a closed linear operator. Consider the graph of T. Obviously $G(T)$ is a subspace because T is linear. Since $T0 = 0$, this implies that $(0,0) \in G(T)$. Let $(u_1, v_1), (u_2, v_2) \in G(T)$, then $Tu_1 = v_1$, $Tu_2 = v_2$, and $T(c_1 u_1 + c_2 u_2) = c_1 v_1 + c_2 v_2$ for any scalars c_1, c_2. This shows that $c_1(u_1, v_1) + c_2(u_2, v_2) \in G(T)$, and therefore $G(T)$ is a subspace. Now let $(u_n, v_n) \in G(T)$ be a convergent sequence, that is $(u_n, v_n) \to (u, v)$. Since $(u_n, v_n) \in G(T)$ this implies that $u_n \in D(T)$ and $v_n = Tu_n$. Also since $(u_n, v_n) \to (u, v)$ this implies that $u_n \to u$ and $v_n = Tu_n \to v$. Therefore since T is closed this implies that $u \in D(T)$ and $Tu = v$. This shows that $(u, v) \in G(T)$, and thus $G(T)$ is closed.

Let $T : D(T) \subset \mathbf{H} \to \mathbf{H}$ be a linear operator and let $G(T)$, the graph of T, be a closed subspace. Let $u_n \in D(T)$, such that $u_n \to u$ and $Tu_n \to v$. This implies that $(u_n, Tu_n) \in G(T)$ and that $(u_n, Tu_n) \to (u, v)$. Since $G(T)$ is closed this implies that $(u, v) \in G(T)$, and thus $u \in D(T)$ and $Tu = v$. Therefore T is a closed operator. $\square$

#5 If $T : D(T) \subset \mathbf{H} \to \mathbf{H}$ is a densely defined linear operator, $v \in \mathbf{H}$ and the map $u \to \langle Tu, v\rangle$ is bounded on $D(T)$, show that there exists $v^* \in \mathbf{H}$ such that (v, v^*) is an admissible pair for T^*.

Proof. Let $T : D(T) \subset \mathbf{H} \to \mathbf{H}$ be a densely defined linear operator, let $v \in \mathbf{H}$ and let $\phi_v(u) = \langle Tu, v\rangle$ be a bounded functional on $D(T)$. Note that since $\phi_v(u) = \langle Tu, v\rangle$ is bounded it is also continous and there exists a continuous extension of ϕ_v onto $\overline{D(T)}$ by Proposition 10.1. Let $S : \overline{D(T)} \to \mathbf{H}$ be this continuous extension. Since T is densely defined, $\overline{D(T)} = \mathbf{H}$. Therefore $S \in \mathcal{B}(\mathbf{H}, \mathbb{C})$ so the Riesz Representation Theorem states that there exists $v^* \in \mathbf{H}$ such that $Su = \langle u, v^*\rangle$ for all $u \in \mathbf{H}$. Now note that for $u \in D(T)$, $Su = \langle Tu, v\rangle$ and $Su = \langle u, v^*\rangle$, therefore $\langle Tu, v\rangle = \langle u, v^*\rangle$. This implies that (v, v^*) is an admissable pair for T^*. $\square$

#8 Show that if T is self-adjoint and one-to-one then T^{-1} is also self-adjoint.

Proof. Let T be self-adjoint and one-to-one. This implies that T is linear and densely defined as the adjoint of T isn't defined for T not linear and densely defined. Also $T = T^*$, and $N(T) = \{0\}$ because T is one-to-one. By Theorem 11.3

$$\overline{R(T)} = N(T^*)^{\perp} = N(T)^{\perp} = \{0\}^{\perp} = \mathbf{H}$$

This shows that the range of T is dense in $\mathbf{H}$. Since $D(T^{-1}) = R(T)$ this also shows that T^{-1} is densely defined on $\mathbf{H}$. Now since the range of T is dense, Proposition 11.6 implies that T^* is one-to-one and $(T^*)^{-1} = (T^{-1})^*$. However since T is self-adjoint this is equivalent to $T^{-1} = (T^{-1})^*$, which states that T^{-1} is self-adjoint. $\square$

#11 Assume that T is closed and S is bounded

(a) Show that $S+T$ is closed

Proof. First note that since S is bounded there exists a continuous extension of S onto the closure of its domain, by Proposition 10.1. Also we can assume that S has been replaced by this extension, and therefore $D(S)$ is closed. Now let $u_n \in D(S+T)$ and let $u_n \to u$ and $(S+T)u_n \to v$. This implies that $u_n \in D(S) \cap D(T)$ and that $Su_n + Tu_n \to v$. Since $u_n \in D(S)$ and $D(S)$ is closed, $u \in D(S)$. Also since S is bounded it is continuous, therefore $Su_n \to Su$. Note that

$$\lim_{n\to\infty}(Su_n + Tu_n) = Su + \lim_{n\to\infty}(Tu_n) = v$$

Therefore $Tu_n \to v - Su$. Also since $u_n \in D(T)$ and $u_n \to u$ by the closedness of T this implies that $u \in D(T)$ and $Tu = v - Su$. Therefore $u \in D(S+T)$ and $Su + Tu = (S+T)u = v$, so $S+T$ is closed. □

(b) Show that TS is closed, but that ST is not closed, in general.

Proof. As in part (a) it can be assumed that S is replaced with its unique continuous extension to $\overline{D(S)}$. Let $u_n \in D(TS)$ such that $u_n \to u$ and $TSu_n \to v$. Note that the domain of TS can be expressed as follows.

$$D(TS) = \{x \in D(S) : Sx \in D(T)\}$$

This implies that $u_n \in D(S)$ and since $D(S)$ is closed $u \in D(S)$. Therefore because S is bounded and continous $Su_n \to Su$. Now note that since $Su_n \in D(T)$ and $Su_n \to Su$ and $T(Su_n) \to v$, by the closedness of T, $Su \in D(T)$ and $TSu = v$. Therefore $u \in D(TS)$ and $TSu = v$, so TS is closed.

However the operator ST is not closed in general. Let $\mathbf{H} = L^2(0,1)$ and consider the closed operator $Tu = u'$ on

$$D(T) = \left\{u \in H^1(0,1) : u(0) = 0\right\}$$

and let $S : \mathbf{H} \to \mathbf{H}$ be the zero operator that is $Su = 0$ for all $u \in \mathbf{H}$. Note that S is clearly bounded. Let $u_n(x) = \frac{1}{1+e^{-n(x-1/2)}}$ be the logistic function. As $n \to \infty$ this logistic function gets steeper and steeper. In $L^2(0,1)$ this sequence of function converges to the Heaviside function at $x = 1/2$, that is $u_n(x) \to H(x - 1/2) \in L^2(0,1)$. All the assumptions of closedness are met $u_n \in D(ST)$, $u_n \to u \in L^2(0,1)$, and $STu_n \to 0$, but $u \notin D(T)$, as $u \notin H^1(0,1)$. This implies that $u \notin D(ST)$ and therefore ST is not closed. □

#14 If T is closable, show that T and $\overline{T}$ have the same adjoint.

Proof. Since T has an adjoint, T must be densely defined. By Theorem 11.5 this implies that T^* is densely defined and $\overline{T} = T^{**}$. Also since T^* is closed it is also closable so applying Theorem 11.5 to T^* implies that T^{**} is densely defined and $\overline{T^*} = T^{***}$. Since T^* is closed $T^* = \overline{T^*}$, so $T^* = T^{***} = \overline{T}^*$. This shows that T and $\overline{T}$ have the same adjoint. $\square$