# Caleb Logemann
# MATH 520 Methods of Applied Math II
# Homework 7

## Section 13.6

#7 Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of nonzero real numbers satisfying

$$\sum_{j=1}^{\infty} \left(\lambda_j^2\right) < \infty$$

Construct a symmetric Hilbert Schmidt kernel $K$ such that the corresonding integral operator has eigenvalues $\lambda_j,\ j = 1, 2, \ldots$ and for which 0 is an eigenvalue of infinite multiplicity. ( Suggestion: look for such a $K$ in the form $K(x, y) = \sum_{j=1}^{\infty} \left(\lambda_j u_j(x)\overline{u_j(y)}\right)$ where $\{u_j\}$ are orthonormal but not complete in $L^2(\Omega)$.)

Let $u_j = e^{1/\lambda_j x}$ and let $K(x, y) = \sum_{j=1}^{\infty} \left(\lambda_j u_j(x)\overline{u_j(y)}\right)$.

#12 Compute the singular value decomposition of the Volterra operator

$$Tu(x) = \int_0^x u(s)\,\mathrm{d}s$$

in $L^2(0,1)$ and use it to find $\|T\|$. Is $T$ normal? (Suggestion: The equation $T^*Tu = \lambda u$ is equivalent to an ODE eigenvalue problem which you can solve explicitly.)

In order to compute the singular value decomposition of the Volterra operator, we must first find the operator $T^*$, so that we may construct $S = T^*T$. Since the Volterra operator is an integral operator in may be rewritten as

$$Tu(x) = \int_0^1 K(x,y)u(y)\,\mathrm{d}y$$

where the kernel $K(x,y)$ is

$$K(x,y) = \begin{cases} 1 & y < x \\ 0 & y > x \end{cases}.$$

The adjoint of any integral operator is another integral operator with kernel, $\overline{K(y,x)}$. In this case

$$\overline{K(y,x)} = \begin{cases} 1 & x < y \\ 0 & x > y \end{cases}.$$

Therefore

$$\begin{aligned} T^*u(x) &= \int_0^1 \overline{K(y,x)}u(y)\,\mathrm{d}y \\ &= \int_x^1 u(y)\,\mathrm{d}y. \end{aligned}$$

Now I will define $S = T^*T$ which can be expressed as

$$\begin{aligned} Su(x) &= T^*Tu(x) \\ &= T^* \int_0^x u(s)\,\mathrm{d}s \\ &= \int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

In order to compute the singular value decomposition of $T$ we must first find the eigenvalues and eigenfunctions of $S$. This means solving $Su = \lambda u$. Note that we have previously shown that $\lambda \geq 0$ because $S = T^*T$. Also note that if $\lambda = 0$, then

$$\int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y = 0$$

which implies that $u = 0$. Therefore 0 cannot be an eigenvalue, and also $\lambda > 0$. In order to find the eigenvalues, this integral equation can be changed into a differential equation as follows.

$$\begin{aligned} \lambda u(x) &= Su(x) \\ \lambda u(x) &= \int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \\ \lambda u(x) &= -\int_1^x \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \\ \lambda u'(x) &= -\int_0^x u(s)\,\mathrm{d}s \\ \lambda u''(x) &= -u(x) \end{aligned}$$

The boundary conditions for this differential equation can be found by evaluating $Su(x)$ at $x = 1$. Clearly $Su(1) = 0$ so this implies that $u(1) = 0$. The other boundary condition can be found by evaluating $\frac{\mathrm{d}}{\mathrm{d}x}(Su(x))$ at $x = 0$. In this case the result is $u'(0) = 0$.

So now the original integral equation is identical to the following differential equation.

$$\lambda u''(x) + u(x) = 0$$
$$u(1) = 0 \qquad u'(0) = 0$$

This differntial equation can be solved using the characteristic polynomial

$$\lambda r^2 + 1 = 0$$

The roots of this polynomial are $r = \sqrt{1/\lambda}i$ and $r = -\sqrt{1/\lambda}i$. Since these are complex roots the solutions will be of the form

$$u(x) = c_1 \sin\left(\sqrt{1/\lambda x}\right) + c_2 \cos\left(\sqrt{1/\lambda x}\right)$$

Applying the boundary conditions can find the appropriate constants

$$\begin{aligned}
u'(0) &= c_1\sqrt{1/\lambda}\cos\left(\sqrt{1/\lambda 0}\right) - c_2\sqrt{1/\lambda}\sin\left(\sqrt{1/\lambda 0}\right)\\
0 &= c_1\sqrt{1/\lambda}\\
0 &= c_1\\
u(1) &= c_2\cos\left(\sqrt{1/\lambda}\right)\\
0 &= c_2\cos\left(\sqrt{1/\lambda}\right)
\end{aligned}$$

Since $c_2$ can not equal zero, otherwise $u = 0$, this implies that $\cos\left(\sqrt{1/\lambda}\right) = 0$. This is equivalent to $\sqrt{1/\lambda} = \pi/2 + k\pi$, or the eigenvalues are $\lambda_k = \frac{1}{(\pi(1/2+k))^2}$, which can be indexed by $k \geq 0$. The eigenfunctions then are $u_k(x) = c\cos\left(\sqrt{1/\lambda_k x}\right)$. We wish to normalize these eigenfunctions in order to create the singular value decomposition of $T$.

$$\begin{aligned}
\|u_k\|^2 &= \int_0^1 c^2 \cos\left(\frac{1}{\sqrt{\lambda_k}}x\right)^2 \mathrm{d}x\\
1 &= c^2 \int_0^1 \frac{1+\cos\left(\frac{2}{\sqrt{\lambda_k}}x\right)}{2}\mathrm{d}x\\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin\left(\frac{2}{\sqrt{\lambda_k}}x\right)\Big|_{x=0}^{1}\right)\\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin\left(\frac{2}{\sqrt{\lambda_k}}\right)\right)\\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin(\pi + 2k\pi)\right)\\
1 &= c^2\frac{1}{2}\\
2 &= c^2\\
c &= \sqrt{2}
\end{aligned}$$

The eigenfunctions are thus $u_k(x) = \sqrt{2}\cos\left(\frac{1}{\sqrt{\lambda_k}}x\right)$. By Theorem 13.10 these eigenfunctions form an orthonormal basis of $L^2(0,1)$.

The singular values are the square roots of the eigenvalues, that is $\sigma_k = \sqrt{\lambda_k} = \frac{1}{\pi(1/2+k)}$.

Lastly we can compute $v_k = Tu_k/\sigma_k$.

$$\begin{aligned}
v_k &= \frac{1}{\sigma_k}Tu_k \\
&= \frac{1}{\sigma_k}\int_0^x u(y)\,\mathrm{d}y \\
&= \frac{1}{\sigma_k}\int_0^x \sqrt{2}\cos\left(\frac{1}{\sqrt{\lambda_k}}y\right)\mathrm{d}y \\
&= \frac{\sqrt{2\lambda_k}}{\sigma_k}\sin\left(\frac{1}{\sqrt{\lambda_k}}y\right)\Big|_{y=0}^{x} \\
&= \sqrt{2}\sin\left(\frac{1}{\sqrt{\lambda_k}}x\right)
\end{aligned}$$

Now that we have found $\sigma_k$, $u_k$, and $v_k$, the singular value decomposition of $T$ is $\sum_{n=0}^{\infty}(\sigma_n\langle u, u_n\rangle v_n)$.

The norm of $T$ is the largest singular value, that is $\|T\| = \sigma_0 = \frac{2}{\pi}$. Also $T$ is not normal because if $T$ was normal then $Tu_n = \sigma_n u_n$ or $v_n = u_n$. However this is not the case so $T$ is not normal.

## Section 14.5

#1 Let $Lu = (x-2)u'' + (1-x)u' + u$ on $(0, 1)$.

(a) Find the Green's function for

$$Lu = f \qquad u'(0) = 0 \quad u(1) = 0$$

(Hint First show that $x-1$, $e^x$ are linearly independent solutions of $Lu = 0$.)
First consider $Lu = 0$. Let $u(x) = x - 1$, and consider $Lu$.

$$\begin{aligned} Lu(x) &= (x-2)u''(x) + (1-x)u'(x) + u(x) \\ &= (x-2)0 + (1-x)1 + (x-1) \\ &= 0 \end{aligned}$$

Let $u(x) = e^x$, then

$$\begin{aligned} Lu(x) &= (x-2)e^x + (1-x)e^x + e^x \\ &= (x-2+1-x+1)e^x \\ &= 0 \end{aligned}$$

Also these two functions are linearly independent because they are not multiples of one another. The boundary conditions for this problem are $B_1u = u'(0) = 0$ and $B_2u = u(1) = 0$. Let $\phi_1(x) = e^x - (x-1)$ and $\phi_2(x) = x - 1$ and note that $L\phi_1 = 0$, $B_1\phi_1 = 0$, $L\phi_2 = 0$ and $B_2\phi_2 = 0$. Thus we can use $\phi_1$ and $\phi_2$ to build the Green's function for this differential equation.
The next thing to do is compute $C_1(y)$ and $C_2(y)$, which first requires computing $W(y)$, the Wronskian of $\phi_1$ and $\phi_2$.

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x - (x-1) & x-1 \\ e^x - 1 & 1 \end{vmatrix} \\ &= e^x - (x-1) - (x-1)(e^x - 1) \\ &= (2-x)e^x \end{aligned}$$

Now we can compute $C_1$ and $C_2$ as

$$\begin{aligned} C_1(y) &= \frac{\phi_1(y)}{a_2(y)W(y)} \\ &= \frac{y-1}{(y-2)(2-y)e^y} \\ &= \frac{1-y}{(y-2)^2e^y} \\ C_2(y) &= \frac{\phi_2(y)}{a_2(y)W(y)} \\ &= \frac{e^y - (y-1)}{(y-2)(2-y)e^y} \\ &= \frac{y-1-e^y}{(y-2)^2e^y} \end{aligned}$$

Finally the Green's Function for this differential equation is

$$G(x,y) = \begin{cases} C_1(y)\phi_1(x) & 0 < x < y < 1 \\ C_2(y)\phi_2(y) & 0 < y < x < 1 \end{cases}$$
$$= \begin{cases} \frac{1-y}{(y-2)^2 e^y}(e^x - (x-1)) & 0 < x < y < 1 \\ \frac{y-1-e^y}{(y-2)^2 e^y}(x-1) & 0 < y < x < 1 \end{cases}$$

(b) Find the adjoint operator and boundary conditions.
The adjoint operator is given by

$$L^*u = (\overline{a_2}u)'' - (\overline{a_1}u)' + \overline{a_0}u$$

Therefore

$$L^*u = (x-2)u'' + (3-x)u'$$

The boundary functional for the adjoint is

$$J(\phi, \psi) = (x-2)(\phi'\overline{\psi} - \phi\overline{\psi}') - x\phi\overline{\psi}$$

where $\phi \in D(T)$ and $\psi \in D(T^*)$. The boundary conditions are equivalent to $J(\phi, \psi)|_{x=0}^{1} = 0$.

$$-1(\phi'(1)\overline{\psi(1)} - \phi(1)\overline{\psi}'(1)) - \phi(1)\overline{\psi(1)} + 2(\phi'(0)\overline{\psi(0)} - \phi(0)\overline{\psi(0)}') = 0$$

Since $\phi'(0) = 0$ and $\phi(1) = 0$, this is equivalent to

$$-\phi'(1)\overline{\psi(1)} - 2\phi(0)\overline{\psi(0)}') = 0$$

Now since $\phi'(1)$ and $\phi(0)$ can be anything the two boundary conditions are

$$B_1\psi = -2\psi'(0) = 0$$
$$B_2\psi = -\psi(1) = 0$$

#2 Let

$$Tu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

on the domain

$$D(T) = \left\{u \in H^2(1,2) : u(1) = u(2) = 0\right\}$$

(a) Show that $N(T) = \{0\}$.

*Proof.* First note that clearly $0 \in N(T)$. Now consider $u \neq 0$ such that $u \in N(T)$, then

$$\begin{aligned}
Tu &= 0 \\
-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= 0 \\
x\frac{\mathrm{d}u}{\mathrm{d}x} &= c_1 \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{c_1}{x} \\
u(x) &= \int \frac{c_1}{x}\,\mathrm{d}x + c_2 \\
u(x) &= c_1\ln(x) + c_2
\end{aligned}$$

If $u \in D(T)$, then $u$ must satisfy $u(1) = u(2) = 0$.

$$\begin{aligned}
u(1) &= c_1\ln(1) + c_2 \\
0 &= c_2 \\
u(2) &= c_1\ln(2) \\
0 &= c_1
\end{aligned}$$

This shows that $u = 0$, which contradicts that $u \neq 0$. This shows that $N(T) = \{0\}$. □

(b) Find the Green's function for the boundary value problem $Tu = f$.

In order to find the Green's function for the boundary value problem, we must first find $\phi_1$ and $\phi_2$ where $L\phi_1 = 0$, $B_1\phi_1 = 0$, $L\phi_2 = 0$, and $B_2\phi_2 = 0$. For this problem $Lu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$, $B_1u = u(1)$, and $B_2u = u(2)$. We have already shown in part (a) that any function that satisfies $Lu = 0$ is of the form $u(x) = c_1\ln(x) + c_2$. Therefore let $\phi_1(x) = c_1\ln(x) + c_2$ and let $B_1\phi_1 = 0$, then $\phi_1(1) = c_1\ln(1) + c_2 = 0$ or $c_2 = 0$. Also let $c_1 = 1$, then $\phi_1(x) = \ln(x)$. Now $\phi_2(x)$ is of the form $c_1\ln(x) + c_2$. If $\phi_2(2) = 0$, then $c_1\ln(2) + c_2 = 0$ and $c_2 = -c_1\ln(2)$. Then let $c_1 = 1$ and $\phi_2(x) = \ln(x) - \ln(2)$.

The next step is to compute the Wronskian.

$$\begin{aligned}
W(y) &= \begin{vmatrix} \ln(y) & \ln(y) - \ln(2) \\ 1/y & 1/y \end{vmatrix} \\
&= \frac{\ln(y)}{y} - \frac{\ln(y) - \ln(2)}{y} \\
&= \frac{\ln(2)}{y}
\end{aligned}$$

We also need to find $a_2(x)$, which can be found using the product rule

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = -x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{\mathrm{d}u}{\mathrm{d}x}$$

Therefore $a_2(x) = -x$.
Now $C_1(y)$ and $C_2(y)$ are

$$\begin{aligned} C_1(y) &= \frac{\phi_2(y)}{a_2(y)W(y)} \\ &= \frac{\ln(y) - \ln(2)}{-y(\ln(2)/y)} \\ &= \frac{\ln(2) - \ln(y)}{\ln(2)} \\ C_2(y) &= \frac{\phi_1(y)}{a_2(y)W(y)} \\ &= \frac{\ln(y)}{-y(\ln(2)/y)} \\ &= -\frac{\ln(y)}{\ln(2)} \end{aligned}$$

Finally the Green's function is

$$\begin{aligned} G(x,y) &= \begin{cases} C_1(y)\phi_1(x) & 0 < x < y < 1 \\ C_2(y)\phi_2(x) & 0 < y < x < 1 \end{cases} \\ &= \begin{cases} \frac{\ln(2)-\ln(y)}{\ln(2)}\ln(y) & 0 < x < y < 1 \\ -\frac{\ln(y)}{\ln(2)}(\ln(y) - \ln(2)) & 0 < y < x < 1 \end{cases} \end{aligned}$$

(c) State and prove a result about the continuous dependence of the solution $u$ on $f$ in part (b).
Let $f_1, f_2 \in R(T)$, then there exists $u_1, u_2 \in D(T)$ such that $Tu_1 = f_1$ and $Tu_2 = f_2$ or equivalently $u_1 = Sf_1$ and $u_2 = Sf_2$, where $Sf = \int_1^2 G(x,y)f(y)\,\mathrm{d}y$. I will define $\Delta f = f_1 - f_2$ and $\Delta u = u_1 - u_2$ and since $S$ is linear this implies that $\Delta u = S\Delta f$. The magnitude of $\Delta u$ satisfies

$$\|\Delta u\| \le M\|\Delta f\|$$

where $M > 0$.

*Proof.* This can be proved by noting that $G(x,y)$ is bounded on $[1,2]\times[1,2]$. Since $G(x,y)$ is bounded this implies that the integral operator $S$ is bounded, and therefore

$$\|\Delta u\| = \|S\Delta f\| \le \|S\|\|\Delta f\|.$$

□

#4 Prove the validity of (14.1.22). (Suggestions: start by writing $u(x)$ in the form

$$u(x) = \phi_2(x)\int_a^x C_2(y)f(y)\,\mathrm{d}y + \phi_1(x)\int_x^b C_1(y)f(y)\,\mathrm{d}y$$

and note that some of the terms that arise in the expression for u'(x) will cancel.)

*Proof.* First let $u(x) = Sf = \int_a^b G(x,y)f(y)\,\mathrm{d}y$, and then I will show that $u$ satisfies $Lu = f$. Using the piecewise definition of $G(x,y)$, it is possible to rewrite $u$ as

$$u(x) = \phi_2(x)\int_a^x C_2(y)f(y)\,\mathrm{d}y + \phi_1(x)\int_x^b C_1(y)f(y)\,\mathrm{d}y$$

In order to compute $Lu$, we must first compute $u'$ and $u''$. For simplicity $I_2(x) = \int_a^x C_2(y)f(y)\,\mathrm{d}y$ and $I_1 = \int_x^b C_1(y)f(y)\,\mathrm{d}y$.

$$u'(x) = \phi_2'(x)I_2(x) + \phi_2(x)C_2(x)f(x) + \phi_1'(x)I_1(x) - \phi_1(x)C_1(x)f(x)$$

Since $\phi_2(X)C_2(x) = \phi_1(x)C_1(x)$

$$\begin{aligned}
u'(x) &= \phi_2'(x)I_2(x) + \phi_1'(x)I_1(x)\\
u''(x) &= \phi_2''(x)I_2(x) + \phi_2'(x)C_2(x)f(x) + \phi_1''(x)I_1(x) - \phi_1'(x)C_1(x)f(x)
\end{aligned}$$

Now $Lu$ can be found.

$$\begin{aligned}
Lu(x) &= a_2(x)u''(x) + a_1u'(x) + a_0u(x)\\
&= a_2(x)\big(\phi_2''(x)I_2(x) + \phi_2'(x)C_2(x)f(x) + \phi_1''(x)I_1(x) - \phi_1'(x)C_1(x)f(x)\big)\\
&+ a_1(x)\big(\phi_2'(x)I_2(x) + \phi_1'(x)I_1(x)\big) + a_0\big(\phi_2(x)I_2(x) + \phi_1(x)I_1(x)\big)\\
&= I_2(x)\big(a_2(x)\phi_2''(x) + a_1(x)\phi_2'(x) + a_0\phi_2(x)\big)\\
&+ I_1(x)\big(a_2(x)\phi_1''(x) + a_1(x)\phi_1'(x) + a_0(x)\phi_1(x)\big) + \phi_2'(x)C_2(x)f(x) - \phi_1'(x)C_1(x)f(x)\\
&= I_2(x)L\phi_2(x) + I_1(x)L\phi_1(x) + a_2(x)\phi_2'(x)C_2(x)f(x) - a_2(x)\phi_1'(x)C_1(x)f(x)
\end{aligned}$$

Since $L\phi_2(x) = 0$ and $L\phi_1(x) = 0$.

$$\begin{aligned}
&= a_2(x)\phi_2'(x)C_2(x)f(x) - a_2(x)\phi_1'(x)C_1(x)f(x)\\
&= a_2(x)\left(\frac{\phi_2'(x)\phi_1(x) - \phi_1'(x)\phi_2(x)}{a_2(x)W(x)}\right)f(x)\\
&= \left(\frac{\phi_2'(x)\phi_1(x) - \phi_1'(x)\phi_2(x)}{W(x)}\right)f(x)\\
&= f(x)
\end{aligned}$$

This shows that $Lu = f$. Finally I will show that $B_1u = 0$ and $B_2u = 0$.

$$\begin{aligned}
B_1u &= c_1u(a) + c_2u'(a)\\
&= c_1\big(\phi_2(a)I_2(a) + \phi_1(a)I_1(a)\big) + c_2\big(\phi_2'(a)I_2(a) + \phi_1'(a)I_1(a)\big)
\end{aligned}$$

Since $I_2(a) = 0$

$$\begin{aligned}
&= c_1\phi_1(a)I_1(a) + c_2\phi_1'(a)I_1(a)\\
&= I_1(a)\big(c_1\phi_1(a) + c_2\phi_1'(a)\big)\\
&= I_1(a)B_1\phi_1
\end{aligned}$$

Since $B_1\phi_1 = 0$

$$\begin{aligned}
B_1u &= 0 \\
B_2u &= c_3u(b) + c_4u'(b) \\
&= c_3(\phi_2(b)I_2(b) + \phi_1(b)I_1(b)) + c_4(\phi_2'(b)I_2(b) + \phi_1'(b)I_1(b))
\end{aligned}$$

Since $I_1(b) = 0$

$$\begin{aligned}
&= c_3\phi_2(b)I_2(b) + c_4\phi_2'(b)I_2(b) \\
&= I_2(b)(c_3\phi_2(b) + c_4\phi_2'(b)) \\
&= I_2(b)B_2\phi_2
\end{aligned}$$

Since $B_2\phi_2 = 0$

$$B_2u = 0$$

This shows that $u$ satisfies the boundary conditions and the differential equation. $\square$