Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 7

Section 13.6

#6 Let

$$Tu(x) = \frac{1}{x}\int_0^x u(y)\,\mathrm{d}y \qquad u \in L^2(0,1)$$

Show that $(0,2) \subset \sigma_p(T)$ and that T is not compact.

Proof. Let $\lambda \in (0,2)$ and define $\alpha = \frac{1}{\lambda} - 1$. Since $\lambda \in (0,2)$ this is well-defined and this implies that $\alpha \in (-1/2, \infty)$. Now let $u(x) = x^\alpha$, and first note that $u \in L^2(0,1)$ for any $\alpha \in (-1/2, \infty)$.

$$\begin{aligned}\int_0^1 |u(x)|^2\,\mathrm{d}x &= \int_0^1 |x^\alpha|^2\,\mathrm{d}x \\ &= \int_0^1 x^{2\alpha}\,\mathrm{d}x\end{aligned}$$

Since $\alpha > -1/2$, this implies that $2\alpha > -1$ and this integral can be evaluated using the power rule

$$\begin{aligned}&= \frac{1}{2\alpha+1}\left.x^{2\alpha+1}\right|_{x=0}^{1} \\ &= \frac{1}{2\alpha+1} \\ &\leq \infty\end{aligned}$$

This shows that $u \in L^2(0,1)$.

Now $Tu(x)$,

$$\begin{aligned}Tu(x) &= \frac{1}{x}\int_0^x u(y)\,\mathrm{d}y \\ &= \frac{1}{x}\int_0^x y^\alpha\,\mathrm{d}y \\ &= \frac{1}{x}\left.\frac{1}{\alpha+1}y^{\alpha+1}\right|_{y=0}^{x} \\ &= \frac{1}{x}\frac{1}{\alpha+1}x^{\alpha+1} \\ &= \frac{1}{\alpha+1}x^\alpha \\ &= \frac{1}{\alpha+1}u(x) \\ &= \lambda u(x)\end{aligned}$$

This shows that λ and $u(x)$ are an eigenvalue and eigenfunction pair for T. Thus $\lambda \in \sigma_p(T)$ and $(0,2) \subset \sigma_p(T)$ □

#9 Let T be the integral operator with kernel $K(x,y) = e^{-|x-y|}$ on $L^2(-1,1)$. Find all the eigenvalues and eigenfunctions of T. Suggestion: $Tu = \lambda u$ is equivalent to an ODE problem. Don't forget about boundary conditions. the eigenvalues may need to be characterized in terms of the roots of a certain nonlinear functions.

In order to find the eigenvalues and eigenfunctions of T we must find solutions to $Tu = \lambda u$.

$$\begin{aligned}
\lambda u(x) &= Tu(x) \\
\lambda u(x) &= \int_{-1}^{1} e^{-|x-y|} u(y) \, \mathrm{d}y \\
\lambda u(x) &= \int_{-1}^{x} e^{-|x-y|} u(y) \, \mathrm{d}y + \int_{x}^{1} e^{-|x-y|} u(y) \, \mathrm{d}y \\
\lambda u(x) &= \int_{-1}^{x} e^{-|x-y|} u(y) \, \mathrm{d}y - \int_{1}^{x} e^{-|x-y|} u(y) \, \mathrm{d}y
\end{aligned}$$

In the first integral $y < x$ and in the second integral $x < y$

$$\begin{aligned}
\lambda u(x) &= \int_{-1}^{x} e^{-x+y} u(y) \, \mathrm{d}y - \int_{1}^{x} e^{x-y} u(y) \, \mathrm{d}y \\
\lambda u(x) &= e^{-x} \int_{-1}^{x} e^{y} u(y) \, \mathrm{d}y - e^{x} \int_{1}^{x} e^{-y} u(y) \, \mathrm{d}y
\end{aligned}$$

Differentiating both sides

$$\begin{aligned}
\lambda u'(x) &= -e^{-x} \int_{-1}^{x} e^{y} u(y) \, \mathrm{d}y + e^{-x} e^{x} u(x) - e^{x} \int_{1}^{x} e^{-y} u(y) \, \mathrm{d}y - e^{x} e^{-x} u(x) \\
\lambda u'(x) &= -e^{-x} \int_{-1}^{x} e^{y} u(y) \, \mathrm{d}y + u(x) - e^{x} \int_{1}^{x} e^{-y} u(y) \, \mathrm{d}y - u(x) \\
\lambda u'(x) &= -e^{-x} \int_{-1}^{x} e^{y} u(y) \, \mathrm{d}y - e^{x} \int_{1}^{x} e^{-y} u(y) \, \mathrm{d}y
\end{aligned}$$

#10 We say that $T \in \mathcal{B}(\mathbf{H})$ is a positive operator if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathbf{H}$. If T is a positive self-adjoint compact operator show that T has a square root, more precisely there exists a compact self-adjoint operator S such that $S^2 = T$. (Suggestion: If $T = \sum_{n=1}^{\infty} (\lambda_n P_n)$ try $S = \sum_{n=1}^{\infty} \left(\sqrt{\lambda_n} P_n\right)$. In a similar manner, one can define other fractional powers of T.)

Proof. Let T be a positive self-adjoint compact operator. Note that the eigenvalues of T must all be nonnegative. To see this, let u be an eigenvector of T with eigenvalue λ, then

$$0 < \langle Tu, u \rangle = \langle \lambda u, u \rangle = \lambda \|u\|^2$$

This implies that $\lambda \geq 0$, since $\|u\|^2 \geq 0$. Also since T is self-adjoint and compact, the set of eigenvectors make an orthonormal basis of $\mathbf{H}$ and T can be expressed as the following infinite sum

$$T = \sum_{n=1}^{\infty} (\lambda_n P_n)$$

where P_n is the projection operator onto the eigenvector u_n and λ_n is the corresponding eigenvalue. Since $\lambda_n \geq 0$ for all n, we can define the following operator.

$$S = \sum_{n=1}^{\infty} \left(\sqrt{\lambda} P_n\right)$$

This operator is compact as can be seen by noting that

$$S_N = \sum_{n=1}^{N} \left(\sqrt{\lambda} P_n\right)$$

is a sequence of compact operators that converge to S. That is $S_N \in \mathcal{K}(\mathbf{H})$ as it has a finite dimensional range. Also $S_N \to S$ so $S \in \mathcal{K}(\mathbf{H})$ as $\mathcal{K}(\mathbf{H})$ is a closed subspace.

Also S is self adjoint because it is a sum of self-adjoint operators. Lastly $S^2 = T$ because $P_n P_n = P_n$ and $P_n P_m = 0$ when $n \neq m$. □

#14 If $Q \in \mathcal{B}(\mathbf{H})$ is a Fredholm operator of index zero, show that there exists a one-to-one operator $S \in \mathcal{B}(\mathbf{H})$ and $T \in \mathcal{K}(\mathbf{H})$ such that $Q = S + T$. (Hint: Define $T = AP$ where P is the orthogonal projecion onto $N(Q)$ and $A : N(Q) \to N(Q^*)$ is one-to-one and onto.)

Proof. Let $Q \in \mathcal{B}(\mathbf{H})$ be a Fredholm operator of index zero. This implies that $\dim(N(Q)) = \dim(N(Q^*)) < \infty$ and that $R(Q)$ is closed. Since $\dim(N(Q)) = \dim(N(Q^*))$ there exists a one-to-one and onto operator $A : N(Q) \to N(Q^*)$. Note that since A is bounded because it is a linear operator from one finite dimensional space to another finite dimensional space, see exercise 10.9.6. Now let P be the projection from H to $N(Q)$. Since P is a projection it is bounded and since $\dim(N(Q)) < \infty$, P is compact. Define $T = AP$, which is compact as it is the composition of a bounded operator with a compact operator.

Next let $S = Q - T$. Since $Q, T \in \mathcal{B}(\mathbf{H})$ this implies that $S \in \mathcal{B}(\mathbf{H})$. Lastly I will show that S is one-to-one. Let $u \in \mathbf{H}$ such that $Su = 0$. This implies that $Qu - Tu = 0$ or that $Qu = Tu$. Since $Qu = Tu$, this implies that $Tu \in R(Q)$ or since the range of Q is closed, $T \in \overline{R(Q)}$. However it is known that $\overline{R(Q)} = N(Q^*)^{\perp}$, so therefore $Tu \in N(Q^*)^{\perp}$. One the other hand $Tu = APu$ and therefore $Tu \in R(A) = N(Q^*)$. Since $Tu \in N(Q^*)$ and $Tu \in N(Q^*)^{\perp}$ this implies that $Tu = 0$ and therefore $Qu = 0$ because $Qu = Tu$. Since $Qu = 0$ this shows that $u \in N(Q)$. Also with $Tu = 0$, this implies that $APu = 0$ and since A is one-to-one, $Pu = 0$. Because $Pu = 0$ this implies that u must be orthogonal to $N(Q)$ as P is the projection onto $N(Q)$, therefore $u \in N(Q)^{\perp}$. Now that $u \in N(Q)$ and $u \in N(Q)^{\perp}$, this shows that $u = 0$. We have now shown that $N(S) = \{0\}$ and therefore S is one-to-one.

Thus there exists a one-to-one and bounded function S and a compact function T, such that $Q = S + T$ for any Fredholm operator of index 0, Q. □