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MATH 520 Methods of Applied Math II
Homework 4

Section 11.4

#6 Let $\mathbf{H} = L^2(0,1)$ and $T_1 u = T_2 u = iu'$ with domains

$$D(T_1) = \left\{u \in H^1(0,1) : u(0) = u(1)\right\} D(T_2) \qquad = \left\{u \in H^1(0,1) : u(0) = u(1) = 0\right\}$$

Show that T_1 is self-adjoint, and that T_2 is closed and symmetric but not self-adjoint. What is T_2^*?

Proof. □

#7 If T is symmetric with $R(T) = \mathbf{H}$ show that T is self-adjoint.

Proof. $\square$

#16 We say that a linear operator on a Hilbert space **H** is bounded below if there exists a constant $c_0 > 0$ such that

$$\langle Tu, u \rangle \geq -c_0 \|u\|^2 \quad \forall u \in D(T)$$

Show that Theorem 11.6 remains valid if the condition that T be positive is replaced by the assumption that T is bounded below.

Proof. $\square$

Section 12.4

#3 Recall that the resolvent operator of T is defined to be $R_\lambda = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$.

(a) Prove the resolvant identity (12.1.3).

Proof. □

(b) Deduce from this that R_λ and R_μ commute.

Proof. □

(c) Show also that T and R_λ commute for $\lambda \in \rho(T)$.

Proof. □

#4 Show that $\lambda \to R_\lambda$ is continuously differentiable, regarded as a mapping from $\rho(T) \subset \mathbb{C}$ into $\mathcal{B}(\mathbf{H})$, with

$$\frac{\mathrm{d}R_\lambda}{\mathrm{d}\lambda} = -R_\lambda^2$$

Proof. $\square$