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MATH 520 Methods of Applied Math II
Homework 9

Section 14.5

#5 Let $Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$ with $a_2' = a$, so that L is formally self adjoint. If $B_1 u = C_1 u(a) + C_2 u'(a)$, $B_2 u = C_3 u(b) + C_4 u'(b)$, show that $\{B_1^*, B_2^*\} = \{B_1, B_2\}$.

Proof. Let $\{B_1^*, B_2^*\}$ be the set of boundary operators adjoint to $\{B_1, B_2\}$. This implies that

$$J(\phi, \psi)\big|_a^b = 0$$

whenever $B_1\phi = B_2\phi = B_1^*\psi = B_2^*\psi = 0$. The boundary function J can be expressed as

$$J(\phi, \psi) = a_2\left(\phi'\overline{\psi} - \phi\overline{\psi'}\right) + (a_1 - a_2')\phi\overline{\psi}$$

and since L is formally self-adjoint, this implies that $a_1 - a_2' = 0$, so the boundary functional can be simplified to

$$J(\phi, \psi) = a_2\left(\phi'\overline{\psi} - \phi\overline{\psi'}\right).$$

□

#8 When we rewrite $a_2(x)u'' + a_1(x)u' + a_0(x)u = \lambda u$ as

$$-(p(x)u')' + q(x)u = \lambda\rho(x)u$$

the latter is often referred to as the *Liouville normal form.* Consider the eigenvalue problem

$$x^2u'' + xu' + u = \lambda u \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

(a) Find the Liouville normal form.
In order to find the Liouville normal form, the function $a_2(x)$ must be strictly less than zero, so I will first rewrite this eigenvalue problem as

$$-x^2u'' - xu' - u = -\lambda u \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

The functions $p(x)$, $\rho(x)$, and $q(x)$ can be found as follows.

$$\begin{aligned}
p(x) &= \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\,\mathrm{d}s\right) \\
&= \exp\left(\int_a^x \frac{-s}{-s^2}\,\mathrm{d}s\right) \\
&= \exp\left(\int_a^x \frac{1}{s}\,\mathrm{d}s\right) \\
&= \exp(\ln(s)|_{s=a}^x) \\
&= e^{\ln(x)-\ln(a)} \\
&= e^{\ln\left(\frac{x}{a}\right)} \\
&= \frac{x}{a} \\
\rho(x) &= -\frac{p(x)}{a_2(x)} \\
&= -\frac{x/a}{-x^2} \\
&= \frac{1}{ax} \\
q(x) &= a_0(x)\rho(x) \\
&= (-1)\frac{1}{ax} \\
&= -\frac{1}{ax}
\end{aligned}$$

Therefore the Liouville normal form of this eigenvalue problem is

$$-\left(\frac{x}{a}\phi'\right)' - \frac{1}{ax}\phi = -\lambda\frac{1}{ax}\phi$$

or

$$\left(\frac{x}{a}\phi'\right)' + \frac{1}{ax}\phi = \lambda\frac{1}{ax}\phi$$

(b) What is the orthogonality relationship satisfied by the eigenfunctions?

The eigenfunctions of this linear operator satisfy an orthogonality relationship with respect to the weight ρ. In mathematical terms,

$$\int_a^b \phi_n(x)\phi_m(x)\rho(x)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

or

$$\int_a^b \frac{\phi_n(x)\phi_m(x)}{ax}\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

(c) Find the eigenvalues and eigenfunctions. (You may find the original form of the equation easier to work with than the Liouville normal form when computing the eigenvalues and eigenfunctions.)

#10 Consider the Sturm-Liouville problem

$$u'' + \lambda u = 0 \qquad 0 < x < 1$$
$$u(0) - u'(0) = u(1) = 0$$

(a) Multiply the equation by u and integrate by parts to show that any eigenvalue is positive.
First I will note a few useful facts, first since $u(0) - u'(0) = 0$, this implies that $u(0) = u'(0)$. Also if u is nontrivial this guarantees that $\int_0^1 u^2(x)\,\mathrm{d}x > 0$. Finally if u is a nontrivial solution then $u'(x) \neq 0$ as $u(1) = 0$ makes any constant function is zero. This shows that $\int_0^1 (u'(x))^2\,\mathrm{d}x > 0$ as well.
Multiplying by u gives the following equation

$$uu'' + \lambda u^2 = 0.$$

Integrating both sides over $[0, 1]$ gives

$$\int_0^1 u(x)u''(x)\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x$$

This can be simplified using integration by parts

$$\begin{aligned}
\int_0^1 u(x)u''(x)\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0 \\
u(x)u'(x)\big|_{x=0}^1 - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0 \\
u(1)u'(1) - u(0)u'(0) - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x &= 0
\end{aligned}$$

Since $u(1) = 0$ and $u(0) = u'(0)$

$$-u^2(0) - \int_0^1 (u'(x))^2\,\mathrm{d}x + \lambda \int_0^1 u^2(x)\,\mathrm{d}x = 0.$$

Since $\int_0^1 u^2(x)\,\mathrm{d}x > 0$

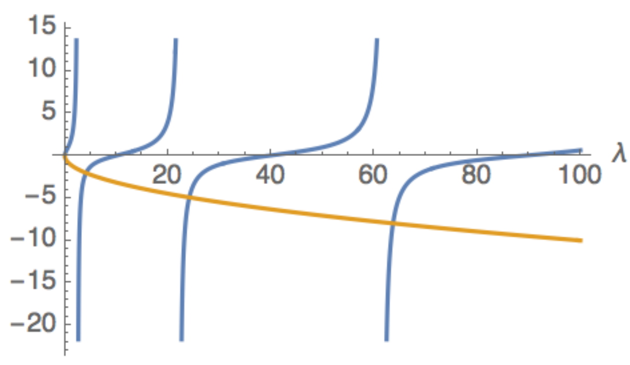
$$\lambda = \frac{u^2(0) + \int_0^1 (u'(x))^2\,\mathrm{d}x}{\int_0^1 u^2(x)\,\mathrm{d}x} > 0.$$

(b) Show that the eigenvalues are the positive solutions of $\tan\left(\sqrt{\lambda}\right) = -\sqrt{\lambda}$.

(c) Show graphically that such roots exist, and form an infinite sequence λ_k such that $(k - 1/2)\pi < \sqrt{\lambda_k} < k\pi$ and

$$\lim_{k \to \infty} \left(\sqrt{\lambda_k} - (k - 1/2)\pi\right) = 0$$

First this graph shows that solutions to the equation $\tan\left(\sqrt{\lambda}\right) = -\sqrt{\lambda}$ exist.



Next

#14 If $\{\psi_n\}_{n=1}^{\infty}$ are Dirichlet eigenfunctions of the Laplacian making up an orthonormal basis of $L^2(\Omega)$, let $\xi_n = \psi_n/\sqrt{\lambda_n}$ (λ_n the corresponding eigenvalue).

(a) Show that $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$.

(b) Show that $f \in H_0^1(\Omega)$ if and only if $\sum_{n=1}^{\infty} \left(\lambda_n |\langle f, \psi_n \rangle|^2\right) < \infty$

#15 If $\Omega < \mathbb{R}^n$ is a bounded open set with smooth enough boundary, find a solution of the wave equation problem

$$\begin{aligned} u_{tt} - \Delta u = 0 \qquad & x \in \Omega \quad t > 0 \\ u(x,t) = 0 \qquad & x \in \partial\Omega \quad t > 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \qquad & x \in \Omega \end{aligned}$$

in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(c_n(t)\psi_n(x)\right)$$

where $\{\psi_n\}_{n=1}^{\infty}$ are the Dirichlet eigenfunctions of $-\Delta$ in Ω.

#16 Derive formally that
$$G(x, y) = \sum_{n=1}^{\infty} \left(\frac{\psi_n(x)\psi_n(y)}{\lambda_n} \right)$$
where λ_n, ψ_n are the Dirichlet eigenvalues and normalized eigenfunctions for the domain Ω, and $G(x, y)$ is the corresponding Green's function in (14.4.96). (Suggestion: if $-\Delta u = f$, expand both u and f in the ψ_n basis.)