Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 2

Section 10.9

#10 Let S_+ and S_- be the left and right shift operators on ℓ^2. Show that $S_- = S_+^*$ and $S_+ = S_-^*$.

Proof. Both S_+ and S_- are in $\mathcal{B}(\ell^2)$ therefore they both have unique adjoints. Consider $x, y \in \ell^2$, then

$$\begin{aligned}
\langle S_+ x, y \rangle &= \sum_{n=1}^{\infty} \left((S_+ x)_n \cdot \overline{y_n} \right) \\
&= \sum_{n=2}^{\infty} \left(x_{n-1} \cdot \overline{y_n} \right) \\
&= \sum_{n=1}^{\infty} \left(x_n \cdot \overline{y_{n+1}} \right) \\
&= \sum_{n=1}^{\infty} \left(x_n \cdot \overline{(S_- y)_n} \right) \\
&= \langle x, S_- y \rangle
\end{aligned}$$

This shows that $S_+^* = S_-$. Now since $S_+, S_- \in \mathcal{B}(\ell^2)$ it is true that $\left(S_+^*\right)^* = S_+$ or $S_-^* = S_+$. □

#11 Let T be the Volterra integral operator $Tu = \int_0^x u(y)\,\mathrm{d}y$ considered as an operator on $L^2(0,1)$. Find T^* and $N(T^*)$.

Consider $u, v \in L^2(0,1)$.

$$\begin{aligned}
\langle Tu, v\rangle &= \int_0^1 Tu(x)\overline{v(x)}\,\mathrm{d}x \\
&= \int_0^1 \int_0^x u(y)\,\mathrm{d}y\overline{v(x)}\,\mathrm{d}x \\
&= \int_0^1 \int_y^1 \overline{v(x)}\,\mathrm{d}x u(y)\,\mathrm{d}y \\
&= \int_0^1 \overline{\int_y^1 v(x)\,\mathrm{d}x} u(y)\,\mathrm{d}y \\
&= \langle u, T^* v\rangle
\end{aligned}$$

where

$$T^* v(y) = \int_y^1 v(x)\,\mathrm{d}x$$

Since $T \in \mathcal{B}(L^2(0,1))$ this is the unique adjoint of T.

In order to find $N(T^*)$ consider $u \in L^2(0,1)$ such that

$$T^* u = 0$$

This implies that

$$T^* v(y) = \int_y^1 v(x)\,\mathrm{d}x = 0$$

for every $y \in (0,1)$. Using the Fundamental Theorem of Calculus for L^2 functions it can be seen that

$$0 = -v(y) + v(1)$$

This implies that $v(y) = v(1)$ for every $y \in (0,1)$ or equivalently that v is a constant function. Therefore the $N(T^*) = \{v \in L^2(0,1) : v = c \text{ for some } c \in \mathbb{R}\}$.

#12 Suppose $T \in \mathcal{B}(\mathbf{H})$ is self-adjoint and there exists a constant $c > 0$ such that $\|Tu\| \geq c\|u\|$ for all $u \in \mathbf{H}$. Show that there exists a solution of $Tu = f$ for all $f \in \mathbf{H}$. Show by example that the conclusion may be false if the assumption of self-adjointedness is removed.

Proof. First note that $N(T) = \{0\}$. Assume that $u \in N(T)$, then $\|u\| \leq \frac{\|Tu\|}{c} = 0$. Since $\|u\| = 0$, this implies that $u = 0$. Also since T is self-adjoint, $N(T^*) = \{0\}$. This implies by Proposition 10.3 that $\overline{R(T)} = \mathbf{H} = \{0\}^{\perp}$. Therefore the range of T is dense in $\mathbf{H}$, and thus if we can show that the range is in fact closed then $Tu = f$ will have a solution for all $f \in \mathbf{H}$.

In order to show that $R(T)$ is closed I will first consider T^{-1}. Since $N(T) = \{0\}$ this inverse is well defined for all $u \in R(T)$. Consider some sequence $u_n \in R(T)$ such that $u_n \to 0$ as $n \to \infty$. Now consider the sequence $v_n = T^{-1}u_n$. Using the property that $\|Tu\| \geq c\|u\|$, it can be seen that

$$\|v_n\| \leq \frac{\|Tv_n\|}{c} = \frac{\|u_n\|}{c}$$

Therefore $\lim_{n\to\infty}(\|v_n\|) = 0$ or equivalently $T^{-1}u_n \to 0$, this shows that T^{-1} is continous at 0. Also T^{-1} is bounded because $\frac{\|T^{-1}u\|}{\|u\|} \leq \frac{1}{c}$ for all $u \in R(T)$. Since T^{-1} is bounded and continous at 0, T^{-1} is continous everywhere. Now it is known that continuity implies that for $F \subset \mathbf{H}$ closed $\left(T^{-1}\right)^{-1}(F)$ is also closed. Since $\left(T^{-1}\right)^{-1} = T$ and $\mathbf{H}$ is closed when considered as a subset of itself, this implies that $T(\mathbf{H}) = R(T)$ is closed. Now that we have shown that $R(T)$ is closed and dense, this implies that $R(T) = \mathbf{H}$.

This conclusion may be false if that operator is not self-adjoint. Consider the operator S_+ on ℓ^2. We have already shown that $S_+^* = S_-$ so S_+ is not self-adjoint. However $\|S_+x\| = \|x\|$ for all $x \in \ell^2$, so with $c = 1$ S_+ satisfies $\|S_+x\| \geq c\|x\|$ for all $x \in \ell^2$. However $R(S_+) = \left\{x \in \ell^2 : x_1 = 0\right\}$, so $S_+u = x$ will not have a solution if $x_1 \neq 0$. $\square$

#13 Let M be the multiplication operator $Mu(x) = xu(x)$ in $L^2(0,1)$. Show that $R(M)$ is dense but not closed.

Proof. First of all note that M is self adjoint, this is because $M^*u(x) = \overline{x}u(x) = xu(x) = Mu(x)$ on $x \in (0,1)$. Therefore $N(M^*) = N(M)$. By definition

$$N(M) = \left\{x \in L^2(0,1) : \|xu(x)\|_{L^2} = 0 \forall x \in 0,1\right\}$$

Equivalently this implies that

$$\int_0^1 |xu(x)|^2 \, \mathrm{d}x = 0$$

which is the same as saying that $u(x) = 0$ almost everywhere. This shows that $N(M) = \{0\} = N(M^*)$. Now by proposition 10.3, we know that $\overline{R(M)} = N(M^*)^\perp = L^2(0,1)$. Thus the range of M is dense in $L^2(0,1)$.

Next I will construct a sequence in $R(M)$ that does not converge to a point in $R(M)$. That is I will find $u_n \in L^2(0,1)$ such that $Mu_n \to v \notin R(M)$. This will show that $R(M)$ is not closed. Let

$$u_n(x) = \begin{cases} 0 & x < \frac{1}{n} \\ x^{\frac{1}{n}-1} & x \geq \frac{1}{n} \end{cases}$$

First I will show that $u_n(x) \in L^2(0,1)$ for every $n \in \mathbb{N}$, $n \geq 3$. The fact that $n \geq 3$ is used when taking the antiderivative.

$$\begin{aligned}
\int_0^1 |u_n(x)|^2 \, \mathrm{d}x &= \int_{\frac{1}{n}}^1 \left|x^{\frac{1}{n}-1}\right|^2 \mathrm{d}x \\
&= \int_{\frac{1}{n}}^1 x^{\frac{2}{n}-2} \, \mathrm{d}x \\
&= \left(\frac{1}{\frac{2}{n}-1} x^{\frac{2}{n}-1}\right)\Bigg|_{x=\frac{1}{n}}^1 \\
&= \frac{1}{\frac{2}{n}-1}\left(1 - \left(\frac{1}{n}\right)^{\frac{2}{n}-1}\right) \\
&= \frac{n}{2-n}\left(1 - \left(\frac{1}{n}\right)^{\frac{2-n}{n}}\right) < \infty
\end{aligned}$$

Second I will show that $Mu_n(x) \to 1$ in $L^2(0,1)$. Consider

$$\begin{aligned}
\|Mu_n - 1\|_{L^2(0,1)}^2 &= \int_0^1 (Mu_n(x) - 1)^2 \, \mathrm{d}x \\
&= \int_0^1 1 - 2Mu_n(x) + Mu_n(x)^2 \, \mathrm{d}x \\
&= \int_0^1 1 \, \mathrm{d}x - 2\int_0^1 Mu_n(x) \, \mathrm{d}x + \int_0^1 Mu_n(x)^2 \, \mathrm{d}x \\
&= 1 - 2\int_{\frac{1}{n}}^1 x^{\frac{1}{n}} \, \mathrm{d}x + \int_{\frac{1}{n}}^1 x^{\frac{2}{n}} \, \mathrm{d}x \\
&= 1 - 2\left(\frac{1}{\frac{1}{n}+1} x^{\frac{1}{n}+1}\right)\Bigg|_{x=\frac{1}{n}}^1 + \left(\frac{1}{\frac{2}{n}+1} x^{\frac{2}{n}+1}\right)\Bigg|_{x=\frac{1}{n}}^1 \\
&= 1 - \frac{2}{\frac{1}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{1}{n}+1}\right) + \frac{1}{\frac{2}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{2}{n}+1}\right)
\end{aligned}$$

Now consider the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty}\left(\|Mu_n - 1\|_{L^2(0,1)}^2\right) &= \lim_{n\to\infty}\left(1 - \frac{2}{\frac{1}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{1}{n}+1}\right) + \frac{1}{\frac{2}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{2}{n}+1}\right)\right) \\
&= 1 - \lim_{n\to\infty}\left(\frac{2}{\frac{1}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{1}{n}+1}\right)\right) + \lim_{n\to\infty}\left(\frac{1}{\frac{2}{n}+1}\left(1 - \left(\frac{1}{n}\right)^{\frac{2}{n}+1}\right)\right) \\
&= 1 - 2 + 1 = 0
\end{aligned}$$

Therefore $Mu_n \to 1$ in $L^2(0,1)$. Lastly I will show that $1 \notin R(M)$ in order to show that $R(M)$ is not closed. If $1 \in R(M)$ then there exists $u \in L^2(0,1)$ such that $xu(x) = 1$. This would imply that $u(x) = \frac{1}{x}$, however $\frac{1}{x}$ is not in $L^2(0,1)$. To verify this consider the following.

$$\begin{aligned}
\int_0^1 \left|x^{-1}\right|^2 \,\mathrm{d}x &= \int_0^1 x^{-2}\,\mathrm{d}x \\
&= -x^{-1}\Big|_{x=0}^{1} \\
&= -1 + \infty = \infty
\end{aligned}$$

This shows that x^{-1} is not in $L^2(0,1)$. $\square$

#15 An operator $T \in \mathcal{B}(\mathbf{H})$ is said to be normal if it commutes with its adjoint, i.e. $TT^* = T^*T$. Thus, for example, any sefl-adjoint, skey-adjoint, or unitary operator is normal. For a normal operator T show that

(a) $\|Tu\| = \|T^*u\|$ for every $u \in \mathbf{H}$.

Proof. Let $u \in \mathbf{H}$ and consider the following.

$$\begin{aligned}
\|Tu\|^2 &= \langle Tu, Tu\rangle \\
&= \langle u, T^*Tu\rangle \\
&= \langle u, TT^*u\rangle \\
&= \langle u, (T^*)^*T^*u\rangle \\
&= \langle T^*u, T^*u\rangle \\
&= \|T^*u\|^2
\end{aligned}$$

Therefore $\|Tu\| = \|T^*u\|$ for every $u \in \mathbf{H}$. □

(b) T is one to one if and only if it has dense range.

Proof. First note that when T is normal $N(T) = N(T^*)$. This can be see by using part (a) and by letting $Tu = 0$, then $0 = \|Tu\| = \|T^*u\|$. This implies that $T^*u = 0$. Thus if $u \in N(T)$ then $n \in N(T^*)$. The opposite direction is equivalent, that is when $u \in N(T^*)$, then $u \in N(T)$. Now assume that T is one to one or equivalently $N(T) = \{0\}$. As was shown earlier this implies that $N(T^*) = \{0\}$ and by proposition 10.3 it is clear that

$$\overline{R(T)} = N(T^*)^{\perp} = \{0\}^{\perp} = \mathbf{H}$$

Therefore the range of T is dense in $\mathbf{H}$.
Finally let T have dense range, that is $\overline{R(T)} = \mathbf{H}$. This implies that $N(T^*)^{\perp} = \mathbf{H}$. Therefore for all $u \in \mathbf{H}$, $u \perp v$ for all $v \in N(T^*)$. The only element of $\mathbf{H}$ that is orthogonal to all of $\mathbf{H}$ is the zero element. Therefore $N(T^*) = \{0\}$ and it follows that $N(T) = \{0\}$, which shows that T is one to one. □

(c) Show that any multiplication operator or Fourier multiplication operator is normal in L^2.
Let S be a multiplication operator on L^2, then $Su(x) = w(x)u(x)$ for some $w \in L^{\infty}$. We have also shown that $S^*u(x) = \overline{w(x)}u(x)$. Now consider any $u \in L^2$.

$$\begin{aligned}
S^*Su(x) &= S^*w(x)u(x) \\
&= \overline{w(x)}w(x)u(x) \\
&= w(x)\overline{w(x)}u(x) \\
&= S\overline{w(x)}u(x) \\
&= SS^*u(x)
\end{aligned}$$

Therefore $S^*S = SS^*$ and S is normal.
Now let T be a Fourier multiplication operator. Then $T = F^{-1}SF$ where F is the Fourier Transform and S is some multiplication operator. Now all of these operators are in $\mathcal{B}(L^2)$ so $T^* = \left(F^{-1}SF\right)^* = F^*S^*\left(F^{-1}\right)^*$. Also note that F is unitary so $F^* = F^{-1}$ and $\left(F^{-1}\right)^* = F$. Thus $T^* = F^{-1}S^*F$. Now consider

$$\begin{aligned}
TT^* &= F^{-1}SFF^{-1}S^*F \\
&= F^{-1}SS^*F
\end{aligned}$$

Since S as a multiplication operator is normal

$$\begin{aligned}
&= F^{-1}S^*SF \\
&= F^{-1}S^*FF^{-1}SF \\
&= T^*T
\end{aligned}$$

Thus T is a normal operator.

(b) Show that the shift operators S_+ and S_- are not normal in ℓ^2.

Consider $x \in \ell^2$ such that $x_1 \neq 0$, then $S_-S_+x = x$ however $S_+S_-x = (0, x_2, x_3, \cdots)$. Thus $S_+S_+^*x = S_+S_-x \neq S_-S_+x = S_+^*S_+x$. Therefore S_+ is not normal. Also $S_-S_-^*x = S_-S_+x \neq S_+S_-x = S_-^*S_-x$, so S_- is not normal either.

#19 If $T_n \in \mathcal{B}(X)$ and $\sum_{n=1}^{\infty} (\|T_n\|) < \infty$, show that the series $\sum_{n=1}^{\infty} (T_n)$ is uniformly convergent.

Proof. To show that the series $\sum_{n=1}^{\infty} (T_n)$ is uniformly convergent is equivalent to showing that the series of partial sums converges uniformly to the infinite sum. Consider the following

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} (T_n) - \sum_{n=1}^{N-1} (T_n) \right\| &= \left\| \sum_{n=N}^{\infty} (T_n) \right\| \\ &\leq \sum_{n=N}^{\infty} (\|T_n\|) \end{aligned}$$

However since this is part of a convergent sum as N goes to infinity this sum goes to 0, therefore

$$\lim_{N \to \infty} \left(\left\| \sum_{n=1}^{\infty} (T_n) - \sum_{n=1}^{N-1} (T_n) \right\| \right) = 0$$

This shows that the series $\sum_{n=1}^{\infty} (T_n)$ is uniformly convergent. □