Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 10

Section 16.8

#2 Consider the Sturm-Liouville problem

$$u'' + \lambda u = 0 \qquad 0 < x < 1$$

$$u'(0) = u(1) = 0$$

whose eigenvalues are the critical points of

$$J(u) = \frac{\int_0^1 u'(x)\,\mathrm{d}x}{\int_0^1 u(x)^2\,\mathrm{d}x}$$

on the space $H = \{u \in H^1(0,1) : u(1) = 0\}$. Use the Rayleigh-Ritz method to estimate the first two eigenvalues, and compare to the exact values. For best results, choose polynomial trial functions which resemble what the first two eigenfunctions should look like.

#3 Use the result of Exercise 14 in Chapter 14 to give an alternate derivation of the fact the Dirichlet quotient achieves its minimum at ψ_1. (Hint: For $u \in H_0^1(\Omega)$ compute $\|u\|_{H_0^1(\Omega)}^2$ and $\|u\|_{L^2(\Omega)}^2$ by expanding in the eigenfunction basis.)

Proof. Let $\{\psi_n\}$ be a set of eigenfunctions for the Laplacian on Ω be chosen such that $\{\psi_n\}$ form an orthonormal basis for $L^2(\Omega)$. From exercise 14 in chapter 14, we know that the set of functions $\{\psi_n/\sqrt{\lambda_n}\}$ form an orthonormal basis for $H_0^1(\Omega)$. Using these basis we can rewrite u as

$$u = \sum_{n=1}^{\infty} \left(\langle u, \psi_n/\sqrt{\lambda} \rangle_{H_0^1(\Omega)} \psi_n/\sqrt{\lambda_n} \right)$$

and

$$u = \sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)} \psi_n \right)$$

Now the norms $\|u\|_{H_0^1(\Omega)}^2$ and $\|u\|_{L^2(\Omega)}^2$ can be simplified by noting that the norms squared are the sums of the coefficients.

$$\|u\|_{H_0^1(\Omega)}^2 = \left\| \sum_{n=1}^{\infty} \left(\langle u, \psi_n/\sqrt{\lambda_n} \rangle_{H_0^1(\Omega)} \psi_n/\sqrt{\lambda_n} \right) \right\|_{H_0^1(\Omega)}^2$$

$$= \sum_{n=1}^{\infty} \left(\langle u, \psi_n/\sqrt{\lambda_n} \rangle_{H_0^1(\Omega)}^2 \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left(\int_{\Omega} \nabla \psi_n \cdot \nabla u \, \mathrm{d}x \right)^2 \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \left(\lambda_n \int_{\Omega} \psi_n u \, \mathrm{d}x \right)^2 \right)$$

$$= \sum_{n=1}^{\infty} \left(\lambda_n \langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)$$

$$\|u\|_{L^2(\Omega)}^2 = \left\| \sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)} \psi_n \right) \right\|_{L^2(\Omega}^2$$

$$= \sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)$$

Now the value of $J(u)$ can be rewritten using the eigenfunction basis as

$$J(u) = \frac{\sum_{n=1}^{\infty} \left(\lambda_n \langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}{\sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}$$

Since $\lambda_1 \leq \lambda_i$ for all other eigenvalues we can bound the value of $J(u)$.

$$J(u) = \frac{\sum_{n=1}^{\infty} \left(\lambda_n \langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}{\sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}$$

$$\leq \frac{\sum_{n=1}^{\infty} \left(\lambda_1 \langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}{\sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}$$

$$= \lambda_1 \frac{\sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}{\sum_{n=1}^{\infty} \left(\langle u, \psi_n \rangle_{L^2(\Omega)}^2 \right)}$$

$$= \lambda_1$$

Also since the functions ψ_n are orthonormal $J(\psi_1) = \lambda_1$, which implies that $J(\psi_1) \leq J(u)$ for all $u \in H_0^1(\Omega)$ so the Dirichlet quotient does achieve its minimum. $\qquad \square$

#5 Let A be an $m \times n$ real matrix, $b \in \mathbb{R}^m$ and define $J(x) = \|Ax - b\|_2$ for $x \in \mathbb{R}^n$. (Here $\|x\|_2$ denotes the 2 norm, the usual Euclidean distance on $\mathbb{R}^m$)

(a) What is the Euler-Lagrange equation for the problem of minimizing J?

The Euler-Lagrange equation for this problem is

$$\frac{\mathrm{d}}{\mathrm{d}t}(J(x + ty))\Big|_{t=0} = 0$$

for all $y \in \mathbb{R}^n$. This can be simplified as follows.

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}(J(x + ty))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\|A(x + ty) - b\|_2)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{(A(x + ty) - b)^T(A(x + ty) - b)}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{((x + ty)^T A^T - b^T)(A(x + ty) - b)}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{(x + ty)^T A^T A(x + ty) - (x + ty)^T A^T b - b^T A(x + ty) + b^T b}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{(x^T + ty^T)A^T A(x + ty) - (x^T + ty^T)A^T b - b^T Ax - tb^T Ay + b^T b}\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{y^T A^T Ay t^2 + (2x^T A^T Ay - 2y^T A^T b)t + x^T A^T Ax - 2x^T A^T b + b^T b}\right)\Big|_{t=0} \\
&= \frac{1}{2}\frac{2y^T A^T Ay t + 2x^T A^T Ay - 2y^T A^T b}{\sqrt{y^T A^T Ay t^2 + (2x^T A^T Ay - 2y^T A^T b)t + x^T A^T Ax - 2x^T A^T b + b^T b}}\Bigg|_{t=0} \\
&= \frac{1}{2}\frac{2x^T A^T Ay - 2y^T A^T b}{\sqrt{x^T A^T Ax - 2x^T A^T b + b^T b}} \\
&= \frac{x^T A^T Ay - y^T A^T b}{\sqrt{x^T A^T Ax - 2x^T A^T b + b^T b}}
\end{aligned}
$$

So the final Euler-Lagrage equation is

$$\frac{x^T A^T Ay - y^T A^T b}{\sqrt{x^T A^T Ax - 2x^T A^T b + b^T b}} = 0$$

In order for this equation to be satisfied the numerator must be zero, or equivalently

$$y^T A^T Ax = y^T A^T b$$

for all $y \in \mathbb{R}^n$.

(b) Under what circumstances does the Euler-Lagrange equation have a unique solution?

When A has full rank, then the solution to the Euler-Lagrange will have a unique solution. In this case there is a unique solution to $Ax = b$ as well.

(c) Under the circumstances will the solution of the Euler-Lagrange equation also be a solution of $Ax = b$.

The solution to the Euler Lagrange equation will also be a solution to the equation $Ax = b$, when b is in the range of A. That is if a solution to $Ax = b$ exists, then the minimizer of $J(x)$ will be the solution to $Ax = b$, so that $J(x) = 0$.

#13 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $\rho \in C(\overline{\Omega})$, $\rho(x) > 0$ in Ω, and

$$J(u) = \frac{\int_\Omega \|\nabla u(x)\|^2 \, \mathrm{d}x}{\int_\Omega \rho(x) u(x)^2 \, \mathrm{d}x}$$

(a) Show that any nonzero critical point $u \in H_0^1(\Omega)$ of J is a solution of the eigenvalue problem

$$-\Delta u = \lambda \rho(x) u \qquad x \in \Omega$$

$$u = 0 \qquad x \in \partial\Omega$$

Proof. Let $u \in H_0^1(\Omega)$ be a nonzero critical point of J. This means that for any $v \in H_0^1(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(J(u + tv))\Big|_{t=0} = 0$$

This condition can be simplified in many ways. First I will compute the derivatives evaluated at $t = 0$ of the numerator and denominator in order to use the quotient rule.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \|\nabla(u + tv)\|^2 \, \mathrm{d}x\right)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \|\nabla u + t\nabla v\|^2 \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (\nabla u + t\nabla v) \cdot (\nabla u + t\nabla v) \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \nabla u \cdot \nabla u + 2t\nabla u \cdot \nabla v + t^2 \nabla v \cdot \nabla v \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \nabla u \cdot \nabla u \, \mathrm{d}x + 2t\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + t^2\int_\Omega \nabla v \cdot \nabla v \, \mathrm{d}x\right)\Big|_{t=0} \\
&= 2\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + 2t\int_\Omega \nabla v \cdot \nabla v \, \mathrm{d}x\Big|_{t=0} \\
&= 2\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho(x)(u + tv)^2 \, \mathrm{d}x\right)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho(x)(u^2 + 2tuv + t^2 v^2) \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho(x)u^2 \, \mathrm{d}x + 2t\int_\Omega \rho(x)uv \, \mathrm{d}x + t^2\int_\Omega \rho(x)v^2 \, \mathrm{d}x\right)\Big|_{t=0} \\
&= 2\int_\Omega \rho(x)uv \, \mathrm{d}x + 2t\int_\Omega \rho(x)v^2 \, \mathrm{d}x\Big|_{t=0} \\
&= 2\int_\Omega \rho(x)uv \, \mathrm{d}x
\end{aligned}
$$

Now the full derivative of the quotient evaluated at $t = 0$ can be computed as

$$\frac{\mathrm{d}}{\mathrm{d}t}(J(u + tv))\Big|_{t=0} = \frac{2\int_\Omega \rho(x)u(x)^2 \, \mathrm{d}x \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - 2\int_\Omega \rho(x)u(x)v(x) \, \mathrm{d}x \int_\Omega \nabla u \cdot \nabla u \, \mathrm{d}x}{\left(\int_\Omega \rho(x)(u(x))^2 \, \mathrm{d}x\right)^2}$$

The condition that this be equal to zero is equivalent to the numerator being equal to zero. Note that the denominator is strictly positive as $\rho(x)$ is strictly positive and u is nonzero. The

condition can now be simplified to

$$2 \int_\Omega \rho(x)u(x)^2 \, dx \int_\Omega \nabla u \cdot \nabla v \, dx = 2 \int_\Omega \rho(x)u(x)v(x) \, dx \int_\Omega \nabla u \cdot \nabla u \, dx$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \frac{\int_\Omega \rho(x)u(x)^2 \, dx}{\int_\Omega \nabla u \cdot \nabla u \, dx} \int_\Omega \rho(x)u(x)v(x) \, dx$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = J(u) \int_\Omega \rho(x)u(x)v(x) \, dx$$

This last statement is exactly the weak formulation of the following PDE

$$-\Delta u = J(u)\rho(x)u$$

This shows that the critical points of J are solutions to the eigenvalue problem where $J(u)$ is the eigenvalue. □

(b) Show that the eigenvalues are positive.

Proof. As was shown in part (a) the eigenvalues of this problem are all values of the weighted Dirichlet quotient J. Therefore if $J(u)$ is positive for all $u \in H_0^1(\Omega)$, then this shows that all the eigenvalues of this problem must be positive. Since $\rho(x) > 0$ and $u(x) \neq 0$, this implies that

$$\int_\Omega \rho(x)u(x)^2 \, dx > 0$$

and

$$\|u\|_{L^2(\Omega)}^2 = \int_\Omega u(x)^2 \, dx > 0$$

Therefore the denominator of $J(u)$ is always positive. Now consider the numerator of $J(u)$, which is $\|u\|_{H_0^1(\Omega)}^2$. By the Poincare inequality it is known that

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}$$

for some $C > 0$. This implies that

$$\|u\|_{H_0^1(\Omega)} \geq \frac{1}{C}\|u\|_{L^2(\Omega)} > 0$$

This shows that the numerator is also greater then zero, so $J(u) > 0$ for all $u \in H_0^1(\Omega)$. So all the eigenvalues of this problem are also positive. □

(c) If $\rho(x) \geq 1$ in Ω and λ_1 denotes the smallest eigenvalue, show that $\lambda_1 < \lambda_1^*$ where λ_1^* is the corresponding first eigenvalue of $-\Delta$ in Ω.

Proof. I believe that this question should ask to show that $\lambda_1 \leq \lambda_1^*$ as if $\rho(x) = 1$, then $\lambda_1 = \lambda_1^*$. We have previously shown that

$$\lambda_1^* = \min_{u \in H_0^1(\Omega)} \left\{ \frac{\int_\Omega \|\nabla u(x)\|^2 \, dx}{\int_\Omega u(x)^2 \, dx} \right\}$$

and part (a) showed that

$$\lambda_1 = \min_{u \in H_0^1(\Omega)} \left\{ \frac{\int_\Omega \|\nabla u(x)\|^2 \, dx}{\int_\Omega \rho(x)u(x)^2 \, dx} \right\}$$

Now note that if $\rho(x) \geq 1$ then

$$\int_\Omega \rho(x)u(x)^2 \,\mathrm{d}x \geq \int_\Omega u(x)^2 \,\mathrm{d}x$$

for a given $u \in H_0^1(\Omega)$. If the denominator is larger, this implies that the quotient is smaller or that

$$\frac{\int_\Omega \|\nabla u(x)\|^2 \,\mathrm{d}x}{\int_\Omega \rho(x)u(x)^2 \,\mathrm{d}x} \leq \frac{\int_\Omega \|\nabla u(x)\|^2 \,\mathrm{d}x}{\int_\Omega u(x)^2 \,\mathrm{d}x}$$

Minimizing both sides of this inequality shows that

$$\lambda_1 \leq \lambda_1^*$$

$\square$

#14 Define
$$J(u) = \frac{1}{2} \int_\Omega (\Delta u)^2 \, \mathrm{d}x + \int_\Omega fu \, \mathrm{d}x$$

What PDE problem is satisfied by a critical point of J over $\chi = H^2(\Omega) \cap H_0^1(\Omega)$? Make sure to specify any relevant boundary conditions. What is different if instead we let $\chi = H_0^2(\Omega)$?

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a critical point of $J(u)$. This implies that for every $v \in C_0^\infty(\Omega)$,
$$\frac{\mathrm{d}}{\mathrm{d}t}(J(u+tv))\Big|_{t=0} = 0.$$

This condition can be simplified as follows.

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}(J(u+tv))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega (\Delta(u+tv))^2 \, \mathrm{d}x + \int_\Omega f(u+tv) \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega (\Delta u + t\Delta v)^2 \, \mathrm{d}x + \int_\Omega fu + tfv \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega (\Delta u)^2 + 2t\Delta u \Delta v + t^2 (\Delta u)^2 \, \mathrm{d}x + \int_\Omega fu \, \mathrm{d}x + t \int_\Omega fv \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega (\Delta u)^2 \, \mathrm{d}x + t\int_\Omega \Delta u \Delta v \, \mathrm{d}x + \frac{1}{2}t^2 \int_\Omega (\Delta u)^2 \, \mathrm{d}x + \int_\Omega fu \, \mathrm{d}x + t \int_\Omega fv \, \mathrm{d}x\right)\Big|_{t=0} \\
&= \int_\Omega \Delta u \Delta v \, \mathrm{d}x + t \int_\Omega (\Delta u)^2 \, \mathrm{d}x + \int_\Omega fv \, \mathrm{d}x\Big|_{t=0} \\
&= \int_\Omega \Delta u \Delta v \, \mathrm{d}x + \int_\Omega fv \, \mathrm{d}x
\end{aligned}$$

This condition is equivalent to
$$\int_\Omega \Delta u \Delta v \, \mathrm{d}x = -\int_\Omega fv \, \mathrm{d}x$$

This can be simplified by integrating by parts.

$$\int_\Omega \Delta u \Delta v \, \mathrm{d}x = -\int_\Omega fv \, \mathrm{d}x$$
$$-\int_\Omega \nabla u \cdot \nabla \Delta v \, \mathrm{d}x + \frac{\partial u}{\partial n}\Delta v\Big|_{\partial\Omega} = -\int_\Omega fv \, \mathrm{d}x$$
$$-\int_\Omega u\Delta^2 v \, \mathrm{d}x + \frac{\partial u}{\partial n}\Delta v\Big|_{\partial\Omega} = -\int_\Omega fv \, \mathrm{d}x$$
$$\int_\Omega u\Delta^2 v \, \mathrm{d}x - u\frac{\partial \Delta v}{\partial n}\Big|_{\partial\Omega} + \frac{\partial u}{\partial n}\Delta v\Big|_{\partial\Omega} = -\int_\Omega fv \, \mathrm{d}x$$

This last statement is equivalent to the following PDE problem in the distributional sense.
$$-\Delta^2 u = f$$
$$u = \frac{\partial u}{\partial n} = 0 \qquad x \in \partial\Omega$$

When $u \in H_0^2(\Omega)$ instead the partial differential equation is the same, but the boundary conditions change. $\qquad \square$