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MATH 520 Methods of Applied Math II
Homework 4

Section 11.4

#6 Let $\mathbf{H} = L^2(0,1)$ and $T_1 u = T_2 u = iu'$ with domains

$$D(T_1) = \left\{ u \in H^1(0,1) : u(0) = u(1) \right\}$$
$$D(T_2) = \left\{ u \in H^1(0,1) : u(0) = u(1) = 0 \right\}$$

Show that T_1 is self-adjoint, and that T_2 is closed and symmetric but not self-adjoint. What is T_2^*?

Proof. First I will show that T_1 is self-adjoint. Let $v \in D(T_1)$, then (v, iv') is an admissable pair for T_1^*. To see this note that

$$\begin{aligned}
\langle T_1 u, v \rangle &= \int_0^1 iu'(x)\overline{v(x)}\,\mathrm{d}x \\
&= iu(x)\overline{v(x)}\Big|_{x=0}^1 - \int_0^1 iu(x)\overline{v'(x)}\,\mathrm{d}x \\
&= iu(x)\overline{v(x)}\Big|_{x=0}^1 + \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x \\
&= iu(1)\overline{v(1)} - iu(0)\overline{v(0)} + \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x
\end{aligned}$$

Since $u(0) = u(1)$ and $v(0) = v(1)$.

$$\begin{aligned}
&= \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x \\
&= \langle u, T_1 v \rangle
\end{aligned}$$

This shows that $D(T_1) \subset D(T_1^*)$ and that $T_1 u = T_1^* u = iu'$ for $u \in D(T)$.

Now let $v \in D(T_1^*)$, if we can show that $v \in D(T)$ and $T_1^* v = iv'$, then we know that $T_1 = T_1^*$. Since $v \in D(T_1^*)$ then there exists $g \in L^2(0,1)$ such that (v, g) is an admissable pair for T_1^* and $\int_0^1 g(x)\,\mathrm{d}x = 0$ Now by definition $T_1^* v = g$. Also since (v, g) is an admissable pair,

$$\langle Tu, v \rangle = \langle u, g \rangle$$

for all $u \in D(T)$. Next I will define the following function

$$G(x) = \int_0^x g(y)\,\mathrm{d}y + \alpha$$

where

$$\alpha = i\int_0^1 v(s)\,\mathrm{d}s - \int_0^1 \int_0^s g(y)\,\mathrm{d}y\,\mathrm{d}s.$$

Since $v, g \in L^2(0,1)$, this function is well-defined. Note that by the Fundamental Theorem of Calculus for L^2 functions, $G'(x) = g(x)$. Also note that since $\int_0^1 g(x)\,\mathrm{d}x = 0$,

$$G(0) = \int_0^0 g(y)\,\mathrm{d}y + \alpha = \alpha = \int_0^1 g(y)\,\mathrm{d}y + \alpha = G(1)$$

Now reconsider the inner product $\langle u, g \rangle$.

$$\begin{aligned}\langle u, g \rangle &= \int_0^1 u(x)\overline{g(x)} \, \mathrm{d}x \\ &= \int_0^1 u(x)\overline{G'(x)} \, \mathrm{d}x \\ &= u(x)G(x)|_{x=0}^1 - \int_0^1 u'(x)\overline{G(x)} \, \mathrm{d}x \\ &= (u(1)G(1) - u(0)G(0)) - \int_0^1 u'(x)\overline{G(x)} \, \mathrm{d}x\end{aligned}$$

Since $u \in D(T)$, $u(0) = u(1)$ and as shown before $G(0) = G(1)$, the first term is zero.

$$\langle u, g \rangle = -\int_0^1 u'(x)\overline{G(x)} \, \mathrm{d}x \qquad = -\langle u', G \rangle$$

Now using the definition of admissible pair it is possible to see that

$$\langle Tu, v \rangle = \langle u, g \rangle = -\langle u', G \rangle$$

for all $u \in D(T)$. Equivalently this is

$$\begin{aligned}\int_0^1 iu'(x)\overline{v(x)} \, \mathrm{d}x &= -\int_0^1 u'(x)\overline{G(x)} \, \mathrm{d}x \\ \int_0^1 u'(x)\overline{G(x) - iv(x)} \, \mathrm{d}x &= 0\end{aligned}$$

Since this is true for any $u \in D(T_1)$ this is true in particular for

$$u(x) = \int_0^x G(y) - iv(y) \, \mathrm{d}y$$

In order to verify that $u \in D(T_1)$, note that $u(x) \in L^2(0,1)$ and that $u'(x) = G(x) - iv(x) \in L^2(0,1)$. Also $u(0) = 0$ and

$$\begin{aligned}u(1) &= \int_0^1 G(y) - iv(y) \, \mathrm{d}y \\ &= \int_0^1 \int_0^y g(s) \, \mathrm{d}s + \alpha - iv(y) \, \mathrm{d}y \\ &= \alpha - i\int_0^1 v(y) \, \mathrm{d}y + \int_0^1 \int_0^y g(s) \, \mathrm{d}s \, \mathrm{d}y\end{aligned}$$

Substituting in for α

$$\begin{aligned}&= i\int_0^1 v(y) \, \mathrm{d}y - \int_0^1 \int_0^y g(s) \, \mathrm{d}s \, \mathrm{d}y - i\int_0^1 v(y) \, \mathrm{d}y + \int_0^1 \int_0^y g(s) \, \mathrm{d}s \, \mathrm{d}y \\ &= 0\end{aligned}$$

Now using this function we see that

$$\begin{aligned}\int_0^1 |G(x) - iv(x)|^2 \, \mathrm{d}x &= 0 \\ \|G - iv\| &= 0 \\ G - iv &= 0 \\ G &= iv\end{aligned}$$

Since $G' = g \in L^2(0,1)$ is differentiable this implies that $v' = -iG' \in L^2(0,1)$. Also $v(0) = -iG(0) = -iG(1) = v(1)$ because $G(0) = G(1)$. This shows that $v \in D(T_1)$ and that $T_1^* v = g = iv'$. Therefore $T_1^* = T_1$.

Next I will consider T_2. To see that T_2 is symmetric, let $u, v \in D(T^2)$.

$$\begin{aligned}\langle T_2 u, v\rangle &= \int_0^1 iu'(x)\overline{v(x)}\,\mathrm{d}x \\ &= iu(x)\overline{v(x)}\Big|_{x=0}^1 - \int_0^1 iu(x)\overline{v'(x)}\,\mathrm{d}x\end{aligned}$$

Since $u(0) = u(1) = 0$ and $v(0) = v(1) = 0$

$$\begin{aligned}&= 0 - \int_0^1 iu(x)\overline{v'(x)}\,\mathrm{d}x \\ &= \int_0^1 u(x)\overline{iv'(x)}\,\mathrm{d}x \\ &= \langle u, T_2 v\rangle\end{aligned}$$

Thus T_2 is symmetric.

To see that T_2 is closed let $u_n \in D(T_2)$ such that $u_n \to u \in L^2(0,1)$ and $T_2 u_n \to v \in L^2(0,1)$. Since $u_n(0) = u_n(1) = 0$ for all n this implies that $u(0) = u(1) = 0$. Also since $T_2 u_n \to v$, this implies that $iu_n' \to v$ Therefore $u_n' \to -iv$ which shows that $u' = -iv$, So $u \in D(T_2)$ and $T_2 u = iu' = v$ and T_2 is closed. $\square$

#7 If T is symmetric with $R(T) = \mathbf{H}$ show that T is self-adjoint.

Proof. Let T be symmetric with $R(T) = \mathbf{H}$. Let $u, v \in D(T)$, then

$$\langle Tu, v \rangle = \langle u, Tv \rangle.$$

This shows that (v, Tv) is an admissable pair for T^* and that $T^*v = Tv$. Thus for any $v \in D(T)$, $v \in D(T^*)$, which shows that $D(T) \subset D(T^*)$.

Now let $v \in D(T^*)$, so that there exists $v^* \in \mathbf{H}$ such that (v, v^*) is an admissable pair for T^*. Also since $v^* \in \mathbf{H}$, there exists some $w \in D(T)$ such that $Tw = v^*$. Now using the definition of admissible pair

$$\begin{aligned} \langle Tu, v \rangle &= \langle u, v^* \rangle \\ &= \langle u, Tw \rangle \\ &= \langle Tu, w \rangle \end{aligned}$$

for all $u \in D(T)$. This implies that

$$\begin{aligned} \langle Tu, v - w \rangle &= 0 \\ v - w &\perp R(T) = \mathbf{H} \\ v - w &= 0 \\ v &= w \end{aligned}$$

This shows that $v \in D(T)$ and that $Tv = v^* = T^*v$. Thus $D(T) = D(T^*)$ and $T = T^*$, so T is self-adjoint. $\square$

#16 We say that a linear operator on a Hilbert space $\mathbf{H}$ is bounded below if there exists a constant $c_0 > 0$ such that

$$\langle Tu, u\rangle \geq -c_0\|u\|^2 \quad \forall u \in D(T)$$

Show that Theorem 11.6 remains valid if the condition that T be positive is replaced by the assumption that T is bounded below.

Proof. Let T be densely defined, symmetric, and bounded below by $c_0 > 0$. Consider the operator $T + c_0 I$, where I is the identity operator on $\mathbf{H}$. Note that $D(T + c_0 I) = D(T)$ so that $T + c_0 I$ is densely defined. Also $T + c_0 I$ is positive. To see this let $u \in D(T + c_0 I)$, then

$$\begin{aligned}
\langle (T + c_0 I)u, u\rangle &= \langle Tu + c_0 u, u\rangle \\
&= \langle Tu, u\rangle + c_0\langle u, u\rangle \\
&\geq -c_0\|u\|^2 + c_0\|u\|^2 \\
&= 0
\end{aligned}$$

The operator $T + c_0 I$ is also symmetric, to show this let $u, v \in D(T + c_0 I)$, then

$$\begin{aligned}
\langle (T + c_0 I)u, v\rangle &= \langle Tu + c_0 u, v\rangle \\
&= \langle Tu, v\rangle + c_0\langle u, v\rangle.
\end{aligned}$$

Since T is symmetric

$$\langle (T + c_0 I)u, v\rangle = \langle u, Tv\rangle + c_0\langle u, v\rangle.$$

Also since $c_0 > 0$, $c_0 \in \mathbb{R}$, so

$$\begin{aligned}
\langle (T + c_0 I)u, v\rangle &= \langle u, Tv\rangle + \langle u, c_0 v\rangle \\
&= \langle u, Tv + c_0 v\rangle \\
&= \langle u, (T + c_0 I)v\rangle.
\end{aligned}$$

Since $T + c_0 I$ is densely defined, positive, and symmetric, by Theorem 11.6 there exists a positive self-adjoint extension, S, of $T + c_0 I$. Next I will define the operator $R = S - c_0 I$ and claim that this is a bounded below self adjoint extension of T. First note that $D(R) = D(S) \supset D(T + c_0 I) = D(T)$. Let $u \in D(T)$, then

$$\begin{aligned}
Ru &= (S - c_0 I)u \\
&= Su - c_0 u
\end{aligned}$$

Since $u \in D(T + c_0 I)$ and S is an extension of $T + c_0 I$

$$\begin{aligned}
Ru &= (T + c_0 I)u - c_0 u \\
&= Tu + c_0 u - c_0 u \\
&= Tu
\end{aligned}$$

This shows that R is an extension of T.

Also R is bounded below, to see this let $u \in D(R)$, then

$$\begin{aligned}
\langle Ru, u\rangle &= \langle (S - c_0 I)u, u\rangle \\
&= \langle Su, u\rangle - c_0\langle u, u\rangle
\end{aligned}$$

Since S is positive

$$\langle Ru, u \rangle \geq -c_0 \|u\|^2$$

Finally I will show that R is self-adjoint. Since S and I are self-adjoint.

$$R^* = (S - c_0 I)^* = S^* - c_0 I^* = S - c_0 I = R$$

Therefore for any densely defined, bounded below, symmetric operator there exists a bounded below self-adjoint extension. $\square$

Section 12.4

#3 Recall that the resolvent operator of T is defined to be $R_\lambda = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$.

(a) Prove the resolvant identity (12.1.3).

Proof. Let $\lambda, \mu \in \rho(T)$. Note that $R_\lambda^{-1} = \lambda I - T$ and $R_\mu^{-1} = \mu I - T$ are both defined and $R_\lambda R_\lambda^{-1} = R_\mu R_\mu^{-1} = I$.

$$\begin{aligned}
R_\lambda - R_\mu &= R_\lambda I - I R_\mu \\
&= R_\lambda R_\mu^{-1} R_\mu - R_\lambda R_\lambda^{-1} R_\mu \\
&= R_\lambda \left(R_\mu^{-1} R_\mu - R_\lambda^{-1} R_\mu \right) \\
&= R_\lambda \left(R_\mu^{-1} - R_\lambda^{-1} \right) R_\mu \\
&= R_\lambda ((\mu I - T) - (\lambda I - T)) R_\mu \\
&= R_\lambda (\mu I - T - \lambda I + T) R_\mu \\
&= R_\lambda (\mu I - \lambda I) R_\mu \\
&= R_\lambda (\mu - \lambda) I R_\mu \\
&= (\mu - \lambda) R_\lambda R_\mu
\end{aligned}$$

This shows that

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

for all $\lambda, \mu \in \rho(T)$. □

(b) Deduce from this that R_λ and R_μ commute.

Proof. Let $\lambda, \mu \in \mathbb{C}$. If $\lambda = \mu$, then $R_\lambda = R_\mu$ so

$$R_\lambda R_\mu = R_\lambda^2 = R_\mu R_\lambda.$$

In this case R_λ and R_μ commute trivially. Now let $\lambda \neq \mu$, in this case the resolvant identity states that

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\lambda - \mu}.$$

Now consider the following

$$\begin{aligned}
R_\lambda R_\mu &= \frac{R_\lambda - R_\mu}{\lambda - \mu} \\
&= \frac{R_\mu - R_\lambda}{\mu - \lambda} \\
&= R_\mu R_\lambda
\end{aligned}$$

This shows that R_λ and R_μ commute. □

(c) Show also that T and R_λ commute for $\lambda \in \rho(T)$.

Proof. Let $\lambda \in \rho(T)$, so that $R_\lambda^{-1} = \lambda I - T$ is well-defined.

$$\begin{aligned}
TR_\lambda &= ITR_\lambda \\
&= R_\lambda R_\lambda^{-1} T R_\lambda \\
&= R_\lambda(\lambda I - T)TR_\lambda \\
&= R_\lambda(\lambda T - T^2)R_\lambda \\
&= R_\lambda T(\lambda I - T)R_\lambda \\
&= R_\lambda T R_\lambda^{-1} R_\lambda \\
&= R_\lambda T
\end{aligned}$$

This shows that T and R_λ commute for $\lambda \in \rho(T)$. □

#4 Show that $\lambda \to R_\lambda$ is continuously differentiable, regarded as a mapping from $\rho(T) \subset \mathbb{C}$ into $\mathcal{B}(\mathbf{H})$, with

$$\frac{\mathrm{d}R_\lambda}{\mathrm{d}\lambda} = -R_\lambda^2$$

Proof. First I will show that this mapping is continous. Let $\lambda_n \in \rho(T)$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Consider

$$\begin{aligned} \lim_{n \to infty}(\|R_{\lambda_n} - R_\lambda\|) &= \lim_{n\to\infty}(\|(\lambda - \lambda_n)R_{\lambda_n}R_\lambda\|) \\ &\leq \lim_{n\to\infty}(\langle\lambda_n - \lambda\rangle\|R_{\lambda_n}\|\|R_\lambda\|) \end{aligned}$$

Since R_λ and R_{λ_n} are bounded

$$\lim_{n \to infty}(\|R_{\lambda_n} - R_\lambda\|) = 0$$

Therefore R_λ is continuous when seen as a function of λ.

Now consider the derivative of R_λ with respect to λ.

$$\begin{aligned} \frac{\mathrm{d}R_\lambda}{\mathrm{d}\lambda} &= \lim_{n\to\infty}\left(\frac{R_{\lambda_n} - R_\lambda}{\lambda_n - \lambda}\right) \\ &= \lim_{n\to\infty}\left(\frac{(\lambda - \lambda_n)R_\lambda R_{\lambda_n}}{\lambda_n - \lambda}\right) \\ &= \lim_{n\to\infty}(-R_\lambda R_{\lambda_n}) \end{aligned}$$

Since R_λ is continuous

$$= -R_\lambda^2$$

Finally note that $-R_\lambda^2$ is continuous as a function of λ because negation and squaring are continuous operations. $\square$