Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 12

Section 16.8

#20 Show that the Fréchet derivative, if it exists, must be unique.

Proof. Let X, Y be Banach spaces and let $F : D(F) \subset X \to Y$. Now suppose that $A_1, A_2 \in B(X, Y)$ exist such that $A_1 \neq A_2$ and they both are the Fréchet derivative of F at some $x_0 \in D(F)$. This means that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_1(x - x_0)\|}{\|x - x_0\|} = 0$$

and

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_2(x - x_0)\|}{\|x - x_0\|} = 0$$

$\square$

#21 If $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

show that F is Gâteaux differentiable but not Fréchet differentiable at the origin.

The Gâteaux derivative of F at the origin can be computed as follows.

$$\begin{aligned} DF(0,0)(u,v) &= \frac{\mathrm{d}}{\mathrm{d}t}(F(0+tu, 0+tv))\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tu(tv)^2}{(tu)^2+(tv)^4}\right)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tuv^2}{u^2+t^2v^4}\right)\bigg|_{t=0} \\ &= \frac{(u^2+t^2v^4)uv^2 + tuv^2(2v^4t)}{(u^2+t^2v^4)^2}\bigg|_{t=0} \\ &= \frac{u^3v^2}{u^4} \\ &= \frac{v^2}{u} \end{aligned}$$

This shows that the Gâteaux derivative of F is $A(u, v) = \frac{v^2}{u}$

Now we will consider the Fréchet derivative of F at $(0, 0)$. If the Fréchet derivative exists, then $A \in B(X, Y)$ will exist such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0.$$

This can be simplified by noting that $x_0 = (0, 0)$ and using the definition of F.

$$\lim_{(u,v)\to(0,0)} \frac{\left|\frac{uv^2}{u^2+v^4} - A(u, v)\right|}{\sqrt{u^2+v^2}} = 0.$$

#27 Let X, Y be Banach spaces, $F : D(F) \subset X \to Y$, and let $x, x_0 \in D(F)$ be such that $tx + (1-t)x_0 \in D(F)$ for $t \in [0,1]$. If

$$M := \sup_{0 \le t \le 1} \|DF(tx + (1-t)x_0)\|$$

show that

$$\|F(x) - F(x_0)\| \le M\|x - x_0\|$$

(Suggestion: justify and use a suitable version of the fundamental theorem of calculus.)

Section 17.5

#2 Let λ_1 be the smallest Dirichlet eigenvalue for $-\Delta$ in Ω, assume that $c \in C(\overline{\Omega})$ and $c(x) > -\lambda_1$ in $\overline{\Omega}$. If $f \in L^2(\Omega)$ prove the existence of a solution of

$$-\Delta u + c(x)u = f \quad x \in \Omega \qquad u = 0 \quad \forall x \in \partial\Omega$$

Proof. Even though it isn't stated I will operate in the space $H_0^1(\Omega)$ as this is the natural Hilbert space for the Dirichlet Laplacian. First I will rewrite this PDE in weak form as

$$\int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in H_0^1(\Omega)$. Next I will define the following function

$$A[u,v] = \int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x.$$

To see that this function is bilinear, note that

$$\begin{aligned}
A[u_1 + u_2, v] &= \int_\Omega \nabla(u_1 + u_2)\nabla v + c(x)(u_1 + u_2)v \,\mathrm{d}x \\
&= \int_\Omega \nabla u_1 \nabla v + \nabla u_2 \nabla v + c(x)u_1 v + c(x)u_2 v \,\mathrm{d}x \\
&= \int_\Omega \nabla u_1 \nabla v + c(x)u_1 v \,\mathrm{d}x + \int_\Omega \nabla u_2 \nabla v + c(x)u_2 v \,\mathrm{d}x \\
&= A[u_1, v] + A[u_2, v]
\end{aligned}$$

and that

$$\begin{aligned}
A[u, v_1 + v_2] &= \int_\Omega \nabla u \nabla(v_1 + v_2) + c(x)u(v_1 + v_2) \,\mathrm{d}x \\
&= \int_\Omega \nabla u \nabla v_1 + \nabla u \nabla v_2 + c(x)uv_1 + c(x)uv_2 \,\mathrm{d}x \\
&= \int_\Omega \nabla u \nabla v_1 + c(x)uv_1 \,\mathrm{d}x + \int_\Omega \nabla u \nabla v_2 + c(x)uv_2 \,\mathrm{d}x \\
&= A[u, v_1] + A[u, v_2].
\end{aligned}$$

This shows that A is bilinear.

Next I will show that A is bounded.

$$A[u,v] =$$

Finally I will show that A is coercive.

$$A[u,u] =$$

Now Lax-Milgram's Theorem states that there exists a unique solution to

$$\int_\Omega \nabla u \nabla v + c(x)uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x$$

for all $v \in H_0^1(\Omega)$. This also implies that there is a weak solution to

$$-\Delta u + c(x)u = f \quad x \in \Omega \qquad u = 0 \quad \forall x \in \partial\Omega$$

$\square$

#3 Let $\lambda > 0$ and define

$$A[u,v] = \int_{\Omega} a_{jk}(x) u_{x_k}(x) v_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} uv \, \mathrm{d}x$$

for all $u, v \in H^1(\Omega)$. Assume the ellipticity property (17.1.3) and that $a_{jk} \in L^{\infty}(\Omega)$. If $f \in L^2(\Omega)$ show that there exists a unique solution of

$$u \in H^1(\Omega) \qquad A[u,v] = \int_{\Omega} f v \, \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

Justify that u may be regarded as the weak solution of

$$-(a_{jk} u_{x_k})_{x_j} + \lambda u = f(x) \quad x \in \Omega \qquad a_{jk} u_{x_k} n_j = 0 \quad x \in \partial\Omega$$

The above boundary condition is said to be of conormal type.

Proof. Lax-Milgram's Theorem can be used to show that there exists a unique solution to

$$u \in H^1(\Omega) \qquad A[u,v] = \int_{\Omega} f v \, \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

First I will show that A is bilinear.

$$\begin{aligned}
A[u_1 + u_2, v] &= \int_{\Omega} a_{jk}(x)(u_1 + u_2)_{x_k}(x) v_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} (u_1 + u_2) v \, \mathrm{d}x \\
&= \int_{\Omega} a_{jk}(x)(u_1)_{x_k}(x) v_{x_j}(x) + a_{jk}(x)(u_2)_{x_k}(x) v_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} u_1 v + u_2 v \, \mathrm{d}x \\
&= \int_{\Omega} a_{jk}(x)(u_1)_{x_k}(x) v_{x_j}(x) \, \mathrm{d}x + \int_{\Omega} a_{jk}(x)(u_2)_{x_k}(x) v_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} u_1 v \, \mathrm{d}x + \lambda \int_{\Omega} u_2 v \, \mathrm{d}x \\
&= A[u_1, v] + A[u_2, v] \\
A[u, v_1 + v_2] &= \int_{\Omega} a_{jk}(x) u_{x_k}(x)(v_1 + v_2)_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} u(v_1 + v_2) \, \mathrm{d}x \\
&= \int_{\Omega} a_{jk}(x) u_{x_k}(x)(v_1)_{x_j}(x) + a_{jk}(x) u_{x_k}(x)(v_2)_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} uv_1 + uv_2 \, \mathrm{d}x \\
&= \int_{\Omega} a_{jk}(x) u_{x_k}(x)(v_1)_{x_j}(x) \, \mathrm{d}x + \int_{\Omega} a_{jk}(x) u_{x_k}(x)(v_2)_{x_j}(x) \, \mathrm{d}x + \lambda \int_{\Omega} uv_1 \, \mathrm{d}x + \lambda \int_{\Omega} uv_2 \, \mathrm{d}x \\
&= A[u, v_1] + A[u, v_2]
\end{aligned}$$

This shows that A is bilinear.

Next I will show that A is bounded.

$$A[u,v] =$$

Lastly I will show that A is coercive.

$$A[u,u] =$$

Now Lax-Milgram's Theorem states that there exists a unique $u \in H^1(\Omega)$ such that

$$A[u,v] = \int_{\Omega} f v \, \mathrm{d}x \quad \forall v \in H^1(\Omega).$$

□

#6 Let f and g be in $L^2(0,1)$. Use the Lax-Milgram Theorem to prove there is a unique weak solution $\{u, v\} \in H_0^1(0,1)$ to

$$\begin{aligned} -u'' + u + v' &= f \\ -v'' + v + u' &= g, \end{aligned}$$

where $u(0) = v(0) = 0$ and $u(1) = v(1) = 0$. (Hint: Start by defining the bilinear form

$$A[(u,v),(\phi,\psi)] = \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x$$

on $H_0^1(0,1) \times H_0^1(0,1)$.

Proof. First I will rewrite this system of PDEs in weak form.

$$\int_0^1 -u''\phi + u\phi + v'\phi - v''\psi + v\psi + u'\psi \,\mathrm{d}x = \int_0^1 f\phi + g\psi \,\mathrm{d}x$$

for all $\phi, \psi \in H_0^1(0,1)$. Integrating by parts were necessary gives

$$\int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x = \int_0^1 f\phi + g\psi \,\mathrm{d}x.$$

Now I will define the following bilinear function

$$A[(u,v),(\phi,\psi)] = \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi \,\mathrm{d}x.$$

This function is bilinear because differentiation and integration are both linear operations. To verify this note that

$$\begin{aligned} A[(u_1+u_2, v_1+v_2),(\phi,\psi)] &= \int_0^1 (u_1+u_2)'\phi' + (u_1+u_2)\phi + (v_1+v_2)'\phi + (v_1+v_2)'\psi' + (v_1+v_2)\psi + (u_1+u_2) \\ &= \int_0^1 u_1'\phi + u_2'\phi' + u_1\phi + u_2\phi + v_1'\phi + v_2'\phi + v_1'\psi + v_2'\psi' + v_1\psi + v_2\psi + u_1'\psi + u_2'\psi \,\mathrm{d} \\ &= \int_0^1 u_1'\phi + u_1\phi + v_1'\phi + v_1'\psi + v_1\psi + u_1'\psi \,\mathrm{d}x + \int_0^1 u_2'\phi' + u_2\phi + v_2'\phi + v_2'\psi' + v_2\psi + \\ &= A[(u_1,v_1),(\phi,\psi)] + A[(u_2,v_2),(\phi,\psi)] \end{aligned}$$

and the same can be shown for the second argument.

Next I will show that A is bounded.

$$A[(u,v),(\phi,\psi)]$$

Lastly I will show that A is coercive. Let $u, v \in H_0^1(0,1)$, then

$$\begin{aligned} A[(u,v),(u,v)] &= \int_0^1 (u')^2 + u^2 + uv' + (v')^2 + v^2 + u'v \,\mathrm{d}x \\ &= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x + \int_0^1 uv' \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x + \int_0^1 u'v \,\mathrm{d}x \end{aligned}$$

Integrating by parts

$$\begin{aligned} &= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x - \int_0^1 u'v \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x + \int_0^1 u'v \,\mathrm{d}x \\ &= \int_0^1 (u')^2 \,\mathrm{d}x + \int_0^1 u^2 \,\mathrm{d}x + \int_0^1 (v')^2 \,\mathrm{d}x + \int_0^1 v^2 \,\mathrm{d}x \\ &= \|u\|_{H_0^1}^2 + \|u\|_{L^2}^2 + \|v\|_{H_0^1}^2 + \|v\|_{L^2}^2 \end{aligned}$$

$\square$