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MATH 520 Methods of Applied Math II
Homework 11

Section 16.8

#6 Prove the version of the Poincaré inequality stated in Proposition 16.1. (Suggestions: If no such C exists show that we can find sequence $u_k \in H^1_*(\Omega)$ with $\|u_k\|_{L^2(\Omega)} = 1$ such that $\|\nabla u_k\|_{L^2(\Omega)} < \frac{1}{k}$. Using Rellich's theorem obtain a convergent subsequence whose limit must have contradictory properties.

Proof. Suppose to the contrary that no such C exists, such that

$$\|u\|_{L^2(\Omega)} < C\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H^1_*(\Omega).$$

This implies that for every $k \in \mathbb{N}$ there exists some $v_k \in H^1_*(\Omega)$ such that

$$\|v_k\|_{L^2(\Omega)} \geq k\|\nabla v_k\|_{L^2(\Omega)}$$

Note that this inequality still holds for $u_k = v_k/\|v_k\|_{L^2(\Omega)}$ as

$$\begin{aligned}
\|u_k\|_{L^2(\Omega)} &\geq k\|\nabla u_k\|_{L^2(\Omega)} \\
\left\|v_k/\|v_k\|_{L^2(\Omega)}\right\|_{L^2(\Omega)} &\geq k\left\|\nabla v_k/\|v_k\|_{L^2(\Omega)}\right\|_{L^2(\Omega)} \\
\frac{1}{\|v_k\|_{L^2(\Omega)}}\|v_k\|_{L^2(\Omega)} &\geq \frac{1}{\|v_k\|_{L^2(\Omega)}}k\|\nabla v_k\|_{L^2(\Omega)} \\
\|v_k\|_{L^2(\Omega)} &\geq k\|\nabla v_k\|_{L^2(\Omega)}.
\end{aligned}$$

This now shows that

$$\|u_k\|_{L^2(\Omega)} = 1$$

and that

$$\|\nabla u_k\|_{L^2(\Omega)} \leq \frac{1}{k}.$$

We can now compute the $H^1(\Omega)$ norm of u_k to be

$$\begin{aligned}
\|u\|_{H^1(\Omega)} &= \sqrt{\|u_k\|^2_{L^2(\Omega)} + \|\nabla u_k\|^2_{L^2(\Omega)}} \\
&\leq \sqrt{1 + \frac{1}{k^2}} \\
&\leq \sqrt{2}
\end{aligned}$$

This shows that the set $\{u_k\}$ is bounded in $H^1(\Omega)$. Since $\{u_k\}$ is bounded in $H^1(\Omega)$, there exists a weakly convergent subsequence $\left\{u_{k_j}\right\}$.

Now assuming that Ω has smooth enough boundaries, the Rellich-Kondrachov theorem as well as the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ implies that $\left\{u_{k_j}\right\}$ is strongly convergent in $L^2(\Omega)$. Let $u \in L^2(\Omega)$ be the limit of the subsequence $\left\{u_{k_j}\right\}$, that is $u_{k_j} \to u$ in $L^2(\Omega)$. Note that this also implies that $u_{k_j} \xrightarrow{w} u$ in $L^2(\Omega)$. Therefore

$$\left\langle u_{k_j}, 1\right\rangle_{L^2(\Omega)} \to \langle u, 1\rangle_{L^2(\Omega)} = \int_\Omega u \, \mathrm{d}x$$

but note that since $u_{k_j} \in H^1_*(\Omega)$ as well

$$\left\langle u_{k_j}, 1 \right\rangle_{L^2(\Omega)} = \int_\Omega u_{k_j} \, \mathrm{d}x = 0.$$

This shows that $\int_\Omega u \, \mathrm{d}x = 0$.

Since $\left\{u_{k_j}\right\} \xrightarrow{w} u$ in $H^1(\Omega)$, Proposition 13.1 states that

$$\|u\|_{H^1(\Omega)} \le \liminf_{j \to \infty} \left(\left\| u_{k_j} \right\|_{H^1(\Omega)} \right)$$

This can be simplified as follows.

$$\|u\|_{H^1(\Omega)} \le \liminf_{j \to \infty} \left(\left\| u_{k_j} \right\|_{H^1(\Omega)} \right)$$
$$\|u\|_{H^1(\Omega)} \le \liminf_{j \to \infty} \left(\left\| u_{k_j} \right\|_{H^1(\Omega)} \right)$$

Now consider $\|\|_{H^1(\Omega)}$. Note that since $\left\|\nabla u_{k_j}\right\|_{L^2(\Omega)} \le \frac{1}{k_j}$, this implies that $\left\|\nabla u_{k_j}\right\|_{L^2(\Omega)} \to 0$. Now since $u_{k_j} \to u$ this implies that $\nabla u_{k_j} \to \nabla u$. This along with the fact that $\left\|\nabla u_{k_j}\right\|_{L^2(\Omega)} \to 0$, shows that $\nabla u_{k_j} \to 0 = \nabla u$. This shows that u is a cons □

#8 Consider a Lagrangian of the form $\mathcal{L} = \mathcal{L}(u, p)$ (i.e. it happens not to depend on the space variable x) when $N = 1$. Show that if u is a solution of the Euler-Lagrange equation then

$$\mathcal{L}(u, u') - u' \frac{\partial \mathcal{L}}{\partial p}(u, u') = C$$

for some constant C. In this way we are able to achieve a reduction of order from a second order ODE to a first order ODE. Use this observation to redo the derivation of the solution of the hanging chain problem.

Proof. Let u be a solution of the Euler-Lagrange equation, where the Lagrangian is of the form $\mathcal{L}(u, p)$. □

#9 Find the function $u(x)$ which minimizes

$$J(u) = \int_0^1 (u'(x) - u(x))^2 \, \mathrm{d}x$$

among all functions $u \in H^1(0,1)$ satisfying $u(0) = 0$, $u(1) = 1$.

The function $u(x)$ which minimizes $J(u)$ will be the function that satisfies the Euler-Lagrange equation where the Lagrangian is $L(x,u,p) = (p-u)^2$. The Euler-Lagrange equation in this case is

$$-2(u'-u) - 2\frac{\mathrm{d}}{\mathrm{d}x}(u'-u) = 0$$

This is a second order differential equation which can be solved as follows.

$$\begin{aligned}
0 &= -2(u'-u) - 2\frac{\mathrm{d}}{\mathrm{d}x}(u'-u) \\
0 &= -2(u'-u) - 2(u''-u') \\
0 &= -2u' + 2u - 2u'' + 2u' \\
0 &= 2u - 2u'' \\
2u'' &= 2u \\
u'' &= u
\end{aligned}$$

The solutions to this differential equation will be of the form

$$u(x) = c_1 e^x + c_2 e^{-x}$$

The constants c_1 and c_2 can be found by using the boundary conditions $u(0) = 0$ and $u(1) = 1$.

$$\begin{aligned}
u(0) = c_1 + c_2 &= 0 \\
c_1 &= -c_2 \\
u(1) = c_1 e + c_2 e^{-1} &= 1 \\
1 &= -c_2 e + c_2 e^{-1} \\
1 &= c_2(-e + e^{-1}) \\
c_2 &= \frac{1}{e^{-1} - e} \\
c_2 &= \frac{e}{1 - e^2} \\
c_1 &= \frac{e}{e^2 - 1}
\end{aligned}$$

Thus the solution to this minimization problem that satisfies $u(0) = 0$ and $u(1) = 1$ is

$$u(x) = \frac{e}{e^2 - 1} e^x + \frac{e}{1 - e^2} e^{-x}$$

#10 The area of a surface obtained by revolving the graph of $y = u(x)$, $0 < x < 1$, about the x axis, is

$$J(u) = 2\pi \int_0^1 u(x)\sqrt{1 + u'(x)^2}\, \mathrm{d}x$$

Assume that u is required to satisfy $u(0) = a$, $u(1) = b$ where $0 < a < b$.

(a) Find the Euler-Lagrange equation for the problem of minimizing this surface area.
In this case the Lagrangian is

$$\mathcal{L}(x, u, p) = 2\pi u\sqrt{1 + p^2}$$

The Euler-Lagrange equation is then

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial p} = 0$$

or more specifically

$$2\pi\sqrt{1 + p} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial p} = 0$$

(b) Show that

$$\frac{u(u')^2}{\sqrt{1 + (u')^2}} - u\sqrt{1 + (u')^2}$$

is a constant function for any such minimal surface (Hint: use Exercise 8).

(c) Solve the first order ODE in part b) to find the minimal surface. Make sure to compute all constants of integration.

#18 Show that if Ω is a bounded domain in $\mathbb{R}^N$ and $f \in L^2(\Omega)$, then the problem of minimizing

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 \,\mathrm{d}x - \int_{\Omega} f u \,\mathrm{d}x$$

over $H_0^1(\Omega)$ satisfies all of the conditions of Theorem 16.8. What goes wrong if we replace $H_0^1(\Omega)$ by $H^1(\Omega)$.

#19 We say that $J : \mathcal{X} \to \mathbb{R}$ is stricly convex if

$$J(tx + (1-t)y) < tJ(x) + (1-t)J(y) \qquad x, y \in \chi \quad 0 < t < 1$$

If J is strictly convex show that the minimization problem (16.6.89) has at most one solution.

Proof. First note that in order for J to be strictly convex χ must be a convex set as well. This means that if $x, y \in \chi$, then $tx + (1-t)y \in \chi$ for all $t \in [0, 1]$. Suppose that there exists two distinct solutions to the minimization problem (16.6.89), that is there exists $x, y \in \chi$ such that $x \neq y$ and $J(x) = J(y) = \min_{z \in \chi} J(z)$. Now since J is strictly convex it is known that $x/2 + y/2 \in \chi$ and

$$J(x/2 + y/2) < J(x)/2 + J(y)/2 = J(x)$$

This is a contradiction as $J(x) = \min_{z \in \chi} J(x)$. Therefore J has a most one solution to the minimization problem. $\square$