Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 7

Section 13.6

#7 Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of nonzero real numbers satisfying

$$\sum_{j=1}^{\infty} \left(\lambda_j^2\right) < \infty$$

Construct a summetric Hilber Schmidt kernel K such that the corresonding integral operator has eigenvalues λ_j, $j = 1, 2, \ldots$ and for which 0 is an eigenvalue of infinite multiplicity. (Suggestion: look for such a K in the form $K(x, y) = \sum_{j=1}^{\infty} \left(\lambda_j u_j(x)\overline{u_j(y)}\right)$ where $\{u_j\}$ are orthonormal but not complete in $L^2(\Omega)$.)

#12 Compute the singular value decomposition of the Volterra operator

$$Tu(x) = \int_0^x u(s)\,\mathrm{d}s$$

in $L^2(0,1)$ and use it to find $\|T\|$. Is T normal? (Suggestion: The equation $T^*Tu = \lambda u$ is equivalent to an ODE eigenvalue problem which you can solve explicitly.)

In order to compute the singular value decomposition of the Volterra operator, we must first find the operator T^*, so that we may construct $S = T^*T$. Since the Volterra operator is an integral operator in may be rewritten as

$$Tu(x) = \int_0^1 K(x,y)u(y)\,\mathrm{d}y$$

where the kernel $K(x,y)$ is

$$K(x,y) = \begin{cases} 1 & y < x \\ 0 & y > x \end{cases}.$$

The adjoint of any integral operator is another integral operator with kernel, $\overline{K(y,x)}$. In this case

$$\overline{K(y,x)} = \begin{cases} 1 & x < y \\ 0 & x > y \end{cases}.$$

Therefore

$$\begin{aligned} T^*u(x) &= \int_0^1 \overline{K(y,x)}u(y)\,\mathrm{d}y \\ &= \int_x^1 u(y)\,\mathrm{d}y. \end{aligned}$$

Now I will define $S = T^*T$ which can be expressed as

$$\begin{aligned} Su(x) &= T^*Tu(x) \\ &= T^*\int_0^x u(s)\,\mathrm{d}s \\ &= \int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

In order to compute the singular value decomposition of T we must first find the eigenvalues and eigenfunctions of S. This means solving $Su = \lambda u$. Note that we have previously shown that $\lambda \geq 0$ because $S = T^*T$. Also note that if $\lambda = 0$, then

$$\int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y = 0$$

which implies that $u = 0$. Therefore 0 cannot be an eigenvalue, and also $\lambda > 0$. In order to find the eigenvalues, this integral equation can be changed into a differential equation as follows.

$$\begin{aligned} \lambda u(x) &= Su(x) \\ \lambda u(x) &= \int_x^1 \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \\ \lambda u(x) &= -\int_1^x \int_0^y u(s)\,\mathrm{d}s\,\mathrm{d}y \\ \lambda u'(x) &= -\int_0^x u(s)\,\mathrm{d}s \\ \lambda u''(x) &= -u(x) \end{aligned}$$

The boundary conditions for this differential equation can be found by evaluating $Su(x)$ at $x = 1$. Clearly $Su(1) = 0$ so this implies that $u(1) = 0$. The other boundary condition can be found by evaluating $\frac{\mathrm{d}}{\mathrm{d}x}(Su(x))$ at $x = 0$. In this case the result is $u'(0) = 0$.

So now the original integral equation is identical to the following differential equation.

$$\lambda u''(x) + u(x) = 0$$
$$u(1) = 0 \qquad u'(0) = 0$$

This differntial equation can be solved using the characteristic polynomial

$$\lambda r^2 + 1 = 0$$

The roots of this polynomial are $r = \sqrt{1/\lambda}i$ and $r = -\sqrt{1/\lambda}i$. Since these are complex roots the solutions will be of the form

$$u(x) = c_1 \sin\left(\sqrt{1/\lambda}x\right) + c_2 \cos\left(\sqrt{1/\lambda}x\right)$$

Applying the boundary conditions can find the appropriate constants

$$\begin{aligned}
u'(0) &= c_1\sqrt{1/\lambda}\cos\left(\sqrt{1/\lambda}0\right) - c_2\sqrt{1/\lambda}\sin\left(\sqrt{1/\lambda}0\right) \\
0 &= c_1\sqrt{1/\lambda} \\
0 &= c_1 \\
u(1) &= c_2\cos\left(\sqrt{1/\lambda}\right) \\
0 &= c_2\cos\left(\sqrt{1/\lambda}\right)
\end{aligned}$$

Since c_2 can not equal zero, otherwise $u = 0$, this implies that $\cos\left(\sqrt{1/\lambda}\right) = 0$. This is equivalent to $\sqrt{1/\lambda} = \pi/2 + k\pi$, or the eigenvalues are $\lambda_k = \frac{1}{(\pi(1/2+k))^2}$, which can be indexed by $k \geq 0$. The eigenfunctions then are $u_k(x) = c\cos\left(\sqrt{1/\lambda_k}x\right)$. We wish to normalize these eigenfunctions in order to create the singular value decomposition of T.

$$\begin{aligned}
\|u_k\|^2 &= \int_0^1 c^2 \cos\left(\frac{1}{\sqrt{\lambda_k}}x\right)^2 \mathrm{d}x \\
1 &= c^2 \int_0^1 \frac{1 + \cos\left(\frac{2}{\sqrt{\lambda_k}}x\right)}{2} \mathrm{d}x \\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin\left(\frac{2}{\sqrt{\lambda_k}}x\right)\bigg|_{x=0}^1\right) \\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin\left(\frac{2}{\sqrt{\lambda_k}}\right)\right) \\
1 &= c^2\left(\frac{1}{2} + \frac{\sqrt{\lambda_k}}{4}\sin(\pi + 2k\pi)\right) \\
1 &= c^2\frac{1}{2} \\
2 &= c^2 \\
c &= \sqrt{2}
\end{aligned}$$

The eigenfunctions are thus $u_k(x) = \sqrt{2}\cos\left(\frac{1}{\sqrt{\lambda_k}}x\right)$. By Theorem 13.10 these eigenfunctions form an orthonormal basis of $L^2(0, 1)$.

The singular values are the square roots of the eigenvalues, that is $\sigma_k = \sqrt{\lambda_k} = \frac{1}{\pi(1/2+k)}$.

Lastly we can compute $v_k = Tu_k/\sigma_k$.

$$\begin{aligned} v_k &= \frac{1}{\sigma_k} T u_k \\ &= \frac{1}{\sigma_k} \int_0^x u(y)\,\mathrm{d}y \\ &= \frac{1}{\sigma_k} \int_0^x \sqrt{2}\cos\left(\frac{1}{\sqrt{\lambda_k}}y\right)\mathrm{d}y \\ &= \frac{\sqrt{2\lambda_k}}{\sigma_k} \sin\left(\frac{1}{\sqrt{\lambda_k}}y\right)\Big|_{y=0}^{x} \\ &= \sqrt{2}\sin\left(\frac{1}{\sqrt{\lambda_k}}x\right) \end{aligned}$$

Now that we have found σ_k, u_k, and v_k, the singular value decomposition of T is $\sum_{n=0}^{\infty} (\sigma_n \langle u, u_n\rangle v_n)$.

The norm of T is the largest singular value, that is $\|T\| = \sigma_0 = \frac{2}{\pi}$. Also T is not normal because if T was normal then $Tu_n = \sigma_n u_n$ or $v_n = u_n$. However this is not the case so T is not normal.

Section 14.5

#1 Let $Lu = (x-2)u'' + (1-x)u' + u$ on $(0,1)$.

(a) Find the Green's function for

$$Lu = f \qquad u'(0) = 0 \quad u(1) = 0$$

(Hint First show that $x-1$, e^x are linearly independent solutions of $Lu = 0$.)
First consider $Lu = 0$. Let $u(x) = x - 1$, and consider Lu.

$$\begin{aligned} Lu(x) &= (x-2)u''(x) + (1-x)u'(x) + u(x) \\ &= (x-2)0 + (1-x)1 + (x-1) \\ &= 0 \end{aligned}$$

Let $u(x) = e^x$, then

$$\begin{aligned} Lu(x) &= (x-2)e^x + (1-x)e^x + e^x \\ &= (x-2+1-x+1)e^x \\ &= 0 \end{aligned}$$

Also these two functions are linearly independent because they are not multiples of one another.
The boundary conditions for this problem are $B_1 u = u'(0) = 0$ and $B_2 u = u(1) = 0$. Let $\phi_1(x) = e^x - (x-1)$ and $\phi_2(x) = x - 1$ and note that $L\phi_1 = 0$, $B_1\phi_1 = 0$, $L\phi_2 = 0$ and $B_2\phi_2 = 0$. Thus we can use ϕ_1 and ϕ_2 to build the Green's function for this differential equation.
The next thing to do is compute $C_1(y)$ and $C_2(y)$, which first requires computing $W(y)$, the Wronskian of ϕ_1 and ϕ_2.

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x - (x-1) & x-1 \\ e^x - 1 & 1 \end{vmatrix} \\ &= e^x - (x-1) - (x-1)(e^x - 1) \\ &= (2-x)e^x \end{aligned}$$

Now we can compute C_1 and C_2 as

$$\begin{aligned} C_1(y) &= \frac{\phi_1(y)}{a_2(y)W(y)} \\ &= \frac{y-1}{(y-2)(2-y)e^y} \\ &= \frac{1-y}{(y-2)^2 e^y} \\ C_2(y) &= \frac{\phi_2(y)}{a_2(y)W(y)} \\ &= \frac{e^y - (y-1)}{(y-2)(2-y)e^y} \\ &= \frac{y-1-e^y}{(y-2)^2 e^y} \end{aligned}$$

Finally the Green's Function for this differential equation is

$$
\begin{aligned}
G(x, y) &= \begin{cases} C_1(y)\phi_1(x) & 0 < x < y < 1 \\ C_2(y)\phi_2(y) & 0 < y < x < 1 \end{cases} \\
&= \begin{cases} \frac{1-y}{(y-2)^2 e^y}\left(e^x - (x-1)\right) & 0 < x < y < 1 \\ \frac{y-1-e^y}{(y-2)^2 e^y}(x-1) & 0 < y < x < 1 \end{cases}
\end{aligned}
$$

(b) Find the adjoint operator and boundary conditions.

#2 Let

$$Tu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

on the domain

$$D(T) = \left\{u \in H^2(1,2) : u(1) = u(2) = 0\right\}$$

(a) Show that $N(T) = \{0\}$.

Proof. First note that clearly $0 \in N(T)$. Now consider $u \neq 0$ such that $u \in N(T)$, then

$$\begin{aligned}
Tu &= 0 \\
-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= 0 \\
x\frac{\mathrm{d}u}{\mathrm{d}x} &= c_1 \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{c_1}{x} \\
u(x) &= \int \frac{c_1}{x}\,\mathrm{d}x + c_2 \\
u(x) &= c_1\ln(x) + c_2
\end{aligned}$$

If $u \in D(T)$, then u must satisfy $u(1) = u(2) = 0$.

$$\begin{aligned}
u(1) &= c_1\ln(1) + c_2 \\
0 &= c_2 \\
u(2) &= c_1\ln(2) \\
0 &= c_1
\end{aligned}$$

This shows that $u = 0$, which contradicts that $u \neq 0$. This shows that $N(T) = \{0\}$. □

(b) Find the Green's function for the boundary value problem $Tu = f$.

In order to find the Green's function for the boundary value problem, we must first find ϕ_1 and ϕ_2 where $L\phi_1 = 0$, $B_1\phi_1 = 0$, $L\phi_2 = 0$, and $B_2\phi_2 = 0$. For this problem $Lu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$, $B_1u = u(1)$, and $B_2u = u(2)$. We have already shown in part (a) that any function that satisfies $Lu = 0$ is of the form $u(x) = c_1\ln(x) + c_2$. Therefore let $\phi_1(x) = c_1\ln(x) + c_2$ and let $B_1\phi_1 = 0$, then $\phi_1(1) = c_1\ln(1) + c_2 = 0$ or $c_2 = 0$. Also let $c_1 = 1$, then $\phi_1(x) = \ln(x)$. Now $\phi_2(x)$ is of the form $c_1\ln(x) + c_2$. If $\phi_2(2) = 0$, then $c_1\ln(2) + c_2 = 0$ and $c_2 = -c_1\ln(2)$. Then let $c_1 = 1$ and $\phi_2(x) = \ln(x) - \ln(2)$.

The next step is to compute the Wronskian.

$$\begin{aligned}
W(y) &= \begin{vmatrix} \ln(y) & \ln(y) - \ln(2) \\ 1/y & 1/y \end{vmatrix} \\
&= \frac{\ln(y)}{y} - \frac{\ln(y) - \ln(2)}{y} \\
&= \frac{\ln(2)}{y}
\end{aligned}$$

We also need to find $a_2(x)$, which can be found using the product rule

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = -x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{\mathrm{d}u}{\mathrm{d}x}$$

Therefore $a_2(x) = -x$.
Now $C_1(y)$ and $C_2(y)$ are

$$\begin{aligned}
C_1(y) &= \frac{\phi_2(y)}{a_2(y)W(y)} \\
&= \frac{\ln(y) - \ln(2)}{-y(\ln(2)/y)} \\
&= \frac{\ln(2) - \ln(y)}{\ln(2)} \\
C_2(y) &= \frac{\phi_1(y)}{a_2(y)W(y)} \\
&= \frac{\ln(y)}{-y(\ln(2)/y)} \\
&= -\frac{\ln(y)}{\ln(2)}
\end{aligned}$$

Finally the Green's function is

$$\begin{aligned}
G(x,y) &= \begin{cases} C_1(y)\phi_1(x) & 0 < x < y < 1 \\ C_2(y)\phi_2(x) & 0 < y < x < 1 \end{cases} \\
&= \begin{cases} \frac{\ln(2)-\ln(y)}{\ln(2)}\ln(y) & 0 < x < y < 1 \\ -\frac{\ln(y)}{\ln(2)}(\ln(y) - \ln(2)) & 0 < y < x < 1 \end{cases}
\end{aligned}$$

(c) State and prove a result about the continuous dependence of the solution u on f in part (b).

#4 Prove the validity of (14.1.22). (Suggestions: start by writing $u(x)$ in the form

$$u(x) = \phi_2(x) \int_a^x C_2(y) f(y) \, \mathrm{d}y + \phi_1(x) \int_x^b C_1(y) f(y) \, \mathrm{d}y$$

and note that some of the terms that arise in the expression for u'(x) will cancel.)