# Caleb Logemann
# MATH 520 Methods of Applied Math II
# Homework 12

## Section 16.8

#20 Show that the Fréchet derivative, if it exists, must be unique.

*Proof.* Let $X, Y$ be Banach spaces and let $F : D(F) \subset X \to Y$. Now suppose that $A_1, A_2 \in B(X, Y)$ exist such that $A_1 \neq A_2$ and they both are the Fréchet derivative of $F$ at some $x_0 \in D(F)$. This means that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_1(x - x_0)\|}{\|x - x_0\|} = 0$$

and

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A_2(x - x_0)\|}{\|x - x_0\|} = 0$$

□

#21 If $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

show that $F$ is Gâteaux differentiable but not Fréchet differentiable at the origin.

The Gâteaux derivative of $F$ at the origin can be computed as follows.

$$\begin{aligned} DF(0,0)(u,v) &= \frac{\mathrm{d}}{\mathrm{d}t}(F(0 + tu, 0 + tv))\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tu(tv)^2}{(tu)^2 + (tv)^4}\right)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tuv^2}{u^2 + t^2v^4}\right)\bigg|_{t=0} \\ &= \frac{(u^2 + t^2v^4)uv^2 + tuv^2(2v^4t)}{(u^2 + t^2v^4)^2}\bigg|_{t=0} \\ &= \frac{u^3v^2}{u^4} \\ &= \frac{v^2}{u} \end{aligned}$$

This shows that the Gâteaux derivative of $F$ is $A(u, v) = \frac{v^2}{u}$

Now we will consider the Fréchet derivative of $F$ at $(0, 0)$. If the Fréchet derivative exists, then $A \in B(X, Y)$ will exist such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0.$$

This can be simplified by noting that $x_0 = (0, 0)$ and using the definition of $F$.

$$\lim_{(u,v) \to (0,0)} \frac{\left|\frac{uv^2}{u^2+v^4} - A(u, v)\right|}{\sqrt{u^2 + v^2}} = 0.$$

#27 Let $X, Y$ be Banach spaces, $F : D(F) \subset X \to Y$, and let $x, x_0 \in D(F)$ be such that $tx + (1-t)x_0 \in D(F)$ for $t \in [0,1]$. If

$$M := \sup_{0 \le t \le 1} \|DF(tx + (1-t)x_0)\|$$

show that

$$\|F(x) - F(x_0)\| \le M\|x - x_0\|$$

(Suggestion: justify and use a suitable version of the fundamental theorem of calculus.)

## Section 17.5

#2 Let $\lambda_1$ be the smallest Dirichlet eigenvalue for $-\Delta$ in $\Omega$, assume that $c \in C(\overline{\Omega})$ and $c(x) > -\lambda_1$ in $\overline{\Omega}$. If $f \in L^2(\Omega)$ prove the existence of a solution of

$$-\Delta u + c(x)u = f \quad x \in \Omega \qquad u = 0 \quad \forall x \in \partial\Omega$$

*Proof.* □

#3 Let $\lambda > 0$ and define

$$A[u,v] = \int_\Omega a_{jk}(x)u_{x_k}(x)v_{x_j}(x)\,\mathrm{d}x + \lambda\int_\Omega uv\,\mathrm{d}x$$

for all $u, v \in H^1(\Omega)$. Assume the ellipticity property (17.1.3) and that $a_{jk} \in L^\infty(\Omega)$. If $f \in L^2(\Omega)$ show that there exists a unique solution of

$$u \in H^1(\Omega) \qquad A[u,v] = \int_\Omega fv\,\mathrm{d}x \forall v \in H^1(\Omega).$$

Justify that $u$ may be regarded as the weak solution of

$$-(a_{jk}u_{x_k})_{x_j} + \lambda u = f(x) \quad x \in \Omega \qquad a_{jk}u_{x_k}n_j = 0 \quad x \in \partial\Omega$$

The above boundary condition is said to be of conormal type.

#6 Let $f$ and $g$ be in $L^2(0,1)$. Use the Lax-Milgram Theorem to prove there is a unique weak solution $\{u, v\} \in H_0^1(0,1)$ to

$$\begin{aligned} -u'' + u + v' &= f \\ -v'' + v + u' &= g, \end{aligned}$$

where $u(0) = v(0) = 0$ and $u(1) = v(1) = 0$. (Hint: Start by defining the bilinear form

$$A[(u,v),(\phi,\psi)] = \int_0^1 u'\phi' + u\phi + v'\phi + v'\psi' + v\psi + u'\psi\,\mathrm{d}x$$

on $H_0^1(0,1) \times H_0^1(0,1)$.