Caleb Logemann
MATH 520 Methods of Applied Math II
Homework 1

Section 10.9

#3 Prove Proposition 10.1. Proposition 10.1 states that if T is bounded on its domain then it has a unique norm preserving extension to $\overline{D(T)}$. That is to say there exists a unique linear operator $S : \overline{D(T)} \subset X \to Y$ such that $Sx = Tx$ for $x \in D(T)$ and $\|S\| = \|T\|$.

Proof. Let $S : \overline{D(T)} \subset X \to Y$ be defined as follows.

$$Sx = \lim_{n \to \infty} (Tx_n)$$

where the sequence $\{x_n\}_{n=1}^{\infty}$ is any sequence in $D(T)$ that converges to x. Note that for any $x \in \overline{D(T)}$, x is a limit point of $D(T)$ so the sequence $\{x_n\}$ exists. Also since T is bounded it is also continuous, so the limit always exists.

Next I will show that S is linear. Consider $x_1, x_2 \in \overline{D(T)}$ and $c_1, c_2 \in \mathbb{C}$. Then there exists sequences in $D(T)$, $\{a_n\}_{n=1}^{\infty}$ that converges to x_1 and $\{b_n\}_{n=1}^{\infty}$ that converges to x_2. Now note that the sequence $c_1 a_n + c_2 b_{n}{}_{n=1}^{\infty}$ converges to $c_1 x_1 + c_2 x_2$ by the linearity of limits. Therefore

$$S(c_1 x_1 + c_2 x_2) = \lim_{n \to \infty} (T(c_1 a_n + c_2 b_n))$$

Because T is linear

$$S(c_1 x_1 + c_2 x_2) = \lim_{n \to \infty} (c_1 T(a_n) + c_2 T(b_n))$$

By the linearity of limits

$$S(c_1 x_1 + c_2 x_2) = c_1 \lim_{n \to \infty} (T(a_n)) + c_2 \lim_{n \to \infty} (T(b_n))$$
$$S(c_1 x_1 + c_2 x_2) = c_1 S(x_1) + c_2 S(x_2)$$

This shows that S is a linear operator.

Next I will show that $Sx = Tx$ for $x \in D(T)$. Let $\{x_n\}_{n=1}^{\infty}$ converge to x in $D(T)$, then because T is continuous, $\lim_{n \to \infty}(T(x_n)) = T(x)$. Therefore $Sx = Tx$.

Now I will show that $\|S\| = \|T\|$. Consider the following, note that supremums are taken excluding the zero vector.

$$\begin{aligned}
\|S\| &= \sup *[x \in \overline{D(T)}] \frac{\|Sx\|_Y}{\|x\|_X} \\
&= \sup *[x \in \overline{D(T)}] \frac{\|\lim_{n\to\infty}(Tx_n)\|_Y}{\|x\|_X} \\
&= \sup *[x \in \overline{D(T)}] \frac{\lim_{n\to\infty}(\|Tx_n\|_Y)}{\|x\|_X} \\
&\leq \sup *[x \in \overline{D(T)}] \frac{\lim_{n\to\infty}(\|T\| \|x_n\|_X)}{\|x\|_X} \\
&= \sup *[x \in \overline{D(T)}] \frac{\|T\| \|x\|_X}{\|x\|_X} \\
&= \sup *[x \in \overline{D(T)}] \|T\| \\
&= \|T\|
\end{aligned}$$

Therefore $\|S\| \le \|T\|$. Also $D(T) \subset \overline{D(T)}$, therefore

$$\begin{aligned}
\|S\| &= \sup*[x \in \overline{D(T)}] \frac{\|Sx\|_Y}{\|x\|_X} \\
&\ge \sup*[x \in D(T)] \frac{\|Sx\|_Y}{\|x\|_X} \\
&= \sup*[x \in D(T)] \frac{\|Tx\|_Y}{\|x\|_X} \\
&= \|T\|
\end{aligned}$$

Therefore $\|S\| \ge \|T\|$, and both of these inequalities imply that $\|S\| = \|T\|$.

Finally note that this extension is unique suppose that there exists some other extension S' with the same properties. Consider $x \in \overline{D(T)}$, then there exists some sequence $\{x_n\}_{n=1}^{\infty} \in D(T)$ that converges to x. Now consider $S'x$ since S' is linear and bounded it is continuous, so

$$S'x = \lim_{n \to \infty}(S'x_n) = \lim_{n \to \infty}(Tx_n) = Sx$$

because $S'x = Tx$ for $x \in D(T)$. Therefore the extension is unique. $\square$

#6 Show that a linear operator $T : \mathbb{C}^N \to \mathbb{C}^M$ is always bounded for any choice of norms on $\mathbb{C}^N$ and $\mathbb{C}^M$.

Proof. Let $T : \mathbb{C}^N \to \mathbb{C}^M$ be a linear operator. It is known that any linear operator from $\mathbb{C}^N \to \mathbb{C}^M$ can be expressed as a matrix multiplication, that is there exists matrix $A \in \mathbb{C}^{m \times n}$ such that $Tx = Ax$ for every $x \in \mathbb{C}^N$.

Consider the alternative definition of the norm of an operator

$$\|T\| = \sup[\|x\| = 1]\|Tx\|$$

I will let $S = \left\{x \in \mathbb{C}^N : \|x\| = 1\right\}$, which is commonly known as the unit sphere of the vector space. It is well known that the unit sphere of a finite dimensional vector space is a compact set and this is true for any norm chosen on that space. It is also well known that a continous function is bounded on every compact set. Since any norm on $\mathbb{C}^M$ is continuous, the value of $\|Ax\|$ is bounded for any $x \in S$. This implies that

$$\|T\| = \sup[x \in S]\|Tx\| = \sup[x \in S]\|Ax\| < \infty$$

Therefore the T is bounded for any choice of norms on $\mathbb{C}^N$ and $\mathbb{C}^M$. $\square$

#7 If $T, T^{-1} \in \mathcal{B}(\mathbf{H})$ show that $(T^*)^{-1} \in \mathcal{B}(\mathbf{H})$ and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let $\mathbf{H}$ be a Hilbert Space and let $T, T^{-1} \in \mathcal{B}(\mathbf{H})$. Since $T \in \mathcal{B}(\mathbf{H})$ this implies that there exists a unique linear operator $T^* \in \mathcal{B}(\mathbf{H})$ such that $\langle Tu, v\rangle = |u, T^*v|$ for all $u, v \in \mathbf{H}$. First I will show that T^* is one-to-one so that $(T^*)^{-1}$ exists. Consider $x, y \in \mathbf{H}$ such that $T^*x = T^*y$ or equivalently $T^*(x-y) = 0$. Now consider any $u \in \mathbf{H}$.

$$0 = \langle u, T^*(x-y)\rangle = \langle Tu, x-y\rangle$$

Since $\langle Tu, x-y\rangle = 0$ for any $u \in \mathbf{H}$ this implies that $x - y = 0$ or $x = y$. Therefore T^* is one-to-one and $(T^*)^{-1}$ exists. Secondly I will show that $(T^*)^{-1} = (T^{-1})^*$. Consider the identity operator I which is defined as $Ix = x$ for all $x \in \mathbf{H}$. Note that $T^{-1}T = I$. Also note that $I^* = I$ because $I \in \mathcal{B}(\mathbf{H})$ and

$$\langle Iu, v\rangle = \langle u, v\rangle = \langle u, Iv\rangle$$

for all $u, v \in \mathbf{H}$. This implies that $(T^{-1}T)^* = I$ or equivalently $T^*(T^{-1})^* = I$ because $T, T^{-1} \in \mathcal{B}(\mathbf{H})$. Now consider any $x \in \mathcal{B}(\mathbf{H})$

$$x = T^*\left(T^{-1}\right)^* x$$
$$(T^*)^{-1}x = \left(T^{-1}\right)^* x$$

This is because $(T^*)^{-1}$ is defined for any $x \in \mathbf{H}$. Since this is true for any x, this implies that $(T^*)^{-1} = (T^{-1})^*$. Now it is easy to see that since $\|T\| = \|T^*\|$ this implies that $\left\|(T^*)^{-1}\right\| = \|T^{-1}\| < \infty$. Also since $(T^*)^{-1} = (T^{-1})^*$, this implies that $(T^*)^{-1}$ is linear. Therefore $(T^*)^{-1} \in \mathcal{B}(\mathbf{H})$. $\square$

#14 If $T \in \mathcal{B}(\mathbf{H})$ show that T^* restricted to $R(T)$ is one-to-one.

Proof. Let $\mathbf{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathbf{H})$. Consider T^* restricted to $R(T)$. In other words let S be a linear function such that $D(S) = R(T)$ and $Sx = T^*x$ for every $x \in D(S)$. Let $x, y \in D(S) = R(T)$ such that $Sx = Sy$. Since S is linear this is equivalent to $S(x-y) = 0$, which implies that $x - y \in N(S)$. Note that $N(S) \subseteq N(T^*)$, so $x - y \in N(T^*)$. Also note that $R(T)$ is a subspace of $\mathbf{H}$ so that $x - y \in R(T)$. It has already been stated that $R(T) \subset N(T^*)^\perp$, which implies that $x - y \in N(T^*)^\perp$. Since $x - y \in N(T^*)^\perp$ and $x - y \in N(T^*)$, $x - y$ must be orthogonal to itself.

$$\begin{aligned} \langle x - y, x - y \rangle &= 0 \\ \|x - y\|^2 &= 0 \\ \|x - y\| &= 0 \\ x - y &= 0 \\ x &= y \end{aligned}$$

This follows since the only element of $\mathbf{H}$ with norm 0 is the zero element. This shows that when $Sx = Sy$ then $x = y$. Therefore S is one to one or T^* is one to one when restricted to the range of T. $\square$