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MATH 520 Methods of Applied Math II
Homework 9

Section 14.5

#5 Let $Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$ with $a_2' = a$, so that L is formally self adjoint. If $B_1u = C_1u(a) + C_2u'(a)$, $B_2u = C_3u(b) + C_4u'(b)$, show that $\{B_1^*, B_2^*\} = \{B_1, B_2\}$.

Proof. Let $\{B_1^*, B_2^*\}$ be the set of boundary operators adjoint to $\{B_1, B_2\}$. This implies that

$$J(\phi, \psi)\big|_a^b = 0$$

whenever $B_1\phi = B_2\phi = B_1^*\psi = B_2^*\psi = 0$. The boundary function J can be expressed as

$$J(\phi, \psi) = a_2\left(\phi'\overline{\psi} - \phi\overline{\psi'}\right) + (a_1 - a_2')\phi\overline{\psi}$$

and since L is formally self-adjoint, this implies that $a_1 - a_2' = 0$, so the boundary functional can be simplified to

$$J(\phi, \psi) = a_2\left(\phi'\overline{\psi} - \phi\overline{\psi'}\right).$$

□

#8 When we rewrite $a_2(x)u'' + a_1(x)u' + a_0(x)u = \lambda u$ as

$$-(p(x)u')' + q(x)u = \lambda\rho(x)u$$

the latter is often referred to as the *Liouville normal form.* Consider the eigenvalue problem

$$x^2u'' + xu' + u = \lambda u \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

(a) Find the Liouville normal form.
The functions $p(x)$, $\rho(x)$, and $q(x)$ can be found as follows.

$$\begin{aligned}
p(x) &= \exp\left(\int_a^x \frac{a_1(s)}{a_2(s)}\,\mathrm{d}s\right)\\
&= \exp\left(\int_a^x \frac{s}{s^2}\,\mathrm{d}s\right)\\
&= \exp\left(\int_a^x \frac{1}{s}\,\mathrm{d}s\right)\\
&= \exp(\ln(s)|_{s=a}^x)\\
&= e^{\ln(x)-\ln(a)}\\
&= e^{\ln\left(\frac{x}{a}\right)}\\
&= \frac{x}{a}\\
\rho(x) &= -\frac{p(x)}{a_2(x)}\\
&= -\frac{x/a}{x^2}\\
&= -\frac{1}{ax}\\
q(x) &= a_0(x)\rho(x)\\
q(x) &= -\frac{1}{ax}
\end{aligned}$$

Therefore the Liouville normal form of this eigenvalue problem is

$$-\left(\frac{x}{a}\phi'\right)' - \frac{1}{ax}\phi = \lambda\left(-\frac{1}{ax}\right)\phi$$

(b) What is the orthogonality relationship satisfied by the eigenfunctions?

(c) Find the eigenvalues and eigenfunctions. (You may find the original form of the equation easier to work with than the Liouville normal form when computing the eigenvalues and eigenfunctions.)

#10 Consider the Sturm-Liouville problem

$$\begin{aligned} u'' + \lambda u = 0 \qquad 0 < x < 1 \\ u(0) - u'(0) = u(1) = 0 \end{aligned}$$

(a) Multiply the equation by u and integrate by parts to show that any eigenvalue is positive.

(b) Show that the eigenvalues are the positive solutions of $\tan\left(\sqrt{\lambda}\right) = -\sqrt{\lambda}$.

(c) Show graphically that such roots exist, and form an infinite sequence λ_k such that $(k-1/2)\pi < \sqrt{\lambda_k} < k\pi$ and

$$\lim_{k \to \infty} \left(\sqrt{\lambda_k} - (k - 1/2)\pi\right) = 0$$

#14 If $\{\psi_n\}_{n=1}^{\infty}$ are Dirichlet eigenfunctions of the Laplacian making up an orthonormal basis of $L^2(\Omega)$, let $\xi_n = \psi_n/\sqrt{\lambda_n}$ (λ_n the corresponding eigenvalue).

(a) Show that $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$.

(b) Show that $f \in H_0^1(\Omega)$ if and only if $\sum_{n=1}^{\infty}\left(\lambda_n|\langle f, \psi_n\rangle|^2\right) < \infty$

#15 If $\Omega < \mathbb{R}^n$ is a bounded open set with smooth enough boundary, find a solution of the wave equation problem

$$\begin{aligned} u_{tt} - \Delta u = 0 \qquad x \in \Omega \quad t > 0 \\ u(x,t) = 0 \qquad x \in \partial\Omega \quad t > 0 \\ u(x,0) = f(x) \quad u_t(x,0) = g(x) \qquad x \in \Omega \end{aligned}$$

in the form

$$u(x,t) = \sum_{n=1}^{\infty} (c_n(t)\psi_n(x))$$

where $\{\psi_n\}_{n=1}^{\infty}$ are the Dirichlet eigenfunctions of $-\Delta$ in Ω.

#16 Derive formally that

$$G(x, y) = \sum_{n=1}^{\infty} \left(\frac{\psi_n(x)\psi_n(y)}{\lambda_n} \right)$$

where λ_n, ψ_n are the Dirichlet eigenvalues and normalized eigenfunctions for the domain Ω, and $G(x, y)$ is the corresponding Green's function in (14.4.96). (Suggestion: if $-\Delta u = f$, expand both u and f in the ψ_n basis.)