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MATH 520 Methods of Applied Math II
Homework 5

Section 12.4

#6 Let T denote the right shift operator on ℓ^2.

(a) Show that $\sigma_p(T) = \varnothing$.

(b) Show that $\sigma_c(T) = \{\lambda : |\lambda| = 1\}$.

(c) Show that $\sigma_r(T) = \{\lambda : |\lambda| < 1\}$.

#7 If $\lambda \neq \pm 1, \pm i$ show that λ is in the resolvant set of the Fourier Transform $\mathcal{F}$. (Suggestion: Assuming that a solution of $\mathcal{F}u - \lambda u = f$ exists, derive an explicit formula for it by justifying and using the identity

$$\mathcal{F}^4 u = \lambda^4 u + \lambda^3 f + \lambda^2 \mathcal{F} f + \mathcal{F}^3 f$$

together with the fact that $\mathcal{F}^4 = I$.)

#8 Let $\mathbf{H} = L^2(0,1)$, $T_1 u = T_2 u = T_3 u = u'$ on the domains

$$D(T_1) = H^1(0,1)$$
$$D(T_2) = \left\{u \in H^1(0,1) : u(0) = 0\right\}$$
$$D(T_3) = \left\{u \in H^1(0,1) : u(0) = u(1) = 0\right\}$$

(i) Show that $\sigma(T_1) = \sigma_p(T_1) = \mathbb{C}$.

Proof. $\square$

(ii) Show that $\sigma(T_2) = \varnothing$.

Proof. $\square$

(iii) Show that $\sigma(T_3) = \sigma_r(T_3) = \mathbb{C}$.

Proof. $\square$

#10 Let $Tu(x) = \int_0^x K(x,y)u(y)\,\mathrm{d}u$ be a Volterra intergral operator on $L^2(0,1)$ with a bounded kernel, $|K(x,y)| \leq M$. Show that $\sigma(T) = \{0\}$. (There are several ways to show that T has no nonzero eigenvalues. Here is one approach: Define the equivalent norm on $L^2(0,1)$

$$\|u\|_\theta^2 = \int_0^1 |u(x)|^2 e^{-2\theta x}\,\mathrm{d}x$$

and show that the supremum of $\frac{\|Tu\|_\theta}{\|u\|_\theta}$ can be made arbitrarily small by choosing θ sufficiently large.

#11 If T is a symmetric operator, show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \mathbb{R}$$

(IT is almost the same as showing that $\sigma(T) \subset \mathbb{R}$ for a self-adjoint operator.)