Chapter 2
Approximation and Interpolation

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Approximation of Functions

E.g., evaluation of elementary or transcendental functions: $\cos(x), e^x, \ldots$ etc.

E.g., function values are given at a finite set of points, want function values at other points ...

Approximation

Given a function f and a class Φ of "approximating functions" ϕ and a norm $\|\cdot\|$. A function $\hat{\phi} \in \Phi$ is called the best approximation of f from the class Φ relative to the norm $\|\cdot\|$ if

$$\|f - \hat{\phi}\| \leqslant \|f - \phi\| \text{ for all } \phi \in \Phi.$$

Approximating Functions

- Assume Φ is a (real) linear space with "basis functions" $\pi_j \in \Phi,\ j = 1, 2, \dots, n$, i.e.,
$$\Phi = \Phi_n = \{\phi : \phi(t) = \sum_{j=1}^{n} c_j \pi_j(t), c_j \in \mathbf{R}\}.$$
- Example: $\Phi = \mathbf{P}_m$: polynomials of degree $\leqslant m$. A basis is $\{\pi_j(t) = t^{j-1},\ j = 1, 2, \dots, m+1\}$. $n = m + 1$.
- Example: $\Phi = \mathbf{S}_m^k(\Delta)$: (polynomial) spline functions of degree m and smoothness class k on the subdivision $\Delta : a = t_1 < t_2 < \cdots < t_{N-1} < t_N = b$ of the interval $[a,\ b]$. $n = (m-k)(N-2) + m + 1$.
- Example: $\Phi = \mathbf{T}_m[0,\ 2\pi]$: trigonometric polynomials of degree $\leqslant m$ on $[0,\ 2\pi]$. A basis is $\pi_k(t) = \cos(k-1)t,\ k = 1, \dots, m+1$; $\pi_{m+1+k}(t) = \sin kt,\ k = 1, \dots, m$. $n = 2m + 1$.

Choices of Norms

Types of Approximations and Associated Norms

On interval $[a,\ b]$ or N distinct points $t_1, \dots, t_N \in [a,\ b]$ along with a weight factor $w(t)$, or $w_1, \dots, w_N$.

Continuous norm	Approximation	Discrete norm
$\\|u\\|_\infty = \max_{a \leqslant t \leqslant b} \|u(t)\|$	L_∞	$\\|u\\|_\infty = \max_{1 \leqslant i \leqslant N} \|u(t_i)\|$
$\\|u\\|_1 = \int_a^b \|u(t)\|dt$	L_1	$\\|u\\|_1 = \sum_{i=1}^N \|u(t_i)\|$
$\\|u\\|_{1,w} = \int_a^b \|u(t)\|w(t)dt$	Weighted L_1	$\\|u\\|_{1,w} = \sum_{i=1}^N w_i\|u(t_i)\|$
$\\|u\\|_2 = (\int_a^b \|u(t)\|^2 dt)^{1/2}$	L_2	$\\|u\\|_2 = (\sum_{i=1}^N \|u(t_i)\|^2)^{1/2}$
$\\|u\\|_{2,w} = (\int_a^b \|u(t)\|^2 w(t)dt)^{1/2}$	Weighted L_2	$\\|u\\|_{2,w} = (\sum_{i=1}^N w_i\|u(t_i)\|^2)$

Examples for approximating problems: interpolation problem, least squares problem, etc..

A Uniform Notation

Define

$$\lambda(t) = \begin{cases} 0 & \text{if } t < a (\text{ whenever } -\infty < a) \\ \displaystyle\int_a^t w(\tau) d\tau & \text{if } a \leqslant t \leqslant b \\ \displaystyle\int_a^b w(\tau) d\tau & \text{if } t > b (\text{ whenever } b < \infty). \end{cases}$$

For continuous case:

$$\int_{\mathbf{R}} u(t) d\lambda(t) = \int_a^b u(t) w(t) dt.$$

For discrete case:

$$\int_{\mathbf{R}} u(t) d\lambda(t) = \sum_{i=1}^{N} w_i u(t_i).$$

Least Square Approximation

Formulate the best approximation problem with L_2 norm

$$\|u\|_{2,d\lambda} = (\int_{\mathbf{R}} |u(t)|^2 d\lambda(t))^{1/2},$$

over an n-dimensional linear space of "approximating functions"

$$\Phi = \Phi_n = \{\phi : \phi(t) = \sum_{j=1}^{n} c_j \pi_j(t),\ c_j \in \mathbf{R}\},$$

with an orthogonal basis $\{\pi_j,\ j = 1, \ldots, n\}$.
The least squares problem:

$$\min_{\phi \in \Phi_n} \|\phi - f\|_{2,d\lambda}.$$

Inner Product

Inner Product

- Inner product
$$(u, v) = \int_{\mathbf{R}} u(t)v(t)d\lambda(t).$$
- properties:
  - symmetry: $(u, v) = (v, u)$
  - homogeneity: $(\alpha u, v) = \alpha(u, v), \alpha \in \mathbf{R}$
  - additivity: $(u + v, w) = (u, w) + (v, w)$
  - positivity: $(u, u) \geqslant 0$; $(u, u) = 0$ iff $u = 0$.
  - Then, linearity: $(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1(u_1, v) + \alpha_2(u_2, v)$.
- We say u and v are orthogonal if $(u, v) = 0$.

Generalized Theorem of Pythagoras

For an orthogonal system $\{u_k\}_{k=1}^n$, $\|\sum_{k=1}^n \alpha_k u_k\|^2 = \sum_{k=1}^n |\alpha_k|^2 \|u_k\|^2$.

Least Squares Problem; Normal Equations

Least squares approximation over Φ_n with L_2 norm:

$$\min_{\phi \in \Phi_n} E[\phi] \equiv \min_{\phi \in \Phi_n} \|\phi - f\|.$$

$$\begin{aligned} E^2[\phi] &\equiv \|\phi - f\|^2 = (\phi, \phi) - 2(\phi, f) + (f, f) \\ &= \int_{\mathbf{R}} (\sum_{j=1}^{n} c_j \pi_j(t))^2 d\lambda(t) - 2 \int_{\mathbf{R}} (\sum_{j=1}^{n} c_j \pi_j(t)) f(t) d\lambda(t) + \int_{\mathbf{R}} f^2(t) d\lambda(t) \end{aligned}$$

Minimizing $E^2[\phi]$ over Φ_n is equivalent to considering $E^2[\phi]$ as a function of n variables $\{c_1, \ldots, c_n\}$ and minimizing the function over the n variables.

Using Calculus (find critical points),

$$\frac{\partial}{\partial c_i} E^2[\phi] = 0, \; i = 1, 2, \ldots, n.$$

Normal Equations

$$\frac{\partial}{\partial c_i} E^2[\phi] = 2\int_{\mathbf{R}} (\sum_{j=1}^{n} c_j \pi_j(t))\pi_j(t)d\lambda(t) - 2\int_{\mathbf{R}} \pi_i(t)f(t)d\lambda(t) = 0$$

Normal Equations

- The normal equations:

$$\sum_{j=1}^{n} (\pi_i, \pi_j)c_j = (\pi_i, f), \; i = 1, 2, \ldots, n.$$

- In a compact form:

$$\mathbf{Ac} = \mathbf{b},$$

with $\mathbf{A} = [a_{ij}] = [(\pi_i, \pi_j)], \; \mathbf{b} = [b_i] = [(\pi_i, f)], \; \mathbf{c} = [c_i]$.

Normal Equations $\mathbf{Ac} = \mathbf{b}$

A is symmetric and positive definite (SPD). $\Rightarrow$ invertible $\Rightarrow$ a unique solution $\hat{\mathbf{c}}$ $\Rightarrow$ the least squares approximating problem has a unique solution

$$\hat{\phi}(t) = \sum_{j=1}^{n} \hat{c}_j \pi_j(t).$$

Good in theory! How about in practice?

- **A** may be ill-conditioned with non-orthogonal basis; nonpermanence of the coefficients $\hat{c}_j$.
- **A** becomes diagonal with orthogonal basis; permanence of the coefficients $\hat{c}_j$ with $\hat{c}_j = (\pi_j, f)/(\pi_j, \pi_j),\ j = 1, \ldots, n$. To alleviate cancellation error:

$$\hat{c}_j = \frac{1}{(\pi_j, \pi_j)}(f - \sum_{k=1}^{j-1}, \pi_j),\ j = 1, \ldots, n.$$

- Gram-Schmidt orthogonalization ...

Least Squares Error

We know

$$\min_{\phi \in \Phi_n} \|f - \phi\|_{2,d\lambda} = \|f - \hat{\phi}_n\|_{2,d\lambda}$$

with

$$\hat{\phi}_n(t) = \sum_{j=1}^{n} \hat{c}_j \pi_j(t), \; \hat{c}_j = \frac{(\pi_j, f)}{(\pi_j, \pi_j)}.$$

- $f - \hat{\phi}_n \perp \Phi_n$: $\hat{\phi}_n$ is orthogonal projection of f onto Φ_n.
- $\|f\|^2 = \|f - \hat{\phi}_n\|^2 + \|\hat{\phi}_n\|^2 \Rightarrow$ Least Square error

$$\|f - \hat{\phi}_n\| = \{\|f\|^2 - \sum_{j=1}^{n} |\hat{c}_j|^2 \|\pi_j\|^2\}^{1/2}$$

- Or by definition: Least Squares error

$$\|f - \hat{\phi}_n\| = \{\int_{\mathbf{R}} (f(t) - \hat{\phi}_n(t))^2 d\lambda(t)\}^{1/2}.$$

Convergence

Given a sequence of linear spaces:

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_n \subset \cdots$$

Clearly,

$$\|f - \hat{\phi}_1\| \geqslant \|f - \hat{\phi}_2\| \geqslant \cdots \geqslant \|f - \hat{\phi}_n\| \geqslant \cdots$$

So

$$\lim_{n\to\infty} \|f - \hat{\phi}_n\| \text{ exists}$$

Convergence

- If the limit is 0, we say the least squares approximation process converges as $n \to \infty$.
- Given any $\epsilon > 0$, there $\exists n_\epsilon, \phi^* \in \Phi_{n_\epsilon}$ s.t., $\|f - \phi^*\| \leqslant \epsilon$. ($\{\Phi_n\}$ is said to be complete w.r.t. $\|\cdot\|$.)

Polynomial Interpolation

Given $\{x_i\}_{i=0}^n$ and $\{f_i = f(x_i)\}_{i=1}^n$ of function f, find a polynomial $p \in \mathbf{P}_n$ s.t.,

$$p(x_i) = f_i,\ i = 0, 1, \ldots, n.$$

Questions of Interest:
existence and uniqueness of p;
error $e(x) = f(x) - p(x)$;
what if more points and higher degrees are allowed?

Example: linear interpolation $p \in \mathbf{P}_1$ $(n = 1)$

$$\begin{aligned} p(x) &= \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \text{ (Lagrange)} \\ &= f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) \text{ (Taylor)} \end{aligned}$$

Lagrange Interpolation; Existence; Uniqueness

Lagrange polynomials

$$p(x) = \sum_{i=0}^{n} f_i l_i(x),$$

with Lagrange basis function (elementary Lagrange interpolation polynomial)

$$l_i(x) = \prod_{j=0;j\neq i}^{n} \frac{x - x_j}{x_i - x_j},\ i = 0, 1, 2, \ldots, n.$$

- Existence: $p(x_i) = f_i,\ i = 0, 1, \ldots, n.$
- Uniqueness: by Fundamental Theorem of Algebra. Denote p as

$$p_n(f; x_0, x_1, \ldots, x_n; x) = p_n(f; x) = \sum_{i=0}^{n} f(x_i) l_i(x).$$

Interpolation Operator

Consider Lagrange interpolation as a linear operator

$$P_n : C[a,\ b] \mapsto \mathbf{P}_n, \quad P_n(\cdot) = p_n(f;\cdot)$$

- $P_n(\alpha f) = \alpha P_n(f)$
- $P_n(f+g) = P_n(f) + P_n(g)$
- $P_n(f) = f,\ \forall f \in \mathbf{P}_n$
- $\|P_n\| = \max_{f\in C[a,\ b]} \frac{\|P_n(f)\|}{\|f\|}$; e.g., $\|P_n\|_\infty = \Lambda_n \equiv \|\lambda_n(x)\|_\infty \equiv \|\sum_{i=0}^n |l_i(x)|\|_\infty$.
- If use best approximation of f on $[a,\ b]$ by polynomials of degree $\leqslant n$ defined as

  $$\mathcal{E}_n(f) \equiv \min_{p\in\mathbf{P}_n} \|f-p\|_\infty = \|f-\hat{p}_n\|_\infty,$$

  we see

  $$\|f-p_n(f;\cdot)\|_\infty = \|f-\hat{p}_n-p_n(f-\hat{p}_n;\cdot)\|_\infty \leqslant \|f-\hat{p}_n\|_\infty + \Lambda_n\|f-\hat{p}_n\|_\infty$$

Interpolation Error

Error of interpolation: $f(x) - p_n(f;x)$ for any $x \neq x_i$ in $[a,\ b]$.

$$f(x) - p_n(f;x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{i=0}^{n}(x - x_i),\ x \in [a,\ b],$$

for some $\xi(x) \in (a,\ b)$.

- Define $F(t) = f(t) - p_n(f;t) - \frac{f(x)-p_n(f;x)}{\prod_{i=0}^{n}(x-x_i)}\prod_{i=0}^{n}(t - x_i)$
- $F \in C^{n+1}[a,\ b]$ (assume $f \in C^{n+1}[a,\ b]$)
- $F(x_i) = 0,\ i = 0, 1, \ldots, n$; $F(x) = 0$.
- By Rolle's Theorem, $F^{n+1}(\xi(x)) = 0$ for some $\xi(x) \in (a,\ b) \Rightarrow$

$$0 = f^{n+1}(\xi(x)) - \frac{f(x) - p_n(f;x)}{\prod_{i=0}^{n}(x - x_i)}(n+1)!.$$

Lagrange Interpolation; Convergence

Convergence

For a triangular array of interpolation nodes on $[a,\ b]$:

$$\begin{array}{llll}
x_0^{(0)} & & & \\
x_0^{(1)} & x_1^{(1)} & & \\
x_0^{(2)} & x_1^{(2)} & x_2^{(2)} & \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots} \\
x_0^{(n)} & x_1^{(n)} & x_2^{(n)} & \cdots x_n^{(n)} \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

define $p_n(x) = p_n(f; x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}; x),\ x \in [a,\ b]$. We say Lagrange interpolation based on the triangular array of nodes converges if $p_n(x) \to f(x)$ as $n \to \infty$, uniformly for $x \in [a,\ b]$.

Convergence

By the interpolation error,

$$|f(x) - p_n(x)| \leqslant (b-a)^{n+1}\frac{M_{n+1}}{(n+1)!},\ x \in [a,\ b],$$

where $|f^{(k)}(x)| \leqslant M_k$ for $a \leqslant x \leqslant b,\ k = 0, 1, 2, \ldots$
then convergence is obtained if $\lim_{k\to\infty} \frac{(b-a)^k}{k!} M_k = 0$.

Convergence

Lagrange interpolation converges (uniformly on $[a,\ b]$) for an arbitrary triangular set of nodes in $[a,\ b]$ if f is analytic in the circular disk C_r centered at $(a+b)/2$ and having radius r s.t., $r > \frac{3}{2}(b-a)$.

Proof by Cauchy's Formula,

$$f^{(k)}(x) = \frac{k!}{2\pi i}\oint_{\partial C_r} \frac{f(z)}{(z-k)^{k+1}} dz,\ x \in [a,\ b].$$

Chebyshev Polynomial Inerpolation; Chebyshev Nodes

Different choice of interpolation nodes: Chebyshev nodes.
Assume considering interval $[-1,\ 1]$.

Chebyshev Polynomial of First Kind

- Chebyshev polynomials of first kind

$$T_n(x) = \cos(n\cos^{-1}(x)).$$

- one can show

$$T_0(x) = 1,\ T_1(x) = x,$$

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),\ k = 1, 2, 3, \ldots$$

- Leading coefficient of T_n is 2^{n-1}; monic Chebyshev polynomial of degree n

$$\mathring{T}_n(x) = \frac{1}{2^{n-1}}T_n(x),\ n \geqslant 1; \mathring{T}_0 = T_0.$$

Chebyshev Nodes

- Zeros of T_n:

$$x_k^{(n)} = \cos\theta_k^{(n)},\ \theta_k^{(n)} = \frac{2k-1}{2n},\ k = 1, 2, \ldots, n.$$

- Extrema of T_n:

$$y_k^{(n)} = \cos\eta_k^{(n)},\ \eta_k^{(n)} = k\frac{\pi}{n},\ k = 0, 1, 2, \ldots, n.$$

Theorem

For any arbitrary monic polynomial $\mathring{p}_n$ of degree n, there holds

$$\max_{-1\leqslant x\leqslant 1} |\mathring{p}_n(x)| \ \geqslant \max_{-1\leqslant x\leqslant 1} |\mathring{T}_n(x)| = \frac{1}{2^{n-1}},\ n \geqslant 1$$

Chebyshev Nodes

- By above theorem, $\|\prod_{i=0}^{n}(x - x_i)\|_\infty$ is minimized with x_i being the zeros of T_{n+1}, given as,

$$\hat{x}_i^{(n)} = \cos\frac{2i+1}{2n+2}\pi, \; i = 0, 1, 2, \ldots, n.$$

With these nodes interpolation error:

$$\|f(\cdot) - p_n(f; \cdot)\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\frac{1}{2^n}.$$

- On triangular array of Chebyshev nodes in $[-1,\ 1]$,

$$p_n(f; \hat{x}_0^{(n)}, \hat{x}_1^{(n)}, \ldots, \hat{x}_n^{(n)}; x) \to f(x) \text{ as } n \to \infty,$$

uniformly on $[-1,\ 1]$ provided that $f \in C^1[-1,\ 1]$.

Fourier Expansion in Chebyshev Polynomial

Orthogonality

$$\int_{-1}^{1} T_k(x)T_l(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, \text{ if } k \neq l \\ \pi, \text{ if } k = l = 0 \\ \frac{\pi}{2}, \text{ if } k = l > 0 \end{cases}$$

Fourier Expansion

$$f(x) = {\sum_{j=0}^{\infty}}' c_j T_j(x) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} c_j T_j(x),$$

where

$$c_j = \frac{2}{\pi}\int_{-1}^{1} f(x)T_j(x)\frac{dx}{\sqrt{1-x^2}}, \; j = 0, 1, 2, \ldots.$$

Truncated sum $\tau_n(x) = {\sum_{j=0}^{n}}' c_j T_j(x)$ approximate f with error

$$f(x) - \tau_n(x) = \sum_{j=n+1}^{\infty} c_j T_j(x) \approx c_{n+1}T_{n+1}(x).$$