# MATH 561 Fall 2015 – Homework Set # 2

Last Name: ______________________ First Name: ______________________

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1. (10 points) Given an integer $n \geq 1$, consider the subdivision $\Delta_n$ of the interval $[0,1]$ into $n$ equal subintervals of length $1/n$. Let $\pi_j(t), j = 0, 1, \ldots, n$ be the function having value 1 at $t = j/n$, decreasing on either side linearly to zero at the neighboring subdivision points (if any), and being zero elsewhere.
(a) Draw a picture of these functions.
(b) Determine $\pi_j(k/n)$ for $j, k = 0, 1, \ldots, n$.
(c) Show that the system $\{\pi_j(t)\}_{j=0}^{n}$ is linearly independent on the interval $0 \leq t \leq 1$. Is it also linearly independent on the set of subdivision points $0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1$ of $\Delta_n$? Explain.
(d) Compute the matrix of the normal equations for $\{\pi_j\}$, assuming $d\lambda(t) = dt$ on $[0,1]$. That is, compute the $(n+1) \times (n+1)$ matrix $\mathbf{A} = [a_{ij}]$, where $a_{ij} = \int_0^1 \pi_i(t)\pi_j(t)dt$.

2. (10 points) (a) For quadratic interpolation on equally spaced points $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h$, derive an upper bound for $\|f - p_2(f; \cdot)\|_\infty$ involving $\|f'''\|_\infty$ and $h$. (Here $\|u\|_\infty = \max_{x_0 \leq x \leq x_2} |u(x)|$.)
(b) Compare the bound obtained in (a) with the analogous one for interpolation at the three Chebyshev points on $[x_0, x_2]$.

3. (20 points) (a) Determine

$$\min \max_{a \leq x \leq b} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n|, \quad n \geq 1,$$

where the minimum is taken over all real $a_0, a_1, \ldots, a_n$ with $a_0 \neq 0$. (Hint: use Chebyshev's Theorem 2.2.1)
(b) Let $a > 1$ and $\mathbf{P}_n^a = \{p \in \mathbf{P}_n : p(a) = 1\}$. Define $\hat{p}_n \in \mathbf{P}_n^a$ by $\hat{p}_n(x) = T_n(x)/T_n(a)$, where $T_n$ is the Chebyshev polynomial of degree $n$, and let $\|\cdot\|_\infty$ denote the maximum norm on the interval $[-1,1]$. Prove:

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty \text{ for all } p \in \mathbf{P}_n^a.$$

(Hint: imitate the proof of Chebyshev's Theorem 2.2.1.)
(c) Let $f$ be a positive function defined on $[a,b]$ and assume

$$\min_{a \leq x \leq b} |f(x)| = m_0, \; \max_{a \leq x \leq b} |f^{(k)}(x)| = M_k, \; k = 0, 1, 2, \ldots.$$

(c.1) Denote by $p_{n-1}(f; \cdot)$ the polynomial of degree $\leq n-1$ interpolating $f$ at the $n$ Chebyshev points (relative to the interval $[a,b]$). Estimate the maximum relative error

$r_n = \max_{a \le x \le b} |(f(x) - p_{n-1}(f;x))/f(x)|$.
(c.2) Apply the result of (c.1) to $f(x) = \ln x$ on $I_r = \{e^r \le x \le e^{r+1}\}$, $r \ge 1$ an integer. In particular, show that $r_n \le \alpha(r,n)c^n$, where $0 < c < 1$ and $\alpha$ is slowly varying. Exhibit $c$.

4. (10 points) Consider $f(t) = \cos^{-1} t, \ -1 \le t \le 1$. Obtain least squares approximation $\hat{\phi}_n \in \mathbf{P}_n$ of $f$ relative to the weight function $w(t) = (1-t)^{-1/2}$; that is find the solution $\phi = \hat{\phi}_n$ of
$$\text{minimize } \{\int_{-1}^{1} [f(t) - \phi(t)]^2 \frac{dt}{\sqrt{1-t^2}} : \phi \in \mathbf{P}_n\}.$$
Express $\hat{\phi}_n$ in terms of Chebyshev polynomials $\pi_j(t) = T_j(t)$.

5. (10 points) (a) Let $x_i^C = \cos(\frac{2i+1}{2n+2}\pi), \ i = 0, 1, \ldots, n$, be Chebyshev points on $[-1, 1]$. Obtain the analogous Chebyshev points $t_i^C$ on $[a, b]$ (where $a < b$) and find an upper bound of $\prod_{i=0}^{n}(t - t_i^C)$ for $a \le t \le b$.
(b) Consider $f(t) = \ln t$ on $[a, b], \quad 0 < a < b$, and let $p_n(t) = p_n(f; t_0^{(n)}, \ldots, t_n^{(n)}; t)$. Given $a > 0$, how large can $b$ be chosen such that $\lim_{n\to\infty} p_n(t) = f(t)$ for arbitrary nodes $t_i^{(n)} \in [a, b]$ and arbitrary $t \in [a, b]$?
(c) Repeat (b), but with $t_i^{(n)} = t_i^C$.

6. (10 points) (a) Use Hermite interpolation to find a polynomial of lowest degree satisfying $p(-1) = p'(-1) = 0, \ p(0) = 1, p(1) = p'(1) = 1$. Simplify your expression for $p$ as much as possible.
(b) Suppose the polynomial $p$ of (a) is used to approximate the function $f(x) = \cos^2(\pi x/2)$ on $-1 \le x \le 1$.
(b.1) Express the error $e(x) = f(x) - p(x)$ (for some fixed $x$ in $[-1, 1]$) in terms of an appropriate derivative of $f$.
(b.2) Find an upper bound for $|e(x)|$ (still for a fixed $x \in [-1, 1]$).
(b.3) Estimate $\max_{-1 \le x \le 1} |e(x)|$.

7. (20 points) Let $\Delta : a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a subdivision of $[a, b]$ into $n-1$ subintervals. Suppose we are given $f_i = f(x_i)$ of some function $f(x)$ at the points $x = x_i, \ i = 1, 2, \ldots, n$. In this problem $s \in \mathbf{S}_2^1$ is a quadratic spline in $C^1[a, b]$ that interpolate $f$ on $\Delta$, that is, $s(x_i) = f_i, \ i = 1, 2, \ldots, n$.
(a) Explain why one expects an additional condition to be required in order to determine $s$ uniquely.
(b) Define $m_i = s'(x_i), \ i = 1, 2, \ldots, n-1$. Determine $p_i \equiv= s|_{[x_i, x_{i+1}]}, \ i = 1, 2, \ldots, n-1$, in terms of $f_i, f_{i+1}$ and $m_i$.
(c) Suppose one takes $m_1 = f'(a)$. (According to (a), this determines $s$ uniquely.) Show how $m_2, m_3, \ldots, m_{n-1}$ can be computed.

8. (10 points) (MATLAB Problem)
(a) Write a Matlab function $y = tridiag(n, a, b, c, v)$ for solving a tridiagonal (nonsym-

metric) system

$$\begin{bmatrix} a_1 & c_1 & & & & 0 \\ b_1 & a_2 & c_2 & & & \\ & b_2 & a_3 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & c_{n-1} \\ 0 & & & & b_{n-1} & a_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$$

by Gaussian elimination without pivoting. Keep the program short.
(b) Write a program for computing the natural spline interpolant $s_{nat}(f;\cdot)$ on an arbitrary partition $a = x_1 < x_2 < \cdots < x_{n-1} < x_n$ of $[a,b]$. Print $\{i, errmaxi; i = 1,2,\ldots,n-1\}$, where

$$errmax(i) = \max_{1\le j\le N} |s_{nat}(f;x_{i,j}) - f(x_{i,j})|,\ x_{i,j} = x_i + \frac{j-1}{N-1}\Delta x_i.$$

(You will need the function *tridiag*.) Test the program for cases in which the error is zeros (what are these, and why?).
(c) Run the program in (b) for $[a,b] = [0,1],\ n = 11,\ N = 51$, and

$$(1)\ \ x_i = \frac{i-1}{n-1},\ i = 1,2,\ldots,n;\ f(x) = e^{-x} \text{ and } f(x) = x^{5/2};$$

$$(2)\ \ x_i = (\frac{i-1}{n-1})^2,\ i = 1,2,\ldots,n;\ f(x) = x^{5/2}.$$

Comment on the results.