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MATH 561 Numerical Analysis I
Final Assignment

1. Let $x_1, x_2, \dots, x_n$, for $n > 1$, be machine numbers. Their product can be computed by the alogirithm

$$\begin{aligned} p_1 &= x_1 \\ p_k &= fl(x_k p_{k-1}), k = 2, 3, \dots, n \end{aligned}$$

   (a) Find an upper bound for the relative error in terms of the machine precision *eps* and n.
   The relative error is given by
$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$
   First consider p_k.
$$\begin{aligned} p_k &= fl(x_k p_{k-1}) \\ &= x_k p_{k-1}(1 + \epsilon_k) \end{aligned}$$
   Where $|\epsilon_k| < eps$, for $k = 1, \cdots, n$
$$< x_k p_{k-1}(1 + eps)$$
   Applying this recursively to p_n, we see that
$$\begin{aligned} p_n &< x_n p_{n-1}(1 + eps) \\ &< x_n x_{n-1} p_{n-2}(1 + eps)^2 \\ &< x_n x_{n-1} x_{n-2} p_{n-3}(1 + eps)^3 \\ &\vdots \\ &< x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} \end{aligned}$$
   Therefore the relative error can be bounded as follows
$$\begin{aligned} E &= \left| \frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\ &< \left| \frac{x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\ &= \left| \frac{x_1 x_2 \cdots x_n \left((1 + eps)^{n-1} - 1\right)}{x_1 x_2 \cdots x_n} \right| \\ &= (1 + eps)^{n-1} - 1 \end{aligned}$$
   Therefore the upper bound for the relative error is $E < (1 + eps)^{n-1} - 1$.

   (b) For any integer r that satisfies $r \times eps < \frac{1}{10}$, show that
$$(1 + eps)^r - 1 < 1.06 \times r \times eps$$
   Hence for n not too large, simplify the answer given in (a).
   Using the Binomial Thereom, $(1 + eps)^r$ can be expanded.
$$(1 + eps)^r - 1 = \sum_{i=0}^{r} \left(\binom{r}{i} 1^{r-i} eps^i \right) - 1$$

$$
\begin{aligned}
&= \sum_{i=1}^{r}\left(\binom{r}{i}eps^i\right)\\
&= r\cdot eps + \binom{r}{2}eps^2 + \binom{r}{3}eps^3 + \cdots + eps^r\\
&= r\cdot eps + \frac{r(r-1)}{2}eps^2 + \frac{r(r-1)(r-2)}{3!}eps^3 + \cdots + eps^r\\
&= r\cdot eps\Big(1 + \frac{r-1}{2}eps + \frac{(r-1)(r-2)}{3!}eps^2 + \cdots + \frac{(r-1)(r-2)\cdots(1)}{r!}eps^{r-1}\Big)
\end{aligned}
$$

Since $r \times eps < \frac{1}{10}$, $(r-i)eps < \frac{1}{10}$ for any $0 < i < r$

$$
\begin{aligned}
&< r\cdot eps\left(1 + \frac{1}{2}\frac{1}{10} + \frac{1}{3!}\left(\frac{1}{10}\right)^2 + \cdots + \frac{1}{r!}\left(\frac{1}{10}\right)^{r-1}\right)\\
&= r\cdot eps\sum_{k=0}^{r-1}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k-1}\right)\\
&= r\cdot eps\cdot 10\sum_{k=1}^{r-1}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k}\right)
\end{aligned}
$$

This expression is certainly less than extending the sum to infinity because all of the terms are postive. Also this sum is the Taylor series for $e^x - 1$.

$$
\begin{aligned}
&< r\cdot eps\cdot 10\sum_{k=1}^{\infty}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k}\right)\\
&= r\cdot eps\cdot 10\Big(e^{1/10} - 1\Big)\\
&\approx 1.05171 r\cdot eps\\
&< 1.06 r\cdot eps
\end{aligned}
$$

This result can now be used to simplify the result of part (a). Now if n is not too large, then $|E| < 1.06(n-1)eps$.

2. (a) Determine

$$\min\max *_{a\le x\le b}\Big|a_0x^n + a_1x^{n-1} + \cdots + a_n\Big|$$

for $n \ge 1$ where the minimum is taken over the coefficients $a_0, a_1, \ldots, a_n$ with $a_0 \ne 0$.
First lets apply a linear transformation from the interval $[a, b]$ to $[-1, 1]$, by letting $x = \frac{b-a}{2}t + \frac{b+a}{2}$. This is then equivalent to

$$
\begin{aligned}
&\min\max *_{-1\le t\le 1}\left|a_0\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) + a_1\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)^{n-1} + \cdots + a_n\right|\\
&= \min\max *_{-1\le t\le 1}\left|a_0\left(\frac{b-a}{2}\right)^n t^n + b_1t^{n-1} + \cdots + b_n\right|\\
&= |a_0|\left(\frac{b-a}{2}\right)^n \min\max *{-1\le t\le 1}\Big|t^n + b_1t^{n-1} + \cdots + b_n\Big|
\end{aligned}
$$

From Chebychev's Theorem the monic polynomial with minimum maximum value over $[-1, 1]$ is the monic Chebychev polynomial

$$= |a_0|\left(\frac{b-a}{2}\right)^n \max *{-1\le t\le 1}\Big|\mathring{T}_n(x)\Big|$$

Also from Chebyshev's Theorem, $\max *{-1 \le t \le 1}\left|\mathring{T}_n(x)\right| = \frac{1}{2^{n-1}}$

$$= |a_0|\left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}}$$
$$= 2|a_0|\left(\frac{b-a}{4}\right)^n$$

Thus given an arbitrary choice of $a_0 \neq 0$,

$$\min \max *_{a \le x \le b}\left|a_0 x^n + a_1 x^{n-1} + \cdots + a_n\right| = 2|a_0|\left(\frac{b-a}{4}\right)^n$$

(b) Let $a > 1$ and $\mathbb{P}_n^a = \{p \in \mathbb{P}_n | p(a) = 1\}$. Define $\hat{p}_n \in \mathbb{P}_n^a$ by $\hat{p}_n = T_n(x)/T_n(a)$, where $T_n(x)$ is the Chebyshev polynomial of degree n. Prove that $\|\hat{p}_n\|_\infty \le \|p\|_\infty$ for all $p \in \mathbb{P}_n^a$.

Proof. Assume to the contrary that there exists $p \in \mathbb{P}_n^a$ such that $\|p\|_\infty < \|\hat{p}_n\|_\infty$. Define the polynomial $d(x) = \hat{p}_n(x) - p(x)$. Since d is the difference of two degree n polynomials, the degree of d can be at most n.

Let $\{y_k\}_{k=0}^n$ denote the $n+1$ extrema points for the Chebyshev polynomial $T_n(x)$, that is $T_n(y_k) = (-1)^k$. Obviously $\hat{p}_n$ is just a scaling of $T_n(x)$, so $\|\hat{p}_n\|_\infty = \|T_n(x)\|_\infty / |T_n(a)| = |T_n(y_k)/T_n(a)| = |\hat{p}_n(y_k)|$ □

(c)

3.

4. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Consider a function $f \in C^\infty[a, b]$.

(a) Define what it means for a function S to be a linear spline that interpolates f at all the points x_i for $i = 0, 1, \ldots, n$. Give a formula for S in terms of the point values of f.

In order to define the linear spline, I will first define a set of linear basis functions. Let B_i for $i = 1, 2, \ldots, n-1$ be defined on $[a, b]$ as follows.

$$B_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Also let B_1 and B_n be defined as follows

$$B_1(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & a = x_0 \le x \le x_1 \\ 0 & x > x_1 \end{cases}$$
$$B_n(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0} & x_{n-1} \le x \le x_n = b \\ 0 & x < x_{n-1} \end{cases}$$

A linear spline on $[a, b]$ that interpolates f on the partition $\{x_i\}_{i=0}^n$ is a function $S(x)$ that is a linear combination of the basis functions B_i such that $S(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

Thus a formula for $S(x)$ could be written as $S(x) = \sum_{i=0}^n (f(x_i)B_i(x))$.

(b) Let $h = \max *_{0 \le i \le n-1}(x_{i+1} - x_i)$. Derive an upper bound on $|f(x) - S(x)|$ on $x \in [a, b]$. Use this to prove that $\lim_{h \to 0}(|f(x) - S(x)|) = 0$ for $x \in [a, b]$ and state the rate of convergence.

(c) Define what it means for S to be a clamped cubic spline that interpolates f at all the points x_i, for $i = 0, 1, \ldots, n$.

5.