MATH 561 Fall 2015 – Homework Set # 1

Last Name: ____________________ First Name: ____________________

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1. (10 points) Let $f(x) = \sqrt{1+x^2} - 1$.
(a) Explain the difficulty of computing $f(x)$ for a small value of $|x|$ and show how it can be circumvented.
(b) Compute $(cond\ f)(x)$ and discuss the conditioning of $f(x)$ for small $|x|$.
(c) How can the answers to (a) and (b) be reconciled?

2. (10 points) Let $f(x) = (1 - \cos x)/x,\ x \neq 0$.
(a) Show that direct evaluation of f is inaccurate if $|x|$ is small; assume $fl(f(x)) = fl((1 - fl(\cos x))/x)$, where $fl(\cos x) = (1 + \epsilon_r)\cos x$, and estimate the relative error ϵ_f of $fl(f(x))$ as $x \to 0$.
(b) A mathematically equivalent form of f is $f(x) = \sin^2 x/(x(1 + \cos x))$. Carry out a similar analysis as in (a), based on $fl(f(x)) = fl([fl(\sin x)]^2/fl(x(1 + fl(\cos x))))$, assuming $fl(\cos x) = (1 + \epsilon_c)\cos x,\ fl(\sin x) = (1 + \epsilon_s)\sin x$ and retaining only first-order terms in ϵ_s and ϵ_c. Discuss the result.
(c) Determine the condition of $f(x)$. Indicate for what values of x (if any) $f(x)$ is ill-conditioned. ($|x|$ is no longer small, necessarily.).

3. (20 points) Consider the algebraic equation
$$x^n + ax - 1 = 0,\ a > 0,\ n \geq 2.$$
(a) Show that the equation has exactly one positive root $\xi(a)$.
(b) Obtain a formula for $(cond\ \xi)(a)$.
(c) Obtain (good) upper and lower bounds for $(cond\ \xi)(a)$.

4. (20 points) In the theory of Fourier series, the numbers
$$\lambda_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k} \tan\frac{k\pi}{2n+1},\ n = 1, 2, 3, \ldots,$$
known as Lebesgue constants, are of some importance.
(a) Show that the terms in the sum increases monotonically with k. How do the terms for k near n behave when n is large?
(b) Compute λ_n for $n = 1, 10, 10^2, \ldots, 10^5$ in MATLAB single and double precision and compare the results. Do the same with n replaced by $\lceil n/2 \rceil$. Explain what you observe.

5. (20 points) Let $x_0, x_1, \ldots, x_n$ be pairwise distinct points in $[a,\ b]$, $-\infty < a < b < \infty$, and $f \in C^1[a,\ b]$. Show that, given any $\epsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and, at the same time, $p(x_i) = f(x_i)$, $i = 0, 1, \ldots, n$. (Hint: write $p = p_n(f; \cdot) + \omega_n q$ where $\omega_n(x) = \prod_{i=0}^{n}(x - x_i)$, $\quad q \in \mathbf{P}$, and apply Weierstrass's approximation theorem.)

6. (20 points) Suppose you want to approximate the step function

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t > 1 \end{cases}$$

on the positive line $\mathbf{R}_+$ by a linear combination of exponentials $\pi_j(t) = e^{-jt}$, $j = 1, 2, \ldots, n$, in the (continuous, equally weighted) least squares sense.
(a) Derive the normal equations. How is the matrix related to the Hilbert matrix?
(b) Use MATLAB to solve the normal equations for $n = 1, 2, \ldots, 8$. Print n, the Euclidean condition number of the matrix, along with the solution. Plot the approximations vs. the exact function for $1 \le n \le 4$.