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MATH 561 Numerical Analysis I
Homework 1

1. Let $f(x) = \sqrt{1 + x^2} - 1$

   (a) For small values of $|x|$, $f(x)$ can be difficult to compute becuase $x^2 \approx 0$ and $\sqrt{1 + x^2} \approx 1$. This causes $f(x)$ to be taking the difference to two numbers that are approximately equal, which can cause a loss of accuracy. This can be circumvented by noting that $f(x)$ can be expressed as follows.

$$\begin{aligned} f(x) &= \sqrt{1+x^2} - 1 \\ &= \sqrt{1+x^2} - 1 \times \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} \\ &= \frac{x^2}{\sqrt{1+x^2}+1} \end{aligned}$$

   (b) The condition number of $f(x)$ can be determined as follows

$$\begin{aligned} (condf)(x) &= \left|\frac{xf'(x)}{f(x)}\right| \\ &= \left|\frac{x^2}{\sqrt{1+x^2}\left(\sqrt{1+x^2}-1\right)}\right| \\ &= \left|\frac{x^2}{1+x^2-\sqrt{1+x^2}}\right| \end{aligned}$$

   As $|x| \to 0$, the use of L'Hopital's rule is necessary.

$$\lim_{x\to 0}((cond\, f)(x)) = \left|\frac{2x}{2x - \frac{x}{\sqrt{1+x^2}}}\right|\Big|_{x=0}$$

   L'Hopital's rule can be applied again

$$\begin{aligned} &= \left|\frac{2}{2 - \frac{\sqrt{1+x^2}-x^2}{(1+x^2)\sqrt{1+x^2}}}\right|\Big|_{x=0} \\ &= \left|\frac{2}{2 - \frac{\sqrt{1}}{(1)\sqrt{1}}}\right| \\ &= 2 \end{aligned}$$

Therefore for small x the $(cond\, f)(x) \approx 2$.

(c) The condition number of $f(x)$ doesn't take into account taking the difference of two numbers that are approximately equal.

2. Let $f(x) = (1 - \cos(x))/x$, $x \neq 0$.

(a)

$$
\begin{aligned}
fl(f(x)) &= fl\left(\frac{1 - fl(\cos(x))}{x}\right) \\
&= fl\left(\frac{1 - (1 + \epsilon_r)\cos(x)}{x}\right) \\
&= fl\left(\frac{fl(1 - (1 + \epsilon_r)\cos(x))}{fl(x)}\right) \\
&= fl\left(\frac{(1 - \cos(x))\left(1 - \frac{\cos(x)}{1 - \cos(x)}\epsilon_r\right)}{x(1 + \epsilon_x)}\right) \\
&= \left(\frac{1 - \cos(x)}{x}\right)\left(1 - \frac{\cos(x)}{1 - \cos(x)}\epsilon_r - \epsilon_x\right) \\
\epsilon_f &= -\frac{\cos(x)}{1 - \cos(x)}\epsilon_r - \epsilon_x \\
\lim_{x \to 0}(|\epsilon_f|) &= \left|-\frac{\cos(0)}{1 - \cos(0)}\epsilon_r - \epsilon_x\right| \\
&= \infty
\end{aligned}
$$

(b)

$$\begin{aligned}
fl(f(x)) &= fl\left(\frac{fl(\sin(x))^2}{fl(x(1+fl(\cos(x))))}\right) \\
&= fl\left(\frac{((1+\epsilon_s)\sin(x))^2}{fl(x(1+(1+\epsilon_c)\cos(x)))}\right) \\
&= fl\left(\frac{(1+2\epsilon_s)\sin(x)^2}{fl(x(1+\cos(x))(1+\frac{\cos(x)}{1+\cos(x)}\epsilon_c)}\right) \\
&= fl\left(\frac{(1+2\epsilon_s)\sin(x)^2}{x(1+\cos(x))(1+\frac{\cos(x)}{1+\cos(x)}\epsilon_c+\epsilon_x)}\right) \\
&= \left(\frac{\sin(x)^2}{x(1+\cos(x))}(1+2\epsilon_s-\frac{\cos(x)}{1+\cos(x)}\epsilon_c-\epsilon_x)\right) \\
\epsilon_f &= 2\epsilon_s-\frac{\cos(x)}{1+\cos(x)}\epsilon_c-\epsilon_x \\
\lim_{x\to 0}(|\epsilon_f|) &= \left|2\epsilon_s-\frac{1}{2}\epsilon_c-\epsilon_x\right|
\end{aligned}$$

(c)

$$\begin{aligned}
(cond\, f)(x) &= \left|\frac{xf'(x)}{f(x)}\right| \\
&= \left|\frac{\frac{x\sin(x)-(1-\cos(x))}{x}}{\frac{(1-\cos(x))}{x}}\right| \\
&= \left|\frac{x\sin(x)-(1-\cos(x))}{(1-\cos(x))}\right|
\end{aligned}$$

As $x \to 0$, both the numerator and the denominator go to 0 so L'Hopital's rule must be used

$$\lim_{x\to 0}((cond\, f)(x)) = \left|\frac{\sin(x)+x\cos(x)-\sin(x)}{\sin(x)}\right|\Bigg|_{x=0} \quad = \left|\frac{1+0-1}{1}\right| = 0$$

3. Let $f(x) = x^n + ax - 1$, $a > 0$, $n \geq 2$

(a) Show that $f(x)$ has exactly one positive root $\xi(a)$. First note that $f(0) = -1$ and $f(1) = a > 0$. Since f is a polynomial and is continuous, by the Intermediate Value Theorem, there must exist $c \in (0,1)$, such that $f(c) = 0$. Therefore f has at least on root in the interval $(0,1)$. Also $f'(x) = nx^{n-1} + a$, for $x \geq 0$, $f'(x) > 0$, so f is a strictly increasing function on the interval $[0,\infty)$. Therefore there is only one positive root of $f(x)$ and it is in the interval $(0,1)$. Let $\xi(a)$ be this root.

(b) Obtain a formula for $(cond\,\xi)(a)$. The derivitive of $\xi(a)$ can be found by implicit differentiation of $f(\xi(a))$.

$$\begin{aligned} f(\xi(a)) &= 0 \\ \xi(a)^n + a\xi(a) - 1 &= 0 \end{aligned}$$

By differentiating with respect to a

$$\begin{aligned} n\xi(a)^{n-1}\xi'(a) + a\xi'(a) + \xi(a) &= 0 \\ \xi'(a) &= \frac{-\xi(a)}{n\xi(a)^{n-1} + a} \end{aligned}$$

Also it can be noted that

$$\begin{aligned} \xi(a)^n + a\xi(a) - 1 &= 0 \\ \xi(a)^n &= 1 - a\xi(a) \\ \xi(a)^{n-1} &= \frac{1 - a\xi(a)}{\xi(a)} \end{aligned}$$

Then $\xi'(a)$ can be expressed as

$$\begin{aligned} \xi'(a) &= \frac{-\xi(a)}{n\frac{1-a\xi(a)}{\xi(a)} + a} \\ \xi'(a) &= \frac{-\xi(a)^2}{n - an\xi(a) + a\xi(a)} \end{aligned}$$

The condition number of $\xi(a)$ can then be found

$$\begin{aligned} (cond\,\xi)(a) &= \left|\frac{a\xi'(a)}{\xi(a)}\right| \\ &= \left|\frac{a\frac{-\xi(a)^2}{n-an\xi(a)+a\xi(a)}}{\xi(a)}\right| \\ &= \left|\frac{-a\xi(a)}{n - an\xi(a) + a\xi(a)}\right| \\ &= \frac{a\xi(a)}{n + (1-n)a\xi(a)} \end{aligned}$$

(c) Since $0 < \xi(a) < 1$, bounds for the condition number of $\xi(a)$ can be found.

$$\lim_{\xi(a)\to 0}\left(\frac{a\xi(a)}{n+(1-n)a\xi(a)}\right) = 0$$
$$\lim_{\xi(a)\to 1}\left(\frac{a\xi(a)}{n+(1-n)a\xi(a)}\right) = \frac{a}{n+(1-n)a}$$

Therefore $0 < (cond\,\xi)(a) < \frac{a}{n+(1-n)a}$.

4. (a) Let $x = \frac{k\pi}{2n+1}$, then the terms of the summation are $f(x) = \frac{1}{x}\tan(x)$ to a positive constant. Note that $xf(x) = tan(x)$ and $[xf(x)]' = \sec(x)^2 = \frac{1}{\cos(x)^2}$. Also by the product rule $[xf(x)]' = xf'(x) + f(x)$. Therefore

$$\begin{aligned} xf'(x) &= \frac{1}{\cos(x)^2} - f(x) \\ &= \frac{1}{\cos(x)^2} - \frac{\sin(x)}{x\cos(x)} \\ &= \frac{1}{\cos(x)^2}\left(1 - \frac{\sin(x)\cos(x)}{x}\right) \\ &= \frac{1}{\cos(x)^2}\left(1 - \frac{\sin(2x)}{2x}\right) \end{aligned}$$

for $0 \le x \le \pi/2$

$$xf'(x) > 0$$

Therefore the $f(x)$ is monotonically increasing and the terms of the sum are monotonically increasing. When n is large and k approaches n, $\tan\left(\frac{k\pi}{2n+1}\right) \to \tan\left(\frac{\pi}{2}\right)$. So the terms of the sum become very large and dominate the overall sum.

(b)

n	Single	Double	Difference
1	1.43599117	1.4359911241769170	4.3845e-08
10	2.22335672	2.2233569241536824	2.0037e-07
100	3.13877439	3.1387800926548399	5.6977e-06
1000	4.07021761	4.0701636043526701	5.4005e-05
10000	5.00338697	5.0031838616315740	2.0311e-04
100000	5.93930912	5.9363682124964559	2.9409e-03

```
function [lambdaN] = calculateLebesgueConstant(n, varargin)
    p = inputParser;
    p.addRequired('n', @Utils.checkInteger);
    p.addParameter('UseSinglePrecision', false, @islogical);
    p.parse(n, varargin{:});

    k = 1:n;
    if(p.Results.UseSinglePrecision)
        lambdaN = single(1/(2*n + 1)) + ...
            single((2/pi))*sum(single(tan(k*pi/single((2*n+1)))./k));
    else
        lambdaN = 1/(2*n + 1) + 2/pi*sum(tan(k*pi/(2*n+1))./k);
    end
end
```

```
% Problem #4
for i=0:5
    n = 10^i;
    %n = ceil(n/2);
    lambda = calculateLebesgueConstant(n);
    lambdaSingle = calculateLebesgueConstant(n, ...
        'UseSinglePrecision', true);
    d = abs(lambda - lambdaSingle);
    fprintf('%10.0f %12.8f %19.16f %12.4e\n', n, lambdaSingle, ...
        lambda, d);
end
```

5. Let $x_0, x_1, \ldots, x_n$ be pairwise distinct points in $[a, b]$, $-\infty < a < b < \infty$, and $f \in C^1[a, b]$. Show that given any $\epsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and at the same time $p(x_i) = f(x_i)$, for $i = 0, 1, \ldots, n$.

   Proof. Let $p = p_n(f; \cdot) + \omega_n q$, where $p_n(f; \cdot)$ is the Lagrange interpolation of f at $x_1, x_2, \ldots, x_n$, $\omega_n = \prod_{i=0}^{n}(x - x_i)$, and q is some polynomial. Firstly note that $p(x_i) = p_n(x_i) + 0q = f(x_i)$, so the condition of equality at the points x_i is met. Secondly note $\|f - p\|_\infty = \|f - p_n - \omega_n q\|_\infty = \|\omega_n\|_\infty \left\|\frac{f - p_n}{\omega_n} - q\right\|_\infty$. Consider the function $g(x) = \frac{f(x) - p_n(x)}{\omega_n(x)}$. Since $g(x)$ is composed of continuous functions on $[a, b]$, $g(x)$ is continuous on $[a, b]$ everywhere $\omega_n(x) \neq 0$. The function $\omega_n(x) = 0$ at x_i for $i = 1, 2, \ldots, n$. Therefore the limit of $g(x)$ as $x \to x_i$ needs to be considered. At $x = x_i$, $f(x) - p_n(x) = 0$ and $\omega_n(x) = 0$, therefore L'Hopital's rule can be employed. Therefore $\lim_{x \to x_i}(g(x)) = \lim_{x \to x_i}\left(\frac{f'(x) - p_n'(x)}{w_n'(x)}\right)$. Remember that $f \in C^1[a, b]$, so f is differentiable, and p_n is trivially differentiable. Also ω_n is differentiable and $\omega_n'(x) = \sum_{i=1}^{n}\left(\prod_{k=1, k \neq i}^{n}(x - x_k)\right)$ by repeated use of the product rule. Therefore $\omega_n'(x_i) =$

$\prod_{k=1,k\neq i}^{n}(x_i - x_k) \neq 0$. Thus $\lim_{x\to x_i}\left(\frac{f'(x)-p_n'(x)}{w_n'(x)}\right)$ exists so $g(x)$ is continuous at x_i. Then by the Weierstrass Approximation Theorem there exists a polynomial q such that $\left\|\frac{f-p_n}{\omega_n} - q\right\|_\infty < \epsilon/\|\omega_n\|_\infty$. Thus $\|f-p\|_\infty < \|\omega_n\|_\infty \epsilon/\|\omega_n\|_\infty = \epsilon$. $\square$

6. (a) The normal equations are $\sum_{j=1}^{n}((\pi_i,\pi_j)c_j) = (\pi_i, f)$.

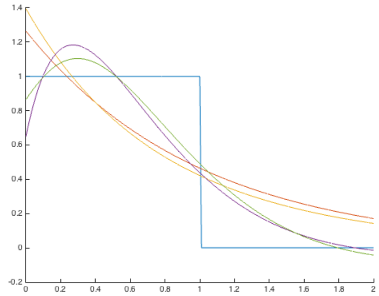
$$
\begin{aligned}
(\pi_i,\pi_j) &= \int_{-\infty}^{\infty}\pi_i\pi_j \,\mathrm{d}\lambda \\
&= \int 0\infty e^{-it}e^{-jt}dt \\
&= \int 0\infty e^{-t(i+j)}dt \\
&= -\frac{1}{i+j}e^{-t(i+j)}\Big|_{x=0}^{\infty} \\
&= \frac{1}{i+j}
\end{aligned}
$$

$$
\begin{aligned}
(\pi_i, f) &= \int_{-\infty}^{\infty}\pi_i f\,\mathrm{d}\lambda \\
&= \int_0^1 e^{-it}\,\mathrm{d}t \\
&= -\frac{1}{i}e^{-it}\Big|_{x=0}^{1} \\
&= -\frac{1}{i}e^{-i} + \frac{1}{i} \\
&= \frac{1}{i}\left(1 - e^{-i}\right)
\end{aligned}
$$

Therefore the normal equations are $\sum_{j=1}^{n}\left(\frac{1}{i+j}c_j\right) = \frac{1}{i}(1-e^{-i})$. The matrix is related to the Hilbert matrix, in that it is a Hilbert matrix with the first column and last row removed.

(b)
```
>> H01
  n      cond(A)      solution
  1    1.0000e+00
                   1.26424111765712e+00
  2    3.8474e+01
                   1.00219345775339e+00
                   3.93071489855588e-01
  3    1.3533e+03
                  -1.23430987802221e+00
```

		9.33908483295806e+00
		-7.45501111925202e+00
4	4.5880e+04	
		-2.09728726098058e+00
		1.58114152051498e+01
		-2.03996718636354e+01
		7.55105210089050e+00
5	1.5350e+06	
		2.95960905277525e-01
		-1.29075627900274e+01
		8.01167511168169e+01
		-1.26470845210468e+02
		6.03098537894039e+01
6	5.1098e+07	
		2.68879580081284e+00
		-5.47821734658646e+01
		3.03448008134088e+02
		-6.28966173619730e+02
		5.62805181711912e+02
		-1.84248287003429e+02
7	1.6978e+09	
		1.19410817469907e+00
		-1.89096708116122e+01
		3.44041842399165e+01
		2.67845885790884e+02
		-9.16936654202640e+02
		9.99543946154416e+02
		-3.66411997771822e+02
8	5.6392e+10	
		-2.39677067760203e+00
		9.42030349713750e+01
		-1.09671436566487e+03
		5.45233178120852e+03
		-1.33586106370687e+04
		1.71754822449684e+04
		-1.11496309593916e+04
		2.88841221308708e+03




```
% Problem #6
figure
hold on
x = 0:.01:2;
y = ones(1,length(x));
y(x > 1) = 0;
plot(x, y);

disp('        n      cond(A)       solution');
for n=1:8
    A = 1./(repmat((1:n), n, 1) + repmat((1:n)', 1, n));
    i = (1:n)';
    b = 1./i .* (1 - exp(-i));
    c = inv(A) * b;
    con = cond(A);
    fprintf('%8.0f %12.4e\n', n, con);
    fprintf('%45.14e\n', c);
    if(i ≤ 4)
        y = c' * exp(-i*x);
        plot(x, y);
    end
end
hold off
```