MATH 561 Fall 2015 – Final Assignment

Last Name: ______________________ First Name: ______________________

UID: ______________________ Siganuture: ______________________

1. (10 points) Let $x_1, x_2, \ldots, x_n$, $n > 1$, be machine numbers. Their product can be computed by the algorithm

$$p_1 = x_1,$$
$$p_k = fl(x_k p_{k-1}),\ \ k = 2, 3, \ldots, n.$$

(a) Find an upper bound for the relative error in terms of the machine precision eps and n,

$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$

(b) For any integer $r \geq 1$ not too large so as to satisfy $r \cdot eps < 1/10$, show that

$$(1 + eps)^r - 1 < 1.06 \cdot r \cdot eps.$$

Hence, for n not too large, simplify the answer given in (a). (Hint: use the binomial theorem)

2. (30 points) (a) Determine

$$\min \max_{a \leq x \leq b} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n|, \quad n \geq 1,$$

where the minimum is taken over all real $a_0, a_1, \ldots, a_n$ with $a_0 \neq 0$. (Hint: use Chebyshev's Theorem 2.2.1)
(b) Let $a > 1$ and $\mathbf{P}_n^a = \{p \in \mathbf{P}_n : p(a) = 1\}$. Define $\hat{p}_n \in \mathbf{P}_n^a$ by $\hat{p}_n(x) = T_n(x)/T_n(a)$, where T_n is the Chebyshev polynomial of degree n, and let $\|\cdot\|_\infty$ denote the maximum norm on the interval $[-1, 1]$. Prove:

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty \text{ for all } p \in \mathbf{P}_n^a.$$

(Hint: imitate the proof of Chebyshev's Theorem 2.2.1.)
(c) Let f be a positive function defined on $[a, b]$ and assume

$$\min_{a \leq x \leq b} |f(x)| = m_0,\ \max_{a \leq x \leq b} |f^{(k)}(x)| = M_k,\ \ k = 0, 1, 2, \ldots.$$

(c.1) Denote by $p_{n-1}(f; \cdot)$ the polynomial of degree $\leq n-1$ interpolating f at the n Chebyshev points (relative to the interval $[a, b]$). Estimate the maximum relative error $r_n = \max_{a \leq x \leq b} |(f(x) - p_{n-1}(f; x))/f(x)|$.
(c.2) Apply the result of (c.1) to $f(x) = \ln x$ on $I_r = \{e^r \leq x \leq e^{r+1}\}$, $r \geq 1$ an integer. In particular, show that $r_n \leq \alpha(r, n)c^n$, where $0 < c < 1$ and α is slowly varying. Exhibit c.

3. (20 points) Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined and integrable on $[-1, 1]$. Let

$$-1 = x_0 < x_1 < \cdots < x_n = 1$$

be a partition of $[-1, 1]$. Consider the following numerical quadrature

$$I(f) \equiv \int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \equiv I_n(f),$$

where

$$w_i = \int_{-1}^{1} L_i(x)dx \text{ with } L_i(x) = \prod_{k=0,k\neq i}^{n} \frac{x - x_k}{x_i - x_k} \text{ for } i = 0, 1, \ldots, n$$

(a) Prove that if n is even and the quadrature points are evenly spaced: $x_i = -1 + ih$ with $h = 2/n$, then the numerical quadrature is exact for polynomial of degree $n + 1$.

(b) Let $n = 2$ and let $x_0 = -1, x_1 = 0$, and $x_2 = 1$. Compute w_0, w_1, and w_2, and explicitly write out the numerical quadrature formula in this case.

(c) When $n = 2$ and let $x_0 = -1, x_1 = 0$, and $x_2 = 1$, what is the degree of precision of the numerical quadrature formula? (Completely justify your answer).

4. (20 points) Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$. Consider a function $f \in C^\infty[a, b]$.

(a) Define what it means for a S to be a linear spline that interpolates f at all the points x_i for $i = 0, 1, \ldots, n$. Give a formula for S in terms of the point values of f.

(b) Let

$$h = \max_{0\leq i\leq n-1} (x_{i+1} - x_i).$$

Derive an upper bound on

$$|f(x) - S(x)| \text{ for } x \in [a, b].$$

Use this to prove that

$$\lim_{h\to 0} |f(x) - S(x)| = 0 \text{ for } x \in [a, b],$$

and state the rate of convergence.

(c) Define what it means for a S to be a clamped cubic spline that interpolates f at all the points x_i, $i = 0, 1, \ldots, n$. (You must include a full definition for a cubic spline, including the clamped part.)

5. (20 points)

(a) Prove the following theorem: consider the system of initial value problems:

$$\mathbf{y}' = \mathbf{f}(\mathbf{y})$$

and apply to it the forward Euler method:

$$\mathbf{u}_{n+1} = \mathbf{F}(\mathbf{u}_n) \equiv \mathbf{u}_n + h\mathbf{f}(\mathbf{u}_n)$$

Then

- $\boldsymbol{\alpha}$ is a fixed point of the Euler method $\mathbf{F}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ if and only if $\boldsymbol{\alpha}$ is a fixed point of the initial value problem ($\mathbf{f}(\boldsymbol{\alpha}) = \mathbf{0}$).
- If $\boldsymbol{\alpha}$ is a linearly stable fixed point of the initial value problem (i.e., all the eigenvalues of the matrix $\partial\mathbf{f}/\partial\mathbf{y}(\boldsymbol{\alpha})$ have negative real parts) and if $|1 + h\lambda_p| < 1$ for each eigenvalue λ_p of $\partial\mathbf{f}/\partial\mathbf{y}(\boldsymbol{\alpha})$, then $\boldsymbol{\alpha}$ is also a linearly stable fixed point of the Euler method.

(b) The fixed points of the Logistic growth equation:

$$y' = f(y) = 2y(1 - y)$$

are $y = 0$ (unstable since $f'(0) = 2$) and $y = 1$ (stable since $f'(1) = -2$). Apply the Euler method to this equation and find and classify all fixed points of the Euler method as a function of the time-step parameter h