MATH 561 Fall 2015 – Homework Set # 3

Last Name: ______________________ First Name: ______________________

UID: ______________________ Siganuture: ______________________

1. (10 points)
   (a) Use the central quotient approximation $f'(x) \approx [f(x+h) - f(x-h)]/(2h)$ of the first derivative to obtain an approximation of $\dfrac{\partial^2 u}{\partial x \partial y}(x, y)$ for a function of u of two variables.
   (b) Use Taylor expansion of a function of two variables to show that the error of the approximation derived in (a) is $O(h^2)$.

2. (20 points) Let s be the function defined by

$$s(x) = \begin{cases} (x+1)^3 & \text{if } -1 \leq x \leq 0, \\ (1-x)^3 & \text{if } 0 \leq x \leq 1. \end{cases}$$

   (a) With Δ denoting the subdivision of $[-1, 1]$ into the two subintervals $[-1, 0]$ and $[0, 1]$, to what class $\mathbf{S}_m^k(\Delta)$ does the spline s belong?
   (b) Estimate the error of the composite trapezoidal rule applied to $\int_{-1}^{1} s(x)dx$, when $[-1, 1]$ is divided into n subintervals of equal length $h = 2/n$ and n is even.
   (c) What is the error of the composite Simpson's rule applied to $\int_{-1}^{1} s(x)dx$, with the same subdivision of $[-1, 1]$ as in (b)?
   (d) What is the error resulting from applying the 2-point Gauss-Legendre rule to $\int_{-1}^{0} s(x)dx$ and $\int_{0}^{1} s(x)dx$ separately and summing?

3. (20 points)
   (a) Determine by Newton's interpolation formula the quadratic polynomial p interpolating f at $x = 0$ and $x = 1$ and f' at $x = 0$. Also, express the error in terms of an appropriate derivative (assumed continuous on $[0, 1]$).
   (b) Based on the result of (a), derive an integration formula of the type

$$\int_0^1 f(x)dx = a_0 f(0) + a_1 f(1) + b_0 f'(0) + E(f).$$

   Determine a_0, a_1, b_0 and an appropriate expression for $E(f)$.
   (c) Transform the result of (b) to obtain an integration rule, with remainder, for $\int_c^{c+h} y(t)dt$, where $h > 0$. (Do not rederive this rule from scratch).

4. (20 points)
(a) Construct the quadratic (monic) polynomial $\pi_2(t;w)$ orthogonal on $(0,\infty)$ with respect to the weight function $w(t)=e^{-t}$. (Hint: use $\int_0^\infty t^m e^{-t}dt = m!$.)
(b) Obtain the two-point Gauss-Laguerre quadrature formula,

$$\int_0^\infty f(t)e^{-t}dt = w_1 f(t_1) + w_2 f(t_2) + E_2(f),$$

including a representation for the remainder $E_2(f)$.
(c) Apply the formula in (b) to approximate $I=\int_0^\infty d^{-t}dt/(t+1)$. Use the remainder term $E_2(f)$ to estimate the error, and compare your estimate with the true error (use $I = 0.596347361\ldots$). Knowing the true error, identify the unknown quantity $\xi > 0$ contained in the error term $E_2(f)$.

5. (20 points) Consider a quadrature formula of the type

$$\int_0^\infty e^{-x} f(x)dx = af(0) + bf(c) + E(f).$$

(a) Find a, b, c such that the formula has degree of exactness $d = 2$. Can you identify the formula so obtained?
(b) Let $p_2(x) = p_2(f;0,2,2;x)$ be the Hermite interpolation polynomial interpolating f at the (simple) point $x=0$ and the double point $x=2$. Determine $\int_0^\infty e^{-x}p_2(x)dx$ and compare with the result in (a).
(c) Obtain the remainder $E(f)$ in the form $E(f) = const \cdot f'''(\xi)$, $\xi > 0$.

6. (10 points) MATLAB
(a) Let $h_k = (b-a)/2^k$, $k = 0, 1, \ldots$. Denote by

$$T_{h_k}(f) = h_k\left(\frac{1}{2}f(a) + \sum_{r=1}^{2^k-1} f(a+rh_k) + \frac{1}{2}f(b)\right)$$

the composite trapezoidal rule and by

$$M_{h_k}(f) = h_k \sum_{r=1}^{2^k} f\left(a + \left(r-\frac{1}{2}\right)h_k\right)$$

the composite midpoint rule, both relative to a subdivision of $[a,b]$ into 2^k subintervals. Show that the first column $T_{k,0}$ of the Romberg array $\{T_{k,m}\}$ can be generated recursively as follows:

$$\begin{aligned} T_{0,0} &= \frac{b-a}{2}[f(a)+f(b)], \\ T_{k+1,0} &= \frac{1}{2}[T_{k,0} + M_{h_k}(f)], \; k = 0,1,2,\ldots. \end{aligned}$$

(b) Write a Matlab function for computing $\int_a^b f(x)dx$ by the Romberg integration scheme,

with $h_k = (b-a)/2^k, k = 0, 1, \ldots, n-1$.
Formal parameters: a, b, n; include f as a subfunction.
Output variable: the $n \times n$ Romberg array T
Order of computation:Generate T row by row; generate the trapezoidal sums recursively as in part (a).
Program size: Keep it down to about 20 lines of Matlab code.
Output: $T_{k,0}, T_{k,k}, k = 0, 1, \ldots, n-1$.
(c) Call your subroutine (with $n = 10$) to approximate the following integral and comment on the behavior of the Romberg scheme.

$$\int_1^2 \frac{e^x}{x} dx,$$