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MATH 561 Numerical Analysis I
Homework 5

1. (a) Consider an explicit multistep method of the form

$$u_{n+2} + \alpha u_{n+1} - \alpha u_{n-1} - u_{n-2} = h(\beta f_{n+1} + \gamma f_n + \beta f_{n-1})$$

Show that the parameters α, β, γ can be chosen uniquely so that the method has order $p = 6$.
In order to examine the order of the given multistep method, the algebraic degree of this method can be found using the linear operator shown below.

$$Lu = \sum_{s=-2}^{2} (\alpha_s u(s) - \beta_s u'(s)) Lu \quad = -u(-2) - \alpha u(-1) + \alpha u(1) + u(2) - \beta u'(-1) - \gamma u'(0) - \beta u'(1)$$

Since we are considering the algebraic degree and are working with polynomials, this linear operator is equivalent to the linear operator defined on $[0, 4]$. This is a result of the set of polynomials being closed under translation.
In order for this method to have order $p = 6$, $Lu = 0$ for $u = 1, t, t^2, t^3, t^4, t^5, t^6$. This result in the following set of equations.

$$\begin{aligned}
0 &= -1 - \alpha + \alpha + 1 = 0 \\
0 &= 2 + \alpha + \alpha + 2 - \beta - \gamma - \beta = 4 + 2\alpha - 2\beta - \gamma \\
0 &= -4 - \alpha + \alpha + 4 + 2\beta - 2\beta = 0 \\
0 &= 8 + \alpha + \alpha + 8 - 3\beta - 3\beta = 16 + 2\alpha - 3\beta \\
0 &= -16 - \alpha + \alpha + 16 + 4\beta - 4\beta = 0 \\
0 &= 32 + \alpha + \alpha + 32 - 5\beta - 5\beta = 32 + 2\alpha - 10\beta \\
0 &= -64 - \alpha + \alpha + 64 + 6\beta - 6\beta = 0
\end{aligned}$$

There are only three nontrivial equations. These are solved for the three variables below.

$$\begin{aligned}
0 &= 16 + 2\alpha - 3\beta \\
\beta &= \frac{16 + 2\alpha}{3} \\
0 &= 32 + 2\alpha - 10\beta \\
0 &= 32 + 2\alpha - 10\frac{16 + 2\alpha}{3} \\
0 &= 32 + 2\alpha - \frac{20}{3}\alpha - \frac{160}{3} \\
0 &= -\frac{64}{3} - \frac{14}{3}\alpha \\
\alpha &= -\frac{32}{7} \\
\beta &= \frac{16}{7} \\
0 &= 4 + 2\alpha - 2\beta - \gamma \\
0 &= 4 + \frac{64}{7} - \frac{32}{7} - \gamma \\
\gamma &= \frac{244}{21}
\end{aligned}$$

Therefore this method has order $p = 6$ if and only if $\alpha = -\frac{32}{7}$, $\beta = \frac{16}{7}$, and $\gamma = \frac{244}{21}$. This method can then be written as

$$u_{n+2} - \frac{32}{7}u_{n+1} + \frac{32}{7}u_{n-1} - u_{n-2} = h\Big(\frac{16}{7}f_{n+1} + \frac{244}{21}f_n + \frac{16}{7}f_{n-1}\Big)$$

(b) Discuss the stability properties of the method obtained in (a).

The stability of the method found (a) can be determined by examining the characteristic polynomial and determining if it meets the root condition.

The characteristic polynomial of this method is

$$\alpha(\xi) = \xi^4 - \frac{32}{7}\xi^3 + \frac{32}{7}\xi - 1$$

The roots of this polynomial can be found by factoring

$$\begin{aligned}\alpha(\xi) &= \xi^4 - \frac{32}{7}\xi^3 + \frac{32}{7}\xi - 1\\ &= \frac{1}{7}\Big(7\xi^4 - 32\xi^3 + 32\xi - 7\Big)\\ &= \frac{1}{7}(\xi+1)(\xi-1)\Big(7\xi^2 - 32\xi + 7\Big)\end{aligned}$$

The polynomial $7\xi^2 - 32\xi + 7$ has roots $\frac{32 \pm \sqrt{32^2 - 4\times 7\times 7}}{14} = \frac{16}{7} \pm \frac{6\sqrt{23}}{7}$ The root $\frac{16}{7} + \frac{6\sqrt{23}}{7} > 1$, therefore $\alpha(\xi)$ does not satisfy the root condition and this method is unstable.

2. Construct a pair of four-step methods, one explicit, the other implicit, both have $\alpha(\xi) = \xi^4 - \xi^3$ and order $p = 4$, but global error constants that are equal in magnitude but opposite in sign.

Since $\alpha(\xi) = \xi^4 - \xi^3$, this implies that $\alpha_4 = 1$ and $\alpha_3 = -1$. Thus these four-step methods can also be expressed as

$$u_{n+4} - u_{n+3} = h(\beta_4 f_{n+4} + \beta_3 f_{n+3} + \beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n)$$

For the explicit method $\beta_4 = 0$ and for the implicit method $\beta_4 \neq 0$.

In order for these four step methods to have order $p = 4$, the function $\delta(\xi) = \frac{\alpha(\xi)}{\ln(\xi)} - \beta(\xi)$ must have a zero at $\xi = 1$ of multiplicity 4.

First consider the Taylor expansion of $\frac{\alpha(\xi)}{\ln(\xi)}$.

$$\begin{aligned}\ln(\xi) &= \ln(1 + \xi - 1)\\ &= (\xi-1) - \frac{1}{2}(\xi-1)^2 + \frac{1}{3}(\xi-1)^3 - \frac{1}{4}(\xi-1)^4 + \cdots\\ \frac{\alpha(\xi)}{\ln(\xi)} &= \frac{\xi^3(\xi-1)}{(\xi-1) - \frac{1}{2}(\xi-1)^2 + \frac{1}{3}(\xi-1)^3 - \frac{1}{4}(\xi-1)^4 + \cdots}\\ &= 1 + \frac{7}{2}(\xi-1) + \frac{53}{12}(\xi-1)^2 + \frac{55}{24}(\xi-1)^3 + \frac{251}{720}(\xi-1)^4 + \cdots\end{aligned}$$

For $\delta(\xi)$ to have a root of order 4 at $\xi = 1$, the Taylor expansion of $\delta(\xi)$ about $\xi = 1$ must have 4 as its least power. Therefore $\beta(\xi)$ must equal the first four terms in the Taylor expansion of $\alpha(\xi)/\ln(()\xi)$, that is

$$\beta(\xi) = 1 + \frac{7}{2}(\xi-1) + \frac{53}{12}(\xi-1)^2 + \frac{55}{24}(\xi-1)^3$$

$$
\begin{aligned}
&= 1 - \frac{7}{2} + \frac{7}{2}\xi + \frac{53}{12}(\xi^2 - 2\xi + 1) + \frac{55}{24}(\xi^3 - 3\xi^2 + 3\xi - 1) \\
&= 1 - \frac{7}{2} + \frac{53}{12} - \frac{55}{24} + \left(\frac{7}{2} - \frac{53}{6} + \frac{55}{8}\right)\xi + \left(\frac{53}{12} - \frac{55}{8}\right)\xi^2 + \frac{55}{24}\xi^3 \\
&= -\frac{3}{8} + \frac{37}{24}\xi - \frac{59}{24}\xi^2 + \frac{55}{24}\xi^3
\end{aligned}
$$

Using this polynomial for $\beta(\xi)$ results in an explicit method because $\beta_4 = 0$. The global error constant is the coefficient of $(\xi - 1)^4$ divided by the constant coefficient, so in this case the global error constant is $\frac{251}{720}$. Thus we have created the explicit four step method shown below with order $p = 4$.

$$
u_{n+4} - u_{n+3} = h\left(\frac{55}{24}f_{n+3} + -\frac{59}{24}f_{n+2} + \frac{37}{24}f_{n+1} - \frac{3}{8}f_n\right)
$$

To create an implicit four-step method, the following polynomial must be used to describe β instead

$$
\beta(\xi) = 1 + \frac{7}{2}(\xi - 1) + \frac{53}{12}(\xi - 1)^2 + \frac{55}{24}(\xi - 1)^3 + b(\xi - 1)^4
$$

where $b \neq 0$ and $b \neq \frac{251}{720}$. If $b = \frac{251}{720}$, then the method would have order $p = 5$. In this case, $\delta(\xi) = \left(\frac{251}{720} - b\right)(\xi - 1)^4 + \cdots$. Therefore the global error constant for this method is $\frac{251}{720} - b$. In order for this global error constant to be equal in modulus and opposite in sign to the previous global error constant $\frac{251}{720} - b = -\frac{251}{720}$. This implies that $b = \frac{251}{360}$.

Therefore the β-polynomial can be simplied as follows

$$
\begin{aligned}
\beta(\xi) &= 1 + \frac{7}{2}(\xi - 1) + \frac{53}{12}(\xi - 1)^2 + \frac{55}{24}(\xi - 1)^3 + \frac{251}{360}(\xi - 1)^4 \\
&= -\frac{3}{8} + \frac{37}{24}\xi - \frac{59}{24}\xi^2 + \frac{55}{24}\xi^3 + \frac{251}{360}(\xi - 1)^4 \\
&= -\frac{3}{8} + \frac{37}{24}\xi - \frac{59}{24}\xi^2 + \frac{55}{24}\xi^3 + \frac{251}{360}(\xi^4 - 4\xi^3 + 6\xi^2 - 4\xi + 1) \\
&= \frac{29}{90} - \frac{449}{360}\xi + \frac{69}{40}\xi^2 - \frac{179}{360}\xi^3 + \frac{251}{360}\xi^4
\end{aligned}
$$

This results in the following implicit four-step method of order $p = 4$.

$$
u_{n+4} - u_{n+3} = h\left(\frac{251}{360}f_{n+4} - \frac{179}{360}f_{n+3} + \frac{69}{40}f_{n+2} - \frac{448}{360}f_{n+1} + \frac{29}{90}f_n\right)
$$

3. Consider the model problem

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega(y - a(x)), 0 \leq x \leq 1, y(0) = y_0
$$

where $\omega > 0$ and (i) $a(x) = x^2$, $y_0 = 0$; and (ii) $a(x) = e^x$, $y_0 = 1$

(a) In each of the cases (i) and (ii), solve the differential equation exactly.
For case (i) this differential equation can be rewritten as

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega\left(y - x^2\right)
$$

for $0 \leq x \leq 1$ and $y(0) = 0$. This is equivalent to

$$
\omega\left(y - x^2\right) + \frac{\mathrm{d}y}{\mathrm{d}x} = 0
$$

This is not an exact differential equation but it can be made into an exact differential equation by multiplying by an integrating factor, $\mu(x)$.

$$\begin{aligned}
\mu(x)\omega\left(y - x^2\right) + \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\partial}{\partial y}\left(\mu(x)\omega\left(y - x^2\right)\right) &= \mu(x)\omega \\
\frac{\partial}{\partial x}(\mu(x)) &= \mu'(x) \\
\mu(x)\omega &= \frac{\mathrm{d}\mu}{\mathrm{d}x} \\
\mu(x) &= e^{\omega x}
\end{aligned}$$

This results in the equivalent exact differential equation

$$e^{\omega x}\omega\left(y - x^2\right) + e^{\omega x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

To solve this type of equation a function $\Psi(x, y)$ must be found such that

$$\frac{\partial}{\partial x}(\Psi(x, y)) = e^{\omega x}\omega\left(y - x^2\right)$$

and

$$\frac{\partial}{\partial y}(\Psi(x, y)) = e^{\omega x}$$

Solving for Ψ

$$\begin{aligned}
\Psi(x, y) &= \int e^{\omega x}\omega\left(y - x^2\right)\mathrm{d}x \\
&= \int \omega y e^{\omega x}\,\mathrm{d}x - \int \omega e^{\omega x}x^2\,\mathrm{d}x + h(y) \\
&= ye^{\omega x} - e^{\omega x}\left(\frac{\omega^2x^2 - 2\omega x + 2}{\omega^2}\right) + h(y) \\
\frac{\partial}{\partial y}(\Psi(x, y)) &= e^{\omega x} + h'(y) \\
e^{\omega x} + h'(y) &= e^{\omega x} \\
h(y) &= C \\
\Psi(x, y) &= ye^{\omega x} - e^{\omega x}\left(\frac{\omega^2x^2 - 2\omega x + 2}{\omega^2}\right) + C
\end{aligned}$$

Now this exact differential equation is equivalent to the equation

$$ye^{\omega x} - e^{\omega x}\left(\frac{\omega^2x^2 - 2\omega x + 2}{\omega^2}\right) = C$$

$$\begin{aligned}
ye^{\omega x} &= e^{\omega x}\left(\frac{\omega^2x^2 - 2\omega x + 2}{\omega^2}\right) + C \\
y(x) &= \frac{\omega^2x^2 - 2\omega x + 2}{\omega^2} + e^{-\omega x}C
\end{aligned}$$

$$y(x) = x^2 - \frac{2}{\omega}x + \frac{2}{\omega^2} + e^{-\omega x}C$$

Now the initial conditions must be used to solve for C.

$$\begin{aligned} y(0) &= \frac{2}{\omega^2} + C \\ 0 &= \frac{2}{\omega^2} + C \\ C &= -\frac{2}{\omega^2} \end{aligned}$$

Therefore the exact solution to (i) is

$$y(x) = x^2 - \frac{2}{\omega}x + \frac{2}{\omega^2}(1 - e^{-\omega x})$$

For case (ii) this differential equation can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega(y - e^x)$$

for $0 \leq x \leq 1$ and $y(0) = 1$.
The same integrating factor as in part (i) can be used

$$\omega e^{\omega x}(y - e^x) + e^{\omega x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

To solve this type of equation a function $\Psi(x, y)$ must be found such that

$$\frac{\partial}{\partial x}(\Psi(x, y)) = e^{\omega x}\omega(y - e^x)$$

and

$$\frac{\partial}{\partial y}(\Psi(x, y)) = e^{\omega x}$$

Solving for Ψ

$$\begin{aligned} \Psi(x, y) &= \int e^{\omega x}\,\mathrm{d}y \\ &= ye^{\omega x} + h(x) \\ \frac{\partial}{\partial x}(\Psi(x, y)) &= \omega ye^{\omega x} + h'(x) \\ \omega ye^{\omega x} + h'(x) &= e^{\omega x}\omega(y - e^x) \\ \omega ye^{\omega x} + h'(x) &= e^{\omega x}\omega y - e^{\omega x}\omega e^x \\ h'(x) &= -\omega e^{(\omega+1)x} \\ h(x) &= -\frac{\omega}{\omega+1}e^{(\omega+1)x} + C \\ \Psi(x, y) &= ye^{\omega x} - \frac{\omega}{\omega+1}e^{(\omega+1)x} + C \end{aligned}$$

Now this exact differential equation is equivalent to the equation

$$ye^{\omega x} - \frac{\omega}{\omega+1}e^{(\omega+1)x} = C$$

$$\begin{aligned}
ye^{\omega x} &= \frac{\omega}{\omega+1}e^{(\omega+1)x} + C \\
y(x) &= e^{-\omega x}\left(\frac{\omega}{\omega+1}e^{(\omega+1)x} + C\right) \\
y(x) &= \frac{\omega}{\omega+1}e^{x} + Ce^{-\omega x}
\end{aligned}$$

Then the initial conditions can be used to solve for C.

$$\begin{aligned}
y(0) &= \frac{\omega}{\omega+1} + C \\
1 &= \frac{\omega}{\omega+1} + C \\
C &= 1 - \frac{\omega}{\omega+1} \\
C &= \frac{1}{\omega+1}
\end{aligned}$$

Thus the exact solution to this differential equation is

$$\begin{aligned}
y(x) &= \frac{\omega}{\omega+1}e^{x} + \frac{1}{\omega+1}e^{-\omega x} \\
y(x) &= \frac{\omega e^{x} + e^{-\omega x}}{\omega+1}
\end{aligned}$$

(b) In each of the cases (i) and (ii), apply the kth-order Adams-Bashford method and kth-order Adams predictor/corrector method, for $k = 4$, using exact starting values and step lengths h = 1/20 , 1/40 , 1/80 , 1/160. Plot the exact values y_n and numerical solution u_n, and check the accuracy of the methods. Try $\omega = 1, 10, 50$. Summarize your results.

First I created two functions to implement the 4 step Adams-Bashforth method and the 4 step Adams predictor corrector method. These functions are shown below.

```
function [u] = adamsBashforth4(f, a, b, y0, N)
    p = inputParser;
    p.addRequired('f', @Utils.isFunctionHandle);
    p.addRequired('a', @isnumeric);
    p.addRequired('b', @isnumeric);
    p.addRequired('y0', @isnumeric);
    p.addRequired('N', @Utils.isPositiveInteger);
    p.parse(f, a, b, y0, N);

    % error checking
    if(a ≥ b)
        error('a must be less than b in order to define the interval');
    end
    if(N < 4)
        error('adamsBashforth4 must be applied on more than 4 steps');
    end

    % find even spacing
```

```
    h = (b - a)/(N-1);

    % create column vector to store solution
    u = zeros(N, 1);

    % create vector for x values
    x = a:h:b;

    % initialize first 4 values with RK4 method
    rk4 = NumericalAnalysis.ODES.standardRK4Method();
    u(1:4) = rk4.solveSystem(f, x(1:4), y0);

    % perform 4-th order Adams Bashforth for remainder of interval
    f4 = f(x(1), u(1));
    f3 = f(x(2), u(2));
    f2 = f(x(3), u(3));
    for n=5:N
        % calculate f1
        % f2, f3, f4 were calculated previously
        f1 = f(x(n-1), u(n-1));

        % calculate next value of u
        u(n) = u(n-1) + h/24*(55*f1 - 59*f2 + 37*f3 - 9*f4);

        % move f values back one
        % reassign so that the values don't have to be recomputed
        f4 = f3;
        f3 = f2;
        f2 = f1;
    end
end
```

```
function [u] = adamsPredictorCorrector(f, a, b, y0, N)
    p = inputParser;
    p.addRequired('f', @Utils.isFunctionHandle);
    p.addRequired('a', @isnumeric);
    p.addRequired('b', @isnumeric);
    p.addRequired('y0', @isnumeric);
    p.addRequired('N', @Utils.isPositiveInteger);
    p.parse(f, a, b, y0, N);

    % error checking
    if(a ≥ b)
        error('a must be less than b in order to define the interval');
    end
    if(N < 4)
        error('adamsPredictorCorrector4 must be applied on more than 4 steps');
    end

    % find even spacing
    h = (b - a)/(N-1);

    % create column vector to store solution
    u = zeros(N, 1);

    % create vector for x values
    x = a:h:b;

    % initialize first 4 values with RK4 method
    rk4 = NumericalAnalysis.ODES.standardRK4Method();
```

```
    u(1:4) = rk4.solveSystem(f, x(1:4), y0);

    % perform 4-th order Adams Bashforth for remainder of interval
    f4 = f(x(1), u(1));
    f3 = f(x(2), u(2));
    f2 = f(x(3), u(3));
    for n=5:N
        % calculate f1
        % f2, f3, f4 were calculated previously
        f1 = f(x(n-1), u(n-1));

        % calculate next predicted value of u
        uStar = u(n-1) + h/24*(55*f1 - 59*f2 + 37*f3 - 9*f4);

        fStar = f(x(n), uStar);
        % correct next value of u with Adams Moulton method
        u(n) = u(n-1) + h/24*(9*fStar + 19*f1 - 5*f2 + f3);

        % move f values back one
        % reassign so that the values don't have to be recomputed
        f4 = f3;
        f3 = f2;
        f2 = f1;
    end
end
```

As the value of ω increases each of the cases become more stiff, that is they become more susceptible to instability. This is especially visible in the Adams-Bashforth method at $\omega = 50$. It is interesting to note that the Adams Predictor Corrector method experiences much less instability with the introduction of the implicit correction step. This can be seen as the Adams Predictor Corrector method is still of the same magnitude as the actual solution. The predictor corrector method oscillates around the actual solution but does not have extremely large error like the Adams-Bashforth method.

For $\omega = 1$, it is clear when examining the error that these methods are of order 4, as the error is decreasing by a factor of 16 as the step size is cut in half. These errors and ratios are shown below.

```
ErrorAB =

   1.0e-05 *

    0.1472
    0.0096
    0.0006
    0.0000
    0.0000
    0.0000


ans =

   15.2793
   15.6750
   15.8457
   15.9249
   15.9644
```

```
ErrorPC =

   1.0e-06 *

    0.1395
    0.0082
    0.0005
    0.0000
    0.0000
    0.0000


ans =

   17.0364
   16.5952
   16.3166
   16.1630
   16.0878


ErrorAB =

   1.0e-06 *

    0.7470
    0.0508
    0.0033
    0.0002
    0.0000
    0.0000


ans =

   14.7062
   15.3715
   15.6898
   15.8452
   15.9210


ErrorPC =

   1.0e-07 *

    0.8152
    0.0466
```

```
    0.0028
    0.0002
    0.0000
    0.0000


ans =

   17.4984
   16.8734
   16.4716
   16.2270
   15.9863
```

Generally the predictor corrector method's error is decreasing slightly more than by a factor of 16. Also the overall error is an order of magnitude less than the error for the regular Adams-Bashforth method.

A printout of the script and the error when $\omega = 50$ is attached.