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MATH 561 Numerical Analysis I
Final Assignment

1. Let x_1, x_2, \dots, x_n , for $n > 1$, be machine numbers. Their product can be computed by the algorithm

$$\begin{aligned} p_1 &= x_1 \\ p_k &= fl(x_k p_{k-1}), k = 2, 3, \dots, n \end{aligned}$$

- (a) Find an upper bound for the relative error in terms of the machine precision eps and n .
 The relative error is given by

$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$

First consider p_k .

$$\begin{aligned} p_k &= fl(x_k p_{k-1}) \\ &= x_k p_{k-1} (1 + \epsilon_k) \end{aligned}$$

Where $|\epsilon_k| < eps$, for $k = 1, \dots, n$

$$< x_k p_{k-1} (1 + eps)$$

Applying this recursively to p_n , we see that

$$\begin{aligned} p_n &< x_n p_{n-1} (1 + eps) \\ &< x_n x_{n-1} p_{n-2} (1 + eps)^2 \\ &< x_n x_{n-1} x_{n-2} p_{n-3} (1 + eps)^3 \\ &\vdots \\ &< x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} \end{aligned}$$

Therefore the relative error can be bounded as follows

$$\begin{aligned} E &= \left| \frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\ &< \left| \frac{x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\ &= \left| \frac{x_1 x_2 \cdots x_n ((1 + eps)^{n-1} - 1)}{x_1 x_2 \cdots x_n} \right| \\ &= (1 + eps)^{n-1} - 1 \end{aligned}$$

Therefore the upper bound for the relative error is $E < (1 + eps)^{n-1} - 1$.

- (b) For any integer r that satisfies $r \times eps < \frac{1}{10}$, show that

$$(1 + eps)^r - 1 < 1.06 \times r \times eps$$

Hence for n not too large, simplify the answer given in (a).

Using the Binomial Theorem, $(1 + eps)^r$ can be expanded.

$$(1 + eps)^r - 1 = \sum_{i=0}^r \left(\binom{r}{i} 1^{r-i} eps^i \right) - 1$$

$$\begin{aligned}
&= \sum_{i=1}^r \left(\binom{r}{i} eps^i \right) \\
&= r \cdot eps + \binom{r}{2} eps^2 + \binom{r}{3} eps^3 + \cdots + eps^r \\
&= r \cdot eps + \frac{r(r-1)}{2} eps^2 + \frac{r(r-1)(r-2)}{3!} eps^3 + \cdots + eps^r \\
&= r \cdot eps \left(1 + \frac{r-1}{2} eps + \frac{(r-1)(r-2)}{3!} eps^2 + \cdots + \frac{(r-1)(r-2) \cdots (1)}{r!} eps^{r-1} \right)
\end{aligned}$$

Since $r \times eps < \frac{1}{10}$, $(r-i)eps < \frac{1}{10}$ for any $0 < i < r$

$$\begin{aligned}
&< r \cdot eps \left(1 + \frac{1}{2} \frac{1}{10} + \frac{1}{3!} \left(\frac{1}{10} \right)^2 + \cdots + \frac{1}{r!} \left(\frac{1}{10} \right)^{r-1} \right) \\
&= r \cdot eps \sum_{k=0}^{r-1} \left(\frac{1}{k!} \left(\frac{1}{10} \right)^k \right) \\
&= r \cdot eps \cdot 10 \sum_{k=1}^{r-1} \left(\frac{1}{k!} \left(\frac{1}{10} \right)^k \right)
\end{aligned}$$

This expression is certainly less than extending the sum to infinity because all of the terms are postive. Also this sum is the Taylor series for $e^x - 1$.

$$\begin{aligned}
&< r \cdot eps \cdot 10 \sum_{k=1}^{\infty} \left(\frac{1}{k!} \left(\frac{1}{10} \right)^k \right) \\
&= r \cdot eps \cdot 10 (e^{1/10} - 1) \\
&\approx 1.05171r \cdot eps \\
&< 1.06r \cdot eps
\end{aligned}$$

This result can now be used to simplify the result of part (a). Now if n is not too large, then $|E| < 1.06(n-1)eps$.

2. (a) Determine

$$\min \max_{a \leq x \leq b} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n|$$

for $n \geq 1$ where the minimum is taken over the coefficients a_0, a_1, \dots, a_n with $a_0 \neq 0$.

First lets apply a linear transformation from the interval $[a, b]$ to $[-1, 1]$, by letting $x = \frac{b-a}{2}t + \frac{b+a}{2}$. This is then equivalent to

$$\begin{aligned}
&\min \max_{-1 \leq t \leq 1} \left| a_0 \left(\frac{b-a}{2}t + \frac{b+a}{2} \right) + a_1 \left(\frac{b-a}{2}t + \frac{b+a}{2} \right)^{n-1} + \cdots + a_n \right| \\
&= \min \max_{-1 \leq t \leq 1} \left| a_0 \left(\frac{b-a}{2} \right)^n t^n + b_1 t^{n-1} + \cdots + b_n \right| \\
&= |a_0| \left(\frac{b-a}{2} \right)^n \min \max_{-1 \leq t \leq 1} |t^n + b_1 t^{n-1} + \cdots + b_n|
\end{aligned}$$

From Chebychev's Theorem the monic polynomial with minimum maximum value over $[-1, 1]$ is the monic Chebychev polynomial

$$= |a_0| \left(\frac{b-a}{2} \right)^n \max_{-1 \leq t \leq 1} |T_n(x)|$$

Also from Chebyshev's Theorem, $\max_{-1 \leq t \leq 1} |\dot{T}_n(x)| = \frac{1}{2^{n-1}}$

$$\begin{aligned} &= |a_0| \left(\frac{b-a}{2} \right)^n \frac{1}{2^{n-1}} \\ &= 2|a_0| \left(\frac{b-a}{4} \right)^n \end{aligned}$$

Thus given an arbitrary choice of $a_0 \neq 0$,

$$\min \max_{a \leq x \leq b} |a_0 x^n + a_1 x^{n-1} + \dots + a_n| = 2|a_0| \left(\frac{b-a}{4} \right)^n$$

- (b) Let $a > 1$ and $\mathbb{P}_n^a = \{p \in \mathbb{P}_n | p(a) = 1\}$. Define $\hat{p}_n \in \mathbb{P}_n^a$ by $\hat{p}_n = T_n(x)/T_n(a)$, where $T_n(x)$ is the Chebyshev polynomial of degree n . Prove that $\|\hat{p}_n\|_\infty \leq \|p\|_\infty$ for all $p \in \mathbb{P}_n^a$.

Proof. Assume to the contrary that there exists $p \in \mathbb{P}_n^a$ such that $\|p\|_\infty < \|\hat{p}_n\|_\infty$. Define the polynomial $d(x) = \hat{p}_n(x) - p(x)$. Since d is the difference of two degree n polynomials, the degree of d can be at most n .

Let $\{y_k\}_{k=0}^n$ denote the $n+1$ extrema points for the Chebyshev polynomial $T_n(x)$, that is $T_n(y_k) = (-1)^k$. Obviously \hat{p}_n is just a scaling of $T_n(x)$, so $\|\hat{p}_n\|_\infty = \|T_n(x)\|_\infty / |T_n(a)| = |T_n(y_k)/T_n(a)| = |\hat{p}_n(y_k)|$. Also since $\|p\|_\infty < \|\hat{p}_n\|_\infty$, then $|p(y_k)| < |\hat{p}_n(y_k)|$.

Now consider $d(y_k) = \hat{p}_n(y_k) - p(y_k)$. Since the magnitude of $p(y_k)$ is less than the magnitude of $\hat{p}_n(y_k)$, $d(y_k)$ has the same sign as $\hat{p}_n(y_k)$. Also as previously noted the sign of $\hat{p}_n(y_k)$ alternates for $k = 0, 1, \dots, n$. Therefore since d is polynomial and continuous and alternates sign $n+1$ times, d must have at least n zeros. Note that these occur in the interval $[-1, 1]$ as all the extreme values of $T_n(x)$ are in $[-1, 1]$. Now consider $d(a) = \hat{p}_n(a) - p(a) = 1 - 1 = 0$. Therefore d also has a zero at $x = a$. This totals $n+1$ zeros as $a > 1$. This contradicts the fact that d is at most a degree n polynomial. Therefore our initial assumption must be incorrect, and in fact $\|p\|_\infty \geq \|\hat{p}_n\|_\infty$ for all $p \in \mathbb{P}_n^a$. \square

- (c) Let f be a positive function defined on $[a, b]$ and assume

$$\begin{aligned} \min_{a \leq x \leq b} |f(x)| &= m_0 \\ \max_{a \leq x \leq b} |f^{(k)}(x)| &= M_k, k = 0, 1, 2, \dots \end{aligned}$$

- (c.1) Let $p_{n-1}(x)$ denote the polynomial of degree at most $n-1$ interpolating f at the n Chebyshev points on $[a, b]$. Estimate the maximum relative error $r_n = \max_{a \leq x \leq b} \left| \frac{f(x) - p_{n-1}(x)}{f(x)} \right|$. First from the error formula for an interpolating polynomial we know that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi(x))}{n!} \prod_{i=1}^n (x - x_i)$$

where x_i for $i = 1, 2, \dots, n$ are the Chebyshev nodes on $[a, b]$. By transforming to the interval $[-1, 1]$, a better approximation can be made. Let $x = \frac{b-a}{2}t + \frac{b+a}{2}$, then $x_i = \frac{b-a}{2}t_i + \frac{b+a}{2}$ where t_i are the Chebyshev nodes on $[-1, 1]$. More specifically $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$. Now we note that

$$\begin{aligned} \prod_{i=1}^n (x - x_i) &= \prod_{i=1}^n \left(\frac{b-a}{2}t + \frac{b+a}{2} - \left(\frac{b-a}{2}t_i + \frac{b+a}{2} \right) \right) \\ &= \prod_{i=1}^n \left(\frac{b-a}{2}t - \frac{b-a}{2}t_i \right) \end{aligned}$$

$$= \left(\frac{b-a}{2}\right)^n \prod_{i=1}^n (t - t_i)$$

Note that $\prod_{i=1}^n (t - t_i)$ is the monic Chebyshev polynomial of degree n , that is $\prod_{i=1}^n (t - t_i) = \mathring{T}_n(t)$. From Chebyshev's Theorem we know that $\|\mathring{T}_n(t)\|_\infty = \frac{1}{2^{n-1}}$. Therefore we know that

$$\begin{aligned} \left| \prod_{i=1}^n (x - x_i) \right| &\leq \left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}} \\ &= 2 \left(\frac{b-a}{4}\right)^n \end{aligned}$$

Also we have bound on $f^{(n)}(\xi(x))$, that is $f^{(n)}(\xi(x)) \leq M_n$. Therefore we can bound the absolute error as

$$\begin{aligned} |f(x) - p_{n-1}(x)| &= \left| \frac{f^{(n)}(\xi(x))}{n!} \prod_{i=1}^n (x - x_i) \right| \\ &\leq 2 \frac{M_n}{n!} \left(\frac{b-a}{4}\right)^n \end{aligned}$$

Lastly the relative error can be bounded because $|f(x)|$ has a lower bound, that is $|f(x)| > m_0$

$$\left| \frac{f(x) - p_{n-1}(x)}{f(x)} \right| \leq 2 \frac{M_n}{m_0 n!} \left(\frac{b-a}{4}\right)^n$$

Thus the maximum relative error is $2 \frac{M_n}{m_0 n!} \left(\frac{b-a}{4}\right)^n$.

- (c.2) Apply the result of (c.1) to $f(x) = \ln(x)$ on $I_r = [e^r, e^{r+1}]$, for an integer $r \geq 1$. In particular, show that $r_n \leq \alpha(r, n)c^n$, where $0 < c < 1$ and α is slowly varying. Exhibit c . First we must find bounds for $f(x)$ and its derivatives.

$$f^{(n)}(x) = (-1)^{n+1} (n-1)! \frac{1}{x^n}$$

Therefore on the interval $[e^r, e^{r+1}]$

$$|f^{(n)}(x)| \leq (n-1)! e^{-rn}$$

Also $|f(x)| < \ln(e^r) = r$ on $[e^r, e^{r+1}]$. Thus $m_0 = r$ and $M_n = (n-1)! e^{-rn}$. We can now construct an upper bound on the relative error

$$\begin{aligned} \left| \frac{f(x) - p_{n-1}(x)}{f(x)} \right| &\leq 2 \frac{M_n}{m_0 n!} \left(\frac{b-a}{4}\right)^n \\ &= 2 \frac{(n-1)! e^{-rn}}{r n!} \left(\frac{e^{r+1} - e^r}{4}\right)^n \\ &= 2 \frac{e^{-rn}}{r n} \left(\frac{e^{r+1} - e^r}{4}\right)^n \\ &= \frac{2}{r n} \left(\frac{e^{-r}(e^{r+1} - e^r)}{4}\right)^n \\ &= \frac{2}{r n} \left(\frac{e - 1}{4}\right)^n \end{aligned}$$

Thus $r_n \leq \alpha(r, n)c^n$, where $\alpha(r, n) = \frac{2}{r n}$ and $c = \frac{e-1}{4}$.

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined and integrable on $[-1, 1]$. Let $-1 = x_0 < x_1 < \dots < x_n = 1$ be a partition of $[-1, 1]$. Consider the following numerical quadrature

$$I(f) = \int_{-1}^1 f(x) dx \approx \sum_{i=0}^n (w_i f(x_i)) = I_n(f)$$

where

$$w_i = \int_{-1}^1 L_i(x) dx \text{ with } L_i(x) = \prod_{k=0, k \neq i}^n \left(\frac{x - x_k}{x_i - x_k} \right)$$

for $i = 0, 1, 2, \dots, n$.

- (a) Prove that if n is even and the quadrature points are evenly spaced: $x_i = -1 + ih$ and $h = 2/n$, then the numerical quadrature is exact for polynomials of degree $n + 1$.

Proof. First note that this numerical quadrature is equivalent to $I_n(f) = \int_{-1}^1 p(x) dx$, where $p(x)$ is the unique interpolating polynomial of f on the points x_i for $i = 0, 1, \dots, n$. Thus the error of the numerical quadrature is equal to the integral of the error of the interpolating polynomial.

$$E(f) = \int_{-1}^1 \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x - x_i) dx$$

If f is a polynomial of degree n or less than $f^{(n+1)}(\xi(x)) = 0$ for any x . Thus the error is zero for $f \in \mathbb{P}_n$, and so the numerical quadrature is exact for polynomials of degree at most n . If f is a polynomial of degree $n + 1$, then $f^{(n+1)}(\xi(x))$ is constant for all x . Thus the error becomes

$$E(f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^1 \prod_{i=0}^n (x - x_i) dx$$

Now consider the polynomial $p(x) = \prod_{i=0}^n (x - x_i)$. Note that since the set of interpolating points are spaced evenly around zero, if the point $x \in \{x_i\}$, then $-x \in \{x_i\}$. Since n is even, then $n/2$ is an integer and $x_{n/2} = 0$. Also for any k such that $-n/2 \leq k \leq n/2$,

$$\begin{aligned} x_{n/2-k} &= -1 + (n/2 - k)h \\ &= -1 + 1 - kh \\ &= -(1 - 1 + kh) \\ &= -(-1 + (n/2 + k)h) \\ &= -x_{n/2+k} \end{aligned}$$

Similarly $x_i = -x_{n-i}$. Now consider $p(-x)$

$$\begin{aligned} p(-x) &= \prod_{i=0}^n (-x - x_i) \\ &= (-1)^{n+1} \prod_{i=0}^n (x + x_i) \end{aligned}$$

Since n is even $(-1)^{n+1} = -1$

$$= - \prod_{i=0}^n (x + x_i)$$

Since $x_i = -x_{n-i}$

$$= - \prod_{i=0}^n (x - x_{n-i})$$

This product is multiplying the same terms as $p(x)$, so this product is equivalent to $p(x)$.

$$= -p(x)$$

Therefore $p(x) = \prod_{i=0}^n (x - x_i)$ is an odd function, and so the integral $\int_{-1}^1 p(x) dx = 0$. Therefore $E(f) = 0$ for $f \in \mathbb{P}_{n+1}$, and so this numerical quadrature is exact for all polynomials whose degree is at most $n + 1$. \square

- (b) Let $n = 2$ and let $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Compute w_0 , w_1 , and w_2 , and explicitly write out the numerical quadrature formula in this case.

First I will compute $L_i(x)$ for $i = 0, 1, 2$.

$$\begin{aligned} L_0(x) &= \frac{x - 0}{-1 - 0} \cdot \frac{x - 1}{-1 - 1} \\ &= \frac{x(x - 1)}{2} \\ &= \frac{1}{2}(x^2 - x) \\ L_1(x) &= \frac{x - -1}{0 - -1} \cdot \frac{x - 1}{0 - 1} \\ &= \frac{(x + 1)(x - 1)}{-1} \\ &= -x^2 + 1 \\ L_2(x) &= \frac{x - -1}{1 - -1} \cdot \frac{x - 0}{1 - 0} \\ &= \frac{(x + 1)x}{2} \\ &= \frac{1}{2}(x^2 + x) \end{aligned}$$

Now the values of w_0 , w_1 , and w_2 can be found by computing the integrals over $[-1, 1]$ for these functions.

$$\begin{aligned} w_0 &= \int_{-1}^1 L_0(x) dx \\ &= \frac{1}{2} \int_{-1}^1 x^2 - x dx \\ &= \frac{1}{2} \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 \right) \Big|_{x=-1}^1 \\ &= \frac{1}{2} \left(\left(\frac{1}{3} - \frac{1}{2} \right) - \left(-\frac{1}{3} - \frac{1}{2} \right) \right) \\ &= \frac{1}{2} \cdot 2 \\ &= \frac{1}{2} \\ w_1 &= \int_{-1}^1 L_1(x) dx \end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^1 -x^2 + 1 \, dx \\
&= -\frac{1}{3}x^3 + x \Big|_{x=-1}^1 \\
&= \left(-\frac{1}{3} + 1\right) - \left(\frac{1}{3} - 1\right) \\
&= 2 - \frac{2}{3} \\
&= \frac{4}{3} \\
w_2 &= \int_{-1}^1 L_2(x) \, dx \\
&= \frac{1}{2} \int_{-1}^1 x^2 + x \, dx \\
&= \frac{1}{2} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) \Big|_{x=-1}^1 \\
&= \frac{1}{2} \left(\left(\frac{1}{3} + \frac{1}{2} \right) - \left(-\frac{1}{3} + \frac{1}{2} \right) \right) \\
&= \frac{1}{2} \cdot \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

Thus the numerical quadrature can be written explicitly as

$$\begin{aligned}
I_n(f) &= \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) \\
I_n(f) &= \frac{1}{3}(f(-1) + 4f(0) + f(1))
\end{aligned}$$

- (c) When $n = 2$ and $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, what is the degree of precision of the numerical quadrature formula?

We have shown in part (a) that this numerical quadrature is exact for polynomials of degree $n + 1 = 3$ or less. If the quadrature formula is not exact for polynomials of degree 4, then the degree of precision is 3. Consider $f(x) = x^4$, then

$$\begin{aligned}
E(f) &= \int_{-1}^1 f(x) \, dx - \frac{1}{3}(f(-1) + 4f(0) + f(1)) \\
&= \int_{-1}^1 x^4 \, dx - \frac{1}{3}(1 + 4 \times 0 + 1) \\
&= \frac{1}{5}x^5 \Big|_{x=-1}^1 - \frac{2}{3} \\
&= \frac{1}{5} - \frac{1}{5} - \frac{2}{3} \\
&= \frac{2}{5} - \frac{2}{3} \\
&= -\frac{4}{15}
\end{aligned}$$

Since the error does not equal 0, the quadrature formula is not exact for polynomials of degree 4. Therefore the degree of precision is 3 for this quadrature formula.

4. Let $a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$. Consider a function $f \in C^\infty[a, b]$.

- (a) Define what it means for a function S to be a linear spline that interpolates f at all the points x_i for $i = 0, 1, \dots, n$. Give a formula for S in terms of the point values of f .

In order to define the linear spline, I will first define a set of linear basis functions. Let B_i for $i = 1, 2, \dots, n-1$ be defined on $[a, b]$ as follows.

$$B_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Also let B_1 and B_n be defined as follows

$$B_1(x) = \begin{cases} \frac{x-x_{n-1}}{x_n-x_{n-1}} & a = x_0 \leq x \leq x_1 \\ 0 & x > x_1 \end{cases}$$

$$B_n(x) = \begin{cases} \frac{x_1-x}{x_1-x_0} & x_{n-1} \leq x \leq x_n = b \\ 0 & x < x_{n-1} \end{cases}$$

A linear spline on $[a, b]$ that interpolates f on the partition $\{x_i\}_{i=0}^n$ is a function $S(x)$ that is a linear combination of the basis functions B_i such that $S(x_i) = f(x_i)$ for $i = 0, 1, \dots, n$.

Thus a formula for $S(x)$ could be written as $S(x) = \sum_{i=0}^n (f(x_i)B_i(x))$.

- (b) Let $h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$. Derive an upper bound on $|f(x) - S(x)|$ on $x \in [a, b]$. Use this to prove that $\lim_{h \rightarrow 0} (|f(x) - S(x)|) = 0$ for $x \in [a, b]$ and state the rate of convergence.
- (c) Define what it means for S to be a clamped cubic spline that interpolates f at all the points x_i , for $i = 0, 1, \dots, n$.

A function $S(x)$ is a clamped cubic spline that interpolates f at the points x_i for $i = 0, 1, \dots, n$ if $S(x)$ is piecewise cubic. $S(x)$ can be expressed as $a_3x^3 + a_2x^2 + a_1x + a_0$ on each interval $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, n-1$. I will denote each of these pieces as $S_i(x)$. Furthermore $S(x)$ must satisfy some other properties. $S(x)$ must match the function values of f at each x_i , that is $S_i(x_i) = f(x_i)$ and $S_i(x_{i+1}) = f(x_{i+1})$ for $i = 0, 1, \dots, n-1$. Furthermore $S(x)$ must have a continuous first and second derivative, this can be written as $S_i^{(k)}(x_{i+1}) = S_{i+1}^{(k)}(x_{i+1})$ for $k = 1, 2$ and $i = 0, 1, \dots, n-2$. Lastly for $S(x)$ to be clamped we require that the first derivatives of $S(x)$ match the derivatives of f at the endpoints, that is $S'_0(x_0) = f'(x_0)$ and $S'_{n-1}(x_n) = f'(x_n)$. These conditions provide $4n$ equations for the $4n$ coefficients of the cubic pieces. Thus these conditions fully define the clamped cubic spline.

5. (a) Prove the following theorem: Consider the system of initial value problems:

$$\mathbf{y}' = f(\mathbf{y})$$

and apply it to the forward Euler method:

$$\mathbf{u}_{n+1} = F(\mathbf{u}_n) = \mathbf{u}_n + hf(\mathbf{u}_n)$$

Then

- $\boldsymbol{\alpha}$ is a fixed point of the Euler method, that is $F(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ if and only if $\boldsymbol{\alpha}$ is a fixed point of the initial value problem, that is $f(\boldsymbol{\alpha}) = \mathbf{0}$.
- If $\boldsymbol{\alpha}$ is a linearly stable fixed point of the initial value problem (i.e. all the eigenvalues of the matrix $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$ have negative real parts) and if $|1 + h\lambda_p| < 1$ for each eigenvalue λ_p of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$, then $\boldsymbol{\alpha}$ is also a linearly stable fixed point of the Euler method.

Proof. To prove the first point, suppose α is a fixed point of the Euler method, then

$$\begin{aligned} F(\alpha) &= \alpha \\ \alpha + f(\alpha) &= \alpha \\ f(\alpha) &= 0 \end{aligned}$$

Thus α is also a fixed point the initial value problem. Reversing this procedure shows that is α is a fixed point of the initial value problem it is also a fixed point of the Euler method. Therefore α is a fixed point for the Euler method if and only if α is a fixed point for the initial value problem.

To prove the second point, suppose α is a linearly fixed point of the initial value problem. This implies that $f(\alpha) = 0$ and the eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ have negative real parts. In order for α to be a linearly stable fixed point of the Euler method F , then the magnitude of the eigenvalues of $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)$ must be less than 1. Note that $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha) = I + h \frac{\partial}{\partial \mathbf{u}}(f)(\alpha)$, where I is the identity matrix. Suppose λ_p is an eigenvalue of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ with eigenvector \mathbf{x} . Now consider $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)\mathbf{x}$.

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}}(F)(\alpha)\mathbf{x} &= \left(I + h \frac{\partial}{\partial \mathbf{u}}(f)(\alpha) \right) \mathbf{x} \\ &= \mathbf{x} + h \frac{\partial}{\partial \mathbf{y}}(f)(\alpha)\mathbf{x} \end{aligned}$$

Since \mathbf{x} is an eigenvector of $\frac{\partial}{\partial \mathbf{u}}(f)(\alpha)$

$$\begin{aligned} &= \mathbf{x} + h\lambda_p\mathbf{x} \\ &= (1 + h\lambda_p)\mathbf{x} \end{aligned}$$

Thus $1 + h\lambda_p$ is an eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)$. Therefore for all eigenvalues λ_p of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$, $1 + h\lambda_p$ is an eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)$. Also any eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)$ must be of this form. Now since all eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ have negative real parts and satisfy $|1 + h\lambda_p| < 1$, the eigenvalues λ_F of $\frac{\partial}{\partial \mathbf{u}}(F)(\alpha)$ must satisfy $|\lambda_F| = |1 + h\lambda_p| < 1$. Therefore α is a linearly stable fixed point of the Euler method. \square

(b) The fixed points of the Logistic growth equation

$$y' = f(y) = 2y(1 - y)$$

are $y = 0$ (unstable since $f'(0) = 2$) and $y = 1$ (stable since $f'(1) = -2$). Apply the Euler method to this equation and find and classify all fixed points of the Euler method as a function of the time step parameter h .

According to the previous theorem the fixed points of the Euler method must also be fixed points of the initial value problems. We can see that $f(y) = 0$ for $y = 0$ and $y = 1$. Therefore the fixed points of the Euler method are $y = 0$ and $y = 1$. If the eigenvalues, λ_p , of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ are negative and satisfy $|1 + h\lambda_p| < 1$, then the fixed point α of the Euler method is linearly stable. In this one-dimensional case, the eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ are simply $f'(\alpha)$. The derivative of f is $f'(y) = 2(1 - y) - 2y$. For $\alpha = 0$, $f'(0) = 2$, therefore this fixed point is unstable in both the initial value problem and the Euler method, for all values of h . For $\alpha = 1$, $f'(1) = -2$, and if $|1 + -2h| < 1$, then this fixed point will be linearly stable for the Euler method. If $0 < h < 1$, then $|1 - 2h| < 1$. Therefore this fixed point is linearly stable for $h \in (0, 1)$.