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MATH 561 Numerical Analysis I
Final Assignment

1. Let $x_1, x_2, \dots, x_n$, for $n > 1$, be machine numbers. Their product can be computed by the alogirithm

$$
\begin{aligned}
p_1 &= x_1 \\
p_k &= fl(x_k p_{k-1}), k = 2, 3, \dots, n
\end{aligned}
$$

   (a) Find an upper bound for the relative error in terms of the machine precision *eps* and n.
   The relative error is given by

$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$

   First consider p_k.

$$
\begin{aligned}
p_k &= fl(x_k p_{k-1}) \\
&= x_k p_{k-1}(1 + \epsilon_k)
\end{aligned}
$$

   Where $|\epsilon_k| < eps$, for $k = 1, \cdots, n$

$$< x_k p_{k-1}(1 + eps)$$

   Applying this recursively to p_n, we see that

$$
\begin{aligned}
p_n &< x_n p_{n-1}(1 + eps) \\
&< x_n x_{n-1} p_{n-2}(1 + eps)^2 \\
&< x_n x_{n-1} x_{n-2} p_{n-3}(1 + eps)^3 \\
&\vdots \\
&< x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1}
\end{aligned}
$$

   Therefore the relative error can be bounded as follows

$$
\begin{aligned}
E &= \left| \frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\
&< \left| \frac{x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\
&= \left| \frac{x_1 x_2 \cdots x_n((1 + eps)^{n-1} - 1)}{x_1 x_2 \cdots x_n} \right| \\
&= (1 + eps)^{n-1} - 1
\end{aligned}
$$

   Therefore the upper bound for the relative error is $E < (1 + eps)^{n-1} - 1$.

   (b) For any integer r that satisfies $r \times eps < \frac{1}{10}$, show that

$$(1 + eps)^r - 1 < 1.06 \times r \times eps$$

   Hence for n not too large, simplify the answer given in (a).
   Using the Binomial Thereom, $(1 + eps)^r$ can be expanded.

$$(1 + eps)^r - 1 = \sum_{i=0}^{r} \left(\binom{r}{i} 1^{r-i} eps^i \right) - 1$$

$$
\begin{aligned}
&= \sum_{i=1}^{r}\left(\binom{r}{i} eps^i\right) \\
&= r\cdot eps + \binom{r}{2} eps^2 + \binom{r}{3} eps^3 + \cdots + eps^r \\
&= r\cdot eps + \frac{r(r-1)}{2} eps^2 + \frac{r(r-1)(r-2)}{3!} eps^3 + \cdots + eps^r \\
&= r\cdot eps\left(1 + \frac{r-1}{2} eps + \frac{(r-1)(r-2)}{3!} eps^2 + \cdots + \frac{(r-1)(r-2)\cdots(1)}{r!} eps^{r-1}\right)
\end{aligned}
$$

Since $r \times eps < \frac{1}{10}$, $(r-i)eps < \frac{1}{10}$ for any $0 < i < r$

$$
\begin{aligned}
&< r\cdot eps\left(1 + \frac{1}{2}\frac{1}{10} + \frac{1}{3!}\left(\frac{1}{10}\right)^2 + \cdots + \frac{1}{r!}\left(\frac{1}{10}\right)^{r-1}\right) \\
&= r\cdot eps \sum_{k=0}^{r-1}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k-1}\right) \\
&= r\cdot eps \cdot 10 \sum_{k=1}^{r-1}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k}\right)
\end{aligned}
$$

This expression is certainly less than extending the sum to infinity because all of the terms are postive. Also this sum is the Taylor series for $e^x - 1$.

$$
\begin{aligned}
&< r\cdot eps \cdot 10 \sum_{k=1}^{\infty}\left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k}\right) \\
&= r\cdot eps \cdot 10\left(e^{1/10} - 1\right) \\
&\approx 1.05171 r \cdot eps \\
&< 1.06 r \cdot eps
\end{aligned}
$$

This result can now be used to simplify the result of part (a). Now if n is not too large, then $|E| < 1.06(n-1)eps$.

2. (a) Determine

$$
\min \max_{a\le x\le b}\left|a_0x^n + a_1x^{n-1} + \cdots + a_n\right|
$$

for $n \ge 1$ where the minimum is taken over the coefficients $a_0, a_1, \ldots, a_n$ with $a_0 \neq 0$.
First lets apply a linear transformation from the interval $[a, b]$ to $[-1, 1]$, by letting $x = \frac{b-a}{2}t + \frac{b+a}{2}$. This is then equivalent to

$$
\begin{aligned}
&\min \max_{-1\le t\le 1}\left|a_0\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) + a_1\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)^{n-1} + \cdots + a_n\right| \\
&= \min \max_{-1\le t\le 1}\left|a_0\left(\frac{b-a}{2}\right)^n t^n + b_1t^{n-1} + \cdots + b_n\right| \\
&= |a_0|\left(\frac{b-a}{2}\right)^n \min \max_{-1\le t\le 1}\left|t^n + b_1t^{n-1} + \cdots + b_n\right|
\end{aligned}
$$

From Chebychev's Theorem the monic polynomial with minimum maximum value over $[-1, 1]$ is the monic Chebychev polynomial

$$
= |a_0|\left(\frac{b-a}{2}\right)^n \max_{-1\le t\le 1}\left|\mathring{T}_n(x)\right|
$$

Also from Chebyshev's Theorem, $\max_{-1\le t\le 1}\left|\mathring{T}_n(x)\right| = \frac{1}{2^{n-1}}$

$$\begin{aligned}
&= |a_0|\left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}} \\
&= 2|a_0|\left(\frac{b-a}{4}\right)^n
\end{aligned}$$

Thus given an arbitrary choice of $a_0 \neq 0$,

$$\min \max_{a\le x\le b}\left|a_0x^n + a_1x^{n-1} + \cdots + a_n\right| = 2|a_0|\left(\frac{b-a}{4}\right)^n$$

(b) Let $a > 1$ and $\mathbb{P}_n^a = \{p \in \mathbb{P}_n | p(a) = 1\}$. Define $\hat{p}_n \in \mathbb{P}_n^a$ by $\hat{p}_n = T_n(x)/T_n(a)$, where $T_n(x)$ is the Chebyshev polynomial of degree n. Prove that $\|\hat{p}_n\|_\infty \le \|p\|_\infty$ for all $p \in \mathbb{P}_n^a$.

Proof. Assume to the contrary that there exists $p \in \mathbb{P}_n^a$ such that $\|p\|_\infty < \|\hat{p}_n\|_\infty$. Define the polynomial $d(x) = \hat{p}_n(x) - p(x)$. Since d is the difference of two degree n polynomials, the degree of d can be at most n.

Let $\{y_k\}_{k=0}^n$ denote the $n+1$ extrema points for the Chebyshev polynomial $T_n(x)$, that is $T_n(y_k) = (-1)^k$. Obviously $\hat{p}_n$ is just a scaling of $T_n(x)$, so $\|\hat{p}_n\|_\infty = \|T_n(x)\|_\infty/|T_n(a)| = |T_n(y_k)/T_n(a)| = |\hat{p}_n(y_k)|$. Also since $\|p\|_\infty < \|\hat{p}_n\|_\infty$, then $|p(y_k)| < |\hat{p}_n(y_k)|$.

Now consider $d(y_k) = \hat{p}_n(y_k) - p(y_k)$. Since the magnitude of $p(y_k)$ is less than the magnitude of $\hat{p}_n(y_k)$, $d(y_k)$ has the same sign as $\hat{p}_n(y_k)$. Also as previously noted the sign of $\hat{p}_n(y_k)$ alternates for $k = 0, 1, \ldots, n$. Therefore since d is polynomial and continuous and alternates sign $n+1$ times, d must have at least n zeros. Note that these occur in the interval $[-1, 1]$ as all the extreme values of $T_n(x)$ are in $[-1, 1]$. Now consider $d(a) = \hat{p}_n(a) - p(a) = 1 - 1 = 0$. Therefore d also has a zero at $x = a$. This totals $n+1$ zeros as $a > 1$. This contradicts the fact that d is at most a degree n polynomial. Therefore our initial assumption must be incorrect, and in fact $\|p\|_\infty \ge \|\hat{p}_n(y_k)\|_\infty$ for all $p \in \mathbb{P}_n^a$. □

(c) Let f be a positive function defined on $[a, b]$ and assume

$$\min_{a\le x\le b}|f(x)| = m_0$$
$$\max_{a\le x\le b}\left|f^{(k)}(x)\right| = M_k, k = 0, 1, 2\ldots$$

(c.1) Let $p_{n-1}(x)$ denote the polynomial of degree at most $n-1$ interpolating f at the n Chebyshev points on $[a, b]$. Estimate the maximum relative error $r_n = \max_{a\le x\le b}\left|\frac{f(x)-p_{n-1}(x)}{f(x)}\right|$.
First from the error formula for an interpolating polynomial we know that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi(x))}{n!}\prod_{i=1}^n (x - x_i)$$

where x_i for $i = 1, 2, \ldots, n$ are the Chebyshev nodes on $[a, b]$. By transforming to the interval $[-1, 1]$, a better approximation can be made. Let $x = \frac{b-a}{2}t + \frac{b+a}{2}$, then $x_i = \frac{b-a}{2}t_i + \frac{b+a}{2}$ where t_i are the Chebyshev nodes on $[-1, 1]$. More specifically $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$. Now we note that

$$\begin{aligned}
\prod_{i=1}^n (x - x_i) &= \prod_{i=1}^n \left(\frac{b-a}{2}t + \frac{b+a}{2} - \left(\frac{b-a}{2}t_i + \frac{b+a}{2}\right)\right) \\
&= \prod_{i=1}^n \left(\frac{b-a}{2}t - \frac{b-a}{2}t_i\right)
\end{aligned}$$

$$= \left(\frac{b-a}{2}\right)^n \prod_{i=1}^{n} (t - t_i)$$

Note that $\prod_{i=1}^{n} (t - t_i)$ is the monic Chebyshev polynomial of degree n, that is $\prod_{i=1}^{n} (t - t_i) = \mathring{T}_n(t)$. From Chebyshev's Theorem we know that $\left\|\mathring{T}_n(t)\right\|_\infty = \frac{1}{2^{n-1}}$. Therefore we know that

$$\left|\prod_{i=1}^{n} (x - x_i)\right| \le \left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}}$$
$$= 2\left(\frac{b-a}{4}\right)^n$$

Also we have bound on $f^{(n)}(\xi(x))$, that is $f^{(n)}(\xi(x)) \le M_n$. Therefore we can bound the absolute error as

$$|f(x) - p_{n-1}(x)| = \left|\frac{f^{(n)}(\xi(x))}{n!} \prod_{i=1}^{n} (x - x_i)\right|$$
$$\le 2\frac{M_n}{n!}\left(\frac{b-a}{4}\right)^n$$

Lastly the relative error can be bounded because $|f(x)|$ has a lower bound, that is $|f(x)| > m_0$

$$\left|\frac{f(x) - p_{n-1}(x)}{f(x)}\right| \le 2\frac{M_n}{m_0 n!}\left(\frac{b-a}{4}\right)^n$$

Thus the maximum relative error is $2\frac{M_n}{m_0 n!}\left(\frac{b-a}{4}\right)^n$.

(c.2) Apply the result of (c.1) to $f(x) = \ln(x)$ on $I_r = [e^r, e^{r+1}]$, for an integer $r \ge 1$. In particular, show that $r_n \le \alpha(r, n)c^n$, where $0 < c < 1$ and α is slowly varying. Exhibit c.
First we must find bounds for $f(x)$ and its derivatives.

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\frac{1}{x^n}$$

Therefore on the inteval $[e^r, e^{r+1}]$

$$\left|f^{(n)}(x)\right| \le (n-1)!e^{-rn}$$

Also $|f(x)| < \ln(e^r) = r$ on $[e^r, e^{r+1}]$. Thus $m_0 = r$ and $M_n = (n-1)!e^{-rn}$. We can now construct an upper bound on the relative error

$$\left|\frac{f(x) - p_{n-1}(x)}{f(x)}\right| \le 2\frac{M_n}{m_0 n!}\left(\frac{b-a}{4}\right)^n$$
$$= 2\frac{(n-1)!e^{-rn}}{rn!}\left(\frac{e^{r+1} - e^r}{4}\right)^n$$
$$= 2\frac{e^{-rn}}{rn}\left(\frac{e^{r+1} - e^r}{4}\right)^n$$
$$= \frac{2}{rn}\left(\frac{e^{-r}(e^{r+1} - e^r)}{4}\right)^n$$
$$= \frac{2}{rn}\left(\frac{e-1}{4}\right)^n$$

Thus $r_n \le \alpha(r, n)c^n$, where $\alpha(r, n) = \frac{2}{rn}$ and $c = \frac{e-1}{4}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined and integrable on $[-1, 1]$. Let $-1 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[-1, 1]$. Consider the following numerical quadrature

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{n} (w_i f(x_i)) = I_n(f)$$

where

$$w_i = \int_{-1}^{1} L_i(x)\,\mathrm{d}x \text{ with } L_i(x) = \prod_{k=0,k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k}\right)$$

for $i = 0, 1, 2, \ldots, n$.

(a) Prove that if n is even and the quadrature points are evenly spaced: $x_i = -1 + ih$ and $h = 2/n$, then the numerical quadrature is exact for polynomials of degree $n + 1$.

Proof. First note that this numerical quadrature is equivalent to $I_n(f) = \int_{-1}^{1} p(x)\,\mathrm{d}x$, where $p(x)$ is the unique interpolating polynomial of f on the points x_i for $i = 0, 1, \ldots, n$. Thus the error of the numerical quadrature is equal to the integral of the error of the interpolating polynomial.

$$E(f) = \int_{-1}^{1} \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)\,\mathrm{d}x$$

If f is a polynomial of degree n or less than $f^{(n+1)}(\xi(x)) = 0$ for any x. Thus the error is zero for $f \in \mathbb{P}_n$, and so the numerical quadrature is exact for polynomials of degree at most n. If f is a polynomial of degree $n + 1$, then $f^{(n+1)}(\xi(x))$ is constant for all x. Thus the error becomes

$$E(f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{1} \prod_{i=0}^{n} (x - x_i)\,\mathrm{d}x$$

Now consider the polynomial $p(x) = \prod_{i=0}^{n} (x - x_i)$. Note that since the set of interpolating points are spaced evenly around zero, if the point $x \in \{x_i\}$, then $-x \in \{x_i\}$. Since n is even, then $n/2$ is an integer and $x_{n/2} = 0$. Also for any k such that $-n/2 \leq k \leq n/2$,

$$\begin{aligned}
x_{n/2-k} &= -1 + (n/2 - k)h \\
&= -1 + 1 - kh \\
&= -(1 - 1 + kh) \\
&= -(-1 + (n/2 + k)h) \\
&= -x_{n/2+k}
\end{aligned}$$

Similarly $x_i = -x_{n-i}$. Now consider $p(-x)$

$$\begin{aligned}
p(-x) &= \prod_{i=0}^{n} (-x - x_i) \\
&= (-1)^{n+1} \prod_{i=0}^{n} (x + x_i)
\end{aligned}$$

Since n is even $(-1)^{n+1} = -1$

$$= -\prod_{i=0}^{n} (x + x_i)$$

Since $x_i = -x_{n-i}$

$$= -\prod_{i=0}^{n} (x - x_{n-i})$$

This product is multiplying the same terms as $p(x)$, so this product is equivalent to $p(x)$.

$$= -p(x)$$

Therefore $p(x) = \prod_{i=0}^{n} (x - x_i)$ is an odd function, and so the integral $\int_{-1}^{1} p(x)\,\mathrm{d}x = 0$. Therefore $E(f) = 0$ for $f \in \mathbb{P}_{n+1}$, and so this numerical quadrature is exact for all polynomials whose degree is at most $n+1$. $\square$

(b) Let $n = 2$ and let $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Compute w_0, w_1, and w_2, and explicitly write out the numerical quadrature formula in this case.
First I will compute $L_i(x)$ for $i = 0, 1, 2$.

$$\begin{aligned}
L_0(x) &= \frac{x-0}{-1-0} \cdot \frac{x-1}{-1-1} \\
&= \frac{x(x-1)}{2} \\
&= \frac{1}{2}\left(x^2 - x\right) \\
L_1(x) &= \frac{x - -1}{0 - -1} \cdot \frac{x-1}{0-1} \\
&= \frac{(x+1)(x-1)}{-1} \\
&= -x^2 + 1 \\
L_2(x) &= \frac{x - -1}{1 - -1} \cdot \frac{x-0}{1-0} \\
&= \frac{(x+1)x}{2} \\
&= \frac{1}{2}\left(x^2 + x\right)
\end{aligned}$$

Now the values of w_0, w_1, and w_2 can be found by computing the integrals over $[-1, 1]$ for these functions.

$$\begin{aligned}
w_0 &= \int_{-1}^{1} L_0(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} x^2 - x\,\mathrm{d}x \\
&= \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{x=-1}^{1} \\
&= \frac{1}{2}\left(\left(\frac{1}{3} - \frac{1}{2}\right) - \left(-\frac{1}{3} - \frac{1}{2}\right)\right) \\
&= \frac{1}{2}\frac{2}{3} \\
&= \frac{1}{3} \\
w_1 &= \int_{-1}^{1} L_1(x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{1} -x^2 + 1 \, \mathrm{d}x \\
&= -\frac{1}{3}x^3 + x \Big|_{x=-1}^{1} \\
&= \left(-\frac{1}{3} + 1\right) - \left(\frac{1}{3} - 1\right) \\
&= 2 - \frac{2}{3} \\
&= \frac{4}{3} \\
w_2 &= \int_{-1}^{1} L_2(x) \, \mathrm{d}x \\
&= \frac{1}{2} \int_{-1}^{1} x^2 + x \, \mathrm{d}x \\
&= \frac{1}{2} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right) \Big|_{x=-1}^{1} \\
&= \frac{1}{2}\left(\left(\frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right)\right) \\
&= \frac{1}{2}\frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

Thus the numerical quadrature can be written explicitly as

$$
\begin{aligned}
I_n(f) &= \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) \\
I_n(f) &= \frac{1}{3}(f(-1) + 4f(0) + f(1))
\end{aligned}
$$

(c) When $n = 2$ and $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, what is the degree of precision of the numerical quadrature formula?

We have shown in part (a) that this numerical quadrature is exact for polynomials of degree $n + 1 = 3$ or less. If the quadrature formula is not exact for polynomials of degree 4, then the degree of precision is 3. Consider $f(x) = x^4$, then

$$
\begin{aligned}
E(f) &= \int_{-1}^{1} f(x) \, \mathrm{d}x - \frac{1}{3}(f(-1) + 4f(0) + f(1)) \\
&= \int_{-1}^{1} x^4 \, \mathrm{d}x - \frac{1}{3}(1 + 4 \times 0 + 1) \\
&= \frac{1}{5}x^5 \Big|_{x=-1}^{1} - \frac{2}{3} \\
&= \frac{1}{5} - -\frac{1}{5} - \frac{2}{3} \\
&= \frac{2}{5} - \frac{2}{3} \\
&= -\frac{4}{15}
\end{aligned}
$$

Since the error does not equal 0, the quadrature formula is not exact for polynomials of degree 4. Therefore the degree of precision is 3 for this quadrature formula.

4. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Consider a function $f \in C^\infty[a, b]$.

(a) Define what it means for a function S to be a linear spline that interpolates f at all the points x_i for $i = 0, 1, \ldots, n$. Give a formula for S in terms of the point values of f.

In order to define the linear spline, I will first define a set of linear basis functions. Let B_i for $i = 1, 2, \ldots, n-1$ be defined on $[a, b]$ as follows.

$$B_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Also let B_1 and B_n be defined as follows

$$B_1(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & a = x_0 \leq x \leq x_1 \\ 0 & x > x_1 \end{cases}$$

$$B_n(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0} & x_{n-1} \leq x \leq x_n = b \\ 0 & x < x_{n-1} \end{cases}$$

A linear spline on $[a, b]$ that interpolates f on the partition $\{x_i\}_{i=0}^n$ is a function $S(x)$ that is a linear combination of the basis functions B_i such that $S(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

Thus a formula for $S(x)$ could be written as $S(x) = \sum_{i=0}^n (f(x_i)B_i(x))$.

(b) Let $h = \max_{0 \leq i \leq n-1}(x_{i+1} - x_i)$. Derive an upper bound on $|f(x) - S(x)|$ on $x \in [a, b]$. Use this to prove that $\lim_{h \to 0}(|f(x) - S(x)|) = 0$ for $x \in [a, b]$ and state the rate of convergence.

On each interval $[x_i, x_{i+1}]$, the error $f(x) - S(x)$ is given by the error for an interpolating polynomial, $\frac{f''(\xi)}{2!}(x - x_i)(x - x_{i+1})$. Note that since $h = \max_{0 \leq i \leq n-1}(x_{i+1} - x_i)$, $|(x - x_i)| \leq h$ and $|(x - x_{i+1})| < h$. Also since $f(x) \in C^\infty[a, b]$ $f''(x)$ is bounded, that is there exists $M \in \mathbb{R}$, such that $|f''(x)| < M$. The error $|f(x) - S(x)|$ on $[a, b]$ is less than or equal to $\max_{0 \leq i \leq n}|f(x) - S(x)|$ on $[x_i, x_{i+1}]$, where the max is taken over all the intervals.

$$\begin{aligned} \max_{0 \leq i \leq n}|f(x) - S(x)| &= \left| \frac{f''(\xi)}{2!}(x - x_i)(x - x_{i+1}) \right| \\ &\leq \frac{M}{2}h^2 \end{aligned}$$

We can now consider the limit $\lim_{h \to 0}(|f(x) - S(x)|)$.

$$\begin{aligned} \lim_{h \to 0}(|f(x) - S(x)|) &\leq \lim_{h \to 0}\left(\frac{M}{2}h^2\right) \\ &= 0 \end{aligned}$$

Also we can see that the error converges to 0 with h^2. In other words the rate of convergence is h^2.

(c) Define what it means for S to be a clamped cubic spline tat interpolates f at all the points x_i, for $i = 0, 1, \ldots, n$.

A function $S(x)$ is a clamped cubic spline that interpolates f at the points x_i for $i = 0, 1, \ldots, n$ if $S(x)$ is piecewise cubic. $S(x)$ can be expressed as $a_3x^3 + a_2x^2 + a_1x + a_0$ on each interval $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$. I will denote each of these pieces as $S_i(x)$. Furthermore $S(x)$ must satisfy some other properties. $S(x)$ must be match the function values of f at each x_i, that is $S_i(x_i) = f(x_i)$ and $S_i(x_{i+1}) = f(x_{i+1})$ for $i = 0, 1, \ldots, n-1$. Furthermore $S(x)$ must have a continuous first and second derivative, this can be written at $S_i^{(k)}(x_{i+1}) = S_{i+1}^{(k)}(x_{i+1})$ for $k = 1, 2$ and $i = 0, 1, \ldots, n-2$. Lastly for $S(x)$ to be clamped we require that the first derivatives of $S(x)$ match the derivatives of f at the endpoints, that is $S_0'(x_0) = f'(x_0)$ and $S_{n-1}(x_n) = f'(x_n)$. These conditions provide $4n$ equations for the $4n$ coefficients of the cubic pieces. Thus these conditions fully define the clamped cubic spline.

5. (a) Prove the following theorem: Consider the system of initial value problems:

$$\mathbf{y}' = f(\mathbf{y})$$

and apply it to the forward Euler method:

$$\mathbf{u}_{n+1} = F(\mathbf{u}_n) = \mathbf{u}_n + hf(\mathbf{u}_n)$$

Then

- $\boldsymbol{\alpha}$ is a fixed point of the Euler method, that is $F(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ if and only if $\boldsymbol{\alpha}$ is a fixed point of the initial value problem, that is $f(\boldsymbol{\alpha}) = \mathbf{0}$.
- If $\boldsymbol{\alpha}$ is a linearly stable fixed point of the initial value problem (i.e. all the eigenvalues of the matrix $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$ have negative real parts) and if $|1 + h\lambda_p| < 1$ for each eigenvalue λ_p of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$, then $\boldsymbol{\alpha}$ is also a linearly stable fixed point of the Euler method.

Proof. To prove the first point, suppose α is a fixed point of the Euler method, then

$$\begin{aligned} F(\boldsymbol{\alpha}) &= \boldsymbol{\alpha} \\ \boldsymbol{\alpha} + f(\boldsymbol{\alpha}) &= \boldsymbol{\alpha} \\ f(\boldsymbol{\alpha}) &= \mathbf{0} \end{aligned}$$

Thus $\boldsymbol{\alpha}$ is also a fixed point the initial value problem. Reversing this procedure shows that is $\boldsymbol{\alpha}$ is a fixed point of the initial value problem it is also a fixed point of the Euler method. Therefore $\boldsymbol{\alpha}$ is a fixed point for the Euler method if and only if $\boldsymbol{\alpha}$ is a fixed point for the initial value problem.

To prove the second point, suppose $\boldsymbol{\alpha}$ is a linearly fixed point of the initial value problem. This implies that $f(\boldsymbol{\alpha}) = 0$ and the eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$ have negative real parts. In order for $\boldsymbol{\alpha}$ to be a linearly stable fixed point of the Euler method F, then the magnitude of the eigenvalues of $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})$ must be less than 1. Note that $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha}) = I + h\frac{\partial}{\partial \mathbf{u}}(f)(\boldsymbol{\alpha})$, where I is the identity matrix. Suppose λ_p is an eigenvalue of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$ with eigenvector $\mathbf{x}$. Now consider $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})\mathbf{x}$.

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})\mathbf{x} &= \left(I + h\frac{\partial}{\partial \mathbf{u}}(f)(\boldsymbol{\alpha})\right)\mathbf{x} \\ &= \mathbf{x} + h\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})\mathbf{x} \end{aligned}$$

Since $\mathbf{x}$ is an eigenvector of $\frac{\partial}{\partial \mathbf{u}}(f)(\boldsymbol{\alpha})$

$$\begin{aligned} &= \mathbf{x} + h\lambda_p\mathbf{x} \\ &= (1 + h\lambda_p)\mathbf{x} \end{aligned}$$

Thus $1 + h\lambda_p$ is an eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})$. Therefore for all eigenvalues λ_p of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$, $1 + h\lambda_p$ is an eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})$. Also any eigenvalue of $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})$ must be of this form. Now since all eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\boldsymbol{\alpha})$ have negative real parts and satisfy $|1 + h\lambda_p| < 1$, the eigenvalues λ_F of $\frac{\partial}{\partial \mathbf{u}}(F)(\boldsymbol{\alpha})$ must satisfy $|\lambda_F| = |1 + h\lambda_p| < 1$. Therefore $\boldsymbol{\alpha}$ is a linearly stable fixed point of the Euler method. $\square$

(b) The fixed points of the Logistic growth equation

$$y' = f(y) = 2y(1-y)$$

are $y = 0$ (unstable since $f'(0) = 2$) and $y = 1$ (stable since $f'(1) = -2$). Apply the Euler method to this equation and find and classify all fixed points of the Euler method as a function of the time step parameter h.

According to the previous theorem the fixed points of the Euler method must also be fixed points of the initial value problems. We can see that $f(y) = 0$ for $y = 0$ and $y = 1$. Therefore the fixed points of the Euler method are $y = 0$ and $y = 1$. If the eigenvalues, λ_p, of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ are negative and satisfy $|1 + h\lambda_p| < 1$, then the fixed point α of the Euler method is linearly stable. In this one-dimensional case, the eigenvalues of $\frac{\partial}{\partial \mathbf{y}}(f)(\alpha)$ are simply $f'(\alpha)$. The derivative of f is $f'(y) = 2(1-y) - 2y$. For $\alpha = 0$, $f'(0) = 2$, therefore this fixed point is unstable in both the initial value problem and the Euler method, for all values of h. For $\alpha = 1$, $f'(1) = -2$, and if $|1 + -2h| < 1$, then this fixed point will be linearly stable for the Euler method. If $0 < h < 1$, then $|1 - 2h| < 1$. Therefore this fixed point is linearly stable for $h \in (0, 1)$.