Caleb Logemann
MATH 561 Numerical Analysis I
Homework 1

1. Let $f(x) = \sqrt{1+x^2} - 1$

   (a) For small values of x, $f(x)$ can be difficult to compute becuase $x^2 \approx 0$ and $\sqrt{1+x^2} \approx 1$. This causes $f(x)$ to be taking the difference to two numbers that are approximately equal, which can cause a loss of accuracy. This can be circumvented by noting that $f(x)$ can be expressed as follows. f(x) = $\sqrt{1+x^2} - 1$
   $= \sqrt{1+x^2} - 1 \times \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}$
   $= \frac{x^2}{\sqrt{1+x^2}+1}$

   (b) The condition number of $f(x)$ can be determined as follows (cond f)(x) = x f'(x)$\frac{}{f(x) = \frac{x^2}{\sqrt{1+x^2}\sqrt{1+x^2}-1} = \frac{x^2}{1+x^2-\sqrt{1+x^2}}}$ As $x \to 0$, the use of L'Hopital's rule is necessary. $\lim x \to 0 (cond\, f)(x) =$

   (c) The condition number of $f(x)$ doesn't take into account taking the difference of two numbers that are approximately equal.

2. Let $f(x) = (1 - \cos x)/x$, $x \neq 0$.

   (a)

   (b)

   (c)

3. Let $f(x) = x^n + ax - 1$, $a > 0$, $n \geq 2$

   (a) Show that $f(x)$ has exactly one positive root $\xi(a)$. First note that $f(0) = -1$ and $f(1) = a > 0$. Since f is a polynomial and is continuous, by the Intermediate Value Theorem, there must exist $c \in (0,1)$, such that $f(c) = 0$. Therefore f has at least on root in the interval $(0,1)$. Also $f'(x) = nx^{n-1} + a$, for $x \geq 0$, $f'(x) > 0$, so f is a strictly increasing function on the interval $[0, \infty)$. Therefore there is only one positive root of $f(x)$ and it is in the interval $(0,1)$. Let $\xi(a)$ be this root.

   (b) Obtain a formula for $(cond\, \xi)(a)$. The derivitive of $\xi(a)$ can be found by implicit differentiation of $f(\xi(a))$. f($\xi(a)$) = 0
   $\xi(a)^n + a\xi(a) - 1 = 0 Bydifferentiatingwithrespecttoan\xi(a)^{n-1}\xi'(a) + a\xi'(a) + \xi(a) = 0$
   $\xi'(a) = \frac{-\xi(a)}{n\xi(a)^{n-1}+a}$
   $Alsoitcanbenotedthat \xi(a)^n + a\xi(a) - 1 = 0$

$\xi(a)^n = 1 - a\xi(a)$
$\xi(a)^{n-1} = \frac{1-a\xi(a)}{\xi(a)} Then \xi'(a)$ can be expressed as $\xi'(a) = \frac{-\xi(a)}{n\frac{1-a\xi(a)}{\xi(a)}+a}$
$\xi'(a) = \frac{-\xi(a)^2}{n-an\xi(a)+a\xi(a)}$
The condition number of $\xi(a)$ can then be found $(\operatorname{cond}\xi)(a) = \frac{a\xi'(a)}{\xi(a)}$
$= \frac{a\frac{-\xi(a)^2}{n-an\xi(a)+a\xi(a)}}{\xi(a)}$
$= \frac{-a\xi(a)}{n-an\xi(a)+a\xi(a)}$
$= \frac{a\xi(a)}{n+(1-n)a\xi(a)}$

(c) Since $0 < \xi(a) < 1$, bounds for the condition number of $\xi(a)$ can be found. $\lim \xi(a) \to 0 \frac{a\xi(a)}{n+(1-n)a\xi(a)} = 0$
$\lim \xi(a) \to 1 \frac{a\xi(a)}{n+(1-n)a\xi(a)} = \frac{a}{n+(1-n)a}$
Therefore $0 < (cond\,\xi)(a) < \frac{a}{n+(1-n)a}$.

4. (a)

   (b)

5. Let $x_0, x_1, \ldots, x_n$ be pairwise distinct points in a, b, $-\infty < a < b < \infty$, and $f \in C^1 a, b$. Show that given any $\epsilon > 0$, there exists a polynomial p such that $[\infty] f - p < \epsilon$ and at the same time $p(x_i) = f(x_i)$, for $i = 0, 1, \ldots, n$. Let $p = p_n(f; \cdot) + \omega_n q$, where $p_n(f; \cdot)$ is the Lagrange interpolation of f at $x_1, x_2, \ldots, x_n$, $\omega_n = i = 0nx - x_i$, and q is some polynomial. Firstly note that $p(x_i) = p_n(x_i) + 0q = f(x_i)$, so the condition of equality at the points x_i is met. Secondly note $[\infty] f - p = [\infty] f - p_n - \omega_n q = [\infty]\omega_n[\infty]\frac{f-p_n}{\omega_n} - q$. Consider the function $g(x) = \frac{f(x)-p_n(x)}{\omega_n(x)}$. Since $g(x)$ is composed of continuous functions on a, b, $g(x)$ is continuous on a, b everywhere $\omega_n(x) \neq 0$. The function $\omega_n(x) = 0$ at x_i for $i = 1, 2, \ldots, n$. Therefore the limit of $g(x)$ as $x \to x_i$ needs to be considered. At $x = x_i$, $f(x) - p_n(x) = 0$ and $\omega_n(x) = 0$, therefore L'Hopital's rule can be employed. Therefore $\lim x \to x_i g(x) = \lim x \to x_i \frac{f'(x)-p_n'(x)}{w_n'(x)}$. Remember that $f \in C^1 a, b$, so f is differentiable, and p_n is trivially differentiable. Also ω_n is differentiable and $\omega_n'(x) = \sum i = 1nk = 1, k \neq inx - x_k$ by repeated use of the product rule. Therefore $\omega_n'(x_i) = k = 1, k \neq inx_i - x_k \neq 0$. Thus $\lim x \to x_i \frac{f'(x)-p_n'(x)}{w_n'(x)}$ exists so $g(x)$ is continuous at x_i. Then by the Weierstrass Approximation Theorem there exists a polynomial q such that $[\infty]\frac{f-p_n}{\omega_n} - q < \epsilon/[\infty]\omega_n$. Thus $[\infty] f - p < [\infty]\omega_n \epsilon/[\infty]\omega_n = \epsilon$.

6. (a) The normal equations are $\sum j = 1n\pi_i, \pi_j c_j = \pi_i, f$. $\pi_i, \pi_j = \int -\infty\infty \pi_i \pi_j d\lambda$
$= \int 0\infty e^{-it} e^{-jt} dt$
$= \int 0\infty e^{-t(i+j)} dt$

$$= -\frac{1}{i+j} e^{-t(i+j)} \Big|_{x=0}^{\infty}$$

(b)