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MATH 561 Numerical Analysis I
Homework 5

1. (a) Consider an explicit multistep method of the form

$$u_{n+2} + \alpha u_{n+1} - \alpha u_{n-1} - u_{n-2} = h(\beta f_{n+1} + \gamma f_n + \beta f_{n-1})$$

Show that the parameters α, β, γ can be chosen uniquely so that the method has order $p = 6$.

(b) Discuss the stability properties of the method obtained in (a).

2. Construct a pair of four-step methods, one explicit, the other implicit, both have $\alpha(\xi) = \xi^4 - \xi^3$ and order $p = 4$, but global error constants that are equal in magnitude but opposite in sign.

3. Consider the model problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega(y - a(x)), 0 \leq x \leq 1, y(0) = y_0$$

where $\omega > 0$ and (i) $a(x) = x^2$, $y_0 = 0$; and (ii) $a(x) = e^x$, $y_0 = 1$

(a) In each of the cases (i) and (ii), solve the differential equation exactly.

For case (i) this differential equation can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega\left(y - x^2\right)$$

for $0 \leq x \leq 1$ and $y(0) = 0$. This is equivalent to

$$\omega\left(y - x^2\right) + \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

This is not an exact differential equation but it can be made into an exact differential equation by multiplying by an integrating factor, $\mu(x)$.

$$\begin{aligned}
\mu(x)\omega\left(y - x^2\right) + \mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\partial}{\partial y}\left(\mu(x)\omega\left(y - x^2\right)\right) &= \mu(x)\omega \\
\frac{\partial}{\partial x}(\mu(x)) &= \mu'(x) \\
\mu(x)\omega &= \frac{\mathrm{d}\mu}{\mathrm{d}x} \\
\mu(x) &= e^{\omega x}
\end{aligned}$$

This results in the equivalent exact differential equation

$$e^{\omega x}\omega\left(y - x^2\right) + e^{\omega x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

To solve this type of equation a function $\Psi(x, y)$ must be found such that

$$\frac{\partial}{\partial x}(\Psi(x, y)) = e^{\omega x}\omega\left(y - x^2\right)$$

and

$$\frac{\partial}{\partial y}(\Psi(x,y)) = e^{\omega x}$$

Solving for Ψ

$$\begin{aligned}
\Psi(x,y) &= \int e^{\omega x}\omega\left(y - x^2\right) \mathrm{d}x \\
&= \int \omega y e^{\omega x} \,\mathrm{d}x - \int \omega e^{\omega x} x^2 \,\mathrm{d}x + h(y) \\
&= y e^{\omega x} - e^{\omega x}\left(\frac{\omega^2 x^2 - 2\omega x + 2}{\omega^2}\right) + h(y) \\
\frac{\partial}{\partial y}(\Psi(x,y)) &= e^{\omega x} + h'(y) \\
e^{\omega x} + h'(y) &= e^{\omega x} \\
h(y) &= C \\
\Psi(x,y) &= y e^{\omega x} - e^{\omega x}\left(\frac{\omega^2 x^2 - 2\omega x + 2}{\omega^2}\right) + C
\end{aligned}$$

Now this exact differential equation is equivalent to the equation

$$y e^{\omega x} - e^{\omega x}\left(\frac{\omega^2 x^2 - 2\omega x + 2}{\omega^2}\right) = C$$

$$\begin{aligned}
y e^{\omega x} &= e^{\omega x}\left(\frac{\omega^2 x^2 - 2\omega x + 2}{\omega^2}\right) + C \\
y(x) &= \frac{\omega^2 x^2 - 2\omega x + 2}{\omega^2} + e^{-\omega x} C \\
y(x) &= x^2 - \frac{2}{\omega}x + \frac{2}{\omega^2} + e^{-\omega x} C
\end{aligned}$$

Now the initial conditions must be used to solve for C.

$$\begin{aligned}
y(0) &= \frac{2}{\omega^2} + C \\
0 &= \frac{2}{\omega^2} + C \\
C &= -\frac{2}{\omega^2}
\end{aligned}$$

Therefore the exact solution to (i) is

$$y(x) = x^2 - \frac{2}{\omega}x + \frac{2}{\omega^2}\left(1 - e^{-\omega x}\right)$$

For case (ii) this differential equation can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega(y - e^x)$$

for $0 \leq x \leq 1$ and $y(0) = 1$.

The same integrating factor as in part (i) can be used

$$\omega e^{\omega x}(y - e^x) + e^{\omega x}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

To solve this type of equation a function $\Psi(x, y)$ must be found such that

$$\frac{\partial}{\partial x}(\Psi(x, y)) = e^{\omega x}\omega(y - e^x)$$

and

$$\frac{\partial}{\partial y}(\Psi(x, y)) = e^{\omega x}$$

Solving for Ψ

$$\begin{aligned}
\Psi(x, y) &= \int e^{\omega x}\,\mathrm{d}y \\
&= ye^{\omega x} + h(x) \\
\frac{\partial}{\partial x}(\Psi(x, y)) &= \omega ye^{\omega x} + h'(x) \\
\omega ye^{\omega x} + h'(x) &= e^{\omega x}\omega(y - e^x) \\
\omega ye^{\omega x} + h'(x) &= e^{\omega x}\omega y - e^{\omega x}\omega e^x \\
h'(x) &= -\omega e^{(\omega+1)x} \\
h(x) &= -\frac{\omega}{\omega+1}e^{(\omega+1)x} + C \\
\Psi(x, y) &= ye^{\omega x} - \frac{\omega}{\omega+1}e^{(\omega+1)x} + C
\end{aligned}$$

Now this exact differential equation is equivalent to the equation

$$ye^{\omega x} - \frac{\omega}{\omega+1}e^{(\omega+1)x} = C$$

$$\begin{aligned}
ye^{\omega x} &= \frac{\omega}{\omega+1}e^{(\omega+1)x} + C \\
y(x) &= e^{-\omega x}\left(\frac{\omega}{\omega+1}e^{(\omega+1)x} + C\right) \\
y(x) &= \frac{\omega}{\omega+1}e^x + Ce^{-\omega x}
\end{aligned}$$

Then the initial conditions can be used to solve for C.

$$\begin{aligned}
y(0) &= \frac{\omega}{\omega+1} + C \\
1 &= \frac{\omega}{\omega+1} + C \\
C &= 1 - \frac{\omega}{\omega+1} \\
C &= \frac{1}{\omega+1}
\end{aligned}$$

Thus the exact solution to this differential equation is

$$y(x) = \frac{\omega}{\omega+1}e^x + \frac{1}{\omega+1}e^{-\omega x}$$

$$y(x) = \frac{\omega e^x + e^{-\omega x}}{\omega + 1}$$

(b) In each of the cases (i) and (ii), apply the kth-order Adams-Bashford method and kth-order Adams predictor/corrector method, for $k = 4$, using exact starting values and step lengths h = 1/20 , 1/40 , 1/80 , 1/160. Plot the exact values y_n and numerical solution u_n, and check the accuracy of the methods. Try $\omega = 1, 10, 50$. Summarize your results.