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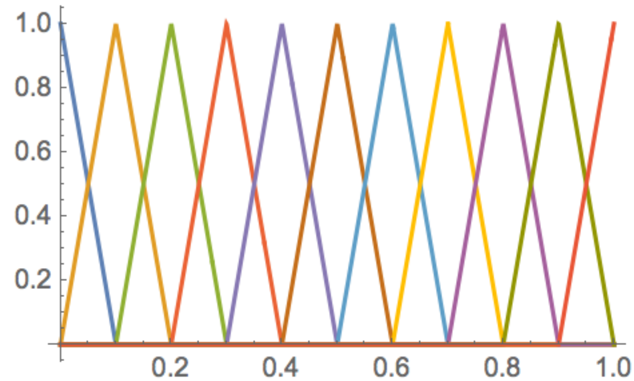
MATH 561 Numerical Analysis I
Homework 2

1. (a)



(b) If $j = k$, then $\pi_j(k/n) = \pi_j j/n = 1$. If $j \neq k$, then $\pi_j(k/n) = 0$, because k/n is outside of the division that is the support of π_j

(c) Let $c_0, c_1, \dots, c_n \in \mathbb{R}$ be given such that $\sum_{i=0}^{n} (c_i \pi_i(t)) = 0$ for all $t \in (0, 1)$ Consider $t = k/n$ for some $k \in \{0, 1, \dots, n\}$, then $\sum_{i=0}^{n} (c_i \pi_i(k/n)) = c_k \pi_k(k/n)$, because $\pi_j k/n = 0$ for all $j \neq k$. Also $\pi_k(k/n) = 1$, so $c_k \pi_k(k/n) = c_k$. However this sum must be equal to 0 at $t = k/n$, so $c_k = 0$. This implies that $c_0 = c_1 = \cdots = c_n = 0$. Thus the $\{\pi_j\}_{j=0}^{n}$ is linearly independent over the interval $(0, 1)$. This also implies that $\{\pi_j\}_{j=0}^{n}$ is linearly independent over the points $\left\{0, 1/n, \dots, \frac{n-1}{n}, 1\right\}$, because at these points only one of the functions contributes to the overall sum.

(d) For $|i - j| > 1$, $\pi_i(t)\pi_j(t) = 0$ for $t \in (0, 1)$. Therefore $\int_0^1 \pi_i(t)\pi_j(t)\,\mathrm{d}t = 0$ and $a_{ij} = 0$ for $|i - j| > 1$.

For $|i - j| = 1$, without loss of generality assume $j = i+1$. Note that $(\pi_i(t)\pi_j(t)) = (i/n, i + 1/n) = (i/n, j/n)$

$$\int_0^1 \pi_i(t)\pi_j(t)\,\mathrm{d}t = \int_0^1 \pi_i(t)\pi_{i+1}(t)\,\mathrm{d}t$$

Since $(\pi_i(t)\pi_{i+1}(t)) = (i/n, i + 1/n)$

$$= \int_{i/n}^{(i+1)/n} \pi_i(t)\pi_{i+1}(t)\,\mathrm{d}t$$

On this interval $\pi_i(t) = -nt + i + 1$ and $\pi_{i+1}(t) = nt - i$

$$
\begin{aligned}
&= \int_{i/n}^{(i+1)/n} (-nt + i + 1)(nt - i)\,\mathrm{d}t \\
&= \int_{i/n}^{(i+1)/n} -n^2t^2 + int + int + nt - i^2 - i\,\mathrm{d}t \\
&= \int_{i/n}^{(i+1)/n} -n^2t^2 + (2i+1)nt - i^2 - i\,\mathrm{d}t \\
&= -\frac{n^2}{3}t^3 + \frac{(2i+1)n}{2}t^2 - (i^2+i)t\Big|_{t=i/n}^{(i+1)/n} \\
&= -\frac{n^2}{3}t^3 + \frac{(2i+1)n}{2}t^2 - (i^2+i)t\Big|_{t=i/n}^{(i+1)/n}
\end{aligned}
$$

2. (a)
   (b)
3.
4.
5.
6. (a)
   (b)(b.1)
      (b.2)
      (b.3)
7.
8.