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MATH 561 Numerical Analysis I
Final Assignment

1. Let $x_1, x_2, \dots, x_n$, for $n > 1$, be machine numbers. Their product can be computed by the alogirithm

$$\begin{aligned} p_1 &= x_1 \\ p_k &= fl(x_k p_{k-1}), k = 2, 3, \dots, n \end{aligned}$$

(a) Find an upper bound for the relative error in terms of the machine precision eps and n.
The relative error is given by

$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$

First consider p_k.

$$\begin{aligned} p_k &= fl(x_k p_{k-1}) \\ &= x_k p_{k-1}(1 + \epsilon_k) \end{aligned}$$

Where $\epsilon_k < eps$, for $k = 1, \cdots, n$

$$< x_k p_{k-1}(1 + eps)$$

Applying this recursively to p_n, we see that

$$\begin{aligned} p_n &< x_n p_{n-1}(1 + eps) \\ &< x_n x_{n-1} p_{n-2}(1 + eps)^2 \\ &< x_n x_{n-1} x_{n-2} p_{n-3}(1 + eps)^3 \\ &\vdots \\ &< x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} \end{aligned}$$

Therefore the relative error can be bounded as follows

$$\begin{aligned} E &= \frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \\ &< \frac{x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \\ &= \frac{x_1 x_2 \cdots x_n \left((1 + eps)^{n-1} - 1\right)}{x_1 x_2 \cdots x_n} \\ &= (1 + eps)^{n-1} - 1 \end{aligned}$$

Therefore the upper bound for the relative error is $|E| < (1 + eps)^{n-1} - 1$.

(b) For any integer r that satisfies $r \times eps < \frac{1}{10}$, show that

$$(1 + eps)^r - 1 < 1.06 \times r \times eps$$

Hence for n not too large, simplify the answer given in (a).
Using the Binomial Thereom, $(1 + eps)^r$ can be expanded.

$$(1 + eps)^r - 1 = \sum_{i=0}^{r} \left(\binom{r}{i} 1^{r-i} eps^i \right) - 1$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \left(\binom{r}{i} eps^i \right) \\
&= r \cdot eps + \binom{r}{2} eps^2 + \binom{r}{3} eps^3 + \cdots + eps^r \\
&= r \cdot eps + \frac{r(r-1)}{2} eps^2 + \frac{r(r-1)(r-2)}{3!} eps^3 + \cdots + eps^r \\
&= r \cdot eps \left(1 + \frac{r-1}{2} eps + \frac{(r-1)(r-2)}{3!} eps^2 + \cdots + \frac{(r-1)(r-2)\cdots(1)}{r!} eps^{r-1} \right)
\end{aligned}
$$

Since $r \times eps < \frac{1}{10}$, $(r-i)eps < \frac{1}{10}$ for any $0 < i < r$

$$
\begin{aligned}
&< r \cdot eps \left(1 + \frac{1}{2}\frac{1}{10} + \frac{1}{3!}\left(\frac{1}{10}\right)^2 + \cdots + \frac{1}{r!}\left(\frac{1}{10}\right)^{r-1} \right) \\
&= r \cdot eps \sum_{k=0}^{r-1} \left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k-1} \right) \\
&= r \cdot eps \cdot 10 \sum_{k=1}^{r-1} \left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k} \right)
\end{aligned}
$$

This expression is certainly less than extending the sum to infinity because all of the terms are postive. Also this sum is the Taylor series for $e^x - 1$.

$$
\begin{aligned}
&< r \cdot eps \cdot 10 \sum_{k=1}^{\infty} \left(\frac{1}{k!}\left(\frac{1}{10}\right)^{k} \right) \\
&= r \cdot eps \cdot 10 \left(e^{1/10} - 1 \right) \\
&\approx 1.05171 r \cdot eps \\
&< 1.06 r \cdot eps
\end{aligned}
$$

This result can now be used to simplify the result of part (a). Now if n is not too large, then $|E| < 1.06(n-1)eps$.

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