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MATH 561 Numerical Analysis I
Homework 1

1. Let $f(x) = \sqrt{1 + x^2} - 1$

   (a) For small values of $|x|$, $f(x)$ can be difficult to compute becuase $x^2 \approx 0$ and $\sqrt{1+x^2} \approx 1$. This causes $f(x)$ to be taking the difference to two numbers that are approximately equal, which can cause a loss of accuracy. This can be circumvented by noting that $f(x)$ can be expressed as follows.

   $$\begin{aligned} f(x) &= \sqrt{1+x^2} - 1 \\ &= \sqrt{1+x^2} - 1 \times \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} \\ &= \frac{x^2}{\sqrt{1+x^2}+1} \end{aligned}$$

   (b) The condition number of $f(x)$ can be determined as follows

   $$\begin{aligned} (condf)(x) &= \left|\frac{xf'(x)}{f(x)}\right| \\ &= \left|\frac{x^2}{\sqrt{1+x^2}\left(\sqrt{1+x^2}-1\right)}\right| \\ &= \left|\frac{x^2}{1+x^2-\sqrt{1+x^2}}\right| \end{aligned}$$

   As $|x| \to 0$, the use of L'Hopital's rule is necessary.

   $$\lim_{x \to 0}((cond\, f)(x)) = ||$$

   (c) The condition number of $f(x)$ doesn't take into account taking the difference of two numbers that are approximately equal.

2. Let $f(x) = (1 - \cos(x))/x$, $x \neq 0$.

   (a)

   (b)

   (c)

3. Let $f(x) = x^n + ax - 1$, $a > 0$, $n \geq 2$

   (a) Show that $f(x)$ has exactly one positive root $\xi(a)$. First note that $f(0) = -1$ and $f(1) = a > 0$. Since f is a polynomial and is continuous, by the Intermediate Value Theorem, there must exist $c \in (0, 1)$, such that $f(c) = 0$. Therefore f has at least on root in the interval $(0, 1)$. Also $f'(x) = nx^{n-1} + a$, for $x \geq 0$, $f'(x) > 0$, so f is a strictly increasing function on the interval $[0, \infty)$. Therefore there is only one positive root of $f(x)$ and it is in the interval $(0, 1)$. Let $\xi(a)$ be this root.

   (b) Obtain a formula for $(cond\,\xi)(a)$. The derivitive of $\xi(a)$ can be found by implicit differentiation of $f(\xi(a))$.

$$\begin{aligned} f(\xi(a)) &= 0 \\ \xi(a)^n + a\xi(a) - 1 &= 0 \end{aligned}$$

By differentiating with respect to a

$$\begin{aligned} n\xi(a)^{n-1}\xi'(a) + a\xi'(a) + \xi(a) &= 0 \\ \xi'(a) &= \frac{-\xi(a)}{n\xi(a)^{n-1} + a} \end{aligned}$$

Also it can be noted that

$$\begin{aligned} \xi(a)^n + a\xi(a) - 1 &= 0 \\ \xi(a)^n &= 1 - a\xi(a) \\ \xi(a)^{n-1} &= \frac{1 - a\xi(a)}{\xi(a)} \end{aligned}$$

Then $\xi'(a)$ can be expressed as

$$\begin{aligned} \xi'(a) &= \frac{-\xi(a)}{n\frac{1-a\xi(a)}{\xi(a)} + a} \\ \xi'(a) &= \frac{-\xi(a)^2}{n - an\xi(a) + a\xi(a)} \end{aligned}$$

The condition number of $\xi(a)$ can then be found

$$
\begin{aligned}
(cond\,\xi)(a) &= \left|\frac{a\xi'(a)}{\xi(a)}\right| \\
&= \left|\frac{a\frac{-\xi(a)^2}{n-an\xi(a)+a\xi(a)}}{\xi(a)}\right| \\
&= \frac{-a\xi(a)}{n-an\xi(a)+a\xi(a)}
\end{aligned}
$$

(c)