1.7.4 B-splines

We describe a system of B-splines (B stands for basis) from which other splines can be obtained.

We first start with B-splines of degree 0, i.e., piecewise constant splines defined as

$$B_i^0(x) = \begin{cases} 1 & x_i \le x < x_{i+1} \\ 0 & otherwise \end{cases}, i \in \mathbf{Z} \tag{1.1.90}$$

Properties of $B_i^0(x)$:

1. $B_i^0(x) \ge 0$, for all x and i
2. $\sum\limits_{i=-\infty}^{\infty} B_i^0(x) = 1$, for all x
3. The support of $B_i^0(x)$ is $[x_i, x_{i+1})$
4. $B_i^0(x) \in C^{-1}$.

We show property (2) by noting that for arbitrary x there exists m such that $x \in [x_m, x_{m+1})$ then write

$$\sum_{i=-\infty}^{\infty} B_i^0(x) = B_m^0(x) = 1.$$

Use the recurrence formula to generate the next basis functions

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x), \quad for\ k > 0. \tag{1.1.91}$$

Properties of B_i^1:

1. $B_i^1(x)$ are the classical piecewise linear hat functions equal to 1 at x_{i+1} and zero all other nodes.
2. $B_i^1(x) \in C^0$
3. $\sum\limits_{i=-\infty}^{\infty} B_i^1(x) = 1$ for all x

4. The support of $B_i^1(x)$ is (x_i, x_{i+2})
5. $B_i^1(x) \geq 0$ for all x and i

In general for arbitrary k one can show that:

1. $B_i^k(x)$ are piecewise polynomials of degree k
2. $B_i^k(x) \in C^{k-1}$
3. $\sum\limits_{i=-\infty}^{\infty} B_i^k(x) = 1$ for all x
4. The support of $B_i^k(x)$ is (x_i, x_{i+k+1})
5. $B_i^k(x) \geq 0$ for all x and i
6. $B_i^k(x)$, $-\infty < i < \infty$ are linearly independent, i.e., they form a basis.

See Figure 1.7.4 for plots of the first four b-splines.

Interpolation using B-splines:

(i) For $k = 0$, we construct a piecewise constant spline interpolation by writing

$$f(x) \approx P_0(x) = \sum_{i=-\infty}^{\infty} c_i B_i^0(x) \tag{1.1.92}$$

Using the properties of $B_i^0(x)$ we show that $c_i = f(x_i)$.

(ii) For $k = 1$, we construct a piecewise linear spline interpolation by writing

$$f(x) \approx P_1(x) = \sum_{i=-\infty}^{\infty} c_i B_i^1(x) \tag{1.1.93}$$

Again using $B_i^1(x_{j+1}) = \delta_{ij}$ we show that $c_i = f(x_{i+1})$.

(ii) For $k = 3$, we construct a piecewise cubic spline interpolation by writing

$$f(x) \approx P_3(x) = \sum_{i=-\infty}^{\infty} c_i B_i^3(x) \tag{1.1.94}$$

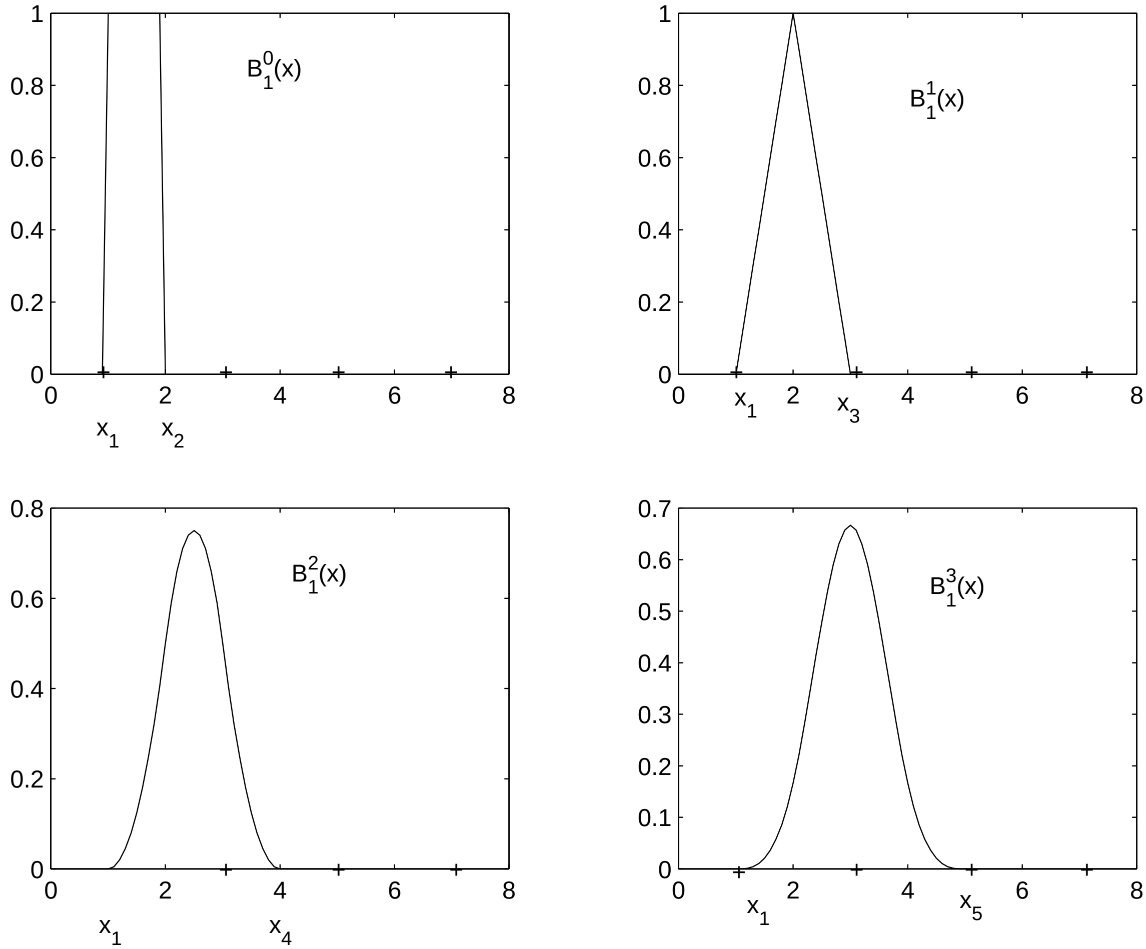

Figure 1.1: B-splines of degree $k = 0, 1, 2, 3$, upper left to lower right.

We recall that $B_i^3 \in C^2$ are piecewise cubic polynomials with support in (x_i, x_{i+3}).

In order to interpolate f at $x_i,\ i = 0, \cdots, n$, we

1. Write $S(x) = \sum_{i=-3}^{n-1} c_i B_i^3(x)$
   (include basis functions whose support intersect $[x_0, x_n]$).

2. Set $n+1$ equations

$$f(x_i) = c_{i-3}B_{i-3}^3(x_i) + c_{i-2}B_{i-2}^3(x_i) + c_{i-1}B_{i-1}^3(x_i), \quad i \;=\; 0, 1, \cdot, n, \tag{1.1.95a}$$

   where $c_{-3}, c_{-2}, c_{-1}, c_0, \cdots, c_n$ are the unknowns.

3. Close the system, for natural Spline, by setting

$$S''(x_0) = 0, \quad S''(x_n) = 0, \tag{1.1.95b}$$

4. Solve the system (1.1.95).

Remarks:

1. If x_i are uniformly distributed we have
   $B_i^2(x_j) = 0,\ j \leq i \ or \ j \geq i+3,$
   $B_i^2(x_{i+1}) = B_i^2(x_{i+2}) = 1/2$
   $B_i^3(x_j) = 0,\ j \leq i \ or \ j \geq i+4,$
   $B_i^3(x_{i+1}) = B_i^3(x_{i+3}) = 1/6,\ B_i^3(x_{i+2}) = 2/3$

2. The system (1.1.95) has a unique solution

3. B-splines may be used to construct clamped splines

1.8 Interpolation in multiple dimensions

Read section of 6.10 of textbook (Kincaid and Cheney).

1.9 Least-squares Approximations

Read section 6.8 of Textbook (Kincaid and Cheney).