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MATH 561 Numerical Analysis I
Homework 4

#1 (a) Determine the principle error function of the general explicit two-stage Runge-Kutta method.

The general explicit two-stage Runge-Kutta method can be described as follows.

$$\begin{aligned} k_1 &= f(x, y) \\ k_2 &= f(x + \mu h, y + \mu h k_1) \\ \Phi(x, y; h) &= \alpha_1 k_1 + \alpha_2 k_2 \end{aligned}$$

To find the priniciple error function, first the local truncation error must be found. The local truncation error is defined as

$$T(x, y; h) = \Phi(x, y; h) - \frac{1}{h}(y(x + h) - y(x))$$

The principle error function is the functional coefficient of h^p in the local truncation error, when p is the order of the method. Two-stage Runge-Kutta methods have in general an order of $p = 2$, so the principle error function is the coefficient of h^2. In order to find this the Taylor expansion of $\Phi(x, y; h)$ and $\frac{1}{h}(y(x + h) - y(x)))$ must be found, at least to the h^2 term.
First I will find the Taylor expansion of $\Phi(x, y; h) = \alpha_1 k_1 + \alpha_2 k_2$. The Taylor expansion of $k_1 = f(x, y)$ is just $f(x, y)$. The Taylor expansion of k_2 can be found as follows.

$$\begin{aligned} k_2 &= f(x + \mu h, y + \mu h k_1) \\ &= f(x + \mu h, y + \mu h f(x, y)) \\ &= f(x, y) + f_x(x, y)(\mu h) + f_y(x, y)(\mu h f(x, y)) \\ &+ \frac{1}{2}\Big(f_{xx}(x, y)(\mu h)^2 + 2f_{xy}(x, y)(\mu^2 h^2 f(x, y)) + f_{yy}(x, y)(\mu^2 h^2 f(x, y)^2\Big) + O(h^3) \\ &= f(x, y) + \mu(f_x(x, y) + f(x, y)f_y(x, y))h \\ &+ \frac{1}{2}\mu^2\Big(f_{xx}(x, y) + 2f(x, y)f_{xy}(x, y) + f(x, y)^2 f_{yy}(x, y)\Big)h^2 + O(h^3) \end{aligned}$$

Now the Taylor expansion of $\Phi(x, y; h)$ can be expressed as follows. Note that moving forward all value or derivatives of f will be evaluated at (x, y). Thus $f = f(x, y)$, $f_x = f_x(x, y)$, $f_y = f_y(x, y)$, and so on.

$$\Phi(x, y; h) = \alpha_1 k_1 + \alpha_2 k_2$$

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\begin{aligned}
&= \alpha_1 f + \alpha_2 \left(f + \mu(f_x + f f_y)h + \frac{1}{2}\mu^2 \Big(f_{xx} + f f_{xy} + f^2 f_{yy} \Big) h^2 + O(h^3) \right) \\
&= (\alpha_1 + \alpha_2) f + \mu\alpha_2 (f_x + f f_y) h + \frac{1}{2}\alpha_2 \mu^2 \Big(f_{xx} + f f_{xy} + f^2 f_{yy} \Big) h^2 + O(h^3)
\end{aligned}
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(b)

(c)