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MATH 561 Numerical Analysis I
Homework 3

1. (a) Use the central quotient approximation $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ to obtain an approximation of $\frac{\partial^2}{\partial x \partial y} u(x, y)$, for a function u of two variables.

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y} u(x, y) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} u(x, y)\right) \\
&\approx \frac{\partial}{\partial x}\left(\frac{u(x, y+h) - u(x, y-h)}{2h}\right) \\
&= \frac{1}{2h}\left(\frac{\partial}{\partial x} u(x, y+h) - \frac{\partial}{\partial x} u(x, y-h)\right) \\
&\approx \frac{1}{2h}\left(\frac{u(x+h, y+h) - u(x-h, y+h)}{2h} - \frac{u(x+h, y-h) - u(x-h, y-h)}{2h}\right) \\
&= \frac{u(x+h, y+h) - u(x-h, y+h) - u(x+h, y-h) + u(x-h, y-h)}{4h^2}
\end{aligned}$$

(b) The fourth order Taylor expansion of $u(v, w)$ approximated at (x, y) is

$$\begin{aligned}
u(v, w) = u(x, y) &+ (v-x)\frac{\partial}{\partial x}(u(x, y)) + (w-y)\frac{\partial}{\partial y}(u(x, y)) + \frac{(v-x)^2}{2}\frac{\partial^2}{\partial x^2}(u(x, y)) \\
&+ (v-x)(w-y)\frac{\partial^2}{\partial x \partial y}(u(x, y)) + \frac{(w-y)^2}{2}\frac{\partial^2}{\partial y^2}(u(x, y)) + \frac{(v-x)^3}{3!}\frac{\partial^3}{\partial x^3}(u(x, y)) \\
&+ \frac{(v-x)^2(w-y)}{2!}\frac{\partial^3}{\partial x^2 \partial y}(u(x, y)) + \frac{(v-x)(w-y)^2}{2!}\frac{\partial^3}{\partial x \partial y^2}(u(x, y)) \\
&+ \frac{(w-y)^3}{3!}\frac{\partial^3}{\partial y^3}(u(x, y)) + \frac{(v-x)^4}{4!}\frac{\partial^4}{\partial x^4}(u(x, y)) + \frac{(v-x)^3(w-y)}{3!}\frac{\partial^4}{\partial x^3 \partial y}(u(x, y)) \\
&+ \frac{(v-x)^2(w-y)^2}{2!2!}\frac{\partial^4}{\partial x^2 \partial y^2}(u(x, y)) + \frac{(v-x)(w-y)^3}{3!}\frac{\partial^4}{\partial x \partial y^3}(u(x, y)) \\
&+ \frac{(w-y)^4}{4!}\frac{\partial^4}{\partial y^4}(u(x, y))
\end{aligned}$$

Taking the fourth order Taylor expansion of the terms found in part (a)

$$\begin{aligned}
&\frac{u(x+h, y+h) - u(x-h, y+h) - u(x+h, y-h) + u(x-h, y-h)}{4h^2} \\
&\approx \frac{1}{4h^2}\left(u(x, y) + h\frac{\partial}{\partial x}(u(x, y)) + h\frac{\partial}{\partial y}(u(x, y)) + \frac{h^2}{2}\frac{\partial^2}{\partial x^2}(u(x, y))\right.
\end{aligned}$$

$$
\begin{aligned}
&+h^2\frac{\partial^2}{\partial x\partial y}(u(x,y))+\frac{h^2}{2}\frac{\partial^2}{\partial y^2}(u(x,y))+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}(u(x,y))\\
&+\frac{h^3}{2!}\frac{\partial^3}{\partial x^2\partial y}(u(x,y))+\frac{h^3}{2!}\frac{\partial^3}{\partial x\partial y^2}(u(x,y))\\
&+\frac{h^3}{3!}\frac{\partial^3}{\partial y^3}(u(x,y))+\frac{h^4}{4!}\frac{\partial^4}{\partial x^4}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x^3\partial y}(u(x,y))\\
&+\frac{h^4}{2!2!}\frac{\partial^4}{\partial x^2\partial y^2}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x\partial y^3}(u(x,y))\\
&+\frac{h^4}{4!}\frac{\partial^4}{\partial y^4}(u(x,y))-u(x,y)+h\frac{\partial}{\partial x}(u(x,y))-h\frac{\partial}{\partial y}(u(x,y))-\frac{h^2}{2}\frac{\partial^2}{\partial x^2}(u(x,y))\\
&+h^2\frac{\partial^2}{\partial x\partial y}(u(x,y))-\frac{h^2}{2}\frac{\partial^2}{\partial y^2}(u(x,y))+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}(u(x,y))\\
&-\frac{h^3}{2!}\frac{\partial^3}{\partial x^2\partial y}(u(x,y))+\frac{h^3}{2!}\frac{\partial^3}{\partial x\partial y^2}(u(x,y))\\
&-\frac{h^3}{3!}\frac{\partial^3}{\partial y^3}(u(x,y))-\frac{h^4}{4!}\frac{\partial^4}{\partial x^4}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x^3\partial y}(u(x,y))\\
&-\frac{h^4}{2!2!}\frac{\partial^4}{\partial x^2\partial y^2}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x\partial y^3}(u(x,y))\\
&-\frac{h^4}{4!}\frac{\partial^4}{\partial y^4}(u(x,y))-u(x,y)-h\frac{\partial}{\partial x}(u(x,y))+h\frac{\partial}{\partial y}(u(x,y))-\frac{h^2}{2}\frac{\partial^2}{\partial x^2}(u(x,y))\\
&+h^2\frac{\partial^2}{\partial x\partial y}(u(x,y))-\frac{h^2}{2}\frac{\partial^2}{\partial y^2}(u(x,y))-\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}(u(x,y))\\
&+\frac{h^3}{2!}\frac{\partial^3}{\partial x^2\partial y}(u(x,y))-\frac{h^3}{2!}\frac{\partial^3}{\partial x\partial y^2}(u(x,y))\\
&+\frac{h^3}{3!}\frac{\partial^3}{\partial y^3}(u(x,y))-\frac{h^4}{4!}\frac{\partial^4}{\partial x^4}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x^3\partial y}(u(x,y))\\
&-\frac{h^4}{2!2!}\frac{\partial^4}{\partial x^2\partial y^2}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x\partial y^3}(u(x,y))\\
&-\frac{h^4}{4!}\frac{\partial^4}{\partial y^4}(u(x,y))+u(x,y)-h\frac{\partial}{\partial x}(u(x,y))-h\frac{\partial}{\partial y}(u(x,y))+\frac{h^2}{2}\frac{\partial^2}{\partial x^2}(u(x,y))\\
&+h^2\frac{\partial^2}{\partial x\partial y}(u(x,y))+\frac{h^2}{2}\frac{\partial^2}{\partial y^2}(u(x,y))-\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}(u(x,y))\\
&-\frac{h^3}{2!}\frac{\partial^3}{\partial x^2\partial y}(u(x,y))-\frac{h^3}{2!}\frac{\partial^3}{\partial x\partial y^2}(u(x,y))\\
&-\frac{h^3}{3!}\frac{\partial^3}{\partial y^3}(u(x,y))+\frac{h^4}{4!}\frac{\partial^4}{\partial x^4}(u(x,y))+\frac{h^4}{3!}\frac{\partial^4}{\partial x^3\partial y}(u(x,y))
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{h^4}{2!2!}\frac{\partial^4}{\partial x^2 \partial y^2}(u(x,y)) + \frac{h^4}{3!}\frac{\partial^4}{\partial x \partial y^3}(u(x,y)) \\
&+ \frac{h^4}{4!}\frac{\partial^4}{\partial y^4}(u(x,y))) \\
&= \frac{1}{4h^2}\left(\frac{2h^4}{3}\frac{\partial^4}{\partial x^3 \partial y}(u(x,y)) + 4h^2\frac{\partial^2}{\partial x \partial y}(u(x,y)) + \frac{2h^4}{3}\frac{\partial^4}{\partial x \partial y^3}(u(x,y))\right) \\
&= \frac{\partial^2}{\partial x \partial y}(u(x,y)) + \frac{h^2}{6}\left(\frac{\partial^4}{\partial x^3 \partial y}(u(x,y)) + \frac{\partial^4}{\partial x \partial y^3}(u(x,y))\right) \\
&= \frac{\partial^2}{\partial x \partial y}(u(x,y)) + O(h^2)
\end{aligned}
$$

2. Let s be a function defined by

$$
s(x) = \begin{cases} (x+1)^3 & -1 \le x \le 0 \\ (1-x)^3 & 0 \le x \le 1 \end{cases}
$$

(a) With Δ denoting the subdivision of $[-1,1]$ into $[-1,0]$ and $[0,1]$, to what class $S_m^k(\Delta)$ does the spline s belong to?

Since each piece of s is degree 3, the degree of s is $m = 3$. Let $s_1(x) = (x+1)^3$ and let $s_2(x) = (1-x)^3$. Then s is continuous because $s_1(0) = (0+1)^3 = 1 = (1-0)^3 = s_2(0)$. Also $s_1'(0) = 3(0+1)^2 = 3$ and $s_2'(0) = -3(1-0)^2 = -3$, therefore the first derivative of s is not continuous. So s belongs to smoothness class $k = 0$.

(b) Estimate the error of the composite trapezoidal rule applied to $\int_{-1}^{1} s(x)\,\mathrm{d}x$, when $[-1,1]$ is divided into n subintervals of equal length $h = 2/n$ and n is even.

(c) What is the error of the composite SimpsonâĂŹs rule applied to $\int_{-1}^{1} s(x)\,\mathrm{d}x$, with the same subdivision of $[-1,1]$ as in (b)?

Simpson's rule has a degree of exactness equal to 3. Simpson's rule is applied to every two intervals, since n is even Simpson's rule can be applied to s over the subdivision Δ.

Since n is even either $n = 4m$ or $n = 4m+2$ for some positive integer m. If $n = 4m$ for some positive integer m, that is n is a multiple of 4, then $\int_{-1}^{1} s(x)\,\mathrm{d}x$ can be approximated by applying Simpson's rule to $\int_{-1}^{0} s(x)\,\mathrm{d}x$ and $\int_{0}^{1} s(x)\,\mathrm{d}x$ separately and summing. This can be done because there $n/2 = 2m$ intervals on $[-1,0]$ and $[0,1]$. Each of these integrals can be evaluated exactly because Simpson's rule has degree of exactness equal to 3. Therefore the total error is 0.

If $n = 4m+2$ for some positive integer m, then $\int_{-1}^{1} s(x)\,\mathrm{d}x$ can be approximated by applying Simpson's rule to $\int_{-1}^{-h} s(x)\,\mathrm{d}x$, $\int_{-h}^{h} s(x)\,\mathrm{d}x$, and $\int_{h}^{1} s(x)\,\mathrm{d}x$ separately and summing. In this situation each interval $[-1,0]$ and $[0,1]$ has an odd number of subintervals, so Simpson's rule must be applied across the interval $[-h,h]$. Simpson's rule evaluates $\int_{-1}^{-h} s(x)\,\mathrm{d}x$ and $\int_{h}^{1} s(x)\,\mathrm{d}x$ exactly because $s(x)$ is a degree 3 polynomial on these intervals. Therefore the error from Simpson's rule comes when approximating the integral $\int_{-h}^{h} s(x)\,\mathrm{d}x$. The error can be found as follows.

$$\begin{aligned}
E &= \int_{-h}^{h} s(x)\,\mathrm{d}x - \frac{h}{3}(s(-h) + 4s(0) + s(h)) \\
&= \int_{-h}^{0} (x+1)^3\,\mathrm{d}x + \int_{0}^{h} (1-x)^3\,\mathrm{d}x - \frac{h}{3}\Big((1-h)^3 + 4 + (1-h)^3\Big) \\
&= \frac{1}{4}(x+1)^4\Big|_{x=-h}^{0} + -\frac{1}{4}(1-x)^4\Big|_{x=0}^{h} - \frac{h}{3}\Big(2(1-h)^3 + 4\Big) \\
&= \frac{1}{4}\Big(1 - (1-h)^4\Big) - \frac{1}{4}\Big((1-h)^4 - 1\Big) - \frac{h}{3}\Big(2(1-h)^3 + 4\Big) \\
&= \frac{1}{2} - \frac{1}{2}(1-h)^4 - \frac{4h}{3} - \frac{2h}{3}(1-h)^3 \\
&= (1-h)^3\left(-\frac{1}{2}(1-h) - \frac{2h}{3}\right) - \frac{4h}{3} + \frac{1}{2} \\
&= (1-h)^3\left(-\frac{h}{6} - \frac{1}{2}\right) - \frac{4h}{3} + \frac{1}{2}
\end{aligned}$$

This is also the total error.

(d) What is the error resulting from applying the 2-point Gauss-Legendre rule to $\int_{-1}^{0} s(x)\,\mathrm{d}x$ and $\int_{0}^{1} s(x)\,\mathrm{d}x$ separately and summing?
The 2-point Gauss-Legendre rule has degree of exactness equal to 3. So on each of these intervals the $s(x)$ is a degree 3 polynomial, therefore the error on each of these intervals will be zero. So the total error is zero.

3. (a) Determine by Hermite interplation the quadractic polynomial p interpolating f at $x = 0$ and $x = 1$ and f' at $x = 0$. Also express the errors in terms of an appropriate derivative.

x	f(x)		
0	f(0)		
0	f(0)	f'(0)	
1	f(1)	f(1) - f(0)	f(1) - f(0) - f'(0)

(b)

(c)

4.

5.

6.