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MATH 561 Numerical Analysis I
Final Assignment

1. Let $x_1, x_2, \ldots, x_n$, for $n > 1$, be machine numbers. Their product can be computed by the alogirithm

$$
\begin{aligned}
p_1 &= x_1 \\
p_k &= fl(x_k p_{k-1}), k = 2, 3, \ldots, n
\end{aligned}
$$

(a) Find an upper bound for the relative error in terms of the machine precision *eps* and n.
The relative error is given by

$$\frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n}$$

First consider p_k.

$$
\begin{aligned}
p_k &= fl(x_k p_{k-1}) \\
&= x_k p_{k-1}(1 + \epsilon_k)
\end{aligned}
$$

Where $|\epsilon_k| < eps$, for $k = 1, \cdots, n$

$$< x_k p_{k-1}(1 + eps)$$

Applying this recursively to p_n, we see that

$$
\begin{aligned}
p_n &< x_n p_{n-1}(1 + eps) \\
&< x_n x_{n-1} p_{n-2}(1 + eps)^2 \\
&< x_n x_{n-1} x_{n-2} p_{n-3}(1 + eps)^3 \\
&\vdots \\
&< x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1}
\end{aligned}
$$

Therefore the relative error can be bounded as follows

$$
\begin{aligned}
E &= \left| \frac{p_n - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\
&< \left| \frac{x_n x_{n-1} \cdots x_1 (1 + eps)^{n-1} - x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n} \right| \\
&= \left| \frac{x_1 x_2 \cdots x_n \left((1 + eps)^{n-1} - 1\right)}{x_1 x_2 \cdots x_n} \right| \\
&= (1 + eps)^{n-1} - 1
\end{aligned}
$$

Therefore the upper bound for the relative error is $E < (1 + eps)^{n-1} - 1$.

(b) For any integer r that satisfies $r \times eps < \frac{1}{10}$, show that

$$(1 + eps)^r - 1 < 1.06 \times r \times eps$$

Hence for n not too large, simplify the answer given in (a).
Using the Binomial Thereom, $(1 + eps)^r$ can be expanded.

$$(1 + eps)^r - 1 = \sum_{i=0}^{r} \left(\binom{r}{i} 1^{r-i} eps^i \right) - 1$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \left(\binom{r}{i} eps^i \right) \\
&= r \cdot eps + \binom{r}{2} eps^2 + \binom{r}{3} eps^3 + \cdots + eps^r \\
&= r \cdot eps + \frac{r(r-1)}{2} eps^2 + \frac{r(r-1)(r-2)}{3!} eps^3 + \cdots + eps^r \\
&= r \cdot eps \left(1 + \frac{r-1}{2} eps + \frac{(r-1)(r-2)}{3!} eps^2 + \cdots + \frac{(r-1)(r-2)\cdots(1)}{r!} eps^{r-1} \right)
\end{aligned}
$$

Since $r \times eps < \frac{1}{10}$, $(r-i)eps < \frac{1}{10}$ for any $0 < i < r$

$$
\begin{aligned}
&< r \cdot eps \left(1 + \frac{1}{2}\frac{1}{10} + \frac{1}{3!}\left(\frac{1}{10}\right)^2 + \cdots + \frac{1}{r!}\left(\frac{1}{10}\right)^{r-1} \right) \\
&= r \cdot eps \sum_{k=0}^{r-1} \left(\frac{1}{k!} \left(\frac{1}{10}\right)^{k-1} \right) \\
&= r \cdot eps \cdot 10 \sum_{k=1}^{r-1} \left(\frac{1}{k!} \left(\frac{1}{10}\right)^{k} \right)
\end{aligned}
$$

This expression is certainly less than extending the sum to infinity because all of the terms are postive. Also this sum is the Taylor series for $e^x - 1$.

$$
\begin{aligned}
&< r \cdot eps \cdot 10 \sum_{k=1}^{\infty} \left(\frac{1}{k!} \left(\frac{1}{10}\right)^{k} \right) \\
&= r \cdot eps \cdot 10 \left(e^{1/10} - 1 \right) \\
&\approx 1.05171 r \cdot eps \\
&< 1.06 r \cdot eps
\end{aligned}
$$

This result can now be used to simplify the result of part (a). Now if n is not too large, then $|E| < 1.06(n-1)eps$.

2.

3.

4. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Consider a function $f \in C^\infty[a, b]$.

   (a) Define what it means for a function S to be a linear spline that interpolates f at all the points x_i for $i = 0, 1, \ldots, n$. Give a formula for S in terms of the point values of f.
   In order to define the linear spline, I will first define a set of linear basis functions. Let B_i for $i = 1, 2, \ldots, n-1$ be defined on $[a, b]$ as follows.

$$
B_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}
$$

   Also let B_1 and B_n be defined as follows

$$
B_1(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & a = x_0 \le x \le x_1 \\ 0 & x > x_1 \end{cases}
$$

$$
B_n(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0} & x_{n-1} \le x \le x_n = b \\ 0 & x < x_{n-1} \end{cases}
$$

(b)

5.