Chapter 2
Approximation and Interpolation

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Approximation of Functions

Approximation

Given a function f and a class Φ of "approximating functions" ϕ and a norm $\|\cdot\|$. A function $\hat{\phi} \in \Phi$ is called the best approximation of f from the class Φ relative to the norm $\|\cdot\|$ if

$$\|f - \hat{\phi}\| \leqslant \|f - \phi\| \text{ for all } \phi \in \Phi.$$

For example:

- Least Squares Approximation: the least squares problem:

$$\min_{\phi \in \Phi_n} \|\phi - f\|_{2,d\lambda}.$$

- Polynomial Interpolatioin: given $\{x_i\}_{i=0}^n$ and $\{f_i = f(x_i)\}_{i=1}^n$ of function f, find a polynomial $p \in \mathbf{P}_n$ s.t.,

$$p(x_i) = f_i, \ i = 0, 1, \ldots, n.$$

Least Squares Problem; Normal Equations

The least squares problem:

$$\min_{\phi \in \Phi_n} \|\phi - f\|_{2,d\lambda}.$$

Normal Equations

- The normal equations:

$$\sum_{j=1}^{n} (\pi_i, \pi_j) c_j = (\pi_i, f), \; i = 1, 2, \ldots, n.$$

- In a compact form:

$$\mathbf{Ac} = \mathbf{b},$$

  with $\mathbf{A} = [a_{ij}] = [(\pi_i, \pi_j)], \; \mathbf{b} = [b_i] = [(\pi_i, f)], \; \mathbf{c} = [c_i]$.
- Unique sol. $\hat{\mathbf{c}} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow$ unique sol. $\hat{\phi}_n(t) = \sum_{j=1}^{n} \hat{c}_j \pi_j(t)$.

Least Squares Problem; Normal Equations $\mathbf{Ac} = \mathbf{b}$

- **A** may be ill-conditioned with non-orthogonal basis; nonpermanence of the coefficients $\hat{c}_j$.
- **A** becomes diagonal with orthogonal basis; permanence of the coefficients $\hat{c}_j$ with $\hat{c}_j = (\pi_j, f)/(\pi_j, \pi_j),\ j = 1, \ldots, n$. To alleviate cancellation error:

$$\hat{c}_j = \frac{1}{(\pi_j, \pi_j)}(f - \sum_{k=1}^{j-1} \pi_k, \pi_j),\ j = 1, \ldots, n.$$

- Gram-Schmidt orthogonalization to obtain an orthogonal basis $\{\pi_j(t)\}$ from a set of linearly independent functions$\{\hat{\pi}_j(t)\}$:

$$\pi_1 = \hat{\pi}_1$$

$$\pi_j = \hat{\pi}_j - \sum_{k=1}^{j-1} c_k \pi_k, \quad c_k = \frac{(\hat{\pi}_j, \pi_k)}{(\pi_k, \pi_k)}, \text{ for } j = 2, 3, \ldots.$$

Least Squares Problem; Least Squares Error

The least squares error:

$$\min_{\phi \in \Phi_n} \|f - \phi\|_{2,d\lambda} = \|f - \hat{\phi}_n\|_{2,d\lambda} = \{\|f\|^2 - \sum_{j=1}^{n} |\hat{c}_j|^2 \|\pi_j\|^2\}^{1/2}$$

Convergence: given a sequence of linear spaces:

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_n \subset \cdots$$

Clearly,

$$\|f - \hat{\phi}_1\|_2 \geqslant \|f - \hat{\phi}_2\|_2 \geqslant \cdots \geqslant \|f - \hat{\phi}_n\|_2 \geqslant \cdots$$

So

$$\lim_{n \to \infty} \|f - \hat{\phi}_n\|_2 \text{ exists!}$$

Question: is $\lim_{n\to\infty} \|f - \hat{\phi}_n\|_2 = 0$ or $\|f\|^2 = \sum_{j=1}^{\infty} |\hat{c}_j|^2 \|\pi_j\|^2$?

Least Squares Problem; Convergence

Definition (Convergence of Least Squares Approximation)

- If the limit is 0, we say the least squares approximation process converges as $n \to \infty$.
- Given any $\epsilon > 0$, there $\exists n_\epsilon, \phi^* \in \Phi_{n_\epsilon}$ s.t., $\|f - \phi^*\| \leqslant \epsilon$. ($\{\Phi_n\}$ is said to be complete w.r.t. $\|\cdot\|$.)

Theorem (Weierstrass's Approximation Theorem)

If f is a continuous real-valued function on $[a,\ b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial p on $[a,\ b]$ such that

$$|f(x) - p(x)| < \epsilon$$

for all x in $[a,\ b]$. In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

Polynomial Interpolation

Problem: given $\{x_i\}_{i=0}^n$ and $\{f_i = f(x_i)\}_{i=0}^n$ of function f, find a polynomial $p \in \mathbf{P}_n$ s.t.,

$$p(x_i) = f_i, \; i = 0, 1, \ldots, n.$$

Existence, Uniqueness

- Lagrange polynomials $p(x) = \sum_{i=0}^n f_i l_i(x)$, with Lagrange basis function (elementary Lagrange interpolation polynomial)

$$l_i(x) = \prod_{j=0; j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \; i = 0, 1, 2, \ldots, n.$$

- Existence: $p(x_i) = f_i, \; i = 0, 1, \ldots, n.$
- Uniqueness proved by Fundamental Theorem of Algebra.
- Denote p as $p_n(f; x_0, x_1, \ldots, x_n; x) = p_n(f; x) = \sum_{i=0}^n f(x_i) l_i(x)$.

Polynomial Interpolation; Interpolation Operator

Consider Lagrange interpolation as a linear operator

$$P_n : C[a,\ b] \mapsto \mathbf{P}_n, \quad P_n(\cdot) = p_n(f;\cdot)$$

- By uniqueness

$$P_n(f) = f,\ \forall f \in \mathbf{P}_n.$$

- Operator norm:

$$\|P_n\| = \max_{f \in C[a,\ b]} \frac{\|P_n(f)\|}{\|f\|}$$

e.g.,

$$\|P_n\|_\infty = \Lambda_n \equiv \|\lambda_n(x)\|_\infty \equiv \|\sum_{i=0}^{n} |l_i(x)|\|_\infty.$$

Polynomial Interpolation; Interpolation Error

Using best approximation of f on $[a,\ b]$ by polynomials of degree $\leqslant n$ defined as

$$\mathcal{E}_n(f) \equiv \min_{p \in \mathbf{P}_n} \|f - p\|_\infty = \|f - \hat{p}_n\|_\infty,$$

we see the interpolation error:

$$\|f - p_n(f;\cdot)\|_\infty = \|f - \hat{p}_n - p_n(f - \hat{p}_n;\cdot)\|_\infty \leqslant \|f - \hat{p}_n\|_\infty + \Lambda_n \|f - \hat{p}_n\|_\infty$$

So

$$\|f - p_n(f;\cdot)\|_\infty \leqslant (1 + \Lambda_n)\mathcal{E}(f)$$

We know

$$\lim_{n\to\infty} \mathcal{E}(f) = 0,$$

is

$$\lim_{n\to\infty} \|f - p_n(f;\cdot)\|_\infty = 0?$$

Polynomial Interpolation; Interpolation Error

Error of interpolation: $f(x) - p_n(f;x)$ for any $x \neq x_i$ in $[a,\ b]$.

Interpolation Error

$$f(x) - p_n(f;x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i),\ x \in [a,\ b],$$

for some $\xi(x) \in (a,\ b)$.

Proof

- Define $F(t) = f(t) - p_n(f;t) - \frac{f(x)-p_n(f;x)}{\prod_{i=0}^{n}(x-x_i)} \prod_{i=0}^{n}(t - x_i)$
- $F \in C^{n+1}[a,\ b]$ (assume $f \in C^{n+1}[a,\ b]$)
- $F(x_i) = 0,\ i = 0, 1, \ldots, n;\ F(x) = 0.$
- By Rolle's Theorem, $F^{(n+1)}(\xi(x)) = 0$ for some $\xi(x) \in (a,\ b) \Rightarrow$

$$0 = f^{(n+1)}(\xi(x)) - \frac{f(x) - p_n(f;x)}{\prod_{i=0}^{n}(x - x_i)}(n+1)!.$$

Lagrange Interpolation; Convergence

Definition (Convergence of Lagrange Interpolation)

For a triangular array of interpolation nodes on $[a,\ b]$:

$$\begin{array}{llll}
x_0^{(0)} & & & \\
x_0^{(1)} & x_1^{(1)} & & \\
x_0^{(2)} & x_1^{(2)} & x_2^{(2)} & \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots} \\
x_0^{(n)} & x_1^{(n)} & x_2^{(n)} & \cdots x_n^{(n)} \\
\multicolumn{4}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

define $p_n(x) = p_n(f; x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}; x),\ x \in [a,\ b]$. We say Lagrange interpolation based on the triangular array of nodes converges if $p_n(x) \to f(x)$ as $n \to \infty$, uniformly for $x \in [a,\ b]$.

Lagrange Interpolation; Convergence

By the interpolation error,

$$|f(x) - p_n(x)| \leqslant (b-a)^{n+1}\frac{M_{n+1}}{(n+1)!},\ x \in [a,\ b],$$

where $|f^{(k)}(x)| \leqslant M_k$ for $a \leqslant x \leqslant b,\ k = 0, 1, 2, \ldots$
then convergence is obtained if $\lim_{k\to\infty} \frac{(b-a)^k}{k!}M_k = 0$.

Theorem (Convergence)

Lagrange interpolation converges (uniformly on $[a,\ b]$) for an arbitrary triangular set of nodes in $[a,\ b]$ if f is analytic in the circular disk C_r centered at $(a+b)/2$ and having radius r s.t., $r > \frac{3}{2}(b-a)$.

Proof by Cauchy's Formula,

$$f^{(k)}(x) = \frac{k!}{2\pi i}\oint_{\partial C_r} \frac{f(z)}{(z-k)^{k+1}}dz,\ x \in [a,\ b].$$

Lagrange Interpolation; Convergence; A Sharp Bound

- Let $d\mu(t)$ be the "limit distribution" of nodes, i.e.,

$$\int_a^x d\mu(t) = \frac{\#\text{ of nodes in } [a,\ b]}{n+1}.$$

- A curve of constant logarithmic potential (constant γ):

$$\{z \in \mathbf{C} : u(z) = \gamma,\ u(x) = \int_a^b \ln \frac{1}{|z-t|} d\mu(t).$$

- Let

$$\Gamma = \sup_{\text{curves } u(z)=\gamma \text{ containing } [a,\ b]} \gamma.$$

Theorem (Convergence)

For the domain $C_\Gamma = \{z \in \mathbf{C} : u(z) \geqslant \Gamma\}$, if f is analytic in any domain C containing C_Γ in its interior, then $|f(z) - p_n(f;z)| \to 0$ as $n \to \infty$ uniformly for $z \in C_\Gamma$.

Chebyshev Polynomial

Choice of nodes influences the convergence. E.g., Chebyshev nodes. Assume considering interval $[-1,\ 1]$ (arbitrary $[a,\ b]$ by linear map).

Chebyshev Polynomial of First Kind

- Chebyshev polynomials of first kind

$$T_n(x) = \cos(n\cos^{-1}(x)).$$

- one can show

$$T_0(x) = 1,\ T_1(x) = x,$$

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),\ k = 1, 2, 3, \ldots$$

- Leading coefficient of T_n is 2^{n-1}; monic Chebyshev polynomial of degree n

$$\mathring{T}_n(x) = \frac{1}{2^{n-1}}T_n(x),\ n \geqslant 1; \mathring{T}_0 = T_0.$$

Chebyshev Nodes

From $\cos n\theta = T_n(\cos\theta)$,

Nodes

- Zeros of T_n:

$$x_k^{(n)} = \cos\theta_k^{(n)},\ \ \theta_k^{(n)} = \frac{2k-1}{2n}\pi,\ \ k = 1, 2, \ldots, n.$$

- Extrema of T_n:

$$y_k^{(n)} = \cos\eta_k^{(n)},\ \ \eta_k^{(n)} = k\frac{\pi}{n},\ \ k = 0, 1, 2, \ldots, n.$$

Thus, $\mathring{T}_n(x) = \prod_{k=1}^{n}(x - x_k^{(n)})$, and

$$T_n(x_k^{(n)}) = 0,\ \ \text{for } x_k^{(n)} = \cos\frac{2k-1}{2n}\pi,\ \ k = 1, 2, \ldots, n;$$
$$T_n(y_k^{(n)}) = (-1)^k,\ \ \text{for } y_k^{(n)} = \cos\frac{k}{n}\pi,\ \ k = 0, 1, 2, \ldots, n;$$

Chebyshev Polynomial

Theorem (Chebyshev)

For any arbitrary monic polynomial $\mathring{p}_n$ of degree n, there holds

$$\max_{-1\leqslant x\leqslant 1} |\mathring{p}_n(x)| \geqslant \max_{-1\leqslant x\leqslant 1} |\mathring{T}_n(x)| = \frac{1}{2^{n-1}},\ n \geqslant 1$$

Proof by contradiction. Assume that $\max_{-1\leqslant x\leqslant 1} |\mathring{p}_n(x)| < \frac{1}{2^{n-1}}$. Then the polynomial $d_n(x) = \mathring{T}_n(x) - \mathring{p}_n(x)$ has degree $\leqslant n-1$ and satisfies

$$(-1)^k d_n(y_k^{(n)}) > 0,\ \text{for } k = 0, 1, 2, \ldots, n.$$

Therefore $d_n(x) \equiv 0$. Contradiction.

From the theorem:

$$\min_{p_{n-1}\in \mathbf{P}_{n-1}} \max_{-1\leqslant x\leqslant 1} |x^n - p_{n-1}(x)| = \max_{-1\leqslant x\leqslant 1} |x^n - (x^n - \mathring{T}_n(x))|.$$

Convergence with Chebyshev Nodes

Convergence

- Recall interpolation error:

$$f(x) - p_n(f;x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n}(x - x_i), \ x \in [-1, \ 1].$$

- By Chebyshev's Theorem, $\| \prod_{i=0}^{n}(x - x_i)\|_\infty$ is minimized with x_i being the zeros of T_{n+1}, given as,
  $\hat{x}_i^{(n)} = \cos \frac{2i+1}{2n+2}\pi, \ i = 0, 1, 2, \ldots, n.$
  With these nodes interpolation error:
  $\|f(\cdot) - p_n(f;\cdot)\|_\infty \leqslant \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \frac{1}{2^n}.$
- On triangular array of Chebyshev nodes in $[-1, \ 1]$,

$$p_n(f; \hat{x}_0^{(n)}, \hat{x}_1^{(n)}, \ldots, \hat{x}_n^{(n)}; x) \to f(x) \text{ as } n \to \infty,$$

uniformly on $[-1, \ 1]$ provided that $f \in C^1[-1, \ 1]$.

Fourier Expansion in Chebyshev Polynomial

Orthogonality

$$\int_{-1}^{1} T_k(x)T_l(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, \text{ if } k \neq l \\ \pi, \text{ if } k = l = 0 \\ \frac{\pi}{2}, \text{ if } k = l > 0 \end{cases}$$

Fourier Expansion

$$f(x) = {\sum_{j=0}^{\infty}}' c_j T_j(x) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} c_j T_j(x),$$

where

$$c_j = \frac{2}{\pi}\int_{-1}^{1} f(x)T_j(x)\frac{dx}{\sqrt{1-x^2}}, \; j = 0, 1, 2, \ldots.$$

Truncated sum $\tau_n(x) = {\sum_{j=0}^{n}}' c_j T_j(x)$ approximate f with error

$$f(x) - \tau_n(x) = \sum_{j=n+1}^{\infty} c_j T_j(x) \approx c_{n+1}T_{n+1}(x).$$

Lagrange Interpolation; Barycentric Formula

For efficient computation, rewrite Lagrange polynomial in a different form. Denote

$$\lambda_0^{(0)} = 1,\ \lambda_i^{(n)} = \prod_{j=0, j\neq i}^{n} \frac{1}{x_i - x_j},\ i = 0, 1, \ldots, n;\ n = 1, 2, 3, \ldots.$$

Then elementary Lagrange interpolation polynomials

$$l_i(x) = \frac{\lambda_i^{(n)}}{x - x_i}\omega_n(x),\ i = 0, 1, \ldots, n;\quad \omega_n(x) = \prod_{j=0}^{n}(x - x_j).$$

By $\sum_{i=0}^{n} l_i(x) \equiv 1$,

$$p_n(f; x) = \sum_{i=0}^{n} f_i l_i(x) = \frac{\sum_{i=0}^{n} f_i l_i(x)}{\sum_{i=0}^{n} l_i(x)} = \frac{\sum_{i=0}^{n} f_i \frac{\lambda_i^{(n)}}{x - x_i}\omega_n(x)}{\sum_{i=0}^{n} \frac{\lambda_i^{(n)}}{x - x_i}\omega_n(x)}.$$

Lagrange Interpolation; Barycentric Formula

Barycentric formula for Lagrange interpolation polynomial:

$$p_n(f;x) = \frac{\sum_{i=0}^{n} f_i \frac{\lambda_i^{(n)}}{x-x_i}}{\sum_{i=0}^{n} \frac{\lambda_i^{(n)}}{x-x_i}}.$$

And

$$l_i(x) = \frac{\frac{\lambda_i^{(n)}}{x-x_i}}{\sum_{i=0}^{n} \frac{\lambda_i^{(n)}}{x-x_i}}, \; i = 0, 1, \ldots, n.$$

Efficient computation of $\lambda_n^{(n)}$:

$$\lambda_0^{(0)} = 1, \text{ for } k = 1, 2, \ldots, n \text{ do}$$

$$\lambda_i^{(k)} = \frac{\lambda_i^{(k-1)}}{x_i - x_k}, \; i = 0, 1, 2, \ldots, k-1,$$

$$\lambda_k^{(k)} = \frac{1}{\prod_{j=0}^{k-1}(x_k - x_j)}.$$

Lagrange Interpolation; Barycentric Formula

- Efficient and stable computation of $\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$: $\frac{1}{2}n(n+1)$ subtractions, $\frac{1}{2}(n-1)n$ multiplications, $\frac{1}{2}n(n+3)$ divisions.

- If to incorporate next point (x_{n+1}, f_{n+1}), generate $\lambda_0^{(n+1)}, \lambda_1^{(n+1)}, \ldots, \lambda_{n+1}^{(n+1)}$, then compute $p_{n+1}(f; x)$.

Lagrange Interpolation; Newton's Formula

Both function values and derivative values can be given. (Denote $p_n(x) = p_n(f; x_0, x_1, \ldots, x_n; x)$, $n = 0, 1, 2, \ldots$.)

Newton's Form

For some constants $a_0, a_1, a_2, \ldots$, one has

$$p_0(x) = a_0,$$
$$p_n(x) = p_{n-1}(x) + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \; n = 1, 2, 3, \ldots,$$

so the interpolation polynomial is

$$\begin{aligned} p_n(f; x) =& a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ & + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

What are the constants $a_0, a_1, \ldots, a_n$? Determined by $f(x_i) = p_n(x_i)$ in principle.

Newton's Formula; Divided Differences

n-th divided difference of f relative to nodes $x_0, x_1, \ldots, x_n$: denoted as

$$a_n = [x_0, x_1, \ldots, x_n]f, \; n = 0, 1, 2, \ldots,$$

computed as

$$a_0 = [x_0]f = f_0,$$

for $k = 1, 2, \ldots, n$

$$a_k = [x_0, x_1, \ldots, x_k]f = \frac{[x_1, x_2, \ldots, x_k]f - [x_0, x_1, \ldots, x_{k-1}]f}{x_k - x_0}$$

A table of divided differences ....
each entry is the difference of the entry immediately to the left and the one above it, divided by the difference of the x-value horizontally to the left and the one opposite the f-value found by going diagonally up.

Newton's Formula; Divided Differences

Table of divided difference:

x	f			
x_0	f_0			
x_1	f_1	$[x_0, x_1]f$		
x_2	f_2	$[x_1, x_2]f$	$[x_0, x_1, x_2]f$	
x_3	f_3	$[x_2, x_3]f$	$[x_1, x_2, x_3]f$	$[x_0, x_1, x_2, x_3]f$
.	..	...	...	...

Diagonal entries are the coefficients in Newton's formula. ($n(n+1)$ additions and $\frac{1}{2}n(n+1)$ divisions for computing first $n+1$ diagonal entries)

Adding one more term (one more node t):

$$p_{n+1}(f; x_0, x_1, \ldots, x_n, t; x) = p_n(f; x) + [x_0, x_1, \ldots, x_n, t]f \cdot \prod_{i=0}^{n}(x - x_i).$$

Newton's Formula; Interpolation Error

The interpolation error:

$$f(x) - p_n(f;x) = [x_0, x_1, \ldots, x_n, x]f \cdot \prod_{i=0}^{n} (x - x_i).$$

From previous result:

$$[x_0, x_1, \ldots, x_n, x]f = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}$$

Letting $x = x_{n+1}$, and replacing $n+1$ by n,

$$[x_0, x_1, \ldots, x_n]f = \frac{1}{n!} f^{(n)}(\xi).$$

Let $x_1, x_2, \ldots, x_n$ tend to x_0, we have

$$[x_0, x_0, \ldots, x_0]f = \frac{1}{n!} f^{(n)}(x_0),$$

i.e., the n-th divided difference at $n+1$ "confluent" points should be defined to be the n-th derivative at this point divided by $n!$.

Hermite Interpolation

Hermite Interpolation Problem

Given $K+1$ distinct points $x_0, x_1, \ldots, x_K \in [a,\ b]$, corresponding integers $m_0, m_1, \ldots, m_K \geqslant 1$, and given a function $f \in C^{M-1}[a,\ b]$ with $M = \max_k m_k$, find a polynomial p of lowest degree such that for $k = 0, 1, \ldots, K$,

$$p^{(\mu)}(x_k) = f_k^{(\mu)} \equiv f^{(\mu)}(x_k),\ \mu = 0, 1, \ldots, m_k - 1.$$

Solved as Lagrange interpolation, with Newton's form. E.g., if $m_k = 3$, then

x	f			
.	.	.	.	.
x_k	f_k			
x_k	f_k	${f'}_k$		
x_k	f_k	${f'}_k$	$\frac{1}{2}{f''}_k$	

Approximation and Interpolation by Spline Functions

Given a partition (subdivision) Δ of $[a,\ b]$,

$$\Delta : a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

with

$$|\Delta| = \max_{1 \leqslant i \leqslant n-1} \Delta x_i,\ \Delta x_i = x_{i+1} - x_i.$$

Recall the spline functions of degree m and smoothness class k relative to the subdivision Δ,

$$\mathbf{S}_m^k(\Delta) = \{s : s \in C^k[a,\ b], s|_{[x_i,\ x_{i+1}]} \in \mathbf{P}_m, i = 1, 2, \ldots, n-1\}.$$

I.e., any function in $\mathbf{S}_m^k$ is piecewise polynomial of degree $\leqslant m$, and upto kth derivative is continuous everywhere including points $x_1, \ldots, x_{n-1}$ of Δ.

Piecewise Linear Interpolation

Interpolation by Piecewise Linear Functions

Find $s \in \mathbf{S}_1^0(\Delta)$ such that for a given function f defined on $[a,\ b]$,

$$s(x_i) = f_i \text{ where } f_i = f(x_i),\ i = 1, 2, \ldots, n.$$

The solution is given by $s(\cdot) = s_1(f; \cdot)$,

$$s_1(f; x) = f_i + (x - x_i)[x_i, x_{i+1}]f \text{ for } x_i \leqslant x \leqslant x_{i+1},\ i = 1, 2, \ldots, n-1.$$

I.e., on each subinterval $[x_i, x_{i+1}]$, s is a linear function. The interpolation error is (from previous results with Newton's form)

$$f(x) - s_1(f; x) = (x - x_i)(x - x_{i+1})[x_i, x_{i+1}, x]f \text{ for } x \in [x_i, x_{i+1}]$$

If $f \in C^2[a,\ b]$,

$$|f(x) - s_1(f; x)| \leqslant \frac{(\Delta x_i)^2}{8} \max_{[x_i, x_{i+1}]} |f''|,\ x \in [x_i, x_{i+1}].$$

Piecewise Linear Functions; Interpolation Error

Interpolation error:

$$\|f(\cdot) - s_1(f;\cdot)\|_\infty \leqslant \frac{1}{8}|\Delta|^2 \|f''\|_\infty$$

Furthermore, with the best approximation of f from $\mathbf{S}_1^0$,

$$dist_\infty(f, \mathbf{S}_1^0) \leqslant \|f(\cdot) - s_1(f;\cdot)\|_\infty \leqslant 2 dist_\infty(f, \mathbf{S}_1^0)$$

where

$$dist_\infty(f, \mathbf{S}) \equiv \inf_{s \in \mathbf{S}} \|f(\cdot) - s\|_\infty$$

is the best approximation to f from $\mathbf{S}$.

Basis for $\mathbf{S}_1^0(\Delta)$

Dimension of $\mathbf{S}_1^0(\Delta)$: n.
A basis: for $i = 1, \ldots, n$, (denote $x_0 = x_1$ and $x_{n+1} = x_n$)

$$B_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leqslant x \leqslant x_i, \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \leqslant x \leqslant x_{i+1}, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly,

$$B_i(x_j) = \delta_{ij} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases}$$

And for any $s\mathbf{S}_1^0(\Delta)$,

$$s(x) = \sum_{i=1}^{n} s(x_i) B_i(x).$$

Least Squares Approximation over $\mathbf{S}_1^0(\Delta)$

Least Squares Approximation

Given $f \in C[a,\ b]$, find $\hat{s}_1(f;\cdot) \in \mathbf{S}_1^0(\Delta)$ such that

$$\|f - \hat{s}_1\|_2 = \min_{s \in \mathbf{S}_1^0(\Delta)} \|f - s\|_2.$$

The unique solution by Normal equations

$$\mathbf{Ac} = \mathbf{b}$$

with $\mathbf{A} = [a_{ij}] = [(B_i, B_j)]$, $\mathbf{b} = [b_i] = [(f, B_i)]$, denoted as $\hat{\mathbf{c}} = \mathbf{A}^{-1}\mathbf{b}$ or $\hat{s}_1(f;x) = \sum_{i=1}^n \hat{c}_i B_i(x)$. Clearly, $(B_i, B_j) = 0$ if $|i-j| > 1$, so $\mathbf{A}$ is tridiagonal, i.e.,

$$\frac{1}{6}\Delta x_{i-1}\hat{c}_{i-1} + \frac{1}{3}(\Delta x_{i-1} + \Delta x_i)\hat{c}_i + \frac{1}{6}\Delta x_i \hat{c}_{i+1} = b_i,\ \ i = 1, 2, \ldots, n.$$

The least squares error:

$$dist_\infty(f, \mathbf{S}_1^0) \leqslant \|f(\cdot) - \hat{s}_1(f;\cdot)\|_\infty \leqslant 4dist_\infty(f, \mathbf{S}_1^0).$$

Interpolation by Cubic Splines

Cubic Splines $\mathbf{S}_3^1(\Delta)$

Given nodes $x_1, \ldots, x_n$, and numbers $m_1, \ldots, m_n$, find $s_3(f; \cdot) \in \mathbf{S}_3^1(\Delta)$ with

$$s_3(f; \cdot)|_{[x_i, x_{i+1}]} \equiv p_i(x), \ i = 1, 2, \ldots, n-1,$$

such that $s_3'(f; x_i) = m_i, \ i = 1, \ldots, n.$
We selecting each piece p_i to be the solution of a Hermite interpolation problem: for $i = 1, 2, \ldots, n-1$,

$$p_i(x_i) = f_i, \ \ p_i(x_{i+1}) = f_{i+1},$$

$$p_i'(x_i) = m_i, \ \ p_i'(x_{i+1}) = m_{i+1}.$$

Cubic Splines $\mathbf{S}_3^1(\Delta)$

Cubic Splines $\mathbf{S}_3^1(\Delta)$

Each piece p_i is given by

- in Newton's form

$$\begin{aligned} p_i(x) =& f_i + (x - x_i)m_i + (x - x_i)^2 \frac{[x_i, x_{i+1}]f - m_i}{\Delta x_i} \\ & + (x - x_i)^2(x - x_{i+1}) \frac{m_{i+1} + m_i - 2[x_i, x_{i+1}]f}{(\Delta x_i)^2} \end{aligned}$$

- in Taylor's form

  $p_i(x) = c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3$, with

$$c_{i,0} = f_i; \quad c_{i,1} = m_i; \quad c_{i,2} = \frac{[x_i, x_{i+1}]f - m_i}{\Delta x_i} - c_{i,3}\Delta x_i;$$

$$c_{i,3} = \frac{m_{i+1} + m_i - 2[x_i, x_{i_1}]f}{(\Delta x_i)^2}$$

Possible Choices of $\{m_i\}$

- Piecewise cubic Hermite interpolation:

$$m_i = f'(x_i).$$

- Cubic spline interpolation: $s_3(f;\cdot) \in \mathbf{S}_3^2(\Delta)$, i.e.,

$$p''_{i-1}(x_i) = p''_i(x_i),\ \ i = 2, 3, \ldots, n-1.$$

Minimality Properties of Cubic Splines

Subdivision Δ:

$$\Delta : a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Subdivision Δ':

$$\Delta' : a = x_0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = x_{n+1} = b$$

This means that whenever we interpolate on Δ', we interpolate not only to function values at all interior points but also to the function as well as first derivative values at the endpoints.

Minimality Properties of Cubic Splines

Theorem (Complete Cubic Spline Interpolant)

For any function $g \in C^2[a,\ b]$ *that interpolates* f *on* Δ', *there holds*

$$\int_a^b [g''(x)]^2 dx \geqslant \int_a^b [s''_{compl}(f;x)]^2 dx$$

with equality iff $g(\cdot) = s_{compl}(f;\cdot)$.

Theorem (Natural Cubic Spline Interpolant)

For any function $g \in C^2[a,\ b]$ *that interpolates* f *on* Δ *(not* Δ'*), there holds*

$$\int_a^b [g''(x)]^2 dx \geqslant \int_a^b [s''_{nat}(f;x)]^2 dx$$

with equality iff $g(\cdot) = s_{nat}(f;\cdot)$.