# MATH 561 Fall 2015 – Homework Set # 4

Last Name: ____________________ First Name: ____________________

UID: ____________________ Siganuture: ____________________

1. (20 points) For the differential equation

$$\frac{dy}{dx} = y^{\lambda}, \ \lambda > 0,$$

(a) determine the principal error function of the general explicit two-stage Runge-Kutta method;
(b) compare the local accuracy of the modified Euler method with that of Heun's method;
(c) determine a $\lambda$-interval such that for each $\lambda$ in this interval, there is a two-stage explicit Runge-Kutta method of order $p = 3$ having parameters $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, and $0 < \mu < 1$.

2. (20 points) Let $\mathbf{f}(x, \mathbf{y})$ satisfy a Lipschitz condition in $\mathbf{y}$ on $[a, b] \times \mathbf{R}^d$, with Lipschitz constant $L$.
(a) Show that the increment function $\mathbf{\Phi}$ of the second-order Runge-Kutta method

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(x, \mathbf{y}), \\ \mathbf{k}_2 &= \mathbf{f}(x + h, \mathbf{y} + h\mathbf{k}_1), \\ \mathbf{\Phi}(x, \mathbf{y}; h) &= \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) \end{aligned}$$

also satisfies a Lipschitz condition whenever $x + h \in [a, b]$, and determine a respective Lipschitz constant $M$.
(b) What would the result be for the classical Runge-Kutta method?
(c) What would it be for the general implicit Runge-Kutta method?

3. (20 points) Consider $y' = \lambda y$ on $[0, \infty)$ for complex $\lambda$ with $Re\lambda < 0$. Let $\{u_n\}$ be the approximations to $\{y(x_n)\}$ obtained by the classical fourth-order Runge-Kutta method with the step $h$ held fixed. (That is, $x_n = nh, h > 0$, and $n = 0, 1, 2, \ldots$.)
(a) Show that $y(x) \to 0$ as $x \to \infty$, for any initial value $y_0$.
(b) Under what condition on $h$ can we assert that $u_n \to 0$ as $n \to \infty$? In particular, what is the condition if $\lambda$ is real (negative)?
(c) What is the analogous result for Euler's method?
(d) Generalize to system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{A}$ a constant matrix all of whose eigenvalues have negative real parts.

4. (20 points) Consider the linear homogeneous system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \ \mathbf{y} \in \mathbf{R}^d, \quad (*)$$

with constant coefficient matrix $\mathbf{A} \in \mathbf{R}^{d \times d}$.
(a) For Euler's method applied to $(*)$, determine $\phi(z)$ and the principal error function.
(b) Do the same for the classical fourth-order Runge-Kutta method.

5. (20 points) (MATLAB)
(a) Write Matlab routines implementing the basic step $(x, \mathbf{y}) \to (x + h, \mathbf{y}_{next})$ in the case of Euler's method and the classical fourth-order Runge-Kutta method, entering the function $\mathbf{f}$ of the differential equation $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ as an input function.
(b) Consider the initial value problem

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \ 0 \le x \le 1, \ \mathbf{y}(0) = \mathbf{1},$$

where

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} \lambda_2 + \lambda_3 & \lambda_3 - \lambda_1 & \lambda_2 - \lambda_1 \\ \lambda_3 - \lambda_2 & \lambda_1 + \lambda_3 & \lambda_1 - \lambda_2 \\ \lambda_2 - \lambda_3 & \lambda_1 - \lambda_3 & \lambda_1 + \lambda_2 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The exact solution is

$$\mathbf{y}(x) = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}, \quad \begin{aligned} y^1 &= -e^{\lambda_1 x} + e^{\lambda_2 x} + e^{\lambda_3 x}, \\ y^2 &= e^{\lambda_1 x} - e^{\lambda_2 x} + e^{\lambda_3 x}, \\ y^3 &= e^{\lambda_1 x} + e^{\lambda_2 x} - e^{\lambda_3 x}, \end{aligned}$$

Integrate the initial value problem with constant step length $h = 1/N$ by

1. Euler's method (order 1)
2. the classical Runge-Kutta method (order 4)

using the programs written in (a). Use $N = 5, 10, 20, 40, 80$. Suggested $\lambda$-values are

1. $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$;
2. $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -10$;

Summarize what you learn from these examples and from others that you may wish to run.