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MATH 561 Numerical Analysis I
Homework 4

#1 (a) Determine the principle error function of the general explicit two-stage Runge-Kutta method.

The general explicit two-stage Runge-Kutta method can be described as follows.

$$\begin{aligned} k_1 &= f(x, y) \\ k_2 &= f(x + \mu h, y + \mu h k_1) \\ \Phi(x, y; h) &= \alpha_1 k_1 + \alpha_2 k_2 \end{aligned}$$

To find the priniciple error function, first the local truncation error must be found. The local truncation error is defined as

$$T(x, y; h) = \Phi(x, y; h) - \frac{1}{h}(y(x + h) - y(x))$$

The principle error function is the functional coefficient of h^p in the local truncation error, when p is the order of the method. Two-stage Runge-Kutta methods have in general an order of $p = 2$, so the principle error function is the coefficient of h^2. In order to find this the Taylor expansion of $\Phi(x, y; h)$ and $\frac{1}{h}(y(x + h) - y(x)))$ must be found, at least to the h^2 term.

First I will find the Taylor expansion of $\Phi(x, y; h) = \alpha_1 k_1 + \alpha_2 k_2$. The Taylor expansion of $k_1 = f(x, y)$ is just $f(x, y)$. The Taylor expansion of k_2 can be found as follows.

$$\begin{aligned} k_2 &= f(x + \mu h, y + \mu h k_1) \\ &= f(x + \mu h, y + \mu h f(x, y)) \\ &= f(x, y) + f_x(x, y)(\mu h) + f_y(x, y)(\mu h f(x, y)) \\ &+ \frac{1}{2}\Big(f_{xx}(x, y)(\mu h)^2 + 2f_{xy}(x, y)(\mu^2 h^2 f(x, y)) + f_{yy}(x, y)(\mu^2 h^2 f(x, y)^2\Big) + O(h^3) \\ &= f(x, y) + \mu(f_x(x, y) + f(x, y) f_y(x, y))h \\ &+ \frac{1}{2}\mu^2\Big(f_{xx}(x, y) + 2f(x, y) f_{xy}(x, y) + f(x, y)^2 f_{yy}(x, y)\Big)h^2 + O(h^3) \end{aligned}$$

Now the Taylor expansion of $\Phi(x, y; h)$ can be expressed as follows. Note that moving forward all values or derivatives of f will be evaluated at (x, y). Thus $f = f(x, y)$, $f_x = f_x(x, y)$, $f_y = f_y(x, y)$, and so on.

$$\begin{aligned} \Phi(x, y; h) &= \alpha_1 k_1 + \alpha_2 k_2 \\ &= \alpha_1 f + \alpha_2\Big(f + \mu(f_x + f f_y)h + \frac{1}{2}\mu^2\Big(f_{xx} + f f_{xy} + f^2 f_{yy}\Big)h^2 + O(h^3)\Big) \\ &= (\alpha_1 + \alpha_2) f + \mu\alpha_2(f_x + f f_y)h + \frac{1}{2}\alpha_2\mu^2\Big(f_{xx} + 2f f_{xy} + f^2 f_{yy}\Big)h^2 + O(h^3) \end{aligned}$$

Now that the Taylor expansion of $\Phi(x, y; h)$ has been found the Taylor expansion of $\frac{1}{h}(y(x + h) - y(x)))$ must be found and put in terms of f.

$$\begin{aligned} y(x + h) &= y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + O(h^4) \\ \frac{1}{h}(y(x + h) - y(x))) &= y'(x) + \frac{h}{2}y''(x) + \frac{h^2}{6}y'''(x) + O(h^3) \end{aligned}$$

Now note that $y'(x) = f(x,y)$, and the other derivatives of y can be put in terms of f as well.

$$\begin{aligned} y''(x) &= f_x(x,y) f_y(x,y) y'(x) = f_x(x,y) + f_y(x,y) f(x,y) = f_x + f_y f \\ y'''(x) &= f_{xx} + f_{xy} f + f_y(f_x + f_y f) + f(f_{yx} + f_{yy} f) \\ &= f_{xx} + 2f f_{xy} + f_x f_y + f f_y^2 + f^2 f_{yy} \end{aligned}$$

Therefore

$$\frac{1}{h}(y(x+h) - y(x))) = f + \frac{h}{2}(f_x + f_y f) + \frac{h^2}{6}\Big(f_{xx} + 2f f_{xy} + f_x f_y + f f_y^2 + f^2 f_{yy}\Big) + O(h^3)$$

Finally the Taylor expansion of the local truncation error can be examined.

$$\begin{aligned} T(x,y;h) &= (\alpha_1 + \alpha_2) f + \mu\alpha_2(f_x + f f_y)h + \frac{1}{2}\alpha_2\mu^2\Big(f_{xx} + 2f f_{xy} + f^2 f_{yy}\Big)h^2 \\ &\quad - \left(f + \frac{h}{2}(f_x + f_y f) + \frac{h^2}{6}\Big(f_{xx} + 2f f_{xy} + f_x f_y + f f_y^2 + f^2 f_{yy}\Big)\right) + O(h^3) \\ &= (\alpha_1 + \alpha_2 - 1) f + \left(\mu\alpha_2 - \frac{1}{2}\right)(f_x + f f_y)h \\ &\quad + \left(\left(\frac{1}{2}\alpha_2\mu^2 - \frac{1}{6}\right)\Big(f_{xx} + 2f f_{xy} + f^2 f_{yy}\Big) - \frac{1}{6}\Big(f_x f_y + f f_y^2\Big)\right)h^2 + O(h^3) \end{aligned}$$

For any general two-stage Runge-Kutta Method, $(\alpha_1 + \alpha_2 - 1) = 0$ and $\Big(\mu\alpha_2 - \frac{1}{2}\Big) = 0$. This implies that $\mu = \frac{1}{2\alpha_2}$. Therefore the principle error function for any general two-stage Runge-Kutta method is

$$\tau(x,y) = \left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\Big(f_{xx} + 2f f_{xy} + f^2 f_{yy}\Big) - \frac{1}{6}\Big(f_x f_y + f f_y^2\Big)$$

(b) Compare the local accuracy of the modified Euler method with that of Heun's method.
For this specific ordinary differential equation, $f(x,y) = y^\lambda$. Thus

$$\begin{aligned} f_x &= 0 \\ f_{xx} &= 0 \\ f_{xy} &= 0 \\ f_y &= \lambda y^{\lambda-1} \\ f_{yy} &= \Big(\lambda^2 - \lambda\Big) y^{\lambda-2} \end{aligned}$$

Therefore the principle error function becomes

$$\begin{aligned} \tau(x,y) &= \left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\Big(y^{2\lambda}\Big(\lambda^2 - \lambda\Big)y^{\lambda-2}\Big) - \frac{1}{6}\Big(y^\lambda \lambda^2 y^{2\lambda-2}\Big) \\ &= \left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\Big(\Big(\lambda^2 - \lambda\Big)y^{3\lambda-2}\Big) - \frac{1}{6}\Big(\lambda^2 y^{3\lambda-2}\Big) \\ &= \left(\left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\Big(\lambda^2 - \lambda\Big) - \frac{1}{6}\lambda^2\right)y^{3\lambda-2} \end{aligned}$$

For the improved Euler method, $\alpha_2 = 1$. Therefore the principle error function for the Euler method, τ_E is

$$\tau_E(x,y) = \left(\left(\frac{1}{8} - \frac{1}{6}\right)\Big(\lambda^2 - \lambda\Big) - \frac{1}{6}\lambda^2\right)y^{3\lambda-2}$$

$$= -\frac{1}{24}\Big(5\lambda^2 - \lambda\Big)y^{3\lambda-2}$$

For Heun's method, $\alpha_2 = \frac{1}{2}$. Therefore the principle error function for Heun's method, τ_H is

$$\begin{aligned}
\tau_H(x,y) &= \left(\left(\frac{1}{4} - \frac{1}{6}\right)\Big(\lambda^2 - \lambda\Big) - \frac{1}{6}\lambda^2\right)y^{3\lambda-2} \\
&= -\frac{1}{12}\Big(\lambda^2 + \lambda\Big)y^{3\lambda-2}
\end{aligned}$$

For what values of λ is the magnitude of the principle error function less Euler's method than Heun's method. For what values of λ is $|\tau_E| < |\tau_H|$

$$\begin{aligned}
|\tau_E(x,y)| &< |\tau_H(x,y)| \\
\left|-\frac{1}{24}\Big(5\lambda^2 - \lambda\Big)y^{3\lambda-2}\right| &< \left|-\frac{1}{12}\Big(\lambda^2 + \lambda\Big)y^{3\lambda-2}\right| \\
\frac{1}{24}\Big|5\lambda^2 - \lambda\Big| &< \frac{1}{12}\Big|\lambda^2 + \lambda\Big| \\
\Big|5\lambda^2 - \lambda\Big| &< 2\Big|\lambda^2 + \lambda\Big| \\
|\lambda(5\lambda - 1)| &< |\lambda(2\lambda + 2)|
\end{aligned}$$

Clearly $|\lambda(5\lambda-1)| = |\lambda(2\lambda+2)|$, when $\lambda = 0$. It is also equal when $(5\lambda - 1) = (2\lambda + 2)$, which implies that $\lambda = 1$. These are the only two points of intersection. When $\lambda = 2$, $|\lambda(5\lambda-1)| > |\lambda(2\lambda+2)|$ and when $\lambda = \frac{1}{2}$, $|\lambda(5\lambda-1)| < |\lambda(2\lambda+2)|$. Therefore $|\tau_E(x,y)| < |\tau_H(x,y)|$ on $\lambda \in (0,1)$, and $|\tau_H(x,y)| < |\tau_E(x,y)|$ on $\lambda \in (1,\infty)$.

(c) Determine an interval of λ such that for each λ in this interval there exists a two-stage explicit Runge-Kutta method of order $p = 3$ having parameters $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ and $0 < \mu < 1$.

In order for a two stage explicit Runge-Kutta method to have order $p = 3$, the principle error function, $\tau(x,y)$, must be zero.

We have previously determined that $\alpha_1 = 1 - \alpha_2$ and $\mu = \frac{1}{2\alpha_2}$. Therefore for $0 < \alpha_1 < 1$, then $0 < \alpha_2 < 1$. Also for $0 < \mu < 1$, then $0 < \frac{1}{2\alpha_2} < 1$ which implies that $\frac{1}{2} < \alpha_2 < \infty$. Therefore if $\frac{1}{2} < \alpha_2 < 1$, all three conditions will be met.

In order for $\tau(x,y) = 0$,

$$\begin{aligned}
0 &= \left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\Big(\lambda^2 - \lambda\Big) - \frac{1}{6}\lambda^2 \\
0 &= \left(\frac{1}{8\alpha_2} - \frac{1}{3}\right)\lambda^2 - \left(\frac{1}{8\alpha_2} - \frac{1}{6}\right)\lambda \\
0 &= (3 - 8\alpha_2)\lambda - 3 + 4\alpha_2 \\
\frac{3 - 4\alpha_2}{3 - 8\alpha_2} &= \lambda
\end{aligned}$$

If $\frac{1}{2} < \alpha_2 < 1$, then $-1 < \lambda < \frac{1}{5}$. Since $\lambda > 0$, then for $0 < \lambda < \frac{1}{5}$ there exists an explicit two-stage Runge-Kutta method with order $p = 3$ and with parameters between 0 and 1.

#2 Let $\mathbf{f}(x, \mathbf{y})$ satisfy a Lipschitz condition in $\mathbf{y}$ on $[a,b] \times \mathbb{R}^d$, with Lipschitz constant L.

(a) Show that the increment function $\mathbf{\Phi}$ of the second order Runge-Kutta method

$$\begin{aligned}\mathbf{k}_1 &= \mathbf{f}(x, \mathbf{y})\\ \mathbf{k}_2 &= \mathbf{f}(x+h, \mathbf{y}+h\mathbf{k}_1)\\ \mathbf{\Phi}(x, \mathbf{y}; h) &= \frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2)\end{aligned}$$

also satisfies a Lipschitz condition whenever $x+h \in [a, b]$ and determine a respective Lipschitz constant M.

To show that $\mathbf{\Phi}(x, \mathbf{y}; h)$ satisfies a Lipschitz condition the value of $\|\mathbf{\Phi}(x, \mathbf{y}; h) - \mathbf{\Phi}(x, \mathbf{y}^*; h)\|$ must be shown to be bounded by a multiple of $\|y - y^*\|$. For notational simplicity, I will define the following values

$$\begin{aligned}\mathbf{k}_1^* &= \mathbf{f}(x, \mathbf{y}^*)\\ \mathbf{k}_2^* &= \mathbf{f}(x+h, \mathbf{y}^*+h\mathbf{k}_1^*)\\ \mathbf{\Phi} &= \mathbf{\Phi}(x, \mathbf{y}; h)\\ \mathbf{\Phi}^* &= \mathbf{\Phi}(x, \mathbf{y}^*; h)\end{aligned}$$

Then

$$\begin{aligned}\|\mathbf{\Phi} - \mathbf{\Phi}^*\| &= \frac{1}{2}\|\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_1^* - \mathbf{k}_2^*\|\\ &\leq \frac{1}{2}(\|\mathbf{k}_1 - \mathbf{k}_1^*\| + \|\mathbf{k}_2 - \mathbf{k}_2^*\|)\end{aligned}$$

Now consider $\|\mathbf{k}_1 - \mathbf{k}_1^*\|$

$$\|\mathbf{k}_1 - \mathbf{k}_1^*\| = \|\mathbf{f}(x, \mathbf{y}) - \mathbf{f}(x, \mathbf{y}^*)\|$$

Since f satisfies the Lipschitz condition

$$\|\mathbf{k}_1 - \mathbf{k}_1^*\| \leq L\|\mathbf{y} - \mathbf{y}^*\|$$

Next consider $\|\mathbf{k}_2 - \mathbf{k}_2^*\|$

$$\|\mathbf{k}_2 - \mathbf{k}_2^*\| = \|\mathbf{f}(x+h, \mathbf{y}+h\mathbf{k}_1) - \mathbf{f}(x+h, \mathbf{y}^*+h\mathbf{k}_1^*)\|$$

Since f satisfies the Lipschitz condition

$$\begin{aligned}\|\mathbf{k}_2 - \mathbf{k}_2^*\| &\leq L\|\mathbf{y} + h\mathbf{k}_1 - \mathbf{y}^* - h\mathbf{k}_1^*\|\\ \|\mathbf{k}_2 - \mathbf{k}_2^*\| &\leq L(\|\mathbf{y} - \mathbf{y}^*\| + h\|\mathbf{k}_1 - \mathbf{k}_1^*\|)\end{aligned}$$

We have already shown that $\|\mathbf{k}_1 - \mathbf{k}_1^*\| \leq L\|\mathbf{y} - \mathbf{y}^*\|$

$$\|\mathbf{k}_2 - \mathbf{k}_2^*\| \leq \left(L + hL^2\right)\|\mathbf{y} - \mathbf{y}^*\|$$

Therefore

$$\|\mathbf{\Phi} - \mathbf{\Phi}^*\| \leq \left(L + \frac{h}{2}L^2\right)\|\mathbf{y} - \mathbf{y}^*\|$$

Therefore $\mathbf{\Phi}$ satisfies a Lipschitz condition and has Lipschitz constant, $M = L + \frac{h}{2}L^2$.

(b) Show that the classical fourth order Runge-Kutta method satisifies a Lipschitz condition.

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(x, \mathbf{y}) \\
\mathbf{k}_2 &= \mathbf{f}(x + \frac{1}{2}h, \mathbf{y} + \frac{1}{2}h\mathbf{k}_1) \\
\mathbf{k}_3 &= \mathbf{f}(x + \frac{1}{2}h, \mathbf{y} + \frac{1}{2}h\mathbf{k}_2) \\
\mathbf{k}_4 &= \mathbf{f}(x + h, \mathbf{y} + h\mathbf{k}_3) \\
\mathbf{\Phi}(x, \mathbf{y}; h) &= \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4
\end{aligned}$$

$$\|\mathbf{\Phi} - \mathbf{\Phi}^*\| \leq \frac{1}{6}\|\mathbf{k}_1 - \mathbf{k}_1^*\| + \frac{1}{3}\|\mathbf{k}_2 - \mathbf{k}_2^*\| + \frac{1}{3}\|\mathbf{k}_3 - \mathbf{k}_3^*\| + \frac{1}{6}\|\mathbf{k}_4 - \mathbf{k}_4^*\|$$

Now consider each of these norms individually

$$\begin{aligned}
\|\mathbf{k}_1 - \mathbf{k}_1^*\| &= \|\mathbf{f}(x, \mathbf{y}) - \mathbf{f}(x, \mathbf{y}^*)\| \\
&\leq L\|\mathbf{y} - \mathbf{y}^*\| \\
\|\mathbf{k}_2 - \mathbf{k}_2^*\| &= \left\|\mathbf{f}(x + \frac{1}{2}h, \mathbf{y} + \frac{1}{2}h\mathbf{k}_1) - \mathbf{f}(x + \frac{1}{2}h, \mathbf{y}^* + \frac{1}{2}h\mathbf{k}_1^*)\right\| \\
&\leq L\left\|\mathbf{y} + \frac{1}{2}h\mathbf{k}_1 - \mathbf{y}^* - \frac{1}{2}h\mathbf{k}_1^*\right\| \\
&\leq L\|\mathbf{y} - \mathbf{y}^*\| + \frac{1}{2}hL\|\mathbf{k}_1 - \mathbf{k}_1^*\| \\
&\leq (L + \frac{1}{2}hL^2)\|\mathbf{y} - \mathbf{y}^*\| \\
\|\mathbf{k}_3 - \mathbf{k}_3^*\| &= \left\|\mathbf{f}(x + \frac{1}{2}h, \mathbf{y} + \frac{1}{2}h\mathbf{k}_2) - \mathbf{f}(x + \frac{1}{2}h, \mathbf{y}^* + \frac{1}{2}h\mathbf{k}_2^*)\right\| \\
&\leq L\left\|\mathbf{y} + \frac{1}{2}h\mathbf{k}_2 - \mathbf{y}^* - \frac{1}{2}h\mathbf{k}_2^*\right\| \\
&\leq L\|\mathbf{y} - \mathbf{y}^*\| + \frac{1}{2}hL\|\mathbf{k}_2 - \mathbf{k}_2^*\| \\
&\leq \left(L + \frac{1}{2}hL^2 + \frac{1}{4}h^2L^3\right)\|\mathbf{y} - \mathbf{y}^*\|
\end{aligned}$$

(c) Show that $\mathbf{\Phi}$ for a general implicit Runge-Kutta method satisfies a Lipschitz condition.
For a general implicit Runge-Kutta method, $\mathbf{\Phi}(x, \mathbf{y}; h) = \sum_{s=1}^{r}(\alpha_s\mathbf{k}_s)$, where $\mathbf{k}_s = f(x + \mu_s h, \mathbf{y} + h\sum_{j=1}^{r}(\lambda_{sj}\mathbf{k}_j))$. I will continue to use the previously established notation.

$$\begin{aligned}
\|\mathbf{\Phi} - \mathbf{\Phi}^*\| &= \left\|\sum_{s=1}^{r}(\alpha_s\mathbf{k}_s) - \sum_{s=1}^{r}(\alpha_s\mathbf{k}_s^*)\right\| \\
&\leq \sum_{s=1}^{r}(\alpha_s\|\mathbf{k}_s - \mathbf{k}_s^*\|)
\end{aligned}$$

Now consider a single value of $\|\mathbf{k}_s - \mathbf{k}_s^*\|$

$$\|\mathbf{k}_s - \mathbf{k}_s^*\| = \left\|f(x + \mu_s h, \mathbf{y} + h\sum_{j=1}^{r}(\lambda_{sj}\mathbf{k}_j)) - f(x + \mu_s h, \mathbf{y}^* + h\sum_{j=1}^{r}\left(\lambda_{sj}\mathbf{k}_j^*\right))\right\|$$

Since f satisfies a Lipschitz condition

$$\begin{aligned}\|\mathbf{k}_s - \mathbf{k}_s^*\| &\leq L\left\|\mathbf{y} + h\sum_{j=1}^{r}(\lambda_{sj}\mathbf{k}_j) - \mathbf{y}^* - h\sum_{j=1}^{r}\left(\lambda_{sj}\mathbf{k}_j^*\right)\right\| \\ &\leq L\|\mathbf{y} - \mathbf{y}^*\| + hL\left\|\sum_{j=1}^{r}(\lambda_{sj}\mathbf{k}_j) - \sum_{j=1}^{r}\left(\lambda_{sj}\mathbf{k}_j^*\right)\right\|\end{aligned}$$

Let Γ be the max of λ_{sj} for $s, j = 0, \ldots, r$

$$\|\mathbf{k}_s - \mathbf{k}_s^*\| \leq L\|\mathbf{y} - \mathbf{y}^*\| + hL\Gamma\sum_{j=1}^{r}\left(\left\|\mathbf{k}_j\mathbf{k}_j^*\right\|\right)$$

Summing both side from $s = 1$ to r results in

$$\begin{aligned}\sum_{s=1}^{r}(\|\mathbf{k}_s - \mathbf{k}_s^*\|) &\leq sL\|\mathbf{y} - \mathbf{y}^*\| + shL\Gamma\sum_{j=1}^{r}\left(\left\|\mathbf{k}_j\mathbf{k}_j^*\right\|\right) \\ \sum_{s=1}^{r}(\|\mathbf{k}_s - \mathbf{k}_s^*\|) &\leq \frac{sL}{1 - shL\Gamma}\|\mathbf{y} - \mathbf{y}^*\|\end{aligned}$$

Now consider $\|\mathbf{\Phi} - \mathbf{\Phi}^*\|$, and let A be the max of α_s for $s = 1, \ldots, n$

$$\begin{aligned}\|\mathbf{\Phi} - \mathbf{\Phi}^*\| &\leq A\sum_{s=1}^{r}(\|\mathbf{k}_s - \mathbf{k}_s^*\|) \\ &\leq \frac{AsL}{1 - shL\Gamma}\|\mathbf{y} - \mathbf{y}^*\|\end{aligned}$$

Therefore $\mathbf{\Phi}$ does satisfy a Lipschitz condition and has a Lipschitz constant of $\frac{AsL}{1-shL\Gamma}$.

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#4

#5