MATH 562 Spring 2016 – Homework 3

1. (10 points) Determine the relative condition number for the following problem. Are there values of x for which the problem is ill-conditioned? Justify your answer.
$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

2. (10 points) Determine whether the calculation of $f(x, y) = (1+x)y^2$ is backward stable by the algorithm
$$\tilde{f}(x, y) = [1 \oplus fl(x)] \otimes [fl(y) \otimes fl(y)]$$

3. (10 points) (a) Compute the LU factorization $\mathbf{A} = \mathbf{LU}$,
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$
Use the factorization to solve the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = [-1\ 1\ 1]^T$.

   (b) Solve the system $\mathbf{Ax} = \mathbf{b}$ by LU factorization with partial pivoting.

4. (10 points) (Problem 20.1, page 154) Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ be nonsingular. Show that $\mathbf{A}$ has an LU factorization if and only if for each k, $1 \leqslant k \leqslant m$, the upper-left $(k \times k)$ block $\mathbf{A}(1 : k, 1 : k)$ of $\mathbf{A}$ is non-singular. (Hint: The row operations of Gaussian elimination of leave determinants unchanged.) Show that this LU factorization is unique.

5. (10 points) Rank Deficient Least Square Problems: Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \geqslant n$, and let $r = rank(\mathbf{A}) < n$, and write SVD of $\mathbf{A}$ as
$$\mathbf{A} = [\mathbf{U}_1, \mathbf{U}_2] \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} [\mathbf{V}_1, \mathbf{V}_2]^T = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^T,$$
where $\mathbf{\Sigma}_1$ is $r \times r$ nonsingular and $\mathbf{U}_1$ and $\mathbf{V}_1$ have r columns. Let $\sigma = \sigma_{\min}(\mathbf{\Sigma}_1)$, the smallest nonzero singular value of $\mathbf{A}$. Consider the following rank deficient least square problem, for some $\mathbf{b} \in \mathbb{R}^m$,
$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_2.$$

Show that:

1. all solutions $\mathbf{x}$ can be written as $\mathbf{x} = \mathbf{V}_1\mathbf{\Sigma}_1^{-1}\mathbf{U}_1^T\mathbf{b} + \mathbf{V}_2\mathbf{z}$, with $\mathbf{z}$ and arbitrary vector;

2. the solution $\mathbf{x}$ has minimal norm $\|\mathbf{x}\|_2$ precisely when $\mathbf{z} = \mathbf{0}$, and in which case, $\|\mathbf{x}\|_2 \leqslant \|\mathbf{b}\|_2/\sigma$.

6. (10 points) Consider matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Using any method you like, determine reduced and full QR factorizations $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ and $\mathbf{A} = \mathbf{QR}$

(b) Using the QR factorization to solve the linear least square problem

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$$

with $\mathbf{b} = [1\ 1\ 0]^T$.

(c) Using the QR factorization to solve a linear least squares problem

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$$

with matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ of rank n and $\mathbf{b} \in \mathbb{R}^m$.

7. (10 points) (Matlab) Consider the least-squares problem $\min_{\mathbf{x}} ||\mathbf{Ax} - \mathbf{b}\|_2$, where $\mathbf{A}$ is the first 5 columns of the 6×6 inverse Hilbert matrix (use Matlab function *invhilb* to generate the matrix), and

$$\mathbf{b} = \begin{bmatrix} 463 \\ -13860 \\ 97020 \\ -258720 \\ 291060 \\ -116424 \end{bmatrix}.$$

The exact solution to this problem is $\mathbf{x} = [1\ 1/2\ 1/3\ 1/4\ 1/5]^T$.

(a) What are the four conditioning numbers (Theorem 18.1) of the problem?

(b) Use all the algorithms in Page 138 – Page 142 to solve the problem.

1. Householder QR;
2. Householder QR of augmented matrix;
3. Modified Gram-Schmidt QR;
4. Modified Gram-Schmite QR of augmented matrix;
5. Normal equation;
6. SVD.

Check the accuray of computed solutions compared to the exact solution, and comment on the computed solutions and the algorithm used. (you must show Matlab codes with solutions and comprarisons)