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MATH 562 Numerical Analysis II
Homework 3

1. Determine the relative condition number for the following problem. Are there values of x for which the problem is ill-conditioned? Justify your answer.

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

Since f is differentiable the relative condition number of f is given by $\kappa = \frac{|f'(x)|}{|f(x)|/|x|}$. For this problem

$$\begin{aligned} f'(x) &= \frac{(1 + e^{-x})e^{-x} - (1 - e^{-x})(-e^{-x})}{(1 + e^{-x})^2} \\ &= \frac{e^{-x} + e^{-2x} + e^{-x} - e^{-2x}}{(1 + e^{-x})^2} \\ &= \frac{2e^{-x}}{(1 + e^{-x})^2} \end{aligned}$$

Thus the relative condition number for this problem is

$$\begin{aligned} \kappa &= \frac{|f'(x)|}{|f(x)|/|x|} \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})^2} / \frac{1 - e^{-x}}{1 + e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})^2} \times \frac{1 + e^{-x}}{1 - e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})} \times \frac{1}{1 - e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 - e^{-2x})} \right| \end{aligned}$$

This problem is not ill-conditioned because for any x this relitive condition number is small. At $x = 0$, this condition number is undefined, but L'Hopital's rule shows that the limit is equal to 1.

$$\lim_{x\to 0}(\kappa) = \lim_{x\to 0}\left(\frac{2e^{-x} - 2xe^{-x}}{2e^{-2x}}\right)$$
$$= \frac{2e^0}{2e^0}$$
$$= 1$$

As $x \to \infty$, $2xe^{-x} \to 0$ and $1 - e^{-2x} \to 1$, therefore $\kappa \to 0$. As $x \to -\infty$, $1 - e^{-2x} > 2xe^{-x}$, so $\kappa \to 0$. In fact $\kappa \leq 1$ for all x, therefore this problem is not ill-conditioned.

2. Determine whether the calculation $f(x,y) = (1+x)y^2$ is backward stable by the alogirithm

$$\tilde{f}(x,y) = [1 \oplus fl(x)] \otimes [fl(y) \otimes fl(y)]$$

The algorithm $\tilde{f}$ is backward stable if there exists $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$ such that $\tilde{f}(x,y) = f(\tilde{x}, \tilde{y})$ and $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} = O(\epsilon_{machine})$ for all $\mathbf{x}$.

$$\begin{aligned}
\tilde{f}(x,y) &= [1 \oplus fl(x)] \otimes [fl(y) \otimes fl(y)] \\
&= [1 \oplus x(1+\epsilon_1)] \otimes [y(1+\epsilon_2) \otimes y(1+\epsilon_3)] \\
&= [1 + x(1+\epsilon_1)](1+\epsilon_4) \otimes [y(1+\epsilon_2) \times y(1+\epsilon_3)](1+\epsilon_5) \\
&= [1 + x(1+\epsilon_1)](1+\epsilon_4) \times [y(1+\epsilon_2) \times y(1+\epsilon_3)](1+\epsilon_5)(1+\epsilon_6) \\
&= [1 + x(1+\epsilon_1)]y^2(1+\epsilon_7)
\end{aligned}$$

where $\epsilon_7 = O(\epsilon_{machine})$

$$\begin{aligned}
\tilde{f}(x,y) &= [1 + x(1+\epsilon_1)]y^2(1+\epsilon_7) \\
&= [1 + x(1+\epsilon_1)]\left(y\sqrt{1+\epsilon_7}\right)^2 \\
&= [1 + x(1+\epsilon_1)](y(1+\epsilon_8))^2 \\
&= f(x(1+\epsilon_1), y(1+\epsilon_8))
\end{aligned}$$

Therefore $\tilde{x} = x(1+\epsilon_1)$ and $\tilde{y} = y(1+\epsilon_8)$. This does satisfy $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} = O(\epsilon_{machine})$, because $\|\mathbf{x} - \tilde{\mathbf{x}}\| = \sqrt{\epsilon_1^2 + \epsilon_8^2} = O(\epsilon_{machine})$. So this algorithm is backward stable.

3. (a) Compute the LU factorization $A = LU$, of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

Use the factorization to solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-111]^T$.

(b) Solve the system $A\mathbf{x} = \mathbf{b}$ by LU factorization with partial pivoting

4. Let $A \in \mathbb{C}^{m\times m}$ be nonsingular. Show that A has an LU factorization if and only if for each k, such that $1 \leq k \leq m$, the upper left $(k \times k)$ block $A(1:k, 1:k)$ of A is nonsingular. Show that this LU factorization is unique.

5. Rank Deficient Least Squares Problem:

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Using any method you like, determine reduced and full QR factorizations.

(b) Use the QR factorization to solve the linear least square problem

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2$$

with $\mathbf{b} = [110]^T$.

(c) Use the QR factorization to solve the linear least squares problem

$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2$$

with matrix $A \in \mathbb{R}^{m\times n}$ with rank n and $\mathbf{b} \in \mathbb{R}^m$.

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