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MATH 562 Numerical Analysis II
Final Exam

1. Let $A \in \mathbb{R}^{m \times m}$ be written in the form $A = L + D + U$, where L is strictly lower triangular, D is the diagonal of A, and U is the strictly upper triangular part of A. Assuming D is invertible, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$. The Jacobi iteration method for solving $A\mathbf{x} = \mathbf{b}$ is defined by

$$\mathbf{x}^{n+1} = -D^{-1}(L+U)\mathbf{x}^n + D^{-1}\mathbf{b}$$

Show that if A is nonsingular and strictly row diagonally dominant:

$$0 < \sum_{j \neq i} (|a_{ij}|) < |a_{ii}|$$

then the Jacobi iteration converges to $\mathbf{x}_* = A^{-1}\mathbf{b}$ for each fixed $\mathbf{b} \in \mathbb{R}^m$.

Proof. □

2. Let $A \in \mathbb{R}^{m \times m}$ be symmetric positive definite (SPD), $\mathbf{b} \in \mathbb{R}^m$ and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

Suppose K is a subspace of $\mathbb{R}^m$. Show that $\hat{\mathbf{x}} \in K$ minimizes $\phi(\mathbf{x})$ over K if and only if $\nabla\phi(\hat{\mathbf{x}}) \perp K$.

Proof. □

3.

4. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Let Gaussian elimination be carried out on A without pivoting. After k steps, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix}$$

where $A_{22}^{(k)}$ is an $(n-k) \times (n-k)$ matrix. Show by induction

(a) $A_{22}^{(k)}$ is symmetric positive definite.

Proof. □

(b) $a_{ii}^{(k)} \leq a_{ii}^{(k-1)}$ for all $k \leq i \leq n$, $k = 1, \cdots, n-1$.

Proof. □

5. Let $A \in \mathbb{R}^{m \times n}$ with $m > n$ and

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 is a nonsingular $n \times n$ matrix, and A_2 is an $(m-n) \times n$ arbitrary matrix.

6. Let $A \in \mathbb{C}^{m \times m}$ with $\mathrm{rank}(A) = r$. Suppose an SVD of A is given by $A = U\Sigma V^*$, where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ denote the columns of U and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ denote the columns of V. Prove that $\langle \mathbf{v}_{r+1}, \ldots, \mathbf{v}_m \rangle = (A)$.

Proof. □

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