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MATH 562 Numerical Analysis II
Final Exam

1. Let $A \in \mathbb{R}^{m \times m}$ be written in the form $A = L + D + U$, where L is strictly lower triangular, D is the diagonal of A, and U is the strictly upper triangular part of A. Assuming D is invertible, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$. The Jacobi iteration method for solving $A\mathbf{x} = \mathbf{b}$ is defined by

$$\mathbf{x}^{n+1} = -D^{-1}(L+U)\mathbf{x}^n + D^{-1}\mathbf{b}$$

Show that if A is nonsingular and strictly row diagonally dominant:

$$0 < \sum_{j \neq i} (|a_{ij}|) < |a_{ii}|$$

then the Jacobi iteration converges to $\mathbf{x}_* = A^{-1}\mathbf{b}$ for each fixed $\mathbf{b} \in \mathbb{R}^m$.

Proof. □

2. Let $A \in \mathbb{R}^{m \times m}$ be symmetric positive definite (SPD), $\mathbf{b} \in \mathbb{R}^m$ and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

Suppose K is a subspace of $\mathbb{R}^m$. Show that $\hat{\mathbf{x}} \in K$ minimizes $\phi(\mathbf{x})$ over K if and only if $\nabla\phi(\hat{\mathbf{x}}) \perp K$.

Proof. □

3.

4. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Let Gaussian elimination be carried out on A without pivoting. After k steps, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix}$$

where $A_{22}^{(k)}$ is an $(n-k) \times (n-k)$ matrix. Show by induction

(a) $A_{22}^{(k)}$ is symmetric positive definite.

Proof. □

(b) $a_{ii}^{(k)} \leq a_{ii}^{(k-1)}$ for all $k \leq i \leq n$, $k = 1, \cdots, n-1$.

Proof. $\square$

5. Let $A \in \mathbb{R}^{m\times n}$ with $m > n$ and

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 is a nonsingular $n \times n$ matrix, and A_2 is an $(m-n) \times n$ arbitrary matrix.

6. Let $A \in \mathbb{C}^{m\times m}$ with $\text{rank}(A) = r$. Suppose an SVD of A is given by $A = U\Sigma V^*$, where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ denote the columns of U and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ denote the columns of V. Prove that $\langle \mathbf{v}_{r+1}, \ldots, \mathbf{v}_m \rangle = (A)$.

Proof. $\square$

7. Problem 33.2 (Page 255)

8. Problem 36.1 (Page 283)

9.

10. MATLAB project

Below are the function for performing the Jacobi method, the Gauss-Seidel Method, and the Conjugate Gradient method.

```
function [x0, k, r] = Jacobi(A, b, tol, maxIter)
    M = diag(diag(A));
    N = M - A;

    x0 = zeros(size(b));
    k = 0;
    r = norm(b - A*x0, inf);
    while(r(end) > tol && k < maxIter)
        k = k + 1;
        x = M\(N*x0) + M\b;
        x0 = x;
        r = [r, norm(b - A*x0, inf)];
    end
end
```

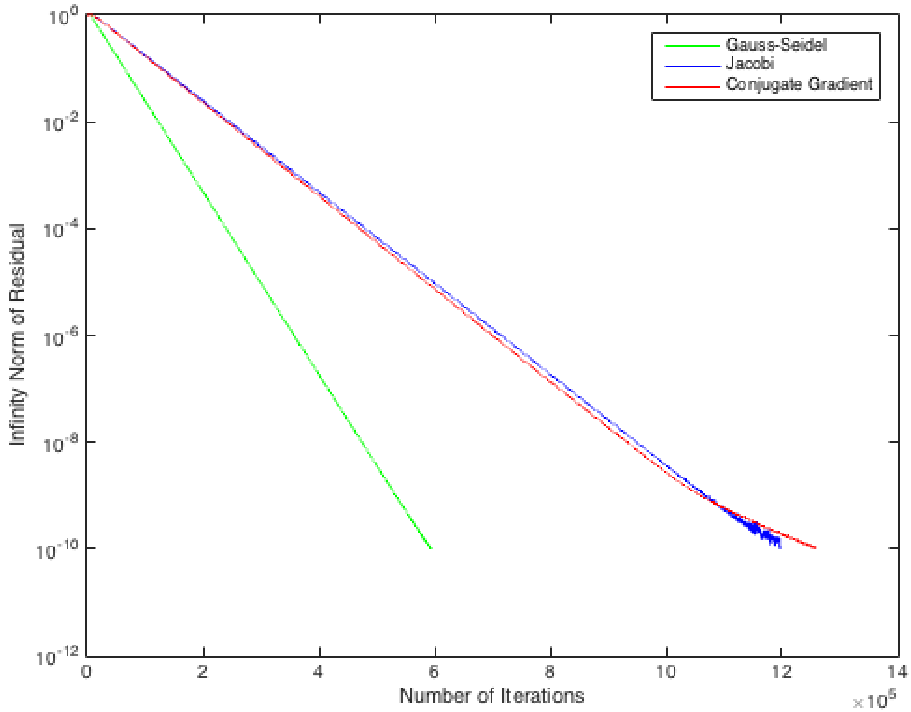
```
function [x0, k, r] = GaussSeidel(A, b, tol, maxIter)
    M = tril(A);
    N = M - A;
```

```
    x0 = zeros(size(b));
    k = 0;
    r = norm(b - A*x0, inf);
    while(r(end) > tol && k < maxIter)
        k = k + 1;
        x = M\(N*x0) + M\b;
        x0 = x;
        r = [r, norm(b - A*x0, inf)];
    end
end
```

```
function [x0, k, r] = ConjugateGradient(A, b, tol, maxIter)
    x0 = zeros(size(b));
    k = 0;
    r0 = b - A*x0;
    r = norm(r0, inf);
    while(r(end) > tol && k < maxIter)
        k = k+1;
        Ar0 = A*r0;
        a = (r0'*Ar0)/(Ar0'*Ar0);
        x = x0 + a*r0;
        x0 = x;
        r0 = b - A*x0;
        r = [r, norm(r0,inf)];
    end
end
```

The following script uses these three methods to solve the diffusion equation $-u_{xx} = 1$ on $x \in (0, 1)$. It also plots the residual against the number of iterations.

```
%% Problem 10
% Initial matrix
tol = 1e-10;
maxIter = 2e6;
m = 500;
h = 1/(m+1);
e = ones(m, 1);
A = (1/h^2)*spdiags([-e, 2*e, -e], -1:1, m,m);
[xGS, kGS, rGS] = GaussSeidel(A, e, tol, maxIter);
[xJ, kJ, rJ] = Jacobi(A, e, tol, maxIter);
[xCG, kCG, rCG] = ConjugateGradient(A, e, tol, maxIter);
figure;
semilogy(0:kGS, rGS, 'g', 0:kJ, rJ, 'b', 0:kCG, rCG, 'r');
xlabel('Number of Iterations');
ylabel('Infinity Norm of Residual');
```


Gauss-Seidel
Jacobi
Conjugate Gradient
Infinity Norm of Residual
Number of Iterations
$\times 10^5$
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
2
4
6
8
10
12
14