MATH 562 Spring 2016 – Homework 4

1. (10 points) For each of the following, show that the statement is correct, or give a counter-example. If nothing else is written, assume that $\mathbf{A} \in \mathbb{C}^{m \times m}$

   1. If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$
   2. If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then $-\lambda$ is an eigenvalue of $\mathbf{A}$.
   3. If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then $\bar{\lambda}$ is an eigenvalue of $\mathbf{A}$.
   4. If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is nonsingular, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$
   5. If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = \mathbf{0}$.
   6. If $\mathbf{A}$ is Hermitian and λ is an eigenvalue of $\mathbf{A}$, then $|\lambda$ is a singular value of $\mathbf{A}$.
   7. If $\mathbf{A}$ is diagonalizable and all eigenvalues are equal, then $\mathbf{A}$ is diagonal.

2. (10 points) (Problem 25.1, page 194)

   (a) Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ be tridiagonal and hermitian, with all of its subdiagonal and superdiagonal entries nonzero. Prove that the eigenvalues of $\mathbf{A}$ are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $\mathbf{A} - \lambda\mathbf{I}$ has rank at least $m - 1$.)

   (b) Let $\mathbf{A}$ be upper-Hessenberg, with all of its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of $\mathbf{A}$ are not necessarily distinct.

3. (10 points) Suppose $\mathbf{A}$ is $m \times m$ and has a complete set of orthonormal eigenvectors, $\mathbf{q}_1, \ldots, \mathbf{q}_m$, and with corresponding eigenvalues $\lambda_1, \ldots, \lambda_m$. Assume that the ordering is such that $|\lambda_j| \geqslant |\lambda_{j+1}|$. Furthermore, assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$. Consider the artificial version of the power method $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}^{(k-1)}/\lambda_1$, with $\mathbf{v}^{(0)} = \alpha_1\mathbf{q}_1 + \cdots + \alpha_m\mathbf{q}_m$ where α_1 and α_2 are both nonzero. (This is artificial since the normalizing factor λ_1 in the iteration would not be known ahead of time. However, this version is easier to analyze.) Show that the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{\infty}$ converges linearly to $\alpha_1\mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

4. (10 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Calculate the eigenvalues and eigenvectors of the matrix $\mathbf{A}^T\mathbf{A}$

(b) Use your results in (a) to compute (by hand) the SVD of $\mathbf{A}$.

(c) Find the 1-, 2-, ∞- and Frobenius norms of $\mathbf{A}$.

5. (10 points) (Matlab) Write a MATLAB function $[v, lam, k] = Pwr(A, v0)$ that uses the method of power iteration (Algorithm 27.1, page 205) to compute the largest eigenvalue "lam", and a corresponding eigenvector v that has length one in the 2-norm. The third argument returned, k, should be the number of iterations used in the computation. The input data is a square matrix $\mathbf{A}$ and a starting vector $v0$. Try out two different stopping criterion. Let $v^{(k)}$ denote the k-th iterate, and $\lambda^{(k)}$ the corresponding Rayleigh quotient.

   (a) (Criterion 1): If $\|v^{(k)} - v^{(k-1)}\|_2 < 10^{-8}$ or $k > 500$ stop the iteration and return $v^{(k)}, \lambda^{(k)}$, and k.

   (b) (Criterion 2): If $\|\lambda^{(k)} - \lambda^{(k-1)}\|_2 < 10^{-8}$ or $k > 500$ stop the iteration and return $v^{(k)}, \lambda^{(k)}$, and k.

   The two versions could be called $Pwr1$ and $Pwr2$. Report on the results you observe for the two sets of data generated by the MATLAB commands below. Include information on the number of iterations used and the accuracy obtained, and any conclusions you have on the performance of the algorithm including the stopping criterion.

   (a) $A = diag([-4, 2, 1, 1, 1]) + triu(rand(5, 5), 1)$; $v0 = ones(5, 1)$;

   (b) $A = diag([9, 2, 1, 5, -8]) + triu(rand(5, 5), 1)$; $v0 = ones(5, 1)$;

6. (10 points) (Matlab) Write a MATLAB function $[v, lam, k] = Inv(A, v0, mu)$ that uses the method of inverse power iteration (Algorithm 27.2, page 206) to compute the eigenvalue "lam" closest to mu, and a corresponding eigenvector v that has length one in the 2-norm. The third argument returned, k, should be the number of iterations used in the computation. The input data is a square matrix A, a starting vector $v0$, and a number mu which could be an approximate eigenvalue. Use Criterion 2 as your stopping criterion. To simply the code use the backslash operator "\" to solve the linear system ($x = A\backslash b$ is the solution of $Ax = b$). How does the performance of this function compare with Pwr if you use it on the second set of data above, with if the additional input parameter mu is taken to be 8.8?

7. (10 points) (Matlab) Write a MATLAB function $[v, lam, k] = Ray(A, v0)$ that uses the method of Rayleigh quotient iteration (Algorithm 27.3, page 207) to

compute the “lam”, and a corresponding eigenvector v that has length one in the 2-norm, given an initial approximation $v0$ of v. The third argument returned, k, should be the number of iterations used in the computation. The input data is a square matrix A, a starting vector $v0$ that is presumably an approximate eigenvector. Use Criterion 2 as your stopping criterion. To simply the code use the backslash operator “\” to solve the linear system Experiment with the matrix from the second set of data above, using various choices for the input vector v0, and comment on what you observe.