Caleb Logemann
MATH 562 Numerical Analysis II
Final Exam

1. Let $A \in \mathbb{R}^{m \times m}$ be written in the form $A = L + D + U$, where L is strictly lower triangular, D is the diagonal of A, and U is the strictly upper triangular part of A. Assuming D is invertible, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = -D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}$. The Jacobi iteration method for solving $A\mathbf{x} = \mathbf{b}$ is defined by

   $$\mathbf{x}^{n+1} = -D^{-1}(L + U)\mathbf{x}^n + D^{-1}\mathbf{b}$$

   Show that if A is nonsingular and strictly row diagonally dominant:

   $$0 < \sum_{j \neq i} (|a_{ij}|) < |a_{ii}|$$

   then the Jacobi iteration converges to $\mathbf{x}_* = A^{-1}\mathbf{b}$ for each fixed $\mathbf{b} \in \mathbb{R}^m$.

   Proof. □

2. Let $A \in \mathbb{R}^{m \times m}$ be symmetric positive definite (SPD), $\mathbf{b} \in \mathbb{R}^m$ and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

   $$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

   Suppose K is a subspace of $\mathbb{R}^m$. Show that $\hat{\mathbf{x}} \in K$ minimizes $\phi(\mathbf{x})$ over K if and only if $\nabla\phi(\hat{\mathbf{x}}) \perp K$.

   Proof. □

3.

4.

5.

6.

7.

8.

9.

10.