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MATH 562 Numerical Analysis II
Final Exam

1. Let $A \in \mathbb{R}^{m \times m}$ be written in the form $A = L + D + U$, where L is strictly lower triangular, D is the diagonal of A, and U is the strictly upper triangular part of A. Assuming D is invertible, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$. The Jacobi iteration method for solving $A\mathbf{x} = \mathbf{b}$ is defined by

$$\mathbf{x}^{(n+1)} = -D^{-1}(L+U)\mathbf{x}^{(n)} + D^{-1}\mathbf{b}$$

Show that if A is nonsingular and strictly row diagonally dominant:

$$0 < \sum_{j \neq i} (|a_{ij}|) < |a_{ii}|$$

then the Jacobi iteration converges to $\mathbf{x}_* = A^{-1}\mathbf{b}$ for each fixed $\mathbf{b} \in \mathbb{R}^m$.

Proof. Let $\mathbf{e}_n$ be the error of the nth iteration of the Jacobi iteration from the actual solution, that is let

$$\mathbf{e}_n = \mathbf{x}^{(n)} - \mathbf{x}_*.$$

The Jacobi iteration converges to the real solution if

$$\lim_{n \to \infty} (\|\mathbf{e}_n\|_\infty) = 0$$

The error vector can be expressed recursively by noting that $\mathbf{x}^{(n)}$ is the Jacobi iteration evaluated on $\mathbf{x}^{(n-1)}$ and that $\mathbf{x}_*$ is a fixed point of the Jacobi iteration as it is the true solution to the linear system. This means that

$$\begin{aligned}
\mathbf{x}^{(n)} &= -D^{-1}(L+U)\mathbf{x}^{(n-1)} + D^{-1}\mathbf{b} \\
\mathbf{x}_* &= -D^{-1}(L+U)\mathbf{x}_* + D^{-1}\mathbf{b}.
\end{aligned}$$

Therefore we can express the error recursively as

$$\begin{aligned}
\mathbf{e}_n &= \mathbf{x}^{(n)} - \mathbf{x}_* \\
\mathbf{e}_n &= \left(-D^{-1}(L+U)\mathbf{x}^{(n-1)} + D^{-1}\mathbf{b}\right) - \left(-D^{-1}(L+U)\mathbf{x}_* + D^{-1}\mathbf{b}\right) \\
\mathbf{e}_n &= -D^{-1}(L+U)\mathbf{x}^{(n-1)} + D^{-1}(L+U)\mathbf{x}_* \\
\mathbf{e}_n &= -D^{-1}(L+U)\left(\mathbf{x}^{(n-1)} - \mathbf{x}_*\right) \\
\mathbf{e}_n &= -D^{-1}(L+U)\mathbf{e}_{n-1}.
\end{aligned}$$

Extrapolating this backwards we see that $\mathbf{e}_n$ can be expressed in terms of $\mathbf{e}_0$

$$\mathbf{e}_n = \left(-D^{-1}(L+U)\right)^n \mathbf{e}_0.$$

Now we can consider the limit of $\|\mathbf{e}_n\|_\infty$ as n goes to infinity.

$$\lim_{n\to\infty}(\|\mathbf{e}_n\|_\infty) = \lim_{n\to\infty}\left(\left\|\left(-D^{-1}(L+U)\right)^n \mathbf{e}_0\right\|_\infty\right)$$
$$\lim_{n\to\infty}(\|\mathbf{e}_n\|_\infty) \leq \|\mathbf{e}_0\|_\infty \lim_{n\to\infty}\left(\left\|D^{-1}(L+U)\right\|_\infty^n\right)$$

Now consider $\|D^{-1}(L+U)\|_\infty$. The infinity norm is the max row sum of the matrix, that is

$$\left\|D^{-1}(L+U)\right\|_\infty = \max *_{1\leq i\leq m} \sum_{j=1}^{m}\left(\left|\left(D^{-1}(L+U)\right)_{ij}\right|\right)$$

Because $L+U = A-D$, $(L+U)_{ij} = a_{ij}$ if $i \neq j$ and $(L+U)_{ii} = 0$. Also D^{-1} is diagonal with $(D^{-1})_{ii} = \frac{1}{D_{ii}} = \frac{1}{a_{ii}}$. Therefore the matrix product $D^{-1}(L+U)$ has entries $(D^{-1}(L+U))_{ij} = \frac{a_{ij}}{a_{ii}}$ if $i \neq j$ or if $i = j$, then $(D^{-1}(L+U))_{ii} = 0$. We can now say that

$$\left\|D^{-1}(L+U)\right\|_\infty = \max *_{1\leq i\leq m} \sum_{j\neq k}\left(\left|\frac{a_{ij}}{a_{ii}}\right|\right)$$
$$\left\|D^{-1}(L+U)\right\|_\infty = \max *_{1\leq i\leq m} \frac{1}{|a_{ii}|}\sum_{j\neq k}(|a_{ij}|)$$

However since A is strictly row diagonally dominant $|a_{ii}| > \sum_{j\neq k}(|a_{ij}|)$, we can conclude that $\frac{1}{|a_{ii}|}\sum_{j\neq k}(|a_{ij}|) < 1$. Therefore

$$\left\|D^{-1}(L+U)\right\|_\infty < 1$$

□

2. Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite (SPD), $\mathbf{b} \in \mathbb{R}^m$ and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

Suppose K is a subspace of $\mathbb{R}^m$. Show that $\hat{\mathbf{x}} \in K$ minimizes $\phi(\mathbf{x})$ over K if and only if $\nabla\phi(\hat{\mathbf{x}}) \perp K$.

Proof. □

3.

4. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Let Gaussian elimination be carried out on A without pivoting. After k steps, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix}$$

where $A_{22}^{(k)}$ is an $(n-k)\times(n-k)$ matrix. Show by induction

   (a) $A_{22}^{(k)}$ is symmetric positive definite.

   Proof. □

   (b) $a_{ii}^{(k)} \le a_{ii}^{(k-1)}$ for all $k \le i \le n$, $k = 1, \cdots, n-1$.

   Proof. □

5. Let $A \in \mathbb{R}^{m\times n}$ with $m > n$ and

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 is a nonsingular $n\times n$ matrix, and A_2 is an $(m-n)\times n$ arbitrary matrix.

6. Let $A \in \mathbb{C}^{m\times m}$ with $\operatorname{rank}(A) = r$. Suppose an SVD of A is given by $A = U\Sigma V^*$, where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ denote the columns of U and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ denote the columns of V. Prove that $\langle \mathbf{v}_{r+1}, \ldots, \mathbf{v}_m \rangle = (A)$.

   Proof. □

7. Problem 33.2 (Page 255)

8. Problem 36.1 (Page 283)

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