Lecture 02
Orthogonal Vectors and Matrices; Vector Norms

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MATH 562 Numerical Analysis II

Outline

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Inner Product

- Inner product (dot product) of two column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ is $\mathbf{u}^*\mathbf{v} = \sum_{i=1}^{m} \bar{u}_i v_i$
- In contrast, outer product of $\mathbf{u}, \mathbf{v}$ is $\mathbf{u}\mathbf{v}^*$.
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- Inner product is bilinear, in the sense that it is linear in each vertex separately:

$$(\mathbf{u}_1 + \mathbf{u}_2)^*\mathbf{v} = \mathbf{u}_1^*\mathbf{v} + \mathbf{u}_2^*\mathbf{v}$$

$$\mathbf{u}^*(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u}^*\mathbf{v}_1 + \mathbf{u}^*\mathbf{v}_2$$

$$(\alpha\mathbf{u})^*(\beta\mathbf{v}) = \bar{\alpha}\beta\mathbf{u}^*\mathbf{v}$$

Orthogonal Vectors

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A set of nonzero vectors S is orthogonal if they are pairwise orthogonal. They are orthonormal if it is orthogonal and in addition each vector has unit Euclidean length.

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Question: If the column vectors of an $m \times n$ matrix **A** are orthogonal, what is the rank of **A**?

Answer: $rank(\mathbf{A}) = \min\{m, n\}$, i.e., **A** has full rank.

Components of Vector

- Given an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\}$ forming a basis of $\mathbb{C}^m$, vector $\mathbf{v}$ can be decomposed into orthogonal components as $\mathbf{v} = \sum_{i=1}^{m} (\mathbf{q}_i^* \mathbf{v}) \mathbf{q}_i$

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- Another way to express the condition is $\mathbf{v} = \sum_{i=1}^{m} (\mathbf{q}_i \mathbf{q}_i^*) \mathbf{v}$
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- $\mathbf{q}_i \mathbf{q}_i^*$ is an orthogonal projection matrix. Note that it is NOT an orthogonal matrix.
- More generally, given an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ with $n \leqslant m$, we have

$$\mathbf{v} = \mathbf{r} + \sum_{i=1}^{n} (\mathbf{q}_i^* \mathbf{v})\mathbf{q}_i = \sum_{i=1}^{n} (\mathbf{q}_i \mathbf{q}_i^*)\mathbf{v}, \text{ and } \mathbf{r}^* \mathbf{q}_i = 0, 1 \leqslant i \leqslant n$$

- Let $\mathbf{Q}$ be composed of column vectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$, $\mathbf{QQ}^* = \sum_{i=1}^n (\mathbf{q}_i \mathbf{q}_i^*)$ is an orthogonal projection matrix.

Unitary Matrices

Definition

A matrix is unitary if $\mathbf{Q}^* = \mathbf{Q}^{-1}$, i.e., if $\mathbf{Q}^*\mathbf{Q} = \mathbf{Q}\mathbf{Q}^* = \mathbf{I}$.

- In the real case, we say the matrix is orthogonal. Its column vectors are orthonormal.
- In other words, $\mathbf{q}_i^*\mathbf{q}_j = \delta_{ij}$, the Kronecker delta.

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Question: What is the geometric meaning of multiplication by a unitary matrix?

Answer: It preserves angles and Euclidean length. In the real case, multiplication by an orthogonal matrix $\mathbf{Q}$ is a rotation (if $det(\mathbf{Q}) = 1$) or reflection (if $det(\mathbf{Q}) = -1$)

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Definition

A norm is a function: $\|\cdot\| : \mathbb{C}^m \to \mathbb{R}$ that assigns a real-valued length to each vector. It must satisfy the following conditions:

- $\|\mathbf{x}\| \geqslant 0$, and $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$
- $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$
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- An example is Euclidean length (i.e., $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{m} |x_i|^2}$)

p-norms

- Euclidean length is a special case of p-norms, defined as

$$\|\mathbf{x}\|_p = (\sum_{i=1}^{m} |x_i|^p)^{1/p}$$

for $1 \leqslant p \leqslant \infty$.

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  - Answer: $\|\mathbf{x}\|_\infty = \max_{1\leqslant i\leqslant m} |x_i|$ (Proof ?)

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  - Answer: $\|\mathbf{x}\|_\infty = \max_{1 \leqslant i \leqslant m} |x_i|$ (Proof ?)
- Why we require $p \geqslant 1$? What happens if $0 < p < 1$?

Weighted p-norms

- A generalization of p-norm is weighted p-norm, which assigns different weights (priorities) to different components. (It is anisotropic instead of isotropic)
- Algebraically, $\|\mathbf{x}\|_W = \|\mathbf{Wx}\|$, where $\mathbf{W}$ is diagonal matrix with i-th diagonal entry $w_i \neq 0$ being weight for i-th component
- In other words,

$$\|\mathbf{x}\|_W = (\sum_{i=1}^{m} |w_i x_i|^p)^{1/p}$$

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- What happens if we allow $w_i = 0$?
- Can we further generalize it to allow $\mathbf{W}$ being arbitrary matrix?
- No. But we can allow $\mathbf{W}$ to be arbitrary nonsingular matrix.