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MATH 562 Numerical Analysis II
Homework 3

1. Determine the relative condition number for the following problem. Are there values of x for which the problem is ill-conditioned? Justify your answer.

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

Since f is differentiable the relative condition number of f is given by $\kappa = \frac{|f'(x)|}{|f(x)|/|x|}$. For this problem

$$\begin{aligned} f'(x) &= \frac{(1 + e^{-x})e^{-x} - (1 - e^{-x})(-e^{-x})}{(1 + e^{-x})^2} \\ &= \frac{e^{-x} + e^{-2x} + e^{-x} - e^{-2x}}{(1 + e^{-x})^2} \\ &= \frac{2e^{-x}}{(1 + e^{-x})^2} \end{aligned}$$

Thus the relative condition number for this problem is

$$\begin{aligned} \kappa &= \frac{|f'(x)|}{|f(x)|/|x|} \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})^2} / \frac{1 - e^{-x}}{1 + e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})^2} \times \frac{1 + e^{-x}}{1 - e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 + e^{-x})} \times \frac{1}{1 - e^{-x}} \right| \\ &= \left| \frac{2xe^{-x}}{(1 - e^{-2x})} \right| \end{aligned}$$

This problem is not ill-conditioned because for any x this relitive condition number is small. At $x = 0$, this condition number is undefined, but L'Hopital's rule shows that the limit is equal to 1.

$$\begin{aligned}\lim_{x\to 0}(\kappa) &= \lim_{x\to 0}\left(\frac{2e^{-x}-2xe^{-x}}{2e^{-2x}}\right)\\ &= \frac{2e^0}{2e^0}\\ &= 1\end{aligned}$$

As $x \to \infty$, $2xe^{-x} \to 0$ and $1 - e^{-2x} \to 1$, therefore $\kappa \to 0$. As $x \to -\infty$, $1 - e^{-2x} > 2xe^{-x}$, so $\kappa \to 0$. In fact $\kappa \leq 1$ for all x, therefore this problem is not ill-conditioned.

2. Determine whether the calculation $f(x,y) = (1+x)y^2$ is backward stable by the alogirithm

$$\tilde{f}(x,y) = [1 \oplus fl(x)] \otimes [fl(y) \otimes fl(y)]$$

The algorithm $\tilde{f}$ is backward stable if there exists $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$ such that $\tilde{f}(x,y) = f(\tilde{x},\tilde{y})$ and $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} = O(\epsilon_{machine})$ for all $\mathbf{x}$.

$$\begin{aligned}\tilde{f}(x,y) &= [1 \oplus fl(x)] \otimes [fl(y) \otimes fl(y)]\\ &= [1 \oplus x(1+\epsilon_1)] \otimes [y(1+\epsilon_2) \otimes y(1+\epsilon_3)]\\ &= [1 + x(1+\epsilon_1)](1+\epsilon_4) \otimes [y(1+\epsilon_2) \times y(1+\epsilon_3)](1+\epsilon_5)\\ &= [1 + x(1+\epsilon_1)](1+\epsilon_4) \times [y(1+\epsilon_2) \times y(1+\epsilon_3)](1+\epsilon_5)(1+\epsilon_6)\\ &= [1 + x(1+\epsilon_1)]y^2(1+\epsilon_7)\end{aligned}$$

where $\epsilon_7 = O(\epsilon_{machine})$

$$\begin{aligned}\tilde{f}(x,y) &= [1 + x(1+\epsilon_1)]y^2(1+\epsilon_7)\\ &= [1 + x(1+\epsilon_1)]\left(y\sqrt{1+\epsilon_7}\right)^2\\ &= [1 + x(1+\epsilon_1)](y(1+\epsilon_8))^2\\ &= f(x(1+\epsilon_1), y(1+\epsilon_8))\end{aligned}$$

Therefore $\tilde{x} = x(1+\epsilon_1)$ and $\tilde{y} = y(1+\epsilon_8)$. This does satisfy $\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} = O(\epsilon_{machine})$, because $\|\mathbf{x}-\tilde{\mathbf{x}}\| = \sqrt{\epsilon_1^2 + \epsilon_8^2} = O(\epsilon_{machine})$. So this algorithm is backward stable.

3. (a) Compute the LU factorization $A = LU$, of

$$A = \begin{bmatrix} 1 & 2 & 4\\ 2 & 3 & 4\\ 2 & 5 & 6\end{bmatrix}$$

Use the factorization to solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-1, 1, 1]^T$. Following algorithm 20.1, the LU factorization can be found as follows.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$$

$$k = 1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$k = 2$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & -6 \end{bmatrix}$$

Therefore the LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & -6 \end{bmatrix}$$

Now this system can be solved using forward and backward substitution. Initially we will solve the system $L\mathbf{y} = \mathbf{b}$ where $\mathbf{y} = U\mathbf{x}$ by forward substitution.

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

Now we can solve the system $U\mathbf{x} = \mathbf{y}$ by backward substitution.

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This is also the solution to the system $A\mathbf{x} = \mathbf{b}$.

(b) Solve the system $A\mathbf{x} = \mathbf{b}$ by LU factorization with partial pivoting

The LU factorization using partial pivoting can be found by following algorithm 21.1.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix} P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$k = 1$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1/2 & 2 \\ 0 & 2 & 2 \end{bmatrix} P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$k = 2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3/2 \end{bmatrix} P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now that we have found this LU factorization with pivoting we can solve the system $A\mathbf{x} = \mathbf{b}$ or the equivalent system $PA\mathbf{x} = P\mathbf{b}$, using forward and backward substitution.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ PA\mathbf{x} &= P\mathbf{b} \\ LU\mathbf{x} &= P\mathbf{b} \end{aligned}$$

Let $\mathbf{y} = U\mathbf{x}$ and solve this system with forward substitution.

$$L\mathbf{y} = P\mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix}$$

Now the system $U\mathbf{x} = \mathbf{y}$ can be solved by backward substitution.

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

This is also the solution to $A\mathbf{x} = \mathbf{b}$ and it is equivalent to the solution found in part (a).

4. Let $A \in \mathbb{C}^{m \times m}$ be nonsingular. Show that A has an LU factorization if and only if for each k, such that $1 \leq k \leq m$, the upper left $(k \times k)$ block $A(1 : k, 1 : k)$ of A is nonsingular. Show that this LU factorization is unique.

5. Rank Deficient Least Squares Problem:

6. Consider the matrix
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   (a) Using any method you like, determine reduced and full QR factorizations.

   (b) Use the QR factorization to solve the linear least square problem
$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2$$
   with $\mathbf{b} = [110]^T$.

   (c) Use the QR factorization to solve the linear least squares problem
$$\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2$$
   with matrix $A \in \mathbb{R}^{m \times n}$ with rank n and $\mathbf{b} \in \mathbb{R}^m$.

7. Consider the least-square problem $\min_{\mathbf{x}} \|A\mathbf{x} - b\|_2$, where A is the first 5 columns of the 6×6 inverse Hilbert matrix and
$$b = \begin{bmatrix} 463 \\ -13860 \\ 97020 \\ -258720 \\ 291060 \\ -116424 \end{bmatrix}$$

   (a) What are the four conditioning numbers (Theorem 18.1) of the problem?
   The following script computes the 4 condition numbers.

```
inverseHilbert = invhilb(6);
A = inverseHilbert(:, 1:5);
[m, n] = size(A);
b = [463; -13860; 97020; -258720; 291060; -116424];
```

```
xExact = [1; 1/2; 1/3; 1/4; 1/5];
y = A*x;
theta = acos(norm(y)/norm(b));
eta = norm(A)*norm(xExact)/norm(y);
kappa = cond(A);

condby = 1/cos(theta)
condbx = kappa/(eta*cos(theta))
condAy = kappa/cos(theta)
condAx = kappa + kappa^2*tan(theta)/eta
```

```
condby =

    1

condbx =

    1.8263e+05

condAy =

    4.6968e+06

condAx =

    4.6968e+06
```

(b) Use all the algorithms on Pages 138-142 to solve the problem.

   i. Householder QR
   ii. Householder QR of augmented matrix
   iii. Modified Gram-Schmidt QR
   iv. Modified Gram-Schmidt QR of augmented matrix
   v. Normal Equation
   vi. SVD

   Check the accuracy of computed solutions as compared to actual solution, and comment on the computed solutions and algorithms used.

```
% Householder QR
[Q, R] = qr(A, 0);
x = R\(Q'*b);
E = norm(x - xExact);
```

```
M = {'Householder QR'};

% Householder QR of augmented matrix
[Q, R] = qr([A, b], 0);
Qb = R(1:n, n+1);
R = R(1:n, 1:n);
x = R\Qb;
E = [E; norm(x - xExact)];
M = [M, {'Householder QR of augmented matrix'}];

% Modified Gram-Schmidt QR
[Q, R] = mgs(A);
x = R\(Q'*b);
E = [E; norm(x - xExact)];
M = [M, {'Modified Gram-Schmidt QR'}];

% Modified Gram-Schmidt QR of augmented matrix
[Q, R] = mgs([A, b]);
Qb = R(1:n, n+1);
R = R(1:n, 1:n);
x = R\Qb;
E = [E; norm(x - xExact)];
M = [M, {'Modified Gram-Schmidt QR of augmented matrix'}];

% Normal Equation
x = (A'*A)\(A'*b);
E = [E; norm(x - xExact)];
M = [M, {'Normal Equations'}];

% SVD
[U, S, V] = svd(A, 0);
x = V*(S\(U'*b));
E = [E; norm(x - xExact)];
M = [M, {'SVD'}];

table(E, 'VariableNames', {'Error'}, 'RowNames', M)
```

```
ans =

                                                       Error
                                                       ----------

    Householder QR                                     8.3668e-11
    Householder QR of augmented matrix                 6.0204e-11
    Modified Gram-Schmidt QR                           4.2067e-06
    Modified Gram-Schmidt QR of augmented matrix        1.253e-12
    Normal Equations                                   1.0259e-05
```

SVD	1.2091e-10