Lecture 17
QR Algorithm and Unnormalized Simultaneous Iteration

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MATH 562 Numerical Analysis II

Outline

1. QR Algorithm and Unnormalized Simultaneous Iteration

Outline

QR Algorithm

- Most basic version of QR algorithm is remarkably simple:

Algorithm: "Pure" QR Algorithm

$\mathbf{A}^{(0)} = \mathbf{A}$
for $k = 1, 2, \ldots$
  $\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)}$
  $\mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)}$

- With some suitable assumptions, $\mathbf{A}^{(k)}$ converge to Schur form of $\mathbf{A}$ (diagonal if $\mathbf{A}$ is symmetric)
- Similarity transformation of $\mathbf{A}$:

$$\mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)} = \left(\mathbf{Q}^{(k)}\right)^T \mathbf{A}^{(k-1)}\mathbf{Q}^{(k)}$$

- But why it works?

Unnormalized Simultaneous Iteration

- To understand QR algorithm, first consider simple algorithm
- Simultaneous iteration is power iteration applied to several vectors
- Start with linearly independent $\mathbf{v}_1^{(0)}, \cdots, \mathbf{v}_n^{(0)}$
- We know from power iteration that $\mathbf{A}^k\mathbf{v}_1^{(0)}$ converges to $\mathbf{q}_1$
- With some assumptions, the space $\langle \mathbf{A}^k\mathbf{v}_1^{(0)}, \cdots, \mathbf{A}^k\mathbf{v}_n^{(0)} \rangle$ converge to $\langle \mathbf{q}_1, \ldots, \mathbf{q}_n \rangle$
- Notation: Define initial matrix $\mathbf{V}^{(0)}$ and matrix $\mathbf{V}^{(k)}$ at step k:

$$\mathbf{V}^{(0)} = [\mathbf{v}_1^{(0)}|\cdots|\mathbf{v}_n^{(0)}], \quad \mathbf{V}^{(k)} = \mathbf{A}^k\mathbf{V}^{(0)} = [\mathbf{v}_1^{(k)}|\cdots|\mathbf{v}_n^{(k)}],$$

Unnormalized Simultaneous Iteration

- Define orthogonal basis for column space of $\mathbf{V}^{(k)}$ by reduced QR factorization $\hat{\mathbf{Q}}^{(k)}\hat{\mathbf{R}}^{(k)} = \mathbf{V}^{(k)}$
- We assume that
  - leading $n+1$ eigenvalues are distinct, and
  - all leading principal submatrices of $\hat{\mathbf{Q}}^T\mathbf{V}^{(0)}$ are nonsingular where $\hat{\mathbf{Q}} = [\mathbf{q}_1|\cdots|\mathbf{q}_n]$.
- We then have columns of $\hat{\mathbf{Q}}^{(k)}$ converge to eigenvectors of $\mathbf{A}$:

$$\|\mathbf{q}_j^{(k)} - (\pm\mathbf{q}_j)\| = O(c^k),$$

  where $c = \max_{1\leqslant k\leqslant n} |\lambda_{k+1}|/|\lambda_k|$
- Proof idea: Show that subspace of any leading j columns of $\mathbf{V}^{(k)} = \mathbf{A}^k\mathbf{V}^{(0)}$ converges to subspace of first j eigenvectors of $\mathbf{A}$, so does the subspace of any leading j columns of $\hat{\mathbf{Q}}^{(k)}$.