Lecture 01
Introduction & Matrix-Vector Multiplication

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MATH 562 Numerical Analysis II

Outline

Course Description

- What is numerical analysis? numerical linear algebra?
  - The study of algorithms for the problems of continuous mathematics.
- Topics: Solutions of linear equations. Matrix factorization and decomposition. Conditioning, stability, and efficiency. Computation of eigenvalues and eigenvectors. Solution of non-linear equations.
- Prerequisite/Co-requisite:
  - Calculus, ODE, PDE, MATH 317 (Linear Algebra).
  - Basic programming tools such as Matlab (or GNU Octave).
- Required Textbook: *Numerical Linear Algebra*, by Lloyd N. Trefethen and David Bau, III, SIAM, 1997, ISBN 0-89871-361-7.
- Classpage (Syllabus): `http://orion.math.iastate.edu/luos/Teaching/MATH562_16SS/MATH562_16SS.html`

Definition

- Matrix-vector product $\mathbf{b} = \mathbf{A}\mathbf{x}$:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

- All entries belong to $\mathbb{C}$, the field of complex numbers. The space of m-vectors is $\mathbb{C}^m$, and the space of $m \times n$ matrices is $\mathbb{C}^{m \times n}$.
- The map $\mathbf{x} \to \mathbf{A}\mathbf{x}$ is linear, which means that for any $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, and any $\alpha \in \mathbb{C}$:

$$\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y},$$

$$\mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{A}\mathbf{x}.$$

- Conversely, every linear map can be expressed as multiplication by a matrix.

Linear Combination

- Alternatively, matrix-vector product can be viewed as

$$\mathbf{b} = \mathbf{A}\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{a}_j$$

  i.e., **b** is a linear combination of column vectors of **A**.
- Two different views of matrix-vector products:
  - scalar operations: **A** acts on **x** to produce **b**: $b_i = \sum_{j=1}^{n} a_{ij} x_j$
  - vector operations: **x** acts on **A** to produce **b**: $\mathbf{b} = \mathbf{A}\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{a}_j$
- If $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{A}\mathbf{x}$ can be viewed as a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Matrix-Matrix Multiplication

- If **A** is $l \times m$ and **C** is $m \times n$, then $\mathbf{b} = \mathbf{AC}$ is $l \times n$, with entries defined by

$$b_{ij} = \sum_{k=1}^{m} a_{ik} c_{kj}$$

- Written in columns, we have

$$\mathbf{b}_j = \mathbf{A}\mathbf{c}_j = \sum_{k=1}^{m} c_{kj} \mathbf{a}_k$$

- In other word, each column of **B** is a linear combination of the columns of **A**.

Perspective: Vector Space

Understanding matrix operations in terms of vector spaces

- Vector space spanned by a set of vectors is composed of linear combinations of these vectors
  - It is closed under addition and scalar multiplication
  - $\mathbf{0}$ is always a member of a subspace
  - Space spanned by m-vectors is subspace of $\mathbb{C}^m$
- If S_1 and S_2 are two subspaces, then $S_1 \bigcap S_2$ is a subspace, so is $S_1 + S_2$, the space of sum of vectors from S_1 and S_2.
  (Note: is $S_1 + S_2$ equivalent to $S_1 \bigcup S_2$?)
- Two subspaces S_1 and S_2 of $\mathbb{C}^m$ are complementary subspaces of each other if $S_1 + S_2 = \mathbb{C}^m$ and $S_1 \bigcap S_2 = \{\mathbf{0}\}$.
  - In other words, $dim(S_1) + dim(S_2) = m$ and $S_1 \bigcap S_2 = \{\mathbf{0}\}$

Range and Null Space

Definition

The range of a matrix $\mathbf{A}$, written as $range(\mathbf{A})$, is the set of vectors that can be expressed as $\mathbf{A}\mathbf{x}$ for some $\mathbf{x}$.

Theorem

$range(\mathbf{A})$ is the space spanned by the columns of $\mathbf{A}$.
(Therefore, the range of $\mathbf{A}$ is also called the column space of $\mathbf{A}$.)

Definition

The null space of $\mathbf{A} \in \mathbb{C}^{m \times n}$, written as $null(\mathbf{A})$, is the set of vectors $\mathbf{x}$ that satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$.
(Entries of $\mathbf{x} \in null(\mathbf{A})$ give coefficients of $\sum x_i \mathbf{a}_i = \mathbf{0}$)

Rank

Definition

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Answer: n (Rank-nullity theorem)

Transpose and Adjoint

- Transpose of $\mathbf{A}$, denoted by $\mathbf{A}^T$, is the matrix $\mathbf{B}$ with $b_{ij} = a_{ji}$
- Adjoint or Hermitian conjugate, denoted by $\mathbf{A}^*$ or $\mathbf{A}^H$, is the matrix $\mathbf{B}$ with $b_{ij} = \bar{a}_{ji}$
- Note that: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
- A matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$. It is Hermitian if $\mathbf{A} = \mathbf{A}^*$.
- A matrix $\mathbf{A}$ is skew-symmetric if $\mathbf{A} = -\mathbf{A}^T$. It is skew-Hermitian if $\mathbf{A} = -\mathbf{A}^*$.
- Diagonal matrix, Upper (Lower) triangular matrix, etc..

Definition

A matrix has full rank if it has the maximal possible rank, i.e., $\min\{m, n\}$

Theorem

A matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ *with* $m \geqslant n$ *has full rank if and only if it maps no two distinct vectors to the same vector.*

Proof.

($\Rightarrow$) Column vectors of $\mathbf{A}$ forms a basis of $range(\mathbf{A})$, so every $\mathbf{b} \in range(\mathbf{A})$ has a unique linear expansion in terms of the columns of $\mathbf{A}$.
($\Leftarrow$) If $\mathbf{A}$ does not have full rank, then its column vectors are linear dependent, so its vectors do not have a unique linear combination.

Definition

A nonsingular or invertible matrix is a square matrix of full rank.

Inverse

Definition

Given a nonsingular matrix $\mathbf{A}$, its inverse is written as $\mathbf{A}^{-1}$, and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

- Note that $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$, and we use $\mathbf{A}^{-*}$ as a shorthand for it

Theorem

The following conditions are equivalent:

(a) $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$

(b) $rank(\mathbf{A})$ is m

(c) $range(\mathbf{A})$ is $\mathbb{C}^m$

(d) $null(\mathbf{A})$ is $\{\mathbf{0}\}$

(e) 0 is not an eigenvalue of $\mathbf{A}$

(f) 0 is not a singular value of $\mathbf{A}$

(g) $det(\mathbf{A}) \neq 0$

Matrix Inverse Times a Vector

- When writing $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, it means $\mathbf{x}$ is the solution of $\mathbf{Ax} = \mathbf{b}$
- In other words, $\mathbf{A}^{-1}\mathbf{b}$ is the vector of coefficients of the expansion of $\mathbf{b}$ in the basis of columns of f $\mathbf{A}$.
- Multiplying $\mathbf{b}$ by $\mathbf{A}^{-1}$ is a change of basis operations to $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ from $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$
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Rank-1 Matrices

- Full-rank matrices are important.
- Another interesting special case is rank-1 matrices.
- A matrix $\mathbf{A}$ is rank-1 if it can be written as $\mathbf{A} = \mathbf{u}\mathbf{v}^*$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors
- $\mathbf{u}\mathbf{v}^*$ is called the outer product of the two vectors, as opposed to the inner product $\mathbf{u}^*\mathbf{v}$