Lecture 14
Eigenvalue Problems

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MATH 562 Numerical Analysis II

Outline

1. Eigenvalue Problems

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Eigenvalue and Eigenvectors

- Eigenvalue problem of $m \times m$ matrix $\mathbf{A}$ is

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

  with eigenvalues λ and eigenvectors $\mathbf{x}$ (nonzero)
- The set of all the eigenvalues of $\mathbf{A}$ is the spectrum of $\mathbf{A}$
- Eigenvalue are generally used where a matrix is to be compounded iteratively
- Eigenvalues are useful for algorithmic and physical reasons
  - Algorithmically, eigenvalue analysis can reduce a coupled system to a collection of scalar problems
  - Physically, eigenvalue analysis can be used to study resonance and stability of physical systems

Eigenvalue Decomposition

- WEigenvalue decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1} \text{ or } \mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda}^{-1}$$

  with eigenvectors $\mathbf{x}_i$ as columns of $\mathbf{X}$ and eigenvalues λ_i along diagonal of $\boldsymbol{\Lambda}$. Alternatively,

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

- Eigenvalue decomposition is change of basis to "eigenvector coordinates"

$$\mathbf{A}\mathbf{x} = \mathbf{b} \rightarrow (\mathbf{X}^{-1}\mathbf{b}) = \boldsymbol{\Lambda}(\mathbf{X}^{-1}\mathbf{x})$$

- Note that eigenvalue decomposition may not exist
- Question: How does eigenvalue decomposition differ from SVD?

Geometric Multiplicity

- Eigenvectors corresponding to a single eigenvalue λ form an eigenspace $E_\lambda \subseteq \mathbb{C}^{m\times m}$
- Eigenspace is invariant in that $\mathbf{A}E_\lambda \subseteq E_\lambda$
- Dimension of E_λ is the maximum number of linearly independent eigenvectors that can be found
- Geometric multiplicity of λ is dimension of E_λ, i.e., $dim(null(\mathbf{A}-\lambda\mathbf{I}))$

Algebraic Multiplicity

- The characteristic polynomial of **A** is degree m polynomial

$$p_{\mathbf{A}}(z) = det(z\mathbf{I} - \mathbf{A}) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_m)$$

  which is monic in that coefficient of z^m is 1
- λ is an eigenvalue of **A** iff $p_{\mathbf{A}}(\lambda) = 0$
  - Proof.
- Algebraic multiplicity of λ is its multiplicity as a root of $p_{\mathbf{A}}$
- Any matrix **A** has m eigenvalues, counted with algebraic multiplicity
- Question: What are the eigenvalues of a triangular matrix?
- Question: How are geometric multiplicity and algebraic multiplicity related?

Similarity Transformations

- The map $\mathbf{A} \to \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}$ is a similarity transformation of $\mathbf{A}$ for any nonsingular $\mathbf{Y}$
- $\mathbf{A}$ and $\mathbf{B}$ are similar if there is a similarity transformation $\mathbf{B} = \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}$

Theorem

If $\mathbf{Y}$ is nonsingular, then $\mathbf{A}$ and $\mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}$ have the same characteristic polynomials, eigenvalues, and algebraic and geometric multiplicities.

- For characteristic polynomial:

$$det(z\mathbf{I} - \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}) = det(\mathbf{Y}^{-1}(z\mathbf{I} - \mathbf{A})\mathbf{Y}) = det(z\mathbf{I} - \mathbf{A})$$

  so algebraic multiplicities remain the same
- If $\mathbf{x} \in E_\lambda$ for $\mathbf{A}$, then $\mathbf{Y}^{-1}\mathbf{x}$ is in eigenspace of $\mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}$ corresponding to λ, and vice versa, so geometric multiplicities remain the same.

Algebraic Multiplicity $\geqslant$ Geometric Multiplicity

- Let n be be geometric multiplicity of λ for **A**. Let $\hat{\mathbf{V}} \in \mathbb{C}^{m\times n}$ constitute of orthonormal basis of the E_λ
- Extend $\hat{\mathbf{V}}$ to unitary $\mathbf{V} = [\hat{\mathbf{V}}, \tilde{\mathbf{V}}] \in \mathbb{C}^{m\times m}$ and form

$$\mathbf{B} = \mathbf{V}^*\mathbf{A}\mathbf{V} = \begin{bmatrix} \hat{\mathbf{V}}^*\mathbf{A}\hat{\mathbf{V}} & \hat{\mathbf{V}}^*\mathbf{A}\tilde{\mathbf{V}} \\ \tilde{\mathbf{V}}^*\mathbf{A}\hat{\mathbf{V}} & \tilde{\mathbf{V}}^*\mathbf{A}\tilde{\mathbf{V}} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$$

- $det(z\mathbf{I} - \mathbf{B}) = det(z\mathbf{I} - \lambda\mathbf{I})det(z\mathbf{I} - \lambda\mathbf{D}) = (z-\lambda)^n det(z\mathbf{I} - \lambda\mathbf{D})$, so the algebraic multiplicity of λ as an eigenvalue of **B** is $\geqslant n$
- **A** and **B** are similar, so the algebraic multiplicity of λ as an eigenvalue of **A** is at least $\geqslant n$
- Examples: $\mathbf{A} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$ Their characteristic polynomial is $(z-2)^3$, so algebraic multiplicity of $\lambda = 2$ is 3. But geometric multiplicity of **A** is 3 and that of **B** is 1.

Defective and Diagonalizable Matrices

- An eigenvalue of a matrix is defective if its algebraic multiplicity $>$ its geometric multiplicity
- A matrix is defective if it has a defective eigenvalue. Otherwise, it is called nondefective.

Theorem

An $m \times m$ matrix $\mathbf{A}$ is nondefective iff it has an eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$

- $(\Leftarrow)$ $\boldsymbol{\Lambda}$ is nondefective, and $\mathbf{A}$ is similar to $\boldsymbol{\Lambda}$, so $\mathbf{A}$ is nondefective.
- $(\Rightarrow)$ $\mathbf{A}$ nondefective matrix has m linearly independent eigenvectors. Take them as columns of $\mathbf{X}$ to obtain $\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$
- Nondefective matrices are therefore also said to be diagonalizable.

Determinant and Trace

- Determinant of $\mathbf{A}$ is $det(\mathbf{A}) = \prod_{j=1}^{m} \lambda_j$, because

$$det(\mathbf{A}) = (-1)^m det(-\mathbf{A}) = (-1)^m p_{\mathbf{A}}(0) = \prod_{j=1}^{m} \lambda_j$$

- Trace of $\mathbf{A}$ is $tr(\mathbf{A}) = \sum_{j=1}^{m} \lambda_j$, since

$$p_{\mathbf{A}}(z) = det(z\mathbf{I} - \mathbf{A}) = z^m - \sum_{j=1}^{m} a_{jj} z^{m-1} + O(z^{m-2})$$

$$p_{\mathbf{A}}(z) = \prod_{j=1}^{m} (z - \lambda_j) = z^m - \sum_{j=1}^{m} \lambda_j z^{m-1} + O(z^{m-2})$$

- Question: Are these results valid for defective or nondefective matrices?

Unitary Diagonalization

- A matrix $\mathbf{A}$ is unitarily diagonalizable if $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^*$ for a unitary matrix $\mathbf{Q}$
- A hermitian matrix is unitarily diagonalizable, with real eigenvalues
- A matrix $\mathbf{A}$ is normal if $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$
    - Examples of normal matrices include hermitian matrices, skew hermitian matrices
    - hermitian $\Leftrightarrow$ matrix is normal and all eigenvalues are real
    - skew hermitian $\Leftrightarrow$ matrix is normal and all eigenvalues are imaginary
    - If $\mathbf{A}$ is both triangular and normal, then $\mathbf{A}$ is diagonal
- Unitarily diagonalizable $\Leftrightarrow$ normal
    - By induction using Schur factorization (next)

Schur Factorization

- Schur factorization is $\mathbf{A} = \mathbf{QTQ}^*$, where $\mathbf{Q}$ is unitary and $\mathbf{T}$ is upper triangular

Theorem

Every square matrix $\mathbf{A}$ *has a Schur factorization.*

Proof

Proof by induction on dimension of $\mathbf{A}$. Case $m = 1$ is trivial.
For $m \geqslant 2$, let $\mathbf{x}$ be any unit eigenvector of $\mathbf{A}$, with corresponding eigenvalue λ. Let $\mathbf{U}$ be unitary matrix with $\mathbf{x}$ as first column. Then $\mathbf{U}^*\mathbf{AU} = \begin{bmatrix} \lambda & \mathbf{w}^* \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$. By induction hypothesis, there is a Schur factorization $\tilde{\mathbf{T}} = \mathbf{V}^*\mathbf{CV}$. Let $\mathbf{Q} = \mathbf{U}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} \lambda & \mathbf{w}^*\mathbf{V} \\ \mathbf{0} & \tilde{\mathbf{T}} \end{bmatrix}$, and then $\mathbf{A} = \mathbf{QTQ}^*$

Eigenvalue Revealing Factorization

- Eigenvalue-revealing factorization of square matrix $\mathbf{A}$
  - Diagonalization $\mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{-1}$ (nondefective $\mathbf{A}$)
  - Unitary Diagonalization $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^*$ (normal $\mathbf{A}$)
  - Unitary triangularization (Schur factorization) $\mathbf{A} = \mathbf{QTQ}^*$ (any $\mathbf{A}$)
  - Jordan normal form $\mathbf{A} = \mathbf{XJX}^{-1}$ where $\mathbf{J}$ block diagonal with

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

- In general, Schur factorization is used, because
  - Unitary matrices are involved, so algorithm tends to be more stable
  - If $\mathbf{A}$ is normal, then Schur form is diagonal