MATH 562   Spring 2016   Homework 1

1. (Problem 1.1, page 9) Let B be a 4×4 matrix to which the following operations are applied in the given order:
   (a) double column 1,
   (b) halve row 3,
   (c) add row 3 to row 1,
   (d) interchange columns 1 and 4,
   (e) subtract row 2 from each of the other rows,
   (f) replace column 4 by column 3
   (g) delete column 1 (so that the final result is a 4×3 matrix).
   The result can be written as a product of eight matrices, one of which is B. a) What are the other 7 matrices and in what order do they appear in the product? b) The result can also be written as a product ABC. What are A and C? (**Suggestion**: Use MATLAB to check your work and do the arithmetic.)

2. (Problem 2.1, page 15) Show that if a matrix $A \in \mathbb{C}^{m \times m}$ is both upper triangular and unitary then it is diagonal.

3. (Problem 2.3, page 15) Let $A \in \mathbb{C}^{m \times m}$ be hermitian (i.e. $A^* = A$. Suppose that $Ax = \lambda x$, where $x \in \mathbb{C}^m$ is a nonzero vector and $\lambda \in \mathbb{C}$, so that λ is an eigenvalue and x is an eigenvector.
   (a) Prove that λ must be real.
   (b) Prove that is x and y are eigenvectors corresponding to distinct eigenvalues then x and y are orthogonal.

4. (Problem 2.6, page 16) If $u, v \in \mathbb{R}^m$, then $A = I + uv^T \in \mathbb{R}^{m \times m}$ is called a rank one perturbation of the identity.
   (a) Show that if A is invertible then its inverse has the form $A^{-1} = I + \alpha uv^T$, for a scalar α, and give an expression for α.
   (b) If A is not invertible what can be said about the vectors u, v? Also describe null(A) in this case.

5. (Problem 3.2, page 24) Let $\| \cdot \|$ denote any norm on $\mathbb{C}^m$ and also the corresponding induced norm on $\mathbb{C}^{m \times m}$, so that $\|Ax\| \leq \|A\| \|x\|$. Show that $\rho(A) \leq \|A\|$, where $\rho(A) = \max\{|\lambda| : \lambda$ is an eigenvalue of $A\}$ is the spectral radius of A

6. Let $\theta \in (0, 2\pi)$ and define the matrix $Q \in \mathbb{R}^{2 \times 2}$ by
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

   Show that $y = Qx$ is the vector obtained by rotating the vector x by θ radians. (**Hint:** Use polar coordinates for the components of your vectors.)

7. (Problem 4.3, page 31) Write a MATLAB function $[U, S, V] = \text{my2dSVD}(A)$ that takes a real 2×2 matrix A as input, computes and returns the matrices in the singular value decompostion of $A = USV^T$, and then plots the right singular vectors v_1, v_2 in the unit circle and the left singular vectors u_1, u_2 in the ellipse that is its image, as in Figure 4.1 on page 26. Make your plot look as much like Figure 4.1 as you can. (See help axis and help subplot for some clues on how to do this.)

There is a built in MATLAB function that computes the SVD of a matrix, but **you should not use it**. Instead follow the strategy suggested by the discussion in class:

(a) Find a vector v_1, that maximizes $\|Av\|_2$ subject to the constraint $\|v\|_2 = 1$. You can use the built in MATLAB function "max" for this purpose (see help max). This will determine a right singular vector and also the singular value σ_1 and a left singular vector u_1.

(b) You can now choose unit vectors orthogonal to v_1 and u_1, being careful of the sign so that $\sigma_2 \geq 0$. (Alternatively, you can find a vector v_2, that minimizes $\|Av\|_2$ subject to the constraint $\|v\|_2 = 1$ (see help min). However, this will not be as accurate or efficient.)

There are sample MATLAB m-files on the course web page that will be helpful in writing your program. In particular, `arrow.m` will draw the vectors for you and also gives you and example of a user defined function (with no outputs). The script `circle_image.m` is the one from the MATLAB demonstration and shows you how to parameterize and plot the unit circle and its image.

For this problem you should hand in your MATLAB code and the output (including the plots) from applying it to the matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

(See help diary to see how to record part of MATLAB session in a file. This would be one way to save the output, although cut and paste is probably easier.)