Caleb Logemann
MATH 562 Numerical Analysis II
Homework 4

1. For each of the following, show that the statement is correct, or give a counter-example. If nothing else is written, assume that $A \in \mathbb{C}^{m \times m}$.

   (a) If λ is an eigenvalue of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.
   Yes this is a true statement.

   Proof. Let $\mathbf{x}$ be the eigenvector for the eigenvalue λ, that is $A\mathbf{x} = \lambda\mathbf{x}$. Thus

   $$\begin{aligned}(A - \mu I)\mathbf{x} &= A\mathbf{x} - \mu I\mathbf{x} \\ &= \lambda\mathbf{x} - \mu\mathbf{x} \\ &= (\lambda - \mu)\mathbf{x}\end{aligned}$$

   Therefore $\mathbf{x}$ is an eigenvector of $A - \mu I$ and the corresponding eigenvalue is $\lambda - \mu$. □

   (b) If A is real and λ is an eigenvalue of A, then $-\lambda$ is an eigenvalue of A.
   This is false. Consider the matrix

   $$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

   The eigenvalues of this matrix are 2 and 3, neither -2 nor -3 are eigenvalues.

   (c) If A is real and λ is an eigenvalue of A, then $\bar{\lambda}$ is an eigenvalue of A.

   (d) If λ is an eigenvalue of A and A is nonsingular, then λ^{-1} is an eigenvalue of A^{-1}.

   (e) If all the eigenvalues of A are zero, than $A = 0$.
   This is false. Consider the matrix

   $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Both of the eigenvalues of this matrix are zero, however $A \neq 0$.

   (f) If A is Hermitian and λ is an eigenvalue of A

   (g) If A is diagonalizable and all eigenvalues are equal, then A is diagonal.

2. (a) Let $A \in \mathbb{C}^{m \times m}$ be tridiagonal and Hermitian, with all of its subdiagonal and superdiagonal entries nonzero. Prove that the eigenvalues of A are distinct.

(b) Let A be upper-Hessenberg, with all of its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of A are not necessarily distinct.

3. Suppose A is $m \times m$ and has a complete set of orthonormal eigenvectors, $\mathbf{q}_1, \dots, \mathbf{q}_m$, and with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$. Assume that the ordering is such that $|\lambda_j| \geq |\lambda_{j+1}|$. Furthermore assume that $|\lambda_1| > |\lambda_2| > |\lambda_3|$. Consider the artificial version of the power method $\mathbf{v}^{(k)} = A\mathbf{v}^{(k-1)}/\lambda_1$ with $\mathbf{v}^{(0)} = \alpha_1\mathbf{q}_1 + \cdots + \alpha_m\mathbf{q}_m$, where α_1 and α_2 are both nonzero. Show that the sequence converges linearly to $\alpha_1\mathbf{q}_1$ with asymptotic constant $C = |\lambda_2/\lambda_1|$.

Proof. $\square$

4. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Calculate the eigenvalues and eigenvectors of A^TA

First we must compute the matrix, A^TA.

$$A^TA = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The eigenvalues can be found by using the characteristic polynomial, that is $p(z) = \det(zI - A^TA)$.

$$\begin{aligned} \det(zI - A^TA) &= \begin{vmatrix} z-3 & 1 & 0 \\ 1 & z-2 & 1 \\ 0 & 1 & z-3 \end{vmatrix} \\ &= (z-3)^2(z-2) - (z-3) - (z-3) \\ &= (z-3)((z-3)(z-2) - 2) \\ &= (z-3)\left(z^2 - 5z + 4\right) \\ &= (z-3)(z-4)(z-1) \end{aligned}$$

The eigenvalues are the zeros of the characteristic polynomial, therefore $\operatorname{spec}(()A) = \{1, 3, 4\}$.

The eigenvectors of A^TA can be found by solving the following systems

$$(I - A^TA)\mathbf{x} = \mathbf{0}$$
$$(3I - A^TA)\mathbf{x} = \mathbf{0}$$
$$(4I - A^TA)\mathbf{x} = \mathbf{0}$$

First I will solve $(I - A^TA)\mathbf{x} = \mathbf{0}$ using the augmented system.

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions to this system are of the form

$$\begin{bmatrix} x \\ 2x \\ x \end{bmatrix}$$

The eigenvector with 2-norm equal to one for eigenvalue 1 is

$$\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

The eigenvector vector for eigenvalue 3 can be found as

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions to this system are of the form

$$\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$$

The eigenvector with 2-norm equal to one for eigenvalue 3 is

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

Lastly the eigenvector for eigenvalue 4 is needed.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions to this system are of the form

$$\begin{bmatrix} x \\ -x \\ x \end{bmatrix}$$

The eigenvector with 2-norm equal to one for eigenvalue 3 is

$$\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Thus the eigenvalue decomposition of $A^T A$ is

$$\begin{aligned} A^T A &= X \Lambda X' \\ X &= \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \\ \Lambda &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

(b) Use your results in (a) to compute (by hand) the SVD of A.

The singular values of A are the nonnegative square roots of the eigenvalues of A^TA. Thus if $A = U\Sigma V^T$ is a singular value decomposition of A, then

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

The unitary matrix U can be found by doing Gram-Schmidt on the columns of AV.

$$AV = \begin{bmatrix} 0 & -2/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -2/\sqrt{3} \\ 2/\sqrt{6} & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \mathbf{u}_2 \qquad = A\mathbf{v}_2 - \mathbf{u}_1^T A\mathbf{v}_2\mathbf{u}_1$$

$$\mathbf{u}_2 = \begin{bmatrix} 0 \end{bmatrix}$$

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(c) Find the 1-, 2-, ∞-, and Frobenius norms of A.

5. Write a MATLAB function $[v, lam, k] = Pwr(A, v0)$ that uses the method of power iteration to compute the largest eigenvalue, "lam", and a corresponding eigenvector v that has length one in the 2-norm. The third argument returned, k, should be the number of iterations used in the computation. The input data is a square matrix A and a starting vector $v0$.

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