MATH 562 Spring 2016 Final Exam

1. (10 points) Let $A \in \mathbf{R}^{m\times m}$ be written in the form $A = L + D + U$ where L is the strictly lower triangular part of A, D is the diagonal part of A, and U is the strictly upper triangular part of A. Assuming $D$is invertible, $A\mathbf{x} = \mathbf{b}$ is equivalent to $\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$. The Jacobi iteration method for solving $Ax = b$ is defined by

$$\mathbf{x}^{(n+1)} = -D^{-1}(L+U)\mathbf{x}^{(n)} + D^{-1}\mathbf{b}.$$

   Show that if A is nonsingular and strictly row diagonally dominant:

$$0 < \sum_{j\neq i} |a_{ij}| < |a_{ii}|, \quad i = 1, ..., m,$$

   then Jacobi iteration converges to $\mathbf{x}_\star = A^{-1}\mathbf{b}$, for each fixed $\mathbf{b} \in \mathbf{R}^m$. (Hint: The norm $||\cdot||_\infty$ is a convenient one to use.)

2. (10 points) Let $A \in \mathbf{R}^{m\times m}$ be symmetric positive definite (SPD), $\mathbf{b} \in \mathbf{R}^m$ and define $\phi : \mathbf{R}^m \to \mathbf{R}$ by

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}.$$

   Suppose K is a subspace of $\mathbf{R}^m$. Show that $\hat{\mathbf{x}} \in K$ minimizes $\phi(\mathbf{x})$ over K if and only if $\nabla\phi(\hat{\mathbf{x}}) \perp K$.

3. (10 points) Show that:
   (a) (Forward-error analysis)

$$|fl(\mathbf{x}^T\mathbf{a}) - \mathbf{x}^T\mathbf{a}| \leq n\epsilon_{machine}|\mathbf{x}|^T|\mathbf{a}| + O(\epsilon_{machine}^2),$$

   where $\mathbf{x}, \mathbf{a}$ are n-dimensional floating point vectors and $fl(\mathbf{x}^T\mathbf{a})$ represents floating point computation of dot product between $\mathbf{x}$ and $\mathbf{a}$. $|\mathbf{x}|$ represents the vector containing absolute values of $\mathbf{x}$. (hint: by induction)

   (b) (Forward-error analysis)

$$\|fl(XA) - XA\|_F \leq n\epsilon_{machine}\|X\|_F\|A\|_F + O(\epsilon_{machine}^2),$$

   where X, A are $n \times n$ dimensional floating point matrices and $fl(XA)$ represents floating point computation of matrix multiplication between X and A.

   (c) (Backward-error analysis) Show that the relative backward error $\frac{\|\delta A\|_F}{\|A\|_F} \leq n\kappa(X)O(\epsilon_{machine})$, where $\kappa(X) = \|X\|_F\|X^{-1}\|_F$, $fl(XA) = X(A + \delta A)$.

4. (10 points) Let $A \in R^{n\times n}$ be a symmetric positive definite matrix. Let Gaussian ellimination be carried out on A without pivoting. After k steps, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix}$$

where $A_{22}^{(k)}$ is an $(n-k)\times(n-k)$ matrix. Show by induction

a). $A_{22}^{(k)}$ is symmetric positive definite,
b). $a_{ii}^{(k)} \le a_{ii}^{(k-1)}$ for all $k \le i \le n$, $k = 1, \dots, n-1$.

5. (5 points) Let $A \in R^{m\times n}$ be an $m \times n$ matrix, $m > n$, and

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 is an $n \times n$ nonsingular matrix, and A_2 an $(m-n)\times n$ arbitrary matrix.

a). What is the pseudo-inverse A^+ of A such that $A^+A = I_n$? Express it explicitly in A_1 and A_2.
b). Prove that $\|A^+\|_2 \le \|A_1^{-1}\|_2$.

6. (10 points) Let $A \in \mathbf{C}^{m\times m}$ with $rank(A) = r$. Suppose and SVD of A is given by $A = U\Sigma V^\star$, where $\mathbf{u}_1, ..., \mathbf{u}_m$ denote the columns of U and $\mathbf{v}_1, ..., \mathbf{v}_m$ denote the columns of V. Prove that $\langle \mathbf{v}_{r+1}, ..., \mathbf{v}_m \rangle = null(A)$.

7. (10 points) Problem 33.2 (Page 255) of the textbook.

8. (5 points) Problem 36.1 (Page 283) of the textbook.

9. (10 points) Let $f: R \to R$ be twice continuously differentiable for all x in the neighborhood $\{x \in R: |x-\xi| < r\}$ of a simple zero ξ of f such that $f(\xi) = 0$. Consider the two-step Newton method:

$$y_k = x_k - f(x_k)/f'(x_k), \qquad x_{k+1} = y_k - f(y_k)/f'(x_k).$$

a). Show that if the method converges, then

$$\lim_{k\to\infty} \frac{x_{k+1}-\xi}{(y_k-\xi)(x_k-\xi)} = \frac{f''(\xi)}{f'(\xi)}.$$

b). The convergence is cubic:

$$\lim_{k\to\infty} \frac{x_{k+1}-\xi}{(x_k-\xi)^3} = \frac{1}{2}\left(\frac{f''(\xi)}{f'(\xi)}\right)^2.$$

10. (30 points) Matlab project.