Lecture 26
Numerical Solutions of Nonlinear Systems of Equations

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MATH 562 Numerical Analysis II

Systems of Nonlinear Equations

A system of nonlinear equations:

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, \ldots, x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
$$

or $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{F}$ maps $\mathbf{R}^n$ to $\mathbf{R}^n$ as

$$
\mathbf{F}(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}
$$

with coordinate functions $f_1, \ldots, f_n$.

Limits and Continuity

Definition

Let f be defined on $D \subset \mathbf{R}^n$ and mapping into $\mathbf{R}$. Then f has the limit L at $\mathbf{x}_0$:

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L,$$

if, given any $\epsilon > 0$, a $\delta > 0$ exists with
$|f(\mathbf{x}) - L| < \epsilon,$ whenever $\mathbf{x} \in D$ and $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$

Definition

Let f be a function from $D \subset \mathbf{R}^n$ to $\mathbf{R}$. Then f is continuous at $\mathbf{x}_0 \in D$ if $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ exists and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

Also, f is continuous on a set D if f is continuous at every point of D: $f \in C(D)$.

Limits and Continuity

Definition

Let $\mathbf{F}$ be a function from $D \subset \mathbf{R}^n$ to $\mathbf{R}^n$:

$$\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^t$$

where f_i is a mapping from $\mathbf{R}^n$ to $\mathbf{R}$. Define

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{F}(\mathbf{x}) = \mathbf{L} = (L_1, L_2, \ldots, L_n)^t$$

if and only if $\lim_{\mathbf{x} \to \mathbf{x}_0} f_i(\mathbf{x}) = L_i$, for $i = 1, 2, \ldots, n$.

$\mathbf{F}$ is continuous at $\mathbf{x}_0 \in D$ if $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{F}(\mathbf{x})$ exists and $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}_0)$.
$\mathbf{F}$ is continuous on a set D if $\mathbf{F}$ is continuous at each $\mathbf{x}$ in D, or $\mathbf{F} \in C(D)$.

Limits and Continuity

Theorem

Let f be a function from $D \subset \mathbf{R}^n$ into $\mathbf{R}$ and $\mathbf{x}_0 \in D$. Suppose all partial derivatives of f exist and $\delta > 0$, $K > 0$ exist so that whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in D$,

$$\left|\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_j}\right| \leqslant K, \quad \text{for } j = 1, 2, \ldots, n$$

Then f is continuous at $\mathbf{x}_0$.

Fixed Points in $\mathbf{R}^n$

Definition

G from $D \subset \mathbf{R}^n$ to $\mathbf{R}^n$ has a fixed point at $\mathbf{p} \in D$ if $G(\mathbf{p}) = \mathbf{p}$.

Theorem

Let $D = \{(x_1, \ldots, x_n)^t | a_i \leqslant x_i \leqslant b_i,$ *for* $i = 1, \ldots, n\}$. *Suppose* **G** *is continuous from* $D \subset \mathbf{R}^n$ *into* $\mathbf{R}^n$ *and* $\mathbf{G}(\mathbf{x}) \in D$ *whenever* $\mathbf{x} \in D$. *Then* **G** *has a fixed point in* D. *Moreover, suppose all components of* **G** *have continuous partial derivatives and* $K < 1$ *exists with* $\frac{\partial g_i(\mathbf{x})}{\partial x_j} \leqslant \frac{K}{n},$ *whenever* $\mathbf{x} \in D$ *for* $j = 1, \ldots, n$ *and each* g_i. *Then* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *defined by an* $\mathbf{x}^{(0)}$ *in* D *and* $\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)})$ *for* $k \geqslant 1$ *converges to the unique fixed point* $\mathbf{p} \in D$ *with*

$$\|\mathbf{x}^{(k)} - \mathbf{p}\|_\infty \leqslant \frac{K^k}{1-K} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty$$

Quadratically Convergent Fixed Point Iterations

Find matrix $A(\mathbf{x})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{x} - A(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$ gives quadratic convergence to the solution of $\mathbf{F}(\mathbf{x})$.

Theorem

Let $\mathbf{p}$ *be a solution of* $\mathbf{G}(\mathbf{x}) = \mathbf{x}$. *Suppose* $\delta > 0$ *exists with*

- $\partial g_i/\partial x_j$ *is continuous on* $N_\delta = \{\mathbf{x} | \|\mathbf{x} - \mathbf{p}\| < \delta\}$ *for all* i, j.
- $\partial^2 g_i/\partial x_j \partial x_k$ *is continuous,* $|\partial^2 g_i/\partial x_j \partial x_k| \leqslant M$ *from some* M *whenever* $\mathbf{x} \in N_\delta$, *for all* i, j, k.
- $\partial g_i(\mathbf{p})/\partial x_j = 0$ *for all* i, j.

Then $\hat{\delta} \leqslant \delta$ *exists such that* $\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)})$ *converges quadratically to* $\mathbf{p}$ *for any* $\mathbf{x}^{(0)}$ *with* $\|\mathbf{x}^{(0)} - \mathbf{p}\| \leqslant \hat{\delta}$. *Moreover,*

$$\|\mathbf{x}^{(k)} - \mathbf{p}\|_\infty \leqslant \frac{n^2 M}{2} \|\mathbf{x}^{(k-1)} - \mathbf{p}\|_\infty^2, \quad \text{for } k \geqslant 1$$

Newton's Method

Newton's method for nonlinear systems:

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} - (J(\mathbf{x}^{(k-1)}))^{-1}\mathbf{F}(\mathbf{x}^{(k-1)})$$

with Jacobian matrix

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$