Caleb Logemann
MATH 562 Numerical Analysis II
Homework 1

1. Problem 1.1 Let B be a 4×4 matrix to which the following operations are applied in the given order.

   1. double column 1
   2. halve row 3
   3. add row 3 to row 1
   4. interchange columns 1 and 4
   5. subtract row 2 from each other rows
   6. replace column 4 by column 3
   7. delete column 1

   The result can be written as a product of 8 matrices one of which is B.

   (a) What are the other 7 matrices and what order do they appear in the matrix? The matrix that doubles column 1 is
   $$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   when right multiplied. The following matrix halves row 3 when left multiplied.
   $$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   The following matrix adds row 3 to the row 1 when left multiplied.
   $$E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   The following matrix interchanges columns 1 and 4 when right multiplied.
   $$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The following matrix subtracts row 2 from every other row, when left multiplied.

$$G = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The following matrix replaces column 4 with column 3 when right multiplied.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrix deletes column 1 when right multiplied.

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix product is given by $GEDBCFHI$, where the matrices are given above.

(b) The result can also be written as a product ABC what are A and C?

In this case A and C are given by the product of the matrices to the left and the right of B in the part (a). Therefore

$$A = \begin{bmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0.5 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Show that if a matrix $A \in \mathbb{C}^{m\times m}$ is upper triangular and unitary, then it is diagonal.

Proof. Let $A \in \mathbb{C}^{m\times m}$ be unitary and upper triangular. I will denote A be its entries a_{ij} and by its columns as vectors $\mathbf{a_j}$. The fact that A is upper triangular implies that $a_{ij} = 0$, for $i > j$. Since A is unitary, $AA^* = I$ or $\mathbf{a}_j^*\mathbf{a}_i = 0$ for $i \neq j$ and $\mathbf{a}_j^*\mathbf{a}_i = 1$ for $i = j$. We will procede by induction. Consider $\mathbf{a}_1^*\mathbf{a}_j = 0$ for $j = 2, \ldots, m$.

However since A is upper triangular $\mathbf{a}_1 = \mathbf{e}_1$, thus $\mathbf{a}_1^*\mathbf{a}_j = a_{1j}$. Therefore $a_{1j} = 0$ for $j = 2, \ldots, m$. Now assume that $a_{ij} = 0$ for $i = 1, \ldots, k-1$ and $j = i+1, \ldots, m$. Consider $\mathbf{a}_k^*\mathbf{a}_j = 0$ for $j = k+1, \ldots, m$. However we know that $a_{ik} = 0$ for $i < k$ and $a_{ik} = 0$ for $i > k$, therefore $\mathbf{a}_k = \mathbf{e}_k$. This implies that $\mathbf{a}_k^*\mathbf{a}_j = a_{kj}$. We can now conclude that $a_{kj} = 0$ for $j = k+1, \ldots, m$. Now mathematical induction implies that $a_{ij} = 0$ for $i = 1, \ldots, m$ and $j = i+1, \ldots, m$. This means that all entries above the diagonal are zero, and thus A is a diagonal matrix. Essential each row can be shown to be zero except on the diagonal by noting that the previous row is zero everywhere except the diagonal. $\square$

3. Let $A \in \mathbb{C}^{m\times m}$ be Hermitian, that is $A = A^*$. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} \in \mathbb{C}^{m\times m}$ and $\lambda \in \mathbb{C}$, so $\mathbf{x}$ is an eigenvector and λ is an eigenvalue.

   (a) Prove that λ must be real.

   Proof. Consider $\mathbf{x}^*A\mathbf{x}$.

$$\begin{aligned}
(\mathbf{x}^*A)\mathbf{x} &= \mathbf{x}^*(A\mathbf{x}) \\
(A^*\mathbf{x})^*\mathbf{x} &= \mathbf{x}^*(A\mathbf{x}) \\
(\lambda\mathbf{x})^*\mathbf{x} &= \mathbf{x}^*(\lambda\mathbf{x}) \\
\overline{\lambda}\mathbf{x}^*\mathbf{x} &= \lambda\mathbf{x}^*\mathbf{x} \\
\overline{\lambda} &= \lambda
\end{aligned}$$

   Since $\overline{\lambda} = \lambda$, λ must be real. $\square$

   (b) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to different eigenvalues, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

   Proof. Let λ_x and λ_y be the eigenvalues corresponding to $\mathbf{x}$ and $\mathbf{y}$ respectively, that is $A\mathbf{x} = \lambda_x\mathbf{x}$, $A\mathbf{y} = \lambda_y\mathbf{y}$, and $\lambda_x \neq \lambda_y$. Since A is Hermitian, we have just shown that $\lambda_x, \lambda_y \in \mathbb{R}$. Also because A is Hermitian

$$\begin{aligned}
A &= A^* \\
\mathbf{x}^*A &= \mathbf{x}^*A^* \\
\mathbf{x}^*A &= (A\mathbf{x})^*.
\end{aligned}$$

   Since $\mathbf{x}$ is an eigenvector

$$\mathbf{x}^*A = (\lambda_x\mathbf{x})^*$$

Since $\lambda_x \in \mathbb{R}$

$$\begin{aligned} \mathbf{x}^* A &= \lambda_x \mathbf{x}^* \\ \mathbf{x}^* A\mathbf{y} &= \lambda_x \mathbf{x}^* \mathbf{y} \\ \mathbf{x}^* \lambda_y \mathbf{y} &= \lambda_x \mathbf{x}^* \mathbf{y} \\ (\lambda_y - \lambda_x)\mathbf{x}^* \mathbf{y} &= 0 \end{aligned}$$

Since $\lambda_x \neq \lambda_y$, $\lambda_y - \lambda_x \neq 0$.

$$\mathbf{x}^* \mathbf{y} = 0$$

Therefore $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. □

4. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, then $A = I + \mathbf{u}\mathbf{v}^T$ is called a rank one perturbation of the identity.

(a) Show that if A is invertible, then it's inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$, and give an expression for the scalar α.
First let us find an expression for α.

$$\begin{aligned} I &= AA^{-1} \\ &= \left(I + \mathbf{u}\mathbf{v}^T\right)\left(I + \alpha\mathbf{u}\mathbf{v}^T\right) \\ &= I + \alpha\mathbf{u}\mathbf{v}^T + \mathbf{u}\mathbf{v}^T + \alpha\mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T \\ &= I + \left(\alpha + 1 + \alpha\mathbf{v}^T\mathbf{u}\right)\mathbf{u}\mathbf{v}^T \end{aligned}$$

In order for this equality to be true

$$\begin{aligned} 0 &= \alpha + 1 + \alpha\mathbf{v}^T\mathbf{u} \\ \alpha &= -\frac{1}{1 + \mathbf{v}^T\mathbf{u}} \end{aligned}$$

Proof. If A is invertible then A^{-1} exists and is unique. Consider the matrix $B = I + \alpha\mathbf{u}\mathbf{v}^T$, where $\alpha = -\frac{1}{1+\mathbf{v}^T\mathbf{u}}$, then the following can be stated.

$$\begin{aligned} AB &= \left(I + \mathbf{u}\mathbf{v}^T\right)\left(I + \alpha\mathbf{u}\mathbf{v}^T\right) \\ &= I + \alpha\mathbf{u}\mathbf{v}^T + \mathbf{u}\mathbf{v}^T + \alpha\mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T \\ &= I + \left(\alpha + 1 + \alpha\mathbf{v}^T\mathbf{u}\right)\mathbf{u}\mathbf{v}^T \\ &= I + \left(\left(1 + \mathbf{v}^T\mathbf{u}\right)\left(-\frac{1}{1 + \mathbf{v}^T\mathbf{u}}\right) + 1\right)\mathbf{u}\mathbf{v}^T \\ &= I + (-1 + 1)\mathbf{u}\mathbf{v}^T \\ &= I \end{aligned}$$

Similarly it can be shown that $BA = I$. Thus B is an inverse of A, and since the inverse is unique $B = I + \alpha \mathbf{u}\mathbf{v}^T$ is the inverse of A. Therefore if A is invertible it's inverse will be of the form $I + \alpha \mathbf{u}\mathbf{v}^T$. $\square$

(b) If A is not invertible than $\mathbf{v}^T\mathbf{u} = -1$, otherwise $I + \alpha \mathbf{u}\mathbf{v}^T$, where $\alpha = -\frac{1}{1+\mathbf{v}^T\mathbf{u}}$ would be an inverse. In this case $\text{null}(A) = \text{span}\{\mathbf{u}\}$. Note that

$$\begin{aligned} A(c\mathbf{u}) &= \left(I + \mathbf{u}\mathbf{v}^T\right)(c\mathbf{u}) \\ &= c\mathbf{u} + c\mathbf{u}\mathbf{v}^T\mathbf{u} \\ &= c\mathbf{u} - c\mathbf{u} \\ &= \mathbf{0} \end{aligned}$$

for any $c \in \mathbb{R}$. Thus any multiple of $\mathbf{u}$ is in $\text{null}(A)$.

5. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the corresponding induced norm on $\mathbb{C}^{m\times m}$, so that $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$. Show that $\rho(A) \leq \|A\|$, where $\rho(A) = \max|\lambda|$: where λ is an eigenvalue of A is the spectral radius of A.

Proof. Let $\mathbf{x}$ be an eigenvector of A with corresponding eigenvalue λ, such that $|\lambda| = \rho(A)$, that is $\mathbf{x}$ is the eigenvector corresponding to the eigenvalue whose magnitude is the spectral radius. Consider $\|A\mathbf{x}\|$

$$\begin{aligned} \|A\mathbf{x}\| &= \|\lambda\mathbf{x}\| \\ &= |\lambda|\|\mathbf{x}\| \\ &= \rho(A)\|\mathbf{x}\| \end{aligned}$$

Also note that $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$, therefore

$$\begin{aligned} \rho(A)\|\mathbf{x}\| &\leq \|A\|\|\mathbf{x}\| \\ \rho(A) &\leq \|A\| \end{aligned}$$

Since $\|\mathbf{x}\| \neq 0$, because $\mathbf{x}$ is an eigenvector. $\square$

6. Let $\theta \in (0, 2\pi)$ and define the matrix $Q \in \mathbb{R}^{2\times 2}$ by

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Show that $\mathbf{y} = Q\mathbf{x}$ is the vector obtained by rotating vector $\mathbf{x}$ by θ radians.

Proof. Let $\mathbf{x} \in \mathbb{R}^2$, have entries x_1 and x_2, then the following statements are true $x_1 = r\cos(\phi)$ and $x_2 = r\sin(\phi)$, where $r = \sqrt{x_1^2 + x_2^2}$ is the radius or length of the vector and $\phi = \arctan\left(\frac{x_1}{x_2}\right)$ is the angle between the vector and the x-axis. If the vector is in the third or fourth quadrants the angle must be shifted as necessary. Now the vector $\mathbf{y}$ can be computed as follows.

$$\begin{aligned}
\mathbf{y} &= Q\mathbf{x} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r\cos(\phi) \\ r\sin(\phi) \end{bmatrix} \\
&= \begin{bmatrix} r(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) \\ r(\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)) \end{bmatrix}
\end{aligned}$$

Using some trigonomtric identities

$$\mathbf{y} = \begin{bmatrix} r\cos(\theta + \phi) \\ r\sin(\theta + \phi) \end{bmatrix}$$

However this is just vector $\mathbf{x}$ except rotated by θ radians. It has the same length or radius, but the angle was changed by θ. $\square$

7. Below I have shown my function for finding the SVD decomposition of a 2×2 real matrix. The code does use the circle_image and arrow functions provided in class.

```
A = [1, 2; 0, 2];
figure;
[U, S, V] = SVD2D(A);

A = [1, 1; 2, 2];
figure;
[U, S, V] = SVD2D(A);
```

```
function [U, S, V] = SVD2D(A)
    % create possible unit vectors v1
    theta = 0:.01:2*pi;
    v = [cos(theta); sin(theta)];

    Av = A*v;
    normAv = arrayfun(@(i) norm(Av(:, i), 2), 1:length(theta));
    [s1, i] = max(normAv);

    % v_i maximizes 2-norm of Av
    v1 = v(:, i);
```

```
    % s1*u1 = A*v1
    Av1 = A*v1;
    s1 = norm(Av1,2);
    u1 = Av1/s1;

    % v2 should be orthogonal to v1
    % generate random vector
    w = rand(2,1);
    v2 = w - (w'*v1)*v1;
    v2 = v2/norm(v2);

    % s2*u2 is A*v2
    Av2 = A*v2;
    s2 = norm(Av2,2);
    u2 = Av2/s2;

    U = [u1, u2];
    V = [v1, v2];
    S = diag([s1, s2]);

    % now plot
    circle_image(A);
    % plot v vectors
    arrow([0, 0], v1, 'b');
    arrow([0, 0], v2, 'b');
    % plot u vectors
    arrow([0, 0], s1*u1, 'r');
    arrow([0, 0], s2*u2, 'r');
end
```

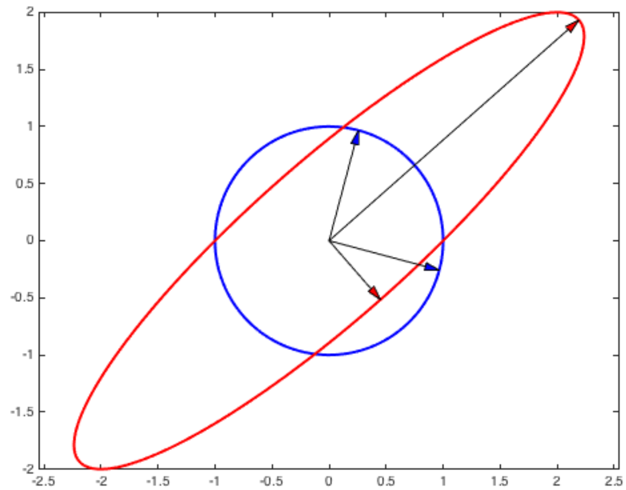
This code produces the following two figures for

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

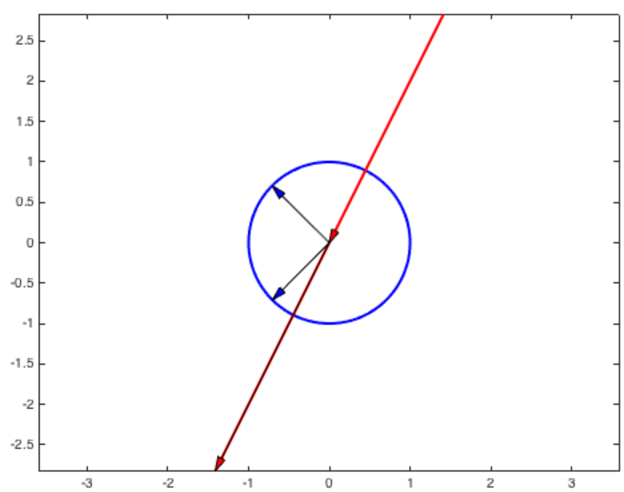
and

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

respectively.



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5



2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3