Caleb Logemann
MATH 562 Numerical Analysis II
Homework 2

1. Let P be an orthogonal projector.

   (a) Prove that $I - 2P$ is unitary.

   Proof. Let P be an orthogonal projector, that is $P^2 = P$ and $P = P^*$. Consider $(I - 2P)^*(I - 2P)$.

   $$\begin{aligned}(I - 2P)^*(I - 2P) &= (I - 2P^*)(I - 2P)\\ &= I - 2P - 2P^* + 4P^*P\end{aligned}$$

   Since $P^* = P$

   $$I - 2P - 2P^* + 4P^*P = I - 4P + 4P^2$$

   Since $P^2 = P$.

   $$I - 4P + 4P^2 = I$$

   Therefore since $(I - 2P)^*(I - 2P) = I$, $I - 2P$ is unitary. □

   (b) Describe the action of $I - 2P$ geometrically.
   Since P is a projector $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto the range of P. Therefore $(I - 2P)\mathbf{x} = \mathbf{x} - 2P\mathbf{x}$, can be thought of as the difference between $\mathbf{x}$ and $2P\mathbf{x}$. Let us first consider the difference $\mathbf{x} - P\mathbf{x}$. This is the projection of $\mathbf{x}$ onto the space orthogonal to the space of P. Since $\mathbf{x} - 2P\mathbf{x} = \mathbf{x} - P\mathbf{x} - P\mathbf{x}$, we can think of this as subtracting the projection again. This results in a reflection over the subspace orthogonal to P. This is in fact a Householder reflector over the subspace orthogonal to the range P.

2. Suppose that $A \in \mathbb{C}^{m \times n}$ with $m \geq n$.

   (a) Show that A^*A is nonsingular if and only if A has full rank.

   Proof. Let $A \in \mathbb{C}^{m \times n}$ with $m \geq n$. Suppose that A^*A is nonsingular, this implies that $\text{null}(A^*A) = \{\mathbf{0}\}$. Now consider $\mathbf{x} \in \text{null}(A)$, that is $A\mathbf{x} = \mathbf{0}$. By right multiplying by A^*, we conclude that $A^*A\mathbf{x} = \mathbf{0}$. This implies that $\mathbf{x} \in \text{null}(A^*A)$ and therefore $\mathbf{x} = \mathbf{0}$. This implies that $\text{null}(A) = \{\mathbf{0}\}$, that is A has full rank.

Now suppose that A full rank, that is $\text{null}(A) = \mathbf{0}$. Now consider $\mathbf{x} \in \text{null}(A^*A)$, that is $A^*A\mathbf{x} = \mathbf{0}$. By right multiplying by $\mathbf{x}^*$, we note that

$$\begin{aligned}\mathbf{x}^*A^*A\mathbf{x} &= 0\\ (A\mathbf{x})^*A\mathbf{x} &= 0\\ \|A\mathbf{x}\|_2^2 &= 0.\end{aligned}$$

The only way the for the 2-norm of a vector to be zero is if that vector is the zero vector, therefore $A\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x} \in \text{null}(A)$ and $\mathbf{x} = \mathbf{0}$. This implies that $\text{null}(A^*A) = \{\mathbf{0}\}$, which implies that A^*A is nonsingular. □

(b) Show that if A has full rank, then $P = A(A^*A)^{-1}A^*$ is an orthogonal projector onto the range of A.

Proof. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ have full rank. Define $P = A(A^*A)^{-1}A^*$. Consider P^2.

$$\begin{aligned}P^2 &= A(A^*A)^{-1}A^*A(A^*A)^{-1}A^*\\ &= A(A^*A)^{-1}(A^*A)(A^*A)^{-1}A^*\\ &= A(A^*A)^{-1}A^*\\ &= P\end{aligned}$$

Therefore P is a projector. Now consider P^*

$$\begin{aligned}P^* &= \Big(A(A^*A)^{-1}A^*\Big)^*\\ &= A\Big(A(A^*A)^{-1}\Big)^*\\ &= A\Big((A^*A)^{-1}\Big)^*A^*\end{aligned}$$

Notice that A^*A is Hermitian, and that it is known that the inverse of a Hermitian matrix is also Hermitian. This implies that $\Big((A^*A)^{-1}\Big)^* = (A^*A)^{-1}$, so

$$\begin{aligned}P^* &= A\Big((A^*A)^{-1}\Big)^*A^*\\ &= A(A^*A)^{-1}A^*\\ &= P\end{aligned}$$

We can now conclude that P is an orthogonal projector. Finally we need to show that P projects onto the range of A, that is for every vector, $\mathbf{x} \in \text{range}(A)$, there exists a vector $\mathbf{y}$ such that $P\mathbf{y} = \mathbf{x}$. Let $\mathbf{x} \in \text{range}(A)$, then there exists $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{x}$. Since (A^*A) is nonsingular $(A^*A)^{-1}$ exists. Therefore $A\mathbf{z} = A(A^*A)^{-1}(A^*A)\mathbf{z}$. This can be rewritten as $A(A^*A)^{-1}A^*\mathbf{x} = P\mathbf{x}$. Therefore $P\mathbf{x} = \mathbf{x}$ and $\mathbf{x} \in \text{range}(P)$. Therefore P projects onto the range of A. $\square$

3. Suppose that $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, and let $A = \hat{Q}\hat{R}$ be the reduced QR factorization of A.

   (a) Show that A has full rank if and only if all the diagonal entries of $\hat{R}$ are nonzero.

   Proof. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ have full rank and let $\hat{Q}\hat{R} = A$ be the reduced QR factorization. Since A has full rank, $\text{null}(A) = \{\mathbf{0}\}$. Let $\mathbf{x} \in \text{null}\left(\hat{R}\right)$, then $\hat{R}\mathbf{x} = \mathbf{0}$. Right multiplying by $\hat{Q}$ results in $\hat{Q}\hat{R}\mathbf{x} = A\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x} = \mathbf{0}$, this implies that $\hat{R}$ has full rank. Since $\hat{R}$ is upper triangular and full rank, the diagonal entries of $\hat{R}$ must all be nonzero. Let $\hat{Q}\hat{R}\mathbf{x} = \mathbf{0}$ Now assume that all the diagonal entries of $\hat{R}$ are nonzero, this implies that $\hat{R}$ has full rank. Therefore $\text{null}\left(\hat{R}\right) = \{\mathbf{0}\}$. Let $\mathbf{x} \in \text{null}(A)$, thus $A\mathbf{x} = \mathbf{0}$. Using the reduced QR factorization implies that $\hat{Q}\hat{R}\mathbf{x} = \mathbf{0}$. Right multiplying by $\hat{Q}^*$ results in

$$\hat{Q}^*\hat{Q}\hat{R}\mathbf{x} = \hat{Q}^*\mathbf{0}$$
$$\hat{R}\mathbf{x} = \mathbf{0}$$

   This implies that $\mathbf{x} \in \text{null}\left(\hat{R}\right)$ and therefore $\mathbf{x} = \mathbf{0}$. Therefore $\text{null}(A) = \{\mathbf{0}\}$, and A has full rank. $\square$

   (b) Suppose that $\hat{R}$ has k nonzero diagonal entries and $n - k$ zero diagonal entries. What does that imply about the rank of A. Justify your answer.
   This implies that $\text{rank}(A) \geq k$. We have shown in part (a) that if A has full rank then $\text{rank}(A) = k = n$. Consider the example

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

   with QR factorization

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$R = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

   In this case $k = 0$ and $\text{rank}(A) = 2$, so $\text{rank}(A) \geq k$.

4. Let A be an $(m \times n)$ matrix. Determine the exact number of floating point additions, subtractions, mutliplications, and divisions involved in computing the reduced QR factorization of A using algorithm 8.1 on page 58.

   The first for loop in algorithm 8.1 contains no floating point operations, and thus does not contribute to the overall flop count. The second for loop does contain floating point operations. This for loop has an inner nested loop, I will first count the number of operations in this inner loop. The first operation in this inner loop is a multiplication of a row vector and column vector each of length m. Therefore this operation includes m multiplications and $m-1$ additions. The second operation $\mathbf{v}_j - r_{ij}\mathbf{q}_i$ is the multiplication of one vector by a constant, m operations, and the subtraction of two vectors, another m operations. Therefore each iteration of this loop includes $m + m - 1 + m + m = 4m - 1$ operations. This loop iterates from $j = i + 1$ to n, so the total number of operations for this loop can be described as the following sum.

$$\sum_{j=i+1}^{n} (4m-1) = (4m-1)n - i$$

   The outer for loop first compute $\|\mathbf{v}_i\|$, assuming this is the two norm then this will take m multiplications, $m-1$ additions, and 1 squareroot operation. Therefore the entire norm computations requires $2m$ flops. Even if this does not refer to the 2-norm, the number of flops will be similar. The next computation divides a vector by a constant, this requires m divisions. Therefore the one iteration of the outer for loop requires $3m$ flops and the number of flops in the inner loop. This loop iterates from $i = 1$ to n, so the total flop count can be expressed as follows.

$$\begin{aligned}
\sum_{i=1}^{n} (3m + (4m-1)(n-i)) &= \sum_{i=1}^{n} (3m) + (4m-1)\sum_{i=1}^{n} (n-i) \\
&= 3mn + (4m-1)\left(\sum_{i=1}^{n} (n) - \sum_{i=1}^{n} (i)\right) \\
&= 3mn + (4m-1)\left(n^2 - \frac{n(n+1)}{2}\right) \\
&= 3mn + (4m-1)\left(\frac{n^2-n}{2}\right) \\
&= 3mn + (4m-1)\left(\frac{n^2-n}{2}\right) \\
&= 2mn^2 + mn - \frac{1}{2}n^2 + \frac{1}{2}n
\end{aligned}$$

   Thus the total number of flops for algorithm 8.1 is $2mn^2 + mn - \frac{1}{2}n^2 + \frac{1}{2}n$.

5. Let $F = I - 2\mathbf{u}\mathbf{u}^T$ be a Householder reflector on $\mathbb{R}^m$. Deterimine the eigenvalues, the determinant, and the singular values of F. Give a geometric argument supporting your algebraic eigenvalue computation.

   First consider $F\mathbf{u}$

$$
\begin{aligned}
F\mathbf{u} &= \left(I - 2\mathbf{u}\mathbf{u}^T\right)\mathbf{u} \\
&= \mathbf{u} - 2\mathbf{u}^T\mathbf{u}\mathbf{u} \\
&= \left(1 - 2\mathbf{u}^T\mathbf{u}\right)\mathbf{u}
\end{aligned}
$$

   Therefore $1 - 2\mathbf{u}^T\mathbf{u}$ is an eigenvalue, when $\|\mathbf{u}\|_2 = 1$, then this eigenvalue is -1. Geometrically this refers to the vectors that are multiples of $\mathbf{u}$, which defines the reflector. When F is applied to these vectors, they are simply reflected over the hyperplane orthogonal to $\mathbf{u}$. This is simply -1 times the original vector.

   Now consider $\mathbf{v}$, such that $\mathbf{v}$ is orthogonal to $\mathbf{u}$, that is $\mathbf{u}^T\mathbf{v} = 0$. In this case

$$
\begin{aligned}
F\mathbf{v} &= \left(I - 2\mathbf{u}\mathbf{u}^T\right)\mathbf{v} \\
&= \mathbf{v} - 2\mathbf{u}^T\mathbf{v}\mathbf{u} \\
&= \mathbf{v} - 2 \times 0 \times \mathbf{u} \\
&= \mathbf{v}
\end{aligned}
$$

   Thus 1 is also an eigenvalue of F. In fact this eigenvalue has multiplicity $m - 1$, because there are $m - 1$ linearly indepenent vectors orthogonal to $\mathbf{u}$. These vectors $\mathbf{v}$ lay in the hyperplane orthogonal to $\mathbf{u}$, thus when F is applied to these vectors they are reflecte onto themselves. These multiplicities account for all of the eigenvalues of F.

   The eigenvalues of F are thus -1 with multiplicity 1 and 1 with multiplicity $m - 1$. The determinant of a matrix is the prouct of its eigenvalues, therefore the determinant of F is -1.

   The singular values of F are the squareroots of the eigenvalues of F^*F. Since F is Hermitian $F^*F = F^2$, and thus the eigenvalues of F^*F are the eigenvalues of F squared. Therefore the eigenvalues of F^*F are all 1. Now the singular values of F are all $\sqrt{1} = 1$.

6. (a) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$ satisfy i) $\mathbf{x} \neq \mathbf{y}$, ii) $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$, iii) $\mathbf{x}^*\mathbf{y} \in \mathbb{R}$. Show that there is a reflector F (satisfying $F^* = F$ and $F^2 = I$) such that $F\mathbf{x} = \mathbf{y}$.

Let $F = I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{(\mathbf{y}-\mathbf{x})^*(\mathbf{y}-\mathbf{x})}$. Since $\mathbf{x}^*\mathbf{y} \in \mathbb{R}$, then $\mathbf{x}^{\bar{*}}\mathbf{y} = \mathbf{y}^*\mathbf{x} \in \mathbb{R}$ and $\mathbf{x}^*\mathbf{y} = \mathbf{x}^*\mathbf{y}$. This allows for F to be simplified as follows

$$
\begin{aligned}
F &= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{(\mathbf{y}-\mathbf{x})^*(\mathbf{y}-\mathbf{x})} \\
&= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\mathbf{y}^*\mathbf{y} - \mathbf{y}^*\mathbf{x} - \mathbf{x}^*\mathbf{y} + \mathbf{x}^*\mathbf{x}} \\
&= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{2\|\mathbf{x}\|_2^2 - 2\mathbf{x}^*\mathbf{y}} \\
&= I - \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}
\end{aligned}
$$

Now consider F^*

$$
\begin{aligned}
F^* &= \left(I - \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}\right)^* \\
&= I^* - \frac{((\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*)^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= I - \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= F
\end{aligned}
$$

Also consider F^2

$$
\begin{aligned}
F^2 &= \left(I - \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}\right)^2 \\
&= I^2 - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} + \left(\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}\right)^2 \\
&= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} + \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\left(\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}\right)^2} \\
&= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} + 2\left(\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}\right)\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\left(\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}\right)^2} \\
&= I - 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} + 2\frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= I
\end{aligned}
$$

Thus F is a reflector. Finally we can examine $F\mathbf{x}$.

$$\begin{aligned}
F\mathbf{x} &= \left(I - \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})^*}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}\right)\mathbf{x} \\
&= \mathbf{x} - (\mathbf{y}-\mathbf{x})\frac{(\mathbf{y}-\mathbf{x})^*\mathbf{x}}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= \mathbf{x} - (\mathbf{y}-\mathbf{x})\frac{\mathbf{y}^*\mathbf{x} - \mathbf{x}^*\mathbf{x}}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= \mathbf{x} + (\mathbf{y}-\mathbf{x})\frac{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}}{\|\mathbf{x}\|_2^2 - \mathbf{x}^*\mathbf{y}} \\
&= \mathbf{x} + \mathbf{y} - \mathbf{x} \\
&= \mathbf{y}
\end{aligned}$$

(b) Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{x} \neq 0$. The polar form of the first component of $\mathbf{x}$ is $x_1 = re^{i\theta}$. Set $\mathbf{y} = \|\mathbf{x}\|_2 e^{i\theta}\mathbf{e}_1$. Assuming $\mathbf{x}$ is not a mutliple of $\mathbf{e}_1$, show that $\mathbf{x}, \mathbf{y}$ satisfy properties i), ii), and iii) above.

Since $\mathbf{x}$ is not a multiple of $\mathbf{e}_1$ there exists an index $i \neq 1$ such that $x_i \neq 0$. However $y_i = 0$ for all $i \neq 1$, therefore $\mathbf{x} \neq \mathbf{y}$ and property i) is satisfied. Consider $\|\mathbf{y}\|_2$

$$\begin{aligned}
\|\mathbf{y}\|_2 &= \left\| \|\mathbf{x}\|_2 e^{i\theta}\mathbf{e}_1 \right\|_2 \\
&= \|\mathbf{x}\|_2 \left| e^{i\theta} \right| \|\mathbf{e}_1\|_2 \\
&= \|\mathbf{x}\|_2 \times 1 \times 1 \\
&= \|\mathbf{x}\|_2.
\end{aligned}$$

Therefore property ii) is satisfied. Now consider $\mathbf{x}^*\mathbf{y}$. Since $\mathbf{y}$ is a multiple $\mathbf{e}_1$,

$$\begin{aligned}
\mathbf{x}^*\mathbf{y} &= \bar{x_1} \times y_1 \\
&= r\bar{e^{i\theta}}\|\mathbf{x}\|_2 e^{i\theta} \\
&= re^{-i\theta}\|\mathbf{x}\|_2 e^{i\theta} \\
&= r\|\mathbf{x}\|_2 e^0 \\
&= r\|\mathbf{x}\|_2 \in \mathbb{R}
\end{aligned}$$

Therefore property iii) is satisfied.

7. Write a MATLAB function $[W,R] = \text{house}(A)$ that takes as input a $(m \times n)$ matrix A and returns an implicit representation of the full QR factorization of A. The matrix W should be the lower triangular matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$

where $\mathbf{v}_k = \|\mathbf{x}_k\|_2 \mathbf{e}_1$âĹŠ$\mathbf{x}_k$ is used to define the Householder reflector F_k at the k-th stage of the process. R should be the triangular factor in the factorization. Some test matrices will be supplied next week. You should turn in your code and the output from the test matrices for this problem.

My function and the code to test the two test matrices are shown below.

```
function [W, R] = HouseholderQR(A)
    p = inputParser;
    p.addRequired('A', @Utils.isMatrix);
    p.parse(A);

    [m, n] = size(A);
    W = zeros(m, n);
    for k=1:n
        x = A(k:m, k);
        %vk = sign(x(1))*norm(x,2)*eye(m-k+1, 1) + x;
        vk = norm(x,2)*eye(m-k+1, 1) + x;
        vk = vk/norm(vk, 2);
        A(k:m, k:n) = A(k:m, k:n) - 2*vk*(vk'*A(k:m, k:n));
        W(k:m, k) = vk;
    end
    R = A;
```

```
A = [1, 2, 3; 4, 5, 6; 7, 8, 7; 4, 2, 3; 4, 2, 2];
[W, R] = HouseholderQR(A)

A = [.5, -2.5; -.5, 2.5; -.5, -7.5; -.5, -7.5];
[W, R] = HouseholderQR(A)
```

This code gives the following output

```
>> H02

W =

    0.7420         0         0
    0.2723    0.7866         0
    0.4765    0.1192    0.1988
    0.2723   -0.4284    0.9081
    0.2723   -0.4284   -0.3685


R =
```

```
   -9.8995   -9.4954   -9.6975
   -0.0000   -3.2919   -3.0129
   -0.0000   -0.0000   -1.9701
   -0.0000    0.0000         0
   -0.0000    0.0000         0


W =

    0.8660         0
   -0.2887    0.8165
   -0.2887   -0.4082
   -0.2887   -0.4082


R =

   -1.0000   -5.0000
   -0.0000  -10.0000
   -0.0000    0.0000
   -0.0000    0.0000
```