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MATH 565 Continuous Optimization
Midterm Exam

1. (a) Derive an expression for the gradient $\underline{\nabla f}$.

   In order to derive an expression for $\underline{\nabla f}$ we must compute all the partial derivatives $\dfrac{\partial f}{\partial x_k}$ and $\dfrac{\partial f}{\partial y_k}$ for all $1 \le k \le N$.

   In order to compute these partial derivatives, I will first rewrite f in a way that seperates the x_k and y_k terms.

$$f(\underline{x}) = \left((x_k - x_k)^2 + (y_k - y_k)^2 - d_{kk}^2\right)^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)^2$$
$$+ \sum_{\substack{i=1 \\ i \neq k}}^{N} \left((x_i - x_k)^2 + (y_i - y_k)^2 - d_{ik}^2\right)^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{\substack{i=1 \\ i \neq k}}^{N} \left((x_i - x_j)^2 + (y_i - y_j)^2 - d_{ik}^2\right)^2$$

   By noting that $(x_i - x_k)^2 = (x_k - x_i)^2$ and that $d_{ik} = d_{ki}$, this can be simplified to

$$f(\underline{x}) = d_{kk}^4 + 2\sum_{\substack{j=1 \\ j \neq k}}^{N} \left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} \sum_{\substack{i=1 \\ i \neq k}}^{N} \left((x_i - x_j)^2 + (y_i - y_j)^2 - d_{ik}^2\right)^2.$$

   Now the partial derivatives of this can be taken easily note that the first and last terms don't contain x_k or y_k. The partial derivatives are

$$\frac{\partial f}{\partial x_k} = 8\sum_{\substack{j=1 \\ j \neq k}}^{N} \left(\left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)(x_k - x_j)\right)$$
$$\frac{\partial f}{\partial y_k} = 8\sum_{\substack{j=1 \\ j \neq k}}^{N} \left(\left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)(y_k - y_j)\right)$$

   When $j = k$ the terms of these sums are zero, so these can also be expressed as

$$\frac{\partial f}{\partial x_k} = 8\sum_{j=1}^{N} \left(\left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)(x_k - x_j)\right)$$
$$\frac{\partial f}{\partial y_k} = 8\sum_{j=1}^{N} \left(\left((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\right)(y_k - y_j)\right).$$

Now the gradient of f can be written using these partial derivatives, as follows

$$\underline{\nabla f}(\underline{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial y_2} \\ \vdots \\ \frac{\partial f}{\partial x_N} \\ \frac{\partial f}{\partial y_N} \end{bmatrix}$$

(b) Derive an expression for the Hessian $\underline{\underline{\nabla^2 f}}$.

The Hessian for this function requires finding all possible partial second derivatives. There are several possible different partial second derivatives. They are $\frac{\partial^2 f}{\partial {x_k}^2}$, $\frac{\partial^2 f}{\partial {y_k}^2}$, $\frac{\partial^2 f}{\partial x_k \partial y_k}$, $\frac{\partial^2 f}{\partial x_k \partial x_i}$, $\frac{\partial^2 f}{\partial y_k \partial y_i}$, and $\frac{\partial^2 f}{\partial x_k \partial y_i}$, where $k \neq i$.

First I will compute $\frac{\partial^2 f}{\partial {x_k}^2}$ and $\frac{\partial^2 f}{\partial {y_k}^2}$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial {x_k}^2} &= \frac{\partial}{\partial x_k}\left(8\sum_{j=1}^{N}\Big((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\Big)(x_k - x_j)\right) \\
&= \frac{\partial}{\partial x_k}\left(8\sum_{j=1}^{N}(x_k - x_j)^3 + \Big((y_k - y_j)^2 - d_{kj}^2\Big)(x_k - x_j)\right) \\
&= 8\sum_{j=1}^{N}\frac{\partial}{\partial x_k}\Big((x_k - x_j)^3\Big) + \Big((y_k - y_j)^2 - d_{kj}^2\Big)\frac{\partial}{\partial x_k}((x_k - x_j)) \\
&= 8\sum_{j=1}^{N}3(x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2 \\
\frac{\partial^2 f}{\partial {y_k}^2} &= \frac{\partial}{\partial y_k}\left(8\sum_{j=1}^{N}\Big((x_k - x_j)^2 + (y_k - y_j)^2 - d_{kj}^2\Big)(y_k - y_j)\right) \\
&= 8\sum_{j=1}^{N}(x_k - x_j)^2 + 3(y_k - y_j)^2 - d_{kj}^2
\end{aligned}$$

Next I will compute $\dfrac{\partial^2 f}{\partial x_k \partial y_k}$.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_k \partial y_k} &= \frac{\partial}{\partial y_k}\left(8\sum_{j=1}^{N}\Big((x_k-x_j)^2+(y_k-y_j)^2-d_{kj}^2\Big)(x_k-x_j)\right) \\
&= 8\sum_{j=1}^{N}(x_k-x_j)\frac{\partial}{\partial y_k}\Big((y_k-y_j)^2\Big) \\
&= 16\sum_{j=1}^{N}(x_k-x_j)(y_k-y_j)
\end{aligned}
$$

Thirdly I will compute $\dfrac{\partial^2 f}{\partial x_k \partial x_i}$ and $\dfrac{\partial^2 f}{\partial y_k \partial y_i}$, for $i \neq k$.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_k \partial x_i} &= \frac{\partial}{\partial x_i}\left(8\sum_{j=1}^{N}\Big((x_k-x_j)^2+(y_k-y_j)^2-d_{kj}^2\Big)(x_k-x_j)\right) \\
&= \frac{\partial}{\partial x_i}\Big(8\Big((x_k-x_i)^2+(y_k-y_i)^2-d_{ki}^2\Big)(x_k-x_i)\Big) \\
&= \frac{\partial}{\partial x_i}\Big(8(x_k-x_i)^3+8\Big((y_k-y_i)^2-d_{ki}^2\Big)(x_k-x_i)\Big) \\
&= -24(x_k-x_i)^2-8\Big((y_k-y_i)^2-d_{ki}^2\Big) \\
&= -8\Big((x_k-x_i)^2+(y_k-y_i)^2-d_{ki}^2\Big)\frac{\partial^2 f}{\partial y_k \partial y_i} \qquad = \frac{\partial}{\partial y_i}\left(8\sum_{j=1}^{N}\Big((x_k-x_j)^2+(y_k-y_j)^2-\right. \\
&= \frac{\partial}{\partial y_i}\Big(8\Big((x_k-x_i)^2+(y_k-y_i)^2-d_{ki}^2\Big)(y_k-y_i)\Big) \\
&= \frac{\partial}{\partial y_i}\Big(8\Big((x_k-x_i)^2-d_{ki}^2\Big)(y_k-y_i)+(y_k-y_i)^3\Big) \\
&= -8\Big((x_k-x_i)^2+3(y_k-y_i)^2-d_{ki}^2\Big)y_i
\end{aligned}
$$

Lastly I will compute $\dfrac{\partial^2 f}{\partial x_k \partial y_i}$ for $i \neq k$. Note that this is equivalent to $\dfrac{\partial^2 f}{\partial y_i \partial x_k}$.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_k \partial y_i} &= \frac{\partial}{\partial y_i}\left(8\sum_{j=1}^{N}\Big((x_k-x_j)^2+(y_k-y_j)^2-d_{kj}^2\Big)(x_k-x_j)\right) \\
&= \frac{\partial}{\partial y_i}\Big(8\Big((x_k-x_i)^2+(y_k-y_i)^2-d_{ki}^2\Big)(x_k-x_i)\Big) \\
&= -16(x_k-x_i)(y_k-y_i)
\end{aligned}
$$

Now with all these partial derivatives the Hessian can be shown as a matrix of partial derivatives.

$$\underline{\underline{\nabla^2 f}} = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} & \frac{\partial^2 f}{\partial x_1 \partial y_N} \\ \frac{\partial^2 f}{\partial y_1 \partial x_1} & \frac{\partial^2 f}{\partial {y_1}^2} & \cdots & \frac{\partial^2 f}{\partial y_1 \partial x_N} & \frac{\partial^2 f}{\partial y_1 \partial y_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \frac{\partial^2 f}{\partial x_N \partial y_1} & \cdots & \frac{\partial^2 f}{\partial {x_N}^2} & \frac{\partial^2 f}{\partial x_N \partial y_N} \\ \frac{\partial^2 f}{\partial y_N \partial x_1} & \frac{\partial^2 f}{\partial y_N \partial y_1} & \cdots & \frac{\partial^2 f}{\partial y_N \partial x_N} & \frac{\partial^2 f}{\partial {y_N}^2} \end{bmatrix}$$

$$\underline{\underline{\nabla^2 f}} = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial y_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial y_N} \\ \frac{\partial^2 f}{\partial y_1 \partial x_1} & \frac{\partial^2 f}{\partial {y_1}^2} & \cdots & \frac{\partial^2 f}{\partial y_1 \partial y_N} \\ \vdots & \ddots & \vdots & \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial {x_N}^2} & \frac{\partial^2 f}{\partial x_N \partial y_N} \\ \frac{\partial^2 f}{\partial y_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial y_N \partial x_N} & \frac{\partial^2 f}{\partial {y_N}^2} \end{bmatrix}$$

(c) Prove that the Hssian, $\underline{\underline{\nabla^2 f}}$ is symmetrix positive definite for all $\underline{x} \in \mathbb{R}^{2N}$.

Proof. $\square$