Fall 2016, MATH-566

Integer Programming - Unimodularity

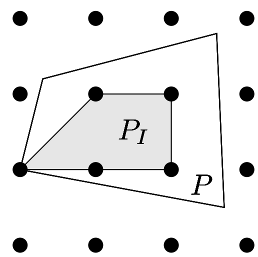
Source: Bill,Bill,Bill,Alex book, chapter 6.5

Problem:

$$(IP) \begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}, \end{cases}$$

where $\mathbf{c} \in \mathbb{Z}^n, \mathbf{b} \in \mathbb{Z}^m, A \in \mathbb{Z}^{m\times n}$, and $\mathbf{x} \in \mathbb{Z}^n$.

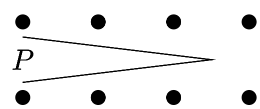
Let $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \le \mathbf{b}\}$ be a polyhedron. Let $P_I = conv(\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} \le \mathbf{b}\})$ be the convex hull of integer points in P. If A and $\mathbf{b}$ are rational, P is called *rational polyhedra.*



Clearly, $P_I \subseteq P$. Polyhedron P is **integral** if $P = P_I$. (or if every face of P contains an integral vector) If P is integral, then (IP) can be solved as linear programming.

1: Why (IP) cannot be always solved by linear program and then rounding?

Solution: Rounding may not be possible.



Theorem 6.22 Rational polytope P is integral iff for all $\mathbf{w} \in \mathbb{Z}^n$, the value of $\max\{\mathbf{w}^T\mathbf{x} : \mathbf{x} \in P\}$ is $\in \mathbb{Z}$.

2: Prove Theorem 6.22. One direction is easy. Other direction: Let $\mathbf{v} \in P$ be the unique optimal solution corresponding to $\mathbf{w}$.... show $\mathbf{v}$ has integer coordinates.

Solution: By multiplying $\mathbf{w}$ by a constant assume that for all other vertices $\mathbf{u} \neq \mathbf{v}$:

$$\mathbf{w}^T\mathbf{v} > \mathbf{w}^T\mathbf{u} + \mathbf{u}_1 - \mathbf{v}_1$$

Hence $\mathbf{v}$ is optimal also for $\mathbf{z} = (\mathbf{w}_1 + 1, \mathbf{w}_2, \ldots)$. Then $\mathbf{z}^T\mathbf{v} = \mathbf{w}^T\mathbf{v} + \mathbf{v}_1$ Since we assumed $\mathbf{z}^T\mathbf{v}$ and $\mathbf{w}^T\mathbf{v}$ are integral, also $\mathbf{v}_1$ is integral. Repeat for other components of $\mathbf{v}$.

What guarantees and integral polyhedra?

Recall $A^{-1} = \frac{1}{\det(A)}A^\star$, where $A^\star_{i,j} = \det(A_{-i,-j})$.

For square matrices:

Theorem 6.23 Let $A \in \mathbb{Z}^{m\times m}$. Then $A^{-1}\mathbf{b}$ is integral for every $\mathbf{b} \in \mathbb{Z}^n$ iff $\det(A) \in \{1, -1\}$.

3: Prove Theorem 6.23

Solution: $\Leftarrow$ Let $\det(A) = \pm 1$. By Cramer's rule, also A^{-1} is integral. Hence $A^{-1}\mathbf{b}$ is integral.
$\Rightarrow$ If $\mathbf{b}$ is ith unit vector, then $A^{-1}\mathbf{b}$ is ith column of A^{-1}. Hence A^{-1} is integral and $\det(A^{-1})$ is an integer. Since $1 = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$ and $\det(A) \in \mathbb{Z}$ we conclude that $\det(A) = \det(A^{-1}) = \pm 1$.

For rectangular matrix:

Matrix $A \in \mathbb{Z}^{m\times n}$ of full row rank is **unimodular** if every $m \times m$ basis of A (full rank square submatrix) has determinant ± 1.

Theorem 6.24 Let $A \in \mathbb{Z}^{m\times n}$ be of full row rank. Polyhedron $P = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is unimodular.

4: Prove Theorem 6.24

Solution: Recall in LP: A solution $\mathbf{x}$ is called basic feasible solution if $\mathbf{x}$ has at most m non-zero entires and the columns of A corresponding to these entries are linearly independent.

$\Leftarrow$ Let $\overline{\mathbf{x}} \in P$ be a vertex (using $\mathbf{x} \geq 0$). Pick basis B corresponding to $\overline{\mathbf{x}}$ - pick columns where $\overline{\mathbf{x}}$ is nonzero (and extend). Use Theorem 6.23 on $B\overline{\mathbf{x}} = \mathbf{b}$.

$\Rightarrow$ Let B be a base of A and pick any $\mathbf{v} \in \mathbb{Z}^n$. Goal is to show that $B^{-1}\mathbf{v} \in \mathbb{Z}^m$ and use Theorem 6.23 to show $\det(B) = \pm 1$. Choose $\mathbf{y} \in \mathbb{Z}^m$ such that $B^{-1}\mathbf{v}+\mathbf{y} \geq \mathbf{0}$. Let $\mathbf{b} = B(B^{-1}\mathbf{v}+\mathbf{y}) = \mathbf{v}+B\mathbf{y} \in \mathbb{Z}^m$. Add zero components to $(B^{-1}\mathbf{v}+\mathbf{y})$, which gives $\mathbf{z} \in \mathbf{R}^n$ such that $A\mathbf{z} = \mathbf{b}$. Now $\mathbf{z} \in P$ and it corresponds to a basic feasible solution, hence $\mathbf{z} \in \mathbb{Z}^n$. Therefore $B^{-1}\mathbf{v} \in \mathbb{Z}^m$.

Matrix $A \in \mathbb{Z}^{m\times n}$ is **totally unimodular** if every square submatrix has determinant in $\{0, 1, -1\}$. (all entries of A are in $\{0, 1, -1\}$.)

HW question: A is totally unimodular iff $[A\ I]$ is unimodular (where I is $m \times m$ unit matrix).

Theorem 6.25 Let $A \in \mathbb{Z}^{m\times n}$. Polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

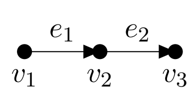
Theorem 6.26 Let $A \in \mathbb{Z}^{m\times n}$. Polyhedron $P = \{A\mathbf{x} \leq \mathbf{b}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^m$ iff A is totally unimodular.

Note: Matrix A being totally unimodular is decidable in polynomial time (algorithm by Seymour).

5: Let A have values $\{0, 1, -1\}$ and every column has at most one 1 and at most one -1. Show that A is totally unimodular. *Hint: induction.*

Solution: Let N be a $k \times k$ submatrix. If $k = 1$ clear. If column with at most one non-zero, expand the determinant. If all columns have 1 and -1, matrix is signular.

Example: Incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of directed graph $G = (V, E)$ is totally unimodular. Matrix M is indexed by V and E. Edge $e = \overrightarrow{uv} \in E$ gives entries $M_{ue} = -1$ and $M_{ve} = 1$.



	e_1	e_2
v_1	-1	
v_2	1	-1
v_3		1

Theorem Matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff for every $R \subseteq \{1, \ldots, m\}$ there is a partition $R = R_1 \cup R_2$ such that for all $j \in 1 \ldots n$.

$$\sum_{i\in R_1} a_{ij} - \sum_{i\in R_2} a_{ij} \in \{-1, 0, 1\}$$

Example: Incidence matrix $M \in \mathbb{R}^{|V|\times|E|}$ of (undirected) bipartite graph $G = (V, E)$ is totally unimodular. $M_{eu} = M_{ev} = 1$ for every $e = uv \in E$.

Next time: Integer Programming - Branch&Bound