Fall 2016, MATH-566

Linear Programming Algorithms - Simplex method

Source: Chapters 4,5 of Matoušek

Assume linear program (P) in *equational* form:

$$(P)\begin{cases}\text{maximize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}=\mathbf{b}\\ & \mathbf{x}\geq\mathbf{0}\end{cases}$$

where $A\in\mathbb{R}^{m\times n}$, $\mathbf{b}\in\mathbb{R}^m$ and $\mathbf{c}\in\mathbb{R}^n$.

1: *Can we assume that rows of A are linearly independent? Can we assume $n\geq m$?*

Solution: Yes - if two rows are linearly dependent, then either there is no solution or one is useless. If the rows are linearly independent, we get for free that $n\geq m$.

A solution $\mathbf{x}$ is called *basic feasible solution* if $n-m$ entries of $\mathbf{x}$ are zero and the columns of A corresponding to these remaining m entries are linearly independent.

2: *Is it possible to find two different basic feasible solutions with the same positions of $n-m$ zero entries?*

Solution: No - if the solution has m nonzeros. The system of equations $m\times m$ has a unique solution since the columns of A are linearly independent. Yes - if the actual optimal solution has more zeros, then some could be replaced by different columns.

Theorem 1. *If program (P) has an optimal solution, it also has an optimal solution that is a basic feasible solution.*

Corollary: Optimal solution to (P) can be found by examining all $\binom{n}{m}$ subsets of columns of A.

A set $B\subset\{1,\ldots,n\}$ is *base* if columns of A indexed by B give a basic feasible solution (denoted by A_B).

Simplex method: Start with a base and alter the base by changing one entry at a time.

Example of simplex method:

$$(P)\begin{cases}\text{maximize} & x_1 + x_2 & \\ \text{subject to} & -x_1 + x_2 & \leq 1\\ & x_1 & \leq 3\\ & x_2 & \leq 2\\ & x_1,x_2\geq 0 &\end{cases}$$

3: Rewrite the program to equational form. *(Hint: use slack variables - that is add 3 more variables)*

Solution:

$$(P)\begin{cases}\text{maximize} & x_1 + x_2 & \\ \text{subject to} & -x_1 + x_2 + x_3 & = 1\\ & x_1 + x_4 & = 3\\ & x_2 + x_5 & = 2\\ & x_1,x_2,x_3,x_4,x_5\geq 0 &\end{cases}$$

4: Is there some obvious basic feasible (not necessarily optimal solution)?

Solution: $x_3 = 1$, $x_4 = 3$, $x_5 = 2$, the base is nice identity matrix.

We create a thing called *simplex tableau* for base $B = \{3, 4, 5\}$:

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & 0 & + & x_1 & + & x_2
\end{array}
$$

Features: A_B is identity matrix, solution is obvious, if non-basis veriables are $= 0$, we keep only "$A - A_B$". z stands for the value of the objective function

5: Will z increase if we increase x_1 or x_2? How much we can increase x_2 if x_1 is kept at zero?

Solution: Both x_1 and x_2 have positive coefficient in the objective function. So increasing any of them will result in increase of the objective function.

If we are increasing x_2, then x_3 and x_5 are decreasing. So we cannot make $x_2 > 1$, otherwise $x_3 < 0$, which is not a feasible solution. Note that increasing x_2 to 1 will make $x_3 = 0$.

6: Increase x_2 as much as you can put it in the base. Use steps like in Gauss elimination to have x_2 instead if x_3 in the left top corner and nowhere else in the tableau. Note that the base will change to $B = \{2, 4, 5\}$.

Solution:

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & 0 & + & x_1 & + & x_2
\end{array}
\sim
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 - x_3 & = & 1 & - & x_1 & & \\
\hline
z + x_3 & = & 1 & + & 2x_1 & &
\end{array}
\sim
\begin{array}{rcrcrcr}
x_2 & = & 1 & + & x_1 & - & x_3 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z & = & 1 & + & 2x_1 & - & x_3
\end{array}
$$

The process when one variable is entering the base and another is leaving is called the **pivot step**.

7: Do another pivot step. Which of the variables in the objective function could be increased next? Increase it as much as possible and do a swap in the tableau as happened for x_2 and x_3.

Solution: Since x_3 has negative coefficient, we should not increase it. But x_2 could be increased. The equation that limits the increase of x_5 is the third one. Hence we take x_1 from there.

$$
\begin{array}{rcrcrcr}
x_2 & = & 1 & + & x_1 & - & x_3 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z & = & 1 & + & 2x_1 & - & x_3
\end{array}
\sim
\begin{array}{rcrcrcr}
x_2 + x_5 & = & 2 & & & & \\
x_4 - x_5 & = & 2 & & & - & x_3 \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z + 2x_5 & = & 3 & & & + & 2x_3
\end{array}
\sim
\begin{array}{rcrcrcr}
x_2 & = & 2 & - & x_5 & & \\
x_4 & = & 2 & + & x_5 & - & x_3 \\
x_1 & = & 1 & - & x_5 & + & x_3 \\
\hline
z & = & 3 & - & 2x_5 & + & x_3
\end{array}
$$

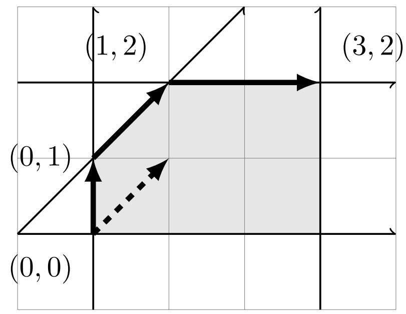
8: Can you do more pivot steps or is this the optimal solution? When is solution optimal?

Solution: We can still increase x_3 and increase z. The bound is x_4.

$$
\begin{array}{rcrcrcr}
x_2 & = & 2 & - & x_5 & & \\
x_4 & = & 2 & + & x_5 & - & x_3 \\
x_1 & = & 1 & - & x_5 & + & x_3 \\
\hline
z & = & 3 & - & 2x_5 & + & x_3
\end{array}
\sim
\begin{array}{rcrcrcr}
x_2 & = & 2 & - & x_5 & & \\
x_4 & = & 2 & + & x_5 & - & x_3 \\
x_1 + x_4 & = & 3 & & & & \\
\hline
z + x_4 & = & 5 & - & x_5 & &
\end{array}
\sim
\begin{array}{rcrcrcr}
x_2 & = & 2 & - & x_5 & & \\
x_3 & = & 2 & + & x_5 & - & x_4 \\
x_1 & = & 3 & & & - & x_4 \\
\hline
z & = & 5 & - & x_5 & - & x_4
\end{array}
$$

Now we have the optimal solution, because increase in any of the variables will not result in increase of the objective function.

9: Draw the polytope of feasible solutions of program (P) (the original program in 2 variables x_1 and x_2. Mark points that correspond to the steps of the solutions using simplex method and draw the direction of the objective function.



10: Use simplex method on the following example:

$$(P)\begin{cases} \text{maximize} & & & x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 0 \\ & x_1 & & & \leq & 2 \\ & & & x_1, x_2 \geq 0 & & \end{cases}$$

That is, convert to the equational form and do iterations until optimum solution is reached.

Solution:

$$\begin{array}{rcrrrrr} x_3 & = & 0 & & x_1 & - & x_2 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & 0 & & & & x_2 \end{array} \sim \begin{array}{rcrrrrr} x_2 & = & 0 & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & 0 & & x_1 & - & x_3 \end{array} \sim \begin{array}{rcrrrrr} x_1 & = & 2 & & & - & x_4 \\ x_2 & = & 2 & - & x_3 & - & x_4 \\ \hline z & = & 2 & - & x_3 & - & x_4 \end{array}$$

11: What happens with the objective function in the first step?

Solution: Nothing. It is staying zero.

- Simplex tableau is usually written in a matrix form (more condensed).
- There are versions - revised simplex method, dual simplex method for minimization, ...
- It is possible to construct an example that simplex method will cycle and never find a solution, if the pivot is chosen badly.
- Smart choice of pivot (Band's pivot rule - lexicographic rule) avoids cycling.
- There are many choices of pivot rules.
- polytopes may have many vertices (see cyclic polytope) but there is a chance of short path between any two vertices (initial solution and optimal solution) - recall Hirsh's conjecture.
- pivot rules can be tricked to walk through all vertices of cube (Klee-Minty cube)