Fall 2016, MATH-566

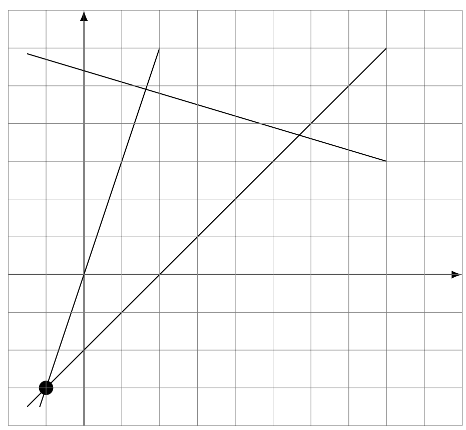
Geometry behing linear programming and basics

Exercise: Solve the following linear program:

$$(LP)\begin{cases}\text{minimize} & x+y\\ \text{s.t.} & x+2y\le 14\\ & 3x-y\ge 0\\ & x-y\le 2,\end{cases}$$

Solution: equations:

$$y\le -\frac{1}{2}x+7 \qquad y\le 3x \qquad y\ge x-2$$



Optimum $(x,y)=(-1,-3)$, value of objective function is -4.

Basic linear programming definitions:

- *feasible solution* is vector $\mathbf{x}$ such that $A\mathbf{x}\le\mathbf{b}$. In other words, point satisfying all the constraints.
- *set of feasible solutions*
- *optimal solution* is a feasible solution that is maximizing/minimizing the objective function.

1: What shape is the set of feasible solutions?

Solution: In the example above a polygon. In general polyhedra, that is unbounded polygons. In 2D we could say intersection of halfplanes.

2: What shape is the set of optimal solutions?

Solution: In the example above, it is a point. But it can be also a line. Or a special case, where it can be every point, but such program is not very interesting.

3: Construct a linear program that has more than one optimal solution.

4: Construct a linear program that has no feasible solution.

5: Construct a linear program that has no optimal solution.

Basic geometric definitions (for LP in higher dimension): Suppose we live in $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

- *hyperplane* is $d-1$ dimensional subspace $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = c\}$, where $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$.
- *closed halfspace* is $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \leq c\}$, where $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$.
- $C \subseteq \mathbb{R}^d$ is *convex* if $\forall \mathbf{x}, \mathbf{y} \in C, \forall t \in [0,1], t\mathbf{x} + (1-t)\mathbf{y} \in C$. (line between x and y is in C)
  Draw figures in class

Note: Intersection of family of convex sets is a convex set (obvious from the definition)

6: Show that the set of feasible solutions to any linear program is a convex set.

Solution: