Fall 2016, MATH-566

Linear Programming Algorithms - Interior point methods

Source: Chapter 11 of Convex Optimization, Stephen Boyd and Lieven Vandenberghe

Let

$$(P) \begin{cases} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \le 0 \text{ for } i = 1, \ldots, m \end{cases}$$

where f, g are convex, twice continuously differentiable and optimal solution $\mathbf{x}^\star$ exists.
Moreover, let (P) be *superconsistent*, that is $\exists \mathbf{x}, \forall i, g_i(\mathbf{x}) < 0$. In other words, the set of feasible solutions have full dimension.

(the setup covers linear, quadratic, geometric, semidefinite, ... programming).

Idea: Change the (P) to a problem without constraints but difficult objective function.

Let

$$(P') = \text{minimize } f(\mathbf{x}) + \sum_{i=1}^{m} I(g_i(\mathbf{x})),$$

where I is an indicator function

$$I(u) = \begin{cases} 0 & \text{if } u \le 0 \\ +\infty & \text{if } u > 0. \end{cases}$$

1: What is the optimal solution to (P')?

Solution: $\mathbf{x}^\star$ The function I works like infinite penalty for violating constraints.

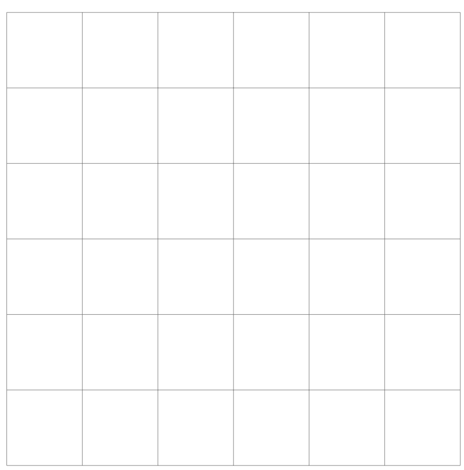
2: Can you solve (P') by methods from calculus?

Solution: No, because I has no derivative.

Use approximation of $I(u) \approx -c \log(-u)$, where $c > 0$.

3: Sketch $I(u)$ and its approximations. Is the approximation better when c is large or small?

Solution:



Approximation improves as $c \to 0$.

For $t > 0$, we consider a smooth unconstrained approximation of (P')

$$\text{minimize } f(\mathbf{x}) - \frac{1}{t} \sum_{i=1}^{m} \log(-g_i(\mathbf{x})).$$

Define *logarithmic barrier function*

$$\Phi(\mathbf{x}) = -\sum_{i=1}^{m} \log(-g_i(\mathbf{x})),$$

for all $\mathbf{x}$ where $g_i(\mathbf{x}) < 0$ (interior of feasible solutions)
Analytic center of the set $S = \{\mathbf{x} : g_i(\mathbf{x}) \le 0\} \subseteq \mathbb{R}^n$ is $\mathbf{x}^\star$ minimizing $\Phi(\mathbf{x})$ over all $\mathbf{x} \in S$.

4: Find the analytic center of a square in $\mathbf{R}^2$ defined by equations

$$x_1 \ge 0, x_2 \ge 0, x_1 \le 1, x_2 \le 1$$

Solution: First rewrite in the form $g_i(\mathbf{x}) \le 0$.

$$-x_1 \le 0, -x_2 \le 0, x_1 - 1 \le 0, x_2 - 1 \le 0.$$

Now we write $\Phi(\mathbf{x})$.

$$\Phi(x_1, x_2) = -\left(\log(x_1) + \log(x_2) + \log(1 - x_1) + \log(1 - x_2)\right)$$

We investigate partial derivatives and let them $= 0$.

$$\begin{aligned} 0 = \frac{\partial \Phi(x_1, x_2)}{\partial x_i} &= -\frac{1}{x_i} + \frac{1}{1 - x_i} \\ 1 - x_i &= x_i \\ x_i &= \frac{1}{2}. \end{aligned}$$

The second derivatives at $(\frac{1}{2}, \frac{1}{2})$ are

$$\begin{aligned} \frac{\partial^2 \Phi(x_1, x_2)}{\partial x_i \partial x_j} &= 0 \\ \frac{\partial^2 \Phi(x_1, x_2)}{\partial^2 x_i} = \frac{1}{x_i^2} + \frac{1}{(1 - x_i)^2} &= 8 \end{aligned}$$

This shows that the Hessian at the critical point

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

which is positive definite. Hence $(\frac{1}{2}, \frac{1}{2})$ is minimum. Notice that $\Phi(\mathbf{x})$ is convex, so we could argue that any critical point is minimum. The analytic center is $(\frac{1}{2}, \frac{1}{2})$.

5: Find the analytic center of a square in $\mathbf{R}^2$ defined by equations

$$x_1 \ge 0, x_2 \ge 0, (1 - x_1)^3 \ge 0, (1 - x_2)^3 \ge 0.$$

Notice it is possible to define center even if functions are not convex everywhere and the center depends on the functions.

Solution:

$$\Phi(x_1, x_2) = -\left(\log(x_1) + \log(x_2) + \log((1 - x_1)^3) + \log((1 - x_2)^3)\right)$$

$$
\begin{aligned}
0 &= \frac{\partial \Phi(x_1, x_2)}{\partial x_i} = -\frac{1}{x_i} + \frac{3}{1 - x_i} \\
x_i &= \frac{1}{4}
\end{aligned}
$$

The analytic center is $(\frac{1}{4}, \frac{1}{4})$.

For $t > 0$ define $\mathbf{x}^\star(t)$ as the optimal solution of

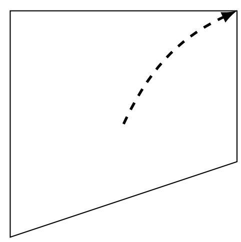
$$(P_t) = \text{ minimize } tf(\mathbf{x}) + \Phi(\mathbf{x}).$$

(assume that the optimal solution is unique)

Central path is $\{\mathbf{x}^\star(t) : t \geq 0\}$.

Interior point method idea: Start in the analytical center and follow the central path.

In iterations increase t and recompute the new optimum using Newton's method. Recall that Newton's method works well if the initial point of Newton's method is close to the optimal solution. With small increases of t, the starting point is close to the optimum.



There exists a notion of dual program (D) for (P), (based on Karush-Kuhn-Tucker theorem). It gives solutions to the dual $\mathbf{y}^\star(t)$ such that

$$f(\mathbf{x}^\star(t)) - h(\mathbf{y}^\star(t)) \leq \frac{m}{t},$$

where h is the objective function of the dual program. Hence the central path converges to $\mathbf{x}^\star$ for (P).