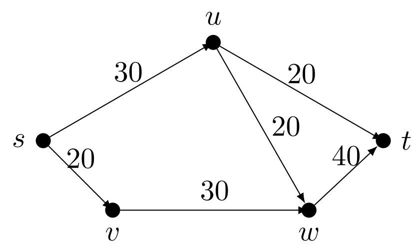
Fall 2016, MATH-566

Network flows - first introduction

Original application (product of cold war)

Suppose country X enters into a war. How quickly can X move tank from storage s to the target t (battle ground)? The tanks are moved on a railroad. Every link gives the capacity how many tank a day can be transported.



1: How many tanks per day can be delivered to the battleground? Is the solution unique?

Solution: 50 tanks. Not unique.

Problem: (Directed) graph G, source s, sink t, capacities $u : E(G) \to \mathbb{R}_+$.
Network is (G, u, s, t).

Input: Network (G, u, s, t)
Output: s-t-flow of maximum value

s-t-flow f is a function $f : E(G) \to \mathbb{R}_+$.
Value of f is $\sum_{su \in E} f(su) - \sum_{us \in E} f(us)$ i.e. leaving − entering to s.

2: How f looks around one vertex of the network? (what must f satisfy?)

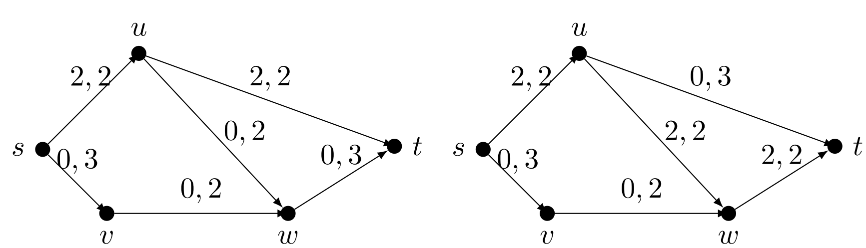
Solution: **Flow conservation rule:** For all $v \in V \setminus \{s, t\}$:

$$\sum_{uv \in E} f(uv) = \sum_{vu \in E} f(uv).$$

and for s, t it satisfies:

$$\sum_{su \in E} f(su) - \sum_{us \in E} f(us) = \sum_{ut \in E} f(ut) - \sum_{tu \in E} f(tu).$$

3: How to improve these flows to be maximum? Description on edges are values of f, u.



Solution: Path s, v, w can be increased by 2. Path s, v, w, u, t can be increased by two.

4: After the improvement, how do you argue that nobody can further improve?

Solution: Consider set $A = \{s, v\}$. The capacity of edges going from A is 4. Hence no flow can route more than 4 through the network.

Let $A \subset V(G)$ such that $s \in A$ and $t \notin A$. Use $\delta^+(A)$ to denote set of edges uv, where $u \in A$ and $v \notin A$ (edges leaving A). Use $\delta^-(A)$ to denote set of edges uv, where $u \notin A$ and $v \in A$ (edges entering A).
Capacity of s-t-cut A is $\sum_{e\in\delta^+(A)} u(e)$.

5: Prove that for A and any flow f holds
(a) value$(f) = \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)$
(b) value$(f) \le \sum_{e\in\delta^+(A)} u(e)$

Solution: (a) Use conservation of flow at vertices in A.

$$\begin{aligned}\text{value}(f) &= \sum_{e\in\delta^+(s)} f(e) - \sum_{e\in\delta^-(s)} f(e)\\ &= \sum_{v\in A}\left(\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e)\right)\\ &= \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)\end{aligned}$$

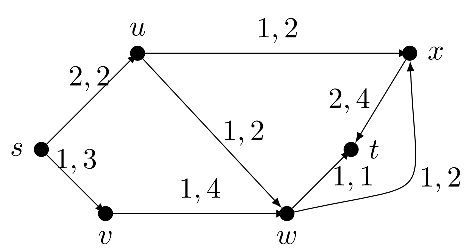
(b) clearly $\sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e) \le \sum_{e\in\delta^+(A)} u(e)$ since $f(e) \le u(e)$.
This proves the *obvious* observation that maximum flow cannot exceed capacity of minimum cut.

Notice in 3. we were improving flow be reducing the flow on uw. We “sent flow in the opposite direction”.
For a digraph G, define $\overleftrightarrow{G}$ by adding for every edge e also its **reverse** $\overleftarrow{e}$.
For f and u define **residual capacities** $u_f : E(\overleftrightarrow{G}) \to \mathbb{R}_+$

$$u_f(e) = u(e) - f(e) \qquad\qquad u_f(\overleftarrow{e}) = f(e)$$

Residual capacities ... how much extra we can send in each direction.
Residual graph G_f is obtained from $\overleftrightarrow{G}$ by removing edges $e \in E(\overleftrightarrow{G})$ with $u_f(e) = 0$.
Augmenting path is an s-t path in G_f.

6: Construct the residual graph for



and find an augmenting path and increase the flow using the augmenting path.

7: How to update (to **augment** the flow f using augmenting path in $\overleftrightarrow{G}$?

Solution: If flow is increasing on edge e by γ, then decrease $u_f(e)$ by γ and increase $u_f(\overleftarrow{e})$ by γ.