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MATH 566 Discrete Optimization
Midterm I

1. Suppose you are making a schedule for an airport. There are n arriving flights. Every airplane j has a possible time arrival in interval $[a_j, b_j]$ (plane can fly faster or slower). Determine the actual arrival schedule for each airplane such that the smallest gap between consecutive flights is maximized and for all j, airplane j arrives before $j + 1$. Formulate a linear program that solves the problem.

   (Example: Suppose there are three airplanes. They have arrival intervals $[1, 5], [2, 7], [6, 7]$*. Then we can assign arrival times to the airplanes, for example* $2, 4.5, 6.2$*. The smallest gap in this schedule is* 1.7 *between the second and third airplane. The number* 1.7 *is the number we want to maximize. Notice that we do not allow schedule* $4, 2, 7$*, where the first airplane arrives AFTER the second one although the it would be feasible with respect to* $[a_i, b_i]$*s (it is easier to solve if the order is fixed).)*

   First I will define a new set of variables, t_j for $1 \le j \le n$, to be the time that plane j arrives at the airport. Clearly we must have the constraints

$$t_j \ge a_j$$
$$t_j \le b_j$$

   for all j, such that $1 \le j \le n$. These constraints force each plane to arrive inside of the allowed interval $[a_j, b_j]$. Also if the order that the planes arrive is to be enforced, the constraints

$$t_{j+1} - t_j \ge 0$$

   are required for $1 \le j \le n - 1$. Lastly we need to set the objective function for this linear program. Our objective function needs to maximize the smallest time gap $t_{j+1} - t_j$ or in other words $\max \min_{1 \le j \le n-1}(t_{j+1} - t_j)$. I am going to create another variable m to be this minimum value, that is

$$m = \min_{1 \le j \le n-1}(t_{j+1} - t_j)$$

   In this case the objective function just becomes $\max m$. However we need constraints to insure that m does in fact equal the minimum value. I will use the following constraints

$$m \le t_{j+1} - t_j$$

   for all j such that $1 \le j \le n - 1$. These constraints guarantee that

$$m \le \min_{1 \le j \le n-1}(t_{j+1} - t_j).$$

   With the addition of the objective function maximizing m, we will achieve equality. Therefore the entire linear program that solves this problem is

$$(P) = \begin{cases} \text{maximiz} & m \\ \text{subject to} & t_j \ge a_j \quad 1 \le j \le n \\ & t_j \le b_j \quad 1 \le j \le n \\ & t_{j+1} - t_j \ge 0 \quad 1 \le j \le n-1 \\ & m \le t_{j+1} - t_j \quad 1 \le j \le n-1 \end{cases}$$

   Note that there is no minimum value of m and we aren't enforcing nonnegativity to the variables t_j as the intervals $[a_j, b_j]$ supersede these conditions.

2. Solve the following linear program (P) using simplex method.

$$(P) = \begin{cases} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 \leq 1 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{cases}$$

Check your solution using computer program (APMonitor, Sage,...). Plot the set of feasible solutions and mark the optimum. Solving using simplex method means make the sequence of simplex tables.

First I will solve this problem using the simplex method. First I will introduce slack variables so that the new linear program is

$$(P) = \begin{cases} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + x_3 \leq 1 \\ & -x_1 + x_2 + x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

The initial basic feasible solution for this linear program is

$$\begin{aligned} x_3 &= 1 - x_1 \\ x_4 &= 1 + x_1 - x_2 \\ z &= 0 + x_1 + x_2 \end{aligned}$$

First x_1 can be increased by 1.

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_4 &= 2 - x_2 - x_3 \\ z &= 1 + x_2 - x_3 \end{aligned}$$

Next x_2 can be increased by 2.

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 2 - x_3 - x_4 \\ z &= 3 - 2x_3 - x_4 \end{aligned}$$

Now all of the coefficients of the variables in the objective function are negative so this is the optimal solution. The optimal solution is $x_1 = 1$, $x_2 = 2$, and $z = 3$.

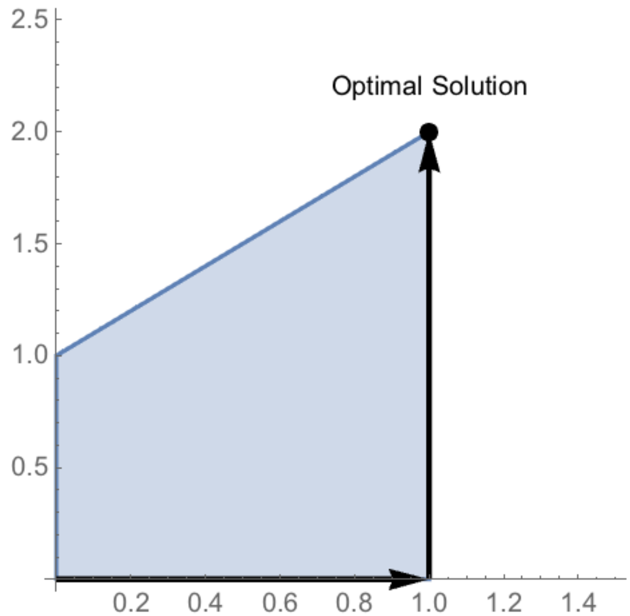
Next I will check this solution using sage. The following script solves this linear program.

```
milp = MixedIntegerLinearProgram(maximization=True)
x = milp.new_variable(nonnegative=True)
milp.set_objective(x[1] + x[2])
milp.add_constraint(x[1] <= 1)
milp.add_constraint(-x[1] + x[2] <= 1)
print('Objective Value: {}'.format(milp.solve()))
for i, v in milp.get_values(x).iteritems():
    print('x_%s = %s' % (i, v))
```

The following is the output of this script.

```
Objective Value: 3.0
x_1 = 1.0
x_2 = 2.0
```

Lastly I will show a plot of the feasible region, along with the optimal solution and the path that I took in the simplex method to get there.



3. Consider the following algorithm. Input is a connected graph $G = (V, E)$ and a cost function $c : E \to \mathbb{R}$. Start with H being a copy of G. First, the edges E are ordered such that $c(e_1) \geq c(e_2) \geq \ldots \geq c(e_m)$. Then process edges one by one according to the ordering. Processing edge e_i means looking if $H - e_i$ connected. If $H - e_i$ is connected, then e_i is removed from H. Otherwise e_i is kept in H. After all edges are processed, the resulting H is the output. Now you can pick what to do. Either a) or b):

   (a) Implement the algorithm and use as inputs the same graph we used for the minimum spanning tree

   (b) Prove that the algorithm produces minimum spanning tree.

   I chose to do part (a). The following script creates random graphs and implements the given algorithm in the function minimumSpanningTree.

```
import itertools as it
load('breadthFirstSearch.sage')

# This plots vertices as red dots and blue edges connecting them
def plot_vertices_edges(vertices, edges):
    drawing = line([])
    for x in vertices:
        drawing = drawing + disk(x, 0.1, (0,2*pi), color='red')
    for e in edges:
        drawing = drawing + line([vertices[e[0]], vertices[e[1]]])
    drawing.show()
```

```python
# Generate 10 random vertices in 10x10 grid
def generate_random_vertices():
    vertexList=[]
    for i in range(10):
        vertexList.append((random()*10, random()*10))
    return vertexList

def generateEdgeList(numVertices):
    return list(it.combinations(range(numVertices), 2))

def generateCostList(vertexList, edgeList):
    costList = []
    for edge in edgeList:
        u = vertexList[edge[0]]
        v = vertexList[edge[1]]
        costList.append(sqrt((u[0]-v[0])^2 + (u[1] - v[1])^2))
    return costList

def minimumSpanningTree(vertexList, edgeList, costList):
    numVertices = len(vertexList)
    numEdges = len(edgeList)

    # sort edges by cost
    s = sorted(zip(edgeList, costList), key=lambda pair:pair[1], reverse
      ↪ =True)
    edgeList = [x for (x, y) in s]
    costList = [y for (x, y) in s]

    treeEdgeList = list(edgeList)

    for edge in edgeList:
        treeEdgeList.remove(edge)
        # if graph without edge is not connected
        if max(breadthFirstSearch(vertexList, treeEdgeList, vertexList
          ↪ [0]).values()) == oo:
            # if not connected add edge back into treeEdgeList
            treeEdgeList.append(edge)

    return treeEdgeList

# generate random graph
vertexList = generate_random_vertices()
edgeList = generateEdgeList(len(vertexList))
costList = generateCostList(vertexList, edgeList)

# find minimum spanning tree
edges = minimumSpanningTree(range(10), edgeList, costList)

# plot minimum spanning tree
```

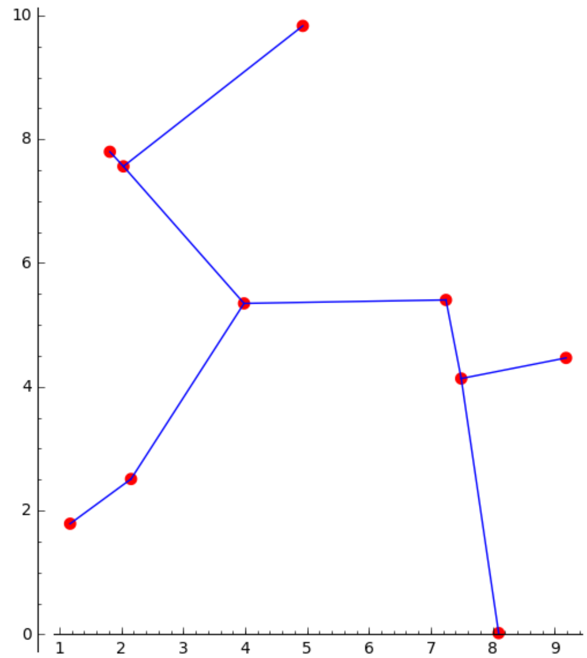
```
plot_vertices_edges(vertexList, edges)
```

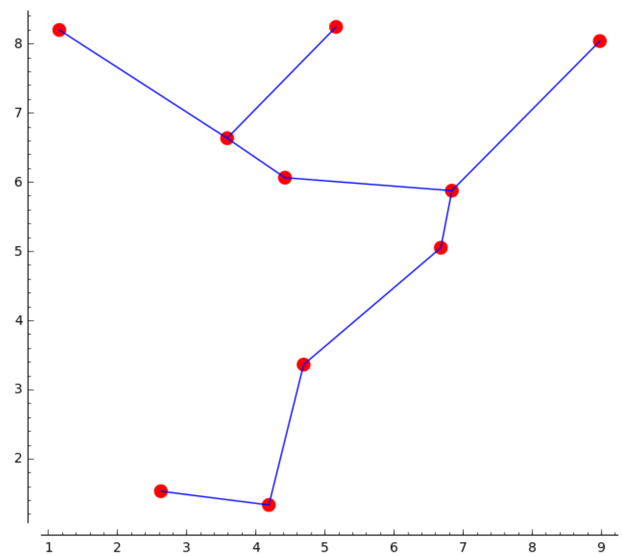
This function requires a breadthSearchFirst algorithm which I implemented as follows.

```
import Queue
def breadthFirstSearch(vertexList, edgeList, rootVertex):
    parentDict = dict()
    distanceDict = dict()
    for vertex in vertexList:
        parentDict[vertex] = None
        distanceDict[vertex] = oo

    # create queue to hold nodes
    q = Queue.Queue()

    distanceDict[rootVertex] = 0
    q.put(rootVertex)

    while not q.empty():
        currentVertex = q.get()
        for edge in edgeList:
            if edge[0] == currentVertex:
                adjacentVertex = edge[1]
            elif edge[1] == currentVertex:
                adjacentVertex = edge[0]
            else:
                continue

            # if we haven't reached adjacentVertex yet
            if distanceDict[adjacentVertex] == oo:
                distanceDict[adjacentVertex] = distanceDict[
                    ↪ currentVertex] + 1
                parentDict[adjacentVertex] = currentVertex
                q.put(adjacentVertex)

    return distanceDict
```

This function checks to see if a graph is connected. If a graph isn't connected it will return with an infinite distance from the root vertex to some other vertex.

The following two plots are the results on running the initial script twice. They show that the algorithm does in fact find the minimum spanning tree.





4. Consider the following problem. Input is a connected graph $G = (V, E)$ and a cost function $c : E \to \mathbb{R}$. Let T be a spanning tree of G. The cost of T is defined as the largest cost of an edge in T:

$$c(T) = \max\{c(e) : e \in E(T)\}.$$

Problem is to find a minimum spanning tree with respect to c
Do both a) and b):

(a) Formulate the problem using integer programming
Let $\{x_e\}$ be a set of binary variables where x_e corresponds to whether $e \in E(T)$, the alternative minimum spanning tree, for every edge, $e \in E$. Let $n = |V|$ and $m = |E|$. First note that the alternative minimum spanning tree must be a spanning tree, so the constraints associated with a spanning tree must also be associated with this alternative spanning tree. Namely this implies that T must have $n - 1$ edges or

$$\sum_{e \in E} (x_e) = n - 1$$

Also T must not contain any cycles. This can be expressed with the constraints

$$\sum_{e \in E(G[X])} (x_e) \leq |X| - 1 \quad \forall \emptyset \neq X \subset V$$

Lastly we need to add the objective function. The objective function is trying to minimize the maximum cost edge in T or

$$\min \max_{e \in E} x_e c(e)$$

I will create a new variable m to represent the maximum cost of an edge in $E(T)$, that is

$$m = \max_{e \in E} x_e c(e)$$

Now in order to enforce this equality certain constraints are needed. First m must be larger than any single cost of an edge in $E(T)$. This requires the constraints

$$m \geq x_e c(e)$$

for all $e \in E$. This makes m larger than $c(e)$ only if $e \in E(T)$. Lastly if we add the objective function $\min m$, this will force equality. These are all the constraints that are required, so the full integer linear program is

$$(P) = \begin{cases} \text{minimize} & m \\ \text{subject to} & \sum_{e \in E} (x_e) = n - 1 \\ & \sum_{e \in E(G[X])} (x_e) \leq |X| - 1 \quad \forall X \subset V, X \neq \emptyset \\ & m \geq x_e c(e) \quad \forall e \in E \\ & x_e \geq 0 \quad \forall e \in E \\ & x_1 \leq 1 \quad \forall e \in E \\ & x_e \in \mathbb{Z} \quad \forall e \in E \end{cases}$$

(b) Find an algorithm for solving this problem in polynomial time and prove its correctness.

Proof. First I will show that any minimum spanning tree is also an alternative minimum spanning tree. Let $G = (V, E)$ be a graph with a minimum spanning tree T. It has been shown previously that if T is a minimum spanning tree, then for every $e \in E(T)$, e is a minimum cost edge of the cut between the connected components in $T - e$. Consider the edge $e_{max} \in E(T)$ such that $c(e_{max}) \geq c(e)$ for all $e \in E(T)$. Therefore e_{max} is the minimum cost edge of the cut between the connected components of $T - e_{max}$. If T was not a alternative minimum tree, then there would be a lower cost edge in the cut between connected components of $T - e_{max}$. Since e_{max} is the minimum cost edge, this implies that T is also a alternative minimum spanning tree. Therefore any minimum spanning tree is also an alternative minimum spanning tree. Clearly not all alternative MSTs are MSTs, but all MSTs are alternative MSTs. This implies that any algorithm that finds a minimum spanning tree also finds an alternative minimum spanning tree. Therefore I can pick any algorithm that finds a minimum spanning tree in polynomial time as my algorithm. Kruskal's algorithm can be implemented in $O(m \log(n))$. This is better than m^2 so it is certainly polynomial. $\square$