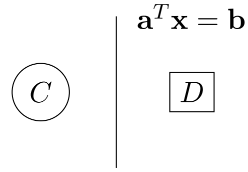
# Separation theorem

How to show that two convex sets are disjoint?

**Theorem 1.** *Let $C, D \subseteq \mathbb{R}^d$ are convex sets and $C \cap D = \emptyset$ then there exists a hyperplane separating $C$ and $D$. That is, exists $\mathbf{a} \in \mathbb{R}^d$, $b \in \mathbb{R}$ such that*
$\forall \mathbf{x} \in C, \mathbf{a}^T\mathbf{x} \leq b$
$\forall \mathbf{x} \in D, \mathbf{a}^T\mathbf{x} \geq b$

Separation can be strict if $C$ and $D$ closed and one bounded.




**1:** Why is the theorem true if $C$ and $D$ are compact?

**Solution:** If both $C$ and $D$ compact, consider $\mathbf{c} \in C$ and $\mathbf{d} \in D$ such that $\|\mathbf{c} - \mathbf{d}\|$ is minimized. That is, $\mathbf{c}$ and $\mathbf{d}$ are the closest points. They exist since $C$ and $D$ are compact and the distance is a continuous function. Let $z = \frac{\mathbf{c}+\mathbf{d}}{2}$, that is, $z$ is in the middle between $\mathbf{c}$ and $\mathbf{d}$. Considet a hyperplane $H$ perpendicular to segment $\mathbf{cd}$ and containing $\mathbf{z}$. Claim is that it is the desired hyperplane $H$. If $H$ is not separating, then we find a contradiction with $\mathbf{c}$ and $\mathbf{d}$ being the closest.

**2:** Why is the theorem true if $C$ compact and $D$ closed?

**Solution:** Assume $D_M$ being $D$ intersected with a huge, but still bounded set $M$. Then we can still argue that there are two closest points and eventually it will be the closest points even if $M$ grows.

**3:** Why is the theorem true in general?

**Solution:** Suppose $C$ and $D$ are general convex sets. We create a sequence of compact sets, that approximate $C_i$ and $D_i$ that are in limit going to $C$ and $D$. We can separate $C_i$ and $D_i$ by the first case by a hyperplane $H_i$. It it possible to show that $H_i$ is converging and it is converging to a separating hyperplane for $C$ and $D$.

Recall that $B(\mathbf{s}, r) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{s}\| \leq r\}$ is a ball centered at $\mathbf{s}$ of radius $r$. Suppose $\mathbf{0} \in C$. We define $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ by letting $C_i = \left(1 - \frac{1}{i}\right) C \cap B(\mathbf{0}, i)$. Notice that $C_i$'s are compact and $C = \cup_i C_i$.
We create similar sets for $D_i$ and then $H_i$ is a hyperplane separating $C_i$ from $D_i$.

**Solution:** Note that compactness of at least one of the sets is needed for for strict separation.