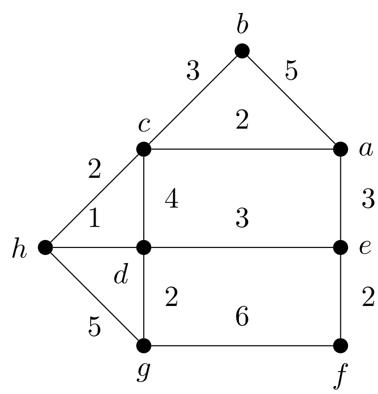
Fall 2016, MATH-566

Minimum Cuts in Unidrected Graphs

Recall: $\delta(S)$ is the set of edges with exactly one endpoint in S.

Problem:
Input: Graph $G = (V, E)$ and cost function $u : E \to \mathbb{R}^+$.
Output: Global Minimum Cut, that is $S \subset V$ such minimizing $u(\delta(S))$

1: Find a minimum cut in the following graph:



Notation:

$\lambda(G)$ is the capacity of minimum cut of G.

$\lambda(G; v, w)$ is the capacity of minimum (v, w)-cut of G.

2: Find an algorithm using network flows.

Solution: Pick a vertex and try to find a network flow to every other vertex.

Node Identification Algorithm:

Let G_{uv} be a graph obtained from G by identification of u and v (delete loops, keep parallel edges).
Main idea:

$$\lambda(G) \le \min(\lambda(G_{vw}), \lambda(G; v, w))$$

How to pick v, w?

A legal ordering starting at v_1 is $v_1, v_2, \ldots, v_n$ if for all i, v_i has the largest capacity of edges joining it to $v_1, \ldots, v_{i-1}$.

3: Find a legal ordering starting with vertex a.

Solution: a, b, c, d, e, h, g, f

Main theorem: If $v_1, \ldots, v_n$ is a legal ordering of G, then $\delta(v_n)$ is a minimum v_n, v_{n-1} cut of G.

1. $M := \infty$ and A undefined
2. while G has more than 1 vertex
3. Find a legal ordering $v_1, v_2, \ldots, v_n$ of G
4. If $u(\delta(v_n)) < M$
5. $M := u(\delta(v_n))$ and $A := \delta(v_n)$

6. $\qquad G := G_{v_n v_{n-1}}$

7. return A

4: Run node identification algorithm.

Random Contraction Algorithm:

1. while G has more than 2 vertices
2. $\qquad$ Choose and edge e of G with probability $u(e)/u(E)$
3. $\qquad G := G_{vw}$, where $e = vw$
4. return the unique cut in G.

5: Let A be a minimum cut of G. Show that the random contraction algorithm returns A with probability at least $2/(n(n-1))$.

Solution: Let $u(A) = \sum_{e \in A} u(A)$. Then

$$P(\text{edge of } A \text{ is picked for contraction}) = \frac{u(A)}{u(E)}$$

Notice that A is the minimum cut in G. Hence $u(A) \leq u(C)$ for any other cut. In particular, we consider cuts around each vertex. A cut around vertex v has capacity $\sum_{e \in \delta(v)} u(e)$. The average cost of a cut around one vertex is

$$\frac{\sum_{e \in \delta(v)} u(e)}{n} = \frac{2 \sum_{e \in E} u(e)}{n} = \frac{2u(E)}{n}.$$

Then picking an edge from A has lower probability than picking an edge from an average cut around a vertex

$$\frac{u(A)}{u(E)} \leq \frac{2u(E)}{n \cdot u(E)} = \frac{2}{n}.$$

After i rounds of the algorithm, G has $n - i$ edges and we get

$$\frac{u(A)}{u(E)} \leq \frac{2}{n-i}.$$

Now the probability that no edge of A was choses is at least

$$1 - \frac{2}{n-i} = \frac{n-i-2}{n-i}$$

The algorithm is running for rounds with $i = 0, \ldots, n-2$ and we get that the probability no edge of A is ever chosen is at least

$$\frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{n(n-1)}.$$

6: Let $k \in \mathbb{N}$. Show that the probability that the random contraction algorithm does not return A is one of kn^2 runs is at most e^{-2k}.

Solution: We use the estimate from previous round kn^2 times.

$$\left(1 - \frac{2}{n(n-1)}\right)^{kn^2} \leq \left(1 - \frac{2}{n^2}\right)^{kn^2} \leq \left(e^{-\frac{2}{n^2}}\right)^{kn^2} = e^{-2k}.$$