

Introduction to the Duality for Linear Programming

Let

$$(P) \left\{ \begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 \\ \text{s.t.} & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array} \right.$$

1: Without solving (P) itself, is it possible to provide an upper bound on the value of (P) by using equation $4x_1 + 8x_2 \leq 12$?

Solution: Yes - easily:

$$2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$$

so the maximum is at most 12. We can even improve it by

$$2x_1 + 3x_2 \leq \frac{1}{2}(4x_1 + 8x_2) \leq 6.$$

This gives maximum is at most 6.

2: Without solving (P) , is it possible to provide an upper bound on the value of (P) using equations $4x_1 + 8x_2 \leq 12$ and $2x_1 + x_2 \leq 3$?

Solution:

$$\begin{array}{r} 4x_1 + 8x_2 \leq 12 \\ 2x_1 + x_2 \leq 3 \\ + \hline 6x_1 + 9x_2 \leq 15 \end{array}$$

Now we get $2x_1 + 3x_2 \leq \frac{1}{3}(6x_1 + 9x_2) \leq 5$.

3: Without solving (P) , how would you try to find the combination of constraints that provides the best upper bound? (solution might be another linear program, call it (D))

Solution: We try to combine the three constraints (not the non-negativity constraints) and obtain an upper bound. Say the first constraints is multiplied by y_1 , the second by y_2 and third by y_3 .

So we have a combination of

$$\begin{array}{l} y_1 \cdot (4x_1 + 8x_2 \leq 12) \\ y_2 \cdot (2x_1 + x_2 \leq 3) \\ y_3 \cdot (3x_1 + 2x_2 \leq 4) \end{array}$$

what else y_i satisfies? If $y_i < 0$, inequality is reversed, so $y_i \geq 0$ We need the left hand sides to be at least $2x_1 + 3x_2$, hence

$$\begin{array}{l} y_1 \cdot 4x_1 + y_2 \cdot 2x_1 + y_3 \cdot 3x_1 \geq 2x_1 \\ y_1 \cdot 8x_2 + y_2 \cdot x_2 + y_3 \cdot 2x_2 \geq 3x_2 \end{array}$$

Next, we want to minimize the right hand side, which is $12y_1 + 3y_2 + 4y_3$. It gives a linear program (D) :

$$(D) \begin{cases} \text{minimize} & 12y_1 + 3y_2 + 4y_3 \\ \text{s.t.} & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{cases}$$

- (D) gives an upper bound on (P)
- (P) gives a lower bound on (D)

4: Are solutions $\mathbf{x} = (\frac{1}{2}, \frac{5}{4})$ of (P) and $\mathbf{y} = (\frac{5}{16}, 0, \frac{1}{4})$ for (D) optimal solutions?

Solution: Yes! They are optimal solutions, because they satisfy constraints and values of the objective functions are the same.

5: Find the dual program (D) to

$$(P) \begin{cases} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

Solution:

$$(D) \begin{cases} \text{minimize} & \mathbf{b}^T \mathbf{y} \\ \text{s.t.} & A^T \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{cases}$$

6: Find the dual program (D) to

$$(P) \begin{cases} \text{minimize} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases}$$

Solution: We first rewrite $A\mathbf{x} = \mathbf{b}$ as $A\mathbf{x} \geq \mathbf{b}$ and $-A\mathbf{x} \geq -\mathbf{b}$. Then we get

$$(P) \begin{cases} \text{minimize} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b} \\ & -A\mathbf{x} \geq -\mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases} \quad (D) \begin{cases} \text{maximize} & \mathbf{b}, -\mathbf{b}^T(\mathbf{u}, \mathbf{v}) \\ \text{s.t.} & A^T \mathbf{u} - A^T \mathbf{v} \leq \mathbf{c} \\ & \mathbf{u}, \mathbf{v} \geq 0 \end{cases}$$

Let $\mathbf{y} = \mathbf{u} - \mathbf{v}$. Then we can write

$$(D) \begin{cases} \text{maximize} & \mathbf{b}, ^T(\mathbf{y}) \\ \text{s.t.} & A^T \mathbf{y} \leq \mathbf{c} \end{cases}$$

Note that \mathbf{y} can be negative.