Fall 2016, MATH-566

Interpretation of Duality and Duality theorem

Dualization for everyone:
$A \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$

	primal	dual
variables	$x_1, \dots, x_n$	$y_1, \dots, y_m$
matrix	A	A^T
right hand	$\mathbf{b}$	$\mathbf{c}$
objective	max $\mathbf{c}^T\mathbf{x}$	min $\mathbf{b}^T\mathbf{y}$
constraint	ith constrain $\leq$	$y_i \geq 0$
	ith constrain $\geq$	$y_i \leq 0$
	ith constrain $=$	$y_i \in \mathbb{R}$
	$x_i \geq 0$	ith constrain $\geq$
	$x_i \leq 0$	ith constrain $\leq$
	$x_i \in \mathbb{R}$	ith constrain $=$

Diet problem: How much apricots (x_1), bananas (x_2) and cucumbers (x_3) one has to eat to get enough of Vit A, B, C? Minimize cost.

Need to know: % of daily value and cost:

	A	C	K	\$	ammount
apricots	60	26	6	1.53	155g
bananas	3	33	1	0.37	225g
cucumbers	2	7	12	0.18	133g

1: Write Linear Program (P) solving the diet problem and write also its dual (D)

Solution:

$$(P)\begin{cases}\text{minimize} & 1.53x_1 + 0.37x_2 + 0.18x_3\\ \text{s.t.} & 60x_1 + 3x_2 + 2x_3 \geq 100\\ & 26x_1 + 33x_2 + 7x_3 \geq 100\\ & 6x_1 + 1x_2 + 12x_3 \geq 100\\ & x_1, x_2, x_3 \geq 0\end{cases} \qquad (D)\begin{cases}\text{maximize} & 100y_1 + 100y_2 + 100y_3\\ \text{s.t.} & 60y_1 + 26y_2 + 6y_3 \leq 1.53\\ & 3y_1 + 33y_2 + 1y_3 \leq 0.37\\ & 2y_1 + 7y_2 + 12y_3 \leq 0.18\end{cases}$$

Solution for (P) is $\mathbf{x} = (1.4, 0.3, 7.6)$ and the cost is \$3.62.
and for (D) is $\mathbf{y} = (0.0209, 0.009122, 0.006191)$ and the cost is \$3.62.

2: What are units of y_i in (D)? (Hint: inequalities need to make sense in units.)

Solution: Units of y_i are $\frac{\$}{\text{1\% of daily intake of vitamin}}$.

3: Imagine you want to create a multivitamine pills ACK. What is the maximum price of one ACK pull if it has to deliver 100% of recommended daily value of vitamins A,C, and K and it must beat any fruit and vegetable in terms of price? *(If you don't manage to beat fruit and vegetable, nobody will buy your fancy ACK pill)* (Compute your price as a combination of prices of each of the vitamines.)

Solution: Same as the dual program (D).

$$(D)\begin{cases} \text{maximize} & 100y_1 + 100y_2 + 100y_3 \\ \text{s.t.} & 60y_1 + 26y_2 + 6y_3 \le 1.53 \\ & 3y_1 + 33y_2 + 1y_3 \le 0.37 \\ & 2y_1 + 7y_2 + 12y_3 \le 0.18 \end{cases}$$

Duality Theorem
For the linear programs

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0} \qquad (P)$$

and

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} \ge \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0} \qquad (D)$$

exactly one of the following possibilities occurs:

1. Neither (P) nor (D) has a feasible solution.
2. (P) is unbounded and (D) has no feasible solution.
3. (P) has no feasible solution and (D) is unbounded.
4. Both (P) and (D) have a feasiblesolution. Then both have an optimal solution, and if $\mathbf{x}^\star$ is an optimal solution of (P) and $\mathbf{y}^\star$ is an optimal solution of (D), then

   $$\mathbf{c}^T\mathbf{x}^\star = \mathbf{b}^T\mathbf{y}^\star.$$

   That is, the maximum of (P) equals the minimum of (D).

Next goal is to prove the duality theorem.