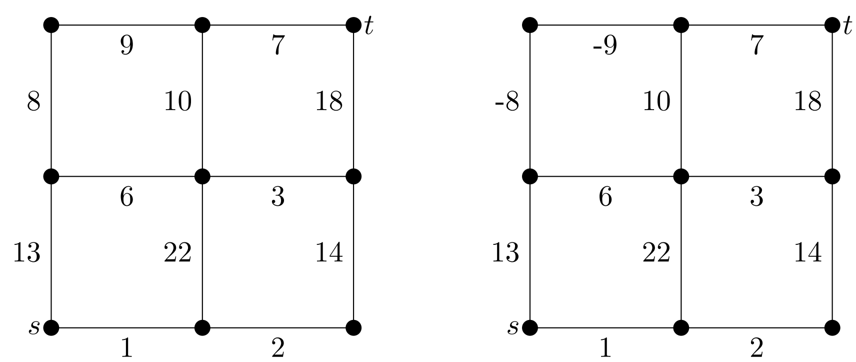
Shortest path

Source: Chapter 2.2 (Bills), Chapter 7 of Combinatorial Optimization (Korte)

Shortest path
Input: Graph $G = (V, E)$, costs $c : E \to \mathbb{R}$, and $s, t \in V$.
Output: s-t-path P, where $\sum_{e \in P} c(e)$ is minimized.

1: Find shortest (lowest cost) s-t-paths in the following graphs



Cost c is called *conservative* if there is no circuit of negative total weight.

Bellman's principle: Let $s, \ldots, v, w$ be the least cost s-w-path of length k. The $s, \ldots, v$ is the least cost s-v-path of length $k-1$.

2: *Prove Bellman's principle.*

Solution: By contradiction. If there is a lesser cost path to v, we could find a lesser cost path to w.

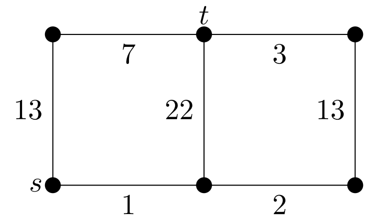
Notice: This gives a recursion for computing the shortest path.

Dijkstra's algorithm
$c : E \to \mathbb{R}_+$, computes shortest s-t-path from s to ALL other vertices $t \in V$.

1. $l(s) := 0$; $\forall v \neq s$ $l(v) = +\infty$
2. $R = \emptyset$
3. while $R \neq V$
4. find $v \in V - R$ with minimum $l(v)$
5. $R := R \cup \{v\}$
6. $\forall vw \in E$, $l(w) = \min\{l(w), l(v) + c(v, w)\}$

$R \ldots$ vertices with final number; $l \ldots$ upper bound on the cost;
Running time $O(n^2)$ easily, $O(m + n \log n)$ with Fibonacci heaps.

3: Run Dijkstra's algorithm on the following graph



4: How to get shortest s-v-path?

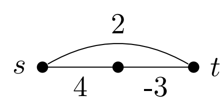
Solution: Remember previous vertex. In step 6. of the algorithm, remember why the value was changed. So called *predecessor*.

5: Why is the algorithm correct? (show that if $v \in R$, then $l(v) =$ cost for s-v-path.)

Solution: Could be done for example by contradiction. Suppose that there is a closer one. Then take the one with lowest s-v-costs that is not the same as the shortest path. And look at the predecessor on the shortest s-v-path. It gives contradiction with the run of the algorithm.

6: Why Dijkstra's algorithm does not work for negative costs?

Solution: For simplicity consider directed graph problem. The assumption that we can fix a cost of the lowest visited so far is not true.



Moore-Bellman-Ford Algorithm
$c : E \to \mathbb{R}$, computes shortest s-t-path from s to ALL other vertices $t \in V$ **OR** finds a cycle of negative cost. Assume $|V(G)| = n$.

1. $l(s) := 0$; $\forall v \neq s$ $l(v) = +\infty$
2. repeat $n - 1$ times: // computes the costs
3. $\quad \forall vw \in E$,
4. $\qquad$ if $l(w) > l(v) + c(v, w)$
5. $\qquad\quad l(w) := l(v) + c(v, w)$; $p(w) = v$
6. $\forall vw \in E$, // check for a negative cycle
7. $\quad$ if $l(w) > l(v) + c(v, w)$ then found negative cycle

Note: l gives the least cost, while p gives the **previous** vertex / **predecesor** on the shortest path from s.

7: What is the time complexity of the algorithm if G has m edges and n vertices?

Solution: $O(nm)$.

8: Why the algorithm detects a negative cycle and why the algorithm works?