Fall 2016, MATH-566

Minimum-Weight Perfect Matching for Point in Plane

Source: Bill,Bill,Bill

Goal: Find a minimum-weight perfect matching algorithm that works for points in the plane. Plan is to find a relaxation of the integer program and guide us to build an algorithm by cleverly interpreting the dual solution.

First we write a good linear program for general matching problem.

Recall that Minimum-weight perfect matching problem can be formulated as an integer programming problem in the following way

$$(IP) \begin{cases} \text{minimize} & \sum_{e \in E} c(e) x_e \\ \text{subject to} & \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V \\ & \mathbf{x} \in \{0,1\}^{|E|}, \end{cases}$$

1: Show that a relaxation of (IP) to linear program may result in optimal solution that is not realizable by a perfect matching. You need to cleverly assign weights!

2: Write a better program (P) that prevents issue from the previous figure and its dual (D). (Hint: Let $\mathcal{C}$ be the set of all odd cuts as set of edges and use $\mathcal{C}$.

Solution: We include constraint that for every odd cut, the edges going across sum to at least one. A cut is *odd* if it contains odd number of vertices on each side. Let $\mathcal{C}$ be the set of all odd cuts, where we consider as a cut the set of edges.

$$(P) \begin{cases} \text{minimize} & \sum_{e \in E} c(e) x_e \\ \text{subject to} & \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V \\ & \sum_{e \in D} x_e \geq 1 \text{ for all } D \in \mathcal{C} \\ & x_e \geq 0 \text{ for all } e \in E \end{cases} \qquad (D) \begin{cases} \text{maximize} & \sum_{v \in V} y_v \\ \text{subject to} & y_u + y_v + \sum_{uv \in D \in \mathcal{C}} Y_D \leq c(uv) \text{ for all } uv \in E \\ & y_v \in \mathbb{R} \text{ for all } v \in V \\ & Y_D \geq 0 \text{ for all } D \in \mathcal{C} \end{cases}$$

Notice that the bad solution from the previous relaxation is gone.

Theorem Edmonds: G has a perfect matching iff (P) has a feasible solution. Moreover, the minimum weight of the perfect matching is equal to the value of optimal solution to (P).

3: Write complementary slackness conditions for (P).

Solution:
$x_e > 0$ implies $y_e + y_e + \sum_{e \in D \in \mathcal{C}} Y_D = c(e)$
$Y_D > 0$ implies $\sum_{e \in D} x_e = 1$ for all D

A family of sets $\mathcal{A}$ is *nested* if for any $A, B \in \mathcal{A}$ exactly one of $A \cap B = \emptyset$, $A \subseteq B$, and $B \subseteq A$ holds.

A solution to (D) is *nested* if the family of Ds corresponding to $Y_D > 0$ is nested.

Theorem 5.17 If an optimal solution to (D) exists, then there exists a nested one.

If c satisfies triangle-inequality, then the dual has a nice solution.

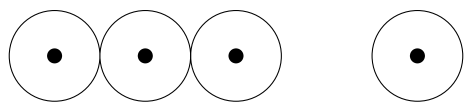
Theorem 5.20 Let G be a complete graph having even number of nodes and $c \geq 0$ satisfy the triangle inequality, then there exists an optimal solution to the dual with $y \geq 0$.

Finally, we are ready for some geometric ideas behind the algorithm that is on the cover of the textbook!

Let V be points in the plane and let $c : uv \to \mathbb{R}^+$ the Euclidean distance of u and v. Find a perfect matching M that is minimizing the sum of costs of the edges in the matching. Let E be the set of all pairs of vertices.

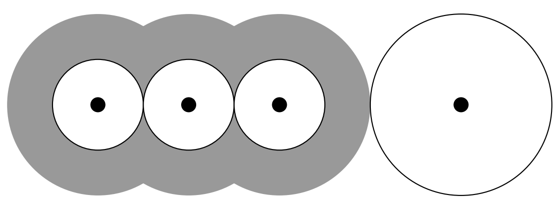
Notice this problem satisfies the triangle inequality.

4: Suppose $|V| = n$ and every point has a disk of radius 1. Assume these disks are disjoint. Can you give a lower bound on the cost of a perfect matching?



Solution: The lower bound is n as for every edge in the matching, it must go trough a radius of two circles.

5: Consider the following extension, where big white disk has radius 2 and the gray area is actually union of three disk of radius 2 without white disk of radius one. Can you provide a lower bound on the cost of a perfect matching?



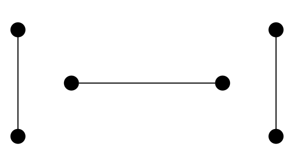
Solution: Lower bound would be 6. This comes from $1 + 1 + 1 + 2$ for the radius of the disks. Then additional $+1$ comes from the fact that they gray area encloses an odd number of points. So at least one edge of the perfect matching must pass through the gray area.

We call the white disks around vertices *control zones.* A pair of compact sets (N, I) is a *moat* if

$$I \subset N, |I \cap V| \text{ is odd, and } N \setminus \text{ interior}(I) \text{ contains no points in } V.$$

Example of a moat is the pair of gray set as N and the three white disks inside as I.

Notice we could interpret y_v as a radius of the control zone and Y_D as a width of a moat. This is a possible interpretation of a dual solution to D - try example, where you find optimal matching and corresponding control zones and moates.



Goal: Algorithm that finds a perfect matching of cost at most twice of the optimum solution.

Algorithm outline:
1) Obtain a forest F, where every component has even number of vertices. So called *even forest*.
2) Transform the forest F into a perfect matching M such that the cost of used edges does not increase.

In order to provide a bound on the cost of the resulting matching M, we build a forest $F^\star$ whose cost would be at most twice the lower bound obtained from control zones and moats.

Definition Let C be a set. Define

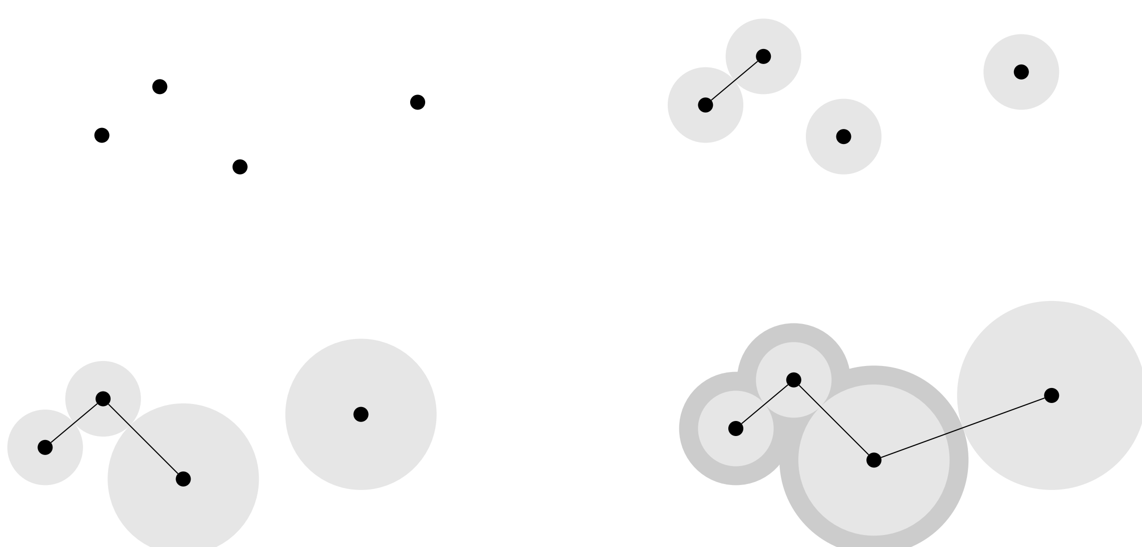
$$parity(C) = \begin{cases} 0 & \text{if } |C| \text{ even} \\ 1 & \text{if } |C| \text{ odd.} \end{cases}$$

Let $\overline{c}_e = c(uv) - y_u + y_v + \sum_{uv \in D \in \mathcal{C}} Y_D$. (Slack in the constraints in D.)

Goemans-Williamson algorithm (sketch)
Goal: Find an even forest F and a feasible solution to dual program (D).

1. $\mathcal{C} = \{\{v\} : v \in V\}$; $F = \emptyset$; $y = 0$; $Y = 0$
2. while exists $C \in \mathcal{C}$ with $|C|$ odd
3. Find an edge $e = uv$, with $u \in C_i$ and $v \in C_j$, $C_i \neq C_j$, that minimizes $\varepsilon = \overline{c}_e/(parity(C_i) + parity(C_j))$. Notice at least one of $|C_i|$ and $|C_j|$ is odd.
4. For all $C \in \mathcal{C}$ where $C = \{x\}$, add ε to y_x.
5. For all $C \in \mathcal{C}$ where $|C| > 1$ and $|C|$ is odd, add ε to Y_C.
6. add e to F and replace C_i and C_j by $C_i \cup C_j$.

Example:



6: Try the algorithm on the following example:

7: Show that there exists a matching M such that $\sum_{e\in M} c(e) \le \sum_{e\in F} c(e)$.

Solution: Use example and previous question to demonstrate the steps!
We provide an algorithmic way and use two steps
(a) If there are is an edge e in F such that $F - e$ is still an even forest, remove e.
(b) Otherwise consider F', which is F without leaves. Let v be a leaf in F'. Since the edge of F' incident with v is not removed in (a), v must have at least two pendant leaves x and y in F that were removed. Replace edges xv and yv by edge xy.
Notice none of these operations increase the cost of F and if none is applicable, F is a matching.

More precise way of computing the cost of the resulting lower bound LB on the matching:

8: Show that

$$LB = \sum_k \varepsilon^k |\{C \in \mathcal{C}^k : |C| \text{ odd }\}|$$

where $\mathcal{C}^k$ and ε^k are from the kth iterations of the algorithm.

Solution: This exactly corresponds to how much each each control zone and moat grows in each iterations and they must all be crossed.

9: Show that the forest $F^\star$ obtained from F by *dropping even edges* has cost at most $2\left(\sum_v y_v + \sum_D Y_D\right)$.
Cost of $F^\star$ is $\sum_{e\in F^\star} c(e)$.
More precisely, consider every edge $e \in F^\star$ of the forest and decompose its cost into small pieces.

$$c_e^k = \begin{cases} 0 & \text{if } C_i = C_j \\ \varepsilon^k(parity(C_i) + parity(C_j)) & \text{otherwise} \end{cases}$$

Now

$$c(e) = \sum_k c_e^k$$

Goal is to show for all k

$$\sum_e c_e^k \le 2\varepsilon^k |\{c \in \mathcal{C}^k : |C| \text{ odd }\}|,$$

which gives

$$cost(M) \le cost(F^\star) = \sum_k \sum_e c_e^k \le 2 \sum_k \varepsilon^k |\{C \in \mathcal{C}^k : |C| \text{ odd }\}| = 2LB.$$

Solution: See the book.