Gomory-Hu Trees

Let $G = (V, E)$ be an **undirected** graph and $u : E \to \mathbb{R}_+$ be capacities on edges.

Problem: Compute minimum s-t-cut for all pairs $(s, t) \in V^2$.

Simple solution: Run $\binom{n}{2}$ times maximum-flow algorithm (it gives minimum cut too).

Better solution: Run $(n - 1)$ times maximum-flow algorithm. Due to Gomory-Hu.

Denote the minimum capacity of s-t-cut by λ_{st}.

1: Show that for any $i, j, k \in V(G)$, $\lambda_{ik} \geq \min\{\lambda_{ij}, \lambda_{jk}\}$.

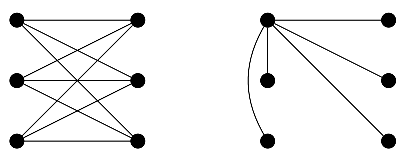
Solution: Consider minimum i-k-cut $\delta(A)$, where $i \in A \subset V$. If $j \in k$, then A is also j-k-cut. Hence $\lambda_{ik} \geq \lambda_{jk}$. The other case is symmetrical.

A tree T is a **Gomory-Hu Tree** for (G, u) if $V(T) = V(G)$ and $\forall s, t \in V$

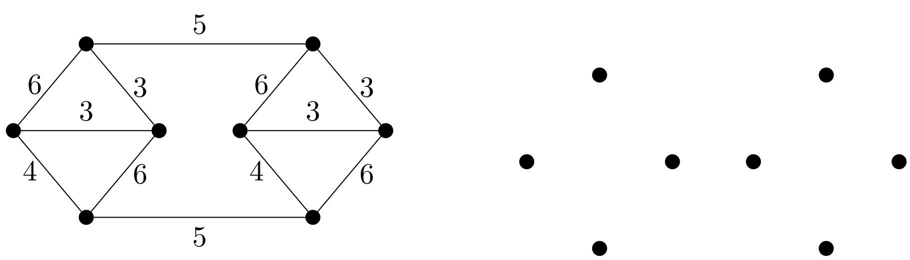
$$\lambda_{st} = \min_{e \in E(P_{st})} u(\delta_G(C_e)),$$

where P_{st} is the unique s-t-path in T, C_e is the set of vertices in the same connected component of $T - e$ as s and $\delta_G(C_e)$ is the cut defined by C_e in G.

Example of G, where $u : E \to 1$ is a constant. The tree T is then a star.



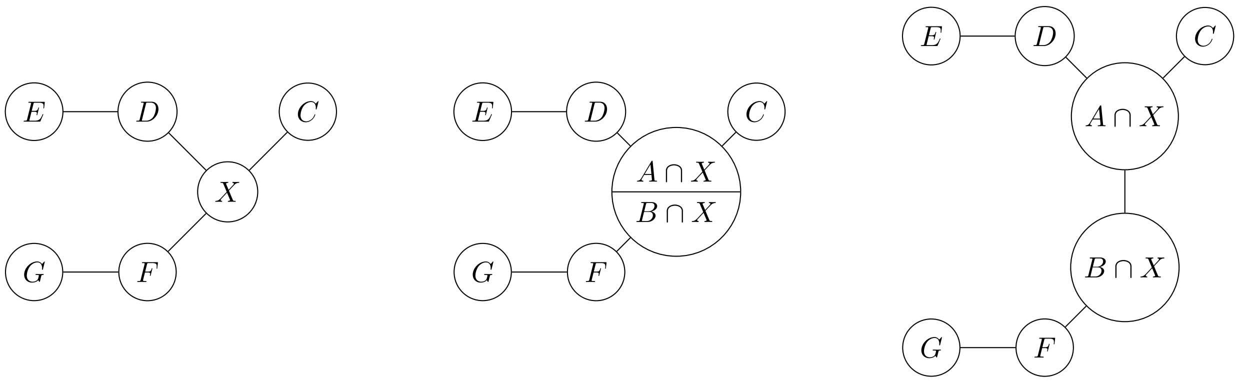
2: Find Gomory-Hu tree for the following graph with weights on edges.



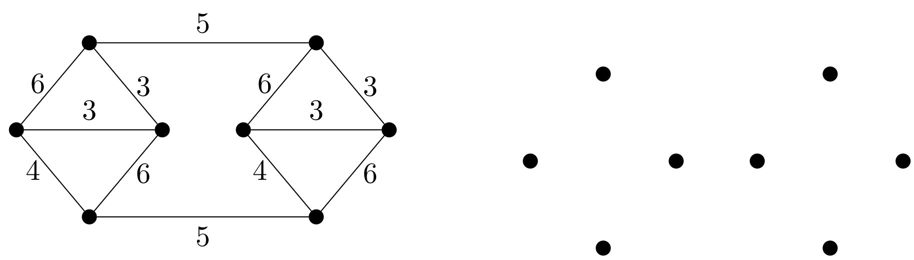
Algorithm:

1. $T = (\{V\}, \emptyset)$
2. while exists $X \in V(T)$, where $|X| \geq 2$,
3. pick any s, t in X
4. contract vertices of all nodes other than X
5. find minimum s-t-cut $A \cup B = V$
6. replace X in $V(T)$ by edge $\{\{A \cap X\}, \{B \cap X\}\}$.

Sketch of one iteration:



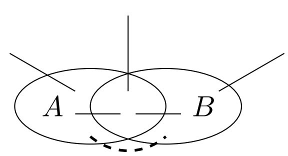
3: Run the algorithm on the graph from question 2.



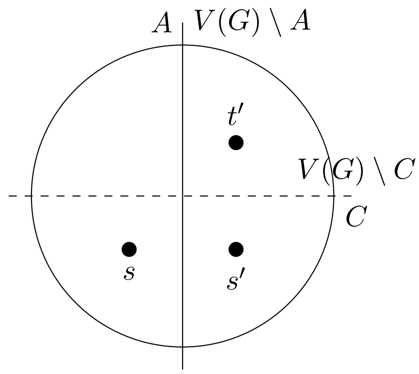
4: *(Cuts are submodular)* That is, let $A, B \subset V$. Show that

$$u(\delta(A \cup B)) + u(\delta(A \cap B)) \leq u(\delta(A)) + u(\delta(B)).$$

Solution: All edges on the left are counted on the right but edges going between $A \setminus B$ and $B \setminus A$ are missing on the left hand side. The dashed edges are missing.



5: *Algorithm creates optimal cuts* Let $s, t \in V$ and let $A \subset V$ such that $\delta(A)$ is a minimum s-t-cut. Let $s', t' \in V \setminus A$. Let (G', u') be obtained from G by contracting vertices of A into one vertex a'. Let $K \subset (G')$ such that $\delta_{G'}(K \cup \{a'\})$ is a minimum s'-t'-cut in G'. Show that $\delta_G(K \cup A)$ is a minimum s'-t'-cut in G.
Proof beginning: Assume $\delta(C)$ is a minimum s'-t' cut in G. Show that $\delta(C \cup A)$ is also a minimum s'-t' cut in G. Wlog $s \in A \cap C$.



Solution: By submodularity, we have

$$u(\delta(A \cup C)) + u(\delta(A \cap C)) \leq u(\delta(A)) + u(\delta(C))$$

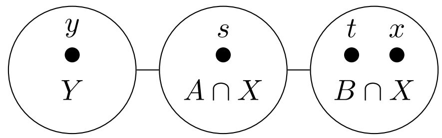
Observe that $\delta(A \cap C)$ is an s-t-cut. Hence $u(\delta(A)) \leq u(\delta(A \cap C))$. Therefore $u(\delta(A \cup C)) \leq u(\delta(C))$ and $\delta(A \cup C)$ is also minimum s'-t'-cut.

6: Let T be a tree during the run of algorithm. Let $e \in E(T)$ be any edge of T. Denote the endpoints of e by X and Y. Show that there are vertices $x \in X$ and $y \in Y$ such that e describes a minimum x-y cut.

The algorithm produces tree that works like Gomory-Hu tree at least for vertices adjacent in the tree.

Proof start: At the beginning of the algorithm (or after the first iteration, the observation is true. Let X and s, t be from step 3. The new edge $A \cap X$ to $B \cap X$ is correct due to s, t. Edges not incident with X are also correct. Remaining are edges that used to be incident with X but now are incident with $A \cap X$ (or $B \cap X$).

Suppose the edge YX had vertices y and x. If Y is in the A part of s-t-cut and x is in the other part, the edge $Y, (X \cap A)$ needs to be verified to satisfy the conclusion.



Solution: We show $\lambda_{sy} = \lambda_{yt}$, which gives the desired conclusion. We get

$$\lambda_{sy} \geq \min\{\lambda_{st}, \lambda_{tx}, \lambda_{xy}\}$$

If we contract $B \cap X$ to one vertex, λ_{sy} does not change. Hence

$$\lambda_{sy} \geq \min\{\lambda_{st}, \lambda_{xy}\}$$

Now A-B-cut separates x and y, hence $\lambda_{st} \geq \lambda_{xy}$ and we have

$$\lambda_{sy} \geq \lambda_{xy}\}$$

On the other hand, X-Y cut has size λ_{xy} and it is also s-y-cut. Hence

$$\lambda_{sy} \leq \lambda_{xy}$$

and the equality is proved.

7: Show that the tree produced by the algorithm is indeed a Gomory-Hu tree.

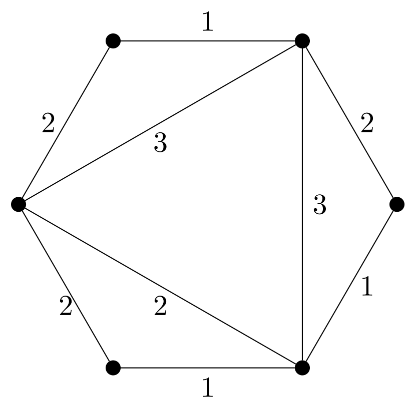
Solution: Let u, v be two vertices. Let $P_{uv} = (u = x_1, x_2, \ldots, x_n = v)$ be the path with endpoints u and v in T.

Observe

$$\lambda_{uv} \geq \min_{1 \leq i \leq n-1} \{\lambda_{x_i x_{i+1}}\}$$

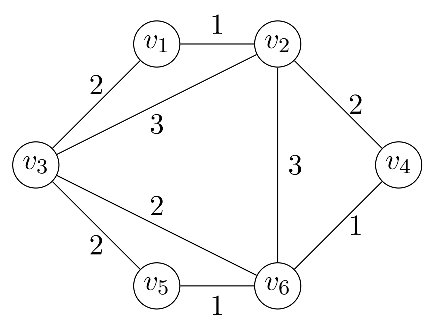
Since $\lambda_{x_i x_{i+1}}$ is obtained by edge $x_i x_{i+1}$ in the tree, we are done.

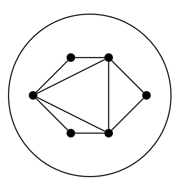
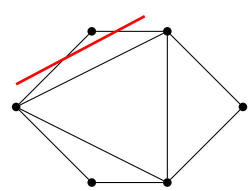
8: *Gomory-Hu Tree* Construct Gomory-Hu Tree for the following graph using the algorithm from the class. Numbers on edges correspond to capacities.

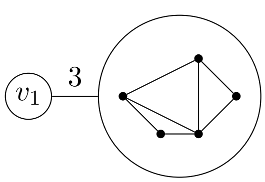


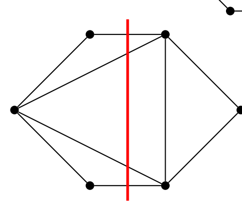
Solution:

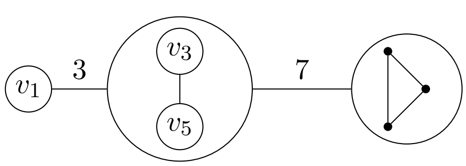
Here is a more detailed outline of the run of the algorithm. First we assign labels to vertices.

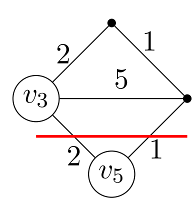


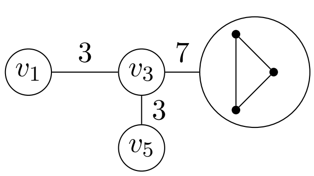
Start with $T = (\{V(G)\}, \emptyset)$:  pick v_1, v_2, then the cut is 

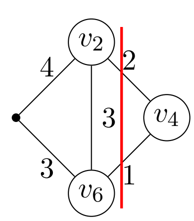
Then T is 

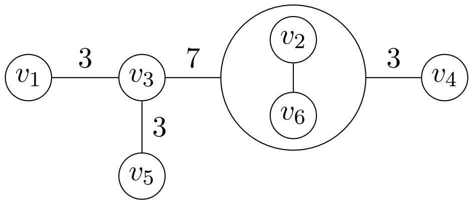
pick v_2, v_3, then the cut is 

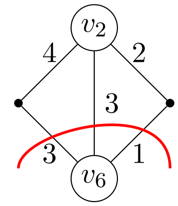
Then T is 

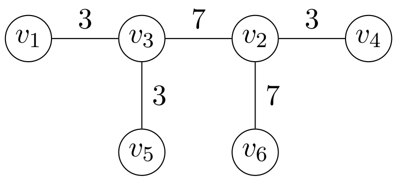
pick v_3, v_5, then the cut is 

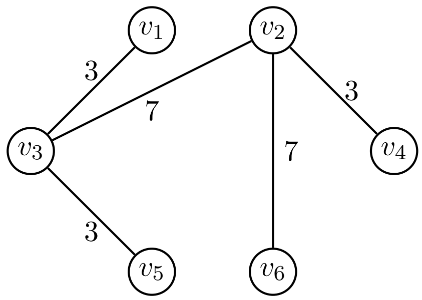
Then T is 

pick v_2, v_4, then the cut is 

Then T is 

pick v_2, v_6, then the cut is 

Then T is 

So in the form of G: 

Note that the tree is not unique and that there are many different ones. The tree depends on the order of cuts. But it should still have a path of length 2 with edges of value 7 and three additional leaves with edges of value 3.