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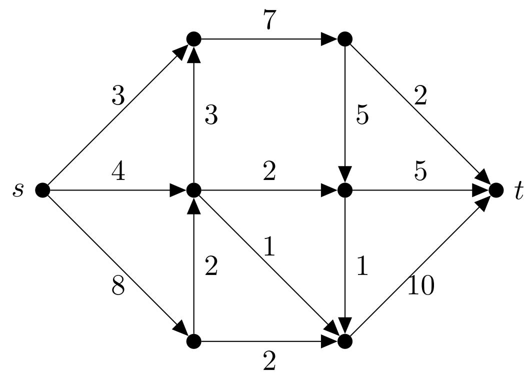
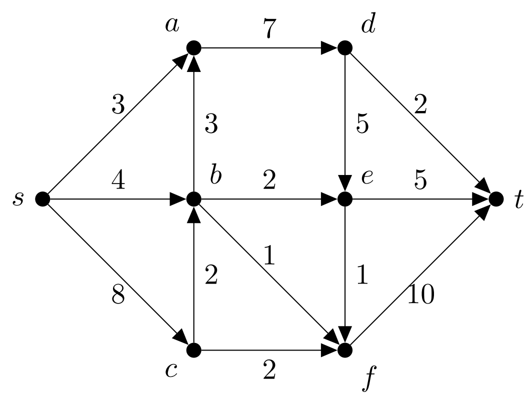
MATH 566 Discrete Optimization
Homework 6

1. Consider the graph below



Find a shortest path and prove optimality using duality (find dual LP and its optimal solution)

First let me redraw the graph with all of the vertices labeled.



From observation it is possible to see that the shortest path is $s \to b \to e \to t$ and it has weight 11.

In order to verify this, the dual of the shortest path linear program can be solved. The dual of the shortest path linear program is shown below

$$\begin{array}{ll} \max & y_t - y_s \\ \text{s.t.} & y_v - y_u \leq c(uv) \quad \forall (u,v) \in E \\ & y_s = 0 \end{array}$$

This linear program can be solved using the following sage script.

```
import operator
M = Matrix([[0, 3, 4, 8, 0, 0, 0, 0],
            [0, 0, 0, 0, 7, 0, 0, 0],
            [0, 3, 0, 0, 0, 2, 1, 0],
            [0, 0, 2, 0, 0, 0, 2, 0],
            [0, 0, 0, 0, 0, 5, 0, 2],
            [0, 0, 0, 0, 0, 0, 1, 5],
            [0, 0, 0, 0, 0, 0, 0,10],
```

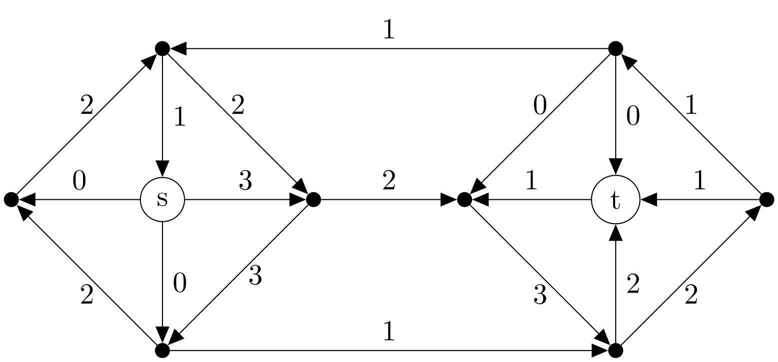
```
                [0, 0, 0, 0, 0, 0, 0, 0]])
G = DiGraph(M, weighted=True)
G.relabel({0:'s', 1:'a', 2:'b', 3:'c', 4:'d', 5:'e', 6:'f', 7:'t'})
s = 's'
t = 't'
milp = MixedIntegerLinearProgram(maximization=True)
y = milp.new_variable(nonnegative=True)
milp.set_objective(y[t] - y[s])
milp.add_constraint(y[s] == 0)
for edge in G.edges():
    milp.add_constraint(y[edge[1]] - y[edge[0]] <= edge[2])


print('Objective Value: {}'.format(milp.solve()))
sol = milp.get_values(y)
sol = sorted(sol.items(), key=operator.itemgetter(0))
for i, v in sol:
    print('y[%s] = %s' % (i, v))
```

The output of this script is as follows

```
Objective Value: 11.0
y[a] = 2.0
y[b] = 4.0
y[c] = 2.0
y[d] = 9.0
y[e] = 6.0
y[f] = 1.0
y[s] = 0.0
y[t] = 11.0
```

This shows that the linear program found a path of length 11 and the results show that $s \to b \to e \to t$ is adding up the weights on their respective edges. In other words $y[b] = 4$ and the weight from $s \to b$ is 4. The weight from $b \to e$ is 2, so $y[e] = 4 + 2 = 6$, and the weight from $e \to t$ is 5, so $y[t] = 6 + 5 = 11$.

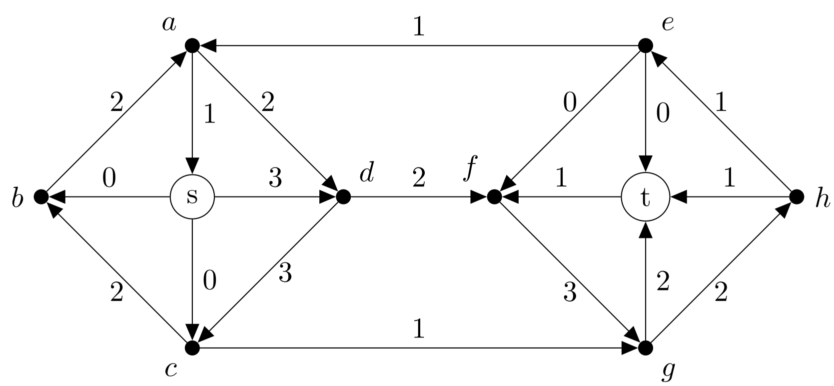
2. Consider the network below with given edge values, forming an integer feasible flow. Find a list of path and cycle flows whose sum is this flow.



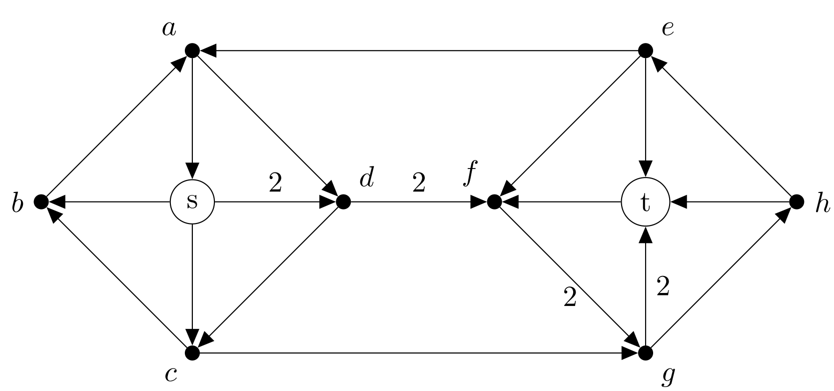
Gallai; Ford and Fulkerson proved a theorem that stated that any flow can be decomposed into s-t paths, $\mathcal{P}$ and circuits $\mathcal{C}$ with weight function $w : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^+$, such that

$$\begin{aligned} f(e) &= \sum_{e \in P \in \mathcal{P}} (w(P)) + \sum_{e \in C \in \mathcal{C}} (w(C)) \\ value(f) &= \sum_{P \in \mathcal{P}} (w(P)) \\ |\mathcal{P} + \mathcal{C}| &\leq |E(G)| \end{aligned}$$

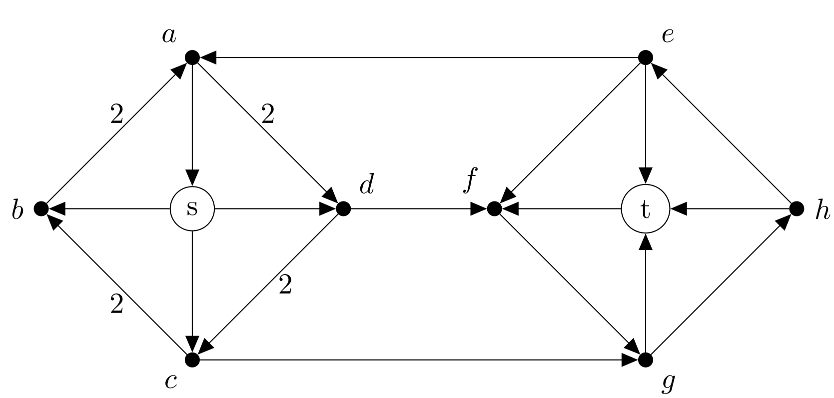
In order to find such a decomposition I will first relabel all of the vertices as follows.



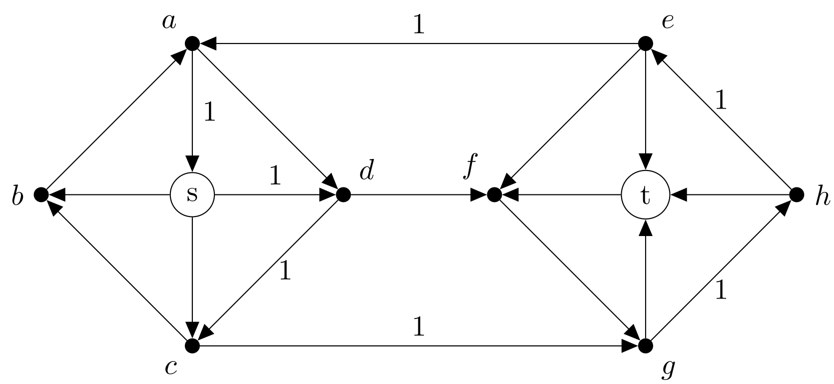
Now one way this flow can be decomposed is by using 1 path and 3 circuits as follows. The one s-t path is $P = s \to d \to f \to g \to t$ with weight $w(P) = 2$.



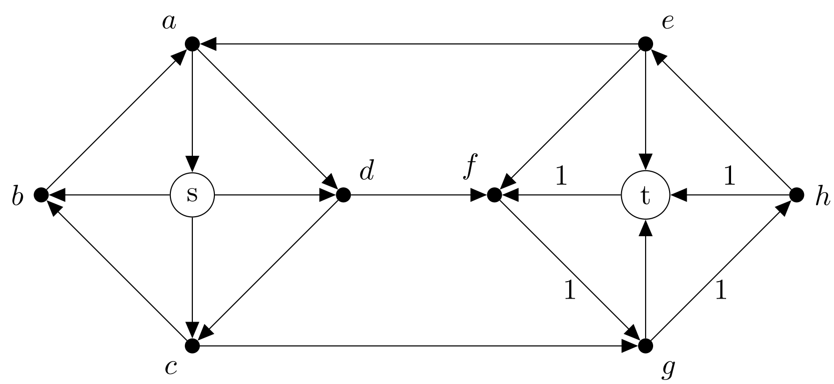
The first circuit is $C_1 = d \to c \to b \to a \to d$ with weight $w(C_1) = 2$.



The second circuit is $C_2 = a \to s \to d \to c \to g \to h \to e \to a$ with weight $w(C_2) = 1$.

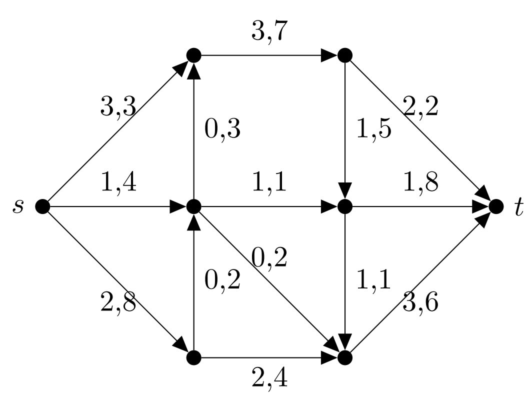


The third and final circuit is $C_3 = f \to g \to h \to t \to f$ with weight $w(C_3) = 1$.

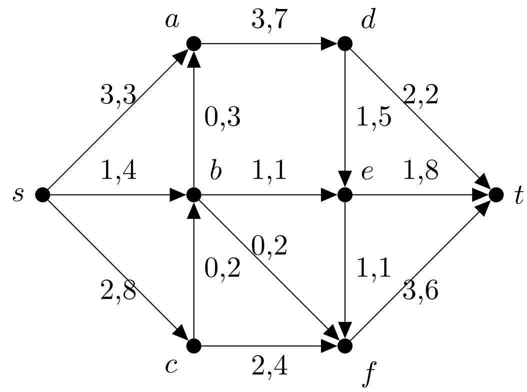


This set of paths and circuits with corresponding weight function satisfies all three of the properties stated in the Theorem. The sum of these paths and circuits results in the original flow. The sum of the weights on the paths is equal to the value of the flow. Also the number of paths and circuits is less than the number of edges of G.

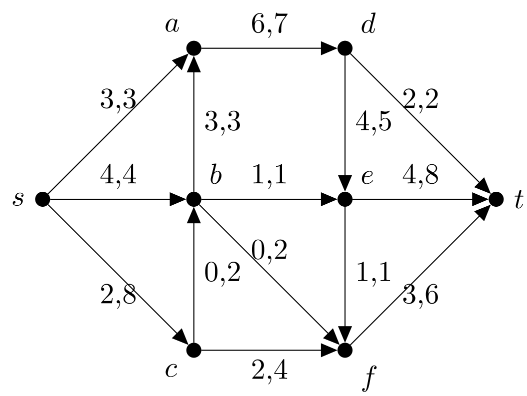
3. Consider the network below with given capacity and flow values. (The edge label f, u means flow-value f and capacity u.) Find augmenting paths and augment the flow to a maximum flow. Provide the list of residual graphs AND augmenting paths. In other words, run Ford-Fulkerson algorithm.



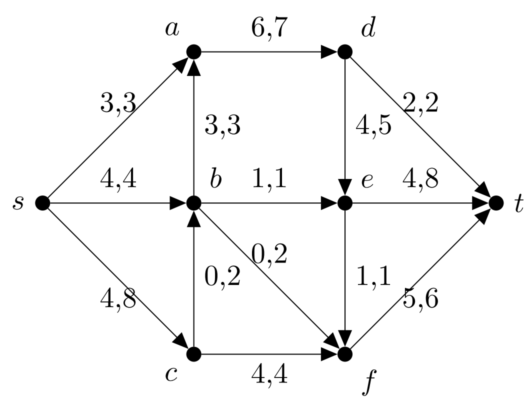
I will show the augmenting paths and the flow on the original graph however I have attached the residual graphs on a separate sheet of paper. First let me relabel the vertices.



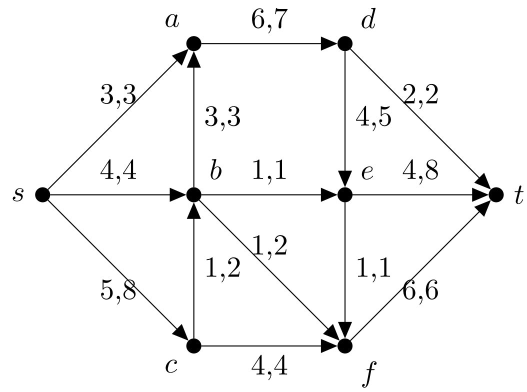
The initial augmenting path will be $P = s \to b \to a \to d \to e \to t$. The minimum capacity over this path is $\gamma = 3$. Augmenting on this path gives the following flow.



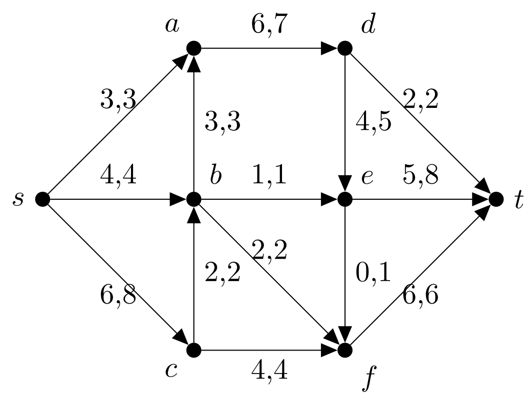
The second augmenting path I will use is $P = s \to c \to f \to t$ with a minimum capacity of $\gamma = 2$. The new flow will be



The third augmenting path is $P = s \to c \to b \to f \to t$ with minimum capacity, $\gamma = 1$. Augmenting along this path gives



The fourth augmenting path is $P = s \to c \to b \to f \to e \to t$ with minimum capacity, $\gamma = 1$. Note that this is decreasing the flow on $f \to e$.



This is last augmenting path, and so this flow is optimal.

4. Let (G, u, s, t) be a network, and let $\delta^+(X)$ and $\delta^+(Y)$ be minimum s-t-cuts in (G, u). Show that $\delta^+(X \cap Y)$ and $\delta^+(X \cup Y)$ are also minimum s-t-cuts in (G, u, s, t).

Proof. Let (G, u, s, t) be a network with edges E and vertices V. Let $X, Y \subset V$ such that $\delta^+(X)$ and $\delta^+(Y)$ are minimum s-t cuts. First let me introduce a notation for summing the capacities of a set of edges.

$$U(Z) = \sum_{e \in Z} (u(e))$$

Note that by a theorem by Ford-Fulkerson the maximum value of the flow is equal to the capacity of $\delta^+(X)$ and $\delta^+(Y)$ as there are both minimum cuts. Let

$$m = U(\delta^+(X))$$

This also implies that

$$m = U(\delta^+(Y))$$

Notice that their are four possibilities for edges in $\delta^+(X)$. Edges can go from $Y \to Y$, from $Y \to Y^c$, from $Y^c \to Y$, and from $Y^c \to Y^c$. Based on this observation consider the following sets of edges.

$$
\begin{aligned}
A &= \{(v,w) : v \in X, v \notin Y, w \notin X, w \in Y\} \\
B &= \{(v,w) : v \in X, v \notin Y, w \notin X, w \notin Y\} \\
C &= \{(v,w) : v \in X, v \in Y, w \notin X, w \in Y\} \\
D &= \{(v,w) : v \in X, v \in Y, w \notin X, w \notin Y\} \\
E &= \{(v,w) : v \in X, v \in Y, w \in X, w \notin Y\} \\
F &= \{(v,w) : v \notin X, v \in Y, w \notin X, w \notin Y\} \\
G &= \{(v,w) : v \notin X, v \in Y, w \in X, w \notin Y\}
\end{aligned}
$$

Note that these sets are mutually disjoint. The sets $\delta^+(X)$ and $\delta^+(Y)$ can be partitioned in terms of these sets. In fact $\delta^+(X) = A \cup B \cup C \cup D$ and $\delta^+(Y) = D \cup E \cup F \cup G$. Since we have partitioned $\delta^+(X)$ and $\delta^+(Y)$, we can sum up the capacities as follows

$$
\begin{aligned}
U(\delta^+(X)) &= U(A) + U(B) + U(C) + U(D) \\
U(\delta^+(Y)) &= U(D) + U(E) + U(F) + U(G)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
m &= U(A) + U(B) + U(C) + U(D) \\
m &= U(D) + U(E) + U(F) + U(G)
\end{aligned}
$$

Now consider the cuts $\delta^+(X \cap Y)$ and $\delta^+(X \cup Y)$. Since these sets are cuts they must be greater than the value of minimum cut, the following inequalities hold.

$$
\begin{aligned}
m &\leq U(\delta^+(X \cap Y)) \\
m &\leq U(\delta^+(X \cup Y))
\end{aligned}
$$

Now consider how we might partition these cuts in terms of the sets that were created earlier. The set $\delta^+(X \cap Y)$ is the set of all edges from both X and Y, to either X^c or Y^c, therefore $\delta^+(X \cap Y) = C \cup D \cup E$. The set $\delta^+(X \cup Y)$ is the set of all edges from X or Y, to $X^c \cap Y^c$, so $\delta^+(X \cup Y) = B \cup D \cup F$. Using these partitions we can consider

$$
\begin{aligned}
2m &\leq U(\delta^+(X \cap Y)) + U(\delta^+(X \cup Y)) \\
&\leq U(C) + U(D) + U(E) + U(B) + U(D) + U(F)
\end{aligned}
$$

Since u is a positive function, $U(A) > 0$ and $U(G) > 0$, so

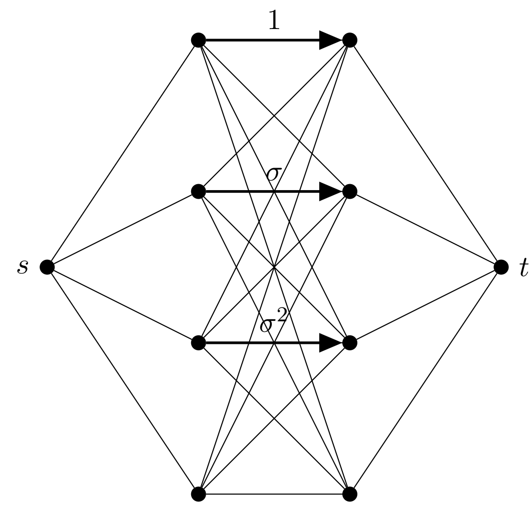
$$
\begin{aligned}
&\leq U(A) + U(B) + U(C) + U(D) + U(D) + U(E) + U(F) + U(D) \\
&\leq U(\delta^+(X)) + U(\delta^+(Y)) \\
&\leq 2m
\end{aligned}
$$

This shows that $U(\delta^+(X \cap Y)) + U(\delta^+(X \cup Y)) = 2m$. Since each cut individually is greater than m it follows that

$$
\begin{aligned}
U(\delta^+(X \cap Y)) &= m \\
U(\delta^+(X \cup Y)) &= m
\end{aligned}
$$

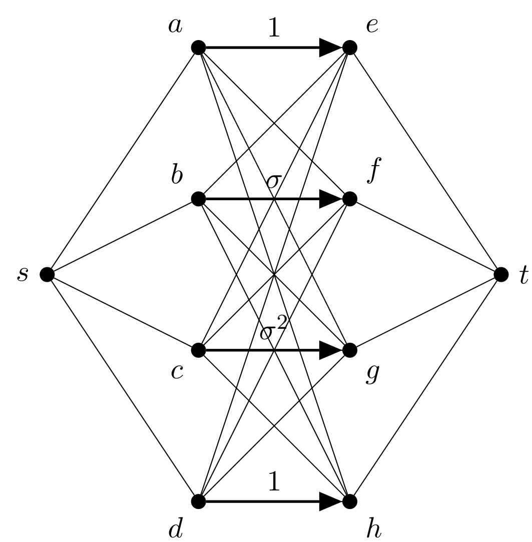
Since these cut's capacity are equal to the maximum flow, they must be minimum cuts. This shows that $\delta^+(X \cap Y)$ and $\delta^+(X \cup Y)$ are both minimum cuts. $\square$

5. Show that in case of irrational capacities, the Ford-Fulkerson algorithm may not terminate at all. Hint: See the Korte book (in particular exercises on page 199.). It contains the following network:



Where $\sigma = \frac{\sqrt{5}-1}{2}$. Note that σ satisfies $\sigma^n = \sigma^{n+1} + \sigma^{n+2}$. All other capacities are 1.

In order to show that the Ford-Fulkerson algorithm may not terminate it must be shown that there is an infinite sequence of augmenting paths. First let me relabel the vertices of the graph.



Also note that I am going to change the capacities on all other edges to 100, except for (d, h) where $u((d, h)) = 1$. This is just to make sure that the capacities on other edges are large enough that the flow on these edges doesn't need to be considered. This way only the flow on the edges (a, e), (b, f), (c, g), and (d, h) needs to be considered, which I will name e_1, e_2, e_3, and e_4 respecitively

Consider the following augmenting paths.

$$
\begin{aligned}
P_1 &= s \to a \to e \to t \\
P_2 &= s \to b \to f \to d \to e \to a \to g \to d \to h \to t \\
P_3 &= s \to a \to e \to d \to f \to b \to e \to t \\
P_4 &= s \to d \to h \to b \to f \to d \to e \to a \to h \to t \\
P_5 &= s \to a \to e \to b \to h \to d \to e \to t
\end{aligned}
$$

Path	$f(e_1)$	$f(e_2)$	$f(e_4)$
P_1	1	0	0
P_2	$1-\sigma$	σ	$1-\sigma^2$
P_3	1	0	$1-\sigma^2$
P_4	$1-\sigma^2$	$\sigma-\sigma^3$	1
P_5	1	$\sigma-\sigma^3$	$1-\sigma^2$
P_2	$1-\sigma^3$	σ	$1-\sigma^4$
P_3	1	$\sigma-\sigma^3$	$1-\sigma^4$
P_4	$1-\sigma^4$	$\sigma-\sigma^5$	1
P_5	1	$\sigma-\sigma^5$	$1-\sigma^4$

Table 1: Flows

These paths have the following effects P_1 initializes the flow by increasing across e_1. The path P_2 increases along e_2 and e_4 and decreases e_1. The path P_3 increases along e_1 and decreases along e_2. The path P_4 increases along e_4 and e_2 and decreases e_1. The path P_5 is similar to P_1 and increases e_1 and decreases e_4. If we augment on P_1 and then cycle P_2 through P_4 the flows on the main edges are described in the following table. This shows that $4k$ augmenting path the flow on e_1 will be $1-\sigma^{2k}$ and $f(e_2) = \sigma - \sigma^{2k+1}$. So this shows that this series of augmenting paths will not terminate, and thus the Ford-Fulkerson algorithm does not necessarily terminate for irrational capacities.

6. Red-Blue meta algorithm for MST. Let G be a graph and w be a weight assignment to $E(G)$. Assume that all weights are distinct. Start with all edges being uncolored. Apply the following rules as long as possible.

   - if $e \in E$ is in a cycle C where e is the heaviest edge, color e red
   - if there is a cut where $e \in E$ is the lightest edge, color e blue.

   Claim is that blue edges form a minimum spanning tree.

   - Show that red edge cannot be in MST.
   - Show that blue edge must be in MST.
   - Show that blue edges form a tree
   - Show that every edge gets colored.
   - Show that no edge satisfies both red and blue criteria. (i.e. every edge has one color).

   I chose not to do this problem.

7. Implement Edmonds-Karp algorithm and run it on the network from question three. Print the sequence of augmenting paths used by your implementation. Print the flow and its value.

   I implemented the Edmonds-Karp algorithm in the following function.

```
def edmondsKarp(G, s, t):
    # Find mazimal flow on G from vertex s to vertex t
    # G weighted digraph - weights represent capacities
    # s - starting/source vertex
    # t - ending/target vertex

    # create residual graph as copy of original graph
    RG = G.copy()
```

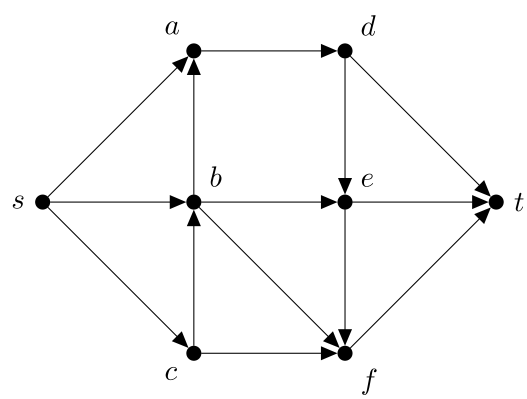
```
    for e in G.edges():
        RG.add_edge(e[1], e[0], 0)

    path = shortestPath(RG, s, t)
    while path != None:
        path.reverse()
        print path
        min_capacity = min({e[2] for e in path})
        # augment flow
        for edge in path:
            RG.add_edge(edge[0], edge[1], edge[2] - min_capacity)
            RG.add_edge(edge[1], edge[0], RG.edge_label(edge[1], edge
                ↪ [0]) + min_capacity)
        path = shortestPath(RG, s, t)

    # uses dictionary to store flow
    # if e is edge in G, then f[e] is flow on e
    # intialize all to have 0 flow
    flow = dict()
    for edge in G.edges():
        flow[edge] = RG.edge_label(edge[1], edge[0])

    return flow

def shortestPath(RG, source, target):
    # G is a graph
    # find the shortest path, P, from s to t or return None
    # shortest path in terms of least number of edges

    path = None
    # remove edges with 0 weight
    G = RG.copy()
    for edge in RG.edges():
        if edge[2] == 0:
            G.delete_edge(edge)
    tree = breadthFirstSearch(G, source)
    if tree.neighbors_in(target):
        path = []
        current_vertex = target
        while tree.neighbors_in(current_vertex):
            edge = tree.incoming_edges(current_vertex)[0]
            path.append(edge)
            current_vertex = edge[0]
    return path
```

This algorithm using a breadth first search which is implemented in the following function.

```
import Queue
def breadthFirstSearch(G, s):
    # G is a graph
    # s is the starting vertex
```

```
    # create empty tree
    T = DiGraph([G.vertices() ,[]])
    R = {s}

    # create queue to hold nodes
    q = Queue.Queue()

    #distanceDict[s] = 0
    q.put(s)

    while not q.empty():
        currentVertex = q.get()
        # iterate over edges incident to currentVertex
        # if G is directed only includes edges going out from
            ↪ currentVertex
        # Don't use neighbors function different for directed and
            ↪ undirected graphs
        for e in G.edges_incident(currentVertex):
            adjacentVertex = e[1]
            # if we haven't reached adjacentVertex yet
            if adjacentVertex not in R:
                q.put(adjacentVertex)
                R.add(adjacentVertex)
                T.add_edge(e)
    return T
```

In order to run this algorithm on the graph from problem 3, I first relabeled the vertices in this graph. The graph was relabeled as shown below.



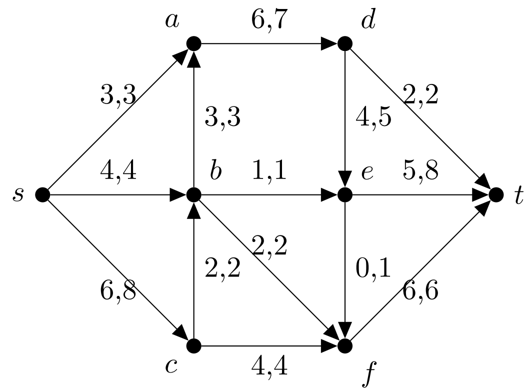

```
load('breadthFirstSearch.sage')
load('edmondsKarp.sage')
M = Matrix([[0, 3, 4, 8, 0, 0, 0, 0],
            [0, 0, 0, 0, 7, 0, 0, 0],
            [0, 3, 0, 0, 0, 1, 2, 0],
            [0, 0, 2, 0, 0, 0, 4, 0],
            [0, 0, 0, 0, 0, 5, 0, 2],
            [0, 0, 0, 0, 0, 0, 1, 8],
            [0, 0, 0, 0, 0, 0, 0, 6],
```

```
                [0, 0, 0, 0, 0, 0, 0, 0]])
G = DiGraph(M, weighted=True)
G.relabel({0:'s', 1:'a', 2:'b', 3:'c', 4:'d', 5:'e', 6:'f', 7:'t'})
s = 's'
t = 't'

flow = edmondsKarp(G, s, t)
print flow
```

This is the output of this script. Each list is the augmenting path. Each tuple is an edge in the augmenting path, with first entry the starting vertex, the second entry the ending vertex, and the third entry the available flow. The dictionary shows the flow on each edge in the form edge:flow.

```
[('s', 'a', 3), ('a', 'd', 7), ('d', 't', 2)]
[('s', 'b', 4), ('b', 'e', 1), ('e', 't', 8)]
[('s', 'b', 3), ('b', 'f', 2), ('f', 't', 6)]
[('s', 'c', 8), ('c', 'f', 4), ('f', 't', 4)]
[('s', 'a', 1), ('a', 'd', 5), ('d', 'e', 5), ('e', 't', 7)]
[('s', 'b', 1), ('b', 'a', 3), ('a', 'd', 4), ('d', 'e', 4), ('e', 't', 6)]
[('s', 'c', 4), ('c', 'b', 2), ('b', 'a', 2), ('a', 'd', 3),
  ('d', 'e', 3), ('e', 't', 5)]
{
  ('b', 'f', 2): 2,
  ('c', 'b', 2): 2,
  ('b', 'a', 3): 3,
  ('f', 't', 6): 6,
  ('s', 'b', 4): 4,
  ('e', 'f', 1): 0,
  ('a', 'd', 7): 6,
  ('s', 'c', 8): 6,
  ('d', 'e', 5): 4,
  ('s', 'a', 3): 3,
  ('b', 'e', 1): 1,
  ('c', 'f', 4): 4,
  ('d', 't', 2): 2,
  ('e', 't', 8): 5
  }
}
```

This flow can also be shown on the graph as follows.


a
6,7
d
3,3
3,3
2,2
4,5
s
4,4
b
1,1
e
5,8
t
2,2
2,2
0,1
6,8
6,6
c
4,4
f