Fall 2016, MATH-566

Minimum Cost Flow

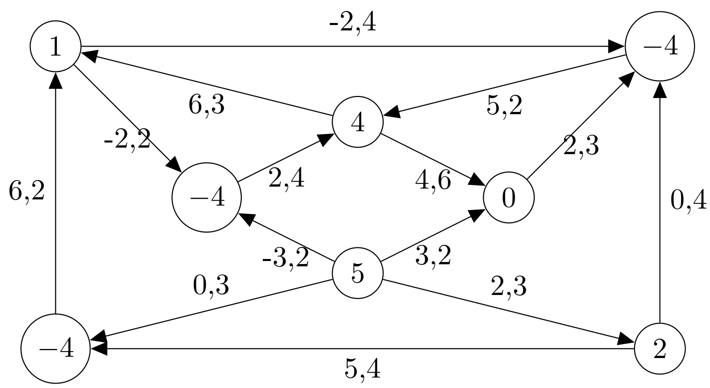
Problem: There are n coal mines and m power plants. Power plants have demands, coal mines supply coal. How to transport coal in order to satisfy the demands and minimize cost of transportation.

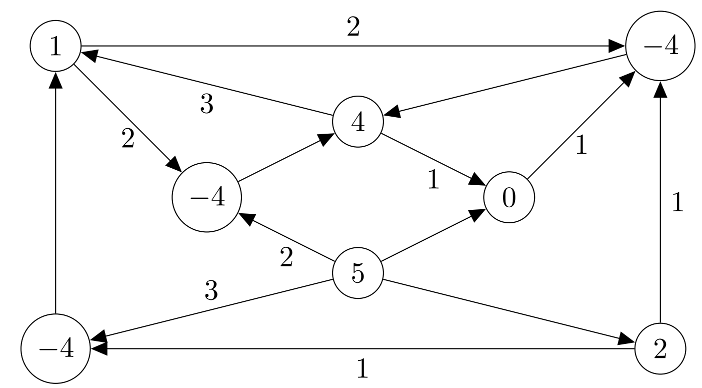
Let $G = (V, E)$ be a directed graph, $u : E \to \mathbb{R}^+$ be capacities on edges and $c : E \to \mathbb{R}$ be costs for every edge.

Let $b : V \to \mathbb{R}$ with $\sum_v b(v) = 0$ be a *supply demand* function. Called *boundary*.

b-flow is $f : E \to \mathbb{R}^+$ such that $f(e) \le u(e)$ and $\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e) = b(v)$.

1: Find a b-flow (that minimizes $\sum_e c(e)f(e)$) in the following network: (b is in every vertex, edges have c, u).





Solution: The value of the flow is 15.

If $b(v) > 0$, then b is *supply*, if $b(v) < 0$, then b is *demand*. Like flows but multiple sources and sinks.

Minimum Cost Flow Problem: find a b-flow f that minimizes $c(f) = \sum_e c(e)f(e)$.

2: Show that b-flow f exists iff

$$\sum_{e\in\delta^+(X)} u(e) \ge \sum_{v\in X} b(v) \text{ for all } X \subseteq V(G).$$

(That is, there is always enough capacity to take excessive flow out of X.)

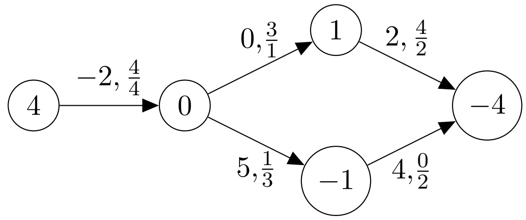
Solution: $\Rightarrow$ If there is a flow, the condition is easily satisfied, since $f(e) \le u(e)$.

$\Leftarrow$ Suppose there is no b-flow f. Add new vertices s and t, where sv is edge for every $v \in V$ with $b(v) > 0$. Assign $u(sv) = b(v)$. Do analogous operation with t. No b-flow implies no flow in the new graph. Hence there is a cut. The cut gives X that violates the big condition.

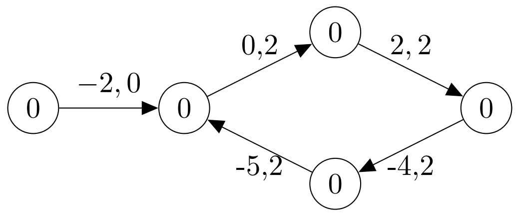
Consequence: It is possible to detect no solution case.

Circulation is a flow in a network where $b(v) = 0$ for every vertex.

3: Let f and f' be two b-flows. Consider their *difference* $f - f'$ and show that it is a circulation. Try on example first: Edge labels are $c, \frac{f}{f'}$. Compute $c(f)$, $c(f')$, find what is the difference.



Solution: $c(f) = 5$, $c(f') = 19$. Cost difference $c(f) - c(f') = -14$. In picture, we assign the difference of the flows, if the difference was negative, flip the edge and count negative cost. The picture gives a circulation with cost -14.



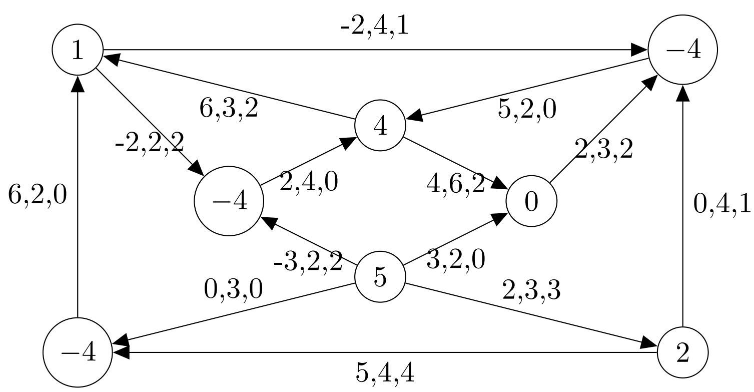
General: Difference between two flows is a circulation, its cost is the difference in the costs of f and f'. Since f and f' are b-flows, we get $\sum_{e\in\delta^+(v)}(f(e) - f'(e)) - \sum_{e\in\delta^-(v)}(f(e) - f'(e)) = b(v) - b(v) = 0$, Hence the difference is a circulation. Summing the difference gives the difference in costs.

Algorithm Minimum Cost Flow:

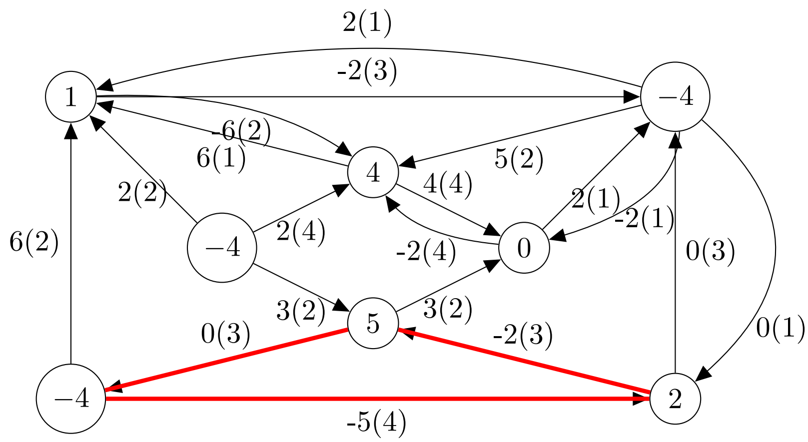
1. f be any b-flow
2. while exists negative cost cycle C in residual graph
3. pick C of minimum mean cost $= \frac{\sum_{e\in C} c(e)}{|C|}$.
4. augment on C

Minimum mean cost cycle gives polynomial time $O(m^2 n^2 \log n)$ (picking any cycle - same problem as Ford-Fulkerson).

4: Find a negative cycle in a residual graph for the following, where on edge is c, u, f in this order.



Solution: Residual graph is created by keeping edges e with nonzero $u(e) - f(e)$ and adding reverse edges if $f(e) > 0$. In order to find a negative cycle, we only need to know the cost. It is depicted in red.



5: Show that the algorithm is correct when it finishes. That is, f is an optimal b-flow iff it has no negative cycle.

Solution: $\Rightarrow$ If there is a negative cycle, augmenting on it, it will decrease the cost of f, contradiction with optimality of f.

$\Leftarrow$ If there is a better flow f', the difference between flows is a circulation. Since the sum of the circulation is negative, it must contain a negative circuit.

6: How to find minimum mean cycle?

Solution: Wait for the next class.