Stochastic Galerkin Method for Hamilton-Jacobi Equations with Uncertainty

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Early Graduate Research

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Introduction

Motivation

- Interest in the development of efficient and robust numerical methods for uncertainty quantification for PDE's with random input
- Randomness manipulated via initial conditions, boundary conditions, etc.
- The *generalized polynomial chaos* method combined with stochastic Galerkin method has resulted in fast convergence for engineering problems where solutions is sufficiently smooth

Stochastic Galerkin Method

- Approximates the stochastic problem via an orthogonal polynomial series.
- Different types of random variables require different families of orthogonal polynomials.
- The method adds dimensions to a given a problem to incorporate the randomness in the solution.

Stochastic Galerkin Example

Problem

Consider
$$\frac{\partial}{\partial t}u(t,Z) = \alpha u(t,Z) \quad u(0,Z) = \beta.$$
Let $\alpha = \mu + \sigma Z$ where Z is random variable, with a standard normal distribution and β is constant.

First we expand the parameters of the problem on the Hermite polynomial basis.

$$H_0 = 1$$
$$H_1 = z$$
$$H_2 = z^2 - 1$$
$$\vdots$$

Stochastic Galerkin Example

Step 1: Project onto basis

Growth Parameter

$$\alpha \approx \alpha_N(Z) = \sum_{i=0}^{N} a_i H_i(Z)$$

where

$$a_0 = \mu, \quad a_1 = \sigma, \quad a_i = 0,\ i > 1.$$

Initial Conditions

$$\beta \approx \beta_N(Z) = \sum_{i=1}^{N} b_i H_i(Z)$$

where

$$b_0 = \beta, \quad b_i = 0,\ i > 0$$

Stochastic Galerkin Example

Nth degree approximation

$$u \approx v_n(t, Z) = \sum_{i=0}^{N} \hat{v}_i(t) H_i(Z)$$

Updated Problem

$$\frac{\partial}{\partial t} v_N(t, Z) = \alpha_N v_N(t, Z), \quad v_N(0, Z) = \beta_N.$$

Stochastic Galerkin Example

Step 2: Use Orthogonality of basis

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} v_N(t, Z) H_k(Z) \mathrm{d}Z = \int_{-\infty}^{\infty} \alpha_N v_N(t, Z) H_k(Z) \mathrm{d}Z$$

$$\hat{v}_i'(t) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_i \hat{v}_j(t) e_{ijk}$$

for $k = 0, 1, 2, \ldots, N$, where

$$e_{ijk} = \int_{-\infty}^{\infty} H_i(Z) H_j(Z) H_k(Z) \mathrm{d}Z$$

Stochastic Galerkin Example

Step 3: Use Initial Conditions

$$v_N(0, Z) = \beta_N$$

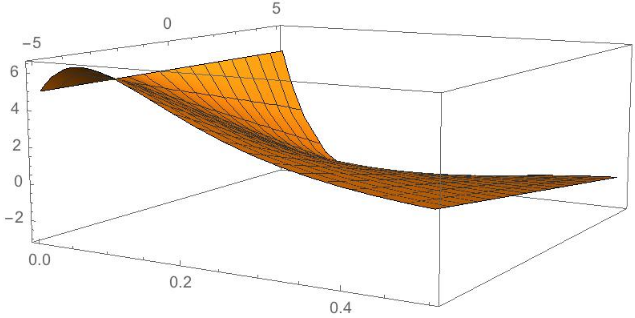
$$\int_{-\infty}^{\infty} v_N(0, Z) H_k(Z) \mathrm{d}Z = \int_{-\infty}^{\infty} \beta_N H_k(Z) \mathrm{d}Z$$

$$\hat{v}_k(0) = b_k$$

Stochastic Galerkin Example

Step 4: Solve the new system of differential equations

$$v_N(t,z) = -\frac{1}{12}e^{-12t}+(-31-30e^{4t}+e^{12t})-\frac{1}{6}e^{-12t}(-31+30e^{4t}+e^{12t})z$$



Applications of HJ

- Game Theory
- Optimal Control
- Geometric Optics
- Front Propagation
- **Quantum Dynamics**

Hamilton-Jacobi Equation

Hamilton-Jacobi Equation

$$\partial_t u + H(\nabla_{\mathbf{x}} u, \mathbf{x}, \mathbf{z}) = 0.$$

Definition

A PDE is hyperbolic if the Jacobian of H has real eigenvalues and a full set of eigenvectors

Remark

This PDE can be transformed into a system of hyperbolic PDEs, by letting $\mathbf{p} = \nabla_{\mathbf{x}} u$ and taking the gradient with respect to $\mathbf{x}$ of both sides:

$$\partial_t \mathbf{p} + \nabla_{\mathbf{x}} H(\mathbf{p}, \mathbf{x}, \mathbf{z}) = \mathbf{0}.$$

Hamilton-Jacobi Hyperbolicity 1D

Remark

The driving force behind this paper is that HJ equation is always hyperbolic for any real Hamiltonian, even after applying Stochastic Galerkin expansion.

For example in 1D we can show hyperbolicity as follows. Letting $p = u_x$, we have

$$u_t + H(p, x, z) = 0$$

Taking the x derivative

$$p_t + \frac{\mathrm{d}}{\mathrm{d}x} H(p, x, z) = 0$$
$$p_t + \frac{\partial H}{\partial p} p_x + \frac{\partial H}{\partial x} = 0$$

Hamilton-Jacobi Hyperbolicity 2D

Now in 2D, letting $p^1 = u_x$ and $p^2 = u_y$, we have

$$u_t + H(p^1, p^2, x, y, z) = 0$$

Taking the x and y derivatives

$$p_t^1 + \frac{\mathrm{d}}{\mathrm{d}x} H(p^1, p^2, x, y, z) = 0$$
$$p_t^2 + \frac{\mathrm{d}}{\mathrm{d}y} H(p^1, p^2, x, y, z) = 0$$

Since $\partial_y p^1 = \partial_x p_x^2$ or $u_{xy} = u_{yx}$, this is equivalent to

$$\begin{bmatrix} p^1 \\ p^2 \end{bmatrix}_t + \begin{bmatrix} H_{p^1} & 0 \\ 0 & H_{p^1} \end{bmatrix} \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}_x + \begin{bmatrix} H_{p^2} & 0 \\ 0 & H_{p^2} \end{bmatrix} \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}_y + \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{bmatrix} = \mathbf{0}$$

Remark

In any dimension, n, this generalizes to the following hyperbolic system.

$$\partial_t \mathbf{p}^{(i)} + \sum_{j=1}^{n} \partial_{p^{(j)}} H \partial_{x_j} p^{(i)} + \partial_{x_i} H = 0 \quad i = 1, \ldots, n.$$

$$\begin{bmatrix} p^1 \\ \vdots \\ p^n \end{bmatrix}_t + \sum_{j=1}^{n} \begin{bmatrix} H_{p^j} & & \\ & \ddots & \\ & & H_{p^j} \end{bmatrix} \partial_{x_j} \begin{bmatrix} p^1 \\ \vdots \\ p^n \end{bmatrix} + \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \vdots \\ \frac{\partial H}{\partial x_n} \end{bmatrix} = \mathbf{0}$$

Remark

Notice this is strongly hyperbolic because the Jacobian matrix is diagonalizable with real eigenvalues.

Jin Xin Relaxation Scheme

Jin Xin Relaxation Scheme

The Jin Xin Relaxation Scheme linearizes our nonlinear system of PDEs.

$$\partial_t \mathbf{p} + \nabla_x w = \mathbf{0} \tag{1}$$

$$\partial_t w + \alpha \nabla_x \cdot \mathbf{p} = -\frac{1}{\epsilon}(w - H(\mathbf{p}, \mathbf{x}, \mathbf{z})) \tag{2}$$

$$\partial_t u + w = 0 \tag{3}$$

Remark

The linearization of this system occurs in 2, because $w - H(\mathbf{p}, \mathbf{x}, \mathbf{z}) = O(\epsilon)$. This still include the nonlinear term $H(\mathbf{p}, \mathbf{x}, \mathbf{z})$.

Expanding Jin Xin Relaxation Scheme

We assume that the solution to this system can be expressed as an expansion on the Galerkin basis.

$$\mathbf{p} \approx \hat{\mathbf{p}}(\mathbf{x}, t, \mathbf{z}) = \sum_{i=0}^{N} \hat{p}_i(\mathbf{x}, t)\Phi_i(\mathbf{z})$$

$$w \approx \hat{\mathbf{w}}(\mathbf{x}, t, \mathbf{z}) = \sum_{i=0}^{N} \hat{w}_i(\mathbf{x}, t)\Phi_i(\mathbf{z})$$

$$H(\mathbf{p}, \mathbf{x}, \mathbf{z}) \approx \hat{\mathbf{H}}(\hat{\mathbf{p}}, \mathbf{x}, \mathbf{z}) = \sum_{i=0}^{N} \hat{H}_i(\hat{\mathbf{p}}, \mathbf{x})\Phi_i(\mathbf{z})$$

Expanding Jin Xin Relaxation Scheme

The value of $\hat{\mathbf{H}}$ is known from the problem and can be solved as follows.

$$H_k(\hat{\mathbf{p}}, \mathbf{x}) = \int H(\hat{\mathbf{p}}, \mathbf{x}, \mathbf{z}) \Phi_k(\mathbf{z}) \mathrm{d}\mu(\mathbf{z})$$

Sometimes this integral can be done exactly and sometimes numerical integration, such as Gauss-Legendre quadrature, is required.

Stochastic Jin Xin Relaxation Scheme

After transforming to the Galerkin basis, the relaxation scheme is as follows

$$\partial_t \hat{\mathbf{p}} + \nabla_x \hat{\mathbf{w}} = \mathbf{0}$$
$$\partial_t \hat{\mathbf{w}} + \alpha \nabla_x \cdot \hat{\mathbf{p}} = -\frac{1}{\epsilon}(\hat{\mathbf{w}} - \hat{\mathbf{H}}(\hat{\mathbf{p}}, \mathbf{x}, \mathbf{z}))$$
$$\partial_t \hat{\mathbf{u}} + \hat{\mathbf{w}} = \mathbf{0}$$

We chose to numerically solve this system of differential equations in two ways.

- 2 step - convection and instant relaxation
- Runge Kutta IMEX schemes

2 - Step Method

We can split the relaxation scheme in two steps via operator splitting:

Convection Step

This can be solved using numerical methods designed for linear hyperbolic systems.

$$\begin{aligned} \partial_t \hat{\mathbf{p}} + \nabla_x \hat{\mathbf{w}} &= \mathbf{0} \\ \partial_t \hat{\mathbf{w}} + \alpha \nabla_x \cdot \hat{\mathbf{p}} &= \mathbf{0} \\ \partial_t \hat{u} + \hat{\mathbf{w}} &= \mathbf{0} \end{aligned}$$

Relaxation Step

This instant relaxation incorporates the nonlinearity of the system.

$$\hat{\mathbf{w}} = \hat{\mathbf{H}}$$

2 - Step Method: Convection Step

Convection Step As A Matrix Equation

$$\mathbf{q}_t + A\mathbf{q}_x = \mathbf{0}$$

where

$$\mathbf{q} = \begin{pmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{w}} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & I \\ \alpha I & 0 \end{pmatrix}$$

2 - Step Method: Convection Step

Discretize space and time and pick a numerical method

Upwind - O(h):

View the equation as modeling the concentration of a tracer the in air blowing at speed a, then we are looking in the correct upwind direction to judge how the concentration will change with time [3].

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}\left(A^+\left(Q_i^n - Q_{i-1}^n\right) - A^-\left(Q_{i+1}^n - Q_i^n\right)\right)$$

Lax-Wendroff - $O(h^2)$

A simple way to achieve a two-level explicit method with higher accuracy is to use the idea of Taylor series methods applied directly to to the linear system of ODEs [3].

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x} A\left(Q_{i+1}^n Q_{i-1}^n\right) + \frac{\Delta t^2}{2\Delta x^2} A^2\left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right)$$

IMEX Runge-Kutta Scheme

The IMEX Runge Kutta Schemes solve ODEs of the form

$$\mathbf{q}'(t) = f(\mathbf{q}) + \frac{1}{\epsilon} g(\mathbf{q}), \quad \mathbf{q}(t_0) = \mathbf{q_0}.$$

For our system

$$\mathbf{q} = \begin{pmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{w}} \end{pmatrix}$$

$$f(\mathbf{q}) = -A\mathbf{q}_x = -\begin{pmatrix} \hat{\mathbf{w}}_x \\ \alpha \hat{\mathbf{p}}_x \end{pmatrix}$$

$$g(\mathbf{q}) = \begin{pmatrix} \mathbf{0} \\ H(\hat{\mathbf{p}}) - \hat{\mathbf{w}} \end{pmatrix}$$

IMEX Runge-Kutta Scheme

An IMEX Runge-Kutta scheme is represented by the following double Butcher's tableau

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{w}^{\mathsf{T}} \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & w^{\mathsf{T}} \end{array}$$

which corresponds to

$$\mathbf{q}^{n+1} = \mathbf{q}^n + \Delta t \sum_{i=1}^{\nu} \tilde{w}_i f(\mathbf{q}^{(i)}) + \frac{\Delta t}{\epsilon} \sum_{i=1}^{\nu} w_i g(\mathbf{q}^{(i)})$$

$$\mathbf{q}^{(1)} = \mathbf{q}^n + \frac{\Delta t}{\epsilon} a_{11} g(\mathbf{q}^{(1)})$$

$$\mathbf{q}^{(i)} = \mathbf{q}^n + \Delta t \sum_{l=1}^{i-1} \tilde{a}_{il} f(\mathbf{q}^{(l)}) + \frac{\Delta t}{\epsilon} \sum_{l=1}^{i} a_{il} g(\mathbf{q}^{(l)})$$

IMEX Runge-Kutta Example Schemes

IMEX1 - $O(h)$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

IMEX2 - $O(h^2)$

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

Burgers' Equation Example

1D Burgers' equation with random Riemann initial data

Consider $u_t + H(u_x) = 0, \quad H(p) = \frac{1}{2}p^2, \quad x \in [0, 1]$

$$p_0(x, z) = \begin{cases} u_l + \sigma z, & x \leq .5 \\ u_r + \sigma z, & x > .5 \end{cases} \text{ with } \sigma = .1.$$

Exact Solution: Case 1 (shock)

$$p(x, z, t) = \begin{cases} u_l + \sigma z, & x \leq .5 + \sigma z t \\ u_r + \sigma z, & x > .5 + \sigma z t \end{cases}$$

$$u(x, z, t) = \begin{cases} (u_l + \sigma z)x - \frac{(u_1 + \sigma z)^2}{2} t, & x \leq .5 + \sigma z t \\ (u_r + \sigma z)x + .5(u_l - u_r) - \frac{(u_r + \sigma z)^2}{2} t, & x > .5 + \sigma z t \end{cases}$$

Exact Solution: Case 2 (rarefaction)

$$p(x,z,t) = \begin{cases} u_l + \sigma z, & x \leq .5 + (u_l + \sigma z)t \\ \frac{x-.5}{t}, & .5 + (u_l + \sigma z)t < x \leq .5 + (u_r + \sigma z)t, \\ u_r + \sigma z, & x > .5 + (u_r + \sigma z)t \end{cases}$$

$$u(x,z,t) = \begin{cases} (u_l + \sigma z)x - \frac{(u_1+\sigma z)^2}{2}t, & x \leq .5 + (u_l + \sigma z)t \\ .5(u_l + \sigma z) + \frac{1}{2t}(z - 0.5)^2, & \\ \qquad .5 + (u_l + \sigma z)t < x \leq .5 + (u_r + \sigma z)t, & \\ (u_r + \sigma z)x + .5(u_l - u_r) - \frac{(u_r+\sigma z)^2}{2}t, & x > .5 + (u_r + \sigma z)t \end{cases}$$

Burgers' Equation Example

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$$u_0(x, z) = \begin{cases} u_l + \sigma z, & x \leq .5 \\ u_r + \sigma z, & x > .5 \end{cases} \quad \text{with} \quad \sigma = .1.$$

Let z be uniformly distributed on $[-1, 1]$. We approximate a solution p as

$$p_N = \sum_{i=0}^{N} \hat{p}_i \Phi_i(z),$$

where $\Phi_i(z)$ is the Legendre polynomial of order i.

Burgers' Equation Example

Remark

Given the approximate of p we also need to approximate $H(p)$ over the Legendre basis.

Burgers' Equation Example

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Given the approximate of p we also need to approximate $H(p)$ over the Legendre basis.

$$H(p_N) = \frac{1}{2} \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{p}_i \hat{p}_j \Phi_i(z) \Phi_j(z)$$

Example

Remark

Given the approximate of p we also need to approximate $H(p)$ over the Legendre basis.

$$H(p_N) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_i\hat{p}_j\Phi_i(z)\Phi_j(z)$$

$$H(p_N)\Phi_k(z) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_i\hat{p}_j\Phi_i(z)\Phi_j(z)\Phi_k(z)$$

Burgers' Equation Example

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$$\int_{-1}^{1} H(p_N)\Phi_k(z)\mathrm{d}z = \int_{-1}^{1}\frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_i\hat{p}_j\Phi_i(z)\Phi_j(z)\Phi_k(z)\mathrm{d}z$$

Burgers' Equation Example

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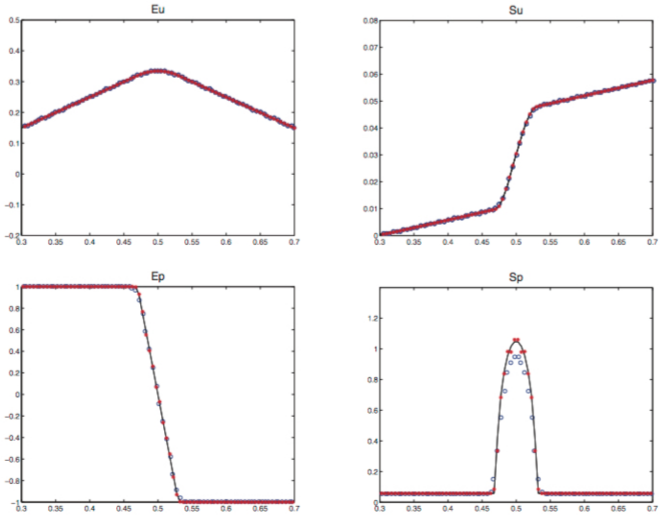
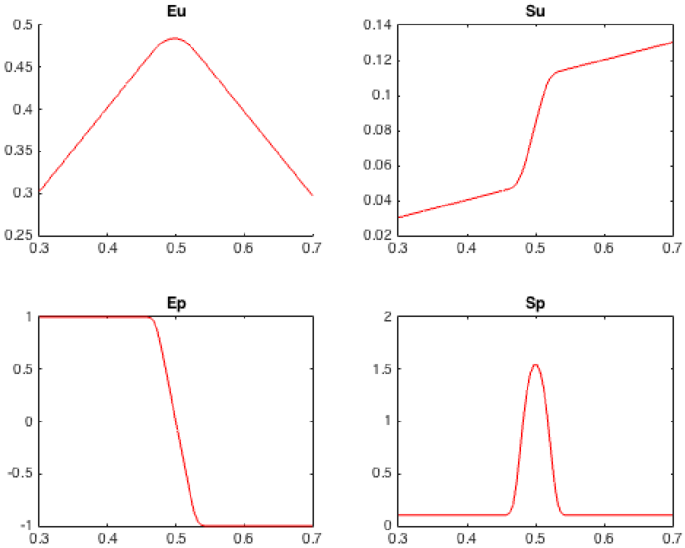
$$\int_{-1}^{1} H(p_N)\Phi_k(z)\mathrm{d}z = \int_{-1}^{1}\frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_i\hat{p}_j\Phi_i(z)\Phi_j(z)\Phi_k(z)\mathrm{d}z$$

$$\hat{H}_k = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{N}\hat{p}_i\hat{p}_j S_{ijk}$$

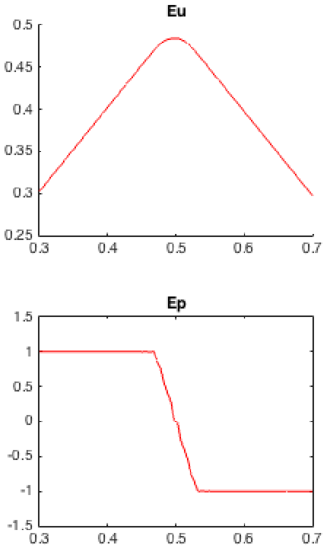
Shock [1]



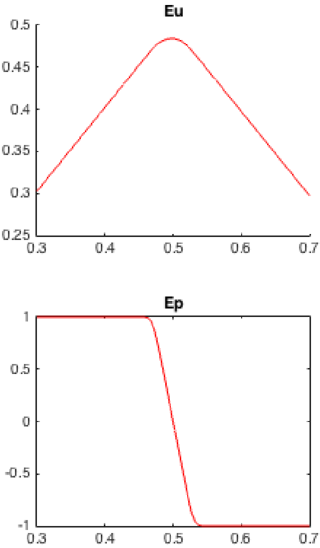
Upwind Shock



Lax Wendroff Shock



IMEX1 Shock



Eu
0.5
0.45
0.4
0.35
0.3
0.25
0.3
0.4
0.5
0.6
0.7
Su
0.14
0.12
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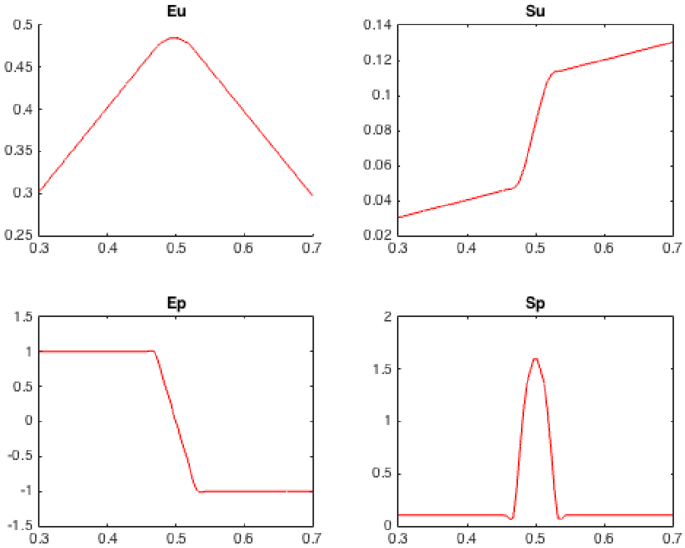
0.08
0.06
0.04
0.02
0.3
0.4
0.5
0.6
0.7
Ep
1
0.5
0
-0.5
-1
0.3
0.4
0.5
0.6
0.7
Sp
2
1.5
1
0.5
0
0.3
0.4
0.5
0.6
0.7

IMEX2 Shock


Eu
0.5
0.45
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0.35
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0.25
0.3 0.4 0.5 0.6 0.7
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0.12
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0.08
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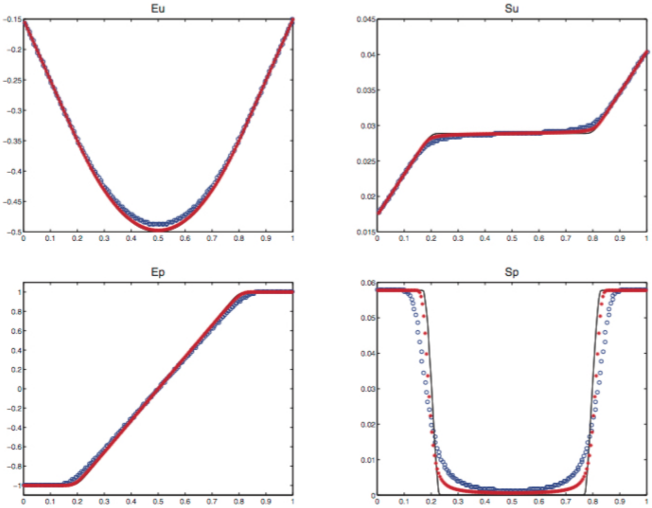
0.04
0.02
0.3 0.4 0.5 0.6 0.7
Ep
1.5
1
0.5
0
-0.5
-1
-1.5
0.3 0.4 0.5 0.6 0.7
Sp
2
1.5
1
0.5
0
0.3 0.4 0.5 0.6 0.7

Courant-Friedrichs-Lewy Condition

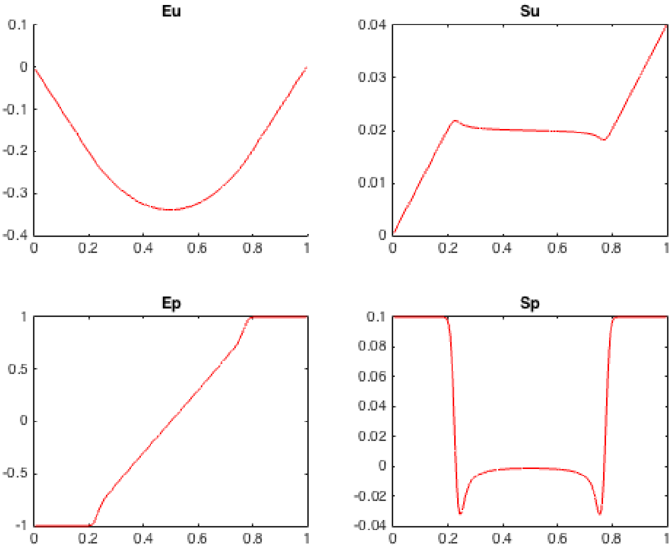
Definition

This is a necessary condition in general to ensure that a method is stable and convergent. Here we have that $\frac{\alpha K}{h} < 1$.

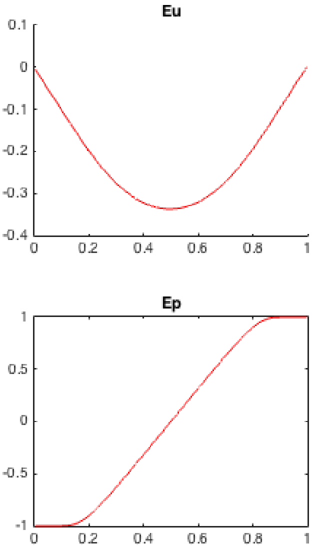
Rarefaction



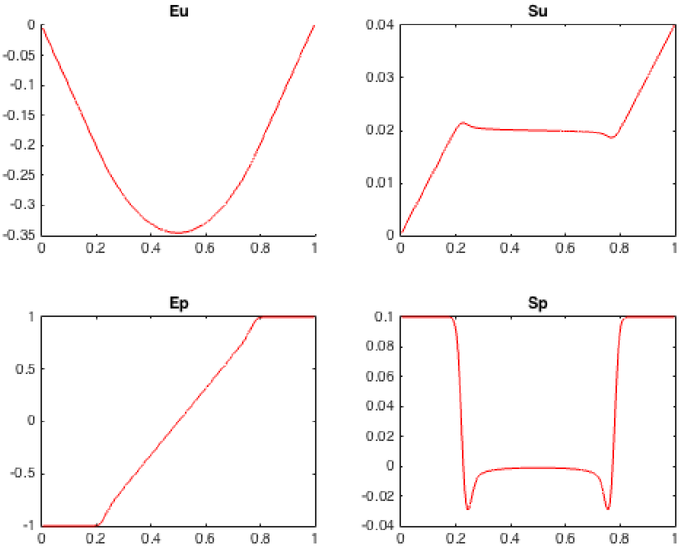
Upwind Rarefaction with CFL = 0.8



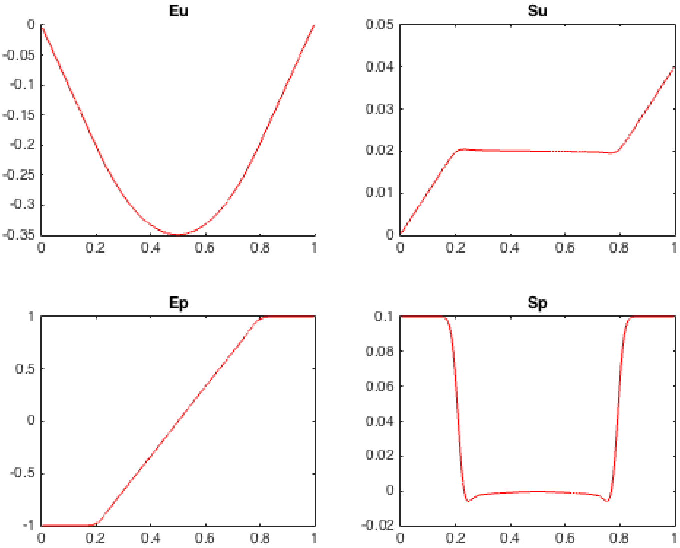
Upwind Rarefaction with CFL = 0.2



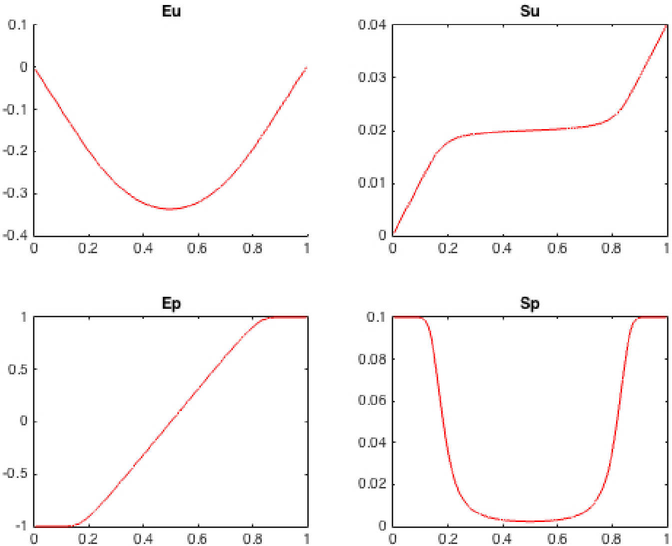
Lax Wendroff Rarefaction with CFL = 0.4



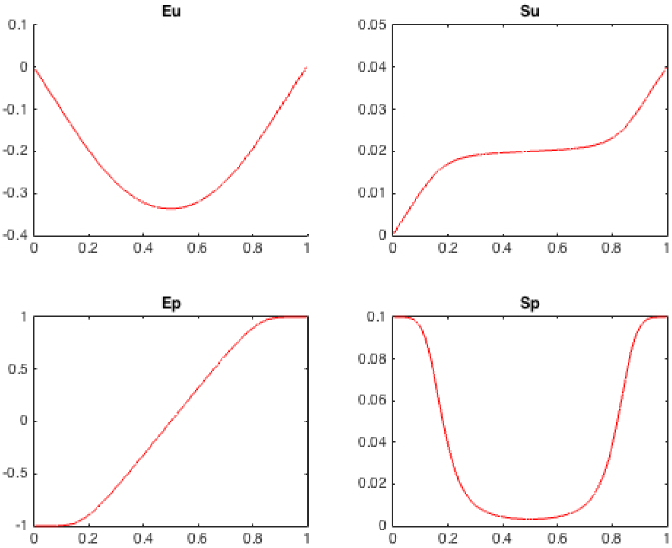
Lax Wendroff Rarefaction with CFL = 0.2



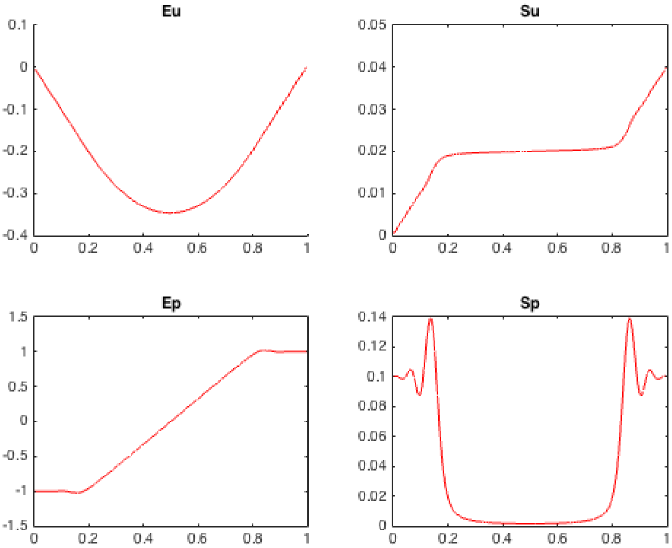
IMEX1 Rarefaction with CFL = 0.8



IMEX1 Rarefaction with CFL = 0.2



IMEX2 Rarefaction with CFL = 0.2



Eikonal Equation Example

The Eikonal equation is a nonlinear weight equation

1D Eikonal equation with random wave speed

Consider

$u_t + H(u_x, x, z) = 0, \quad H(p, x, z) = c(x, z)|p|, \quad x \in [0, 2pi]$, with initial conditions

$$u_0(x, z) = \sin(x)$$
$$p_0(x, z) = \cos(x)$$

Gauss-Legendre Numerical Integration

Expansion of Hamiltonian

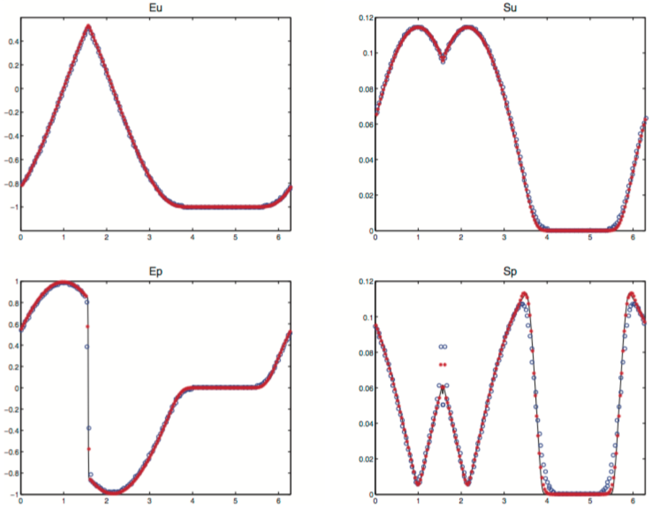
As in the Burgers' Equation example, we need to apply Stochastic Galerking expansion on the Hamiltonian function.

$$\hat{H}_k = \int H\left(\sum_{i=0}^{N} \hat{p}_i \Phi_i(z)\right) \Phi_k(z) \mathrm{d}\mu(z)$$

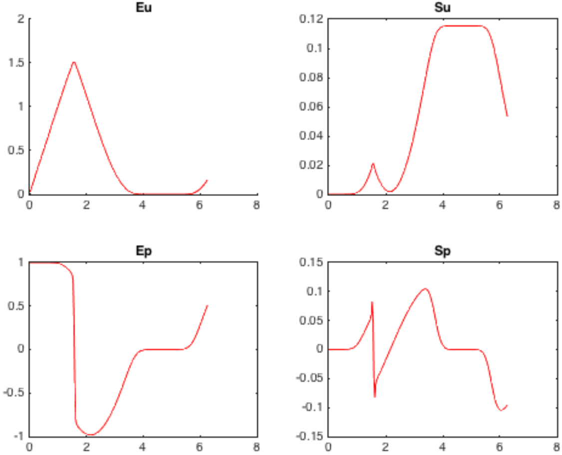
However since $H(p, x, z) = (1 + \sigma z)|p|$, this integral cannot be computed exactly. Therefore some numerical integration or quadrature rule must be applied. We chose to use Gauss-Legendre Quadrature with 20 points.

$$\hat{H}_k \approx \sum_{j=1}^{20} \left(w_j \cdot H\left(\sum_{i=0}^{N} \hat{p}_i \Phi_i(z_j)\right) \Phi_k(z_j)\right)$$

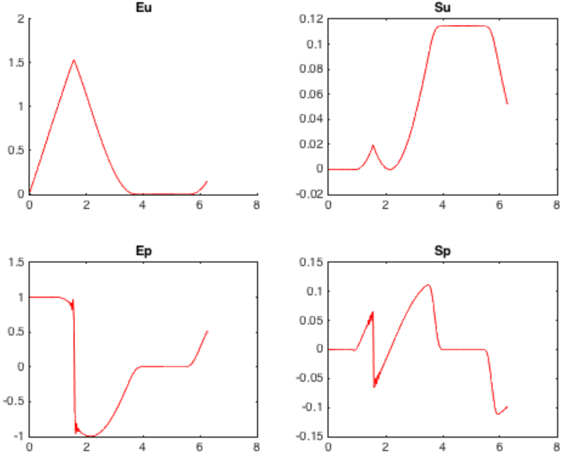
Eikonal Equation [1]



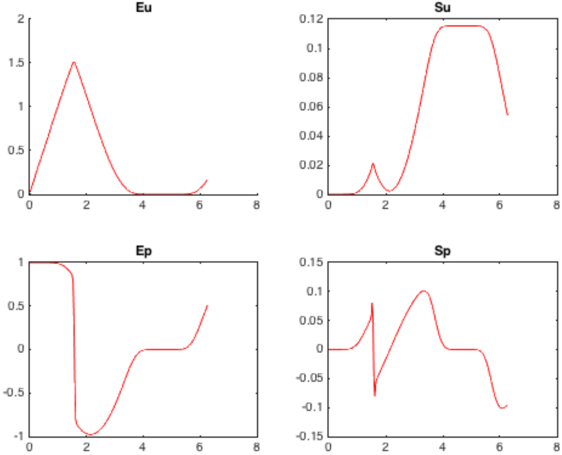
Eikonal Equation Upwind



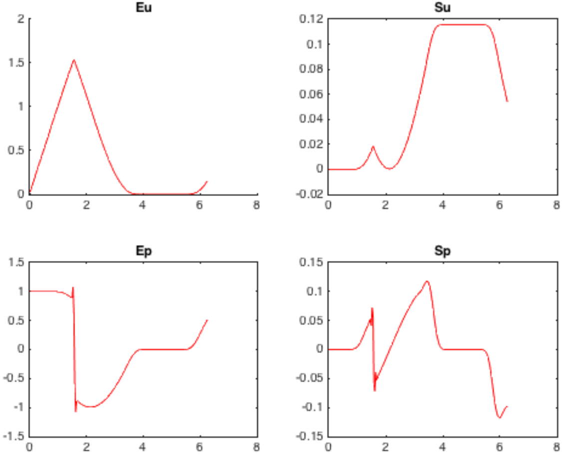
Eikonal Equation Lax-Wendroff



Eikonal Equation IMEX1



Eikonal Equation IMEX2



Future Work

Things left to do...

- 2D Eikonal equations
- 2D Burgers' equation
- IMEX3
- 2D IMEX1/IMEX2

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