AN OVERVIEW OF THE HAMILTON-JACOBI EQUATION

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ABSTRACT. This paper is a survey of the Hamilton-Jacobi partial differential equation. We begin with its origins in Hamilton's formulation of classical mechanics. Next, we show how the equation can fail to have a proper solution. Setting this issue aside temporarily, we move to a problem of optimal control to show another area in which the equation arises naturally. In the final section, we present some modern theory of the equation. We define a viscosity solution to the PDE and describe some of its properties. We conclude by bringing together ideas developed in the earlier sections.

1. INTRODUCTION

1.1. **Prerequisites.** No knowledge of partial differential equations is required. No knowledge of mechanics is assumed either; section 2 can serve as review for readers already familiar with Hamiltonian mechanics, or as an introduction to the area for other readers. We choose to introduce the material in that section through differential forms. Some familiarity with differential forms is recommended, but a quick overview is given in Appendix B. Although the sections of the paper make reference to each other, the theory in one section is developed largely independently of the others.

1.2. **A word on notation.** Notation used for partial differential equations varies greatly across different texts. The conventions adopted by this paper are presented in Appendix A. The reader is highly encouraged to skim through that appendix now, and to refer back to it whenever necessary during the reading of this paper.

2. HAMILTONIAN MECHANICS

2.1. **Overview.** Hamiltonian mechanics is a reformulation of Newtonian mechanics which uses the parameters $(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}}$, where $\mathbf{p} = (p_1, \ldots, p_n)$ is the *generalized momentum* and $\mathbf{x} = (x_1, \ldots, x_n)$ is the *generalized position*.[1] Unlike in Newtonian mechanics, the generalized position $\mathbf{x}$ does not need to be expressed in rectangular coordinates. This allows for a much greater freedom in the choice of coordinates to describe a system.

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[1] In physics texts, the generalized position vector is often denoted $\mathbf{q}$, but we will use $\mathbf{x}$ instead for consistency with the rest of the paper.

One function, the *Hamiltonian* $H : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$, concisely and completely expresses the constraints of the system, via *Hamilton's equations*:

$$\begin{aligned} \dot{\mathbf{p}}(t) &= -\nabla_{\mathbf{x}} H(\mathbf{p}(t), \mathbf{x}(t)) \\ \dot{\mathbf{x}}(t) &= \nabla_{\mathbf{p}} H(\mathbf{p}(t), \mathbf{x}(t)) \end{aligned} \tag{2.1}$$

Since H is given, this is an system of $2n$ ordinary differential equations, with $\mathbf{p}(t)$ and $\mathbf{x}(t)$ as the unknowns. If initial data is given (for example, if $\mathbf{p}(0) = \mathbf{p}_0$ and $\mathbf{x}(0) = \mathbf{x}_0$), then the ODE can be solved and we can describe the motion of the system.

Example 2.1. If we use rectangular coordinates, the Hamiltonian for a particle of mass m in a force field is

$$H(\mathbf{p}, \mathbf{x}) = \frac{1}{2m}|\mathbf{p}|^2 + V(\mathbf{x})$$

where $V : \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$ is the potential energy. Then equations (2.1) reduce to $\dot{\mathbf{p}} = -\nabla_{\mathbf{x}} V(\mathbf{x})$ and $\dot{\mathbf{x}} = \frac{1}{m}\mathbf{p}$. The first equation is a statement of Newton's second law $\mathbf{F} = m\mathbf{a}$. The second equation relates the classical position and momentum vectors. ♢

Remark 2.2. Observe that in Example 2.1, the Hamiltonian is equal to the total energy of the system. This is no coincidence. To see why, fix a Hamiltonian $H : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$, and let $(\mathbf{p}(t), \mathbf{x}(t))$ be a solution to Hamilton's equations. Then set $E(t) = H(\mathbf{p}(t), \mathbf{x}(t))$. We have

$$\dot{E}(t) = \nabla_{\mathbf{p}} H(\mathbf{p}(t), \mathbf{x}(t)) \cdot \dot{\mathbf{p}}(t) + \nabla_{\mathbf{x}} H(\mathbf{p}(t), \mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) = 0$$

where the equality on the right is due to (2.1). Thus, the quantity $E(t)$ is indeed conserved, and the Hamiltonian admits an interpretation as energy.

However, a caution: we have defined a Hamiltonian as a function of $\mathbf{p}$ and $\mathbf{x}$. If the Hamiltonian is also time-dependent, that is, a function of $\mathbf{p}$, $\mathbf{x}$ *and* t, then it is not that case that H is conserved. ♢

2.2. **As a formulation of Lagrangian mechanics.** Another formulation of Newtonian mechanics uses the Lagrangian, which employs the parameters $\mathbf{x}$, the generalized position, and $\mathbf{v}$, the generalized velocity. The *Lagrangian* of a system is a function $L : \mathbb{R}^n_{\mathbf{v}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$. Motion of the system is described by solutions $\mathbf{x}(t)$ to the Euler-Lagrange equation[2]

$$\frac{d}{dt}\nabla_{\mathbf{v}} L(\dot{\mathbf{x}}(t), \mathbf{x}(t)) = \nabla_{\mathbf{x}} L(\dot{\mathbf{x}}(t), \mathbf{x}(t))$$

Any Lagrangian system that is convex in $\mathbf{v}$ can be described with Hamiltonian mechanics. To translate a Lagrangian system into a Hamiltonian system, we need a map $\mathbb{R}^n_{\mathbf{v}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}}$. The proper map to take is $(\mathbf{v}, \mathbf{x}) \mapsto (\nabla_{\mathbf{v}} L(\mathbf{v}, \mathbf{x}), \mathbf{x})$. That is, we take the generalized

[2]It is true that L, like H, is a function of $2n$ independent variables. However, the resulting Euler-Lagrange equation is a system of n second-order ODE whose unknown is $\mathbf{x}(t)$. In contrast, Hamilton's equations are a system of $2n$ first-order ODE whose unknowns are $\mathbf{p}(t)$ and $\mathbf{x}(t)$.

momentum to be $\mathbf{p} = \nabla_{\mathbf{v}} L(\mathbf{v}, \mathbf{x})$. The Hamiltonian H is constructed from L via the *Legendre transform*:

$$H(\mathbf{p}, \mathbf{x}) = \sup_{\mathbf{v} \in \mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{v} - L(\mathbf{v}, \mathbf{x})\}$$

Then the Euler-Lagrange equation translates into Hamilton's equations. It is a fact that the Legendre transform is its own inverse, so

$$L(\mathbf{v}, \mathbf{x}) = \sup_{\mathbf{p} \in \mathbb{R}^n} \{\mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}, \mathbf{x})\}$$

We will not go into detail here, but this relationship between H and L will appear in subsection 5.3 when we discuss the Hopf-Lax formula.

2.3. **The integral invariant of Poincaré-Cartan.** We can view a solution to Hamilton's equations in two different ways: as a path in the *phase space* $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}}$, or to make time explicit, as a path in the *extended phase space* $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t$. (See Figure 2.1.)

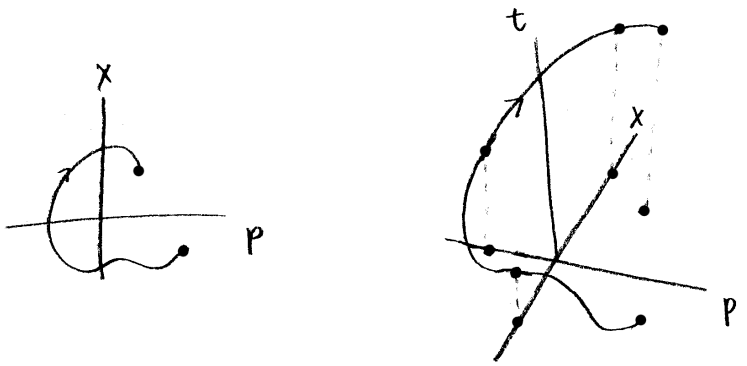

FIGURE 2.1. Left: Path in phase space. Right: Same path in extended phase space.

There is another way to describe solutions to Hamilton's equations. Define ω^1 to be the following differential 1-form on $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t$:

$$\omega^1 = \mathbf{p}\, d\mathbf{x} - H\, dt = \sum_{i=1}^{n} p_i\, dx_i - H\, dt$$

The form ω^1 is called the *integral invariant of Poincaré-Cartan.* (The reader is invited to read Appendix B for a review of differential forms.)

Lemma 2.3. *The solutions to Hamilton's equations, viewed as paths in the extended phase space, are the bi-characteristics of* ω^1.

Proof. The matrix corresponding to the 2-form $\omega^2 := d\omega^1$ is

$$\begin{pmatrix} \mathbf{0}_{n\times n} & -I_{n\times n} & \nabla_{\mathbf{p}} H \\ I_{n\times n} & \mathbf{0}_{n\times n} & \nabla_{\mathbf{x}} H \\ -\nabla_{\mathbf{p}} H & -\nabla_{\mathbf{x}} H & 0 \end{pmatrix}$$

Note that the upper left $2n \times 2n$ submatrix has full rank, so the kernel has dimension at most one. The dimension is in fact one because $\boldsymbol{\xi} = (-\nabla_{\mathbf{x}} H, \nabla_{\mathbf{p}} H, 1)$ lies in the kernel.

Suppose that we start at some point $(\mathbf{p}_0, \mathbf{x}_0, t_0)$ in the extended phase space, and move along a path so that the tangent vector of the path at any point $(\mathbf{p}, \mathbf{x}, t)$ is parallel to the eigenvector $(-\nabla_{\mathbf{x}} H(\mathbf{p}, \mathbf{x}), \nabla_{\mathbf{p}} H(\mathbf{p}, \mathbf{x}), 1)$ at that point. Then, by Definition B.13, our path is a bi-characteristic of ω^1 in the extended phase space. Since we are always moving the positive time direction, we can represent the resulting path by the vector functions $\mathbf{p}(t)$ and $\mathbf{x}(t)$.

To compute, for example, $\dot{p}_1(t)$, we consider the map $(\mathbf{p}(t), \mathbf{x}(t), t) \mapsto (p_1(t), t)$, which is the projection of the path in extended phase space onto the space $\mathbb{R}_{p_1} \times \mathbb{R}_t$. We take the ratio of the first and last (scalar) components of $(-\nabla_{\mathbf{x}} H, \nabla_{\mathbf{p}} H, 1)$, giving us

$$\dot{p}_1(t) = \frac{\text{first component}}{\text{last component}} = \frac{-\partial_{x_1} H(\mathbf{p}(t), \mathbf{x}(t))}{1}$$

By taking ratios of other components with the last component, we have $\dot{p}_i(t) = -\partial_{x_i} H(\mathbf{p}(t), \mathbf{x}(t))$ and $\dot{x}_i(t) = \partial_{p_i} H(\mathbf{p}(t), \mathbf{x}(t))$. Viewed collectively, these $2n$ equations are precisely Hamilton's equations (2.1). □

2.4. **Canonical transformations.** Recall that in Hamiltonian mechanics, we use $2n$ independent parameters to describe the configuration of a system. Sometimes, it would be more useful to describe the system through another set of $2n$ parameters; in that case we perform a change of variables $(\mathbf{p}, \mathbf{x}) \mapsto (\mathbf{P}(\mathbf{p}, \mathbf{x}), \mathbf{X}(\mathbf{p}, \mathbf{x}))$. Note that the new momentum $\mathbf{P}$ depends not only on the old momentum $\mathbf{p}$ but also the old position $\mathbf{x}$. (Similarly, the new position $\mathbf{X}$ depends on both $\mathbf{p}$ and $\mathbf{x}$.)

However, not every possible change of coordinates is useful. For one thing, we require the transformation to be invertible (at least locally), so that if we have a solution $(\mathbf{P}(t), \mathbf{X}(t))$, then we can obtain a solution $(\mathbf{p}(t), \mathbf{x}(t))$.

More importantly, we would like something akin to Hamilton's equations to hold in the new coordinates as well. Thus, we seek to characterize those transformations that do preserve the form of Hamilton's equations.

Definition 2.4. A map $g : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}^n_{\mathbf{P}} \times \mathbb{R}^n_{\mathbf{X}}$ is *canonical* if it preserves the differential form $\omega^2 = d\mathbf{p} \wedge d\mathbf{x} = \sum dp_i \wedge dx_i$, that is, if $g^*(\omega^2) = \omega^2$. ♢

Remark 2.5. The statement $g^*(\omega^2) = \omega^2$ is independent of coordinates, but in terms of our coordinates $(\mathbf{p}, \mathbf{x})$, this statement is equivalent to $\sum dp_i \wedge dx_i = \sum dP_i \wedge dX_i$. ♢

If $(\mathbf{p}, \mathbf{x}) \mapsto (\mathbf{P}, \mathbf{X})$ is a canonical transformation, then

$$d(\mathbf{p}\, d\mathbf{x} - \mathbf{P}\, d\mathbf{X}) = d\mathbf{p} \wedge d\mathbf{x} - d\mathbf{P} \wedge d\mathbf{X} = 0 \tag{2.2}$$

Since (2.2) shows that the differential 1-form $\mathbf{p}\,d\mathbf{x} - \mathbf{P}\,d\mathbf{X}$ is closed, we know the form is exact. That is, there exists a function $S : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$ such that $\mathbf{p}\,d\mathbf{x} - \mathbf{P}\,d\mathbf{X} = dS$.[3] Then

$$\omega^1 = \mathbf{p}\,d\mathbf{x} + H\,dt = \mathbf{P}\,d\mathbf{X} + H\,dt + dS \tag{2.3}$$

Let $K : \mathbb{R}^n_{\mathbf{P}} \times \mathbb{R}^n_{\mathbf{X}} \to \mathbb{R}$ be given by $K(\mathbf{P}, \mathbf{X}) = H(\mathbf{p}(\mathbf{P}, \mathbf{X}), \mathbf{x}(\mathbf{P}, \mathbf{X}))$. Because of (2.3), the bi-characteristics of ω^1 in the new coordinates $(\mathbf{P}, \mathbf{X})$ are given by

$$\begin{aligned} \dot{\mathbf{P}}(t) &= -\nabla_{\mathbf{X}} K(\mathbf{P}(t), \mathbf{X}(t)) \\ \dot{\mathbf{X}}(t) &= \nabla_{\mathbf{P}} K(\mathbf{P}(t), \mathbf{X}(t)) \end{aligned} \tag{2.4}$$

Since (2.1) and (2.4) both describe the bi-characteristics of ω^1, they represent the same path in the phase space, just with different coordinates. Thus, Hamilton's equations hold as well for the new space, with the same Hamiltonian. We have just shown

Theorem 2.6. *Canonical transformations preserve Hamilton's equations.*

2.5. **Time-dependent canonical transformations.** We can consider changes of variables that also depend on time. We can write either $\mathbf{P} = \mathbf{P}_t(\mathbf{p}, \mathbf{x})$ or $\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{x}, t)$ (and do the same for $\mathbf{X}$), depending on whether we want to view the time t as a fixed parameter or not.

Consider maps $g_t : (\mathbf{p}, \mathbf{x}) \mapsto (\mathbf{P}_t(\mathbf{p}, \mathbf{x}), \mathbf{X}_t(\mathbf{p}, \mathbf{x}))$. Suppose that each $g_t : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}^n_{\mathbf{P}} \times \mathbb{R}^n_{\mathbf{X}}$ is a canonical transformation. Then we say that $g(\mathbf{p}, \mathbf{x}, t) := g_t(\mathbf{p}, \mathbf{x})$ is a canonical transformation.

For each g_t (think of t as fixed), we can define a function S_t as in the previous section. By doing so, we obtain

$$dS_t = \mathbf{p}\,d\mathbf{x} - \mathbf{P}_t\,d\mathbf{X}_t \tag{2.5}$$

Note that both sides are differential forms on $\mathbb{R}^n \times \mathbb{R}^n$. We now allow t to vary. Let $S(\mathbf{p}, \mathbf{x}, t) = S_t(\mathbf{p}, \mathbf{x})$. Then[4]

$$\begin{aligned} dS &= \partial_t S\,dt + dS_t \\ d\mathbf{X} &= \partial_t \mathbf{X}\,dt + d\mathbf{X}_t \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we find that

$$\mathbf{p}\,d\mathbf{x} - H\,dt = \mathbf{P}\,d\mathbf{X} - (H + \mathbf{P} \cdot \partial_t \mathbf{X} + \partial_t S)\,dt + dS \tag{2.7}$$

It follows that $K(\mathbf{P}, \mathbf{X}, t) := H + \mathbf{P} \cdot \partial_t \mathbf{X} + \partial_t S$ is the transformed Hamiltonian for the new coordinates, so we have proved the following:

[3] For example, take $S(\mathbf{p}, \mathbf{x}) = \int_{C(\mathbf{p},\mathbf{x})} \mathbf{p}\,d\mathbf{x} - \mathbf{P}\,d\mathbf{X}$ where $C(\mathbf{p}, \mathbf{x})$ is a curve in $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}}$ from some fixed $(\mathbf{p}_0, \mathbf{x}_0)$ to $(\mathbf{p}, \mathbf{x})$. S is well defined because the differential form is closed and the space $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}}$ is simply connected.

[4] Here, we emphasize a point on notation from Appendix A: $\partial_t \mathbf{X}$ is not the same as $\dot{\mathbf{X}}$. The former means: "view $\mathbf{X}$ as a function $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$ and take the partial derivative with respect to t." The latter means: "let $\mathbf{X} : \mathbb{R}_t \to \mathbb{R}^n$ be a solution to Hamilton's equations (together with $\mathbf{P} : \mathbb{R}_t \to \mathbb{R}^n$), and take the derivative."

Theorem 2.7. *If* $(\mathbf{p}, \mathbf{x}, t) \mapsto (\mathbf{P}(\mathbf{p}, \mathbf{x}, t), \mathbf{X}(\mathbf{p}, \mathbf{x}, t))$ *is a time-dependent canonical transformation, then the system in the new coordinates satisfies Hamilton's equation with* $K(\mathbf{P}, \mathbf{X}, t) = H + \mathbf{P} \cdot \partial_t \mathbf{X} + \partial_t S$.

Remark 2.8. The transformed Hamiltonian, K, is also time dependent. Thus, as pointed out in Remark 2.2, K is not necessarily constant along a solution $(\mathbf{P}(t), \mathbf{X}(t))$.) ◊

2.6. **Type-2 generating functions.** We take a brief detour to describe a class of canonical transformations, using a tool called a generating function.

Let's start with any transformation. The transform gives us a function $S(\mathbf{p}, \mathbf{x}, t)$. Observe that $d(\mathbf{P} \cdot \mathbf{X}) = \mathbf{P}\, d\mathbf{X} + X\, d\mathbf{P} + (\partial_t \mathbf{P} \cdot \mathbf{X} + \mathbf{P} \cdot \partial_t \mathbf{X})\, dt$, so (2.7) can be written as

$$d(\mathbf{P} \cdot \mathbf{X} + S) = \mathbf{p}\, d\mathbf{x} + \mathbf{X}\, d\mathbf{P} + C\, dt$$

where we have collected all the dt terms above (since they will not be important). Let $u(\mathbf{P}, \mathbf{x}, t) = \mathbf{P} \cdot \mathbf{X} + S(\mathbf{p}, \mathbf{x}, t)$. That is, we describe our space with the new momenta $\mathbf{P}$ and the old coordinates $\mathbf{x}$, assuming this is possible. We have

$$du = \mathbf{p}\, d\mathbf{x} + \mathbf{X}\, d\mathbf{P} + C\, dt$$

from which it follows that $\nabla_{\mathbf{x}} u = \mathbf{p}$ and $\nabla_{\mathbf{P}} u = \mathbf{X}$.

Let's review what we have done in the discussion above:

(1) We start with a time-dependent canonical transformation $(\mathbf{p}, \mathbf{x}, t) \mapsto (\mathbf{P}, \mathbf{X}, t)$.
(2) From this map, we are able to construct a function $S(\mathbf{p}, \mathbf{x}, t)$.
(3) From S, we construct $u : \mathbb{R}^n_{\mathbf{P}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$ by the relation $u(\mathbf{P}, \mathbf{x}, t) = \mathbf{P} \cdot \mathbf{X} + S(\mathbf{p}, \mathbf{x}, t)$.

Now, we work backwards from what we have done above.

(1) We start with a function $u : \mathbb{R}^n_{\mathbf{P}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$, a function of $2n + 1$ independent coordinates.
(2) We *define* $\mathbf{p}$ to be $\nabla_{\mathbf{x}} u$ and we *define* $\mathbf{X}$ to be $\nabla_{\mathbf{P}} u$.
(3) Then $(\mathbf{p}, \mathbf{x}, t) \mapsto (\mathbf{P}, \mathbf{X}, t)$ is a canonical transformation.

Note that we have described a canonical transformation by a single function u. This function is an example of a *generating function*, that is, a function that "generates" a canonical transformation.[5]

We can write the new Hamiltonian in terms of the old Hamiltonian and the generating function:

Lemma 2.9. *If* $(\mathbf{p}, \mathbf{x}, t) \mapsto (\mathbf{P}(\mathbf{p}, \mathbf{x}, t), \mathbf{X}(\mathbf{p}, \mathbf{x}, t))$ *is a time-dependent canonical transformation generated by* $u(\mathbf{P}, \mathbf{x}, t)$*, then the new Hamiltonian* K*, the old Hamiltonian* H*, and the*

[5]There are 4 types of generating functions, and u falls in the class of *type-2* generating functions. The generating functions used here have no relation to the generating functions used in combinatorics.

generating function u are related by

$$K(\mathbf{P}, \mathbf{X}, t) = H(\mathbf{p}, \mathbf{x}) + \partial_t u(\mathbf{P}, \mathbf{x}, t)$$

Proof. First, we have

$$S(\mathbf{p}, \mathbf{x}, t) = u(\mathbf{P}(\mathbf{p}, \mathbf{x}, t), \mathbf{x}, t) - \mathbf{P}(\mathbf{p}, \mathbf{x}, t) \cdot \mathbf{X}(\mathbf{p}, \mathbf{x}, t)$$

so

$$\begin{aligned} \partial_t S &= \nabla_{\mathbf{p}} u \cdot \partial_t \mathbf{P} + \partial_t u - \mathbf{P} \cdot \partial_t \mathbf{X} - \partial_t \mathbf{P} \cdot \mathbf{X} \\ &= \mathbf{X} \cdot \partial_t \mathbf{P} + \partial_t u - \mathbf{P} \cdot \partial_t \mathbf{X} - \partial_t \mathbf{P} \cdot \mathbf{X} \\ &= \partial_t u - \mathbf{P} \cdot \partial_t \mathbf{X} \end{aligned}$$

Combining the two results with Theorem 2.7 gives $K = H + \partial_t u$. □

2.7. **Hamilton-Jacobi equation.** In this section, we seek a time-dependent canonical transformation for which the transformed Hamiltonian, K, is identically 0. If we are successful, Hamilton's equations in the new coordinates are

$$\begin{aligned} \dot{\mathbf{P}}(t) &= 0 \\ \dot{\mathbf{X}}(t) &= 0 \end{aligned}$$

and solving the system becomes trivial.

We will begin our search with a generating function $u(\mathbf{P}, \mathbf{x}, t)$. Recall that by Lemma 2.9, we have $K = H + \partial_t u$. Also recall that $\mathbf{p} = \nabla_{\mathbf{x}} u$. Thus, in order for K to be zero, we need:

$$\partial_t u(\mathbf{P}, \mathbf{x}, t) + H(\nabla_{\mathbf{x}} u(\mathbf{P}, \mathbf{x}, t), \mathbf{x}) = 0$$

Note that $\mathbf{P}$ has no role in the equation above, so we can assume that u has no explicit dependence on $\mathbf{P}$. Our equation reduces to

$$\partial_t u(\mathbf{x}, t) + H(\nabla_{\mathbf{x}} u(\mathbf{x}, t), \mathbf{x}) = 0$$

This is the *Hamilton-Jacobi equation*. It is a partial differential equation, whose unknown is $u : \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$.

The material developed in this section can be found in many texts on mechanics, such as [Arn89, Chapter 9] or [LL76, Chapter 7].

3. Eikonal Equation and Optics

In this section, we work with $n = 1$ and take $H(p, x) = \frac{1}{2}p^2$. (Recall, from Example 2.1 that this H is the Hamiltonian for a free particle.)

For given initial data $g(x)$, we consider a classical solution to the PDE

$$\begin{aligned} \partial_t u + \tfrac{1}{2}(\partial_x u)^2 &= 0 \quad \text{on } \mathbb{R}_x \times (0,\infty)_t \\ u &= g \quad \text{on } \mathbb{R}_x \times \{0\}_t \end{aligned} \tag{3.1}$$

(By classical, we simply mean that u is differentiable, so that the PDE can be satisfied everywhere on $\mathbb{R}_x \times (0,\infty)_t$.)

3.1. **Characteristic lines.** Suppose u is a classical solution to (3.1). Furthermore, suppose u is twice differentiable in the variable x. Since u is differentiable, we can let $v(x,t) = \partial_x u(x,t)$. Then v satisfies the equation

$$\begin{aligned} \partial_t v + v \cdot \partial_x v &= 0 \quad \text{on } \mathbb{R}_x \times (0,\infty)_t \\ v &= h \quad \text{on } \mathbb{R}_x \times \{0\}_t \end{aligned} \tag{3.2}$$

where $h(x) := g'(x)$

The PDE (3.2) is known as the *inviscid Burgers' equation* and plays an important role in fluid mechanics. We have the following property for solutions of Burgers' equation.

Lemma 3.1. *If v satisfies (3.2), then v is constant along the lines $x = x_0 + t \cdot h(x_0)$ for each $x_0 \in \mathbb{R}$.*

Proof. Suppose $v(x,t)$ is a differentiable solution to (3.2). Consider a function $x(s)$ with $x(0) = x_0$. Then $(x(s), s) \in \mathbb{R}_x \times [0,\infty)_t$ traces out a path through the space, beginning at $(x_0, 0)$. We want to strategically choose the path $x(s)$ so that v behaves "nicely" along the path. To do this, we note that

$$\frac{d}{ds} v(x(s), s) = \partial_x v(x(s), s) \cdot \dot{x}(s) + \partial_t v(x(s), s)$$

If we require that our path $x(s)$ satisfy the ODE $\dot{x}(s) = v(x(s), s)$, then the right hand side above is zero, due to (3.2). In that case v is constant along the path $(x(s), s)$. It follows that $\ddot{x}(s) = 0$, so x is linear.

We have the initial data $x(0) = x_0$ and $\dot{x}(0) = v(x(0), 0) = h(x_0)$, so our path is $x(s) = x_0 + s \cdot h(x_0)$. □

Remark 3.2. This is an application of the *method of characteristics*, which is used to solve first-order nonlinear PDE. The general technique reduces the PDE into a system of ODEs. The lines $x = x_0 + t \cdot h(x_0)$ are called the *characteristic lines* of (3.2). ◊

Example 3.3. Suppose we have initial data $h(x) = 1$. Then the characteristic lines are given by $x = x_0 + t$, and v has the value 1 along each line. (See Figure 3.1.) Thus, we see that $v(x,t) = 1$ is the unique solution to Burgers' equation with the given initial condition. ◊

Example 3.4. Suppose $h(x) = x$. Then v has the value x_0 along the line $x = x_0 + x_0 t$. (See Figure 3.2.) The resulting function $v(x,t) = \frac{x}{1+t}$ is the unique solution to the PDE. ◊

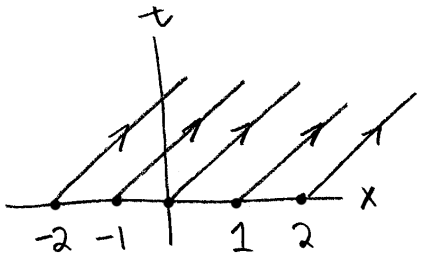

FIGURE 3.1. Characteristic lines for initial data $h(x) = 1$

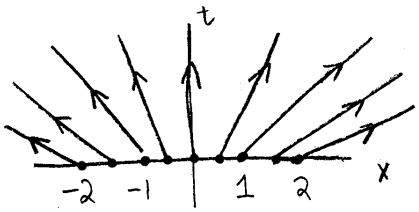

FIGURE 3.2. Characteristic lines for initial data $h(x) = x$

3.2. **Non-existence of a classical solution.** In both Example 3.3 and Example 3.4, there was exactly one unique differentiable solution. However, we can easily come up with initial data for which that is not true.

Suppose we have initial data $g(x) = -|x|$ to the Hamilton-Jacobi equation (3.1).[6] Then v has initial data

$$h(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0 \end{cases}$$

However, we immediately see a problem with our given initial data. Take, for example, the characteristic lines $x = 1 - t$ and $x = -1 + t$. v must be constant along these two lines. This is fine for all $t < 1$, but the two characteristics intersect at $(x, t) = (0, 1)$. (This intersection is called a *shock*.) Since the values of v along the two lines are different, there cannot be a differentiable solution.

3.3. **Falling back to a weak solution.** We can still use the lemma to help us build a plausible solution u to (3.1).

If we return to the two characteristic lines $x = 1-t$ and $x = -1+t$, we note that these lines cause us no problems until we reach $x = 0$. This is, in fact, the case for all characteristic lines: for $a > 0$, the pair of characteristic lines $x = a - t$ and $x = -a + t$ intersect at $(x, t) = (0, a)$. (See Figure 3.3.)

[6]We should not expect lack of regularity of the initial data, by itself, to cause the non-existence of a differentiable solution. For example, let u be the solution to the heat equation $u_t(\mathbf{x}, t) - \Delta_{\mathbf{x}} u(\mathbf{x}, t) = 0$ for initial data $g(\mathbf{x})$. If g is bounded and continuous, then u becomes instantaneously smooth for all $t > 0$, even if g is not smooth (or even differentiable).

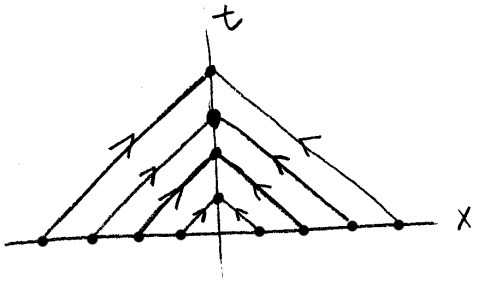

FIGURE 3.3. Characteristic lines intersect at $x = 0$. The points of intersection are called *shocks*.

Thus, we can divide the space $\mathbb{R}_x \times [0, \infty)_t$ along $x = 0$ and consider the function

$$v(x,t) = h(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0 \end{cases}$$

This is a solution to Burgers' equation except when $x = 0$. We get the corresponding solution u to (3.1) by integrating v with respect to x:

$$u(x,t) = \begin{cases} x - \frac{1}{2}t, & \text{if } x < 0 \\ -x - \frac{1}{2}t, & \text{if } x > 0 \end{cases} \qquad = -|x| - \frac{1}{2}t \tag{3.3}$$

The constant of integration is taken to be $-\frac{1}{2}t$ so that u does indeed solve (3.1) when $x \neq 0$.

At this point, we may try to define a "weak solution" to (3.1) to be a function $u(\mathbf{x}, t)$ which satisfies the PDE everywhere except on a single line. However, is this notion adequate?

Suppose at a shock, we choose to "kill off" the characteristic lines emanating from $x > 0$, and to continue the ones from $x < 0$. (See Figure 3.4.)

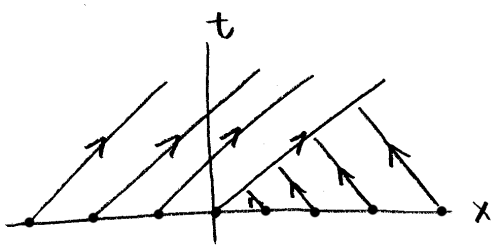

FIGURE 3.4. Another way to resolve the intersection of the characteristic lines.

This produces the solution

$$v(x,t) = \begin{cases} 1, & \text{if } x < t \\ -1, & \text{if } x > t \end{cases}$$

giving us

$$u(x,t) = \begin{cases} x - \frac{1}{2}t, & \text{if } x < t \\ -x - \frac{1}{2}t, & \text{if } x > t \end{cases} \tag{3.4}$$

This solves (3.1) except on the line $x = t$. Hence, it is also a "weak solution," in the sense described above. However, this solution is "worse" than (3.3) because this one is not continuous.

For our given initial data $g(x) = |x|$, we may conclude, based on continuity, that (3.3) is a "better" weak solution than (3.4), but in other situations, it may be difficult to determine the "best" way to resolve the intersection of characteristic lines.

In the following sections, we will develop some theory to give a better notion of a weak solution, and we will return to the particular issue here in subsection 5.3.

4. Control Theory

4.1. **Introduction.** We now move away from physics and towards optimal control theory. Quite interestingly, we will start with the "solution" $u(\mathbf{x}, t)$ and find a PDE that it solves, which turns out to have the form of the Hamilton-Jacobi equation. Quite remarkably, the theory developed here can be used to solve the Hamilton-Jacobi equation for a certain class of Hamiltonians, as we will see in subsection 5.3.

4.2. **Basic problem in control theory.** Consider the ODE

$$\begin{aligned} \dot{\mathbf{y}}(s) &= \mathbf{f}(\mathbf{y}(s)) \quad \text{on } \mathbb{R}_y \times (0,t)_s \\ \mathbf{y}(t) &= \mathbf{x} \end{aligned}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is some fixed function. The goal is to solve for $\mathbf{y} : [0,t] \to \mathbb{R}^n$, a path in the space $\mathbb{R}^n_{\mathbf{y}} \times \mathbb{R}_s$.

We modify this ODE as follows: Let A be some compact subset of, say, $\mathbb{R}^m$. We will call a map $\alpha : [0,t] \to A$ a *control*. Let $\mathcal{A}$ be the set of controls. We let $\mathbf{f} : \mathbb{R}^n \times A \to \mathbb{R}^n$ vary based on the control, and consider the modified ODE

$$\begin{aligned} \dot{\mathbf{y}}(s) &= \mathbf{f}(\mathbf{y}(s), \alpha(s)) \quad \text{on } \mathbb{R}_y \times (0,t)_s \\ \mathbf{y}(t) &= \mathbf{x} \end{aligned} \tag{4.1}$$

In a way, the control α "steers" the resulting path by varying the ODE's solution. We can let $\mathbf{y}^{\alpha(\cdot)}(s)$ denote the solution to the ODE for a particular control $\alpha(\cdot)$.

Example 4.1. Suppose that $n = 1$, that $A = [-1, 1]$, and that $\mathbf{f}(\mathbf{y}, \alpha) = \alpha$. Then our ODE is $\dot{y}(s) = \alpha(s)$, so the control $\alpha(s)$ represents the velocity. Take the particular case $x = 1$ and $t = 1$.

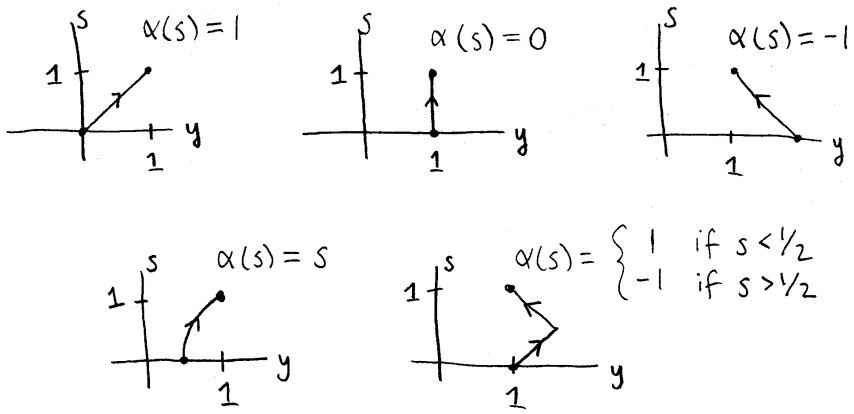

FIGURE 4.1. Solutions $y^{\alpha(\cdot)}(s)$ for some controls $\alpha(s)$.

Figure 4.1 shows the resulting paths $y^{\alpha(\cdot)}(s)$ for some controls $\alpha(\cdot)$. Note that there is no control for which $y(0) < 0$ or $y(0) > 2$; at these starting points, we cannot "run fast enough" to travel to $y = 1$ by time $t = 1$. ◊

By introducing the control as a parameter of our ODE, we can ask the following question: "What choice of $\alpha(\cdot)$ will cause the resulting $\mathbf{y}^{\alpha(\cdot)}(s)$ to be 'optimal' in some way?" We will associate each control to a cost $C_{\mathbf{x},t}[\alpha(\cdot)]$:

$$C_{\mathbf{x},t}[\alpha(\cdot)] = g\left(\mathbf{y}^{\alpha(\cdot)}(0)\right) + \int_0^t r\left(\mathbf{y}^{\alpha(\cdot)}(s), \alpha(s)\right)\, ds$$

where $g : \mathbb{R}^n \to \mathbb{R}$ and $r : \mathbb{R}^n \times A \to \mathbb{R}$ are fixed functions. We can interpret g as the *initial cost* and r as the *running cost per unit time*. That is, the cost to begin at point $\mathbf{x} \in \mathbb{R}^n$ is $g(\mathbf{x})$ and the cost per unit time to pass through point $\mathbf{x} \in \mathbb{R}^n$ with parameter $a \in A$ is $r(\mathbf{x}, a)$.

Then we define

$$u(\mathbf{x}, t) = \inf_{\alpha \in \mathcal{A}} C_{\mathbf{x},t}[\alpha(\cdot)] \tag{4.2}$$

Then $u(\mathbf{x}, t)$ answers the following question: "If we start somewhere in $\mathbb{R}^n_{\mathbf{y}} \times \{0\}_s$ and want to move to the point $(\mathbf{x}, t)$, what is the cheapest way to do so?"

4.3. **Dynamic programming.** Let us suppose that in (4.2), the infimum is actually an minimum, and that the control α attains the minimum cost. In that case, we can write the total cost as

$$C_{\mathbf{x},t}[\alpha(\cdot)] = g\left(\mathbf{y}^{\alpha(\cdot)}(0)\right) + \int_0^{t-h} r\left(\mathbf{y}^{\alpha(\cdot)}(s), \alpha(s)\right)\, ds + \int_{t-h}^t r\left(\mathbf{y}^{\alpha(\cdot)}(s), \alpha(s)\right)\, ds$$

The first two terms represent the cost of moving from $s = 0$ to $(\mathbf{y}^{\alpha(\cdot)}(t-h), t-h)$. The remaining term represents the cost of moving onwards to $(\mathbf{x}, t)$. Employing the principle of

dynamic programming, α should be the control that attains the minimum cost when our target is $(\mathbf{y}^{\alpha(\cdot)}(t-h), t-h)$. That is, if α is the optimal control for time t, then it should be the optimal control for all times earlier than t. Thus, we should have the relation

$$u(\mathbf{x}, t) = u\left(\mathbf{y}^{\alpha(\cdot)}(t-h), t-h\right) + \int_{t-h}^{t} r\left(\mathbf{y}^{\alpha(\cdot)}(s), \alpha(s)\right)\, ds$$

The problem with this argument is that in (4.2), the infinum might not be attainable. The precise statement is actually

Lemma 4.2. *If $0 < t-h < t$, then*

$$u(\mathbf{x}, t) = \inf_{\alpha \in \mathcal{A}} \left\{ u\left(\mathbf{y}^{\alpha(\cdot)}(t-h), t-h\right) + \int_{t-h}^{t} r\left(\mathbf{y}^{\alpha(\cdot)}(s), \alpha(s)\right)\, ds \right\} \tag{4.3}$$

For a proof, see [Eva10, Chapter 10].[7] Once we understand how to interpret the statement Equation 4.3, the proof becomes very straightforward–it is essentially the argument we have presented above, but with some technical modifications.

4.4. **Deriving the Hamilton-Jacobi-Bellman equation.** Lemma 4.2 relates u at t with u at some earlier time $t-h$. We want to consider an infinitesimal version of the relation by letting $h \to 0$.

Suppose again that there exists an $\alpha \in \mathcal{A}$ which solves the problem of minimum cost. Let $\mathbf{y} = \mathbf{y}^{\alpha(\cdot)}$. Then (4.3) can be rearranged to

$$\frac{u(\mathbf{y}(t), t) - u(\mathbf{y}(t-h), t-h)}{h} = \frac{1}{h} \int_{t-h}^{t} r(\mathbf{y}(s), \alpha(s))\, ds$$

The left hand side is a difference quotient. Taking the limit as $h \to 0$ gives us

$$\nabla_{\mathbf{x}} u(\mathbf{y}(t), t) \cdot \dot{\mathbf{y}}(t) + \partial_t u(\mathbf{y}(t), t) = r(\mathbf{y}(t), \alpha(t))$$

Using (4.1), we obtain

$$\nabla_{\mathbf{x}} u(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, \alpha(t)) + \partial_t u(\mathbf{x}, t) = r(\mathbf{x}, \alpha(t))$$

Recall that the relation above is for a particular function $\alpha(\cdot)$. However, because we have taken the limit $h \to 0$, the relation only depends on the value of α at t. Let $a = \alpha(t)$. Once again, because the infinmum might not be actually attained, we have arrived at a slightly modified relation

$$\partial_t u(\mathbf{x}, t) + \max_{a \in A} \{\mathbf{f}(\mathbf{x}, a) \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, t) - r(\mathbf{x}, a)\} = 0 \tag{4.4}$$

[7] Note however, that our setup is slightly different from the setup in [Eva10]. We have chosen to present the control theory problem "backwards," so that our resulting PDE is an initial-value problem, instead of a terminal-value problem (which is what [Eva10] obtains).

Note that the PDE has the form of the Hamilton-Jacobi equation, with Hamiltonian $H(\mathbf{p}, \mathbf{x}) = \max_{a \in A} \{\mathbf{f}(\mathbf{x}, a) \cdot \mathbf{p} - r(\mathbf{x}, a)\}$. This equation in control theory is known as the *Hamilton-Jacobi-Bellman equation*.

We did not, however, provide a rigorous argument that u does solve (4.4). In fact, the function u defined by (4.2) might not even be differentiable anywhere. In the next section, we describe properties that u does satisfy to argue why we should indeed refer to u as *the* solution to (4.4).

5. Introduction to Viscosity Solutions

5.1. **The method of vanishing viscosity.** M. Crandall and P. Lions first introduced the concept of viscosity solutions in their 1983 paper [CL83]. We illustrate the method of vanishing viscosity by applying it to the Hamilton-Jacobi equation. The method works as follows: Instead of seeking a solution u to

$$\begin{aligned} \partial_t u + H(\nabla_{\mathbf{x}} u, \mathbf{x}) &= 0 \quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t \\ u &= g \quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times \{0\}_t \end{aligned} \tag{5.1}$$

we seek a solution u^ϵ to the modified PDE

$$\begin{aligned} \partial_t u^\epsilon + H_\epsilon(\nabla_{\mathbf{x}} u^\epsilon, \mathbf{x}) &= \epsilon \Delta_{\mathbf{x}} u^\epsilon \quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t \\ u^\epsilon &= g \qquad\quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times \{0\}_t \end{aligned} \tag{5.2}$$

where H_ϵ is a sequence of functions that converges uniformly on compact subsets to H.

Suppose for each $\epsilon > 0$, we can find a solution u^ϵ. Also suppose that there is a sequence $\{\epsilon_i\}$ decreasing to 0 such that u^{ϵ_i} converges uniformly on compact subsets to u. We would like to find some properties that characterize u. It turns out one characterization uses the class of smooth test functions $C^\infty(\mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t)$ as well as maximum/minimum principles.[8]

Fix $v \in C^\infty(\mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t)$. Suppose $u - v$ has a local maximum at $(\mathbf{x}_0, t_0) \in \mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t$. Then by continuity, we can find a sequence of points $(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i})$ such that $u^{\epsilon_i} - v$ has a local maximum at $(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i})$, and $(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}) \to (\mathbf{x}_0, t_0)$.

Then the first order conditions give us

$$\begin{aligned} \nabla_{\mathbf{x}}(u^{\epsilon_i} - v)(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}) &= \mathbf{0} \\ \partial_t(u^{\epsilon_i} - v)(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}) &= 0 \end{aligned} \tag{5.3}$$

At a maximum, the Hessian matrix of $u^{\epsilon_i} - v$ is negative-definite. This gives us the second order condition

$$\Delta_{\mathbf{x}}(u^{\epsilon_i} - v)(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}) \leq 0 \tag{5.4}$$

[8] $C^\infty(\mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t)$ denotes the set of smooth $\mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t \to \mathbb{R}$ functions. We call this the class of smooth "test" functions because we will be "testing" our viscosity solutions against this class of functions.

Then by combining (5.2), (5.3), and (5.4), we have

$$\partial_t v(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}) + H_{\epsilon_i}(\nabla_{\mathbf{x}} v(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i}), \mathbf{x}_{\epsilon_i}) \leq \epsilon_i \Delta_{\mathbf{x}} v(\mathbf{x}_{\epsilon_i}, t_{\epsilon_i})$$

Letting $\epsilon_i \to 0$, we conclude that $\partial_t v(\mathbf{x}_0, t_0) + H(\nabla_{\mathbf{x}} v(\mathbf{x}_0, t_0), \mathbf{x}_0) \leq 0$.

We have just shown that if $u - v$ has a local maximum at $(\mathbf{x}_0, t_0)$, then $\partial_t v + H(\nabla_{\mathbf{x}} v, \mathbf{x}) \leq 0$ at that point. Similarly, we can show that if $u - v$ has a local minimum at $(\mathbf{x}_0, t_0)$, then $\partial_t v + H(\nabla_{\mathbf{x}} v, \mathbf{x}) \geq 0$ at that point. (The argument is the same, except the inequality (5.4) is reversed.)

Remark 5.1. The $\epsilon \Delta_{\mathbf{x}} u^\epsilon$ term in (5.2) is the *viscosity* term. The Laplacian appears in numerous PDE with varying interpretations. Intuitively, the Laplacian of a function u at a point $\mathbf{x}$ measures how much $u(\mathbf{x})$ differs from values of u around $\mathbf{x}$. For example, in the heat equation $u_t = \alpha \Delta_{\mathbf{x}} u$, the constant α (the "thermal diffusivity") measures how quickly heat can be transferred. The Navier-Stokes equation contains a term $\mu \Delta \mathbf{v}$, which represents viscosity in the fluid. ♢

The introduction of the viscosity term changes the PDE from being fully nonlinear to being semilinear. Semilinear PDE are usually better behaved than fully nonlinear PDE. We define these classes of PDE below.

Definition 5.2. A PDE of order m is *semilinear* if the derivatives of order m appear linearly with known coefficients. (That is, the coefficients do not contain the unknown function or any of its derivatives.) A PDE of order m is *fully nonlinear* if, when we view all lower order derivatives as constants, the derivatives of order m still appear nonlinearly. ♢

Example 5.3. The equation $\partial_t u + \frac{1}{2}(\partial_x u)^2 = \epsilon \partial_{xx} u$ is a PDE of order $m = 2$. Because the coefficient of $\partial_{xx} u$ is constant, the equation is semilinear. On the other hand, the equation $\partial_t u + \frac{1}{2}(\partial_x u)^2 = 0$ is a fully nonlinear PDE of order $m = 1$, due to the nonlinear term $\frac{1}{2}(\partial_x u)^2$. ♢

5.2. **Viscosity solutions of the Hamilton-Jacobi equation.** The discussion in subsection 5.1 motivates the following definition of a weak solution to the Hamilton-Jacobi equation:

Definition 5.4. We say $u \in C(\mathbb{R}^n_{\mathbf{x}} \times [0, \infty)_t)$ is a *viscosity solution* of the Hamilton-Jacobi equation $\partial_t u + H(\nabla_{\mathbf{x}} u, \mathbf{x}) = 0$ with initial data $g(\mathbf{x})$ if

(1) $u(\mathbf{x}, 0) = g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.
(2) For all $v \in C^\infty(\mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t)$:
   - If $u - v$ has a local maximum at a point $(\mathbf{x}_0, t_0) \in \mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t$, then
   $$\partial_t v(\mathbf{x}_0, t_0) + H(\nabla_{\mathbf{x}} v(\mathbf{x}_0, t_0), \mathbf{x}_0) \leq 0$$
   - If $u - v$ has a local minimum at a point $(\mathbf{x}_0, t_0) \in \mathbb{R}^n_{\mathbf{x}} \times (0, \infty)_t$, then
   $$\partial_t v(\mathbf{x}_0, t_0) + H(\nabla_{\mathbf{x}} v(\mathbf{x}_0, t_0), \mathbf{x}_0) \geq 0$$

♢

As we see from the definition, u can be a viscosity solution even if it is not differentiable; the tradeoff is that we must compare u to an *entire class* of test functions $C^\infty(\mathbb{R}^n_{\mathbf{x}} \times (0,\infty)_t)$. In subsection 5.1, we effectively proved the following property of viscosity solutions:

Theorem 5.5 (stability)**.** *Let H_ϵ converge uniformly on compact subsets to H and suppose $u^\epsilon : \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$ is a classical solution to*

$$\begin{aligned} \partial_t u^\epsilon + H_\epsilon(\nabla_{\mathbf{x}} u^\epsilon, \mathbf{x}) &= \epsilon \Delta_{\mathbf{x}} u^\epsilon \quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times (0,\infty)_t \\ u^\epsilon &= g \qquad\quad \text{on } \mathbb{R}^n_{\mathbf{x}} \times \{0\}_t \end{aligned}$$

Suppose there is some sequence $\{\epsilon_i\}$ that decreases to 0 such that u^{ϵ_i} converges uniformly on compact subsets to u. Then u is a viscosity solution to $\partial_t u + H(\nabla_{\mathbf{x}} u, \mathbf{x}) = 0$ with initial data $g(\mathbf{x})$.

We also have the following:

Theorem 5.6 (consistency)**.** *If u is a classical solution to the Hamilton-Jacobi equation (5.1), then u is also a viscosity solution to (5.1).*

Proof. Suppose v is a smooth test function and $u - v$ has a local extremum at $(\mathbf{x}_0, t_0)$. Then the first order conditions on $u - v$ are $\nabla_{\mathbf{x}}(u-v)(\mathbf{x}_0,t_0) = 0$ and $\partial_t(u-v)(\mathbf{x}_0,t_0) = 0$. Since u satisfies the PDE (5.1) at $(\mathbf{x}_0, t_0)$, it follows from the first order conditions that v does as well. □

Viscosity solutions allow us to conclude our discussion on control theory and the Hamilton-Jacobi-Bellman equation in section 4.

Theorem 5.7. *As in section 4 , let $g : \mathbb{R}^n_{\mathbf{y}} \to \mathbb{R}$ be the initial cost and $r : \mathbb{R}^n_{\mathbf{y}} \times A \to \mathbb{R}$ be the running cost per unit time. Define $u(\mathbf{x}, t)$ by (4.2).*

Furthermore, assume $\mathbf{f}(\mathbf{y}, a)$, $r(\mathbf{y}, a)$ and $g(\mathbf{y})$ are Lipschitz continuous in $\mathbf{y}$ and uniformly bounded.

Then u is a viscosity solution to the Hamilton-Jacobi-Bellman equation

$$\partial_t u(\mathbf{x}, t) + \max_{a \in A} \{\mathbf{f}(\mathbf{x}, a) \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, t) - r(\mathbf{x}, a)\} = 0$$

with initial data $g(\mathbf{x})$.

Remark 5.8. A proof of this theorem is given in [Eva10, Chapter 10]. While the regularity conditions on $\mathbf{f}(\mathbf{y}, a)$, $r(\mathbf{y}, a)$ and $g(\mathbf{y})$ are needed in order to proceed in the proof, they are not the focus. ◊

The result (5.7) in itself is not necessarily a reason for us to consider u to be a legitimate solution; when dealing different PDEs, many notions of "weak" fail to be satisfactory. We would like one and only one solution, but sometimes weak solutions might not exist, or if one does it might not be unique. Viscosity solutions, on the other hand, do in fact satisfy existence and uniqueness properties!

Theorem 5.9 (existence and uniqueness)**.** *Under certain regularity conditions on the Hamiltonian H, there exists a unique viscosity solution to the Hamilton-Jacobi equation* (5.1)*.*

Remark 5.10. The Hopf-Lax formula, which will be presented in subsection 5.3, gives a solution to (5.1) for a narrow class of Hamiltonians. To prove the existence in more general cases, we can use our argument in subsection 5.1, provided we can prove the existence of solutions u^{ϵ_i} to (5.2); see [CL83, Section IV] for a complete discussion. For a proof of uniqueness, see [Eva10, Chapter 10] or [CL83, Theorem V.2]. ◊

5.3. **Hopf-Lax Formula for Hamiltonian Mechanics.** We make the following assumptions about H:

- $H : \mathbb{R}^n_{\mathbf{p}} \to \mathbb{R}$ is a function of $\mathbf{p}$ only
- H is convex.
- $\lim_{|\mathbf{p}|\to\infty} \frac{H(\mathbf{p})}{|\mathbf{p}|} = 0$

Under these assumptions, we can *define* the Lagrangian by $L(\mathbf{v}) = \sup_{\mathbf{p}\in\mathbb{R}^n} \{\mathbf{p}\cdot\mathbf{v} - H(\mathbf{p})\}$. We can show that L will also be convex and it will also satisfy $\lim_{|\mathbf{v}|\to\infty} \frac{L(\mathbf{v})}{|\mathbf{v}|} = 0$.

Using properties of the Legendre transform stated in subsection 2.2, the Hamilton-Jacobi equation takes the form

$$\partial_t u(\mathbf{x},t) + \max_{\mathbf{v}\in\mathbb{R}^n}\{\mathbf{v}\cdot\nabla_{\mathbf{x}} u(\mathbf{x},t) - L(\mathbf{v})\} = 0$$

This has the form of the Hamilton-Jacobi-Bellman equation! Here we use $\mathbf{v}\in\mathbb{R}^n$ in place of $a \in A$. Then, we have $\mathbf{f}(\mathbf{x},\mathbf{v}) = \mathbf{v}$ and $r(\mathbf{x},\mathbf{v}) = L(\mathbf{v})$. If $g(\mathbf{x})$ is our initial data, then our optimization "problem" in this case is

$$u(\mathbf{x},t) = \min_{\mathbf{v}(\cdot)} \left\{ \int_0^t L(\mathbf{v}(s))\,ds + g(\mathbf{y}(0)) \right\} = \min_{\mathbf{y}_0\in\mathbb{R}^n} \left\{ g(\mathbf{y}_0) + \min_{\mathbf{y}(\cdot)} \int_0^t L(\dot{\mathbf{y}}(s))\,ds \right\}$$

where the inner minimum is taken over all paths $\mathbf{y}(s)$ such that $\mathbf{y}(0) = \mathbf{y}_0$ and $\mathbf{y}(t) = \mathbf{x}$. By the theory of Lagrangian mechanics, since our Lagrangian depends on $\mathbf{v}$ and not on $\mathbf{x}$, the minimum of that integral is attained by a straight line path.[9] It follows that

$$u(\mathbf{x},t) = \min_{\mathbf{y}\in\mathbb{R}^n} \left\{ tL\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right) + g(\mathbf{y}) \right\}$$

where we have dropped the subscript 0 from $\mathbf{y}_0$. This formula is the *Hopf-Lax formula*. What this tells us is that the generating function we seek in subsection 2.7 is given by a minimization problem. While the formula can be derived through other means, our path makes the connection to optimization problems clear.

Example 5.11. Take $H(p) = \frac{1}{2}p^2$ and $g(x) = -|x|$. Then $L(v) = \frac{1}{2}v^2$ and the Hopf-Lax formula gives

$$u(x,t) = \min_{y\in\mathbb{R}} \left\{ \frac{(x-y)^2}{2t} - |y| \right\} = -|x| - \frac{1}{2}t$$

[9]The integral itself is called the *action*.

which agrees with our first solution (3.3) in subsection 3.3. By consistency and uniqueness of viscosity solutions, we know there are no differentiable solutions to the PDE (a fact we had already established due to the crossing of characteristics). ◊

5.4. **Generalizations and limitations of viscosity solutions.** As we have already pointed out, the existence and uniqueness of viscosity solutions to the Hamilton-Jacobi equation is quite remarkable. In general, trying to come up with an adequate notion of a weak solution is difficult; the existence of a type of weak solution that satisfies existence and uniqueness properties is rare.

Viscosity solutions first arose from the Hamilton-Jacobi equation, but the theory can be extended to first-order PDE. For a self-contained account, see [CEL84]. [CIL92] extends the theory of viscosity solutions to elliptic second-order PDE.

Viscosity solutions, however, cannot be generalized to systems of PDE. The issue comes not from the general method of vanishing viscosity, which we demonstrated in subsection 5.1, but from a particular step: our derivation relies on the maximum principle, that is, the sign of the Laplacian at a local extremum. This principle does not extend to systems of equations, so there is no analogue to (5.4), a step which was crucial in leading us to the definition of a viscosity solution.

Appendix A. Notation

We employ consistent notation conventions throughout this paper.

A.1. **Functions.**

- When we write $\mathbb{R}^n$ with $\mathbf{x}$ in the subscript (as in $\mathbb{R}^n_{\mathbf{x}}$), we refer to a n-dimensional space whose "typical" element is $\mathbf{x} = (x_1, \ldots, x_n)$.
- If we define a function, we will say what the "typical" arguments are. For example $H : \mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \to \mathbb{R}$ means that H is a function of $2n$ variables "typically" denoted $(\mathbf{p}, \mathbf{x}) = (p_1, \ldots, p_n, x_1, \ldots, x_n)$.

A.2. **Differentiation.**

- The derivative of a function is always taken with respect to one of the "typical variables." We will use ∂ to denote the partial derivative with respect to one scalar variable. We will use ∇ and Δ to denote the gradient and Laplacian, respectively.
- For example, let $u : \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$. Then:
  (1) $\partial_t u$ is a $\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$ function given by

$$\partial_t u(\mathbf{x}, t) = \frac{\partial u}{\partial t}(\mathbf{x}, t)$$

(2) $\nabla_{\mathbf{x}} u$ is a $\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}^n$ function given by

$$\nabla_{\mathbf{x}} u = (\partial_{x_1} u, \dots, \partial_{x_n} u) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$$

(3) $\Delta_{\mathbf{x}} u$ is a $\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}$ function given by

$$\Delta_{\mathbf{x}} u = \partial_{x_1 x_1} u + \ldots + \partial_{x_n x_n} u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

- Using the dot to denote a derivative is reserved for functions of one variable. For example, if $\mathbf{x} : \mathbb{R}_t \to \mathbb{R}^n$, then $\dot{\mathbf{x}}(t) = \frac{d}{dt} x(t)$.
- Note that sometimes a symbol might be used in two different ways. For example $\mathbf{X}$ might denote a $\mathbb{R}^n_{\mathbf{p}} \times \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}_t \to \mathbb{R}^n$ function as well as a (related) $\mathbb{R}_t \to \mathbb{R}^n$ function. $\partial_t \mathbf{X}$ makes sense only for the former and $\dot{\mathbf{X}}(t)$ makes sense only for the latter.

Appendix B. Differential forms

For section 2 (and only section 2), we assume the reader is familiar with elementary techniques and properties of differential forms. Nonetheless, we include this very quick review of differential forms to highlight properties that will be important in our main discussion.

Let V be a n-dimensional vector space over $\mathbb{R}$.

Definition B.1. A *k-form* on the space V is a multilinear, alternating form $\omega : V^k \to \mathbb{R}$. By *multilinear*, we mean that $\omega(v_1, \dots, v_k)$ is linear when viewed as a function in one of its coordinates. By *alternating*, we mean that $\omega(v_{\sigma 1}, \dots, v_{\sigma k}) = \operatorname{sgn}(\sigma)\omega(v_1, \dots, v_k)$, where σ is a permutation of $\{1, \dots, k\}$ and $\operatorname{sgn}(\sigma)$ is the sign of the permutation. ♢

Remark B.2. Let $x_1, \dots, x_n$ be a basis for V. For $i = 1, \dots, n$, let $dx_i : V \to \mathbb{R}$ be the 1-form defined by

$$dx_i : c_1 x_1 + \cdots + c_n x_n \mapsto c_i$$

That is, dx_i denotes projection onto coordinate x_i. Observe that $dx_1, \dots, dx_n$ is a basis for the space of 1-forms on V.[10] ♢

Definition B.3. If ω_1 and ω_2 are 1-forms on V, then the *wedge product* $\omega_1 \wedge \omega_2$ is a 2-form on V, given by

$$\omega_1 \wedge \omega_2 : (\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto \begin{vmatrix} \omega_1(\boldsymbol{\xi}) & \omega_1(\boldsymbol{\eta}) \\ \omega_2(\boldsymbol{\xi}) & \omega_2(\boldsymbol{\eta}) \end{vmatrix}$$ ♢

Remark B.4. It follows from the determinant formula that $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$. ♢

Remark B.5. The set $\{dx_i \wedge dx_j : 1 \le i < j \le n\}$ is a basis for the space of 2-forms on V. ♢

Remark B.6. Also from the determinant formula, $(dx_1 \wedge dx_2)(\boldsymbol{\xi}, \boldsymbol{\eta})$ is the signed area of the projection of the parallelogram spanned by $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ onto the space spanned by x_1 and x_2. ♢

[10] Note that the projection dx_i depends not only on the vector x_i, but on the *entire basis* $x_1, \dots, x_n$.

Remark B.7. Consider the 2-form $\omega^2 = \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$. We can associate to ω^2 the $n \times n$ matrix A given by the following properties:

- If $i < j$, then $(A)_{ij} = a_{ij}$. That is, the upper-diagonal entries of A are given by the values a_{ij}.
- A is skew-symmetric. (Note that this property, when combined with the previous one, completely determines A.)

Then an easy computation shows that

$$\omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = \begin{bmatrix} dx_1(\boldsymbol{\xi}) & \cdots & dx_n(\boldsymbol{\xi}) \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} dx_1(\boldsymbol{\eta}) \\ \vdots \\ dx_n(\boldsymbol{\eta}) \end{bmatrix} \qquad \lozenge$$

Remark B.8. It is possible to extend the definition of the wedge product: If ω_1 is a k_1-form and ω_2 is a k_2-form, then $\omega_1 \wedge \omega_2$ will be a $(k_1 + k_2)$-form. (We will not require this extension.) $\lozenge$

Definition B.9. A differential 1-form on V has the form $\omega = \sum_{i=1}^n f_i\, dx_i$, where each f_i is a $V \to \mathbb{R}$ map. If we take a point $v \in V$ and evaluate each f_i at v, then we get the 1-form $\sum f_i(v)\, dx_i$. $\lozenge$

Remark B.10. If $x_1, \ldots, x_n$ is the standard orthonormal basis of $\mathbb{R}^n$, then $\omega = \sum f_i\, dx_i$ can be interpreted as the vector field over $\mathbb{R}^n$ given by $\mathbf{x} \mapsto (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$. $\lozenge$

Definition B.11. The *exterior derivative* of a differential 1-form is a differential 2-form, given as follows: For the 1-form $\omega^1 = \sum_{j=1}^n f_j\, dx_j$, its exterior derivative is

$$d\omega^1 = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial f_j}{\partial x_i} dx_i \wedge dx_j = \sum_{i<j} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) dx_i \wedge dx_j \qquad \lozenge$$

Remark B.12. If $n = 3$ and $\omega^1 = f_1(\mathbf{x})\, dx_1 + f_2(\mathbf{x})\, dx_2 + f_3(\mathbf{x})\, dx_3$, then $d\omega = g_1(\mathbf{x})\, dx_2 \wedge dx_3 + g_2(\mathbf{x})\, dx_3 \wedge dx_1 + g_3(\mathbf{x})\, dx_1 \wedge dx_2$, where $(g_1, g_2, g_3) = \nabla \times (f_1, f_2, f_3)$. Since $d\omega^1$ is a 2-form, we can associate to it a matrix, in particular:

$$\begin{bmatrix} 0 & g_3 & -g_2 \\ -g_3 & 0 & g_1 \\ g_2 & -g_1 & 0 \end{bmatrix}$$

Observe that $\boldsymbol{\xi} := (g_1, g_2, g_3)$ lies in the kernel of the matrix above. $\lozenge$

Definition B.13. Suppose the matrix corresponding to $d\omega^1$ has a unique eigenvector $\boldsymbol{\xi}(\mathbf{x})$ of eigenvalue 0. Then a *bi-characteristic* of ω^1 is a path through $\mathbb{R}^n$ such that the tangent vector to the path at any point $\mathbf{x}$ is parallel to $\boldsymbol{\xi}(\mathbf{x})$. $\lozenge$

Remark B.14. The bi-characteristics of a differential 1-form are independent of the coordinate system chosen. $\lozenge$

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