Caleb Logemann
MATH666 Finite Element Methods
Homework 1

#1 (a) In order to recast this as a variational problem we will multiply the equation by a test function and integrate.

$$\begin{aligned} -u'' + qu &= f \\ -\int_{-\pi}^{\pi} u''v \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x &= \int_{-\pi}^{\pi} fv \,\mathrm{d}x \\ \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x - (u'v)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} quv \,\mathrm{d}x &= \int_{-\pi}^{\pi} fv \,\mathrm{d}x \end{aligned}$$

If we let $v(\pi) = v(-\pi)$, then the boundary term goes to zero because $u'(\pi) = u'(-\pi)$

$$\int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x = \int_{-\pi}^{\pi} fv \,\mathrm{d}x$$

Thus the variational problem is to find a $u \in V$ where

$$V = \left\{ \int_{-\pi}^{\pi} \left((u')^2 + qu^2 \right) \mathrm{d}x < \infty | u(\pi) = u(-\pi), u'(\pi) = u'(-\pi) \right\}$$

such that

$$\int_{\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x = \int_{-\pi}^{\pi} fv \,\mathrm{d}x$$

for all $v \in V$. The test function v must satisfy the periodic boundary conditions on the function value, but I believe we can also choose to have v satisfy the periodic condition on the derivative as well. This makes the test and trial function spaces the same.

(b) The energy minimization problem that is equivalent to the variational problem is to find a $u \in V$ such that

$$F(u) \leq F(w)$$

for all $w \in V$, where

$$F(w) = \frac{1}{2}\left(\int_{\pi}^{\pi} (w')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} qw^2 \,\mathrm{d}x \right) - \int_{-\pi}^{\pi} fw \,\mathrm{d}x$$

I will now prove that the energy minimization problem and the variational problem are equivalent.

Proof. Let u be a solution to the variational problem, and consider some $w \in V$. Then there exists $v \in V$ such that $u + v = w$. Now consider $F(w)$.

$$\begin{aligned} F(w) &= F(u+v) \\ &= \frac{1}{2}\left(\int_{\pi}^{\pi} (u' + v')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} q(u+v)^2 \,\mathrm{d}x \right) - \int_{-\pi}^{\pi} f(u+v) \,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_{\pi}^{\pi} (u')^2 + 2u'v' + (v')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} qu^2 + 2quv + qv^2 \,\mathrm{d}x \right) - \int_{-\pi}^{\pi} f(u+v) \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\pi}^{\pi} (u')^2 + qu^2 \,\mathrm{d}x - \int_{-\pi}^{\pi} fu \,\mathrm{d}x + \int_{-\pi}^{\pi} u'v' + quv \,\mathrm{d}x - \int_{-\pi}^{\pi} fv \,\mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} (v')^2 + qv^2 \,\mathrm{d}x \\ &= F(u) + \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x - \int_{-\pi}^{\pi} fv \,\mathrm{d}x + \int_{-\pi}^{\pi} (v')^2 + qv^2 \,\mathrm{d}x \end{aligned}$$

Since u is a solution to the variational problem the middle three terms cancel

$$= F(u) + \int_{-\pi}^{\pi} (v')^2 + qv^2 \,\mathrm{d}x$$

Also since $q(x) > 0$ for $x \in [-\pi, \pi]$, this integral is nonnegative

$$\geq F(u)$$

This shows that u is a solution to the energy minimization problem when u is a solution to the variational problem.
Now let u be a solution to the energy minimization problem, then

$$F(u) \leq F(w)$$

for all $w \in V$. Let $v \in V$ and consider $w = u + \varepsilon v$, then $F(u) \leq F(u + \varepsilon v)$. Consider the function of ε,

$$g(\varepsilon) = F(u + \varepsilon v)$$

we know that g has a minimum at $\varepsilon = 0$ thus $g'(0) = 0$.

$$\begin{aligned}
g(\varepsilon) &= \frac{1}{2}\left(\int_{\pi}^{\pi} (u' + \varepsilon v')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} q(u + \varepsilon v)^2 \,\mathrm{d}x\right) - \int_{-\pi}^{\pi} f(u + \varepsilon v) \,\mathrm{d}x \\
g'(\varepsilon) &= \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x - \int_{-\pi}^{\pi} fv \,\mathrm{d}x + 2\varepsilon \int_{-\pi}^{\pi} (v')^2 + qv^2 \,\mathrm{d}x \\
g'(0) &= \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x - \int_{-\pi}^{\pi} fv \,\mathrm{d}x \\
0 &= \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x - \int_{-\pi}^{\pi} fv \,\mathrm{d}x \\
\int_{-\pi}^{\pi} fv \,\mathrm{d}x &= \int_{-\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x
\end{aligned}$$

Since this is true for any $v \in V$, this shows that u is a solution to the variational problem. □

(c) Consider $u, w \in V$ solutions to the variational, then for any $v \in V$ we have

$$\begin{aligned}
\int_{\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x &= \int_{-\pi}^{\pi} fv \,\mathrm{d}x \\
\int_{\pi}^{\pi} w'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} qwv \,\mathrm{d}x &= \int_{-\pi}^{\pi} fv \,\mathrm{d}x \\
\int_{\pi}^{\pi} (u - w)'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} q(u - w)v \,\mathrm{d}x &= 0
\end{aligned}$$

Thus the Galerkin Orthogonality property of this problem is

$$\int_{\pi}^{\pi} (u - w)'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} q(u - w)v \,\mathrm{d}x = 0$$

for u, w solutions to the variational problem and for all $v \in V$.
The energy norm for this problem is found by letting $v = u - w$ and taking the squareroot.

$$\|u\|_E = \sqrt{\int_{\pi}^{\pi} (u')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} qu^2 \,\mathrm{d}x}$$

Note that an energy inner product can be formed as well,

$$[u, v]_E = \int_{\pi}^{\pi} u'v' \,\mathrm{d}x + \int_{-\pi}^{\pi} quv \,\mathrm{d}x \qquad \|u\|_E = \sqrt{[u, u]_E}$$

This satisfies all the properties of an inner product.

$$\begin{aligned}
[u,v]_E &= [v,u]_E \\
[au,v]_E &= a[u,v]_E \\
[u+w,v]_E &= [u,v]_E + [w,v]_E \\
[u,u]_E &\geq 0 \\
[u,u]_E &= 0 \Leftrightarrow u = 0
\end{aligned}$$

Note that $q > 0$ is required for the last two statements. The fact that this forms an inner product allows the Cauchy-Schwarz inequality to be applied directly to the energy norm. Also the Galerkin Orthogonality condition can be expressed as

$$[u-w,v]_E = 0$$

for u, w solutions to the variational problem and for all $v \in V$.

(d) The cG(1) method for this problem is formulated by replacing the test and trial space V with a subspace, V_h^1. Let $-\pi = x_0 < x_1 < \cdots < x_M < x_{M+1} = \pi$ be a partition of $[-\pi, \pi]$, and define $h_j = x_j - x_{j-1}$ and let $h = \max_{1 \leq j \leq M+1}\{h_j\}$. I will also define the functions

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & x_{j-1} \leq x \leq x_j \\ \frac{x - x_{j+1}}{x_j - x_{j+1}} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \cdots, M$ and

$$\phi_0(x) = \phi_{M+1}(x) = \begin{cases} \frac{x - x_0}{x_1 - x_0} & x_0 \leq x \leq x_1 \\ \frac{x - x_{M+1}}{x_M - x_{M+1}} & x_M \leq x \leq x_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

Then the space $V_h^1 = \left\{ \sum_{j=1}^{M+1} (\xi_j \phi_j) | \xi_{M+1} = \frac{h_1 h_{M+1}}{h_1 + h_{M+1}} \left(\frac{\xi_1}{h_1} + \frac{\xi_M}{h_{M+1}} \right) \right\}$. The condition on ξ_{M+1} guarantees the continuity of the derivative at $x_0 = x_{M+1}$. The design of the basis function ϕ_{M+1} guarantees the continuity of the function.

(e) The following will show that the cG(1) method gives the optimal solution in the energy norm.

Proof. Let u be the solution to the original variational and let U be the solution to the cG(1) method. Consider an arbitrary $v \in V_h^1$, and note that Galerkin orthogonality states that $[u-U, U-v]_E = 0$.

$$\begin{aligned}
\|u-U\|_E^2 &= [u-U, u-U]_E \\
&= [u-U, u-U]_E + [u-U, U-v]_E \\
&= [u-U, u-v]_E
\end{aligned}$$

Using Cauchy-Schwarz

$$\leq \|u-U\|_E \|u-v\|_E$$

This shows that $\|u-U\|_E \leq \|u-v\|_E$. □

Now in order to find an error estimate let $v = \pi_h u$.

$$\begin{aligned}\|u - U\|_E^2 &\leq \|u - \pi_h u\|_E^2 \\ &= \int_{-\pi}^{\pi} ((u - \pi_h u)')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} q(u - \pi_h u)^2 \,\mathrm{d}x \\ &\leq Ch^2\|u''\|^2 + Ch^4\|u''\|^2\end{aligned}$$

Thus

$$\|u - U\|_E \leq Ch\|u''\|$$

since the h term dominates the error.

(f) Now I will find an error estimate in the L^2 norm.
First note that this problem is self adjoint, and so the dual problem is

$$\begin{aligned}-\phi'' + q\phi &= e \\ \phi(-\pi) &= \phi(\pi) \\ \phi'(-\pi) &= \phi'(\pi)\end{aligned}$$

Now consider $\|e\|_{L^2}^2$,

$$\|e\|_{L^2}^2 = \int_{-\pi}^{\pi} e^2 \,\mathrm{d}x$$

Using the dual problem

$$\begin{aligned}&= \int_{-\pi}^{\pi} e(-\phi'' + q\phi) \,\mathrm{d}x \\ &= -\int_{-\pi}^{\pi} e\phi'' \,\mathrm{d}x + \int_{-\pi}^{\pi} qe\phi \,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} e'\phi' \,\mathrm{d}x + \int_{-\pi}^{\pi} qe\phi \,\mathrm{d}x\end{aligned}$$

Using Galerkin Orthogonality

$$\begin{aligned}&= \int_{-\pi}^{\pi} e'\phi' \,\mathrm{d}x + \int_{-\pi}^{\pi} qe\phi \,\mathrm{d}x - \int_{-\pi}^{\pi} e'\pi_h\phi' \,\mathrm{d}x - \int_{-\pi}^{\pi} qe\pi_h\phi \,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} e'(\phi - \pi_h\phi)' \,\mathrm{d}x + \int_{-\pi}^{\pi} qe(\phi - \pi_h\phi) \,\mathrm{d}x \\ &= [e, \phi - \pi_h\phi]_E\end{aligned}$$

Now applying Cauchy-Schwarz

$$\leq \|e\|_E\|\phi - \pi_h\phi\|_E$$

From part (e)

$$\leq Ch\|u''\|\|\phi - \pi_h\phi\|_E$$

Now consider $\|\phi - \pi_h\phi\|_E$

$$\begin{aligned}\|\phi - \pi_h\phi\|_E^2 &= \int_{-\pi}^{\pi} ((\phi - \pi_h\phi)')^2 \,\mathrm{d}x + \int_{-\pi}^{\pi} q(\phi - \pi_h\phi)^2 \,\mathrm{d}x \\ &\leq C_1h^2\|\phi''\|^2 + C_2h^4\|\phi''\|^2\end{aligned}$$

Since the h^2 term dominates, the h^4 term can be incorporated into C_1

$$\leq C_1 h^2 \|\phi''\|^2$$

Therefore $\|\phi - \pi_h \phi\|_E \leq Ch\|\phi''\|$.
Next I will show that $\|\phi''\| \leq C\|e\|$. Consider the dual problem again.

$$-\phi'' + q\phi = e$$
$$\int_{-\pi}^{\pi} (-\phi'' + q\phi)^2 \,\mathrm{d}x = \int_{-\pi}^{\pi} e^2 \,\mathrm{d}x$$
$$\int_{-\pi}^{\pi} (\phi'')^2 - 2q\phi''\phi + q^2\phi^2 \,\mathrm{d}x = \int_{-\pi}^{\pi} e^2 \,\mathrm{d}x$$
$$\|\phi''\|^2 + \int_{-\pi}^{\pi} -2q\phi''\phi + q^2\phi^2 \,\mathrm{d}x = \|e\|^2$$
$$\|\phi''\|^2 + \int_{-\pi}^{\pi} 2q(\phi')^2 + q^2\phi^2 \,\mathrm{d}x = \|e\|^2$$
$$\|\phi''\|^2 = \|e\|^2 - \int_{-\pi}^{\pi} 2q(\phi')^2 + q^2\phi^2 \,\mathrm{d}x$$

Since $\int_{-\pi}^{\pi} 2q(\phi')^2 + q^2\phi^2 \,\mathrm{d}x$ is strictly positive

$$\|\phi''\|^2 \leq \|e\|^2$$

Now putting this all together shows that

$$\|e\|_{L^2}^2 \leq Ch\|u''\|_{L^2}\|\phi - \pi_h\phi\|_E \leq Ch^2\|u''\|_{L^2}\|\phi''\|_{L^2} \leq Ch^2\|u''\|_{L^2}\|e\|_{L^2}$$

Dividing both side by $\|e\|_{L^2}$ gives

$$\|e\|_{L^2} \leq Ch^2\|u''\|_{L^2}$$

(g) The following function solves the cG(1) FEM.

```
function [ xi ] = cg1(Bij, Bi0, B00, Li, L0, a, b, nCells )

    A = zeros(nCells);
    for i = 1:nCells
        for j = 1:i
            if( i < nCells && j < nCells)
                B = Bij(i, j);
            elseif ( i == nCells && j < nCells)
                B = Bi0(j);
            elseif ( i == nCells && j == nCells)
                B = B00;
            end
            A(i, j) = B;
            A(j, i) = B;
        end
    end

    rhs = zeros(nCells, 1);
    for i = 1:nCells
        if( i < nCells)
            rhs(i) = Li(i);
```

```
        else
            rhs(i) = L0;
        end
    end

    xi = A\rhs;
end
```

The following script uses this function to solve the given problem.

```
%% Problem 1
a = -pi;
b = pi;
q = @(x) 3 - sin(x) - sin(2*x) - cos(x);
f = @(x) 2*exp(sin(x)).*exp(cos(x));
exactSol = @(x) exp(sin(x)).*exp(cos(x));
exactSolD = @(x) (cos(x) - sin(x)).*exp(sin(x) + cos(x));
B = @(u, uD, v, vD) integral(@(x) uD(x).*vD(x) + q(x).*u(x).*v(x), a, b, 'AbsTol',
    ↪ 1e-10, 'RelTol', 1e-10);
L = @(v) integral(@(x) f(x).*v(x), a, b,'AbsTol', 1e-10, 'RelTol', 1e-10);

EnergyNorm = @(u, uD) sqrt(B(u, uD, u, uD));
L2Norm = @(u) sqrt(integral(@(x) (u(x)).^2, a, b,'AbsTol', 1e-10, 'RelTol', 1e-10))
    ↪ ;

hArray = [];
EnergyErrorArray = [];
L2ErrorArray = [];
for nCells = [10, 20, 40, 80, 160]
    h = (b - a)/nCells;
    hArray = [hArray, h];
    xj = @(j) a + h*j;
    phi = @(x, j) (x - xj(j-1))/h.*(x >= xj(j-1) & x <= xj(j)) - (x - xj(j+1))/h.*(
        ↪ x >= xj(j) & x <= xj(j+1));
    phiD = @(x, j) 1/h.*(x >= xj(j-1) & x <= xj(j)) - 1/h.*(x >= xj(j) & x <= xj(j
        ↪ +1));
    phi0 = @(x) -(x - xj(1))/h.*(x >= xj(0) & x < xj(1)) + (x - xj(nCells-1))/h.*(x
        ↪  >= xj(nCells-1) & x <= xj(nCells));
    phi0D = @(x) -1/h.*(x >= xj(0) & x < xj(1)) + 1/h.*(x >= xj(nCells-1) & x <= xj
        ↪ (nCells));

    Bij = @(i, j) integral(@(x) phiD(x, i).*phiD(x, j) + q(x).*phi(x,i).*phi(x,j),
        ↪ xj(i-1), xj(i+1), 'AbsTol', 1e-10, 'RelTol', 1e-10);
    Bi0 = @(i) integral(@(x) phiD(x, i).*phi0D(x) + q(x).*phi(x,i).*phi0(x), min(a,
        ↪  xj(i-1)), max(b, xj(i+1)), 'AbsTol', 1e-10, 'RelTol', 1e-10);
    B00 = integral(@(x) phi0D(x).*phi0D(x) + q(x).*phi0(x).*phi0(x), xj(0), xj(1),
        ↪ 'AbsTol', 1e-10, 'RelTol', 1e-10) + ...
        integral(@(x) phi0D(x).*phi0D(x) + q(x).*phi0(x).*phi0(x), xj(nCells-1), xj
            ↪ (nCells), 'AbsTol', 1e-10, 'RelTol', 1e-10);

    Li = @(i) integral(@(x) f(x).*(phi(x, i)), xj(i-1), xj(i+1),'AbsTol', 1e-10, '
        ↪ RelTol', 1e-10);
    L0 = integral(@(x) f(x).*phi0(x), xj(0), xj(1), 'AbsTol', 1e-10, 'RelTol', 1e
        ↪ -10) + ...
        integral(@(x) f(x).*phi0(x), xj(nCells-1), xj(nCells), 'AbsTol', 1e-10, '
            ↪ RelTol', 1e-10);
    xi = cg1(Bij, Bi0, B00, Li, L0, a, b, nCells);

    sol = @(x) sum([phi0(x)*xi(end); cell2mat(arrayfun(@(j) phi(x, j).*xi(j), 1:
        ↪ nCells-1, 'UniformOutput', false)')]);
```

```
    solD = @(x) sum([phi0D(x)*xi(end); cell2mat(arrayfun(@(j) phiD(x, j)*xi(j), 1:
        ↪ nCells-1, 'UniformOutput', false)')]);
    EnergyError = EnergyNorm(@(x) exactSol(x) - sol(x), @(x) exactSolD(x) - solD(x)
        ↪ );
    EnergyErrorArray = [EnergyErrorArray, EnergyError];
    L2Error = L2Norm(@(x) exactSol(x) - sol(x));
    L2ErrorArray = [L2ErrorArray, L2Error];
    x = linspace(a, b, nCells*5);
    plot(x, sol(x), x, exactSol(x));
    pause();
end
EnergyOrder = log(EnergyErrorArray(1:end-1)./EnergyErrorArray(2:end))./log(hArray(1
    ↪ :end-1)./hArray(2:end));
L2Order = log(L2ErrorArray(2:end)./L2ErrorArray(1:end-1))./log(hArray(2:end)./
    ↪ hArray(1:end-1));
disp(EnergyOrder);
disp(L2Order);

%% Problem 2
p1 = @(x) -64*pi*(18 - 6*pi^4*x.^2 + 18*pi^2*x.^3 + 2*pi^4*x - 36*pi^2*x.^2 - ...
    27*x + 12*pi^2*x + 4*pi^4*x.^3 + 3*pi^2);
p2 = @(x) 216 - 816*pi^2 - 96*pi^4 + 32*pi^6*x.^2 - 384*pi^4*x.^3 - ...
```

The following table gives the rates of convergence.

M+1	h	Energy Error	Energy Order	L2 Error	L2 Order
10	.628	1.02	-	0.17	-
20	.314	0.52	1	0.04	2
40	.157	0.26	1	0.01	2
80	.078	0.13	1	0.0026	2
160	.039	0.65	1	0.0007	2

#2 (a) In order to recast this as a variational problem I will multiply by a test function and integrate.

$$\begin{aligned} u^{(iv)} &= f \\ \int_0^1 u^{(iv)} v \,\mathrm{d}x &= \int_0^1 f v \,\mathrm{d}x \\ -\int_0^1 u''' v' \,\mathrm{d}x + u''' v|_{x=0}^1 &= \int_0^1 f v \,\mathrm{d}x \\ \int_0^1 u'' v'' \,\mathrm{d}x - u'' v'|_{x=0}^1 + u''' v|_{x=0}^1 &= \int_0^1 f v \,\mathrm{d}x \end{aligned}$$

Letting $v(0) = 0$ and $v'(0) = 0$ gives

$$\int_0^1 u'' v'' \,\mathrm{d}x = \int_0^1 f v \,\mathrm{d}x$$

Thus the Bilinear operator and Linear operator of this varational problem are

$$B(u, v) = \int_0^1 u'' v'' \,\mathrm{d}x \qquad L(v) = \int_0^1 f v \,\mathrm{d}x$$

The test and trial functions must be smooth enough to be well defined for these operators, so u and v must be in

$$V = \left\{ \int_0^1 (u'')^2 + (u')^2 + u^2 \,\mathrm{d}x < \infty | u(0) = 0, u'(0) = 0, u''(1) = 0, u'''(1) = 0 \right\}$$

The test functions must satisfy the same boundary conditions in order for this operator to be self-adjoint.

(b) The energy functional is given by

$$F(w) = \frac{1}{2}B(w,w) - L(w) = \frac{1}{2}\int_0^1 (w'')^2 \,\mathrm{d}x - \int_0^1 fw \,\mathrm{d}x$$

Therefore the energy minimization problem is to find a $u \in V$ such that

$$F(u) \leq F(w)$$

for all $w \in V$. Next I will prove that these two problems are equivalent.

Proof. Let u be a solution to the Energy Minimization problem, and let $v \in V$. Now consider $g(\varepsilon) = F(u + \varepsilon v)$. Since F has a minimum at u, this means that $g'(0) = 0$.

$$\begin{aligned}
g(\varepsilon) &= \frac{1}{2}\int_0^1 ((u+\varepsilon v)'')^2 \,\mathrm{d}x - \int_0^1 f(u+\varepsilon v)\,\mathrm{d}x \\
g'(\varepsilon) &= \int_0^1 ((u+\varepsilon v)'')v'' \,\mathrm{d}x - \int_0^1 fv \,\mathrm{d}x \\
g'(0) &= \int_0^1 u''v'' \,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x \\
0 &= \int_0^1 u''v'' \,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x \\
\int_0^1 u''v''\,\mathrm{d}x &= \int_0^1 fv\,\mathrm{d}x
\end{aligned}$$

Since this is true for any $v \in V$, u is also a solution to the variational problem.
Now let u be a solution to the variational problem and let $w \in V$, then set $v = w - u$, so that $u + v = w$.

$$\begin{aligned}
F(w) &= F(u+v) \\
&= \frac{1}{2}\int_0^1 ((u+v)'')^2 \,\mathrm{d}x - \int_0^1 f(u+v)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (u'')^2 + 2u''v'' + (v'')^2 \,\mathrm{d}x - \int_0^1 f(u+v)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (u'')^2 \,\mathrm{d}x - \int_0^1 fu\,\mathrm{d}x + \int_0^1 u''v''\,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x + \frac{1}{2}\int_0^1 (v'')^2\,\mathrm{d}x
\end{aligned}$$

Since u is a solution to the variational problem

$$\begin{aligned}
&= F(u) + \frac{1}{2}\int_0^1 (v'')^2\,\mathrm{d}x \\
&\geq F(u)
\end{aligned}$$

Thus u is also a solution to the Energy Minimization problem. $\square$

(c) The cG(3) method for this problem is formulated by replacing the test and trial space V with a subspace, V_h^3. Let $-\pi = x_0 < x_1 < \cdots < x_M < x_{M+1} = \pi$ be a partition of $[-\pi, \pi]$, and define $h_j = x_j - x_{j-1}$ and let $h = \max_{1 \leq j \leq M+1}\{h_j\}$. I will also define the functions

$$\phi_j^1(x) = \begin{cases} \left(\frac{1}{h^2} - \frac{2(x-x_j)}{h^3}\right)(x - x_{j-1})^2 & x_{j-1} \leq x \leq x_j \\ \left(\frac{1}{h^2} + \frac{2(x-x_j)}{h^3}\right)(x - x_{j+1})^2 & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi_j^2(x) = \begin{cases} (x - x_j)\frac{(x-x_{j-1})^2}{h^2} & x_{j-1} \leq x \leq x_j \\ (x - x_j)\frac{(x-x_{j+1})^2}{h^2} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \cdots, M$ and

$$\phi_{M+1}^1(x) = \begin{cases} \left(\frac{1}{h^2} - \frac{2(x-x_{M+1})}{h^3}\right)(x - x_M)^2 & x_M \leq x \leq x_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_{M+1}^2(x) = \begin{cases} (x - x_{M+1})\frac{(x-x_M)^2}{h^2} & x_M \leq x \leq x_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

for $j = M + 1$.

Then the space $V_h^3 = \text{span}\left(\phi_j^1, \phi_j^2\right)_{j=1}^{M+1}$. The design of the basis functions guarantees the continuity of the function and its derivative.

Thus the FEM is to find a solution to the variational problem in V_h^3, that is find $u \in V_h^3$ such that

$$B(u, v) = L(v)$$

for all $v \in V_h^3$.

(d) Next I will find a a priori estimate for the error in terms of the energy norm.

$$\begin{aligned} \|e\|_E^2 &= \|u - u_h\|_E^2 \\ &= B(u - u_h, u - u_h) \end{aligned}$$

By Galerkin Orthogonality $B(u - u_h, u_h - \pi_h u) = 0$, so

$$\begin{aligned} &= B(u - u_h, u - u_h) + B(u - u_h, u_h - \pi_h u) \\ &= B(u - u_h, u - \pi_h u) \end{aligned}$$

Using Cauchy-Schwarz

$$\begin{aligned} &\leq \|u - u_h\|_E \|u - \pi_h u\|_E \\ \|e\|_E &\leq \|u - \pi_h u\|_E \\ &\leq \sqrt{\int_0^1 ((u - \pi_h u)'')^2 \, \mathrm{d}x} \\ &\leq \sqrt{\int_0^1 (ch^2 \|u^{(iv)}\|_{L^2})^2 \, \mathrm{d}x} \\ &\leq ch^2 \left\|u^{(iv)}\right\|_{L^2} \end{aligned}$$

Thus this method is second order accurate in the energy norm.

(e) The error estimate in the L^2 norm can be found by considering the dual problem. Since this problem is self-adjoint, the dual problem is

$$\begin{aligned} \phi^{(iv)} &= f \\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) &= 0 \end{aligned}$$

Now consider $\|e\|_{L^2}^2$.

$$\begin{aligned} \|e\|_{L^2}^2 &= \int_0^1 e^2 \, \mathrm{d}x \\ &= \int_0^1 e\phi^{(iv)} \, \mathrm{d}x \\ &= -\int_0^1 e'\phi''' \, \mathrm{d}x \\ &= \int_0^1 e''\phi'' \, \mathrm{d}x \end{aligned}$$

Using Galerkin Orthogonality

$$\begin{aligned} &= \int_0^1 e''\phi'' \, \mathrm{d}x - \int_0^1 e''(\pi_h\phi)'' \, \mathrm{d}x \\ &= \int_0^1 e''(\phi - \pi_h\phi)'' \, \mathrm{d}x \end{aligned}$$

By the Cauchy-Scwarz Inequality

$$\begin{aligned} &\le \|e\|_E \|\phi - \pi_h\phi\|_E \\ &\le Ch^2 \left\|u^{(iv)}\right\|_{L^2} Ch^2 \left\|\phi^{(iv)}\right\|_{L^2} \end{aligned}$$

Using the dual problem and taking the L^2 norm of both sides, we see that $\left\|\phi^{(iv)}\right\|_{L^2} = \|e\|_{L^2}$

$$\begin{aligned} .\|e\|_{L^2}^2 &\le Ch^4 \left\|u^{(iv)}\right\|_{L^2} \|e\|_{L^2} \\ \|e\|_{L^2} &\le Ch^4 \left\|u^{(iv)}\right\|_{L^2} \end{aligned}$$

Thus the FEM is fourth order accurate in the L^2 norm.

(f) The following function solves the cG(3) FEM.

```
function [ xi ] = cg3(B11ij, B11Mp1j, B11Mp1Mp1, B12ij, B12Mp1j, B12iMp1, B12Mp1Mp1
    ↪ , B22ij, B22Mp1j, B22Mp1Mp1, L1i, L1Mp1, L2i, L2Mp1, a, b, M )
    A11 = zeros(M+1);
    A22 = zeros(M+1);
    for i = 1:M+1
        for j = 1:i
            if(i == M+1)
                B11 = B11Mp1j(j);
                B22 = B22Mp1j(j);
            else
                B11 = B11ij(i, j);
                B22 = B22ij(i, j);
            end
```

```
            A11(i, j) = B11;
            A11(j, i) = B11;
            A22(i, j) = B22;
            A22(j, i) = B22;
        end
    end
    A11(M+1,M+1) = B11Mp1Mp1;
    A22(M+1,M+1) = B22Mp1Mp1;

    A12 = zeros(M+1);
    for i = 1:M+1
        for j = 1:M+1
            if (i == M+1)
                B12 = B12Mp1j(j);
            elseif (j == M+1)
                B12 = B12iMp1(i);
            else
                B12 = B12ij(i, j);
            end
            A12(i, j) = B12;
        end
    end
    A12(M+1, M+1) = B12Mp1Mp1;

    A = [A11, A12; A12', A22];
    % zero out small values
    A(abs(A)<1e-10)=0;

    rhs1 = zeros(M+1,1);
    rhs2 = zeros(M+1,1);
    for i = 1:M
            rhs1(i) = L1i(i);
            rhs2(i) = L2i(i);
    end
    rhs1(M+1) = L1Mp1;
    rhs2(M+1) = L2Mp1;
    rhs = [rhs1; rhs2];

    xi = A\rhs;
end
```

The following script uses this function to solve the given problem.

```
%% Problem 2
p1 = @(x) -64*pi*(18 - 6*pi^4*x.^2 + 18*pi^2*x.^3 + 2*pi^4*x - 36*pi^2*x.^2 - ...
    27*x + 12*pi^2*x + 4*pi^4*x.^3 + 3*pi^2);
p2 = @(x) 216 - 816*pi^2 - 96*pi^4 + 32*pi^6*x.^2 - 384*pi^4*x.^3 - ...
    64*pi^6*x.^3 - 288*pi^4*x.^2 + 3456*pi^2*x + 576*pi^4*x - 2592*pi^2*x.^2 + ...
    144*pi^4*x.^4 + 32*pi^6*x.^4;
f = @(x) p1(x).*cos(2*pi*x) + p2(x).*sin(2*pi*x);
exactSol = @(x) ((18 + 2*pi^2)*x.^2 + (-24-4*pi^2)*x.^3 + (9 + 2*pi^2)*x.^4).*sin
    ↪ (2*pi*x);
exactSolDD = @(x) 4.*pi.*(2.*(18+2.*pi.^2).*x+3.*((-24)+(-4).*pi.^2).*x
    ↪ .^2+4.*(9+2.* ...
  pi.^2).*x.^3).*cos(2.*pi.*x)+(2.*(18+2.*pi.^2)+6.*((-24)+(-4).* ...
  pi.^2).*x+12.*(9+2.*pi.^2).*x.^2).*sin(2.*pi.*x)+(-4).*pi.^2.*(( ...
  18+2.*pi.^2).*x.^2+((-24)+(-4).*pi.^2).*x.^3+(9+2.*pi.^2).*x.^4).* ...
  sin(2.*pi.*x);
a = 0;
b = 1;
```

```
B = @(uDD, vDD) integral(@(x) uDD(x).*vDD(x), a, b, 'AbsTol', 1e-10, 'RelTol', 1e
    ↪ -10);
L = @(v) integral(@(x) f(x).*v(x), a, b,'AbsTol', 1e-10, 'RelTol', 1e-10);

EnergyNorm = @(uDD) sqrt(B(uDD, uDD));
L2Norm = @(u) sqrt(integral(@(x) (u(x)).^2, a, b,'AbsTol', 1e-10, 'RelTol', 1e-10))
    ↪ ;

hArray = [];
EnergyErrorArray = [];
L2ErrorArray = [];
for M = [9, 19, 39] % 79, 159]
%for M = [9, 19]
    h = (b - a)/(M+1);
    hArray = [hArray, h];
    xj = @(j) a + h*j;
    phi1 = @(x, j) (3*((x-xj(j-1))/h).^2 - 2*((x - xj(j-1))/h).^3).*(x >= xj(j-1) &
        ↪  x < xj(j)) ...
        + (3*((x-xj(j+1))/h).^2 - 2*((x - xj(j+1))/(-1*h)).^3).*(x >= xj(j) & x <=
            ↪ xj(j+1));
    phi2 = @(x, j) ((x - xj(j)).*(x - xj(j-1)).^2/h^2).*(x >= xj(j-1) & x < xj(j))
        ↪ ...
        + ((x - xj(j)).*(x - xj(j+1)).^2/h^2).*(x >= xj(j) & x <= xj(j+1));
    phi1Mp1 = @(x, j) (3*((x-xj(M))/h).^2 - 2*((x - xj(M))/h).^3).*(x >= xj(M) & x
        ↪ <= xj(M+1));
    phi2Mp1 = @(x) ((x - xj(M+1)).*(x - xj(M)).^2/h^2).*(x >= xj(M) & x <= xj(M+1))
        ↪ ;
    phi1DD = @(x, j) (6/h^2 - 12*(x - xj(j-1))/h^3).*(x >= xj(j-1) & x < xj(j)) ...
        + (6/h^2 + 12*(x - xj(j+1))/h^3).*(x >= xj(j) & x <= xj(j+1));
    phi2DD = @(x, j) (2*(x - xj(j)) + 4*(x - xj(j-1))).*(x >= xj(j-1) & x < xj(j))
        ↪ ...
        + (2*(x - xj(j)) + 4*(x - xj(j+1))).*(x >= xj(j) & x <= xj(j+1));
    phi1DDMp1 = @(x) (6/h^2 - 12*(x - xj(M))/h^3).*(x >= xj(M) & x <= xj(M+1));
    phi2DDMp1 = @(x) (2*(x - xj(M+1)) + 4*(x - xj(M))).*(x >= xj(M) & x <= xj(M+1))
        ↪ ;

    B11ij = @(i, j) integral(@(x) phi1DD(x, i).*phi1DD(x, j) , xj(i-1), xj(i+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B11Mp1j = @(j) integral(@(x) phi1DDMp1(x).*phi1DD(x, j) , xj(M), xj(M+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B11Mp1Mp1 = integral(@(x) phi1DDMp1(x).*phi1DDMp1(x) , xj(M), xj(M+1), 'AbsTol'
        ↪ , 1e-10, 'RelTol', 1e-10);
    B12ij = @(i, j) integral(@(x) phi1DD(x, i).*phi2DD(x, j) , xj(i-1), xj(i+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B12Mp1j = @(j) integral(@(x) phi1DDMp1(x).*phi2DD(x, j) , xj(M), xj(M+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B12iMp1 = @(i) integral(@(x) phi1DD(x,i).*phi2DDMp1(x) , xj(M), xj(M+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B12Mp1Mp1 = integral(@(x) phi1DDMp1(x).*phi2DDMp1(x) , xj(M), xj(M+1), 'AbsTol'
        ↪ , 1e-10, 'RelTol', 1e-10);
    B22ij = @(i, j) integral(@(x) phi2DD(x, i).*phi2DD(x, j) , xj(i-1), xj(i+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B22Mp1j = @(j) integral(@(x) phi2DDMp1(x).*phi2DD(x, j) , xj(M), xj(M+1), '
        ↪ AbsTol', 1e-10, 'RelTol', 1e-10);
    B22Mp1Mp1 = integral(@(x) phi2DDMp1(x).*phi2DDMp1(x) , xj(M), xj(M+1), 'AbsTol'
        ↪ , 1e-10, 'RelTol', 1e-10);
    L1i = @(i) integral(@(x) f(x).*(phi1(x, i)), xj(i-1), xj(i+1),'AbsTol', 1e-10,
        ↪ 'RelTol', 1e-10);
    L2i = @(i) integral(@(x) f(x).*(phi2(x, i)), xj(i-1), xj(i+1),'AbsTol', 1e-10,
        ↪ 'RelTol', 1e-10);
```

```
    L1Mp1 = integral(@(x) f(x).*(phi1Mp1(x)), xj(M), xj(M+1),'AbsTol', 1e-10, '
        ↪ RelTol', 1e-10);
    L2Mp1 = integral(@(x) f(x).*(phi2Mp1(x)), xj(M), xj(M+1),'AbsTol', 1e-10, '
        ↪ RelTol', 1e-10);

    xi = cg3(B11ij, B11Mp1j, B11Mp1Mp1, B12ij, B12Mp1j, B12iMp1, B12Mp1Mp1, B22ij,
        ↪ B22Mp1j, B22Mp1Mp1, L1i, L1Mp1, L2i, L2Mp1, a, b, M);

    sol = @(x) sum([cell2mat(arrayfun(@(j) phi1(x, j).*xi(j), 1:M, 'UniformOutput',
        ↪  false)'); phi1Mp1(x)*xi(M+1); cell2mat(arrayfun(@(j) phi2(x, j).*xi(M
        ↪ +1+j), 1:M, 'UniformOutput', false)'); phi2Mp1(x)*xi(end)]);
    solDD = @(x) sum([cell2mat(arrayfun(@(j) phi1DD(x, j).*xi(j), 1:M, '
        ↪ UniformOutput', false)'); phi1DDMp1(x)*xi(M+1); cell2mat(arrayfun(@(j)
        ↪ phi2DD(x, j).*xi(M+1+j), 1:M, 'UniformOutput', false)'); phi2DDMp1(x)*xi
        ↪ (end)]);
    EnergyError = EnergyNorm(@(x) exactSolDD(x) - solDD(x));
    EnergyErrorArray = [EnergyErrorArray, EnergyError];
    L2Error = L2Norm(@(x) exactSol(x) - sol(x));
    L2ErrorArray = [L2ErrorArray, L2Error];
    x = linspace(a, b, nCells*5);
    plot(x, sol(x), x, exactSol(x));
    pause();
end
EnergyOrder = log(EnergyErrorArray(1:end-1)./EnergyErrorArray(2:end))./log(hArray(1
    ↪ :end-1)./hArray(2:end));
L2Order = log(L2ErrorArray(2:end)./L2ErrorArray(1:end-1))./log(hArray(2:end)./
    ↪ hArray(1:end-1));
disp(EnergyOrder);
disp(L2Order);
```

The following shows the order of convergence

M+1	h	Energy Error	Energy Order	L2 Error	L2 Order
10	0.1	3.6	-	0.0016	-
20	0.05	0.9	1.99	0.0001	3.99
40	0.025	0.22	1.99	0.000006	4.02
80	0.0125	0.057	1.99	0.0000004	3.83
160	0.00625	0.013	2.05	0.0000004	0.065