Discontinuous Galerkin Method for Solving Thin Film Equations

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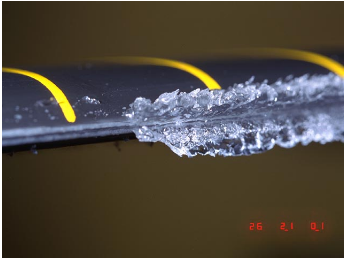
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating

Model Equations

- Navier-Stokes Equation

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g} \\
\partial_t h_s + (u, v)^T \cdot \nabla h_s &= w \\
\partial_t h_b + (u, v)^T \cdot \nabla h_b &= w
\end{aligned}
$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation - 1D with q as fluid height.

$$
q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x
$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

  Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x = 0$$

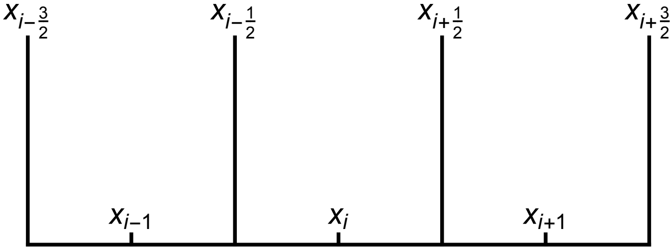
  for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$
\begin{aligned}
\int_{I_j} Q_t v \, \mathrm{d}x = &\int_{I_j} f(Q) v_x \, \mathrm{d}x \\
&- \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)
\end{aligned}
$$

  for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$
\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)
$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$q^\star = q^n + \Delta t L(q^n)$$
$$q^{n+1} = \frac{1}{2}(q^n + q^\star) + \frac{1}{2}\Delta t L(q^\star)$$

Diffusion

- Diffusion Equation

$$q_t + \left(q^3 u_{xxx}\right)_x = 0 \qquad (0, T) \times \Omega$$

- Local Discontinuous Galerkin

$$\begin{aligned} r &= q_x \\ s &= r_x \\ u &= q^3 s_x \\ q_t &= -u_x \end{aligned}$$

Local Discontinuous Galerkin

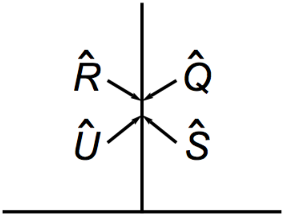
Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t,\cdot), R, S, U \in V_h = V_h = \left\{ v \in L^1(\Omega) : \, v|_{I_j} \in P^k(I_j) \right\}$

$$
\begin{aligned}
\eta &= Q^3 \\
\int_{I_j} R v \,\mathrm{d}x &= -\int_{I_j} Q v_x \,\mathrm{d}x + \left(\hat{Q}_{j+1/2} v_{j+1/2}^{-} - \hat{Q}_{j-1/2} v_{j-1/2}^{+} \right) \\
\int_{I_j} S w \,\mathrm{d}x &= -\int_{I_j} R w_x \,\mathrm{d}x + \left(\hat{R}_{j+1/2} w_{j+1/2}^{-} - \hat{R}_{j-1/2} w_{j-1/2}^{+} \right) \\
\int_{I_j} U y \,\mathrm{d}x &= \int_{I_j} S_x \eta y \,\mathrm{d}x - \left(S_{j+1/2}^{-} \eta_{j+1/2}^{-} y_{j+1/2}^{-} - S_{j-1/2}^{+} \eta_{j-1/2}^{+} y_{j-1/2}^{+} \right) \\
&\quad + \left(\hat{S}_{j+1/2} \hat{\eta}_{j+1/2} y_{j+1/2}^{-} - \hat{S}_{j-1/2} \hat{\eta}_{j-1/2} y_{j-1/2}^{+} \right) \\
\int_{I_j} Q_t z \,\mathrm{d}x &= -\int_{I_j} U z_x \,\mathrm{d}x + \left(\hat{U}_{j+1/2} z_{j+1/2}^{-} - \hat{U}_{j-1/2} z_{j-1/2}^{+} \right)
\end{aligned}
$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$\hat{\eta}_{j+1/2} = \frac{1}{2}\left(\eta^{+}_{j+1/2} + \eta^{-}_{j+1/2}\right)$$
$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$
$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$
$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$
$$\hat{U}_{j+1/2} = U^{-}_{j+1/2}$$



Implicit L-Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$\begin{aligned} q^{\star} &= q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^{\star})) \\ 3q^{n+1} &= 4q^{\star} - q^n + \Delta t L(q^{n+1}) \end{aligned}$$

Linear Solver

- Generalized Minimal Residual (GMRES)

$$\min_{x \in K_n} \{ \|A\mathbf{x} - \mathbf{b}\| \}$$
$$K_n = \operatorname{span}\left(\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{n-1}\mathbf{b}\right)$$

- Preconditioned

$$P = A_0^{-1}$$
$$PAx = Pb$$

Riemann Problem

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Introduction

Method

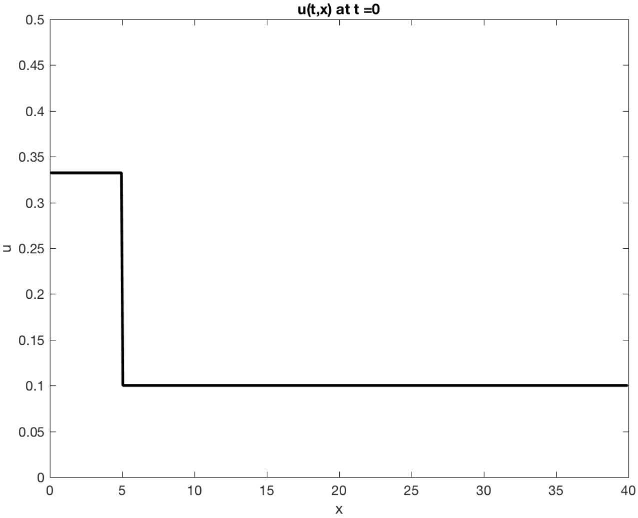
Convection

Diffusion

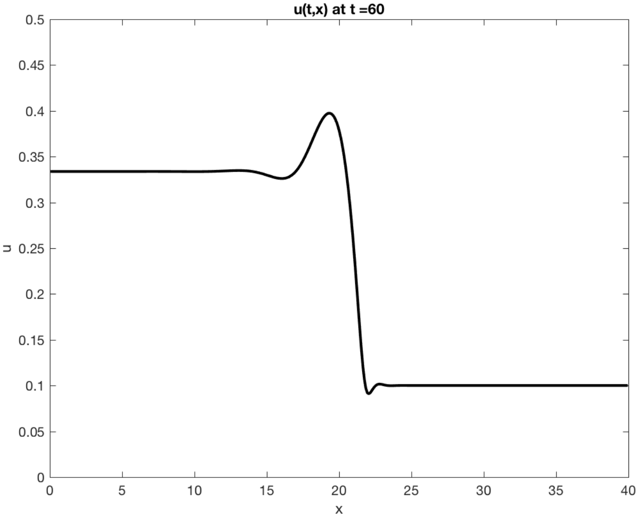
Numerical Results

Conclusion

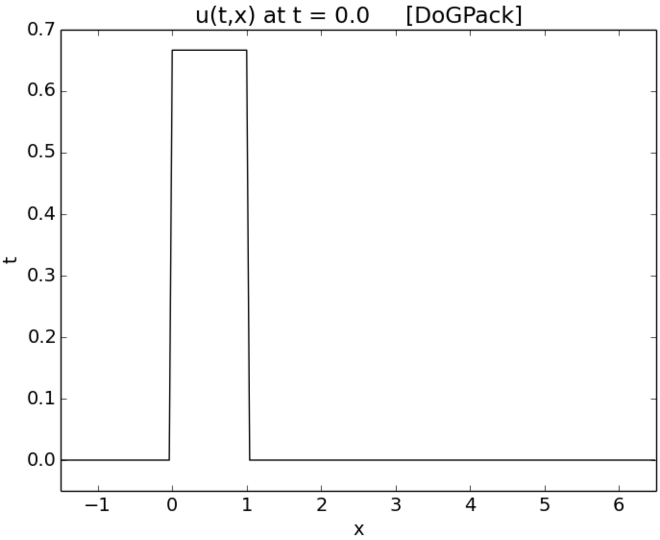
References



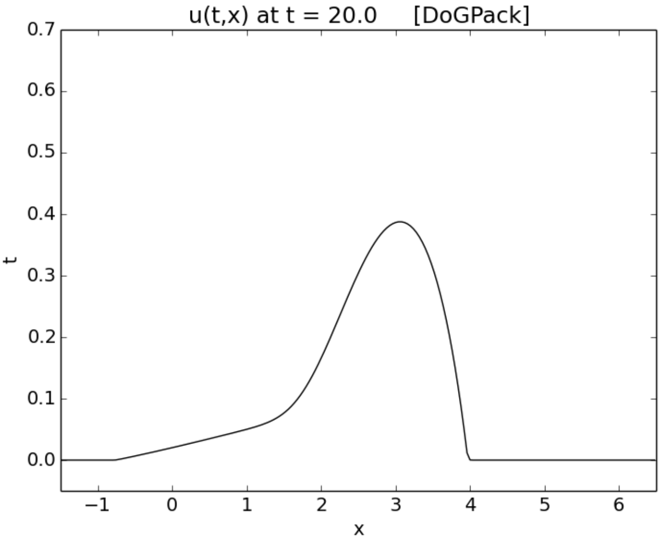
Riemann Problem



Square Wave



Square Wave



Future Work

- Show second order convergence
- Runge Kutta IMEX
- Space and time dependent coefficients

Bibliography

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