MATH 666: Finite Element Methods
Homework 3

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#1 Consider the Crank-Nicolson-cG(1) FEM for the diffusion equation. Find $u_h^n \in V_h^1$ such that

$$\left\langle \frac{u_h^n - u_h^n}{\delta T}, v \right\rangle + \left\langle \nabla \frac{u_h^n + u_h^{n-1}}{2}, \nabla v \right\rangle = \left\langle \frac{f^n + f^{n-1}}{2}, v \right\rangle \quad \forall v \in V_h^1, n = 1, 2, 3, \ldots,$$
$$\left\langle u_h^0, v \right\rangle = \langle u_0, v \rangle$$

Following the convergence proof of the Backward-Euler-cG(1) FEM from class, prove that the Crank-Nicolson-cG(1) FEM satisfies:

$$\|u_h^n - u(t^n)\| = \mathcal{O}\Big(\Delta t^2 + h^2\Big).$$

The initial variational problem is formulated as finding $u \in V$ such that

$$\langle u_t, v \rangle + \langle \nabla u, \nabla v \rangle = \langle f, v \rangle \quad \forall v \in V$$
$$\langle u(0, x), v \rangle = \langle u_0, v \rangle$$

The semidescrete form of the FEM can be expressed as finding $u_h \in V_h^1$ such that

$$\langle (u_h)_t, v \rangle + \langle \nabla u_h, \nabla v \rangle = \langle f, v \rangle \quad \forall v \in V_h^1 \langle u_h(0, x), v \rangle = \langle u_0, v \rangle$$

I will also make use of the following projection $R_h(u)$ which satisfies

$$\langle \nabla R_h(u), \nabla v \rangle = \langle \nabla u, \nabla v \rangle \quad \forall v \in V_h^1$$

Suppose u is the solution to the original variational problem, then for all $v \in V_h^1$,

$$\langle u_t, v \rangle + \langle \nabla u, \nabla v \rangle = \langle f, v \rangle$$
$$\langle u_t, v \rangle + \langle \nabla R_h(u), \nabla v \rangle = \langle f, v \rangle$$

This is true for every time $t > 0$, in particular it is true at times t^n and t^{n-1}, so

$$\langle u_t^n, v \rangle + \langle \nabla R_h(u^n), \nabla v \rangle = \langle f^n, v \rangle$$
$$\left\langle u_t^{n-1}, v \right\rangle + \left\langle \nabla R_h(u^{n-1}), \nabla v \right\rangle = \left\langle f^{n-1}, v \right\rangle$$
$$\left\langle \frac{u_t^n + u_t^{n-1}}{2}, v \right\rangle + \left\langle \nabla R_h\left(\frac{u^n + u^{n-1}}{2}\right), \nabla v \right\rangle = \left\langle \frac{f^n + f^{n-1}}{2}, v \right\rangle$$