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MATH666 Finite Element Methods
Homework 1

#1

(a) In order to recast this as a variational problem we will multiply the equation by a test function and integrate.

$$
\begin{aligned}
-u'' + qu &= f \\
-\int_{-\pi}^{\pi} u''v \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x &= \int_{-\pi}^{\pi} fv \, \mathrm{d}x \\
\int_{-\pi}^{\pi} u'v' \, \mathrm{d}x - \left. (u'v) \right|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} quv \, \mathrm{d}x &= \int_{-\pi}^{\pi} fv \, \mathrm{d}x
\end{aligned}
$$

If we let $v(\pi) = v(-\pi)$, then the boundary term goes to zero because $u'(\pi) = u'(-\pi)$

$$
\int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x = \int_{-\pi}^{\pi} fv \, \mathrm{d}x
$$

Thus the variational problem is to find a $u \in V = \left\{ \int_{-\pi}^{\pi} ((u')^2 + qu^2) \, \mathrm{d}x < \infty | u(\pi) = u(-\pi), u'(\pi) = u'(-\pi) \right\}$

$$
\int_{\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x = \int_{-\pi}^{\pi} fv \, \mathrm{d}x
$$

for all $v \in V$. The test function v must satisfy the periodic boundary conditions on the function value, but I believe we can also choose to have v satisfy the periodic condition on the derivative as well. This makes the test and trial function spaces the same.

(b) The energy minimization problem that is equivalent to the variational problem is to find a $u \in V$ such that

$$
F(u) \leq F(w)
$$

for all $w \in V$, where

$$
F(w) = \frac{1}{2} \left(\int_{\pi}^{\pi} (w')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} qw^2 \, \mathrm{d}x \right) - \int_{-\pi}^{\pi} fw \, \mathrm{d}x
$$

I will now prove that the energy minimization problem and the variational problem are equivalent.

Proof. Let u be a solution to the variational problem, and consider some $w \in V$. Then there exists $v \in V$ such that $u + v = w$. Now consider $F(w)$.

$$
\begin{aligned}
F(w) &= F(u + v) \\
&= \frac{1}{2} \left(\int_{\pi}^{\pi} (u' + v')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} q(u + v)^2 \, \mathrm{d}x \right) - \int_{-\pi}^{\pi} f(u + v) \, \mathrm{d}x \\
&= \frac{1}{2} \left(\int_{\pi}^{\pi} (u')^2 + 2u'v' + (v')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} qu^2 + 2quv + qv^2 \, \mathrm{d}x \right) - \int_{-\pi}^{\pi} f(u + v) \, \mathrm{d}x \\
&= \frac{1}{2} \left(\int_{\pi}^{\pi} (u')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} qu^2 \, \mathrm{d}x \right) - \int_{-\pi}^{\pi} fu \, \mathrm{d}x + \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \int_{-\pi}^{\pi} fv \, \mathrm{d}x + \int_{-\pi}^{\pi} (v')^2 + qv^2 \, \mathrm{d}x \\
&= F(u) + \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \int_{-\pi}^{\pi} fv \, \mathrm{d}x + \int_{-\pi}^{\pi} (v')^2 + qv^2 \, \mathrm{d}x
\end{aligned}
$$

Since u is a solution to the variational problem the middle three terms cancel

$$= F(u) + \int_{-\pi}^{\pi} (v')^2 + qv^2 \, \mathrm{d}x$$

Also since $q(x) > 0$ for $x \in [-\pi, \pi]$, this integral is nonnegative

$$\geq F(u)$$

This shows that u is a solution to the energy minization problem when u is a solution to the variational problem.

Now let u be a solution to the energy minimization problem, then

$$F(u) \leq F(w)$$

for all $w \in V$. Let $v \in V$ and consider $w = u + \varepsilon v$, then $F(u) \leq F(u + \varepsilon v)$. Consider the function of ε,

$$g(\varepsilon) = F(u + \varepsilon v)$$

we know that g has a minimum at $\varepsilon = 0$ thus $g'(0) = 0$.

$$\begin{aligned}
g(\varepsilon) &= \frac{1}{2}\left(\int_{\pi}^{\pi} (u' + \varepsilon v')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} q(u + \varepsilon v)^2 \, \mathrm{d}x\right) - \int_{-\pi}^{\pi} f(u + \varepsilon v) \, \mathrm{d}x \\
&= \frac{1}{2}\left(\int_{\pi}^{\pi} (u')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} qu^2 \, \mathrm{d}x\right) - \int_{-\pi}^{\pi} fu \, \mathrm{d}x + \varepsilon \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \varepsilon \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \varepsilon \int_{-\pi}^{\pi} fv \, \mathrm{d}x + \varepsilon^2 \int_{-\pi}^{\pi} (v')^2 + \\
g'(\varepsilon) &= \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \int_{-\pi}^{\pi} fv \, \mathrm{d}x + 2\varepsilon \int_{-\pi}^{\pi} (v')^2 + qv^2 \, \mathrm{d}x \\
g'(0) &= \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \int_{-\pi}^{\pi} fv \, \mathrm{d}x \\
0 &= \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x - \int_{-\pi}^{\pi} fv \, \mathrm{d}x \\
\int_{-\pi}^{\pi} fv \, \mathrm{d}x &= \int_{-\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x
\end{aligned}$$

Since this is true for any $v \in V$, this shows that u is a solution to the variational problem. $\square$

(c) Consider $u, w \in V$ solutions to the variational, then for any $v \in V$ we have

$$\begin{aligned}
\int_{\pi}^{\pi} u'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} quv \, \mathrm{d}x &= \int_{-\pi}^{\pi} fv \, \mathrm{d}x \\
\int_{\pi}^{\pi} w'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} qwv \, \mathrm{d}x &= \int_{-\pi}^{\pi} fv \, \mathrm{d}x \\
\int_{\pi}^{\pi} (u - w)'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} q(u - w)v \, \mathrm{d}x &= 0
\end{aligned}$$

Thus the Galerkin Orthogonality property of this problem is

$$\int_{\pi}^{\pi} (u - w)'v' \, \mathrm{d}x + \int_{-\pi}^{\pi} q(u - w)v \, \mathrm{d}x = 0$$

for all $v \in V$.

The energy norm for this problem is found by letting $v = u - w$

$$\|u\|_E = \int_{\pi}^{\pi} (u')^2 \, \mathrm{d}x + \int_{-\pi}^{\pi} qu^2 \, \mathrm{d}x$$

(d) The cG(1) method for this problem is formulated by replacing the test and trial space V with a subspace, V_h^1. Let $-\pi = x_0 < x_1 < \cdots < x_M < x_{M+1} = \pi$ be a partition of $[-\pi, \pi]$, and define $h_j = x_j - x_{j-1}$ and let $h = \max_{1 \le j \le M+1}\{h_j\}$. I will also define the functions

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & x_{j-1} \le x \le x_j \\ \frac{x - x_{j+1}}{x_j - x_{j+1}} & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \cdots, M$ and

$$\phi_0(x) = \phi_{M+1}(x) = \begin{cases} \frac{x - x_0}{x_1 - x_0} & x_0 \le x \le x_1 \\ \frac{x - x_{M+1}}{x_M - x_{M+1}} & x_M \le x \le x_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

Then the space $V_h^1 = \left\{ \sum_{j=1}^{M+1} (\xi_j \phi_j) | \xi_{M+1} = \frac{h_1 h_{M+1}}{h_1 + h_{M+1}} \left(\frac{\xi_1}{h_1} + \frac{\xi_M}{h_{M+1}} \right) \right\}$. The condition on ξ_{M+1} guarantees the continuity of the derivative at $x_0 = x_{M+1}$. The design of the basis function ϕ_{M+1} guarantees the continuity of the function.

(e)

(f)

(g)

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