MATH 666: Finite Element Methods
Homework 2

Caleb Logemann

October 23, 2018

#1 Consider the 2D Poisson equation:

$$\begin{aligned} \textbf{PDE}: &\quad -\nabla\cdot\nabla u = f(x,y) \quad \text{in} \quad \Omega = [-1,1]\times[-1,1] \\ \textbf{BC}: &\quad u + \nabla u\cdot\hat{\mathbf{n}} = g \quad \text{on} \quad \partial\Omega \end{aligned}$$

(a) Recast this problem as a variational problem. Clearly state the test and trial function spaces.

In order to recast this as a variational problem, I will multiply by a test function and integrate over Ω.

$$\iint_\Omega -(\nabla\cdot\nabla u)v\,\mathrm{d}\mathbf{x} = \iint_\Omega fv\,\mathrm{d}\mathbf{x}$$

Integrating by parts gives

$$\iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega}(\nabla u\cdot\hat{\mathbf{n}})v\,\mathrm{d}s = \iint_\Omega fv\,\mathrm{d}\mathbf{x}$$

Using the boundary condition we see that $\nabla u\cdot\hat{\mathbf{n}} = g - u$ on $\partial\Omega$

$$\begin{aligned} \iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega}(g-u)v\,\mathrm{d}s &= \iint_\Omega fv\,\mathrm{d}\mathbf{x} \\ \iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} uv\,\mathrm{d}s &= \iint_\Omega fv\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} gv\,\mathrm{d}s \end{aligned}$$

In order for this equation to be well defined u and v must be in the space $H^1(\Omega)$, that is they are square integrable on Ω and the norm of their gradients are square integrable on Ω. Note that being in $H^1(\Omega)$ implies $L^2(\partial\Omega)$, so the boundary integrals are also well defined.
So the bilinear form of this variational problem is to find $u \in H^1(\Omega)$ such that

$$B(u,v) = L(v)$$

for all $v \in H^1(\Omega)$, where

$$B(u,v) = \iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} uv\,\mathrm{d}s$$

and

$$L(v) = \iint_\Omega fv\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} gv\,\mathrm{d}s.$$

(b) Show that the variational problem has a unique solution by showing that it meets all of the criteria of the Lax-Milgram Theorem.

The four conditions of the Lax-Milgram Theorem are symmetry, continuity of B, V-ellipticity, and continuity of L, and are shown below.

$$\begin{aligned} B(u,v) &= B(v,u) \\ |B(u,v)| &\le \gamma\|u\|_V\|v\|_V \\ B(v,v) &\ge \alpha\|v\|_V^2 \\ |L(v)| &\le \Gamma\|v\|_V \end{aligned}$$

I will show these four conditions in order. First, symmetry

$$\begin{aligned} B(u,v) &= \iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} uv\,\mathrm{d}s \\ &= \iint_\Omega \nabla v\cdot\nabla u\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} vu\,\mathrm{d}s \\ &= B(v,u) \end{aligned}$$

Second, boundedness of B

$$\begin{aligned}|B(u,v)| &= \left|\iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} uv\,\mathrm{d}s\right| \\ &\le \left|\iint_\Omega \nabla u\cdot\nabla v\,\mathrm{d}\mathbf{x}\right| + \left|\int_{\partial\Omega} uv\,\mathrm{d}s\right|\end{aligned}$$

Using Cauchy-Schwarz on both integrals gives

$$\le \|\|\nabla u\|\|_{L^2(\Omega)}\|\|\nabla v\|\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}$$

Since $\partial\Omega \subset \Omega$, this implies that $\|u\|_{L^2(\partial\Omega)} \le \|u\|_{L^2(\Omega)}$, so

$$\le \|\|\nabla u\|\|_{L^2(\Omega)}\|\|\nabla v\|\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

Now since $\|u\|_{H^1(\Omega)}$ is greater than both $\|\|\nabla u\|\|_{L^2(\Omega)}$ and $\|u\|_{L^2(\Omega)}$

$$|B(u,v)| \le 2\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

Third I will show V-ellipticity of B. There are two possible cases $v = 0$ on $\partial\Omega$ or $v \neq 0$ on $\partial\Omega$. If $v = 0$ on $\partial\Omega$, then

$$\begin{aligned}B(v,v) &= \iint_\Omega \|\nabla v\|^2\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} v^2\,\mathrm{d}s \\ &= \iint_\Omega \|\nabla v\|^2\,\mathrm{d}\mathbf{x} \\ &= \frac{1}{2}\Big(\iint_\Omega \|\nabla v\|^2\,\mathrm{d}\mathbf{x} + \iint_\Omega \|\nabla v\|^2\,\mathrm{d}\mathbf{x}\Big) \\ &= \frac{1}{2}\Big(\|\|\nabla v\|\|^2_{L^2(\Omega)} + \|\|\nabla v\|\|^2_{L^2(\Omega)}\Big)\end{aligned}$$

Poincare's Inequality states that there exists a constant $C > 0$ such that $C\|v\|^2_{L^2(\Omega)} \le \|\|\nabla v\|\|^2_{L^2(\Omega)}$, therefore

$$\begin{aligned}B(v,v) &\ge \frac{1}{2}\Big(\|\|\nabla v\|\|^2_{L^2(\Omega)} + C\|v\|^2_{L^2(\Omega)}\Big) \\ &\ge \frac{1}{2}\min\{1,C\}\Big(\|\|\nabla v\|\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\Big) \\ &= \frac{1}{2}\min\{1,C\}\|v\|_{H^1(\Omega)}\end{aligned}$$

Lastly I will show that L is bounded,

$$\begin{aligned}|L(v)| &= \left|\iint_\Omega fv\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} gv\,\mathrm{d}s\right| \\ &\le \left|\iint_\Omega fv\,\mathrm{d}\mathbf{x}\right| + \left|\int_{\partial\Omega} gv\,\mathrm{d}s\right|\end{aligned}$$

Using Cauchy-Schwarz on both integrals gives

$$\le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}$$

Now since $\|v\|_{H^1(\Omega)}$ is greater than both $\|v\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\partial\Omega)}$

$$\begin{aligned}&\le \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|v\|_{H^1(\Omega)} \\ &\le 2\max\Big\{\|f\|_{L^2(\Omega)}, \|g\|_{L^2(\partial\Omega)}\Big\}\|v\|_{H^1(\Omega)}\end{aligned}$$

This shows that the variational problem satisfies all of the criteria for the Lax-Milgram theorem, therefore it has a unique solution.

(c)

(d)

#2