Discontinuous Galerkin Method for Solving Thin Film Equations

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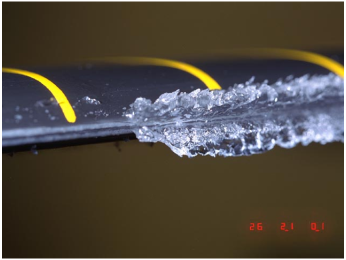
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating
- Paint Drying

Model Equations

- Navier-Stokes Equation

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{u}\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$\partial_t h_s + (u, v)^T \cdot \nabla h_s = w$$

$$\partial_t h_b + (u, v)^T \cdot \nabla h_b = w$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation - 1D with q as fluid height.

$$q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

  Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_\Omega (q_t v - f(q) v_x) \, \mathrm{d}x = 0$$

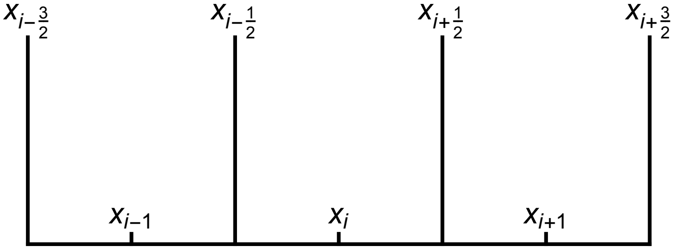
  for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \,\mathrm{d}x = \int_{I_j} f(Q) v_x \,\mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^\star &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^\star) + \frac{1}{2}\Delta t L(q^\star) \end{aligned}$$

Diffusion

- Diffusion Equation

$$q_t + \left(q^3 u_{xxx}\right)_x = 0 \qquad (0, T) \times \Omega$$

- Local Discontinuous Galerkin

$$\begin{aligned} r &= q_x \\ s &= r_x \\ u &= q^3 s_x \\ q_t &= -u_x \end{aligned}$$

Local Discontinuous Galerkin

Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t,\cdot), R, S, U \in V_h = V_h = \left\{v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j)\right\}$

$$\eta = Q^3$$

$$\int_{I_j} Rv \, \mathrm{d}x = -\int_{I_j} Qv_x \, \mathrm{d}x + \left(\hat{Q}_{j+1/2}v^-_{j+1/2} - \hat{Q}_{j-1/2}v^+_{j-1/2}\right)$$

$$\int_{I_j} Sw \, \mathrm{d}x = -\int_{I_j} Rw_x \, \mathrm{d}x + \left(\hat{R}_{j+1/2}w^-_{j+1/2} - \hat{R}_{j-1/2}w^+_{j-1/2}\right)$$

$$\int_{I_j} Uy \, \mathrm{d}x = \int_{I_j} S_x\eta y \, \mathrm{d}x - \left(S^-_{j+1/2}\eta^-_{j+1/2}y^-_{j+1/2} - S^+_{j-1/2}\eta^+_{j-1/2}y^+_{j-1/2}\right) + \left(\hat{S}_{j+1/2}\hat{\eta}_{j+1/2}y^-_{j+1/2} - \hat{S}_{j-1/2}\hat{\eta}_{j-1/2}y^+_{j-1/2}\right)$$

$$\int_{I_j} Q_t z \, \mathrm{d}x = -\int_{I_j} Uz_x \, \mathrm{d}x + \left(\hat{U}_{j+1/2}z^-_{j+1/2} - \hat{U}_{j-1/2}z^+_{j-1/2}\right)$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

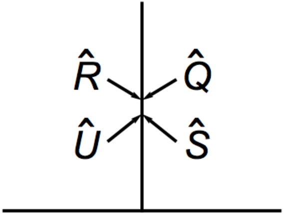
$$\hat{\eta}_{j+1/2} = \frac{1}{2}\left(\eta^{+}_{j+1/2} + \eta^{-}_{j+1/2}\right)$$

$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$

$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$

$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$

$$\hat{U}_{j+1/2} = U^{-}_{j+1/2}$$



Implicit L Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$q^\star = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^\star))$$
$$3q^{n+1} = 4q^\star - q^n + \Delta t L(q^{n+1})$$

Future Work

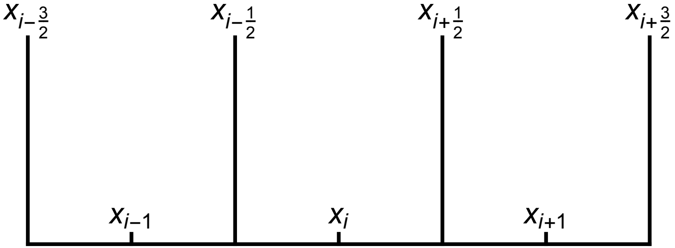
- Show second order convergence

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

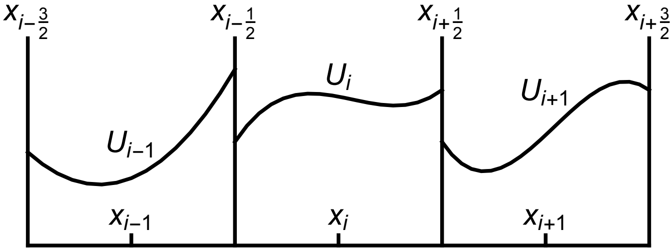
- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$.



Numerical Solutions

- Use canonical variable $\xi \in [-1, 1]$
- Let $\left\{\phi^k(\xi)\right\}$ be the Legendre polynomials.
- Solution of order M on each cell

$$u\big|_{x \in V_i} \approx U_i = \sum_{k=1}^{M} U_i^k \phi^k(\xi)$$



Convection

- Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_\xi = 0$$
$$f(u) = u^2 - u^3$$

- Weak Form

$$\int_{-1}^{1} \left(u_t \phi(\xi) + \frac{2}{\Delta x} f(u)_\xi \phi(\xi) \right) \mathrm{d}\xi = 0$$

- Runge-Kutta Discontinuous Galerkin

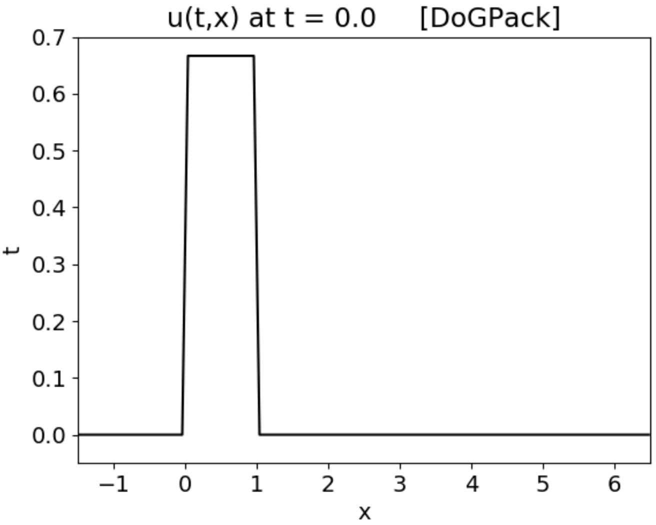
$$\dot{U}_i^\ell = \frac{1}{\Delta x} \int_{-1}^{1} f(U_i) \phi_\xi^\ell \,\mathrm{d}\xi - \frac{1}{\Delta x} \left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2} \right)$$

- Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^\ell(1)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Numerical Example - Square Wave



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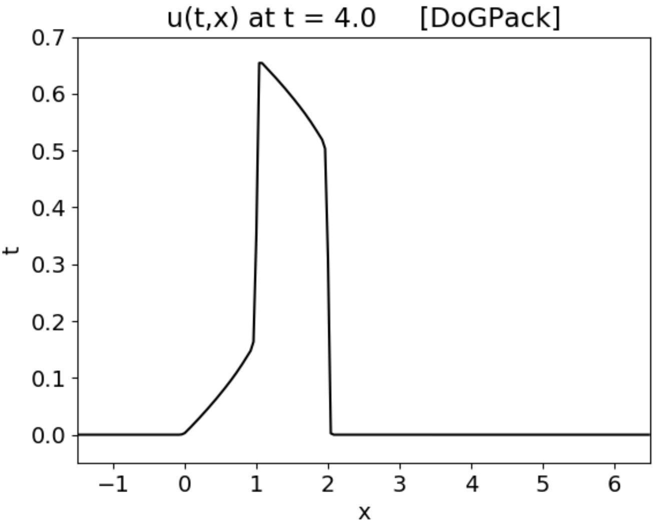

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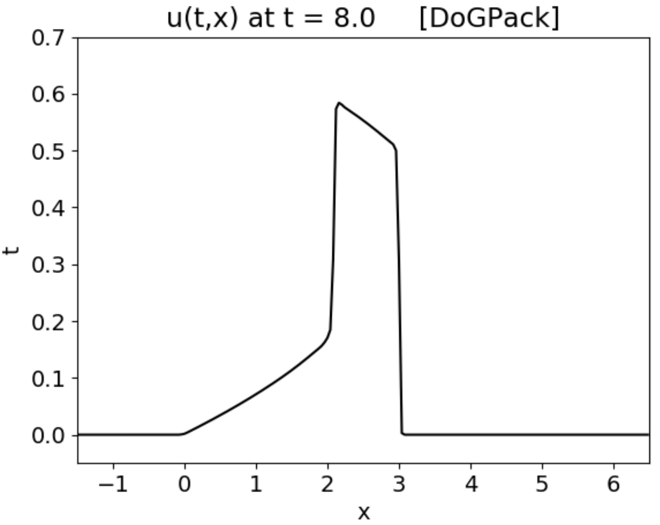

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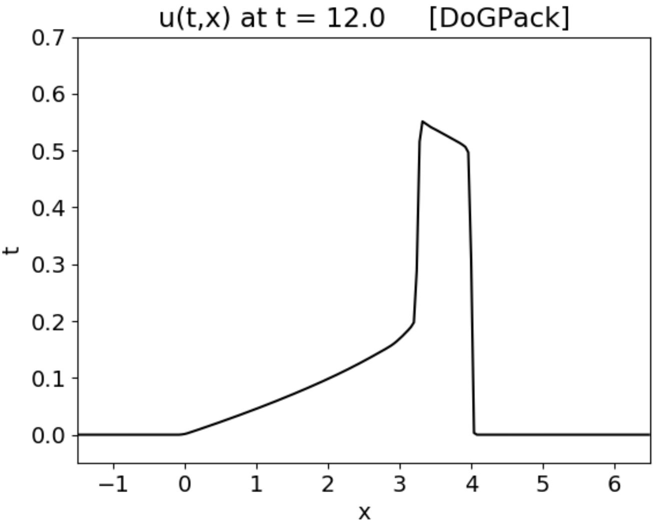

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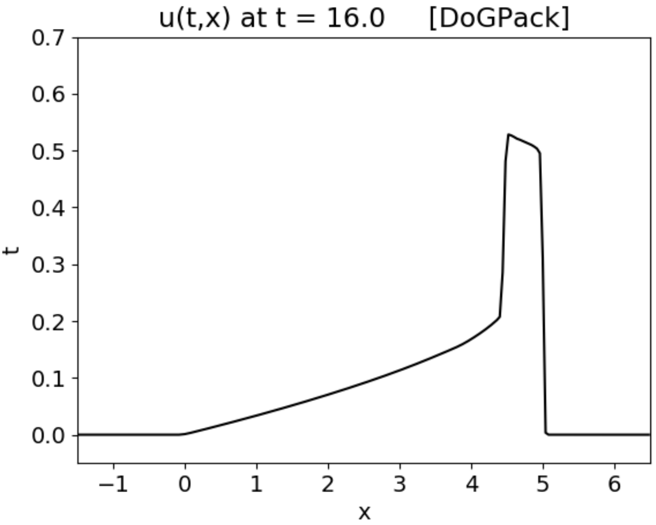

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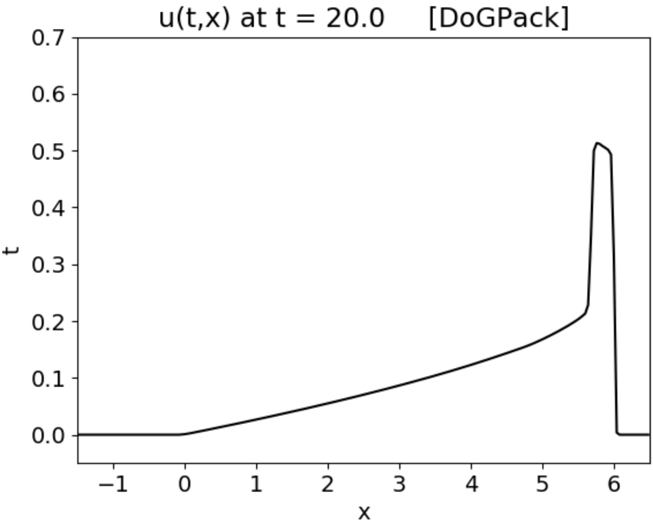


Hyper-Diffusion

- Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4}\left(u^3 u_{\xi\xi\xi}\right)_\xi = 0$$

- Local Discontinuous Galerkin (LDG)

$$\begin{aligned}
q &= \frac{2}{\Delta x} u_\xi \\
r &= \frac{2}{\Delta x} q_\xi \\
s &= \frac{2}{\Delta x} u^3 r_\xi \\
u_t &= -\frac{2}{\Delta x} s_\xi
\end{aligned}$$

Local Discontinuous Galerkin

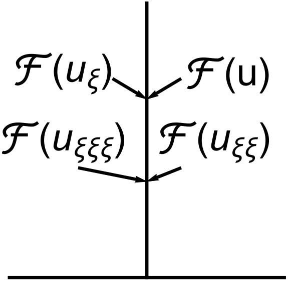
$$
\begin{aligned}
\eta(\xi) &= \left(U_i^n\right)^3 \\
Q_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} U_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(U)_{i+1/2}^\ell + \mathcal{F}(U)_{i-1/2}^\ell\right) \\
R_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} Q_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(Q)_{i+1/2}^\ell + \mathcal{F}(Q)_{i-1/2}^\ell\right) \\
S_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} (R_i)_\xi \eta(\xi) \phi^\ell \,\mathrm{d}\xi\right) \\
&\quad + \frac{1}{\Delta x}\left(\mathcal{F}(\eta)_{i+1/2}\mathcal{F}(R)_{i+1/2}^\ell - \mathcal{F}(\eta)_{i-1/2}\mathcal{F}(R)_{i-1/2}^\ell\right) \\
\dot{U}_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} S_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(S)_{i+1/2}^\ell + \mathcal{F}(S)_{i-1/2}^\ell\right)
\end{aligned}
$$

Local Discontinuous Galerkin

$$\mathcal{F}(\eta)_{i+1/2} = \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1))$$

$$\mathcal{F}(\eta)_{i-1/2} = \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1))$$

$$\mathcal{F}(*)^{\ell}_{i+1/2} = \phi^{\ell}(1)*_{i+1/2}$$

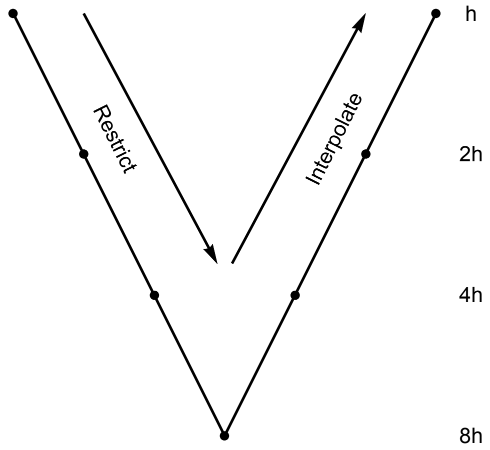


Local Discontinuous Galerkin

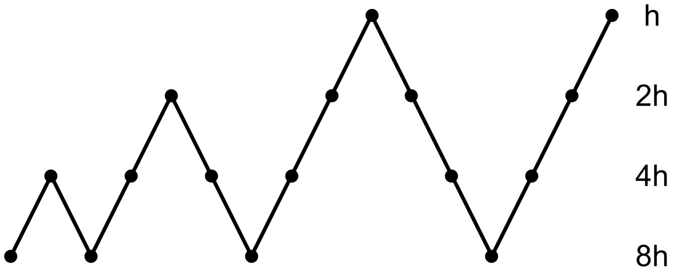
- Explicit SSP Runge Kutta
  - Severe time step restriction
  - $\Delta t \sim \Delta x^4$
  - $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
  - $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
  - Linear System Solver
  - Stabilized BiConjugate Gradient
  - MultiGrid Solver

Multigrid Solver

V-Cycle



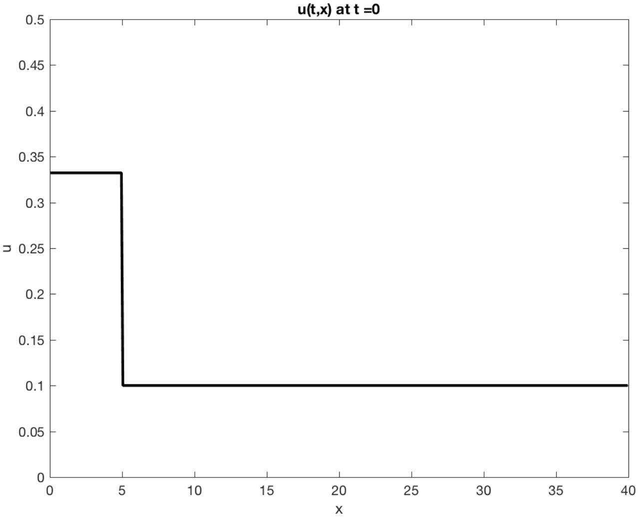
Multigrid Solver



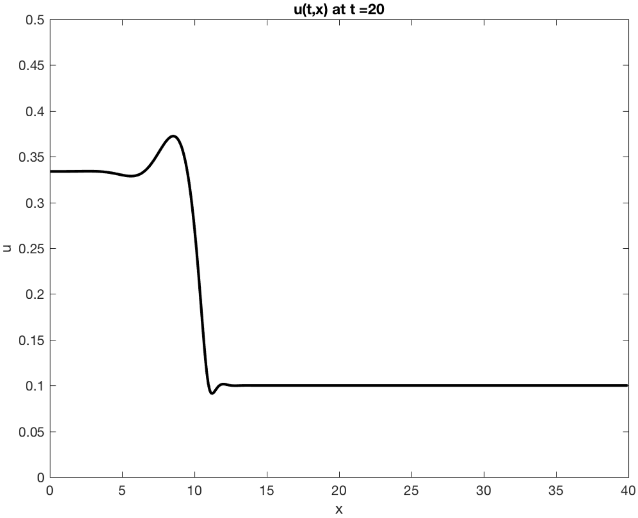
Operator Splitting

- Strang Splitting
  - 1 time step
    - 1/2 time step for convection
    - 1 time step for hyper-diffusion
    - 1/2 time step for convection
  - Second order splitting

Numerical Results - Riemann Problem



Numerical Results - Riemann Problem



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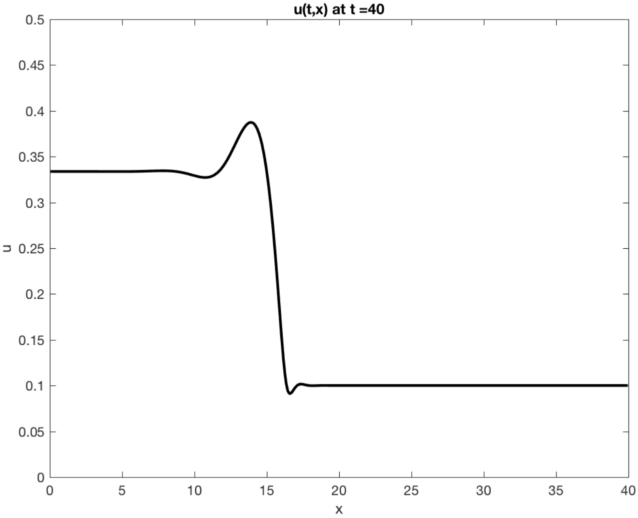

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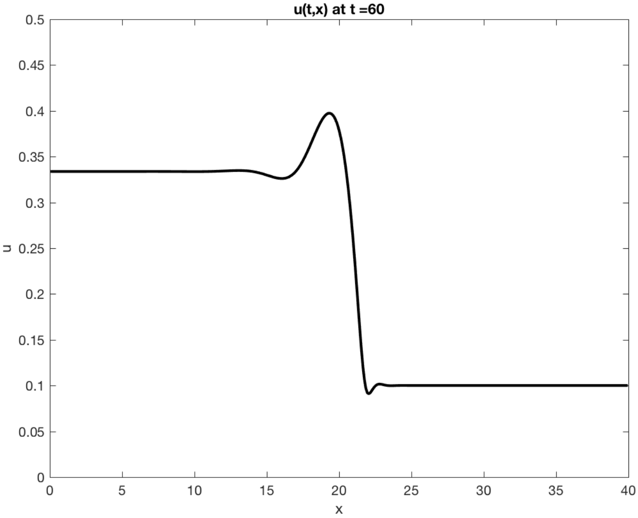

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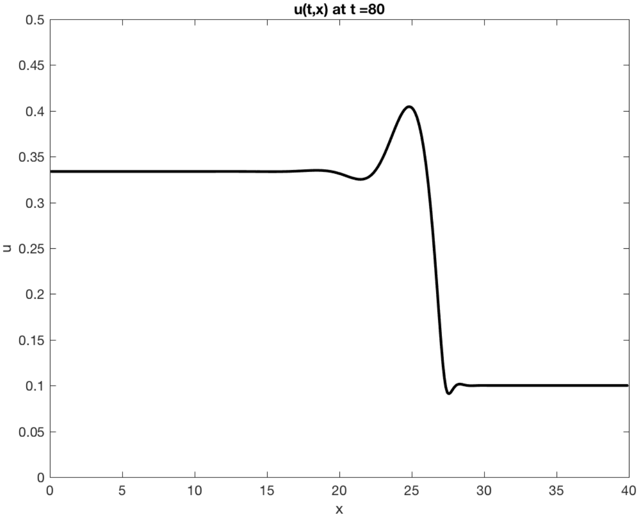

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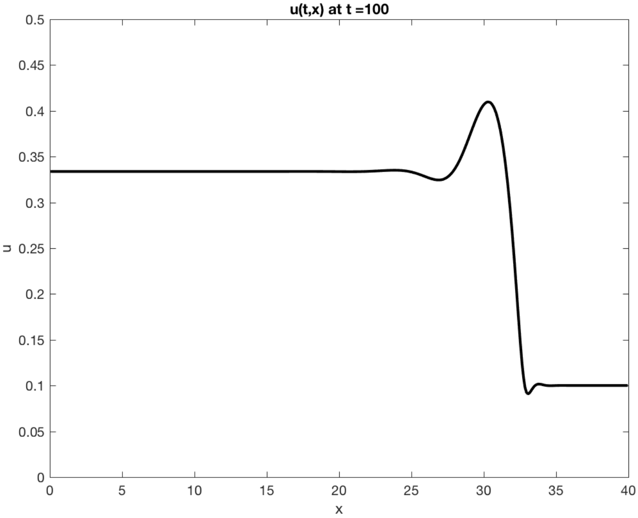

Future Work

- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Runge Kutta IMEX

Conclusion

- Thanks
  - James Rossmanith
  - Alric Rothmayer
- Questions?

Bibliography