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MATH666 Finite Element Methods
Homework 1

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- (a) In order to recast this as a variational problem we will multiply the equation by a test function and integrate.

$$\begin{aligned} -u'' + qu &= f \\ -\int_{-\pi}^{\pi} u''v \, dx + \int_{-\pi}^{\pi} quv \, dx &= \int_{-\pi}^{\pi} fv \, dx \\ \int_{-\pi}^{\pi} u'v' \, dx - \left[u'v \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} quv \, dx &= \int_{-\pi}^{\pi} fv \, dx \end{aligned}$$

If we let $v(\pi) = v(-\pi)$, then the boundary term goes to zero because $u'(\pi) = u'(-\pi)$

$$\int_{-\pi}^{\pi} u'v' \, dx + \int_{-\pi}^{\pi} quv \, dx = \int_{-\pi}^{\pi} fv \, dx$$

Thus the variational problem is to find a $u \in V = \left\{ \int_{-\pi}^{\pi} ((u')^2 + qu^2) \, dx < \infty \mid u(\pi) = u(-\pi), u'(\pi) = u'(-\pi) \right\}$

$$\int_{-\pi}^{\pi} u'v' \, dx + \int_{-\pi}^{\pi} quv \, dx = \int_{-\pi}^{\pi} fv \, dx$$

for all $v \in V$. The test function v must satisfy the periodic boundary conditions on the function value, but I believe we can also choose to have v satisfy the periodic condition on the derivative as well. This makes the test and trial function spaces the same.

- (b) The energy minimization problem that is equivalent to the variational problem is to find a $u \in V$ such that

$$F(u) \leq F(w)$$

for all $w \in V$, where

$$F(w) = \frac{1}{2} \left(\int_{-\pi}^{\pi} (w')^2 \, dx + \int_{-\pi}^{\pi} qw^2 \, dx \right) - \int_{-\pi}^{\pi} fw \, dx$$

I will now prove that the energy minimization problem and the variational problem are equivalent.

Proof. Let u be a solution to the variational problem, and consider some $w \in V$. Then there exists $v \in V$ such that $u + v = w$. Now consider $F(w)$.

$$\begin{aligned} F(w) &= F(u + v) \\ &= \frac{1}{2} \left(\int_{-\pi}^{\pi} (u' + v')^2 \, dx + \int_{-\pi}^{\pi} q(u + v)^2 \, dx \right) - \int_{-\pi}^{\pi} f(u + v) \, dx \\ &= \frac{1}{2} \left(\int_{-\pi}^{\pi} (u')^2 + 2u'v' + (v')^2 \, dx + \int_{-\pi}^{\pi} qu^2 + 2quv + qv^2 \, dx \right) - \int_{-\pi}^{\pi} f(u + v) \, dx \\ &= \frac{1}{2} \left(\int_{-\pi}^{\pi} (u')^2 \, dx + \int_{-\pi}^{\pi} qu^2 \, dx \right) - \int_{-\pi}^{\pi} fu \, dx + \int_{-\pi}^{\pi} u'v' \, dx + \int_{-\pi}^{\pi} quv \, dx - \int_{-\pi}^{\pi} fv \, dx + \int_{-\pi}^{\pi} (v')^2 + qv^2 \, dx \\ &= F(u) + \int_{-\pi}^{\pi} u'v' \, dx + \int_{-\pi}^{\pi} quv \, dx - \int_{-\pi}^{\pi} fv \, dx + \int_{-\pi}^{\pi} (v')^2 + qv^2 \, dx \end{aligned}$$

Since u is a solution to the variational problem the middle three terms cancel

$$= F(u) + \int_{-\pi}^{\pi} (v')^2 + qv^2 dx$$

Also since $q(x) > 0$ for $x \in [-\pi, \pi]$, this integral is nonnegative

$$\geq F(u)$$

This shows that u is a solution to the energy minization problem when u is a solution to the variational problem.

Now let u be a solution to the energy minimization problem, then

$$F(u) \leq F(w)$$

for all $w \in V$. Let $v \in V$ and consider $w = u + \varepsilon v$, then $F(u) \leq F(u + \varepsilon v)$. Consider the function of ε ,

$$g(\varepsilon) = F(u + \varepsilon v)$$

we know that g has a minimum at $\varepsilon = 0$ thus $g'(0) = 0$.

$$\begin{aligned} g(\varepsilon) &= \frac{1}{2} \left(\int_{-\pi}^{\pi} (u' + \varepsilon v')^2 dx + \int_{-\pi}^{\pi} q(u + \varepsilon v)^2 dx \right) - \int_{-\pi}^{\pi} f(u + \varepsilon v) dx \\ &= \frac{1}{2} \left(\int_{-\pi}^{\pi} (u')^2 dx + \int_{-\pi}^{\pi} qu^2 dx \right) - \int_{-\pi}^{\pi} fu dx + \varepsilon \int_{-\pi}^{\pi} u'v' dx + \varepsilon \int_{-\pi}^{\pi} quv dx - \varepsilon \int_{-\pi}^{\pi} fv dx + \varepsilon^2 \int_{-\pi}^{\pi} (v')^2 + qv^2 dx \\ g'(\varepsilon) &= \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx - \int_{-\pi}^{\pi} fv dx + 2\varepsilon \int_{-\pi}^{\pi} (v')^2 + qv^2 dx \\ g'(0) &= \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx - \int_{-\pi}^{\pi} fv dx \\ 0 &= \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx - \int_{-\pi}^{\pi} fv dx \\ \int_{-\pi}^{\pi} fv dx &= \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx \end{aligned}$$

Since this is true for any $v \in V$, this shows that u is a solution to the variational problem. \square

(c) Consider $u, w \in V$ solutions to the variational, then for any $v \in V$ we have

$$\begin{aligned} \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx &= \int_{-\pi}^{\pi} fv dx \\ \int_{-\pi}^{\pi} w'v' dx + \int_{-\pi}^{\pi} qwv dx &= \int_{-\pi}^{\pi} fv dx \\ \int_{-\pi}^{\pi} (u - w)'v' dx + \int_{-\pi}^{\pi} q(u - w)v dx &= 0 \end{aligned}$$

Thus the Galerkin Orthogonality property of this problem is

$$\int_{-\pi}^{\pi} (u - w)'v' dx + \int_{-\pi}^{\pi} q(u - w)v dx = 0$$

for u, w solutions to the variational problem and for all $v \in V$.

The energy norm for this problem is found by letting $v = u - w$ and taking the squareroot.

$$\|u\|_E = \sqrt{\int_{-\pi}^{\pi} (u')^2 dx + \int_{-\pi}^{\pi} qu^2 dx}$$

Note that an energy inner product can be formed as well,

$$\langle u, v \rangle_E = \int_{-\pi}^{\pi} u'v' dx + \int_{-\pi}^{\pi} quv dx$$

This satisfies all the properties of an inner product.

$$\begin{aligned}\langle u, v \rangle_E &= \langle v, u \rangle_E \\ \langle au, v \rangle_E &= a \langle u, v \rangle_E \\ \langle u + w, v \rangle_E &= \langle u, v \rangle_E + \langle w, v \rangle_E \\ \langle u, u \rangle_E &\geq 0 \\ \langle u, u \rangle_E &= 0 \Leftrightarrow u = 0\end{aligned}$$

Note that $q > 0$ is required for the last two statements.

- (d) The cG(1) method for this problem is formulated by replacing the test and trial space V with a subspace, V_h^1 . Let $-\pi = x_0 < x_1 < \dots < x_M < x_{M+1} = \pi$ be a partition of $[-\pi, \pi]$, and define $h_j = x_j - x_{j-1}$ and let $h = \max_{1 \leq j \leq M+1} \{h_j\}$. I will also define the functions

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}} & x_{j-1} \leq x \leq x_j \\ \frac{x-x_{j+1}}{x_j-x_{j+1}} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \dots, M$ and

$$\phi_0(x) = \phi_{M+1}(x) = \begin{cases} \frac{x-x_0}{x_1-x_0} & x_0 \leq x \leq x_1 \\ \frac{x-x_{M+1}}{x_M-x_{M+1}} & x_M \leq x \leq x_{M+1} \\ 0 & \text{otherwise} \end{cases}$$

Then the space $V_h^1 = \left\{ \sum_{j=1}^{M+1} (\xi_j \phi_j) \mid \xi_{M+1} = \frac{h_1 h_{M+1}}{h_1 + h_{M+1}} \left(\frac{\xi_1}{h_1} + \frac{\xi_M}{h_{M+1}} \right) \right\}$. The condition on ξ_{M+1} guarantees the continuity of the derivative at $x_0 = x_{M+1}$. The design of the basis function ϕ_{M+1} guarantees the continuity of the function.

- (e) The following will show that the cG(1) method gives the optimal solution in the energy norm.

Proof. Let u be the solution to the original variational and let U be the solution to the cG(1) method. Consider an arbitrary $v \in V_h^1$, now

$$\|u - U\|_E^2 =$$

□

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