Caleb Logemani

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Jumerical Result

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Hyper-Diffusio

Operator Splitting

Conclusion

Discontinuous Galerkin Method for Solving Thin Film Equations

Caleb Logemann

December 13, 2018

Overview

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Splitting

- 1 Introduction
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 - Convection
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Motivation

Introduction

- Aircraft Icing
- Runback





- Industrial Coating
- Paint Drying

Model Equations

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Conclusio

■ Navier-Stokes Equation

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{u} \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \sigma + \mathbf{g}$$

$$\partial_t h_s + (u, v)^T \cdot \nabla h_s = w$$

$$\partial_t h_b + (u, v)^T \cdot \nabla h_b = w$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation 1D with q as fluid height.

$$q_t + (f(x,t)q^2 - g(x,t)q^3)_x = -(h(x,t)q^3q_{xxx})_x$$

Operator Splitting

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Simplified Model

$$q_t + (q^2 - q^3)_x = -(q^3 q_{xxx})_x \qquad (0, T) \times \Omega$$

Operator Splitting

$$q_t + (q^2 - q^3)_x = 0$$
$$q_t + (q^3 u_{xxx})_x = 0$$

■ Strang Splitting $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

 Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

 $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

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Convection Equation

$$q_t + f(q)_x = 0$$
 $(0, T) \times \Omega$
 $f(q) = q^2 - q^3$

Weak Form Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x = 0$$

for all test functions v

Notation

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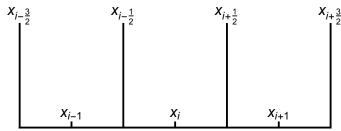
Operator Splitting

Conclusion

■ Partition the domain, [a, b] as

$$a = x_{1/2} < \dots < x_{j-1/2} < x_{j+1/2} < \dots < x_{N+1/2} = b$$

- $I_j = [x_{j-1/2}, x_{j+1/2}]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}.$



Runge Kutta Discontinuous Galerkin

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$$\int_{I_j} Q_t v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x$$
$$- \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

for all $v \in V_h$

Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2} \left(f\left(Q_{j+1/2}^{-}\right) + f\left(Q_{j+1/2}^{+}\right) \right) + \max_{q} \left\{ \left| f'(q) \right| \right\} \left(Q_{j+1/2}^{-} - Q_{j+1/2}^{+}\right)$$

 Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

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■ Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

Second Order

$$egin{aligned} q^\star &= q^n + \Delta t \mathcal{L}(q^n) \ q^{n+1} &= rac{1}{2}(q^n + q^\star) + rac{1}{2}\Delta t \mathcal{L}(q^\star) \end{aligned}$$

Diffusion

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■ Diffusion Equation

$$q_t + (q^3 u_{xxx})_{x} = 0$$
 $(0, T) \times \Omega$

Local Discontinuous Galerkin

$$r = q_{x}$$

$$s = r_{x}$$

$$u = q^{3}s_{x}$$

$$q_{t} = -u_{x}$$

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Find
$$Q(t,x), R(x), S(x), U(x)$$
 such that for all $t \in (0,T)$ $Q(t,\cdot), R, S, U \in V_h = V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$
$$\eta = Q^3$$

$$\int_{I_j} Rv \, dx = -\int_{I_j} Qv_x \, dx + \left(\hat{Q}_{j+1/2} v_{j+1/2}^- - \hat{Q}_{j-1/2} v_{j-1/2}^+ \right)$$

$$\int_{I_j} Sw \, dx = -\int_{I_j} Rw_x \, dx + \left(\hat{R}_{j+1/2} w_{j+1/2}^- - \hat{R}_{j-1/2} w_{j-1/2}^+ \right)$$

$$\int_{I_j} Uy \, dx = \int_{I_j} S_x \eta y \, dx - \left(S_{j+1/2}^- \eta_{j+1/2}^- y_{j+1/2}^- - S_{j-1/2}^+ \eta_{j-1/2}^+ y_{j-1/2}^+ \right)$$

$$+ \left(\hat{S}_{j+1/2} \hat{\eta}_{j+1/2} y_{j+1/2}^- - \hat{S}_{j-1/2} \hat{\eta}_{j-1/2} y_{j-1/2}^+ \right)$$

$$\int_{I_j} Q_t z \, dx = -\int_{I_j} Uz_x \, dx + \left(\hat{U}_{j+1/2} z_{j+1/2}^- - \hat{U}_{j-1/2} z_{j-1/2}^+ \right)$$
for all $I_t \in \Omega$ and all v_t w_t $z \in V_t$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

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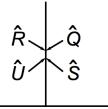
$$\hat{\eta}_{j+1/2} = \frac{1}{2} \left(\eta_{j+1/2}^+ + \eta_{j+1/2}^- \right)$$

$$\hat{Q}_{j+1/2} = Q_{j+1/2}^+$$

$$\hat{R}_{j+1/2} = R_{j+1/2}^-$$

$$\hat{S}_{j+1/2} = S_{j+1/2}^+$$

$$\hat{U}_{j+1/2} = U_{j+1/2}^-$$



Implicit L Stable Runge Kutta

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Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

2nd Order

$$q^* = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^*))$$

 $3q^{n+1} = 4q^* - q^n + \Delta t L(q^{n+1})$

Future Work

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■ Show second order convergence

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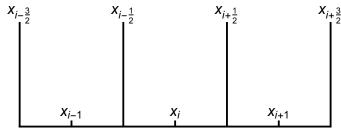
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Conclusion

■ Partition the domain, [a, b] as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = [x_{j-1/2}, x_{j+1/2}]$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}.$



Numerical Solutions

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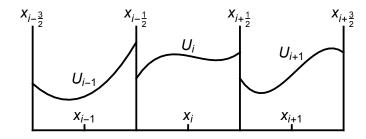
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Operator

- Use canonical variable $\xi \in [-1, 1]$
- Let $\{\phi^k(\xi)\}$ be the Legendre polynomials.
- Solution of order *M* on each cell

$$u|_{x \in V_i} \approx U_i = \sum_{k=1}^M U_i^k \phi^k(\xi)$$



Convection

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Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_{\xi} = 0$$
$$f(u) = u^2 - u^3$$

Weak Form

$$\int_{-1}^{1} \left(u_t \phi(\xi) + \frac{2}{\Delta x} f(u) \xi \phi(\xi) \right) d\xi = 0$$

■ Runge-Kutta Discontinuous Galerkin

$$\dot{U}_i^\ell = \frac{1}{\Delta x} \int_{-1}^1 f(U_i) \phi_{\xi}^\ell d\xi - \frac{1}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2})$$

Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^{\ell}(1)$$

 Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

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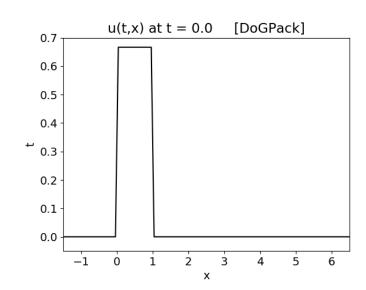
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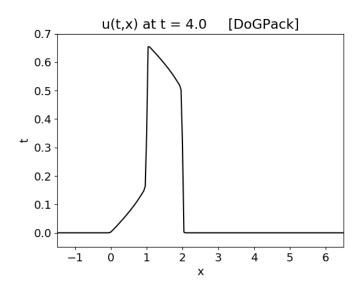
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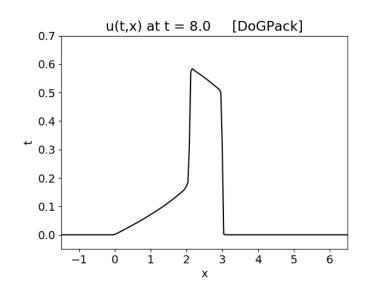
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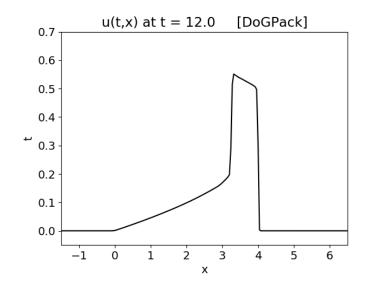
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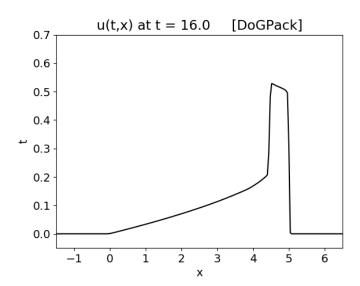
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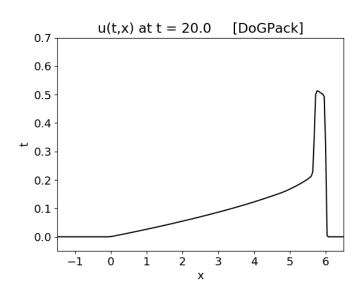
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Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4} \left(u^3 u_{\xi\xi\xi} \right)_{\xi} = 0$$

Local Discontinuous Galerkin (LDG)

$$q = \frac{2}{\Delta x} u_{\xi}$$

$$r = \frac{2}{\Delta x} q_{\xi}$$

$$s = \frac{2}{\Delta x} u^{3} r_{\xi}$$

$$u_{t} = -\frac{2}{\Delta x} s_{\xi}$$

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Splitting

$$\begin{split} &\eta(\xi) = (U_{i}^{n})^{3} \\ &Q_{i}^{\ell} = -\frac{1}{\Delta x} \left(\int_{-1}^{1} U_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(U)_{i+1/2}^{\ell} + \mathcal{F}(U)_{i-1/2}^{\ell} \right) \\ &R_{i}^{\ell} = -\frac{1}{\Delta x} \left(\int_{-1}^{1} Q_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(Q)_{i+1/2}^{\ell} + \mathcal{F}(Q)_{i-1/2}^{\ell} \right) \\ &S_{i}^{\ell} = \frac{1}{\Delta x} \left(\int_{-1}^{1} (R_{i})_{\xi} \eta(\xi) \phi^{\ell} \, \mathrm{d}\xi \right) \\ &+ \frac{1}{\Delta x} \left(\mathcal{F}(\eta)_{i+1/2} \mathcal{F}(R)_{i+1/2}^{\ell} - \mathcal{F}(\eta)_{i-1/2} \mathcal{F}(R)_{i-1/2}^{\ell} \right) \\ &\dot{U}_{i}^{\ell} = \frac{1}{\Delta x} \left(\int_{-1}^{1} S_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(S)_{i+1/2}^{\ell} + \mathcal{F}(S)_{i-1/2}^{\ell} \right) \end{split}$$

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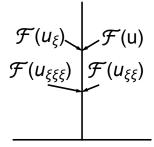
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$$\begin{split} \mathcal{F}(\eta)_{i+1/2} &= \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1)) \\ \mathcal{F}(\eta)_{i-1/2} &= \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1)) \\ \mathcal{F}(*)_{i+1/2}^{\ell} &= \phi^{\ell}(1) *_{i+1/2} \end{split}$$



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■ Explicit SSP Runge Kutta

■ Severe time step restriction

 $\Delta t \sim \Delta x^4$

 $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$

 $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$

■ Implicit SSP Runge Kutta

■ Linear System Solver

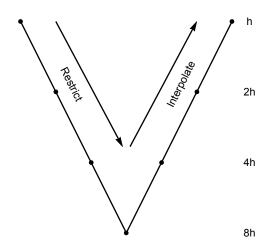
■ Stabilized BiConjugate Gradient

MultiGrid Solver

Multigrid Solver

Hyper-Diffusion

V-Cycle



Multigrid Solver

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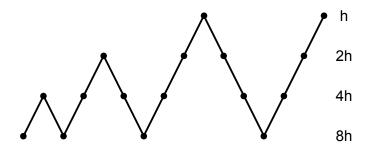
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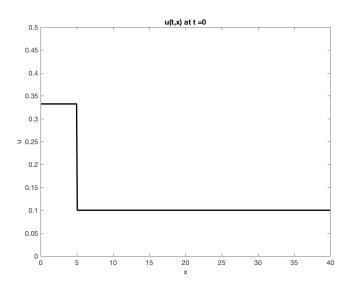
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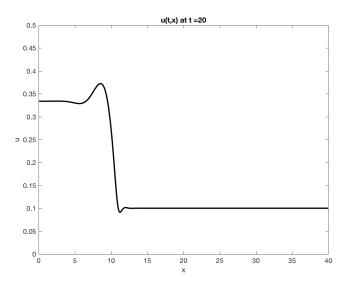
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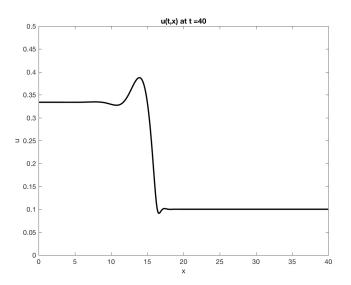
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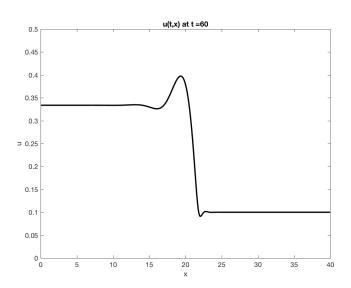
Operator Splitting

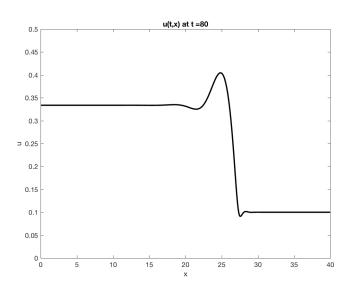
- Strang Splitting
 - 1 time step
 - 1/2 time step for convection
 - 1 time step for hyper-diffusion
 - 1/2 time step for convection
 - Second order splitting

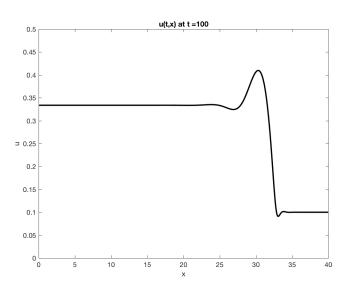












Future Work

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- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Runge Kutta IMEX

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- Thanks
 - James Rossmanith
 - Alric Rothmayer
- Questions?

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