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MATH667 Hyperbolic Partial Differential Equations
Homework 2

Solve Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad x \in [-5, 5] \tag{1}$$

$$u(x, 0) = \begin{cases} u_l & x \le 0 \\ u_r & x > 0 \end{cases} \tag{2}$$

with the following Riemann initial data sets.

(a)

$$u(x, 0) = \begin{cases} 1 & x \le 0 \\ -0.5 & x > 0 \end{cases}$$

(b)

$$u(x, 0) = \begin{cases} -1 & x \le 0 \\ 0.5 & x > 0 \end{cases}$$

1. Determine the exact solutions for all $t > 0$.

   (a) In this case $u_l > u_r$, so the solution is a shock that propogates with speed

   $$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{1}{2}(u_l + u_r) = \frac{1}{2}(1 - 0.5) = 0.25$$

   Therefore the exact solution is

   $$u(x, t) = \begin{cases} 1 & x \le 0.25t \\ -0.5 & x > 0.25t \end{cases}$$

   (b) In this case $u_l < u_r$ so the solution is a rarefaction.

   $$u(x, t) = \begin{cases} -1 & x \le -t \\ x/t & -t < x < 0.5t \\ 0.5 & 0.5t \le x \end{cases}$$

2.