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MATH667 Hyperbolic Partial Differential Equations
Homework 2

1. Determine the exact solution to Burgers' equation for all $t > 0$, with the following initial data sets.

   (a)
$$u(x,0) = \begin{cases} 1 & x < -1 \\ 0 & -1 \le x \le 1 \\ -1 & x > 1 \end{cases}$$

Initially there are two discontinuities with $u_l > u_r$. Using the Rankine-Hugionot condition for Burgers' equation we see that the shock speed for the left discontinuity is $1/2$ and the shock speed for the right discontinuity is $-1/2$. So the shock locations can be described with the following equations

$$x_l(t) = \frac{1}{2}t - 1$$
$$x_r(t) = -\frac{1}{2}t + 1$$

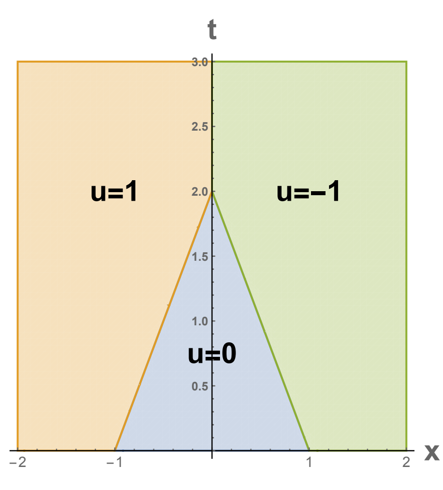
Note that these shocks are moving towards each other. They will meet when

$$\frac{1}{2}t - 1 = -\frac{1}{2}t + 1$$

Solving this shows that the shocks meet when $t = 2$ at $x = 0$. Now a new shock speed must be determined. The average of the left and right states is now 0, so the shock speed is 0. This implies that there is a standing shock wave at $x = 0$, when $t > 2$.
Now the full solution can be expressed as follows

$$u(x,t) = \begin{cases} 1 & t < 2 \text{ and } x < \frac{1}{2}t - 1 \\ 0 & \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1 \\ -1 & t < 2 \text{ and } x > -\frac{1}{2}t + 1 \\ 1 & t > 2 \text{ and } x < 0 \\ -1 & t > 2 \text{ and } x > 0 \end{cases}$$

The following image is a graphical description of this solution.

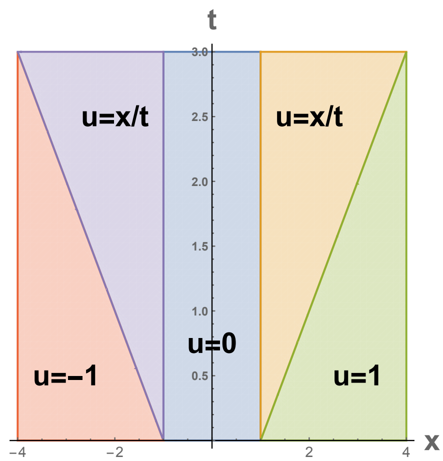


(b)

$$u(x,0) = \begin{cases} -1 & x < -1 \\ 0 & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

In this case we start with two discontinuities with $u_l < u_r$. Both of these discontinuities will result in a rarefaction. For the left rarefaction the boundaries of the rarefaction will be $x = -1$ and $x = -t - 1$. For the right rarefaction the boundaries will be $x = 1$ and $x = t + 1$. On a standard rarefaction centered at $x = 0$ for Burgers' equation the boundaries are $x = u_r t$ and $x = u_l t$, however these cases need to encorporate the shift to $x = \pm 1$. The full solution is thus

$$u(x,t) = \begin{cases} -1 & x < -t - 1 \\ \frac{x}{t} & -t - 1 < x < -1 \\ 0 & -1 < x < 1 \\ \frac{x}{t} & 1 < x < t + 1 \\ 1 & x > 1 \end{cases}$$



(c)

$$u(x,0) = \begin{cases} 12 & x < 0 \\ 8 & 0 \le x < 14 \\ 4 & 14 \le x \le 17 \\ 2 & x \ge 17 \end{cases}$$

This problem starts with three discontinuities all with $u_l > u_r$. The left most shock will propogate with speed $(12 + 8)/2 = 10$, so the location of this shock is $x_l(t) = 10t$. The middle shock will propogate with speed $(8 + 4)/2 = 6$ from initial location $x = 14$. Thus the location of this shock in time is $x_m(t) = 6t + 14$. The rightmost shock will propogate with speed $(4 + 2)/2 = 3$ starting from $x = 17$, so the location of this shock will be $x_r(t) = 3t + 17$.

Now these shocks will at some point in time meet. To find this time we must solve $x_l(t) = x_m(t)$ and $x_m(t) = x_r(t)$.

$$\begin{aligned} 10t &= 6t + 14 \\ 4t &= 14 \\ t &= \frac{7}{2} \end{aligned}$$

and

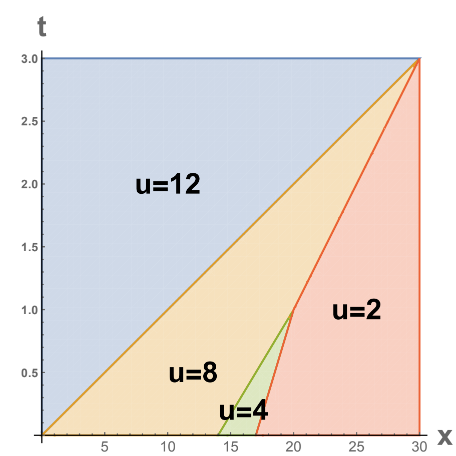
$$
\begin{aligned}
6t + 14 &= 3t + 17 \\
3t &= 3 \\
t &= 1
\end{aligned}
$$

This shows that the middle and rightmost shocks will meet first at $(x,t) = (20,1)$. When the do meet the new shock speed will be $(8+2)/2 = 5$ and the location of this shock will be $x_{mr}(t) = 5(t-1) + 20 = 5t + 15$.
Now the last two shocks will meet when $x_l(t) = x_{mr}(t)$.

$$
\begin{aligned}
10t &= 5(t-1) + 20 \\
5t &= 15 \\
t &= 3
\end{aligned}
$$

These two shocks will meet at $(x,t) = (30,3)$. They will merge into a single final shock with speed $(12+2)/2 = 7$ and location $x_{lmr}(t) = 7(t-3) + 30 = 7t + 9$.
The full solution can now be expressed as

$$
u(x,t) = \begin{cases} 12 & (t < 3 \text{ and } x < 10t) \text{ or } (t > 3 \text{ and } x < 7t + 9) \\ 8 & (t < 1 \text{ and } 10t < x < 6t + 14) \text{ or } (1 < t < 3 \text{ and } 10t < x < 5t + 15) \\ 4 & 6t + 14 < x < 3t + 17 \\ 2 & (t < 1 \text{ and } 3t + 17 < x) \text{ or } (1 < t < 3 \text{ and } x > 5t + 15) \text{ or } (t > 3 \text{ and } x > 7t + 9) \end{cases}
$$



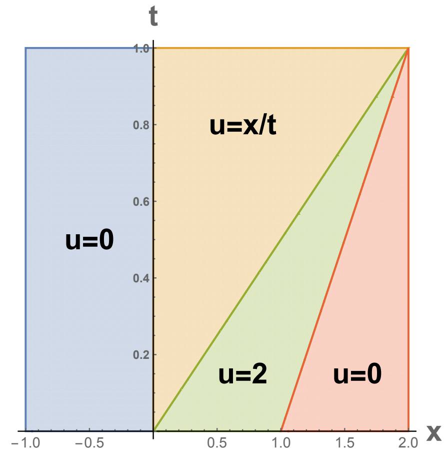
(d)

$$
u(x,0) = \begin{cases} 0 & x < 0 \\ 2 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}
$$

This problem starts with two discontinuities one with $u_l < u_r$ and one with $u_l > u_r$. The left discontinuity will result in a rarefaction with endpoints $x = 0$ and $x = 2t$. The right discontinuity will result in a shock propogating at speed $(2+0)/2 = 1$, with location $x(t) = t+1$. When $2t = t+1$ the rarefaction and shock will meet, that is at $t = 1$ At this time the left hand side of the shock will now be x/t, and this will cause the shock speed to decrease over time.

However we can describe the solution for $t < 1$.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < 2t \\ 2 & 2t < x < t+1 \\ 0 & t+1 < x \end{cases}$$



2. Consider the simple wave equation $u_t + u_x = 0$, derive the local truncation error of the Lax-Friedrichs and Lax-Wendroff schemes:

$$\text{Lax-Friedrichs:} u_j^{n+1} = u_j^n - \Delta t \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{\Delta x^2}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$\text{Lax-Wendroff:} u_j^{n+1} = u_j^n - \Delta t \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{\Delta t^2}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

First I will consider the Lax-Friedrichs method. The truncation error can be expressed as

$$\tau_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\Delta x^2}{2\Delta t} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Next I will substitue in the exact solution to the PDE, $u(x,t)$, and I will shorthand $u = u(x_j, t^n)$. I will simplify using the following Taylor expansions.

$$\begin{aligned} u_j^n &= u(x_j, t^n) = u \\ u_j^{n+1} &= u(x_j, t^n + \Delta t) = u + \Delta t u_t + O(\Delta t^2) \\ u_{j+1}^n &= u(x_j + \Delta x, t^n) = u + \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} + O(\Delta x^3) \\ u_{j-1}^n &= u(x_j - \Delta x, t^n) = u - \Delta x u_x + \frac{1}{2}\Delta x^2 u_{xx} + O(\Delta x^3) \end{aligned}$$

Now I will simplify the three main terms in the truncation error expression.

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{u + \Delta t u_t + O(\Delta t^2) - u}{\Delta t} \\ &= u_t + O(\Delta t) \end{aligned}$$

$$\begin{aligned}\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} &= \frac{2\Delta x u_x + O(\Delta x^3)}{2\Delta x} \\ &= u_x + O(\Delta x^2)\end{aligned}$$

$$\begin{aligned}-\frac{\Delta x^2}{2\Delta t}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} &= -\frac{\Delta x^2}{2\Delta t}\frac{\Delta x^2 u_{xx} + O(\Delta x^3)}{\Delta x^2} \\ &= -\frac{\Delta x^2}{2\Delta t}(u_{xx} + O(\Delta x))\end{aligned}$$

Now the truncation error can be expressed as

$$\tau_j^n = u_t + O(\Delta t) + u_x + O(\Delta x^2) - \frac{\Delta x^2}{2\Delta t}(u_{xx} + O(\Delta x))$$

Since u is a solution to the PDE, $u_t + u_x = 0$ Now the truncation error is

$$\tau_j^n = O(\Delta t + \Delta x^2) - \frac{\Delta x^2}{2\Delta t}(u_{xx} + O(\Delta x))$$

Next I will do the truncation error for the Lax-Wendroff method.

$$\tau_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\Delta t}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

I will use the same notation and Taylor series as earlier. Only the third term is different

$$\begin{aligned}-\frac{\Delta t}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} &= -\frac{\Delta t}{2}\frac{\Delta x^2 u_{xx} + O(\Delta x^3)}{\Delta x^2} \\ &= -\frac{\Delta t}{2}(u_{xx} + O(\Delta x))\end{aligned}$$

Now the local truncation error is

$$\tau_j^n = u_t + O(\Delta t) + u_x + O(\Delta x^2) - \frac{\Delta t}{2}(u_{xx} + O(\Delta x))$$

Again since u is a solution to the PDE, $u_t + u_x = 0$ Now the truncation error is

$$\tau_j^n = O(\Delta t + \Delta x^2) - \frac{\Delta t}{2}(u_{xx} + O(\Delta x))$$

3. The following function implements the Lax-Friedrichs method.

```
function [u] = laxFriedrichs(u0, deltaT, deltaX, nTimeSteps)
    nGridCells = length(u0);
    u = zeros(nTimeSteps+1, nGridCells);
    u(1, :) = u0;
    a = 0.5 * deltaT/deltaX;

    for n = 1:nTimeSteps
        for j = 1:nGridCells
            % periodic boundary conditions
            jm1 = j-1;
```

```
            if (j == 1)
                jm1 = nGridCells;
            end
            jp1 = j+1;
            if (j == nGridCells)
                jp1 = 1;
            end

            % update
            u(n+1, j) = u(n, j) - a*(u(n,jp1) - u(n,jm1)) + 0.5*(u(n,jp1) - 2*u(n,j) +
                ↪ u(n,jm1));
        end
    end
end
```

The following function implements the Lax-Wendroff method.

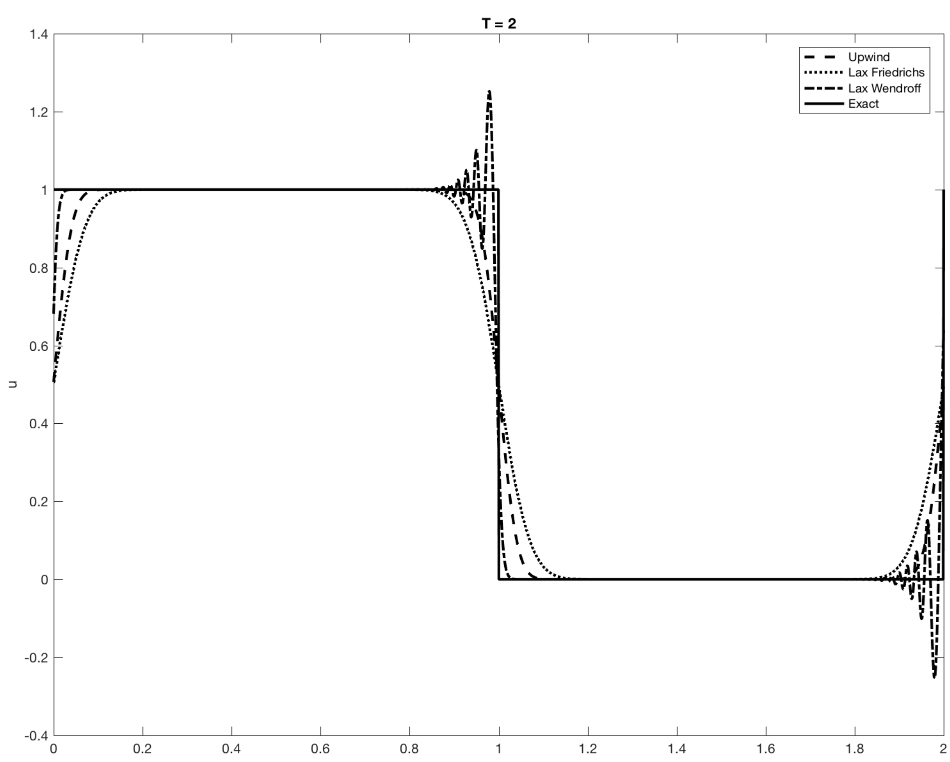
```
function [u] = laxWendroff(u0, deltaT, deltaX, nTimeSteps)
    nGridCells = length(u0);
    u = zeros(nTimeSteps+1, nGridCells);
    u(1, :) = u0;
    a = 0.5 * deltaT/deltaX;
    b = 0.5 * (deltaT/deltaX)^2;

    for n = 1:nTimeSteps
        for j = 1:nGridCells
            % periodic boundary conditions
            jm1 = j-1;
            if (j == 1)
                jm1 = nGridCells;
            end
            jp1 = j+1;
            if (j == nGridCells)
                jp1 = 1;
            end

            % update
            u(n+1, j) = u(n, j) - a*(u(n,jp1) - u(n,jm1)) + b*(u(n,jp1) - 2*u(n,j) + u(
                ↪ n,jm1));
        end
    end
end
```

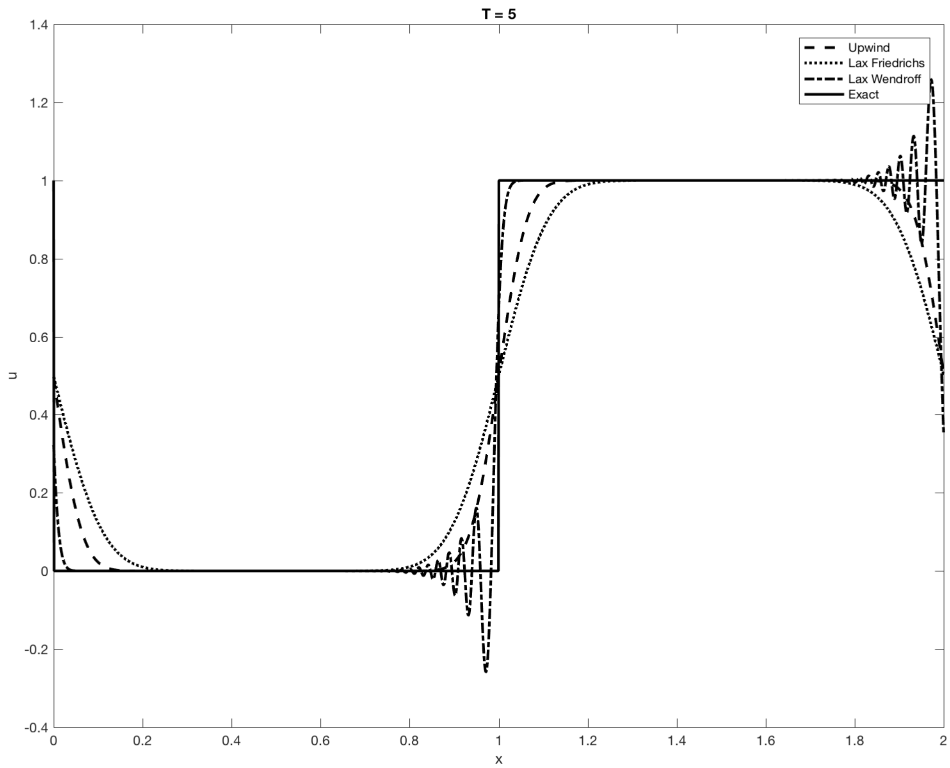
The following script now uses the previous two methods along with the upwind method from homework 1 to compute a solution to the advection equation.

```
u0func = @(x) 1.0*(x <= 1.0);
deltaX = 0.001;
a = 0.0;
b = 2.0;
nGridCells = (b - a)/deltaX;

deltaT = deltaX/2.0;
x = linspace(a, b, nGridCells);
u0 = u0func(x);

tFinal = 5;
nTimeSteps = tFinal/deltaT;
```

```
uExactFunc = @(x, t) 1.0*(mod(x - t, 2) <= 1.0);
upwindSol = upwind(u0, deltaT, deltaX, nTimeSteps);
laxFriedrichsSol = laxFriedrichs(u0, deltaT, deltaX, nTimeSteps);
laxWendroffSol = laxWendroff(u0, deltaT, deltaX, nTimeSteps);

t2 = 2/deltaT;
t5 = nTimeSteps;

uExactSol2 = arrayfun(@(xj) uExactFunc(xj, 2), x);
plot(x, upwindSol(t2+1,:), 'k--', x, laxFriedrichsSol(t2+1,:), 'k:', ...
    x,  laxWendroffSol(t2+1,:), 'k-.', x, uExactSol2, 'k-','LineWidth',2);
xlabel('x');
ylabel('u');
title('T = 2');
legend('Upwind', 'Lax Friedrichs', 'Lax Wendroff', 'Exact');
saveas(gcf, 'Figures/02_01.png', 'png');

uExactSol5 = arrayfun(@(xj) uExactFunc(xj, 5), x);
plot(x, upwindSol(t5+1,:), 'k--', x, laxFriedrichsSol(t5+1,:), 'k:', ...
    x, laxWendroffSol(t5+1,:), 'k-.', x, uExactSol5, 'k-','LineWidth',2);
xlabel('x');
ylabel('u');
title('T = 5');
legend('Upwind', 'Lax Friedrichs', 'Lax Wendroff', 'Exact');
saveas(gcf, 'Figures/02_02.png', 'png');
```

The following two images are produced. Note that the Lax-Wendroff method has the least amount of decay but it does have oscillations around the discontinuites. The upwind and Lax-Friedrichs method decay the solution more, with the Upwind method being slightly better.




T = 5
Upwind
Lax Friedrichs
Lax Wendroff
Exact
u
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2