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MATH667 Hyperbolic Partial Differential Equations Homework 1

1. For constant coefficient linear wave equation initial value problem

$$\begin{cases} u_t + au_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

for all $x \in \mathbb{R}$ and $t \geq 0$, verify that the solution $u(x,t) = u_0(x - at)$ satisfies the following integral form. We assume the initial $u_0(x)$ is a smooth function.

$$\int_{x_1}^{x_2} u(x, t_2) \, \mathrm{d}x = \int_{x_1}^{x_2} u(x, t_1) \, \mathrm{d}x + \int_{t_1}^{t_2} au(x_1, t) \, \mathrm{d}t - \int_{t_1}^{t_2} au(x_2, t) \, \mathrm{d}t, \quad \forall x_1, x_2 \in \mathbb{R}, \forall t_1, t_2 \geq 0.$$

Let U be an antiderivative of u_0, that is $\int_{y_1}^{y_2} u_0(y) \, \mathrm{d}y = U(y_2) - U(y_1)$. This is guaranteed to exist since u_0 is a smooth function. Now we can consider each of the terms of the integral form seperately. The first term can be simplified as follows,

$$\begin{aligned} \int_{x_1}^{x_2} u(x, t_2) \, \mathrm{d}x &= \int_{x_1}^{x_2} u_0(x - at_2) \, \mathrm{d}x \\ &= U(x_2 - at_2) - U(x_1 - at_2). \end{aligned}$$

The second term is

$$\begin{aligned} \int_{x_1}^{x_2} u(x, t_1) \, \mathrm{d}x &= \int_{x_1}^{x_2} u_0(x - at_1) \, \mathrm{d}x \\ &= U(x_2 - at_1) - U(x_1 - at_1). \end{aligned}$$

The third term is

$$\begin{aligned} \int_{t_1}^{t_2} au(x_1, t) \, \mathrm{d}t &= \int_{t_1}^{t_2} au_0(x_1 - at) \, \mathrm{d}t \\ &= \frac{a}{-a}(U(x_1 - at_2) - U(x_1 - at_1)) \\ &= -U(x_1 - at_2) + U(x_1 - at_1). \end{aligned}$$

The last term becomes

$$\begin{aligned} \int_{t_1}^{t_2} au(x_2, t) \, \mathrm{d}t &= \int_{t_1}^{t_2} au_0(x_2 - at) \, \mathrm{d}t \\ &= \frac{a}{-a}(U(x_1 - at_2) - U(x_1 - at_1)) \\ &= -U(x_2 - at_2) + U(x_2 - at_1). \end{aligned}$$

Combining these four terms back into the integral form gives.

$$\begin{aligned} U(x_2 - at_2) - U(x_1 - at_2) &= U(x_2 - at_1) - U(x_1 - at_1) - U(x_1 - at_2) \\ &\quad + U(x_1 - at_1) + U(x_2 - at_2) - U(x_2 - at_1) \\ U(x_2 - at_2) - U(x_1 - at_2) &= -U(x_1 - at_2) + U(x_2 - at_2) \\ -U(x_1 - at_2) &= -U(x_1 - at_2) \\ 0 &= 0 \end{aligned}$$

This shows that the integral form is satisfied for all values of $x_1, x_2 \in \mathbb{R}$ and $t_1, t_2 \geq 0$.

2. For viscous Burger's equation $u_t + uu_x + \varepsilon u_{xx}$ with initial condition

$$u(x,0) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases},$$

verify the traveling wave solution $u_\varepsilon(x,t)$ satisfies the given PDE. We have $u_\varepsilon(x,t) = w(x - \frac{1}{2}t)$, where $w(y) = \frac{1}{2}(1 - \tanh(\frac{y}{4\varepsilon}))$. Graph the solution $u_\varepsilon(x,t)$ at $t = 1$ with $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$.

First note that

$$\begin{aligned} u_{\varepsilon,t} &= -\frac{1}{2}w'\left(x - \frac{1}{2}t\right) \\ u_{\varepsilon,x} &= w'\left(x - \frac{1}{2}t\right) \\ u_{\varepsilon,xx} &= w''\left(x - \frac{1}{2}t\right). \end{aligned}$$

Also note the derivatives of w are

$$\begin{aligned} w'(y) &= -\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{y}{4\varepsilon}\right)\right) \\ w''(y) &= \frac{1}{16\varepsilon^2}\tanh\left(\frac{y}{4\varepsilon}\right)\left(1 - \tanh^2\left(\frac{y}{4\varepsilon}\right)\right) \end{aligned}$$

Now we see that

$$\begin{aligned} u_{\varepsilon,t} &= \frac{1}{16\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ u_{\varepsilon,x} &= -\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ u_{\varepsilon,xx} &= \frac{1}{16\varepsilon^2}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right). \end{aligned}$$

Plugging these into the left hand side and right hand side of the PDE gives the following. First I will simplify the left hand side

$$u_{\varepsilon,t} + u_\varepsilon u_{\varepsilon,x}$$

$$\frac{1}{16\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) + -\frac{1}{2}\left(1 - \tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)$$

$$\frac{1}{16\varepsilon} - \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + \left(-\frac{1}{16\varepsilon} + \frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)$$

$$\frac{1}{16\varepsilon} - \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + -\frac{1}{16\varepsilon} + \frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) - \frac{1}{16\varepsilon}\tanh^3\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)$$

$$\frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) - \frac{1}{16\varepsilon}\tanh^3\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right).$$

Next I will simplify the right hand side

$$\varepsilon u_{\varepsilon,xx} \tag{1}$$

$$\varepsilon\frac{1}{16\varepsilon^2}\tanh\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\left(1-\tanh^2\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\right) \tag{2}$$

$$\frac{1}{16\varepsilon}\left(\tanh\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)-\tanh^3\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\right) \tag{3}$$

$$\tag{4}$$

We see that this is equal to the left hand side shown previously, so $u_\varepsilon(x,t)$ does satisfy the PDE. Also $u_\varepsilon(x,t)$ satisfies the inital conditions as $\varepsilon \to 0$. To see this consider the following

$$\begin{aligned}\lim_{\varepsilon\to 0}(u_\varepsilon(x,0)) &= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\left(1-\tanh\left(\frac{x}{3\varepsilon}\right)\right)\right)\\ &= \frac{1}{2}-\lim_{\varepsilon\to 0}\left(\tanh\left(\frac{x}{4\varepsilon}\right)\right)\end{aligned}$$

If $x \le 0$, then this is equivalent to

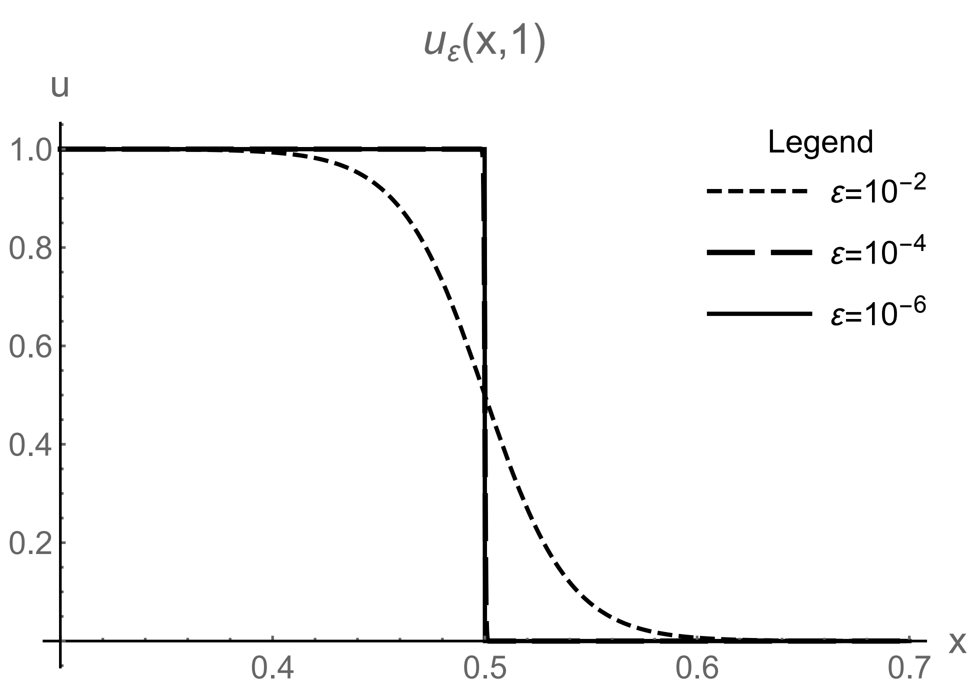
$$\begin{aligned}&= \frac{1}{2}-\lim_{y\to -\infty}(\tanh(y))\\ &= \frac{1}{2}--\frac{1}{2}=1\end{aligned}$$

If $x > 0$, then this limit is equivalent to

$$\begin{aligned}&= \frac{1}{2}-\lim_{y\to \infty}(\tanh(y))\\ &= \frac{1}{2}-\frac{1}{2}=0\end{aligned}$$

This shows that $u_\varepsilon(x,t)$ satisfies the initial conditions as $\varepsilon \to 0$.

The following is a graph of $u_\varepsilon(x,1)$ with $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$. Note that graph for $\varepsilon = 10^{-4}$ and the graph for $\varepsilon = 10^{-6}$ are almost directly on top of on another.



3. For Burger's equation Riemann problem $u(x, 0) =$

4. I implemented a Forward Euler Upwind method in the following MATLAB function.

```
function [u] = upwind(u0, deltaT, deltaX, nTimeSteps)
    nGridCells = length(u0);
    u = zeros(nTimeSteps+1, nGridCells);
    u(1, :) = u0;
    alpha = deltaT/deltaX;

    for n = 1:nTimeSteps
        for j = 1:nGridCells
            % periodic boundary conditions
            jm1 = j-1;
            if (j == 1)
                jm1 = nGridCells;
            end

            % update
            u(n+1, j) = u(n, j) + alpha*(u(n, jm1) - u(n, j));
        end
    end
end
```

The following script uses the previous function to simulate a square wave propagating with periodic boundary conditions.

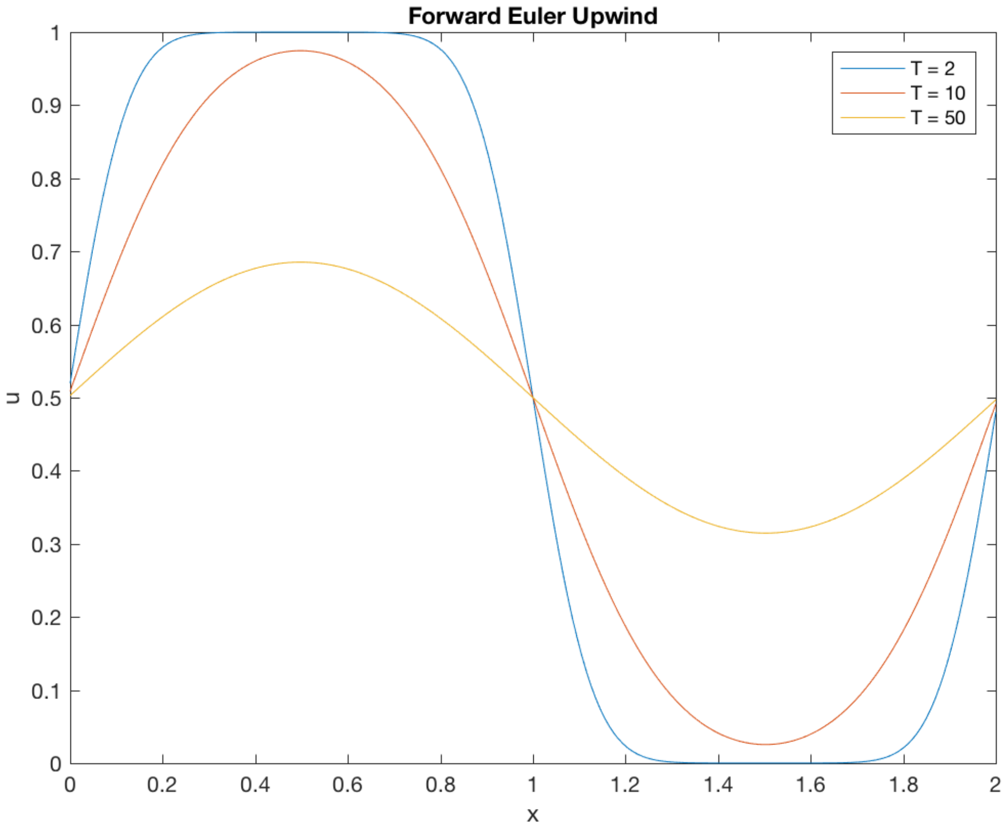
```
u0func = @(x) 1.0*(x <= 1.0);
deltaX = 0.01;
a = 0.0;
b = 2.0;
nGridCells = (b - a)/deltaX;

deltaT = deltaX/2.0;
x = linspace(a, b, nGridCells);
u0 = u0func(x);

tFinal = 50;
nTimeSteps = tFinal/deltaT;
u = upwind(u0, deltaT, deltaX, nTimeSteps);

t2 = 2/deltaT;
t10 = 10/deltaT;
t50 = nTimeSteps;
plot(x, u(t2+1,:), x, u(t10+1,:), x, u(t50+1,:));
legend('T = 2', 'T = 10', 'T = 50');
xlabel('x');
ylabel('u');
title('Forward Euler Upwind');
saveas(gcf, 'Figures/01_02.png', 'png');
```

The following image is produced. Note that as time increases the solution decays more and more from the exact solution which is a perfect square wave.


Forward Euler Upwind
T = 2
T = 10
T = 50
u
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2