Caleb Logemann
MATH667 Hyperbolic Partial Differential Equations
Final Project

1. Write out the Jacobian matrix and derive the eigenvalues and eigenvectors for the Euler equations.

   We have the following vector flux function written in terms of the conserved variables.

$$\mathbf{f}(\mathbf{w}) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + P \\ \frac{m}{\rho}(E + P) \end{bmatrix}$$

   where

$$P = (\gamma - 1)\left(E - \frac{m^2}{2\rho}\right)$$

   or

$$E = \frac{P}{\gamma - 1} + \frac{\rho u^2}{2}$$

   In order to compute the Jacobian of $\mathbf{f}$ I will first compute the partial derivatives of P.

$$P_\rho = (\gamma - 1)\frac{m^2}{2\rho^2}$$
$$P_m = -(\gamma - 1)\frac{m}{\rho}$$
$$P_E = \gamma - 1$$

   Now the Jacobian of $\mathbf{f}$ can expressed as

$$\mathbf{f}'(\mathbf{w}) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + P_\rho & 2\frac{m}{\rho} + P_m & P_E \\ P_\rho \frac{m}{\rho} - (E + P)\frac{m}{\rho^2} & P_m \frac{m}{\rho} + (E + P)\frac{1}{\rho} & \frac{m}{p}(1 + P_E) \end{bmatrix}$$

   Simplifying this results in

$$\mathbf{f}'(\mathbf{w}) = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)\frac{m^2}{2\rho^2} & (3 - \gamma)\frac{m}{\rho} & \gamma - 1 \\ (\gamma - 1)\frac{m^3}{2\rho^3} - m\frac{E+P}{\rho^2} & \frac{E+P}{\rho} - (\gamma - 1)\frac{m^2}{\rho^2} & \gamma\frac{m}{p} \end{bmatrix}$$

   This can also be changed into primitive variables.

$$\mathbf{f}'(\mathbf{w}) = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)\frac{u^2}{2} & (3 - \gamma)u & \gamma - 1 \\ \frac{1}{2}(\gamma - 2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} & \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2} - \gamma\right)u^2 & \gamma u \end{bmatrix}$$

   Now in order to find the eigenvalues and eigenvectors of this matrix we start by subtracting λI and solving $\det(\mathbf{f}'(\mathbf{w}) - \lambda I) = 0$.

$$\begin{aligned} \det(\mathbf{f}'(\mathbf{w}) - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 0 \\ (\gamma - 3)\frac{u^2}{2} & (3 - \gamma)u - \lambda & \gamma - 1 \\ \frac{1}{2}(\gamma - 2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} & \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2} - \gamma\right)u^2 & \gamma u - \lambda \end{vmatrix} \\ &= -\lambda((3 - \gamma)u - \lambda)(\gamma u - \lambda) + (\gamma - 1)\left(\frac{1}{2}(\gamma - 2)u^3 - \frac{\gamma}{\gamma - 1}\frac{uP}{\rho}\right) \\ &\quad + \lambda(\gamma - 1)\left(\frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \left(\frac{3}{2} - \gamma\right)u^2\right) - (\gamma - 3)\frac{u^2}{2}(\gamma u - \lambda) \end{aligned}$$

I will separate out each of these terms as

$$\begin{aligned}
a_1 &= -\lambda((3-\gamma)u-\lambda)(\gamma u-\lambda) \\
a_2 &= (\gamma-1)\left(\frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho}\right) \\
a_3 &= \lambda(\gamma-1)\left(\frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2\right) \\
a_4 &= -(\gamma-3)\frac{u^2}{2}(\gamma u-\lambda)
\end{aligned}$$

so the determinant is $a_1+a_2+a_3+a_4$. Now I will simplify each of these terms

$$\begin{aligned}
a_1 &= -\lambda((3-\gamma)u-\lambda)(\gamma u-\lambda) \\
&= \Big((\gamma-3)u\lambda+\lambda^2\Big)(\gamma u-\lambda) \\
&= (\gamma-3)\gamma u^2\lambda - (\gamma-3)u\lambda^2 + \gamma u\lambda^2 - \lambda^3 \\
&= (\gamma-3)\gamma u^2\lambda + 3u\lambda^2 - \lambda^3
\end{aligned}$$

$$\begin{aligned}
a_2 &= (\gamma-1)\left(\frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho}\right) \\
&= \frac{1}{2}(\gamma-1)(\gamma-2)u^3 - \gamma\frac{uP}{\rho}
\end{aligned}$$

$$\begin{aligned}
a_3 &= \lambda(\gamma-1)\left(\frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2\right) \\
&= \lambda\left(\gamma\frac{P}{\rho} + (\gamma-1)\left(\frac{3}{2}-\gamma\right)u^2\right)
\end{aligned}$$

$$\begin{aligned}
a_4 &= -(\gamma-3)\frac{u^2}{2}(\gamma u-\lambda) \\
&= \lambda(\gamma-3)\frac{u^2}{2} - \gamma(\gamma-3)\frac{u^3}{2}
\end{aligned}$$

Adding these all back together gives

$$a_1+a_2+a_3+a_4 = -\lambda^3 + 3u\lambda^2$$
$$+\left(\gamma\frac{P}{\rho} + (\gamma-1)\left(\frac{3}{2}-\gamma\right)u^2 + (\gamma-3)\left(\gamma+\frac{1}{2}\right)u^2\right)\lambda - \gamma(\gamma-3)\frac{u^3}{2} + \frac{1}{2}(\gamma-1)(\gamma-2)u^3 - \gamma\frac{uP}{\rho}$$

Now I will simplify each of the coefficients of λ.

$$\begin{aligned}
&\gamma\frac{P}{\rho} + (\gamma-1)\left(\frac{3}{2}-\gamma\right)u^2 + (\gamma-3)\left(\gamma+\frac{1}{2}\right)u^2 \\
&= \gamma\frac{P}{\rho} + \left((\gamma-1)\left(\frac{3}{2}-\gamma\right) + (\gamma-3)\left(\gamma+\frac{1}{2}\right)\right)u^2 \\
&= \gamma\frac{P}{\rho} + \left(-\gamma^2 + \frac{5}{2}\gamma - \frac{3}{2} + \gamma^2 - \frac{5}{2}\gamma - \frac{3}{2}\right)u^2 \\
&= \gamma\frac{P}{\rho} + -3u^2
\end{aligned}$$

$$
\begin{aligned}
&-\gamma(\gamma-3)\frac{u^3}{2}+\frac{1}{2}(\gamma-1)(\gamma-2)u^3-\gamma\frac{uP}{\rho}\\
&=(-\gamma(\gamma-3)+(\gamma-1)(\gamma-2))\frac{u^3}{2}-\gamma\frac{uP}{\rho}\\
&=\left(-\gamma^2+3\gamma+\gamma^2-3\gamma+2\right)\frac{u^3}{2}-\gamma\frac{uP}{\rho}\\
&=u^3-\gamma\frac{uP}{\rho}
\end{aligned}
$$

So now simplifying these terms by factoring into a difference cubed

$$
\begin{aligned}
a_1+a_2+a_3+a_4&=-\lambda^3+3u\lambda^2+\left(\gamma\frac{P}{\rho}+-3u^2\right)\lambda+u^3-\gamma\frac{uP}{\rho}\\
&=-\lambda^3+3u\lambda^2-3u^2\lambda+u^3+\gamma\lambda\frac{P}{\rho}-\gamma\frac{uP}{\rho}\\
&=(u-\lambda)^3-(u-\lambda)\frac{\gamma P}{\rho}
\end{aligned}
$$

Thus we have shown that

$$
\det(\mathbf{f}'(\mathbf{w})-\lambda I)=(u-\lambda)^3-(u-\lambda)\frac{\gamma P}{\rho}
$$

Setting this equal to zero and solving for λ gives.

$$
\begin{aligned}
(u-\lambda)^3-(u-\lambda)\frac{\gamma P}{\rho}&=0\\
(u-\lambda)\left((u-\lambda)^2-\frac{\gamma P}{\rho}\right)&=0\\
\lambda&=u\\
(u-\lambda)^2-\frac{\gamma P}{\rho}&=0\\
(u-\lambda)^2&=\frac{\gamma P}{\rho}\\
(u-\lambda)&=\pm\sqrt{\frac{\gamma P}{\rho}}\\
\lambda&=u\pm\sqrt{\frac{\gamma P}{\rho}}
\end{aligned}
$$

So the three eigenvalues of this system are

$$
\lambda_1=u,\quad \lambda_2=u+\sqrt{\frac{\gamma P}{\rho}},\quad \lambda_3=u-\sqrt{\frac{\gamma P}{\rho}}
$$

Now I will just check that we have the correct eigenvectors. The eigenvectors are

$$
\mathbf{w}_1=\begin{bmatrix}1\\u\\\frac{1}{2}u^2\end{bmatrix},\quad \mathbf{w}_2=\begin{bmatrix}1\\u+\sqrt{\frac{\gamma P}{\rho}}\\\frac{\gamma}{\gamma-1}\frac{P}{\rho}+\frac{u^2}{2}+u\sqrt{\frac{\gamma P}{\rho}}\end{bmatrix},\quad \mathbf{w}_3=\begin{bmatrix}1\\u-\sqrt{\frac{\gamma P}{\rho}}\\\frac{\gamma}{\gamma-1}\frac{P}{\rho}+\frac{u^2}{2}-u\sqrt{\frac{\gamma P}{\rho}}\end{bmatrix}
$$

To check

$$
\begin{aligned}
\mathbf{f}'\mathbf{w}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} & \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2 & \gamma u \end{bmatrix} \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} \\
&= \begin{bmatrix} u \\ (\gamma-3)\frac{u^2}{2} - (\gamma-3)u^2 + (\gamma-1)\frac{u^2}{2} \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} + \frac{\gamma}{\gamma-1}\frac{uP}{\rho} + \left(\frac{3}{2}-\gamma\right)u^3 + \gamma\frac{u^3}{2} \end{bmatrix} \\
&= \begin{bmatrix} u \\ u^2 \\ \frac{1}{2}u^3 \end{bmatrix} \\
&= u \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}
\end{aligned}
$$

Let $c = \sqrt{\frac{\gamma P}{\rho}}$

$$
\begin{aligned}
\mathbf{f}'\mathbf{w}_2 &= \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} & \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2 & \gamma u \end{bmatrix} \begin{bmatrix} 1 \\ u+c \\ \frac{1}{\gamma-1}c^2 + \frac{u^2}{2} + uc \end{bmatrix} \\
&= \begin{bmatrix} u+c \\ (\gamma-3)\frac{u^2}{2} + (3-\gamma)u(u+c) + (\gamma-1)\left(\frac{1}{\gamma-1}c^2 + \frac{u^2}{2} + uc\right) \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} + \left(\frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2\right)(u+c) + \gamma u\left(\frac{1}{\gamma-1}c^2 + \frac{u^2}{2} + uc\right) \end{bmatrix} \\
&= \begin{bmatrix} u+c \\ (\gamma-3)\frac{u^2}{2} + (3-\gamma)u^2 + (3-\gamma)uc + c^2 + (\gamma-1)\frac{u^2}{2} + (\gamma-1)uc \\ \frac{1}{2}(\gamma-2)u^3 + uc^2 + \frac{1}{\gamma-1}c^2(u+c) + \left(\frac{3}{2}-\gamma\right)u^3 + \left(\frac{3}{2}-\gamma\right)u^2c + \gamma\frac{u^3}{2} + \gamma u^2 c \end{bmatrix} \\
&= \begin{bmatrix} u+c \\ u^2 + 2uc + c^2 \\ \frac{1}{2}u^3 + \frac{1}{\gamma-1}c^2(u+c) + \frac{3}{2}u^2c + uc^2 \end{bmatrix} \\
&= \begin{bmatrix} u+c \\ (u+c)^2 \\ \left(\frac{u^2}{2} + uc\right)(u+c) + \frac{1}{\gamma-1}c^2(u+c) \end{bmatrix} \\
&= (u+c) \begin{bmatrix} 1 \\ u+c \\ \frac{1}{\gamma-1}c^2 + \frac{u^2}{2} + uc \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{f}'\mathbf{w}_3 &= \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} & \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2 & \gamma u \end{bmatrix} \begin{bmatrix} 1 \\ u-c \\ \frac{1}{\gamma-1}c^2 + \frac{u^2}{2} - uc \end{bmatrix} \\
&= \begin{bmatrix} u-c \\ (\gamma-3)\frac{u^2}{2} + (3-\gamma)u(u-c) + (\gamma-1)\left(\frac{1}{\gamma-1}c^2 + \frac{u^2}{2} - uc\right) \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}\frac{uP}{\rho} + \left(\frac{\gamma}{\gamma-1}\frac{P}{\rho} + \left(\frac{3}{2}-\gamma\right)u^2\right)(u-c) + \gamma u\left(\frac{1}{\gamma-1}c^2 + \frac{u^2}{2} - uc\right) \end{bmatrix} \\
&= \begin{bmatrix} u-c \\ (\gamma-3)\frac{u^2}{2} + (3-\gamma)u^2 - (3-\gamma)uc + c^2 + (\gamma-1)\frac{u^2}{2} - (\gamma-1)uc \\ \frac{1}{2}(\gamma-2)u^3 + uc^2 + \frac{1}{\gamma-1}c^2(u-c) + \left(\frac{3}{2}-\gamma\right)u^3 - \left(\frac{3}{2}-\gamma\right)u^2c + \gamma\frac{u^3}{2} - \gamma u^2 c \end{bmatrix} \\
&= \begin{bmatrix} u-c \\ u^2 - 2uc + c^2 \\ \frac{1}{2}u^3 + \frac{1}{\gamma-1}c^2(u-c) - \frac{3}{2}u^2c + uc^2 \end{bmatrix} \\
&= \begin{bmatrix} u-c \\ (u-c)^2 \\ \left(\frac{u^2}{2} - uc\right)(u-c) + \frac{1}{\gamma-1}c^2(u-c) \end{bmatrix} \\
&= (u-c)\begin{bmatrix} 1 \\ u-c \\ \frac{1}{\gamma-1}c^2 + \frac{u^2}{2} - uc \end{bmatrix}
\end{aligned}
$$

2.