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MATH667 Hyperbolic Partial Differential Equations
Final Project

1. Write out the Jacobian matrix and derive the eigenvalues and eigenvectors for the Euler equations.

   We have the following vector flux function written in terms of the conserved variables.

$$\mathbf{f}(\mathbf{u}) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + P \\ \frac{m}{\rho}(E + P) \end{bmatrix}$$

   where

$$P = (\gamma - 1)\left(E - \frac{m^2}{2\rho}\right)$$

   In order to compute the Jacobian of $\mathbf{f}$ I will first compute the partial derivatives of P.

$$\begin{aligned} P_\rho &= (\gamma - 1)\frac{m^2}{2\rho^2} \\ P_m &= -(\gamma - 1)\frac{m}{\rho} \\ P_E &= \gamma - 1 \end{aligned}$$

   Now the Jacobian of $\mathbf{f}$ can expressed as

$$\mathbf{f}'(\mathbf{u}) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + P_\rho & 2\frac{m}{\rho} + P_m & P_E \\ P_\rho\frac{m}{\rho} - (E + P)\frac{m}{\rho^2} & P_m\frac{m}{\rho} + (E + P)\frac{1}{\rho} & \frac{m}{p}(1 + P_E) \end{bmatrix}$$

   Simplifying this results in

$$\mathbf{f}'(\mathbf{u}) = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)\frac{m^2}{2\rho^2} & (3 - \gamma)\frac{m}{\rho} & \gamma - 1 \\ (\gamma - 1)\frac{m^3}{2\rho^3} - \frac{m}{\rho}H & H - (\gamma - 1)\frac{m^2}{\rho^2} & \gamma\frac{m}{p} \end{bmatrix}$$

   where

$$H = \frac{E + P}{\rho}.$$

   This can also be changed into primitive variables.

$$\mathbf{f}'(\mathbf{u}) = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma - 3)\frac{u^2}{2} & (3 - \gamma)u & \gamma - 1 \\ (\gamma - 1)u^3 - uH & H - (\gamma - 1)u^2 & \gamma u \end{bmatrix}$$

   Now in order to find the eigenvalues and eigenvectors of this matrix we start by subtracting λI and solving $\det(\mathbf{f}'(\mathbf{u}) - \lambda I) = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 \\ (\gamma - 3)\frac{u^2}{2} & (3 - \gamma)u - \lambda & \gamma - 1 \\ (\gamma - 1)u^3 - uH & H - (\gamma - 1)u^2 & \gamma u - \lambda \end{vmatrix}$$

$$= -\lambda((3 - \gamma)u - \lambda)(\gamma u - \lambda) + (\gamma - 1)\Big((\gamma - 1)u^3 - uH\Big) - (\gamma - 3)\frac{u^2}{2}(\gamma u - \lambda) + \lambda(\gamma - 1)\Big(H - (\gamma - 1)u^2\Big)$$

I will separate out each of these terms as

$$\begin{aligned}
a_1 &= -\lambda((3-\gamma)u-\lambda)(\gamma u-\lambda)\\
a_2 &= (\gamma-1)\Big((\gamma-1)u^3-uH\Big)\\
a_3 &= -(\gamma-3)\frac{u^2}{2}(\gamma u-\lambda)\\
a_4 &= \lambda(\gamma-1)\Big(H-(\gamma-1)u^2\Big)
\end{aligned}$$

so the determinant is $a_1+a_2+a_3+a_4$. Now I will simplify each of these terms

$$\begin{aligned}
a_1 &= -\lambda((3-\gamma)u-\lambda)(\gamma u-\lambda)\\
&= \Big((\gamma-3)u\lambda+\lambda^2\Big)(\gamma u-\lambda)\\
&= (\gamma-3)\gamma u^2\lambda-(\gamma-3)u\lambda^2+\gamma u\lambda^2-\lambda^3\\
&= (\gamma-3)\gamma u^2\lambda+3u\lambda^2-\lambda^3
\end{aligned}$$

$$\begin{aligned}
a_2 &= (\gamma-1)\Big((\gamma-1)u^3-uH\Big)\\
&= (\gamma-1)^2u^3-(\gamma-1)uH
\end{aligned}$$

$$\begin{aligned}
a_3 &= -(\gamma-3)\frac{u^2}{2}(\gamma u-\lambda)\\
&= (\gamma-3)\frac{u^2}{2}\lambda-(\gamma-3)\gamma\frac{u^3}{2}
\end{aligned}$$

$$\begin{aligned}
a_4 &= \lambda(\gamma-1)\Big(H-(\gamma-1)u^2\Big)\\
&= \lambda\Big((\gamma-1)H-(\gamma-1)^2u^2\Big)
\end{aligned}$$

Adding these all back together gives

$$a_1+a_2+a_3+a_4 = -\lambda^3+3u\lambda^2$$
$$+\left(\left(\frac{1}{2}+\gamma\right)(\gamma-3)u^2+(\gamma-1)H-(\gamma-1)^2u^2\right)\lambda-(\gamma-2)\gamma\frac{u^3}{2}+(\gamma-1)^2u^3-(\gamma-1)uH$$

2.