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MATH667 Hyperbolic Partial Differential Equations
Homework 5

1. For the following schemes to solve nonlinear conservation laws, show which ones are monotone schemes.

   - Godunov scheme
   - Lax-Friedrichs scheme

     The Lax-Friedrichs scheme is monotone if the CFL condition is meet. To see this note that the Lax-Friedrichs method can be written as $U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j-1}^n)$ where

$$H(U_{j-1}, U_j, U_{j+1}) = \left(1 - \frac{\alpha \Delta t}{\Delta x}\right) U_j + \frac{\alpha \Delta t}{2\Delta x}(U_{j+1} + U_{j-1}) - \frac{\Delta t}{2\Delta x}(f(U_{j+1}) - f(U_{j-1}))$$

and $\alpha = \max_u\{|f'(u)|\}$ To show that this method is monotone, we need to show that $\frac{\partial H}{\partial U_i} \geq 0$ for $i = j-1, j, j+1$.

First consider $\frac{\partial H}{\partial U_j}$.

$$\begin{aligned}
\frac{\partial H}{\partial U_j} &= \left(1 - \frac{\alpha \Delta t}{\Delta x}\right) \\
\left(1 - \frac{\alpha \Delta t}{\Delta x}\right) &\geq 0 \\
1 &\geq \frac{\alpha \Delta t}{\Delta x}
\end{aligned}$$

This is exactly the CFL condition, so this condition is satisfied by this method.

Second consider $\frac{\partial H}{\partial U_{j-1}}$.

$$\begin{aligned}
\frac{\partial H}{\partial U_{j-1}} &= \frac{\alpha \Delta t}{2\Delta x} + \frac{\Delta t}{2\Delta x} f'(U_{j-1}) \\
&= \frac{\Delta t}{2\Delta x}(\alpha + f'(U_{j-1}))
\end{aligned}$$

Since $\alpha = \max_u\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha + f'(U_{j-1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j-1}} \geq 0$$

Finally consider $\frac{\partial H}{\partial U_{j+1}}$.

$$\begin{aligned}
\frac{\partial H}{\partial U_{j+1}} &= \frac{\alpha \Delta t}{2\Delta x} - \frac{\Delta t}{2\Delta x} f'(U_{j+1}) \\
&= \frac{\Delta t}{2\Delta x}(\alpha - f'(U_{j+1}))
\end{aligned}$$

Since $\alpha = \max_u\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha - f'(U_{j+1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j+1}} \geq 0$$

These three conditions are met by the Lax-Friedrichs method, so the scheme is monotone.

- Local Lax-Friedrichs scheme
  The Local Lax-Friedrichs scheme is monotone if the CFL condition is meet. To see this note that the Local Lax-Friedrichs method can be written as $U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j-1}^n)$ where

  $$H(U_{j-1}, U_j, U_{j+1}) = \left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right)U_j + \frac{\Delta t}{2\Delta x}(\alpha_+ U_{j+1} + \alpha_- U_{j-1}) - \frac{\Delta t}{2\Delta x}(f(U_{j+1}) - f(U_{j-1}))$$

  and $\alpha_+ = \max_{(U_j, U_{j+1})}\{|f'(u)|\}$ and $\alpha_- = \max_{(U_j, U_{j-1})}\{|f'(u)|\}$. To show that this method is monotone, we need to show that $\frac{\partial H}{\partial U_i} \geq 0$ for $i = j-1, j, j+1$.

  First consider $\frac{\partial H}{\partial U_j}$.

  $$\frac{\partial H}{\partial U_j} = \left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right)$$
  $$\left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right) \geq 0$$
  $$1 \geq \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}$$

  Since α_+ and α_- are both less than α, this condition is met if the CFL condition is met.

  Second consider $\frac{\partial H}{\partial U_{j-1}}$.

  $$\begin{aligned}\frac{\partial H}{\partial U_{j-1}} &= \frac{\alpha_- \Delta t}{2\Delta x} + \frac{\Delta t}{2\Delta x}f'(U_{j-1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha_- + f'(U_{j-1}))\end{aligned}$$

  Since $\alpha_- = \max_{[U_{j-1}, U_j]}\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha_- + f'(U_{j-1})) \geq 0$ and

  $$\frac{\partial H}{\partial U_{j-1}} \geq 0$$

  Finally consider $\frac{\partial H}{\partial U_{j+1}}$.

  $$\begin{aligned}\frac{\partial H}{\partial U_{j+1}} &= \frac{\alpha_+ \Delta t}{2\Delta x} - \frac{\Delta t}{2\Delta x}f'(U_{j+1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha_+ - f'(U_{j+1}))\end{aligned}$$

  Since $\alpha = \max_{[U_j, U_{j+1}]}\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha_+ - f'(U_{j+1})) \geq 0$ and

  $$\frac{\partial H}{\partial U_{j+1}} \geq 0$$

  These three conditions are met by the Local Lax-Friedrichs method, so the scheme is monotone.
- Lax-Wendroff scheme The Lax Wendroff scheme is not monotone.

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