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MATH667 Hyperbolic Partial Differential Equations
Homework 5

1. For the following schemes to solve nonlinear conservation laws, show which ones are monotone schemes.

   - Godunov scheme
     The Godunov scheme is monotone if the CFL condition is met. To see this note that the Godunov method can be written as $U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j-1}^n)$ where

$$H(U_{j-1}, U_j, U_{j+1}) = U_j - \frac{\Delta t}{\Delta x}\Big(F_{j+1/2} - F_{j-1/2}\Big)$$

and

$$F_{j+1/2} = \begin{cases} \min_{u \in [U_j, U_{j+1}]}\{f(u)\} & U_j \le U_{j+1} \\ \max_{u \in [U_{j+1}, U_j]}\{f(u)\} & U_j > U_{j+1} \end{cases}$$
$$F_{j-1/2} = \begin{cases} \min_{u \in [U_{j-1}, U_j]}\{f(u)\} & U_{j-1} \le U_j \\ \max_{u \in [U_j, U_{j-1}]}\{f(u)\} & U_{j-1} > U_j \end{cases}$$

To show that this method is monotone, we need to show that $\frac{\partial H}{\partial U_i} \ge 0$ for $i = j-1, j, j+1$.
First consider $\frac{\partial H}{\partial U_{j-1}}$.

$$\frac{\partial H}{\partial U_{j-1}} = \frac{\Delta t}{\Delta x}\frac{\partial F_{j-1/2}}{\partial U_{j-1}}$$

Now consider $\frac{\partial F_{j-1/2}}{\partial U_{j-1}}$. If $U_{j-1} \le U_j$ and U_{j-1} increases the size of $[U_{j-1}, U_j]$ decreases. Now we are taking a minimum over a smaller integral so the value of $F_{j-1/2}$ increases. If $U_{j-1} > U_j$ and U_{j-1} increases then the size of $[U_j, U_{j-1}]$ increases. Now the value of $F_{j-1/2}$ would increase as we are taking a maximum over a larger interval. This implies that $\frac{\partial F_{j-1/2}}{\partial U_{j-1}} \ge 0$. This shows that

$$\frac{\partial H}{\partial U_{j-1}} \ge 0.$$

Now consider $\frac{\partial H}{\partial U_{j+1}}$.

$$\frac{\partial H}{\partial U_{j+1}} = -\frac{\Delta t}{\Delta x}\frac{\partial F_{j+1/2}}{\partial U_{j+1}}$$

Now consider $\frac{\partial F_{j+1/2}}{\partial U_{j+1}}$. If $U_j \le U_{j+1}$ and U_{j+1} increases the size of $[U_j, U_{j+1}]$ increases. Now we are taking a minimum over a larger integral so the value of $F_{j+1/2}$ possibly decreases. If $U_j > U_{j+1}$ and U_{j+1} increases then the size of $[U_{j+1}, U_j]$ decreases. Now the value of $F_{j+1/2}$ would increase as we are taking a maximum over a larger interval. This implies that $\frac{\partial F_{j+1/2}}{\partial U_{j+1}} \le 0$. This shows that

$$\frac{\partial H}{\partial U_{j+1}} \ge 0.$$

Last consider $\dfrac{\partial H}{\partial U_j}$.

$$\frac{\partial H}{\partial U_{j+1}} = 1 - \frac{\Delta t}{\Delta x}\left(\frac{\partial F_{j+1/2}}{\partial U_j} - \frac{\partial F_{j-1/2}}{\partial U_j}\right)$$

I will first consider $\dfrac{\partial F_{j+1/2}}{\partial U_j}$ and $\dfrac{\partial F_{j-1/2}}{\partial U_j}$. Note that if U_j increases the value of $F_{j+1/2}$ may increase because either the interval over which a minimum is being taken will decrease of the intervale over which a maximum is being taken will increase. However this increase is bounded by $\alpha = \max\{|f'(u)|\}$ the global maximum of the wave speed. So

$$\frac{\partial F_{j+1/2}}{\partial U_j} \leq \alpha$$

Similarly for $F_{j-1/2}$. If U_j increases then $F_{j-1/2}$ may decrease but it is bounded below by $-\alpha$. So

$$\frac{\partial F_{j-1/2}}{\partial U_j} \geq -\alpha.$$

Therefore

$$\frac{\partial H}{\partial U_{j+1}} \geq 1 - \frac{2\Delta t}{\Delta x}\alpha$$

So if the following CFL condition is met, then the Godunov method is monotone.

$$\frac{\Delta t \alpha}{\Delta x} < \frac{1}{2}$$

- Lax-Friedrichs scheme
  The Lax-Friedrichs scheme is monotone if the CFL condition is met. To see this note that the Lax-Friedrichs method can be written as $U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j-1}^n)$ where

$$H(U_{j-1}, U_j, U_{j+1}) = \left(1 - \frac{\alpha \Delta t}{\Delta x}\right)U_j + \frac{\alpha\Delta t}{2\Delta x}(U_{j+1} + U_{j-1}) - \frac{\Delta t}{2\Delta x}(f(U_{j+1}) - f(U_{j-1}))$$

and $\alpha = \max_u\{|f'(u)|\}$ To show that this method is monotone, we need to show that $\dfrac{\partial H}{\partial U_i} \geq 0$ for $i = j-1, j, j+1$.
First consider $\dfrac{\partial H}{\partial U_j}$.

$$\frac{\partial H}{\partial U_j} = \left(1 - \frac{\alpha\Delta t}{\Delta x}\right)$$
$$\left(1 - \frac{\alpha\Delta t}{\Delta x}\right) \geq 0$$
$$1 \geq \frac{\alpha \Delta t}{\Delta x}$$

This is exactly the CFL condition, so this condition is satisfied by this method.

Second consider $\dfrac{\partial H}{\partial U_{j-1}}$.

$$\begin{aligned}\frac{\partial H}{\partial U_{j-1}} &= \frac{\alpha\Delta t}{2\Delta x} + \frac{\Delta t}{2\Delta x} f'(U_{j-1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha + f'(U_{j-1}))\end{aligned}$$

Since $\alpha = \max_u\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha + f'(U_{j-1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j-1}} \geq 0$$

Finally consider $\dfrac{\partial H}{\partial U_{j+1}}$.

$$\begin{aligned}\frac{\partial H}{\partial U_{j+1}} &= \frac{\alpha\Delta t}{2\Delta x} - \frac{\Delta t}{2\Delta x} f'(U_{j+1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha - f'(U_{j+1}))\end{aligned}$$

Since $\alpha = \max_u\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha - f'(U_{j+1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j+1}} \geq 0$$

These three conditions are met by the Lax-Friedrichs method, so the scheme is monotone.

- Local Lax-Friedrichs scheme

  The Local Lax-Friedrichs scheme is monotone if the CFL condition is meet. To see this note that the Local Lax-Friedrichs method can be written as $U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j-1}^n)$ where

  $$H(U_{j-1}, U_j, U_{j+1}) = \left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right)U_j + \frac{\Delta t}{2\Delta x}(\alpha_+ U_{j+1} + \alpha_- U_{j-1}) - \frac{\Delta t}{2\Delta x}(f(U_{j+1}) - f(U_{j-1}))$$

  and $\alpha_+ = \max_{(U_j, U_{j+1})}\{|f'(u)|\}$ and $\alpha_- = \max_{(U_j, U_{j-1})}\{|f'(u)|\}$. To show that this method is monotone, we need to show that $\dfrac{\partial H}{\partial U_i} \geq 0$ for $i = j-1, j, j+1$.

  First consider $\dfrac{\partial H}{\partial U_j}$.

  $$\frac{\partial H}{\partial U_j} = \left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right)$$

  $$\left(1 - \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}\right) \geq 0$$

  $$1 \geq \frac{(\alpha_+ + \alpha_-)\Delta t}{2\Delta x}$$

  Since α_+ and α_- are both less than α, this condition is met if the CFL condition is met.

  Second consider $\dfrac{\partial H}{\partial U_{j-1}}$.

  $$\begin{aligned}\frac{\partial H}{\partial U_{j-1}} &= \frac{\alpha_-\Delta t}{2\Delta x} + \frac{\Delta t}{2\Delta x} f'(U_{j-1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha_- + f'(U_{j-1}))\end{aligned}$$

Since $\alpha_- = \max_{[U_{j-1},U_j]}\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha_- + f'(U_{j-1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j-1}} \geq 0$$

Finally consider $\dfrac{\partial H}{\partial U_{j+1}}$.

$$\begin{aligned}\frac{\partial H}{\partial U_{j+1}} &= \frac{\alpha_+ \Delta t}{2\Delta x} - \frac{\Delta t}{2\Delta x} f'(U_{j+1}) \\ &= \frac{\Delta t}{2\Delta x}(\alpha_+ - f'(U_{j+1}))\end{aligned}$$

Since $\alpha = \max_{[U_j,U_{j+1}]}\{|f'(u)|\} \geq f'(U_{j-1})$, then $(\alpha_+ - f'(U_{j+1})) \geq 0$ and

$$\frac{\partial H}{\partial U_{j+1}} \geq 0$$

These three conditions are met by the Local Lax-Friedrichs method, so the scheme is monotone.

- Lax-Wendroff scheme
  The Lax-Wendroff scheme is not monotone. This is obvious because the Lax-Wendroff scheme is second order and monotone schemes must be first order at most. Also Lax-Wendroff creates oscillations at shocks, which are clearly not monotone.
  To see this specifically, consider $f(u) = u$. In this case,

  $$H(U_{j-1}, U_j, U_{j+1}) = U_j - \frac{\Delta t}{2\Delta x}(U_{j+1} - U_{j-1}) + \frac{\Delta t^2}{2\Delta x^2}(U_{j+1} - 2U_j + U_{j-1})$$

  Now consider $\dfrac{\partial H}{\partial U_{j+1}}$

  $$\begin{aligned}\frac{\partial H}{\partial U_{j+1}} &= -\frac{\Delta t}{2\Delta x} + \frac{\Delta t^2}{2\Delta x^2} \\ &= \frac{\Delta t}{2\Delta x}\left(\frac{\Delta t}{\Delta x} - 1\right) \\ &\leq 0\end{aligned}$$

  If the CFL condition is satisfied then $\left(\frac{\Delta t}{\Delta x} - 1\right) \leq 0$, so this partial derivative is negative and the method is not monotone.

2. Solve Burger's equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ on $x \in [0, 2\pi]$ with initial data $u(x, 0) = 1 + \frac{1}{2}\sin(x)$. Let's consider the 1st order finite difference Godunov scheme. Implement the scheme to (a) $t = 1.0$ and (b) $t = 3.0$. Apply periodic boundary conditions. For part (a) output L^∞ error/order table with uniform mesh $N = 20, 40, 80, 160$. For part (b) graph the simulation with $N = 80$ and solid line for exact solution and symbols for numerical approximations.

   The following is my implementation of the Godunov scheme.

```
function [u] = godunov(f, u0, deltaT, deltaX, nTimeSteps)
    nGridCells = length(u0);
    u = zeros(nTimeSteps+1, nGridCells);
```

```
    u(1, :) = u0;
    nu = deltaT/deltaX;

    boundaryConditions = 'periodic';

    % flux array, F(i) is flux at i - 1/2 interface
    F = zeros(nGridCells,1);

    for n = 1:nTimeSteps
        % compute fluxes at boundaries
        for j = 1:nGridCells
            % zero flux boundary conditions
            jm1 = j-1;
            if (j == 1)
                if (strcmp(boundaryConditions,'periodic'))
                    jm1 = nGridCells;
                elseif (strcmp(boundaryConditions,'zeroFlux'))
                    jm1 = 1;
                end
            end

            ful = f(u(n,jm1));
            fur = f(u(n, j));
            if (u(n, jm1) <= u(n, j))
                % min[ul < u < ur]{f(u)}
                % specific to burger's equation
                if (u(n, jm1)*u(n, j) < 0)
                    F(j) = 0;
                else
                    F(j) = min(ful, fur);
                end
            else
                % max[ur < u < ul]{f(u)}
                F(j) = max(ful, fur);
            end
        end

        % update solution
        for j = 1:nGridCells
            % zero flux boundary conditions
            jp1 = j+1;
            if (j == nGridCells)
                if (strcmp(boundaryConditions,'periodic'))
                    jp1 = 1;
                elseif (strcmp(boundaryConditions,'zeroFlux'))
                    jp1 = nGridCells;
                end
            end

            u(n+1, j) = u(n, j) + nu*(F(j) - F(jp1));
        end
    end
end
```

(a) The following script uses this method to compute the solutions at $t = 1.0$ for the different values of N and it shows the order of convergence.

```
%% Problem 2 (a)
u0func = @(x) 1 + 0.5*sin(x);
```

```
du0func = @(x) 0.5*cos(x);
a = 0;
b = 2*pi;
tFinal = 1.0;
f = @(u) (u^2)/2;

E = zeros(4, 4);
iter = 0;

for N = [20, 40, 80, 160]
    iter = iter + 1;
    deltaX = (b - a)/N;
    x = linspace(a, b, N);
    u0 = u0func(x);

    deltaT = 0.5*deltaX;
    nTimeSteps = ceil(tFinal/deltaT);
    deltaT = tFinal/nTimeSteps;

    sol = godunov(f, u0, deltaT, deltaX, nTimeSteps);
    exactSol = burgersExactSolution(x, u0func, du0func, tFinal);

    E(iter, 1) = N;
    E(iter, 2) = deltaX;
    E(iter, 3) = max(abs(sol(end,:) - exactSol'));
    if(iter >= 2)
        E(iter, 4) = log(E(iter-1, 3)/E(iter, 3))/log(E(iter-1, 2)/E(iter, 2));
    end
end
disp(latexFileWriter.printMatrix(E,3));
```

The following table is output from this script. Note that the Godunov method converges with order 1, which is what we expect.

N	Δx	L^∞ Error	Order
20.000	0.314	0.098	-
40.000	0.157	0.049	0.997
80.000	0.079	0.025	0.987
160.000	0.039	0.013	0.980

(b) The following script uses the Godunov method to plot the solution at $t = 3.0$.

```
%% Problem 2 (b)
u0func = @(x) 1 + 0.5*sin(x);
a = 0;
b = 2*pi;
tFinal = 3.0;
f = @(u) (u^2)/2;
style = ["k--", "k-"];

iter = 0;
hold on;
for N = [80,800]
    iter = iter+1;
    deltaX = (b - a)/N;
    x = linspace(a, b, N);
    u0 = u0func(x);

    deltaT = 0.5*deltaX;
```

```
    nTimeSteps = ceil(tFinal/deltaT);
    deltaT = tFinal/nTimeSteps;

    sol = godunov(f, u0, deltaT, deltaX, nTimeSteps);
    plot(x, sol(end,:), char(style(iter)), 'LineWidth', 2);
end
xlabel('x');
ylabel('u');
legend('N = 80', 'Exact Solution', 'Location', 'northwest');
hold off;
saveas(gcf, 'Figures/05_01.png', 'png');
```

The following image is produced.