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MATH667 Hyperbolic Partial Differential Equations
Homework 7

1. Derive the following 3rd order accuracy reconstruction for finite volume methods.

$$
\begin{aligned}
u^-_{j+1/2} &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1} \\
u^+_{j+1/2} &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}
\end{aligned}
$$

For this problem we would like to construct a quadractic polynomial $p(x)$ such that p matches the cell average on intervals I_{j-1}, I_j, and I_{j+1}. If we set $p(x) = a\left(x - x_{j+1/2}\right)^2 + b\left(x - x_{j-1/2}\right) + c$, then $p(x_{j-1/2}) = c$. We can then solve the following three equations for c.

$$
\begin{aligned}
\frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j-1} \\
\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j} \\
\frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j+1}
\end{aligned}
$$

These equation can be simplified as follows.

$$
\begin{aligned}
\frac{1}{3h}a\left(-h^3 + 8h^3\right) + \frac{1}{2h}b\left(h^2 - 4h^2\right) + c &= \bar{u}_{j-1} \\
\frac{1}{3h}a\left(h^3\right) + \frac{1}{2h}b\left(-h^2\right) + c &= \bar{u}_{j} \\
\frac{1}{3h}a\left(h^3\right) + \frac{1}{2h}b\left(h^2\right) + c &= \bar{u}_{j+1}
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
\frac{7h^2}{3}a - \frac{3h}{2}b + c &= \bar{u}_{j-1} \\
\frac{h^2}{3}a - \frac{h}{2}b + c &= \bar{u}_{j} \\
\frac{h^2}{3}a + \frac{h}{2}b + c &= \bar{u}_{j+1}.
\end{aligned}
$$

Subtracting three times the second equation to the first equation and adding the last two equations gives

$$
\begin{aligned}
\frac{4h^2}{3}a - 2c &= \bar{u}_{j-1} - 3\bar{u}_j \\
\frac{2h^2}{3}a + 2c &= \bar{u}_j + \bar{u}_{j+1}.
\end{aligned}
$$

Subtracting 2 times the second equation from the first gives

$$
\begin{aligned}
-6c &= \bar{u}_{j-1} - 5\bar{u}_j - 2\bar{u}_{j+1} \\
c &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}
\end{aligned}
$$

Thus we have for the first reconstruction that

$$u^-_{j+1/2} = p(x_{j+1/2}) = c = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}$$

We can now do this process again on the intervals I_j, I_{j+1}, and I_{j+2}. This only changes one of the equations. We now have

$$\begin{aligned}\frac{1}{h}\int_{x_{j+3/2}}^{x_{j+5/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j+2}\\ \frac{1}{3h}a\Big(8h^3 - h^3\Big) + \frac{1}{2h}b\Big(4h^2 - h^2\Big) + c &= \bar{u}_{j+2}\\ \frac{7h^2}{3}a + \frac{3h}{2}b + c &= \bar{u}_{j+2}\end{aligned}$$

We must now solve the following three equations for c

$$\begin{aligned}\frac{h^2}{3}a - \frac{h}{2}b + c &= \bar{u}_j\\ \frac{h^2}{3}a + \frac{h}{2}b + c &= \bar{u}_{j+1}\\ \frac{7h^2}{3}a + \frac{3h}{2}b + c &= \bar{u}_{j+2}.\end{aligned}$$

Adding the first two equations and subtracting 3 times the second equation from the last equation gives

$$\begin{aligned}\frac{2h^2}{3}a + 2c &= \bar{u}_j + \bar{u}_{j+1}\\ \frac{4h^2}{3}a + -2c &= \bar{u}_{j+2} - 3\bar{u}_{j+1}.\end{aligned}$$

Subtracting 2 times the first equation from the second equation gives

$$\begin{aligned}-6c &= -2\bar{u}_j - 5\bar{u}_{j+1} + \bar{u}_{j+2}\\ c &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}\end{aligned}$$

This is the second

$$u^+_{j+1/2} = p(x_{j+1/2}) = c = \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}$$

2.

3.