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MATH667 Hyperbolic Partial Differential Equations
Homework 7

1. Derive the following 3rd order accuracy reconstruction for finite volume methods.

$$
\begin{aligned}
u_{j+1/2}^{-} &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1} \\
u_{j+1/2}^{+} &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}
\end{aligned}
$$

For this problem we would like to construct a quadractic polynomial $p(x)$ such that p matches the cell average on intervals I_{j-1}, I_j, and I_{j+1}. If we set $p(x) = a\left(x - x_{j+1/2}\right)^2 + b\left(x - x_{j-1/2}\right) + c$, then $p(x_{j-1/2}) = c$. We can then solve the following three equations for c.

$$
\begin{aligned}
\frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j-1} \\
\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j} \\
\frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j+1}
\end{aligned}
$$

These equation can be simplified as follows.

$$
\begin{aligned}
\frac{1}{3h}a\left(\left(x_{j-1/2} - x_{j+1/2}\right)^3 - \left(x_{j-3/2} - x_{j+1/2}\right)^3\right) + \frac{1}{2h}b\left(x_{j-1/2}^2 - x_{j-3/2}^2\right) + c &= \bar{u}_{j-1} \\
\frac{1}{3h}a\left(x_{j+1/2}^3 - x_{j-1/2}^3\right) + \frac{1}{2h}b\left(x_{j+1/2}^2 - x_{j-1/2}^2\right) + c &= \bar{u}_{j} \\
\frac{1}{3h}a\left(x_{j+3/2}^3 - x_{j+1/2}^3\right) + \frac{1}{2h}b\left(x_{j+3/2}^2 - x_{j+1/2}^2\right) + c &= \bar{u}_{j+1}
\end{aligned}
$$

2.

3.