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MATH667 Hyperbolic Partial Differential Equations Homework 1

1. For constant coefficient linear wave equation initial value problem

$$\begin{cases} u_t + au_x = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

for all $x \in \mathbb{R}$ and $t \geq 0$, verify that the solution $u(x, t) = u_0(x - at)$ satisfies the following integral form. We assume the initial $u_0(x)$ is a smooth function.

$$\int_{x_1}^{x_2} u(x, t_2)\,\mathrm{d}x = \int_{x_1}^{x_2} u(x, t_1)\,\mathrm{d}x + \int_{t_1}^{t_2} au(x_1, t)\,\mathrm{d}t - \int_{t_1}^{t_2} au(x_2, t)\,\mathrm{d}t, \quad \forall x_1, x_2 \in \mathbb{R}, \forall t_1, t_2 \geq 0.$$

Let U be an antiderivative of u_0, that is $\int_{y_1}^{y_2} u_0(y)\,\mathrm{d}y = U(y_2) - U(y_1)$. This is guaranteed to exist since u_0 is a smooth function. Now we can consider each of the terms of the integral form seperately. The first term can be simplified as follows,

$$\begin{aligned} \int_{x_1}^{x_2} u(x, t_2)\,\mathrm{d}x &= \int_{x_1}^{x_2} u_0(x - at_2)\,\mathrm{d}x \\ &= U(x_2 - at_2) - U(x_1 - at_2). \end{aligned}$$

The second term is

$$\begin{aligned} \int_{x_1}^{x_2} u(x, t_1)\,\mathrm{d}x &= \int_{x_1}^{x_2} u_0(x - at_1)\,\mathrm{d}x \\ &= U(x_2 - at_1) - U(x_1 - at_1). \end{aligned}$$

The third term is

$$\begin{aligned} \int_{t_1}^{t_2} au(x_1, t)\,\mathrm{d}t &= \int_{t_1}^{t_2} au_0(x_1 - at)\,\mathrm{d}t \\ &= \frac{a}{-a}(U(x_1 - at_2) - U(x_1 - at_1)) \\ &= -U(x_1 - at_2) + U(x_1 - at_1). \end{aligned}$$

The last term becomes

$$\begin{aligned} \int_{t_1}^{t_2} au(x_2, t)\,\mathrm{d}t &= \int_{t_1}^{t_2} au_0(x_2 - at)\,\mathrm{d}t \\ &= \frac{a}{-a}(U(x_1 - at_2) - U(x_1 - at_1)) \\ &= -U(x_2 - at_2) + U(x_2 - at_1). \end{aligned}$$

Combining these four terms back into the integral form gives.

$$\begin{aligned} U(x_2 - at_2) - U(x_1 - at_2) &= U(x_2 - at_1) - U(x_1 - at_1) - U(x_1 - at_2) \\ &\quad + U(x_1 - at_1) + U(x_2 - at_2) - U(x_2 - at_1) \\ U(x_2 - at_2) - U(x_1 - at_2) &= -U(x_1 - at_2) + U(x_2 - at_2) \\ -U(x_1 - at_2) &= -U(x_1 - at_2) \\ 0 &= 0 \end{aligned}$$

This shows that the integral form is satisfied for all values of $x_1, x_2 \in \mathbb{R}$ and $t_1, t_2 \geq 0$.

2. For viscous Burger's equation $u_t + uu_x + \varepsilon u_{xx}$ with initial condition

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases},$$

verify the traveling wave solution $u_\varepsilon(x,t)$ satisfies the given PDE. We have $u_\varepsilon(x,t) = w(x - \frac{1}{2}t)$, where $w(y) = \frac{1}{2}(1 - \tanh(\frac{y}{4\varepsilon}))$. Graph the solution $u_\varepsilon(x,t)$ at $t = 1$ with $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$.

First note that

$$\begin{aligned} u_{\varepsilon,t} &= -\frac{1}{2}w'\left(x - \frac{1}{2}t\right) \\ u_{\varepsilon,x} &= w'\left(x - \frac{1}{2}t\right) \\ u_{\varepsilon,xx} &= w''\left(x - \frac{1}{2}t\right). \end{aligned}$$

Also note the derivatives of w are

$$\begin{aligned} w'(y) &= -\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{y}{4\varepsilon}\right)\right) \\ w''(y) &= \frac{1}{16\varepsilon^2}\tanh\left(\frac{y}{4\varepsilon}\right)\left(1 - \tanh^2\left(\frac{y}{4\varepsilon}\right)\right) \end{aligned}$$

Now we see that

$$\begin{aligned} u_{\varepsilon,t} &= \frac{1}{16\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ u_{\varepsilon,x} &= -\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ u_{\varepsilon,xx} &= \frac{1}{16\varepsilon^2}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right). \end{aligned}$$

Plugging these into the left hand side and right hand side of the PDE gives the following. First I will simplify the left hand side

$$\begin{gathered} u_{\varepsilon,t} + u_\varepsilon u_{\varepsilon,x} \\ \frac{1}{16\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) + -\frac{1}{2}\left(1 - \tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)\frac{1}{8\varepsilon}\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ \frac{1}{16\varepsilon} - \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + \left(-\frac{1}{16\varepsilon} + \frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right)\left(1 - \tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right)\right) \\ \frac{1}{16\varepsilon} - \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + -\frac{1}{16\varepsilon} + \frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) + \frac{1}{16\varepsilon}\tanh^2\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) - \frac{1}{16\varepsilon}\tanh^3\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) \\ \frac{1}{16\varepsilon}\tanh\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right) - \frac{1}{16\varepsilon}\tanh^3\left(\frac{x - \frac{1}{2}t}{4\varepsilon}\right). \end{gathered}$$

Next I will simplify the right hand side

$$\varepsilon u_{\varepsilon,xx} \tag{1}$$

$$\varepsilon \frac{1}{16\varepsilon^2} \tanh\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\left(1-\tanh^2\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\right) \tag{2}$$

$$\frac{1}{16\varepsilon}\left(\tanh\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)-\tanh^3\left(\frac{x-\frac{1}{2}t}{4\varepsilon}\right)\right) \tag{3}$$

$$\tag{4}$$

We see that this is equal to the left hand side shown previously, so $u_\varepsilon(x,t)$ does satisfy the PDE. Also $u_\varepsilon(x,t)$ satisfies the inital conditions as $\varepsilon \to 0$. To see this consider the following

$$\begin{aligned}\lim_{\varepsilon\to 0}(u_\varepsilon(x,0)) &= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\left(1-\tanh\left(\frac{x}{3\varepsilon}\right)\right)\right)\\ &= \frac{1}{2}-\lim_{\varepsilon\to 0}\left(\tanh\left(\frac{x}{4\varepsilon}\right)\right)\end{aligned}$$

If $x \leq 0$, then this is equivalent to

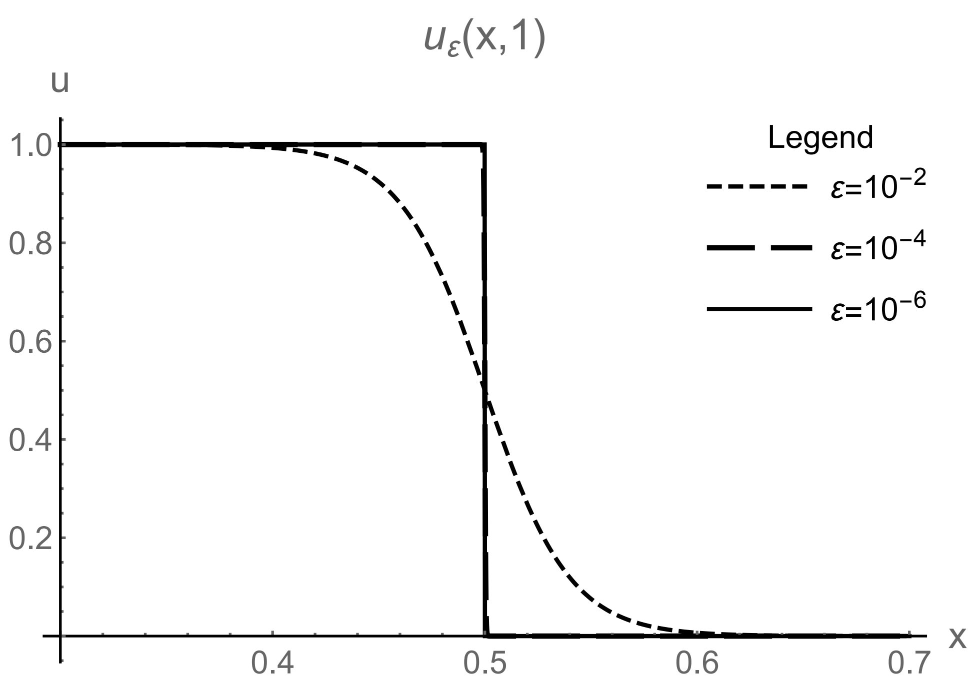
$$\begin{aligned}&= \frac{1}{2}-\lim_{y\to -\infty}(\tanh(y))\\ &= \frac{1}{2}--\frac{1}{2}=1\end{aligned}$$

If $x > 0$, then this limit is equivalent to

$$\begin{aligned}&= \frac{1}{2}-\lim_{y\to \infty}(\tanh(y))\\ &= \frac{1}{2}-\frac{1}{2}=0\end{aligned}$$

This shows that $u_\varepsilon(x,t)$ satisfies the initial conditions as $\varepsilon \to 0$.

The following is a graph of $u_\varepsilon(x,1)$ with $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$. Note that graph for $\varepsilon = 10^{-4}$ and the graph for $\varepsilon = 10^{-6}$ are almost directly on top of on another.



3. For Burger's equation Riemann problem

$$u(x,0) = \begin{cases} u_l & x \le 0 \\ u_r & x > 0 \end{cases}$$

with $u_l < u_r$, show that

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ \frac{x}{t} & u_m t \le x \le u_r t \\ u_r & u_r t \le x \end{cases}$$

is a weak solution. Let $u_l < u_m < u_r$ and $s_m = (u_l + u_r)/2$. Sketch the characteristics for this solution.

Weak solutions of Burger's equation satisfy the following

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + \frac{1}{2}u^2\phi_x \,\mathrm{d}x\,\mathrm{d}t = -\int_{-\infty}^\infty u(x,0)\phi(x,0)\,\mathrm{d}x$$

for all test functions $\phi \in C_0^1(\mathbb{R}^2 \times \mathbb{R}^+)$. I will consider the integral equation in three parts. First consider

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t \,\mathrm{d}x\,\mathrm{d}t$$

This can be simplified by integrating by parts and substituting in for u.

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty u\phi_t \,\mathrm{d}x\,\mathrm{d}t &= \int_{-\infty}^\infty \int_0^\infty u\phi_t \,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{-\infty}^\infty u\phi|_{t=0}^\infty - \int_0^\infty u_t\phi\,\mathrm{d}t\,\mathrm{d}x \end{aligned}$$

Since ϕ has compact support, $\phi|_{t=\infty} = 0$

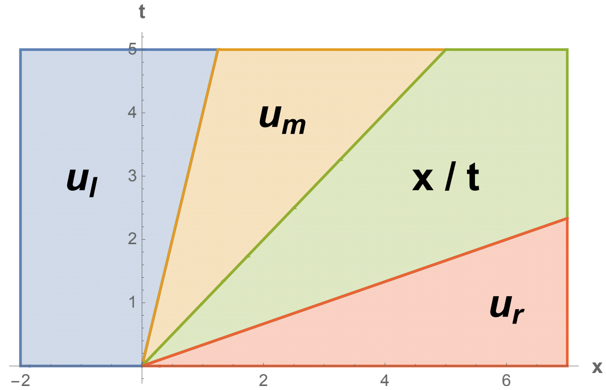
$$= \int_{-\infty}^\infty u(x,0)\phi(x,0)\,\mathrm{d}x - \int_{-\infty}^\infty \int_0^\infty u_t\phi\,\mathrm{d}t\,\mathrm{d}x$$

Now I will just consider

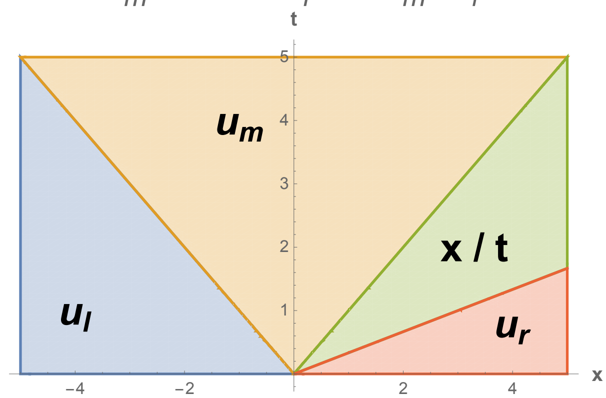
$$\int_{-\infty}^\infty \int_0^\infty u_t\phi\,\mathrm{d}t\,\mathrm{d}x$$

This integral can be broken down in several different ways depending on the values of u_l, u_m, and u_r. There are in fact four different cases, shown below.

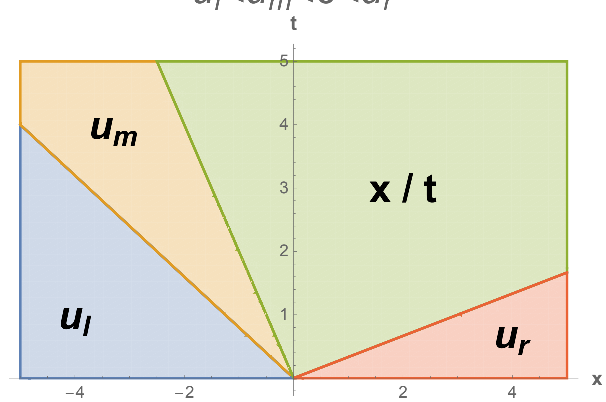
$s_m>0$ and $0<u_m<u_r$



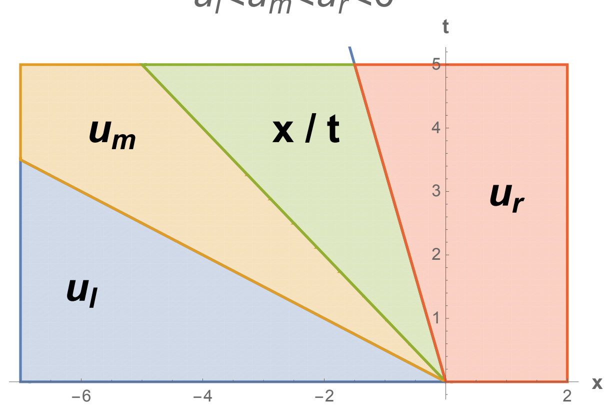
$s_m<0$ and $u_l<0<u_m<u_r$



$u_l<u_m<0<u_r$



$u_l<u_m<u_r<0$



I will only consider the case where $s_m > 0$, this implies that $0 < u_m < u_r$. u_l may be positive or negative but it is large enough that $s_m > 0$. The other cases are roughly equivalent and would be redundant to fully write up. In this case we can revisit the previous integral split it up into left and right quadrants as follows.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{0}^{\infty} u_t\phi \,\mathrm{d}t\,\mathrm{d}x &= \int_{-\infty}^{0}\int_{0}^{\infty} \frac{\partial}{\partial t}(u_l)\phi \,\mathrm{d}t\,\mathrm{d}x + \int_{0}^{\infty}\int_{0}^{\infty} u_t\phi \,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} 0 \times \phi \,\mathrm{d}t\,\mathrm{d}x + \int_{0}^{\infty}\int_{0}^{\infty} u_t\phi \,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{0}^{\infty}\int_{0}^{\infty} u_t\phi \,\mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

Now the right quadrant can be split up into four sections based on time

$$\begin{aligned}
&= \int_{0}^{\infty}\left(\int_{0}^{x/u_r} \frac{\partial}{\partial t}(u_r)\phi \,\mathrm{d}t + \int_{x/u_r}^{x/u_m} \frac{\partial}{\partial t}\left(\frac{x}{t}\right)\phi \,\mathrm{d}t + \int_{x/u_m}^{x/s_m} \frac{\partial}{\partial t}(u_m)\phi \,\mathrm{d}t + \int_{x/s_m}^{\infty} \frac{\partial}{\partial t}(u_l)\phi \,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_{0}^{\infty}\left(\int_{0}^{x/u_r} 0 \times \phi \,\mathrm{d}t + \int_{x/u_r}^{x/u_m} -\frac{x}{t^2}\phi \,\mathrm{d}t + \int_{x/u_m}^{x/s_m} 0 \times \phi \,\mathrm{d}t + \int_{x/s_m}^{\infty} 0 \times \phi \,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_{0}^{\infty}\int_{x/u_r}^{x/u_m} -\frac{x}{t^2}\phi \,\mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

Next I will consider the term with the partial derivative in x.

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}u^2\phi_x \,\mathrm{d}x\,\mathrm{d}t$$

Again this can be simplified by integrating by parts and substituting in for u.

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}u^2\phi_x \,\mathrm{d}x\,\mathrm{d}t = \int_{0}^{\infty} \frac{1}{2}u^2\phi\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} uu_x\phi \,\mathrm{d}x\,\mathrm{d}t$$

Since ϕ has compact support $\phi(\infty, t) = \phi(-\infty, t) = 0$

$$\begin{aligned}
&= \int_{0}^{\infty} 0 - \int_{-\infty}^{\infty} uu_x\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{0}^{\infty}\left(\int_{-\infty}^{s_m t} u_l\frac{\partial}{\partial x}(u_l)\phi \,\mathrm{d}x + \int_{s_m t}^{u_m t} u_m\frac{\partial}{\partial x}(u_m)\phi \,\mathrm{d}x + \int_{u_m t}^{u_r t} \frac{x}{t}\frac{\partial}{\partial x}\left(\frac{x}{t}\right)\phi \,\mathrm{d}x + \int_{u_r t}^{\infty} u_r\frac{\partial}{\partial x}(u_r)\phi \,\mathrm{d}x\right)\mathrm{d}t \\
&= -\int_{0}^{\infty}\left(\int_{-\infty}^{s_m t} u_l \times 0 \times \phi \,\mathrm{d}x + \int_{s_m t}^{u_m t} u_m \times 0 \times \phi \,\mathrm{d}x + \int_{u_m t}^{u_r t} \frac{x}{t}\frac{1}{t}\phi \,\mathrm{d}x + \int_{u_r t}^{\infty} u_r \times 0 \times \phi \,\mathrm{d}x\right)\mathrm{d}t \\
&= -\int_{0}^{\infty}\int_{u_m t}^{u_r t} \frac{x}{t^2}\phi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

4. I implemented a Forward Euler Upwind method in the following MATLAB function.

```
function [u] = upwind(u0, deltaT, deltaX, nTimeSteps)
    nGridCells = length(u0);
    u = zeros(nTimeSteps+1, nGridCells);
    u(1, :) = u0;
    alpha = deltaT/deltaX;

    for n = 1:nTimeSteps
        for j = 1:nGridCells
            % periodic boundary conditions
```

```
            jm1 = j-1;
            if (j == 1)
                jm1 = nGridCells;
            end

            % update
            u(n+1, j) = u(n, j) + alpha*(u(n, jm1) - u(n, j));
        end
    end
end
```

The following script uses the previous function to simulate a square wave propagating with periodic boundary conditions.

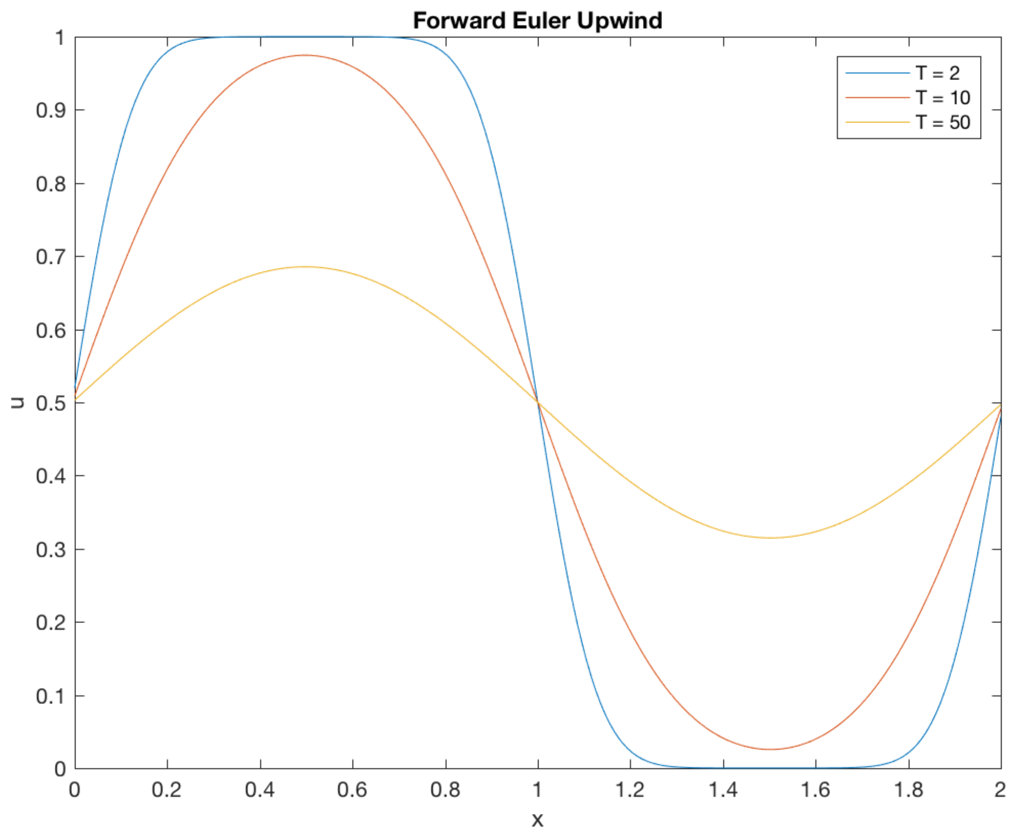
```
u0func = @(x) 1.0*(x <= 1.0);
deltaX = 0.01;
a = 0.0;
b = 2.0;
nGridCells = (b - a)/deltaX;

deltaT = deltaX/2.0;
x = linspace(a, b, nGridCells);
u0 = u0func(x);

tFinal = 50;
nTimeSteps = tFinal/deltaT;
u = upwind(u0, deltaT, deltaX, nTimeSteps);

t2 = 2/deltaT;
t10 = 10/deltaT;
t50 = nTimeSteps;
plot(x, u(t2+1,:), x, u(t10+1,:), x, u(t50+1,:));
legend('T = 2', 'T = 10', 'T = 50');
xlabel('x');
ylabel('u');
title('Forward Euler Upwind');
saveas(gcf, 'Figures/01_02.png', 'png');
```

The following image is produced. Note that as time increases the solution decays more and more from the exact solution which is a perfect square wave.


Forward Euler Upwind
T = 2
T = 10
T = 50
u
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2