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MATH667 Hyperbolic Partial Differential Equations
Homework 7

1. Derive the following 3rd order accuracy reconstruction for finite volume methods.

$$u^-_{j+1/2} = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}$$
$$u^+_{j+1/2} = \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}$$

For this problem we would like to construct a quadractic polynomial $p(x)$ such that p matches the cell average on intervals I_{j-1}, I_j, and I_{j+1}. This means that we need to set $p(x) = ax^2 + bx + c$ and solve the following set of equations for a, b, and c.

$$\frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} p(x)\,\mathrm{d}x = \bar{u}_{j-1}$$
$$\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x)\,\mathrm{d}x = \bar{u}_{j}$$
$$\frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} p(x)\,\mathrm{d}x = \bar{u}_{j+1}$$

These equation can be simplified as follows.

$$\frac{1}{3h}a\left(x^3_{j-1/2} - x^3_{j-3/2}\right) + \frac{1}{2h}b\left(x^2_{j-1/2} - x^2_{j-3/2}\right) + c = \bar{u}_{j-1}$$
$$\frac{1}{3h}a\left(x^3_{j+1/2} - x^3_{j-1/2}\right) + \frac{1}{2h}b\left(x^2_{j+1/2} - x^2_{j-1/2}\right) + c = \bar{u}_{j}$$
$$\frac{1}{3h}a\left(x^3_{j+3/2} - x^3_{j+1/2}\right) + \frac{1}{2h}b\left(x^2_{j+3/2} - x^2_{j+1/2}\right) + c = \bar{u}_{j+1}$$

Factoring the difference of cubes and difference of squares and cancelling h gives

$$\frac{1}{3}a\left(x^2_{j-1/2} + x_{j-1/2}x_{j-3/2} + x^2_{j-3/2}\right) + \frac{1}{2}b\left(x_{j-1/2} + x_{j-3/2}\right) + c = \bar{u}_{j-1}$$
$$\frac{1}{3}a\left(x^2_{j+1/2} + x_{j+1/2}x_{j-1/2} + x^2_{j-1/2}\right) + \frac{1}{2}b\left(x_{j+1/2} + x_{j-1/2}\right) + c = \bar{u}_{j}$$
$$\frac{1}{3}a\left(x^2_{j+3/2} + x_{j+3/2}x_{j+1/2} + x^2_{j+1/2}\right) + \frac{1}{2}b\left(x_{j+3/2} + x_{j+1/2}\right) + c = \bar{u}_{j+1}.$$

Now I will substitute in so that all the x values are in terms of $x_{j+1/2}$ and h, i.e. $x_{j-3/2} = x_{j+1/2} - 2h$, $x_{j-1/2} = x_{j+1/2} - h$, etc.

$$\frac{1}{3}a\left(\left(x_{j+1/2} - h\right)^2 + \left(x_{j+1/2} - h\right)\left(x_{j+1/2} - 2h\right) + \left(x_{j+1/2} - 2h\right)^2\right) + b\left(x_{j+1/2} - \frac{3}{2}h\right) + c = \bar{u}_{j-1}$$
$$\frac{1}{3}a\left(x^2_{j+1/2} + x_{j+1/2}\left(x_{j+1/2} - h\right) + \left(x_{j+1/2} - h\right)^2\right) + b\left(x_{j+1/2} - \frac{1}{2}h\right) + c = \bar{u}_{j}$$
$$\frac{1}{3}a\left(\left(x_{j+1/2} + h\right)^2 + \left(x_{j+1/2} + h\right)x_{j+1/2} + x^2_{j+1/2}\right) + b\left(x_{j+1/2} + \frac{1}{2}h\right) + c = \bar{u}_{j+1}.$$

Simplifying gives

$$\frac{1}{3}a\left(3x^2_{j+1/2} - 9hx_{j+1/2} + 7h^2\right) + b\left(x_{j+1/2} - \frac{3}{2}h\right) + c = \bar{u}_{j-1}$$
$$\frac{1}{3}a\left(3x^2_{j+1/2} - 3hx_{j+1/2} + 1h^2\right) + b\left(x_{j+1/2} - \frac{1}{2}h\right) + c = \bar{u}_{j}$$
$$\frac{1}{3}a\left(3x^2_{j+1/2} + 3hx_{j+1/2} + 1h^2\right) + b\left(x_{j+1/2} + \frac{1}{2}h\right) + c = \bar{u}_{j+1}.$$

Subtracting the last two equations gives

$$\frac{1}{3}a\Big(-6hx_{j+1/2}\Big) - hb = \bar{u}_j - \bar{u}_{j+1}$$

2.

3.