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MATH667 Hyperbolic Partial Differential Equations
Homework 7

1. Derive the following 3rd order accuracy reconstruction for finite volume methods.

$$
\begin{aligned}
u^-_{j+1/2} &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1} \\
u^+_{j+1/2} &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}
\end{aligned}
$$

For this problem we would like to construct a quadractic polynomial $p(x)$ such that p matches the cell average on intervals I_{j-1}, I_j, and I_{j+1}. If we set $p(x) = a\left(x - x_{j+1/2}\right)^2 + b\left(x - x_{j-1/2}\right) + c$, then $p(x_{j-1/2}) = c$. We can then solve the following three equations for c.

$$
\begin{aligned}
\frac{1}{h}\int_{x_{j-3/2}}^{x_{j-1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j-1} \\
\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j} \\
\frac{1}{h}\int_{x_{j+1/2}}^{x_{j+3/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j+1}
\end{aligned}
$$

These equation can be simplified as follows.

$$
\begin{aligned}
\frac{1}{3h}a\left(-h^3 + 8h^3\right) + \frac{1}{2h}b\left(h^2 - 4h^2\right) + c &= \bar{u}_{j-1} \\
\frac{1}{3h}a\left(h^3\right) + \frac{1}{2h}b\left(-h^2\right) + c &= \bar{u}_{j} \\
\frac{1}{3h}a\left(h^3\right) + \frac{1}{2h}b\left(h^2\right) + c &= \bar{u}_{j+1}
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
\frac{7h^2}{3}a - \frac{3h}{2}b + c &= \bar{u}_{j-1} \\
\frac{h^2}{3}a - \frac{h}{2}b + c &= \bar{u}_{j} \\
\frac{h^2}{3}a + \frac{h}{2}b + c &= \bar{u}_{j+1}.
\end{aligned}
$$

Subtracting three times the second equation to the first equation and adding the last two equations gives

$$
\begin{aligned}
\frac{4h^2}{3}a - 2c &= \bar{u}_{j-1} - 3\bar{u}_j \\
\frac{2h^2}{3}a + 2c &= \bar{u}_j + \bar{u}_{j+1}.
\end{aligned}
$$

Subtracting 2 times the second equation from the first gives

$$
\begin{aligned}
-6c &= \bar{u}_{j-1} - 5\bar{u}_j - 2\bar{u}_{j+1} \\
c &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}
\end{aligned}
$$

Thus we have for the first reconstruction that

$$u^-_{j+1/2} = p(x_{j+1/2}) = c = -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1}$$

We can now do this process again on the intervals I_j, I_{j+1}, and I_{j+2}. This only changes one of the equations. We now have

$$\begin{aligned}\frac{1}{h}\int_{x_{j+3/2}}^{x_{j+5/2}} p(x)\,\mathrm{d}x &= \bar{u}_{j+2} \\ \frac{1}{3h}a\Big(8h^3 - h^3\Big) + \frac{1}{2h}b\Big(4h^2 - h^2\Big) + c &= \bar{u}_{j+2} \\ \frac{7h^2}{3}a + \frac{3h}{2}b + c &= \bar{u}_{j+2}\end{aligned}$$

We must now solve the following three equations for c

$$\begin{aligned}\frac{h^2}{3}a - \frac{h}{2}b + c &= \bar{u}_j \\ \frac{h^2}{3}a + \frac{h}{2}b + c &= \bar{u}_{j+1} \\ \frac{7h^2}{3}a + \frac{3h}{2}b + c &= \bar{u}_{j+2}.\end{aligned}$$

Adding the first two equations and subtracting 3 times the second equation from the last equation gives

$$\begin{aligned}\frac{2h^2}{3}a + 2c &= \bar{u}_j + \bar{u}_{j+1} \\ \frac{4h^2}{3}a + -2c &= \bar{u}_{j+2} - 3\bar{u}_{j+1}.\end{aligned}$$

Subtracting 2 times the first equation from the second equation gives

$$\begin{aligned}-6c &= -2\bar{u}_j - 5\bar{u}_{j+1} + \bar{u}_{j+2} \\ c &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}\end{aligned}$$

This is the second reconstruction

$$u^+_{j+1/2} = p(x_{j+1/2}) = c = \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}$$

2. Consider to solve the 1D scalar conservation law. Show the 3rd order finite volume MUSCL scheme is TVD. Use Harten's Thereom.

   First I will write out explicitly what the 3rd order MUSCL scheme is. We started with the following two reconstructions.

$$\begin{aligned}u^-_{j+1/2} &= -\frac{1}{6}\bar{u}_{j-1} + \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1} \\ u^+_{j+1/2} &= \frac{1}{3}\bar{u}_j + \frac{5}{6}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j+2}\end{aligned}$$

   Now define

$$\begin{aligned}\tilde{u}_j &= u^-_{j+1/2} - \bar{u}_j \\ \tilde{\tilde{u}}_{j+1} &= u^+_{j+1/2} + \bar{u}_{j+1}\end{aligned}$$

Now we will do a minmod limiter

$$\begin{aligned}\tilde{u}_j^{\text{mod}} &= \text{minmod}(\tilde{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}) \\ \tilde{\tilde{u}}_{j+1}^{\text{mod}} &= \text{minmod}\big(\tilde{\tilde{u}}_{j+1}, \bar{u}_{j+2} - \bar{u}_{j+1}, \bar{u}_{j+1} - \bar{u}_j\big)\end{aligned}$$

Finally let

$$\begin{aligned}u_{j+1/2}^{-,\text{mod}} &= \bar{u}_j + \tilde{u}_j^{\text{mod}} \\ u_{j+1/2}^{+,\text{mod}} &= \bar{u}_{j+1} - \tilde{\tilde{u}}_{j+1}^{\text{mod}}\end{aligned}$$

Now the full scheme is

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}\Big(\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}}) - \hat{f}(u_{j-1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}})\Big)$$

In order to show that this scheme is TVD, we can rewrite this method as

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}\Big(\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) + \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j-1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}})\Big)$$

Now we can write this method as

$$\bar{u}_j^{n+1} = \bar{u}_j^n - C_{j-1}(\bar{u}_j - \bar{u}_{j-1}) + D_j(\bar{u}_{j+1} - \bar{u}_j)$$

where

$$\begin{aligned}C_{j-1} &= \frac{\Delta t}{\Delta x}\frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j-1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}})}{\bar{u}_j - \bar{u}_{j-1}} \\ D_j &= \frac{\Delta t}{\Delta x}\frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}})}{\bar{u}_{j+1} - \bar{u}_j}\end{aligned}$$

Now Harten's Theorem states that this method is TVD if $C_{j-1} \geq 0$, $D_j \geq 0$, and $C_j + D_j \leq 1$ for all j.

First consider C_{j-1}

$$\begin{aligned}C_{j-1} &= \frac{\Delta t}{\Delta x}\frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j-1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}})}{\bar{u}_j - \bar{u}_{j-1}} \\ &= \frac{\Delta t}{\Delta x}\frac{\hat{f}(\bar{u}_j + \tilde{u}_j^{\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(\bar{u}_{j-1} + \tilde{u}_{j-1}^{\text{mod}}, u_{j-1/2}^{+,\text{mod}})}{\bar{u}_j - \bar{u}_{j-1}} \\ &= \frac{\Delta t}{\Delta x}\hat{f}_1(\xi, u_{j-1/2}^{+,\text{mod}})\frac{\bar{u}_j + \tilde{u}_j^{\text{mod}} - \bar{u}_{j-1} - \tilde{u}_{j-1}^{\text{mod}}}{\bar{u}_j - \bar{u}_{j-1}} \\ &= \frac{\Delta t}{\Delta x}\hat{f}_1(\xi, u_{j-1/2}^{+,\text{mod}})\left(1 + \frac{\tilde{u}_j^{\text{mod}}}{\bar{u}_j - \bar{u}_{j-1}} - \frac{\tilde{u}_{j-1}^{\text{mod}}}{\bar{u}_j - \bar{u}_{j-1}}\right)\end{aligned}$$

Now since $\frac{\Delta t}{\Delta x} > 0$ and $\hat{f}_1(\xi, u_{j-1/2}^{+,\text{mod}}) > 0$ by monotonicity

$$C_{j-1} \geq 1 + \frac{\tilde{u}_j^{\text{mod}}}{\bar{u}_j - \bar{u}_{j-1}} - \frac{\tilde{u}_{j-1}^{\text{mod}}}{\bar{u}_j - \bar{u}_{j-1}}$$

Also $0 \leq \tilde{u}_j^{\text{mod}} \leq \bar{u}_j - \bar{u}_{j-1}$ and $0 \leq \tilde{u}_{j-1}^{\text{mod}} \leq \bar{u}_j - \bar{u}_{j-1}$, so

$$C_{j-1} \geq 1 + 0 - 1 = 0$$

This shows that $C_{j-1} \geq 0$ for all j, so now consider D_j.

$$\begin{aligned}
D_j &= \frac{\Delta t}{\Delta x} \frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}})}{\bar{u}_{j+1} - \bar{u}_j} \\
&= \frac{\Delta t}{\Delta x} \frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, \bar{u}_j - \tilde{\tilde{u}}_j^{\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, \bar{u}_{j+1} - \tilde{\tilde{u}}_{j+1}^{\text{mod}})}{\bar{u}_{j+1} - \bar{u}_j} \\
&= \frac{\Delta t}{\Delta x} \hat{f}_2(u_{j+1/2}^{-,\text{mod}}, \xi) \frac{\bar{u}_j - \tilde{\tilde{u}}_j^{\text{mod}} - \bar{u}_{j+1} + \tilde{\tilde{u}}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \\
&= \frac{\Delta t}{\Delta x} \hat{f}_2(u_{j+1/2}^{-,\text{mod}}, \xi) \left(-1 - \frac{\tilde{\tilde{u}}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} + \frac{\tilde{\tilde{u}}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \right)
\end{aligned}$$

Now since $\frac{\Delta t}{\Delta x} \geq 0$, $\hat{f}_2(u_{j+1/2}^{-,\text{mod}}, \xi) \leq 0$ by monotonicity, $0 \leq \tilde{\tilde{u}}_j^{\text{mod}} \leq \bar{u}_{j+1} - \bar{u}_j$, and $0 \leq \tilde{\tilde{u}}_{j+1}^{\text{mod}} \leq \bar{u}_{j+1} - \bar{u}_j$, this implies that

$$\begin{aligned}
0 &\geq \left(-1 - \frac{\tilde{\tilde{u}}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} + \frac{\tilde{\tilde{u}}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \right) \\
D_j &\geq 0
\end{aligned}$$

Lastly consider $C_j + D_j$,

$$\begin{aligned}
C_j + D_j &= \frac{\Delta t}{\Delta x} \frac{\hat{f}(u_{j+3/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}})}{\bar{u}_{j+1} - \bar{u}_j} + \frac{\Delta t}{\Delta x} \frac{\hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j-1/2}^{+,\text{mod}}) - \hat{f}(u_{j+1/2}^{-,\text{mod}}, u_{j+1/2}^{+,\text{mod}})}{\bar{u}_{j+1} - \bar{u}_j} \\
&= \frac{\Delta t}{\Delta x} \left(\hat{f}_1(\xi, u_{j+1/2}^{+,\text{mod}}) \left(1 + \frac{\tilde{u}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} - \frac{\tilde{u}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \right) + \hat{f}_2(u_{j+1/2}^{-,\text{mod}}, \xi) \left(-1 - \frac{\tilde{\tilde{u}}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} + \frac{\tilde{\tilde{u}}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \right) \right)
\end{aligned}$$

Let $\nu = \max\left\{ \left| \hat{f}_1(\xi, u_{j+1/2}^{+,\text{mod}}) \right|, \left| \hat{f}_2(u_{j+1/2}^{-,\text{mod}}, \xi) \right| \right\}$, then

$$\begin{aligned}
C_j + D_j &\leq \frac{\nu \Delta t}{\Delta x} \left(1 + \frac{\tilde{u}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} - \frac{\tilde{u}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} + 1 + \frac{\tilde{\tilde{u}}_j^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} - \frac{\tilde{\tilde{u}}_{j+1}^{\text{mod}}}{\bar{u}_{j+1} - \bar{u}_j} \right) \\
C_j + D_j &\leq \frac{4\nu \Delta t}{\Delta x}
\end{aligned}$$

If Δt is chosen such that $0 < \Delta t \leq \frac{\Delta x}{4\nu}$, then

$$C_j + D_j \leq 1$$

This show that the third order MUSCL scheme is TVD, by Harten's Theorem.

3. Solve two-phase flow nonconver Buckley-Leverett equation with 3rd order finite volume MUSCL scheme.

$$u_t + \left(\frac{2u^2}{2u^2 + (1-u)^2} \right)_x = 0$$

with initial conditions

$$u(x, 0) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

The solution consists of a rarefaction wave connecting with a shock, ref to Figure 4.7 on page 49. Simulate the evolution of the solution to $T = 1.0$ with total mesh $N = 100$. Output the simulate with one symbol per cell on the figure. Use exact solutions at the boundary (ghost cells) as the given boundary conditions.

The following is my third order MUSCL scheme

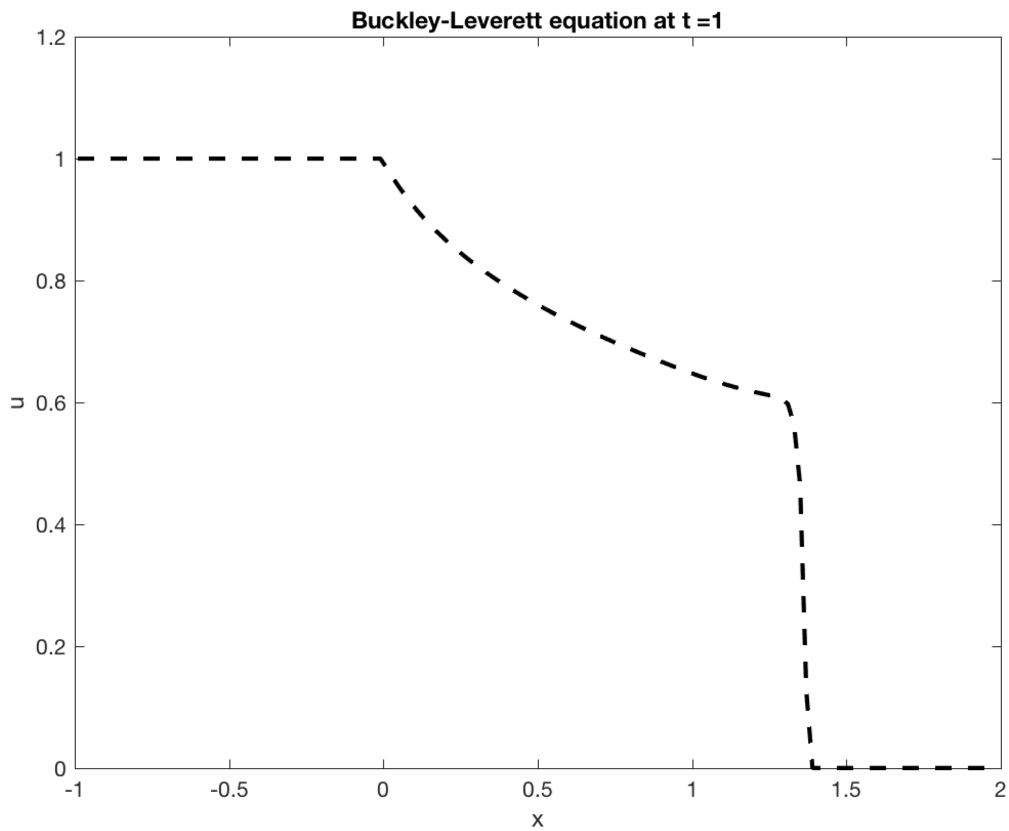
```
function [L] = muscl3(u, f, deltaX)
    nGridCells = length(u);
    L = zeros(nGridCells, 1);
    nu = 1/deltaX;

    boundaryConditions = 'zeroFlux';

    % flux array, F(i) is flux at i + 1/2 interface
    F = zeros(nGridCells,1);

    % compute fluxes at boundaries
    for j = 1:nGridCells
        % boundary conditions
        jm1 = j-1;
        if (j == 1)
            if (strcmp(boundaryConditions,'periodic'))
                jm1 = nGridCells;
            elseif (strcmp(boundaryConditions,'zeroFlux'))
                jm1 = 1;
            end
        end

        jp1 = j+1;
        jp2 = j+2;
        if (j == nGridCells)
            if (strcmp(boundaryConditions,'periodic'))
                jp1 = 1;
                jp2 = 2;
            elseif (strcmp(boundaryConditions,'zeroFlux'))
                jp1 = nGridCells;
                jp2 = nGridCells;
            end
        elseif (j == nGridCells - 1)
            if (strcmp(boundaryConditions,'periodic'))
                jp2 = 1;
            elseif (strcmp(boundaryConditions,'zeroFlux'))
                jp2 = nGridCells;
            end
        end

        uminus = -(1/6)*u(jm1) + (5/6)*u(j) + (1/3)*u(jp1);
        uplus = (1/3)*u(j) + (5/6)*u(jp1) - (1/6)*u(jp2);
        utilde = uminus - u(j);
        udoubletilde = uplus + u(jp1);

        utildemod = minmod3(utilde, u(jp1) - u(j), u(j) - u(jm1));
        udoubletildemod = minmod3(udoubletilde, u(jp2) - u(jp1), u(jp1) - u(j));

        uminusmod = u(j) + utildemod;
        uplusmod = u(jp1) - udoubletildemod;

        u1 = uminusmod - u(j);
        u2 = uplusmod - u(j);
```

```
        F(j) = f(u(j) + minmod(u1, u2));
    end

    % update solution
    for j = 1:nGridCells
        % boundary conditions
        jm1 = j-1;
        if (j == 1)
            if (strcmp(boundaryConditions,'periodic'))
                jm1 = nGridCells;
            elseif (strcmp(boundaryConditions,'zeroFlux'))
                jm1 = 1;
            end
        end

        L(j) = nu*(F(jm1) - F(j));
    end
end
```

The following script now uses this function to solve the Buckley-Leverett equation.

```
u0func = @(x) x < 0;
Iu0func = @(x) x.*(x < 0);
a = -1;
b = 2;
f = @(u) (2*u.^2)./(2*u.^2 + (1 - u).^2);
iter1 = 0;
N = 150;
tFinal = 1.0;
deltaX = (b - a)/N;
x = linspace(a+0.5*deltaX, b-0.5*deltaX, N);

u0 = zeros(N,1);
for i = 1:N
    u0(i) = (1/deltaX)*(Iu0func(x(i) + 0.5*deltaX) - Iu0func(x(i) - 0.5*deltaX));
end

deltaT = 0.5*deltaX;
nTimeSteps = ceil(tFinal/deltaT);
deltaT = tFinal/nTimeSteps;

t = 0:deltaT:tFinal;

rk3 = NumericalAnalysis.ODES.standardRK3Method;
L = @(t, u) muscl3(u, f, deltaX);
sol = rk3.solveSystem(L, t, u0);
plot(x, sol(:, end), 'k--', 'LineWidth', 2);
xlabel('x');
ylabel('u');
title(strcat('Buckley-Leverett equation at t = ', num2str(tFinal)));
saveas(gcf, 'Figures/07_01.png', 'png');
```

The following image is produced.


Buckley-Leverett equation at t =1
1.2
1
0.8
0.6
0.4
0.2
0
u
-1
-0.5
0
0.5
1
1.5
2
x