Nonconservative Discontinuous Galerkin Method for Generalized Shallow Water Equations

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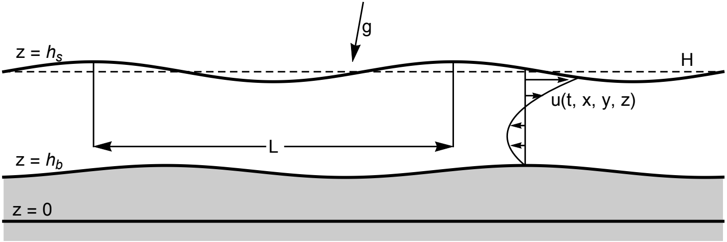
March 05, 2021

Overview

Generalized Shallow Water, (Kowalski and Torrilhon [3])



Navier Stokes Equations with a free surface

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$(h_s)_t + [u(t,x,y,h_s), v(t,x,y,h_s)]^T \cdot \nabla h_s = w(t,x,y,h_s)$$

$$(h_b)_t + [u(t,x,y,h_b), v(t,x,y,h_b)]^T \cdot \nabla h_b = w(t,x,y,h_b)$$

Polynomial Ansatz

$$
\begin{aligned}
\tilde{u}(t,x,y,\zeta) &= u_m(t,x,y) + u_d(t,x,y,\zeta) \\
&= u_m(t,x,y) + \sum_{j=1}^{N} \left(\alpha_j(t,x,y)\phi_j(\zeta)\right) \\
\tilde{v}(t,x,y,\zeta) &= v_m(t,x,y) + v_d(t,x,y,\zeta) \\
&= v_m(t,x,y) + \sum_{j=1}^{N} \left(\beta_j(t,x,y)\phi_j(\zeta)\right)
\end{aligned}
$$

Orthogonality Condition

$$\int_0^1 \phi_j(\zeta)\phi_i(\zeta)\,\mathrm{d}\zeta = 0 \quad \text{for } j \neq i$$

$$\phi_0(\zeta) = 1, \quad \phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$$

Constant Moments

Continuity Equation

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

Conservation of Momentum Equations

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x$$

$$+\left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y = -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg\left(e_x - e_z (h_b)_x\right)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y$$

$$+\left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x = -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\left(e_y - e_z (h_b)_y\right)$$

Higher Order Moments

$$
(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k\right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_y
$$

$$
= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right)
$$

$$
(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y
$$

$$
= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)
$$

Example Systems

1D model with h_b constant, $e_x = e_y = 0$, and $e_z = 1$

Constant System

$$\begin{bmatrix} h \\ hu_m \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = -\frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u_m \pm \sqrt{gh}$

Linear System, $\tilde{u} = u_m + \alpha_1\phi_1$

$$\begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{bmatrix}_x = Q\begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{bmatrix} \quad \mathbf{s} = \frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{bmatrix}$$

Quasilinear Matrix Eigenvalues, $u_m \pm \sqrt{gh + \alpha_1^2}, u_m$

Example Systems

1 dimensional with h_b constant, $e_x = e_y = 0$, and $e_z = 1$
Quadratic Vertical Profile, $\tilde{u} = u + \alpha_1\phi_1 + \alpha_2\phi_2$

$$\begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u + \frac{\alpha_2}{7} \end{bmatrix}, P = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u + \alpha_1 + \alpha_2 \\ 3\left(u + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{bmatrix}$$

Quasilinear Matrix Eigenvalues, $u \pm c\sqrt{gh}$

Nonconservative Products, (Dal Maso, Lefloch, and Murat [2])

Model Equation

$$\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + g_i(\mathbf{q})\mathbf{q}_{x_i} = \mathbf{s}(\mathbf{q}) \quad \text{for } (\mathbf{x}, t) \in \Omega \times [0, T]$$

Traditionally searching for weak solutions, find $\mathbf{q}$ such that

$$\int_0^T \int_\Omega (\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + g_i(\mathbf{q})\mathbf{q}_{x_i}) v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_0^T \int_\Omega \mathbf{s}(\mathbf{q}) v \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

for all $v \in C_0^1(\Omega \times [0, T])$

Regularization Paths

Consider Lipschitz continuous paths, $\psi : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, that satisfy the following properties.

1. $\forall \mathbf{q}_L, \mathbf{q}_R \in \mathbb{R}^p$, $\psi(0, \mathbf{q}_L, \mathbf{q}_R) = \mathbf{q}_L$ and $\psi(1, \mathbf{q}_L, \mathbf{q}_R) = \mathbf{q}_R$
2. $\exists k > 0$, $\forall \mathbf{q}_L, \mathbf{q}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, $\left|\frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R)\right| \leq k|\mathbf{q}_L - \mathbf{q}_R|$ elementwise
3. $\exists k > 0$, $\forall \mathbf{q}_L, \mathbf{q}_R, \mathbf{u}_L, \mathbf{u}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, elementwise

$$\left|\frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R) - \frac{\partial \psi}{\partial s}(s, \mathbf{u}_L, \mathbf{u}_R)\right| \leq k(|\mathbf{q}_L - \mathbf{u}_L| + |\mathbf{q}_R - \mathbf{u}_R|)$$

Let $u = u_0 + H(x - x_0)(u_1 - u_0)$, then regularize

$$u^{\varepsilon}(x) = \begin{cases} u_0 & x < x_0 - \varepsilon \\ \psi\left(\frac{x - x_0 + \varepsilon}{2\varepsilon}, u_0, u_1\right) & x_0 - \varepsilon < x < x_0 + \varepsilon \\ u_1 & x > x_0 + \varepsilon \end{cases}$$

Nonconservative Product Definition

Let $\mathbf{q} \in BV(\Omega \to \mathbb{R}^p)$ and $g \in C^1(\mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p)$, then μ is a Borel measure.

1. If $\mathbf{q}$ is continuous on a Borel set $B \subset \Omega$, then

$$\mu(B) = \int_B g(\mathbf{q}) \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}x} \, \mathrm{d}x$$

2. If $\mathbf{q}$ is discontinuous at a point $x_0 \in \Omega$, then

$$\mu(x_0) = \int_0^1 g(\psi(s; \mathbf{q}(x_0^-), \mathbf{q}(x_0^+))) \frac{\partial \psi}{\partial s}(s; \mathbf{q}(x_0^-), \mathbf{q}(x_0^+)) \, \mathrm{d}s$$

Define

$$\mu = \left[g(\mathbf{q}) \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}x} \right]_\psi$$

Nonconservative Products

If there exists $\mathbf{f}(\mathbf{q})$ such that $\mathbf{f}'(\mathbf{q}) = g(\mathbf{q})$, then

$$[g(\mathbf{q})\mathbf{q}_x]_\psi = \mathbf{f}(\mathbf{q})_x$$

or

$$\int_0^1 \mathbf{f}'(\psi(s, \mathbf{q}_L, \mathbf{q}_R))\frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R)\,\mathrm{d}s = \mathbf{f}(\mathbf{q}_L) - \mathbf{f}(\mathbf{q}_R)$$

Find weak solution $\mathbf{q}$ such that

$$\int_0^T \int_\Omega \mathbf{q} v_t \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_\Omega \mathbf{f}(\mathbf{q})\nabla \cdot v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^T \int_\Omega [g_i(\mathbf{q})\mathbf{q}_{x_i}]_\psi v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{s}(\mathbf{q}) v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for all $v \in C_0^1(\Omega \times [0, T])$.

Nonconservative DG Formulation

Semi Discrete formulation
find $\mathbf{q} \in V_h = \left\{ v \in L^1(\Omega) \middle| \; v|_{K_j} \in \mathbb{P}^M(K_j) \right\}$ such that

$$\int_\Omega v_h \mathbf{q}_t \, \mathrm{d}x + \int_\Omega v_h \nabla \cdot \mathbf{f}(\mathbf{q}) \, \mathrm{d}x + \int_\Omega v_h [g_i(\mathbf{q}) \mathbf{q}_{x_i}]_\psi = \int_\Omega v_h \mathbf{s}(\mathbf{q}) \, \mathrm{d}x$$

or

$$\sum_j \left(\int_{K_j} v_h \mathbf{q}_t \, \mathrm{d}x \right) - \sum_j \left(\int_{K_j} \nabla \cdot v_h \mathbf{f}(\mathbf{q}) \, \mathrm{d}x \right)$$
$$+ \sum_I \left(\left(v_h^L - v_h^R \right) \hat{\mathbf{f}} \right) + \sum_j \left(\int_{K_j} v_h g_i(\mathbf{q}) \mathbf{q}_{x_i} \, \mathrm{d}x \right)$$
$$+ \sum_I \left(\int_I \hat{v}_h \left(\int_0^1 g\left(\psi\left(s, \mathbf{q}_h^L, \mathbf{q}_h^R \right) \right) \psi_s\left(s, \mathbf{q}_h^L, \mathbf{q}_h^R \right) \mathrm{d}s \, \mathbf{n} \right) \mathrm{d}I \right) = \int_\Omega v_h \mathbf{s}(\mathbf{q}) \, \mathrm{d}x$$

for all $v_h \in V_h$.

Nonconservative DG Formulation, (Rhebergen, Bokhove, and Vegt [7])

Test Function Flux,

$$\hat{v}_h = \frac{1}{2}\left(v_h^+ + v_h^-\right)$$

consistent with conservative DG formulation when $\mathbf{h}'(\mathbf{q}) = g(\mathbf{q})$.

Local Lax-Friedrichs Numerical Flux

$$\lambda = \max_{q \in \left[\mathbf{q}_h^-, \mathbf{q}_h^+\right]} \{\rho(\mathbf{f}'(\mathbf{q}) + g(\mathbf{q}))\}$$

$$\hat{\mathbf{f}} = \frac{1}{2}\left(\mathbf{f}\left(\mathbf{q}_h^+\right) + \mathbf{f}\left(\mathbf{q}_h^-\right)\right) - \frac{1}{2}\lambda\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right)$$

Manufactured Solution

Shallow Water Equations, constant vertical velocity profile

	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	0.226	—	8.57×10^{-3}	—	1.67×10^{-4}	—
40	0.117	0.96	2.17×10^{-3}	1.98	2.07×10^{-5}	3.02
80	0.058	1.00	5.40×10^{-4}	2.01	2.57×10^{-6}	3.01
160	0.028	1.06	1.35×10^{-4}	2.00	3.21×10^{-7}	3.00
320	0.014	0.99	3.37×10^{-5}	2.00	4.01×10^{-8}	3.00

	4th Order		5th Order	
n	error	order	error	order
20	3.172×10^{-6}	—	7.606×10^{-8}	0.00
40	1.982×10^{-7}	4.00	2.380×10^{-9}	5.00
80	1.240×10^{-8}	4.00	7.713×10^{-11}	4.95
160	7.755×10^{-10}	4.00	4.035×10^{-11}	0.93
320	4.849×10^{-11}	4.00	8.085×10^{-11}	-1.00

Manufactured Solution

One moment, linear vertical velocity profile

	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	2.53×10^{-1}	—	9.97×10^{-3}	—	1.71×10^{-3}	—
40	1.30×10^{-1}	0.96	2.52×10^{-3}	1.98	3.85×10^{-4}	2.15
80	6.47×10^{-2}	1.00	6.28×10^{-4}	2.00	6.13×10^{-5}	2.65
160	3.13×10^{-2}	1.05	1.57×10^{-4}	2.00	9.09×10^{-6}	2.75
320	1.58×10^{-2}	0.99	3.92×10^{-5}	2.00	1.73×10^{-6}	2.39

	4th Order		5th Order	
n	error	order	error	order
20	1.14×10^{-4}	—	2.68×10^{-5}	—
40	1.74×10^{-5}	2.72	8.01×10^{-7}	5.06
80	7.50×10^{-7}	4.53	1.53×10^{-8}	5.71
160	1.25×10^{-7}	2.59	4.04×10^{-10}	5.25
320	8.79×10^{-9}	3.83	8.40×10^{-11}	2.27

Manufactured Solution

Two moments, quadratic vertical velocity profile

	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	2.778×10^{-1}	—	1.141×10^{-2}	—	5.350×10^{-3}	—
40	1.424×10^{-1}	0.96	2.884×10^{-3}	1.98	6.466×10^{-4}	3.05
80	7.121×10^{-2}	1.00	7.191×10^{-4}	2.00	7.836×10^{-5}	3.04
160	3.454×10^{-2}	1.04	1.797×10^{-4}	2.00	1.270×10^{-5}	2.63
320	1.740×10^{-2}	0.99	4.493×10^{-5}	2.00	2.546×10^{-6}	2.32

	4th Order		5th Order	
n	error	order	error	order
20	3.688×10^{-4}	—	5.194×10^{-5}	—
40	2.461×10^{-5}	3.91	1.121×10^{-6}	5.53
80	1.403×10^{-6}	4.13	1.934×10^{-8}	5.86
160	1.144×10^{-7}	3.62	5.859×10^{-10}	5.04
320	1.092×10^{-8}	3.39	8.791×10^{-11}	2.74

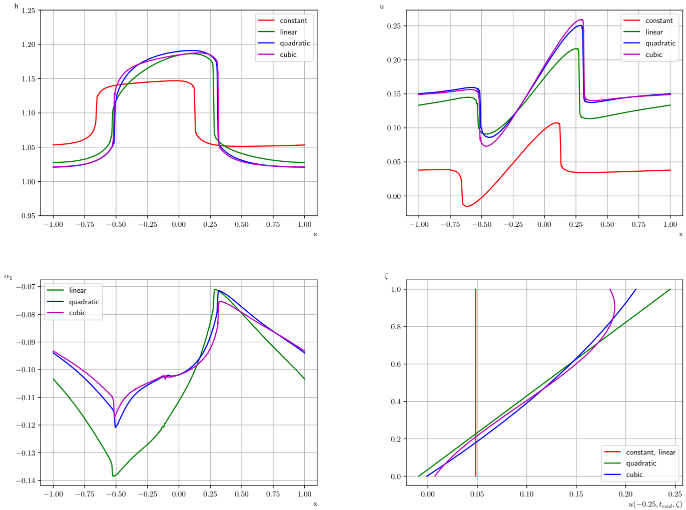
Manufactured Solution

Three moments, cubic vertical velocity profile

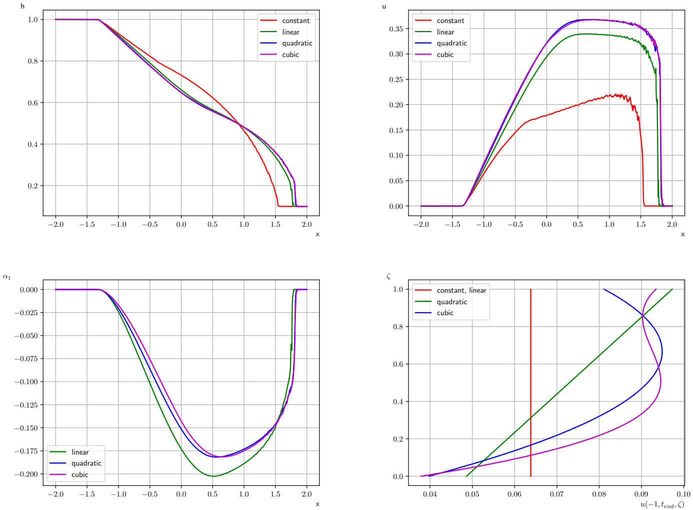
	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	3.024×10^{-1}	—	1.300×10^{-2}	—	7.015×10^{-3}	—
40	1.556×10^{-1}	0.96	3.283×10^{-3}	1.99	6.992×10^{-4}	3.33
80	7.808×10^{-2}	0.99	8.188×10^{-4}	2.00	1.183×10^{-4}	2.56
160	3.802×10^{-2}	1.04	2.046×10^{-4}	2.00	2.545×10^{-5}	2.22
320	1.916×10^{-2}	0.99	5.117×10^{-5}	2.00	5.110×10^{-6}	2.32

	4th Order		5th Order	
n	error	order	error	order
20	3.167×10^{-4}	—	5.571×10^{-5}	—
40	2.384×10^{-5}	3.73	1.099×10^{-6}	5.66
80	2.509×10^{-6}	3.25	2.639×10^{-8}	5.38
160	3.168×10^{-7}	2.99	1.371×10^{-9}	4.27
320	4.675×10^{-8}	2.76	1.171×10^{-10}	3.55

Effect of Higher Moments



Effect of Higher Moments



Conclusions

Results

- Discontinuous Galerkin Method for Generalized Shallow Water Equations
- High Order Method
- Properly Discretized Nonconservative Product

Future Work

- Two Dimensional Meshes
- Icosahedral Spherical Mesh
- Shallow Water test cases on the sphere

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