Introduction to Discontinuous Galerkin Methods

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Goal

- Numerically Solve

$$u_t + f(u)_x = 0$$
$$x \in \Omega \subset \mathbb{R}^d \qquad t \in \mathbb{R}^+$$

- Weak Solution Find u such that for any test function v

$$\int_0^\infty \int_{\mathbb{R}^d} u_t v + f(u)_x v \, \mathrm{d}x \, \mathrm{d}t = 0$$

Goal

- Numerically Solve

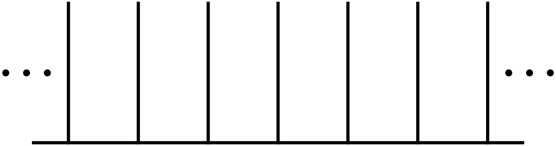
$$u_t + f(u)_x = 0$$
$$x \in [a, b] \quad t \in \mathbb{R}^+$$

- Weak Solution Find u such that for any test function v

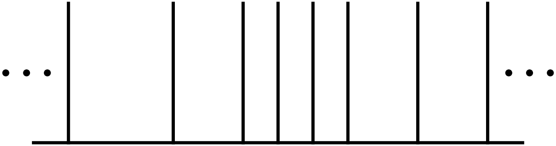
$$\int_0^\infty \int_a^b u_t v + f(u)_x v \,\mathrm{d}x\,\mathrm{d}t = 0$$

Generate Mesh

- 1D Uniform Grid

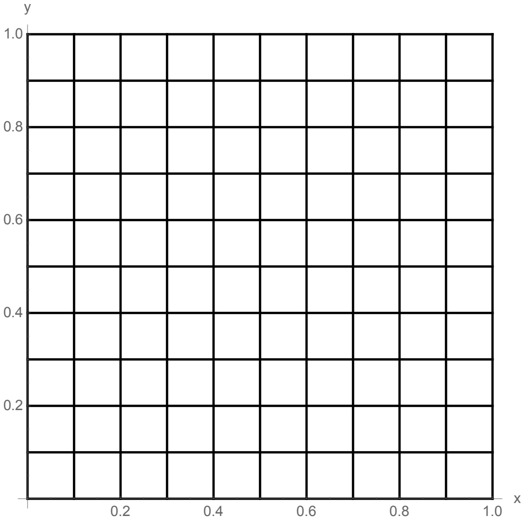


- 1D Variable Grid



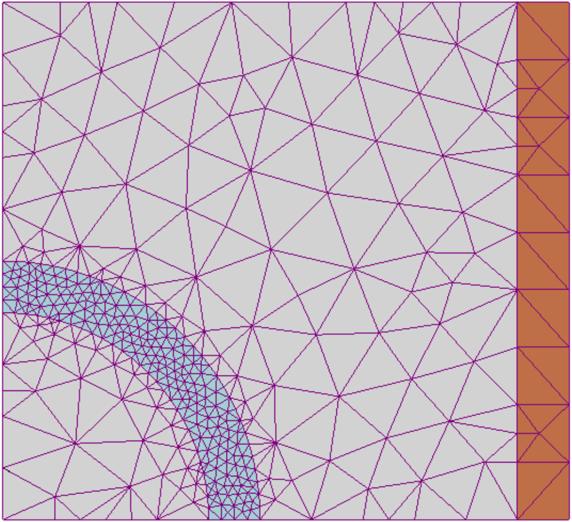
Generate Mesh

- 2D Uniform Grid



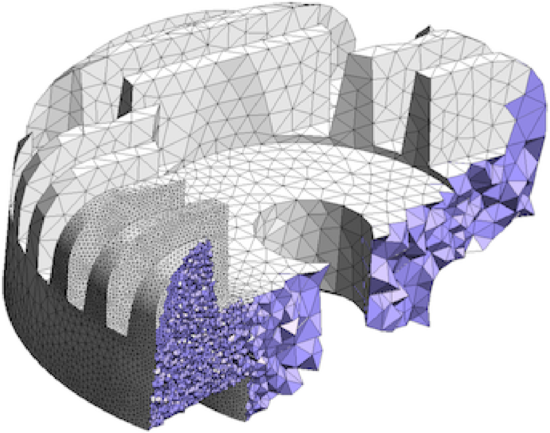
Generate Mesh

- 2D Unstructured Triangulation



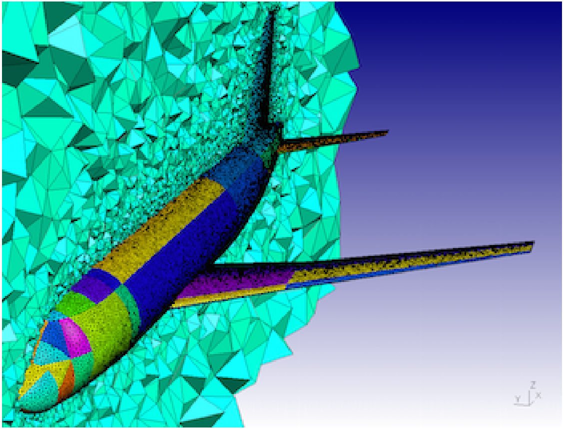
Generate Mesh

- 3D Unstructured Triangulation



Generate Mesh

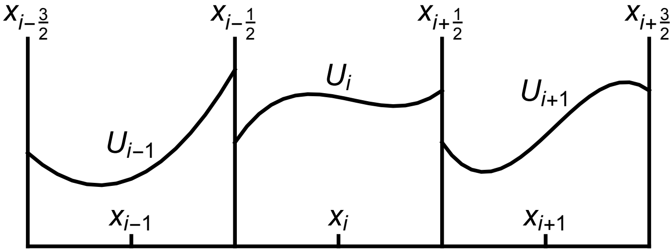
- 3D Unstructured Triangulation



Solution Space

- Label each element in mesh as I_j, $j = 1, \dots N$
- Discontinuous Galerkin Finite Element Space

$$V^M = \left\{ u : \left. u \right|_{I_j} \in P^M(I_j), j = 1, \dots, N \right\}$$



The Method

- At a given time find $u \in V^M$ such that for all $v \in V^M$ and for all $j = 1, \dots, N$.

$$\int_{I_j} u_t v \,\mathrm{d}x + \int_{I_j} f(u)_x v \,\mathrm{d}x = 0$$

- Integrate by parts

$$\int_{I_j} u_t v \,\mathrm{d}x + \hat{f}_{j+1/2} v_{j+1/2}^{-} - \hat{f}_{j-1/2} v_{j-1/2}^{+} - \int_{I_j} f(u) v_x \,\mathrm{d}x = 0$$

- $\hat{f}$ is called the numerical flux
  - Consistent: $\hat{f}\, u, u$

Numerical Flux

- Approximating $f(u(x_{j+1/2}, t))$

$$\hat{f}_{j+1/2} = \hat{f}(u^-_{j+1/2}, u^+_{j+1/2})$$

- Properties
  - Consistent: $\hat{f}(u, u) = f(u)$
  - Lipschitiz Continuous with respect to both arguments
  - Monotone: non-decreasing in first argument, non-increasing with second argument
- Examples
  - Godunov

$$\hat{f}_{j+1/2} = \begin{cases} \min_{u \in [u^-, u^+]}\{f(u)\} & u^- < u^+ \\ \max_{u \in [u^+, u^-]}\{f(u)\} & u^- \geq u^+ \end{cases}$$

  - Rusanov/Local Lax-Friedrichs

$$\hat{f}(u^-, u^+) = \frac{1}{2}\left(f(u^-) + f(u^+) - \alpha(u^+ - u^-)\right)$$

where $\alpha = \max_u\{|f'(u)|\}$.

Implementation

- Linear transformation $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$

$$x = \frac{\Delta x}{2}\xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

- Pick a basis for $P^M([-1, 1])$, e.g. Legendre polynomials

$$\frac{1}{2}\int_{-1}^{1} \phi^j(\xi)\phi^k(\xi) \, \mathrm{d}\xi = \delta_{jk}$$

$$\phi^1(\xi) = 1 \qquad \phi^2(\xi) = \xi \qquad \phi^3(\xi) = \sqrt{5}/2(3\xi^2 - 1)$$

Implementation

- Galerkin Expansion

$$u\big|_{I_j} = \sum_{k=1}^{M} \left(U_k \phi^k(\xi)\right)$$

- Let $v = \phi^j$

$$\int_{-1}^{1} u_t \phi^j \, \mathrm{d}\xi + \frac{2}{\Delta x}\left(\hat{f}_{j+1/2}\phi^j(1) - \hat{f}_{j-1/2}\phi^j(-1)\right) - \frac{2}{\Delta x}\int_{-1}^{1} f(u)\phi^j_\xi \, \mathrm{d}\xi = 0$$

- Using the orthonormality

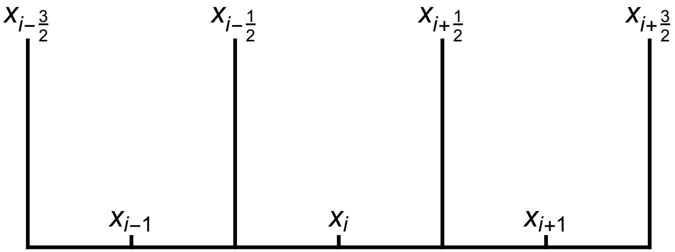
$$(U_k)_t = \frac{1}{\Delta x}\int_{-1}^{1} f(u)\phi^j_\xi \, \mathrm{d}\xi - \frac{1}{\Delta x}\left(\hat{f}_{j+1/2}\phi^j(1) - \hat{f}_{j-1/2}\phi^j(-1)\right)$$

Introduction to Discontinuous Galerkin

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $V_i = \left[x_{i-1/2}, x_{i+1/2}\right]$
- $\Delta x_i = x_{i+1/2} - x_{i-1/2}$
- $x_i = \frac{x_{i+1/2}+x_{i-1/2}}{2}$.

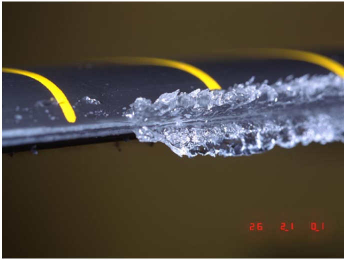


Ongoing Research

- Create DG methods for certain types of equations
- Strong Stability Preserving (SSP)
- Entropy Solutions
- Positivity Preserving
- Slope/Oscillation Limiting

Motivation

- Aircraft Icing
- Runback



- Industrial Coating
- Paint Drying

Model Equations

- Navier-Stokes Equation

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = \frac{4}{3Re} u_{xx}$$

$$E_t + (u(E + p))_x = \frac{1}{Re}\left(\frac{2}{3}\left(u^2\right)_{xx} + \frac{\gamma}{(\gamma - 1)Pr}\left(\frac{p}{\rho}\right)_{xx}\right)$$

- Asymptotic Limit, $\rho << L$
- Thin-Film Equation - 1D with u as fluid height.

$$u_t + \left(f(x,t)u^2 - g(x,t)u^3\right)_x = -\left(h(x,t)u^3 u_{xxx}\right)_x$$

Current Model

- Simplified Expression

$$u_t + \left(u^2 - u^3\right)_x = -\left(u^3 u_{xxx}\right)_x$$

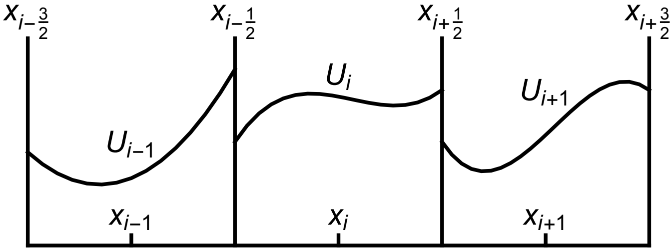
- Operator Splitting

$$\begin{aligned} u_t + \left(u^2 - u^3\right)_x &= 0 \\ u_t + \left(u^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

Numerical Solutions

- Use canonical variable $\xi \in [-1, 1]$
- Let $\left\{\phi^k(\xi)\right\}$ be the Legendre polynomials.
- Solution of order M on each cell

$$u\big|_{x \in V_i} \approx U_i = \sum_{k=1}^{M} U_i^k \phi^k(\xi)$$



Convection

- Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_\xi = 0$$
$$f(u) = u^2 - u^3$$

- Weak Form

$$\int_{-1}^{1} \left(u_t \phi(\xi) + \frac{2}{\Delta x} f(u)_\xi \phi(\xi) \right) \mathrm{d}\xi = 0$$

- Runge-Kutta Discontinuous Galerkin

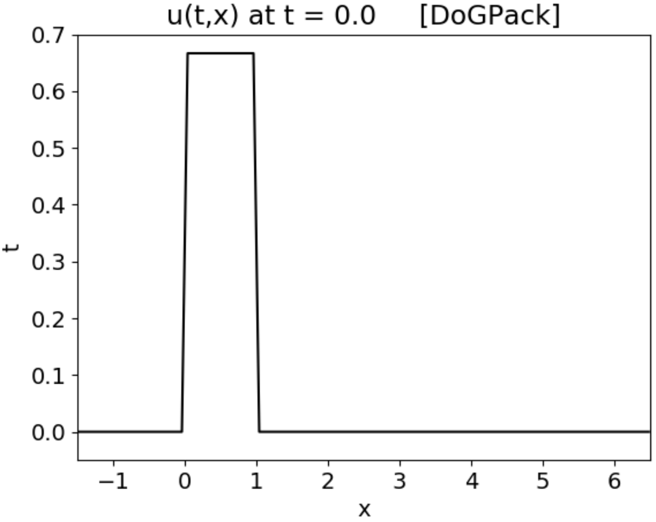
$$\dot{U}_i^\ell = \frac{1}{\Delta x} \int_{-1}^{1} f(U_i) \phi_\xi^\ell \, \mathrm{d}\xi - \frac{1}{\Delta x} \left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2} \right)$$

- Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^\ell(1)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Numerical Example - Square Wave



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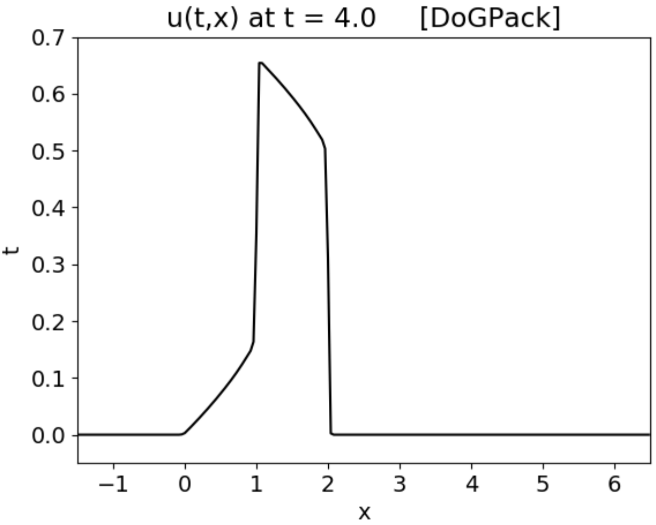

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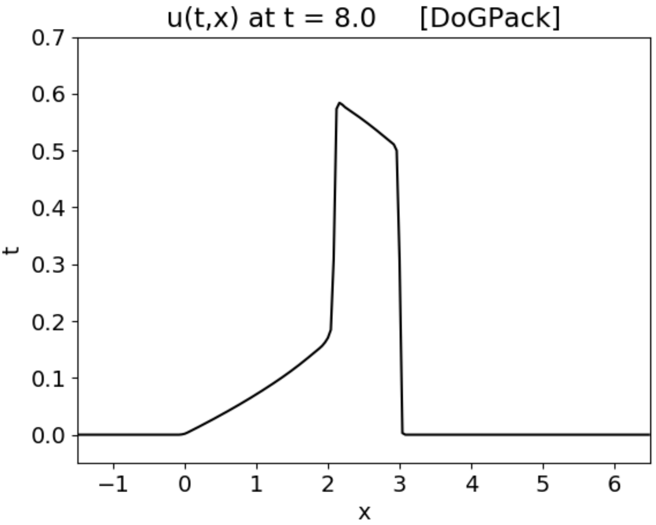

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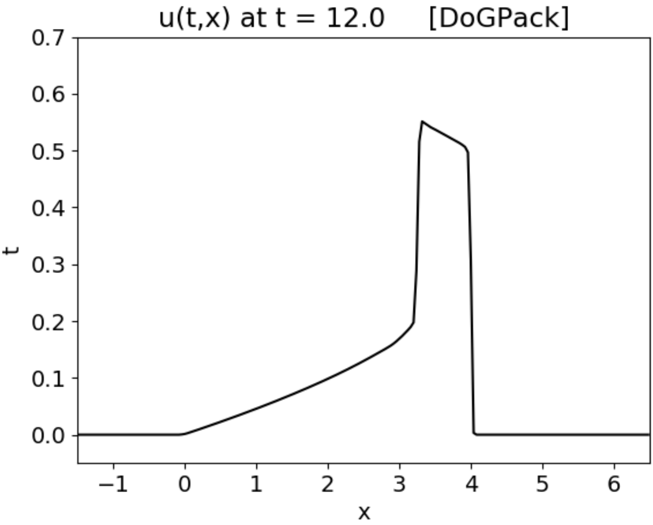

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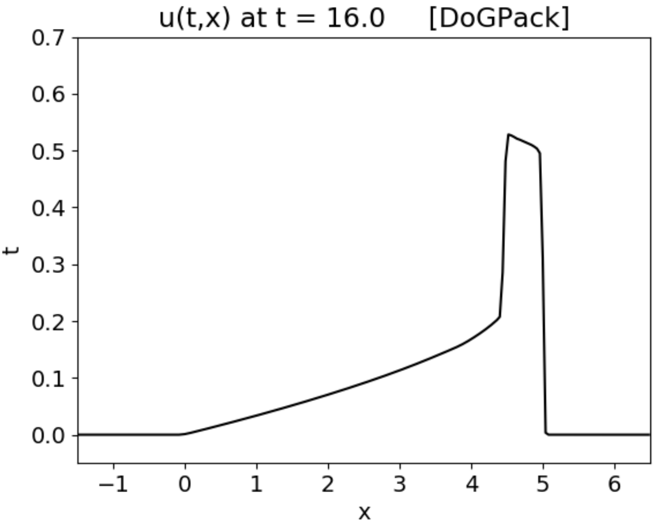

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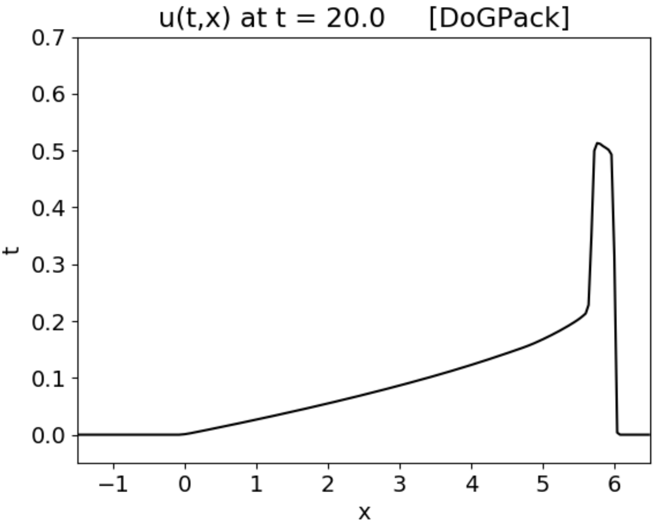


Hyper-Diffusion

- Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4}\left(u^3 u_{\xi\xi\xi}\right)_\xi = 0$$

- Local Discontinuous Galerkin (LDG)

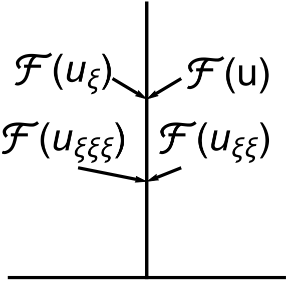
$$\begin{aligned} q &= \frac{2}{\Delta x} u_\xi \\ r &= \frac{2}{\Delta x} q_\xi \\ s &= \frac{2}{\Delta x} u^3 r_\xi \\ u_t &= -\frac{2}{\Delta x} s_\xi \end{aligned}$$

Local Discontinuous Galerkin

$$
\begin{aligned}
\eta(\xi) &= \left(U_i^n\right)^3 \\
Q_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} U_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(U)_{i+1/2}^\ell + \mathcal{F}(U)_{i-1/2}^\ell\right) \\
R_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} Q_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(Q)_{i+1/2}^\ell + \mathcal{F}(Q)_{i-1/2}^\ell\right) \\
S_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} (R_i)_\xi \eta(\xi) \phi^\ell \,\mathrm{d}\xi\right) \\
&\quad + \frac{1}{\Delta x}\left(\mathcal{F}(\eta)_{i+1/2}\mathcal{F}(R)_{i+1/2}^\ell - \mathcal{F}(\eta)_{i-1/2}\mathcal{F}(R)_{i-1/2}^\ell\right) \\
\dot{U}_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} S_i \phi_\xi^\ell \,\mathrm{d}\xi - \mathcal{F}(S)_{i+1/2}^\ell + \mathcal{F}(S)_{i-1/2}^\ell\right)
\end{aligned}
$$

Local Discontinuous Galerkin

$$\mathcal{F}(\eta)_{i+1/2} = \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1))$$
$$\mathcal{F}(\eta)_{i-1/2} = \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1))$$
$$\mathcal{F}(*)^{\ell}_{i+1/2} = \phi^{\ell}(1)*_{i+1/2}$$

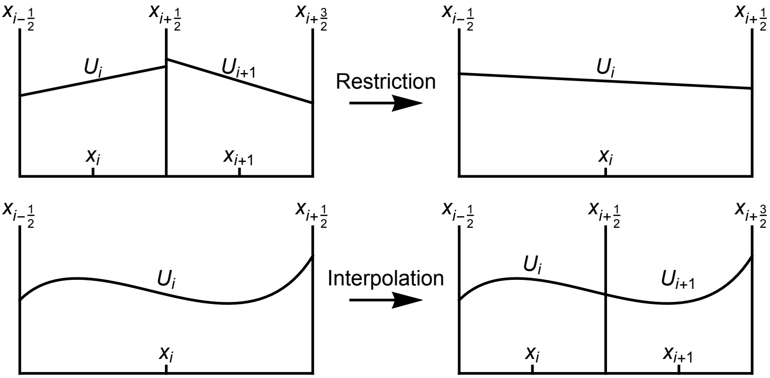


Local Discontinuous Galerkin

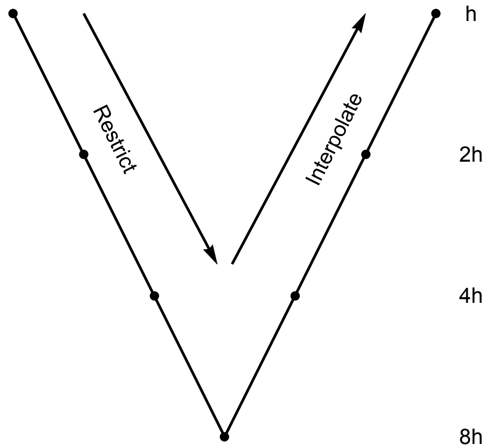
- Explicit SSP Runge Kutta
  - Severe time step restriction
  - $\Delta t \sim \Delta x^4$
  - $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
  - $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
  - Linear System Solver
  - Stabilized BiConjugate Gradient
  - MultiGrid Solver

Multigrid Solver

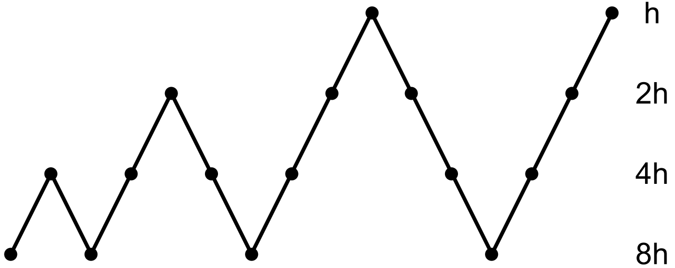
- Relaxation e.g. Jacobi Relaxation



V-Cycle



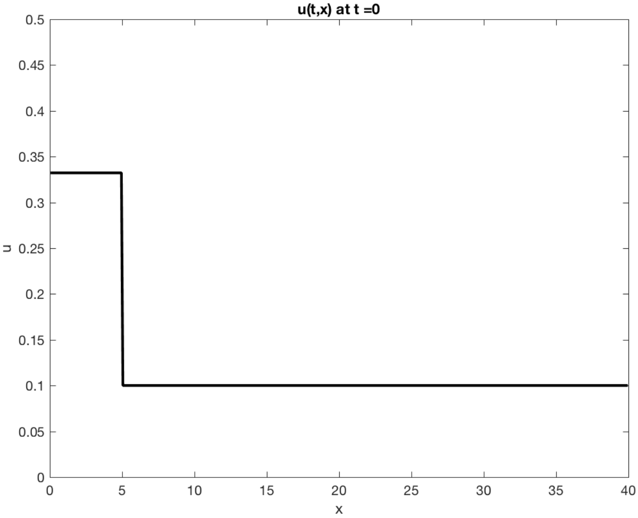
Multigrid Solver



Operator Splitting

- Strang Splitting
  - 1 time step
    - 1/2 time step for convection
    - 1 time step for hyper-diffusion
    - 1/2 time step for convection
  - Second order splitting

Numerical Results - Riemann Problem



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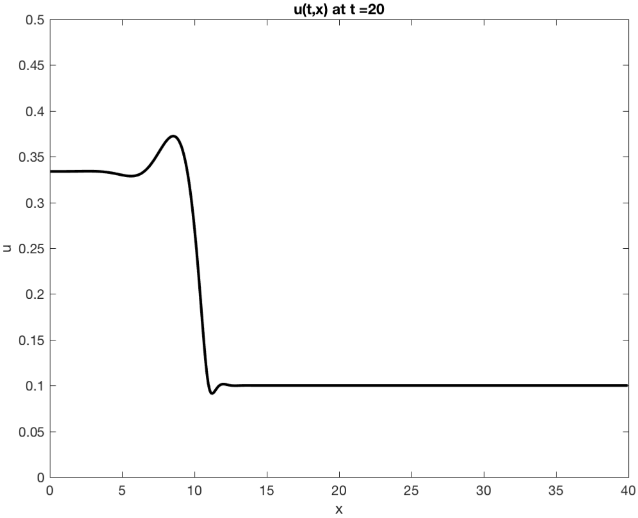

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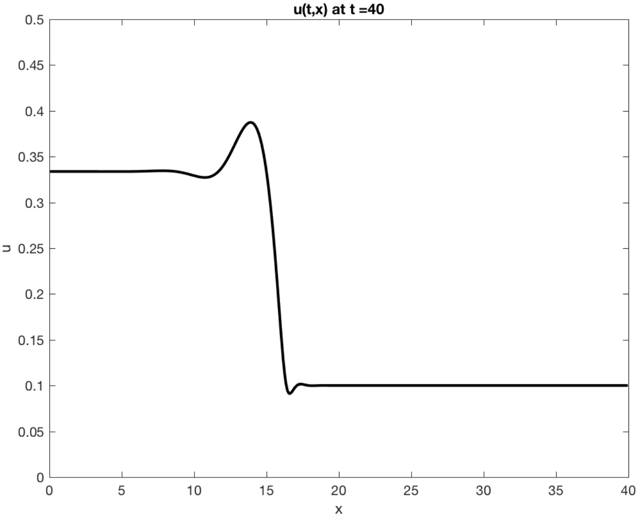

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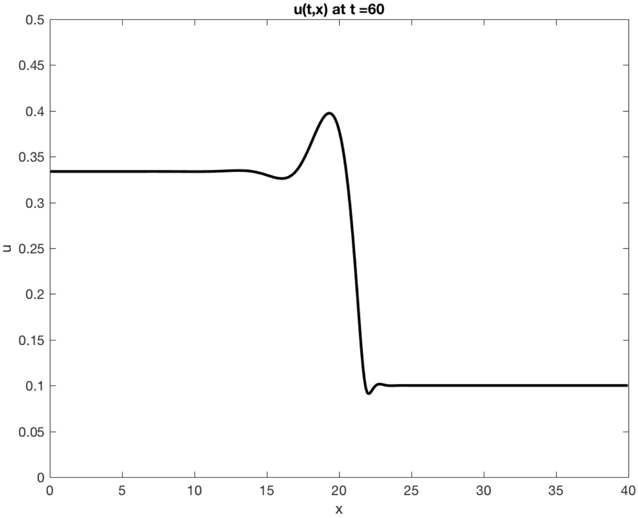

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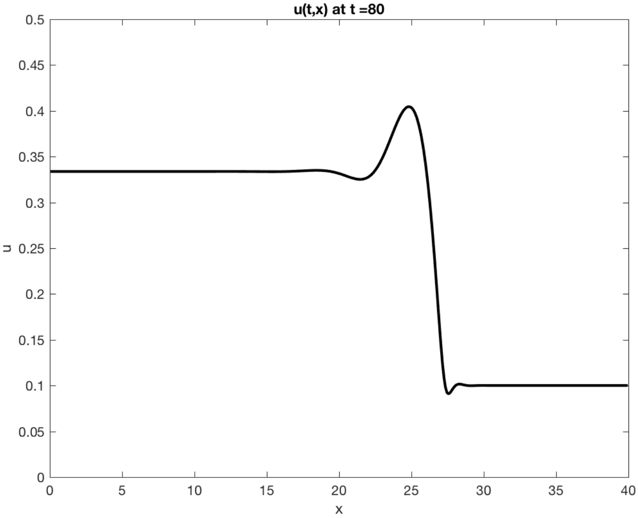

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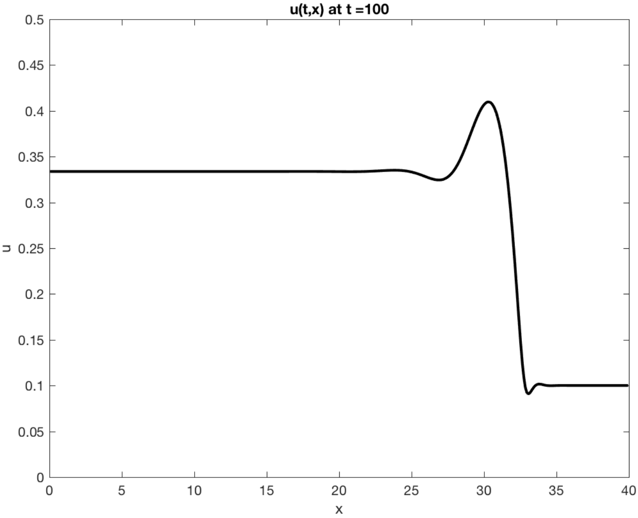

Future Work

- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

Conclusion

- Thanks
  - James Rossmanith
  - Alric Rothmayer
- Questions?

Bibliography

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