

# Derivation of Shallow Water Moment Equations in Spherical Coordinates

First I will define the transformation from cartesian  $\mathbf{x} = [x, y, z]$  to spherical coordinates,  $\mathbf{r} = [r, \theta, \phi]$

$$r = s_1(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\theta = s_2(x, y, z) = \arctan\left(\frac{y}{x}\right) \quad (2)$$

$$\phi = s_3(x, y, z) = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \quad (3)$$

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}(x, y, z) = [s_1(x, y, z), s_2(x, y, z), s_3(x, y, z)] \quad (4)$$

$$x = c_1(r, \theta, \phi) = r \cos(\theta) \sin(\phi) \quad (5)$$

$$y = c_2(r, \theta, \phi) = r \sin(\theta) \sin(\phi) \quad (6)$$

$$z = c_3(r, \theta, \phi) = r \cos(\phi) \quad (7)$$

$$\mathbf{c}(\mathbf{r}) = \mathbf{c}(r, \theta, \phi) = [c_1(r, \theta, \phi), c_2(r, \theta, \phi), c_3(r, \theta, \phi)] \quad (8)$$

The projections from cartesian to spherical coordinates can be computed using the following unit vectors

$$\hat{r} = \cos(\theta) \sin(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + \cos(\phi) \hat{z} \quad (9)$$

$$\hat{\theta} = -\sin(\theta) \hat{x} + \cos(\theta) \hat{y} \quad (10)$$

$$\hat{\phi} = \cos(\theta) \cos(\phi) \hat{x} + \sin(\theta) \cos(\phi) \hat{y} - \sin(\phi) \hat{z} \quad (11)$$

$$\hat{x} = \cos(\theta) \sin(\phi) \hat{r} - \sin(\theta) \hat{\theta} + \cos(\theta) \cos(\phi) \hat{\phi} \quad (12)$$

$$\hat{y} = \sin(\theta) \sin(\phi) \hat{r} + \cos(\theta) \hat{\theta} + \sin(\theta) \cos(\phi) \hat{\phi} \quad (13)$$

$$\hat{z} = \cos(\phi) \hat{r} - \sin(\phi) \hat{\phi} \quad (14)$$

$$(15)$$

The partial derivatives of the transformation to spherical coordinates are needed, when considering differential equations.

$$s_{1,x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2x = \cos(\theta) \sin(\phi) \quad (16)$$

$$s_{1,y} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2y = \sin(\theta) \sin(\phi) \quad (17)$$

$$s_{1,z} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2z = \cos(\phi) \quad (18)$$

$$(19)$$

$$s_{2,x} = \frac{1}{1 + \frac{y^2}{x^2}} \left( -\frac{y}{x^2} \right) \quad (20)$$

$$= \frac{1}{1 + \frac{r^2 \sin^2(\theta) \sin^2(\phi)}{r^2 \cos^2(\theta) \sin^2(\phi)}} \left( -\frac{r \sin(\theta) \sin(\phi)}{r^2 \cos^2(\theta) \sin^2(\phi)} \right) \quad (21)$$

$$= \frac{1}{1 + \frac{\sin^2(\theta)}{\cos^2(\theta)}} \left( -\frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \right) \quad (22)$$

$$= -\frac{1}{1 + \frac{1-\cos^2(\theta)}{\cos^2(\theta)}} \frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \quad (23)$$

$$= -\cos^2(\theta) \frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \quad (24)$$

$$= -\frac{\sin(\theta)}{r \sin(\phi)} \quad (25)$$

$$s_{2,y} = \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right) \quad (26)$$

$$= \cos^2(\theta) \frac{1}{r \cos(\theta) \sin(\phi)} \quad (27)$$

$$= \frac{\cos(\theta)}{r \sin(\phi)} \quad (28)$$

$$s_{2,z} = 0 \quad (29)$$

$$(30)$$

$$s_{3,x} = \frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{1}{2} \frac{2x}{z \sqrt{x^2+y^2}} \quad (31)$$

$$= \frac{1}{1 + \frac{r^2 \cos^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \sin^2(\phi)}{r^2 \cos^2(\phi)}} \frac{r \cos(\theta) \sin(\phi)}{r \cos(\phi) \sqrt{r^2 \cos^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \sin^2(\phi)}} \quad (32)$$

$$= \frac{1}{1 + \frac{\sin^2(\phi)}{\cos^2(\phi)}} \frac{\cos(\theta)}{r \cos(\phi)} \quad (33)$$

$$= \frac{1}{1 + \frac{1-\cos^2(\phi)}{\cos^2(\phi)}} \frac{\cos(\theta)}{r \cos(\phi)} \quad (34)$$

$$= \cos^2(\phi) \frac{\cos(\theta)}{r \cos(\phi)} \quad (35)$$

$$= \frac{\cos(\theta) \cos(\phi)}{r} \quad (36)$$

$$s_{3,y} = \frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{1}{2} \frac{2y}{z \sqrt{x^2+y^2}} \quad (37)$$

$$= \cos^2(\phi) \frac{r \sin(\theta) \sin(\phi)}{r \cos(\phi) r \sin(\phi)} \quad (38)$$

$$= \frac{\sin(\theta) \cos(\phi)}{r} \quad (39)$$

$$s_{3,z} = -\frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{\sqrt{x^2+y^2}}{z^2} \quad (40)$$

$$= -\cos^2(\phi) \frac{r \sin(\phi)}{r^2 \cos^2(\phi)} \quad (41)$$

$$= -\frac{\sin(\phi)}{r} \quad (42)$$

$$(43)$$

The derivative with respect to a cartesian coordinate, can be expressed in terms of the spherical coordinate

derivatives using the chain rule. Consider a function of spherical coordinates,  $f(r, \theta, \phi)$ , then

$$\frac{\partial}{\partial t}(f(\mathbf{s}(\mathbf{x}))) = \frac{\partial f}{\partial r} \frac{\partial s_1}{\partial t} + \frac{\partial f}{\partial \theta} \frac{\partial s_2}{\partial t} + \frac{\partial f}{\partial \phi} \frac{\partial s_3}{\partial t} \quad (44)$$

where  $t = x, y, \text{ or } z$

Now let's consider the Navier Stokes equation. The velocities in cartesian coordinates are given by  $[u, v, w] = \mathbf{u}$ , then the cartesian Navier Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (45)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + g_x \quad (46)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(vw) = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + g_y \quad (47)$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(w^2) = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + g_z \quad (48)$$

$$(49)$$

In spherical coordinates we represent the velocities as  $[u_r, u_\theta, u_\phi] = \mathbf{u}_r$ . Using the unit vectors we can write these velocities in terms of the cartesian velocities.

$$u_r = \cos(\theta) \sin(\phi) u + \sin(\theta) \sin(\phi) v + \cos(\phi) w \quad (50)$$

$$u_\theta = -\sin(\theta) u + \cos(\theta) w \quad (51)$$

$$u_\phi = \cos(\theta) \cos(\phi) u + \sin(\theta) \cos(\phi) v - \sin(\phi) w \quad (52)$$

We can also write the cartesian velocities in terms of the spherical velocities.

$$u = \cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi \quad (53)$$

$$v = \sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi \quad (54)$$

$$w = \cos(\phi) u_r - \sin(\phi) u_\phi \quad (55)$$

Now we can express the continuity equation in spherical coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (56)$$

$$\frac{\partial}{\partial x}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \quad (57)$$

$$+ \frac{\partial}{\partial y}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \quad (58)$$

$$+ \frac{\partial}{\partial z}(\cos(\phi) u_r - \sin(\phi) u_\phi) = 0 \quad (59)$$

$$(60)$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial x} \quad (61)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial x} \quad (62)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial x} \quad (63)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial y} \quad (64)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial y} \quad (65)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial y} \quad (66)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_1}{\partial z} \quad (67)$$

$$+ \frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_2}{\partial z} \quad (68)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_3}{\partial z} = 0 \quad (69)$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \quad (70)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \quad (71)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \quad (72)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \quad (73)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \quad (74)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \quad (75)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \quad (76)$$

$$+ \frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)0 \quad (77)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \quad (78)$$

$$(79)$$

$$\frac{\partial}{\partial r}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \cos(\theta) \sin(\phi) \quad (80)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \frac{-\sin(\theta)}{r \sin(\phi)} \quad (81)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \frac{\cos(\theta) \cos(\phi)}{r} \quad (82)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \sin(\theta) \sin(\phi) \quad (83)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \frac{\cos(\theta)}{r \sin(\phi)} \quad (84)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \frac{\sin(\theta) \cos(\phi)}{r} \quad (85)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi) u_r - \sin(\phi) u_\phi) \cos(\phi) \quad (86)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi) u_r - \sin(\phi) u_\phi) \frac{-\sin(\phi)}{r} = 0 \quad (87)$$

$$(88)$$

Now I will consider the terms of each derivative of each velocity individually,

$$\frac{\partial u_r}{\partial r}$$

$$\cos^2(\theta) \sin^2(\phi) \frac{\partial u_r}{\partial r} + \sin^2(\theta) \sin^2(\phi) \frac{\partial u_r}{\partial r} + \cos^2(\phi) \frac{\partial u_r}{\partial r} = \sin^2(\phi) \frac{\partial u_r}{\partial r} + \cos^2(\phi) \frac{\partial u_r}{\partial r} = \frac{\partial u_r}{\partial r} \quad (89)$$

$$\frac{\partial u_\theta}{\partial r}$$

$$- \sin(\theta) \cos(\theta) \sin(\phi) \frac{\partial u_\theta}{\partial r} + \cos(\theta) \sin(\theta) \sin(\phi) \frac{\partial u_\theta}{\partial r} = 0 \quad (90)$$

$$\frac{\partial u_\phi}{\partial r}$$

$$\cos^2(\theta) \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} + \sin^2(\theta) \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} - \sin(\phi) \cos(\phi) \frac{\partial u_\phi}{\partial r} \quad (91)$$

$$= \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} - \sin(\phi) \cos(\phi) \frac{\partial u_\phi}{\partial r} = 0 \quad (92)$$

$$\frac{\partial u_r}{\partial \theta}$$

$$- \frac{\sin(\theta)}{r} \frac{\partial}{\partial \theta}(\cos(\theta) u_r) + \frac{\cos(\theta)}{r} \frac{\partial}{\partial \theta}(\sin(\theta) u_r) \quad (93)$$

$$= - \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial u_r}{\partial \theta} + \frac{\sin^2(\theta)}{r} u_r + \frac{\cos(\theta) \sin(\theta)}{r} \frac{\partial u_r}{\partial \theta} + \frac{\cos^2(\theta)}{r} u_r = \frac{u_r}{r} \quad (94)$$

$$\frac{\partial u_\theta}{\partial \theta}$$

$$\frac{\sin(\theta)}{r \sin(\phi)} \frac{\partial}{\partial \theta}(\sin(\theta) u_\theta) + \frac{\cos(\theta)}{r \sin(\phi)} \frac{\partial}{\partial \theta}(\cos(\theta) u_\theta) = \quad (95)$$

$$\frac{\sin(\theta)}{r \sin(\phi)} \left( \sin(\theta) \frac{\partial u_\theta}{\partial \theta} + \cos(\theta) u_\theta \right) + \frac{\cos(\theta)}{r \sin(\phi)} \left( \cos(\theta) \frac{\partial u_\theta}{\partial \theta} - \sin(\theta) u_\theta \right) = \quad (96)$$

$$\frac{\sin^2(\theta)}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{\cos^2(\theta)}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} = \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} \quad (97)$$

$$\frac{\partial u_\phi}{\partial \theta}$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\cos(\theta)u_\phi) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\sin(\theta)u_\phi) = \quad (98)$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\phi}{\partial\theta} - \sin(\theta)u_\phi\right) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\phi}{\partial\theta} + \cos(\theta)u_\phi\right) = \quad (99)$$

$$\frac{\sin^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{\cos^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi = \frac{\cos(\phi)}{r\sin(\phi)}u_\phi \quad (100)$$

$$\frac{\partial u_r}{\partial \phi}$$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \quad (101)$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \quad (102)$$

$$\frac{\cos(\phi)}{r}\left(\sin(\phi)\frac{\partial u_r}{\partial\phi} + \cos(\phi)u_r\right) - \frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial u_r}{\partial\phi} - \sin(\phi)u_r\right) = \quad (103)$$

$$\frac{\cos^2(\phi)}{r}u_r + \frac{\sin^2(\phi)}{r}u_r = \frac{1}{r}u_r \quad (104)$$

$$\frac{\partial u_\theta}{\partial \phi}$$

$$-\frac{\cos(\theta)\sin(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} + \frac{\sin(\theta)\cos(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} = 0 \quad (105)$$

$$\frac{\partial u_\phi}{\partial \phi}$$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \quad (106)$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \quad (107)$$

$$\frac{\cos(\phi)}{r}\left(\cos(\phi)\frac{\partial u_\phi}{\partial\phi} - \sin(\phi)u_\phi\right) + \frac{\sin(\phi)}{r}\left(\sin(\phi)\frac{\partial u_\phi}{\partial\phi} + \cos(\phi)u_\phi\right) = \quad (108)$$

$$\frac{\cos^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\sin^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} \quad (109)$$

The simplified continuity equation is thus

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} = 0 \quad (110)$$

$$\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi = 0 \quad (111)$$

Using the product rule this can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = 0 \quad (112)$$

$$(113)$$