Derivation of Shallow Water Linearized Moment Equations

Shallow Water Moment Equations

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2}ge_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y$$

$$= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2}ge_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x$$

$$= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big)$$

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k\right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_y$$

$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right)$$

$$(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y$$

$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \,\mathrm{d}\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right)\phi_k \,\mathrm{d}\zeta$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \,\mathrm{d}\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

To get to the Shallow Water Linearized Moment Equations, we assume that $\alpha_i = O(\varepsilon)$ and $\beta_i = O(\varepsilon)$ are drop all terms of $O(\varepsilon^2)$ in the moment equations. The momentum equations remain the same even though they contain some of these terms.

$$
\begin{aligned}
h_t + (hu_m)_x + (hv_m)_y = 0 \\
(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y \\
= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x) \\
(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x \\
= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big) \\
(h\alpha_i)_t + (2hu_m\alpha_i)_x + (hu_m\beta_i + hv_m\alpha_i)_y = u_m D_i - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j\right) \\
(h\beta_i)_t + (hu_m\beta_i + hv_m\alpha_i)_x + (2hv_m\beta_i)_y = v_m D_i - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right) \\
C_{ij} = \int_0^1 \phi_i' \phi_j' \, \mathrm{d}\zeta \\
D_i = (h\alpha_i)_x + (h\beta_i)_y
\end{aligned}
$$

We can write down the shallow water linearized moments equations in the form

$$\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{1}$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{2}$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $\boldsymbol{n} = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$n_1(\boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q})) + n_2(\boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})).$$

If this matrix is diagonalizable with real eigenvalues for all directions $\boldsymbol{n}$, then this system is considered hyperbolic.

Flux Functions

$$
\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2} e_z g h^2 + hu^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j^2\right) \\ huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\ 2hu\alpha_1 \\ \beta_1 hu + \alpha_1 hv \\ \vdots \\ 2hu\alpha_N \\ hu\beta_N + hv\alpha_N \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\ \frac{1}{2} e_z g h^2 + hv^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\beta_j^2\right) \\ hu\beta_1 + hv\alpha_1 \\ 2hv\beta_1 \\ \vdots \\ hu\beta_N + hv\alpha_N \\ 2hv\beta_N \end{pmatrix}
$$

Nonconservative Matrices

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 &&&&&&& \\ 0 & 0 &&&&&& \\ & 0 & 0 &&&&& \\ && 0 & u &&&& \\ &&& v & 0 &&& \\ &&&& 0 & u && \\ &&&&& v & 0 & \\ &&&&&& 0 & \ddots \\ &&&&&&& \ddots \end{pmatrix}, \quad g_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 &&&&&&& \\ & 0 & 0 &&&&&& \\ && 0 & 0 &&&&& \\ &&& 0 & u &&&& \\ &&&& v & 0 &&& \\ &&&&& 0 & u && \\ &&&&&& v & 0 & \\ &&&&&&& 0 & \ddots \\ &&&&&&&& \ddots \end{pmatrix}$$

Flux Jacobians

$$\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ e_z g h - u^2 - \sum_{i=1}^N \left(\frac{1}{2*i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2N+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\ -2u\alpha_1 & 2\alpha_1 & 0 & 2u &&&& \\ -u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & v & u &&& \\ \vdots & \vdots & \vdots && \ddots & \ddots && \\ -2u\alpha_N & 2\alpha_N & 0 &&& 0 & 2u & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & u \end{pmatrix}$$

$$\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\ e_z g h - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\ -u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & u &&& \\ -2v\beta_1 & 0 & 2\beta_1 && 2v & 0 && \\ \vdots & \vdots & \vdots &&& \ddots & \ddots & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & u \\ -2u\beta_N & 0 & 2\beta_N &&&&& 2v \end{pmatrix}$$

Quasilinear Matrices, $Q_x = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), Q_y = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$Q_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ e_z g h - u^2 - \sum_{i=1}^N \left(\frac{1}{2i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\ -2u\alpha_1 & 2\alpha_1 & 0 & u &&&& \\ -u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 && u &&& \\ \vdots & \vdots & \vdots &&& \ddots && \\ -2u\alpha_N & 2\alpha_N & 0 &&&& u & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&&& u \end{pmatrix}$$

$$Q_y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\ e_z g h - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\ -u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v &&&& \\ -2v\beta_1 & 0 & 2\beta_1 && v &&& \\ \vdots & \vdots & \vdots &&& \ddots && \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & \\ -2u\beta_N & 0 & 2\beta_N &&&&& v \end{pmatrix}$$

Quasilinear Eigenvalues The wavespeeds of this system in direction $\boldsymbol{n} = [n_1, n_2]$ are given by the eigenvalues of the matrix

$$n_1 Q_x + n_2 Q_y.$$

If all of the eigenvalues are real with a full set of eigenvectors, this this system is hyperbolic.

Convenient constants

$$\begin{aligned}
d_0^1 &= n_1\left(e_z g h - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)\\
d_0^2 &= n_1\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2\left(e_z g h - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right)\\
d_i^1 &= n_1(-2u\alpha_i) + n_2(-u\beta_i - \alpha_i v)\\
d_i^2 &= n_1(-u\beta_i - v\alpha_i) + n_2(-2v\beta_i)\\
b_i^1 &= n_1\frac{2}{2i+1}\alpha_i + n_2\frac{1}{2i+1}\beta_i\\
b_i^2 &= n_2\frac{1}{2i+1}\alpha_i\\
b_i^3 &= n_1\frac{1}{2i+1}\beta_i\\
b_i^4 &= n_1\frac{1}{2i+1}\alpha_i + n_2\frac{2}{2i+1}\beta_i\\
c_i^1 &= n_1 2\alpha_i + n_2\beta_i\\
c_i^2 &= n_1\beta_i\\
c_i^3 &= n_2\alpha_i\\
c_i^4 &= n_1\alpha_i + n_2 2\beta_i
\end{aligned}$$

Determinant of Block Matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|\left|A - BD^{-1}C\right|$$

$$\begin{aligned}
\det(n_1 Q_x + n_2 Q_y - \lambda I) &= \begin{vmatrix}
-\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0\\
d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2\\
d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4\\
d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\
d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
d_N^1 & c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\
d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde{\lambda}
\end{vmatrix}\\
&= |D|\left|A - BD^{-1}C\right|
\end{aligned}$$

$$A = \begin{pmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} \end{pmatrix} \tag{3}$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \end{pmatrix} \tag{4}$$

$$C = \begin{pmatrix} d_1^1 & c_1^1 & c_1^3 \\ d_1^2 & c_1^2 & c_1^4 \\ \vdots & \vdots & \vdots \\ d_N^1 & c_N^1 & c_N^3 \\ d_N^2 & c_N^2 & c_N^4 \end{pmatrix} \tag{5}$$

$$D = \begin{pmatrix} -\tilde{\lambda} & & & & \\ & -\tilde{\lambda} & & & \\ & & \ddots & & \\ & & & -\tilde{\lambda} & \\ & & & & -\tilde{\lambda} \end{pmatrix} \tag{6}$$

$$|D| = \tilde{\lambda}^{2N}$$

$$BD^{-1}C = \frac{1}{-\tilde{\lambda}} \begin{pmatrix} 0 & 0 & 0 \\ s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{pmatrix}$$

$$s_1 = \sum_{i=1}^{N} \left(b_i^1 d_i^1 + b_i^2 d_i^2\right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1}(4\alpha_i u n_1 + \beta_i u n_2 + 3\alpha_i v n_2)(\alpha_i n_1 + \beta_i n_2)\right)$$

$$s_2 = \sum_{i=1}^{N} \left(b_i^1 c_i^1 + b_i^2 c_i^2\right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1}(4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2)\right)$$

$$s_3 = \sum_{i=1}^{N} \left(b_i^1 c_i^3 + b_i^2 c_i^4\right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1}\alpha_i n_2(\alpha_i n_1 + \beta_i n_2)\right)$$

$$s_4 = \sum_{i=1}^{N} \left(b_i^3 d_i^1 + b_i^4 d_i^2\right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1}(3\beta_i u n_1 + \alpha_i v n_1 + 4\beta_i v n_2)(\alpha_i n_1 + \beta_i n_2)\right)$$

$$s_5 = \sum_{i=1}^{N} \left(b_i^3 c_i^1 + b_i^4 c_i^2\right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1}\beta_i n_1(\alpha_i n_1 + \beta_i n_2)\right)$$

$$s_6 = \sum_{i=1}^{N} \left(b_i^3 c_i^3 + b_i^4 c_i^4\right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1}(\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2)\right)$$

$$A - BD^{-1}C = \frac{1}{\tilde{\lambda}} \begin{pmatrix} -\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2 & \tilde{\lambda}n_1 & \tilde{\lambda}n_2 \\ \tilde{\lambda}d_0^1 + s_1 & \tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2 & \tilde{\lambda}un_2 + s_3 \\ \tilde{\lambda}d_0^2 + s_4 & \tilde{\lambda}vn_1 + s_5 & \tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6 \end{pmatrix}$$

$$\begin{aligned}
\det\big(\tilde{\lambda}(A - BD^{-1}C)\big) &= \big(-\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2\big)\big(\big(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\big)\big(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\big) - \big(\tilde{\lambda}vn_1 + s_5\big)\big(\tilde{\lambda}un_2 + s_3\big)\big) \\
&\quad -\tilde{\lambda}n_1\big(\big(\tilde{\lambda}d_0^1 + s_1\big)\big(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\big) - \big(\tilde{\lambda}d_0^2 + s_4\big)\big(\tilde{\lambda}un_2 + s_3\big)\big) \\
&\quad +\tilde{\lambda}n_2\big(\big(\tilde{\lambda}d_0^1 + s_1\big)\big(\tilde{\lambda}vn_1 + s_5\big) - \big(\tilde{\lambda}d_0^2 + s_4\big)\big(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\big)\big) \\
&= \big(-\tilde{\lambda}^2 - (un_1 + vn_2)\tilde{\lambda}\big)\big(\tilde{\lambda}^4 + (-un_1 - vn_2)\tilde{\lambda}^3 + (-s_2 - s_6)\tilde{\lambda}^2 + (un_1s_6 + vn_2s_2 - vn_1s_3 - un_2s_5)\tilde{\lambda} + s_2s_6 - s_3s_5\big) \\
&\quad -\tilde{\lambda}n_1\big(-d_0^1\tilde{\lambda}^3 + \big(d_0^1vn_2 - s_1 - d_0^2un_2\big)\tilde{\lambda}^2 + \big(d_0^1s_6 + vn_2s_1 - d_0^2s_3 - un_2s_4\big)\tilde{\lambda} + s_1s_6 - s_3s_4\big) \\
&\quad +\tilde{\lambda}n_2\big(d_0^2\tilde{\lambda}^3 + \big(d_0^1vn_1 - d_0^2un_1 + s_4\big)\tilde{\lambda}^2 + \big(d_0^1s_5 + vn_1s_1 - d_0^2s_2 - un_1s_4\big)\tilde{\lambda} + s_1s_5 - s_2s_4\big) \\
&= -\tilde{\lambda}^6 + \Big(s_2 + s_6 + (un_1 + vn_2)^2\Big)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6)\tilde{\lambda}^3 \\
&\quad +\big(s_3s_5 - s_2s_6 + \big(u^2n_1n_2 + uvn_2^2\big)s_5 + \big(uvn_1^2 + v^2n_1n_2\big)s_3 - \big(u^2n_1^2 + uvn_1n_2\big)s_6 - \big(uvn_1n_2 + v^2n_2^2\big)s_2\big)\tilde{\lambda}^2 + (s_3s_5 - s_2s_6)(un_1 + \\
&\quad +d_0^1n_1\tilde{\lambda}^4 + \big(-d_0^1vn_1n_2 + s_1n_1 + d_0^2un_1n_2\big)\tilde{\lambda}^3 + \big(-d_0^1n_1s_6 - vn_1n_2s_1 + d_0^2n_1s_3 + un_1n_2s_4\big)\tilde{\lambda}^2 + (s_3s_4 - s_1s_6)n_1\tilde{\lambda} \\
&\quad +d_0^2n_2\tilde{\lambda}^4 + \big(d_0^1vn_1n_2 - d_0^2un_1n_2 + n_2s_4\big)\tilde{\lambda}^3 + \big(d_0^1n_2s_5 + vn_1n_2s_1 - d_0^2n_2s_2 - un_1n_2s_4\big)\tilde{\lambda}^2 + (s_1s_5 - s_2s_4)n_2\tilde{\lambda} \\
&= -\tilde{\lambda}^6 + \Big(s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2\Big)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4)\tilde{\lambda}^3 \\
&\quad +\big(s_3s_5 - s_2s_6 + \big(u^2n_1n_2 + uvn_2^2 + d_0^1n_2\big)s_5 + \big(uvn_1^2 + v^2n_1n_2 + d_0^2n_1\big)s_3 - \big(u^2n_1^2 + uvn_1n_2 + d_0^1n_1\big)s_6 - \big(uvn_1n_2 + v^2n_2^2 + d_0^2n \\
&\quad +((s_3s_5 - s_2s_6)(un_1 + vn_2) + (s_3s_4 - s_1s_6)n_1 + (s_1s_5 - s_2s_4)n_2)\tilde{\lambda}
\end{aligned}$$

Look at coefficients of polynomial in $\tilde{\lambda}$.

$$\begin{aligned}
& s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2 \\
&= \sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_in_1 + \beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + \sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_in_1 + 4\beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + (un_1 + vn_2)^2 \\
&\quad +n_1^2\left(e_zgh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_1n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) \\
&\quad +n_1n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2^2\left(e_zgh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right) \\
&= \sum_{i=1}^{N}\left(\frac{5}{2i+1}(\alpha_in_1 + \beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + e_zgh\big(n_1^2 + n_2^2\big) - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(\alpha_i^2n_1^2 + 2\alpha_i\beta_in_1n_2 + \beta_i^2n_2^2\big)\right) \\
&= \sum_{i=1}^{N}\left(\frac{4}{2i+1}(\alpha_in_1 + \beta_in_2)^2\right) + e_zgh\big(n_1^2 + n_2^2\big)
\end{aligned}$$

$$
\begin{aligned}
&vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4 \\
&= \sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_i vn_1n_2(\alpha_in_1+\beta_in_2)\right) \\
&+ \sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_i un_1n_2(\alpha_in_1+\beta_in_2)\right) \\
&+ \sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(4\alpha_i un_1^2+\beta_i un_1n_2\big)(\alpha_in_1+\beta_in_2)\right) \\
&+ \sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(\alpha_i vn_1n_2+4\beta_i vn_2^2\big)(\alpha_in_1+\beta_in_2)\right) \\
&+ \sum_{i=1}^{N}\left(-\frac{1}{2i+1}\big(4\alpha_i un_1^2+\beta_i un_1n_2+3\alpha_i vn_1n_2\big)(\alpha_in_1+\beta_in_2)\right) \\
&+ \sum_{i=1}^{N}\left(-\frac{1}{2i+1}\big(3\beta_i un_1n_2+\alpha_i vn_1n_2+4\beta_i vn_2^2\big)(\alpha_in_1+\beta_in_2)\right) \\
&= \sum_{i=1}^{N}\left(\frac{1}{2i+1}(3\alpha_i vn_1n_2+3\beta_i un_1n_2-3\alpha_i vn_1n_2-3\beta_i un_1n_2)(\alpha_in_1+\beta_in_2)\right) \\
&= 0
\end{aligned}
$$

$$
a_i = \alpha_in_1+\beta_in_2
$$

$$
\begin{aligned}
&s_3s_5 - s_2s_6 \\
&= \sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_in_2a_i\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_in_1a_i\right) - \sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_in_1+\beta_in_2)a_i\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_in_1+4\beta_in_2)a_i\right) \\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{3}{2i+1}\frac{3}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right) - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}(4\alpha_in_1+\beta_in_2)(\alpha_jn_1+4\beta_jn_2)a_ia_j\right)\right) \\
&= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}(9\alpha_i\beta_jn_1n_2-(4\alpha_in_1+\beta_in_2)(\alpha_jn_1+4\beta_jn_2))a_ia_j\right)\right) \\
&= \sum_{i=1}^{N}\left(\frac{1}{(2i+1)^2}(9\alpha_i\beta_in_1n_2-(4\alpha_in_1+\beta_in_2)(\alpha_in_1+4\beta_in_2))a_ia_i\right) \\
&+ \sum_{i=1}^{N}\left(\sum_{j=1}^{i-1}\left(\frac{1}{2i+1}\frac{1}{2j+1}(9\alpha_i\beta_jn_1n_2-(4\alpha_in_1+\beta_in_2)(\alpha_jn_1+4\beta_jn_2)+9\alpha_j\beta_in_1n_2-(4\alpha_jn_1+\beta_jn_2)(\alpha_in_1+4\beta_in_2))a_ia_j\right)\right) \\
&= \sum_{i=1}^{N}\left(-\left(\frac{2}{2i+1}\right)^2(\alpha_in_1+\beta_in_2)^2a_ia_i\right) + \sum_{i=1}^{N}\left(\sum_{j=1}^{i-1}\left(-8\frac{1}{2i+1}\frac{1}{2j+1}(\alpha_in_1+\beta_in_2)(\alpha_jn_1+\beta_jn_2)a_ia_j\right)\right) \\
&= -\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{2}{2i+1}\frac{2}{2j+1}a_i^2a_j^2\right)\right) \\
&= -\left(\sum_{i=1}^{N}\left(\frac{2}{2i+1}a_i^2\right)\right)^2
\end{aligned}
$$

$$\big(u^2n_1n_2+uvn_2^2+d_0^1n_2\big)s_5$$
$$=\left(n_2n_1\left(e_zgh-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right)-n_2^2\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_in_1(\alpha_in_1+\beta_in_2)\right)$$
$$=\left(e_zghn_1n_2-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_in_2(\alpha_in_1+\beta_in_2)\right)\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_in_1(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_1n_2s_5-\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right)$$
$$\big(uvn_1^2+v^2n_1n_2+d_0^2n_1\big)s_3$$
$$=\left(-n_1^2\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)+n_1n_2\left(e_zgh-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right)\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_in_2(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_1n_2s_3-\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_in_1(\alpha_in_1+\beta_in_2)\right)\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_in_2(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_1n_2s_3-\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_j\beta_in_1n_2a_ia_j\right)\right)$$
$$\big(u^2n_1^2+uvn_1n_2+d_0^1n_1\big)s_6$$
$$=\left(n_1^2\left(e_zgh-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right)-n_1n_2\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_in_1+4\beta_in_2)(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_1^2s_6-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_in_1(\alpha_in_1+\beta_in_2)\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_in_1+4\beta_in_2)(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_1^2s_6-\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_in_1(\alpha_jn_1+4\beta_jn_2)a_ia_j\right)\right)$$
$$\big(uvn_1n_2+v^2n_2^2+d_0^2n_2\big)s_2$$
$$=\left(n_2n_1\left(-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)+n_2^2\left(e_zgh-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right)\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_in_1+\beta_in_2)(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_2^2s_2-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_in_2(\alpha_in_1+\beta_in_2)\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_in_1+\beta_in_2)(\alpha_in_1+\beta_in_2)\right)$$
$$=e_zghn_2^2s_2-\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\beta_in_2(4\alpha_jn_1+\beta_jn_2)a_ia_j\right)\right)$$

$$
\begin{aligned}
&s_3s_5 - s_2s_6 + \big(u^2n_1n_2 + uvn_2^2 + d_0^1n_2\big)s_5 + \big(uvn_1^2 + v^2n_1n_2 + d_0^2n_1\big)s_3 \\
&\quad -\big(u^2n_1^2 + uvn_1n_2 + d_0^1n_1\big)s_6 - \big(uvn_1n_2 + v^2n_2^2 + d_0^2n_2\big)s_2 \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zghn_1n_2s_5 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right) \\
&\quad + e_zghn_1n_2s_3 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_j\beta_in_1n_2a_ia_j\right)\right) \\
&\quad - e_zghn_1^2s_6 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_in_1(\alpha_jn_1 + 4\beta_jn_2)a_ia_j\right)\right) \\
&\quad - e_zghn_2^2s_2 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\beta_in_2(4\alpha_jn_1 + \beta_jn_2)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad - 6\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\big(\alpha_i\alpha_jn_1^2 + 8\alpha_i\beta_jn_1n_2 + \beta_i\beta_jn_2^2\big)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\big(\alpha_i\alpha_jn_1^2 + 2\alpha_i\beta_jn_1n_2 + \beta_i\beta_jn_2^2\big)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}a_i^2a_j^2\right)\right) \\
&= -3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big)
\end{aligned}
$$

$$
\begin{aligned}
&n_1n_2(s_3+s_5)-n_1^2s_6-n_2^2s_2\\
&=n_1n_2\sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_i n_2 a_i\right)\\
&+n_1n_2\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_i n_1 a_i\right)\\
&-n_1^2\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_i n_1+4\beta_i n_2)a_i\right)\\
&-n_2^2\sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_i n_1+\beta_i n_2)a_i\right)\\
&=\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(3\alpha_i n_1 n_2^2+3\beta_i n_1^2 n_2-\big(\alpha_i n_1^3+4\beta_i n_1^2 n_2\big)-\big(4\alpha_i n_1 n_2^2+\beta_i n_2^3\big)\big)a_i\right)\\
&=-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(\alpha_i n_1^3+\beta_i n_1^2 n_2+\alpha_i n_1 n_2^2+\beta_i n_2^3\big)a_i\right)\\
&=-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(n_1^2 a_i+n_2^2 a_i\big)a_i\right)\\
&=-\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2+e_z gh\big(n_1n_2(s_3+s_5)-n_1^2s_6-n_2^2s_2\big)\\
&=-3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2-e_z gh\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&(s_3s_5-s_2s_6)(un_1+vn_2)+(s_3s_4-s_1s_6)n_1+(s_1s_5-s_2s_4)n_2\\
&=-4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2(un_1+vn_2)+4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2un_1+4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2vn_2\\
&=0
\end{aligned}
$$

$$\det\big(\tilde\lambda\big(A-BD^{-1}C\big)\big) = -\tilde\lambda^6 + \left(4\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right) + e_z gh\big(n_1^2+n_2^2\big)\right)\tilde\lambda^4$$
$$-\left(3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z gh\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)\tilde\lambda^2$$
$$\det\big(A-BD^{-1}C\big) = -\frac{1}{\tilde\lambda}\left(\tilde\lambda^2 - \left(3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right) + e_z gh\big(n_1^2+n_2^2\big)\right)\right)\left(\tilde\lambda^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)$$
$$\det(n_1Q_x + n_2Q_y + \lambda I) = \det(D)\det\big(A-BD^{-1}C\big)$$
$$= -\tilde\lambda^{2N-1}\left(\tilde\lambda^2 - \left(3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right) + e_z gh\big(n_1^2+n_2^2\big)\right)\right)\left(\tilde\lambda^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)$$

$$\lambda = un_1 + vn_2 \pm \sqrt{e_z gh(n_1^2+n_2^2) + 3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)}$$
$$\lambda = un_1 + vn_2 \pm \sqrt{\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)}$$
$$\lambda = un_1 + vn_2$$

Quasilinear Eigenvectors

$$(n_1Q_x + n_2Q_y - \lambda I)\boldsymbol{q} = 0$$
$$\begin{pmatrix}
-\tilde\lambda - n_1u - n_2v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\
d_0^1 & n_1u - \tilde\lambda & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\
d_0^2 & n_1v & n_2v - \tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\
d_1^1 & c_1^1 & c_1^3 & -\tilde\lambda & & & & \\
d_1^2 & c_1^2 & c_1^4 & & -\tilde\lambda & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
d_N^1 & c_N^1 & c_N^3 & & & & -\tilde\lambda & \\
d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde\lambda
\end{pmatrix}
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \vdots \\ q_{2N+2} \\ q_{2N+3} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Let $\lambda = un_1 + vn_2$

$$\begin{pmatrix}
-n_1u - n_2v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\
d_0^1 & n_1u & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\
d_0^2 & n_1v & n_2v & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\
d_1^1 & c_1^1 & c_1^3 & & & & & \\
d_1^2 & c_1^2 & c_1^4 & & & & & \\
\vdots & \vdots & \vdots & & & & & \\
d_N^1 & c_N^1 & c_N^3 & & & & & \\
d_N^2 & c_N^2 & c_N^4 & & & & &
\end{pmatrix}
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \vdots \\ q_{2N+2} \\ q_{2N+3} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$