Local Discontinuous Galerkin Method for Thin Film Diffusion

We would like to solve the 1D thin film diffusion equation with a Discontinuous Galerkin Method. The equation is given as

$$q_t = -\left(q^3 q_{xxx}\right)_x.$$

Non Dimensional Form Let $q = Hq$, $x = Lx$, and $t = Tt$, then

$$\frac{H}{T} q_t = -\frac{1}{L}\left(H^3 q^3 \frac{H}{L^3} q_{xxx}\right)_x \tag{1}$$

$$q_t = -\frac{TH^3}{L^4}\left(q^3 q_{xxx}\right)_x \tag{2}$$

$$\tag{3}$$

Local Discontinuous Galerkin Method First rewrite the diffusion equation as a system of first order equations.

$$\begin{aligned}
r &= q_x \\
s &= r_x = q_{xx} \\
u &= s_x = q_{xxx} \\
q_t &= -\left(q^3 u\right)_x = -\left(q^3 q_{xxx}\right)_x
\end{aligned}$$

The LDG method becomes the process of finding $q_h, r_h, s_h, u_h \in V_h$ in the DG solution space, such that for all test functions $v_h, w_h, y_h, z_h \in V_h$ and for all j the following equations are satisfied

$$\begin{aligned}
\int_{I_j} r_h w_h \,\mathrm{d}x &= \int_{I_j} (q_h)_x w_h \,\mathrm{d}x \\
\int_{I_j} s_h y_h \,\mathrm{d}x &= \int_{I_j} (r_h)_x y_h \,\mathrm{d}x \\
\int_{I_j} u_h z_h \,\mathrm{d}x &= \int_{I_j} (s_h)_x z_h \,\mathrm{d}x \\
\int_{I_j} (q_h)_t v_h \,\mathrm{d}x &= -\int_{I_j} \left(q_h^3 u_h\right)_x v_h \,\mathrm{d}x
\end{aligned}$$

after integrating by parts, these equations are

$$\begin{aligned}
\int_{i_j} r_h w_h \,\mathrm{d}x &= \left(\left(\hat{q}_h w_h^-\right)_{j+1/2} - \left(\hat{q}_h w_h^+\right)_{j-1/2}\right) - \int_{i_j} q_h (w_h)_x \,\mathrm{d}x \\
\int_{i_j} s_h y_h \,\mathrm{d}x &= \left(\left(\hat{r}_h y_h^-\right)_{j+1/2} - \left(\hat{r}_h y_h^+\right)_{j-1/2}\right) - \int_{i_j} r_h (y_h)_x \,\mathrm{d}x \\
\int_{i_j} u_h z_h \,\mathrm{d}x &= \left(\left(\hat{s}_h z_h^-\right)_{j+1/2} - \left(\hat{s}_h z_h^+\right)_{j-1/2}\right) - \int_{i_j} s_h (z_h)_x \,\mathrm{d}x \\
\int_{i_j} (q_h)_t v_h \,\mathrm{d}x &= -\left(\left(\hat{q^3}_h \hat{u}_h v_h^-\right)_{j+1/2} - \left(\hat{q^3}_h \hat{u}_h v_h^+\right)_{j-1/2}\right) + \int_{i_j} q_h^3 u_h (v_h)_x \,\mathrm{d}x
\end{aligned}$$

A common choice of numerical fluxes are the so-called alternating fluxes.

$$\begin{aligned}
\hat{u}_h &= u_h^- \\
\hat{q}_h &= q_h^+ \\
\hat{r}_h &= r_h^- \\
\hat{s}_h &= s_h^+
\end{aligned}$$

or

$$\hat{u}_h = u_h^+$$
$$\hat{q}_h = q_h^-$$
$$\hat{r}_h = r_h^+$$
$$\hat{s}_h = s_h^-$$

A choice for the flux $\hat{q^3}_h$ must also be made.

Implementation If we consider a single cell I_j, do a linear transformation from $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$, and consider specifically the Legendre polynomial basis $\left\{\phi^k(\xi)\right\}$ with the following orthogonality property

$$\frac{1}{2}\int_{-1}^{1} \phi^j(\xi)\phi^k(\xi)\,\mathrm{d}\xi = \delta_{jk}$$

we can form a more concrete LDG method for implementing. The linear transformation can be expressed as

$$x = \frac{\Delta x}{2}\xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

or

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

After this tranformation the thin film diffusion equation becomes

$$q_t = -\frac{16}{\Delta x^4}\left(q^3 q_{\xi\xi\xi}\right)_\xi$$

on the cell I_j. We can then write this as the following system of first order equations.

$$r = \frac{2}{\Delta x} q_\xi$$
$$s = \frac{2}{\Delta x} r_\xi = \frac{4}{\Delta x^2} q_{\xi\xi}$$
$$u = \frac{2}{\Delta x} s_\xi = \frac{8}{\Delta x^3} q_{\xi\xi\xi}$$
$$q_t = -\frac{2}{\Delta x}\left(q^3 u\right)_\xi = -\frac{16}{\Delta x^4}\left(q^3 q_{\xi\xi\xi}\right)_\xi$$

With the Legendre basis, the numerical solution on I_i can be written as

$$q|_{I_i} \approx q_h|_{I_i} = \sum_{l=1}^{M}\left(Q_i^l \phi^l(\xi)\right)$$
$$r|_{I_i} \approx r_h|_{I_i} = \sum_{l=1}^{M}\left(R_i^l \phi^l(\xi)\right)$$
$$s|_{I_i} \approx s_h|_{I_i} = \sum_{l=1}^{M}\left(S_i^l \phi^l(\xi)\right)$$
$$u|_{I_i} \approx u_h|_{I_i} = \sum_{l=1}^{M}\left(U_i^l \phi^l(\xi)\right)$$

I will use the following shorthand for numerical fluxes using one of the alternating flux options.

$$\begin{aligned}
\hat{Q}_{i+1/2} &= Q^{+}_{i+1/2} = \sum_{l=1}^{M} \left(Q^{l}_{i+1} \phi^{l}(-1) \right) \\
\hat{R}_{i+1/2} &= R^{-}_{i+1/2} = \sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}(1) \right) \\
\hat{S}_{i+1/2} &= S^{+}_{i+1/2} = \sum_{l=1}^{M} \left(S^{l}_{i+1} \phi^{l}(-1) \right) \\
\hat{U}_{i+1/2} &= U^{-}_{i+1/2} = \sum_{l=1}^{M} \left(U^{l}_{i} \phi^{l}(1) \right)
\end{aligned}$$

Now plugging these into the system and multiplying by a Legendre basis function and integrating over cell I_i gives.

$$\begin{aligned}
r &= \frac{2}{\Delta x} q_{\xi} \\
\sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}(\xi) \right) &= \frac{2}{\Delta x} \sum_{l=1}^{M} \left(Q^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \\
\frac{1}{2} \int_{-1}^{1} \sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(Q^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi \\
R^{k}_{i} &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(Q^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi \\
R^{k}_{i} &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(Q^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}_{\xi}(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^{k}(1) \hat{Q}_{i+1/2} - \phi^{k}(-1) \hat{Q}_{i-1/2} \right)
\end{aligned}$$

$$\begin{aligned}
s &= \frac{2}{\Delta x} r_{\xi} \\
\sum_{l=1}^{M} \left(S^{l}_{i} \phi^{l}(\xi) \right) &= \frac{2}{\Delta x} \sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \\
\frac{1}{2} \int_{-1}^{1} \sum_{l=1}^{M} \left(S^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi \\
S^{k}_{i} &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(R^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}_{\xi}(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^{k}(1) \hat{R}_{i+1/2} - \phi^{k}(-1) \hat{R}_{i-1/2} \right)
\end{aligned}$$

$$\begin{aligned}
u &= \frac{2}{\Delta x} s_{\xi} \\
\sum_{l=1}^{M} \left(U^{l}_{i} \phi^{l}(\xi) \right) &= \frac{2}{\Delta x} \sum_{l=1}^{M} \left(S^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \\
\frac{1}{2} \int_{-1}^{1} \sum_{l=1}^{M} \left(U^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(S^{l}_{i} \phi^{l}_{\xi}(\xi) \right) \phi^{k}(\xi) \, \mathrm{d}\xi \\
U^{k}_{i} &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(S^{l}_{i} \phi^{l}(\xi) \right) \phi^{k}_{\xi}(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^{k}(1) \hat{S}_{i+1/2} - \phi^{k}(-1) \hat{S}_{i-1/2} \right)
\end{aligned}$$

$$
\begin{aligned}
q_t &= -\frac{2}{\Delta x}\left(q^3 u\right)_\xi \\
\sum_{l=1}^{M}\left(\dot{Q}_i^l \phi^l(\xi)\right) &= -\frac{2}{\Delta x}\left(\left(\sum_{l=1}^{M}\left(Q_i^l \phi^l(\xi)\right)\right)^3 \sum_{l=1}^{M}\left(U_i^l \phi^l(\xi)\right)\right)_{xi} \\
\frac{1}{2}\int_{-1}^{1}\sum_{l=1}^{M}\left(\dot{Q}_i^l \phi^l(\xi)\right)\phi^k(\xi)\,\mathrm{d}\xi &= -\frac{1}{\Delta x}\int_{-1}^{1}\left(\left(\sum_{l=1}^{M}\left(Q_i^l \phi^l(\xi)\right)\right)^3 \sum_{l=1}^{M}\left(U_i^l \phi^l(\xi)\right)\right)_{xi}\phi^k(\xi)\,\mathrm{d}\xi \\
\dot{Q}_i^k &= \frac{1}{\Delta x}\int_{-1}^{1}\left(\sum_{l=1}^{M}\left(Q_i^l \phi^l(\xi)\right)\right)^3 \sum_{l=1}^{M}\left(U_i^l \phi^l(\xi)\right)\phi_\xi^k(\xi)\,\mathrm{d}\xi \\
&\quad - \frac{1}{\Delta x}\left(\phi^k(1)\hat{Q^3}_{i+1/2}\hat{U}_{i+1/2} - \phi^k(-1)\hat{Q^3}_{i-1/2}\hat{U}_{i-1/2}\right)
\end{aligned}
$$

I will choose to define the flux $\hat{Q^3}_{i+1/2}$ as

$$
\begin{aligned}
\hat{Q^3}_{i+1/2} &= \left(\frac{1}{2}\left(Q_{i+1/2}^- + Q_{i+1/2}^+\right)\right)^3 \\
&= \left(\frac{1}{2}\left(\sum_{l=1}^{M}\left(Q_i^l \phi^l(1)\right) + \sum_{l=1}^{M}\left(Q_{i+1}^l \phi^l(-1)\right)\right)\right)^3
\end{aligned}
$$

Now this is a system of ODEs, there are $M \times N$ ODEs if M is the spatial order and N is the number of cells.

Matrix Representation Some common matrices and vectors that appear in these equations are

$$
\begin{aligned}
\boldsymbol{Q}_i &= \left[Q_i^l\right]_{l=1}^M \\
\boldsymbol{\phi}(\xi) &= \left[\phi^k(\xi)\right]_{k=1}^M \\
\Phi(\xi_1, \xi_2) &= \boldsymbol{\phi}(\xi_1)\boldsymbol{\phi}^T(\xi_2) \\
A &= [a_{kl}]_{k,l=1}^M \\
a_{kl} &= \int_{-1}^{1}\phi_\xi^k(\xi)\phi^l(\xi)\,\mathrm{d}\xi \\
B_i &= [b_{kl}]_{k,l=1}^M \\
b_{kl} &= \int_{-1}^{1} q_i^3(\xi)\phi^k(\xi)\phi_\xi^l(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

For example if $M = 5$, then

$$
\boldsymbol{\phi}(\xi) = \begin{bmatrix} \phi^1(\xi) \\ \phi^2(\xi) \\ \phi^3(\xi) \\ \phi^4(\xi) \\ \phi^5(\xi) \end{bmatrix}
$$

and

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
2\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 2\sqrt{3}\sqrt{5} & 0 & 0 & 0 \\
2\sqrt{7} & 0 & 2\sqrt{5}\sqrt{7} & 0 & 0 \\
0 & 6\sqrt{3} & 0 & 6\sqrt{7} & 0
\end{bmatrix}
$$

Also the numerical fluxes can be written as the following dot products

$$
\begin{aligned}
\hat{Q}_{i+1/2} &= \sum_{l=1}^{M} \left(Q_{i+1}^{l} \phi^{l}(-1) \right) \\
&= \boldsymbol{\phi}^{T}(-1) \boldsymbol{Q}_{i+1} \\
\hat{R}_{i+1/2} &= \sum_{l=1}^{M} \left(R_{i}^{l} \phi^{l}(1) \right) \\
&= \boldsymbol{\phi}^{T}(1) \boldsymbol{R}_{i} \\
\hat{S}_{i+1/2} &= \sum_{l=1}^{M} \left(S_{i+1}^{l} \phi^{l}(-1) \right) \\
&= \boldsymbol{\phi}^{T}(-1) \boldsymbol{S}_{i+1} \\
\hat{U}_{i+1/2} &= \sum_{l=1}^{M} \left(U_{i}^{l} \phi^{l}(1) \right) \\
&= \boldsymbol{\phi}^{T}(1) \boldsymbol{U}_{i}
\end{aligned}
$$

$$
\begin{aligned}
R_i^k &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(Q_i^l \phi^l(\xi) \right) \phi_\xi^k(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^k(1) \hat{Q}_{i+1/2} - \phi^k(-1) \hat{Q}_{i-1/2} \right) \\
R_i^k &= -\frac{1}{\Delta x} \sum_{l=1}^{M} \left(Q_i^l \int_{-1}^{1} \phi^l(\xi) \phi_\xi^k(\xi) \, \mathrm{d}\xi \right) + \frac{1}{\Delta x} \left(\phi^k(1) \hat{Q}_{i+1/2} - \phi^k(-1) \hat{Q}_{i-1/2} \right) \\
R_i^k &= -\frac{1}{\Delta x} (A \boldsymbol{Q}_i)_k + \frac{1}{\Delta x} \left(\phi^k(1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_{i+1} - \phi^k(-1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_i \right) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} A \boldsymbol{Q}_i + \frac{1}{\Delta x} \left(\boldsymbol{\phi}(1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_{i+1} - \boldsymbol{\phi}(-1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_i \right) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} A \boldsymbol{Q}_i + \frac{1}{\Delta x} (\Phi(1,-1) \boldsymbol{Q}_{i+1} - \Phi(-1,-1) \boldsymbol{Q}_i) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} (A + \Phi(-1,-1)) \boldsymbol{Q}_i + \frac{1}{\Delta x} \Phi(1,-1) \boldsymbol{Q}_{i+1}
\end{aligned}
$$

$$
\begin{aligned}
S_i^k &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(R_i^l \phi^l(\xi) \right) \phi_\xi^k(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^k(1) \hat{R}_{i+1/2} - \phi^k(-1) \hat{R}_{i-1/2} \right) \\
S_i^k &= -\frac{1}{\Delta x} (A \boldsymbol{R}_i)_k + \frac{1}{\Delta x} \left(\phi^k(1) \boldsymbol{\phi}^T(1) \boldsymbol{R}_i - \phi^k(-1) \boldsymbol{\phi}^T(1) \boldsymbol{R}_{i-1} \right) \\
\boldsymbol{S}_i &= -\frac{1}{\Delta x} (A - \Phi(1,1)) \boldsymbol{R}_i - \frac{1}{\Delta x} \Phi(-1,1) \boldsymbol{R}_{i-1}
\end{aligned}
$$

Note that I am treating the q^3 fluxes as constants, they don't depend on the unknowns Q_i^l

$$
\begin{aligned}
U_i^k &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(S_i^l \phi_\xi^l(\xi) \right) q_i^3 \phi^k(\xi) \, \mathrm{d}\xi \\
&\quad - \frac{1}{\Delta x} \left(\phi^k(1) \left(q_{i+1/2}^- \right)^3 S_{i+1/2}^- - \phi^k(-1) \left(q_{i-1/2}^+ \right)^3 S_{i-1/2}^+ \right) \\
&\quad + \frac{1}{\Delta x} \left(\phi^k(1) \hat{q}_{i+1/2}^3 \hat{S}_{i+1/2} - \phi^k(-1) \hat{q}_{i-1/2}^3 \hat{S}_{i-1/2} \right) \\
U_i^k &= \frac{1}{\Delta x} \sum_{l=1}^{M} \left(S_i^l \int_{-1}^{1} q_i^3 \phi^k(\xi) \phi_\xi^l(\xi) \, \mathrm{d}\xi \right) \\
&\quad - \frac{1}{\Delta x} \left(\phi^k(1) \left(q_{i+1/2}^- \right)^3 \boldsymbol{\phi}^T(1) \boldsymbol{S}_i - \phi^k(-1) \left(q_{i-1/2}^+ \right)^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
&\quad + \frac{1}{\Delta x} \left(\phi^k(1) \hat{q}_{i+1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_{i+1} - \phi^k(-1) \hat{q}_{i-1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
U_i^k &= \frac{1}{\Delta x} (B_i \boldsymbol{S}_i)_k - \frac{1}{\Delta x} \left(\phi^k(1) \left(q_{i+1/2}^- \right)^3 \boldsymbol{\phi}^T(1) \boldsymbol{S}_i - \phi^k(-1) \left(q_{i-1/2}^+ \right)^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
&\quad + \frac{1}{\Delta x} \left(\phi^k(1) \hat{q}_{i+1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_{i+1} - \phi^k(-1) \hat{q}_{i-1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
\boldsymbol{U}_i &= \frac{1}{\Delta x} B_i \boldsymbol{S}_i - \frac{1}{\Delta x} \left(\boldsymbol{\phi}(1) \left(q_{i+1/2}^- \right)^3 \boldsymbol{\phi}^T(1) \boldsymbol{S}_i - \boldsymbol{\phi}(-1) \left(q_{i-1/2}^+ \right)^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
&\quad + \frac{1}{\Delta x} \left(\boldsymbol{\phi}(1) \hat{q}_{i+1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_{i+1} - \boldsymbol{\phi}(-1) \hat{q}_{i-1/2}^3 \boldsymbol{\phi}^T(-1) \boldsymbol{S}_i \right) \\
\boldsymbol{U}_i &= \frac{1}{\Delta x} B_i \boldsymbol{S}_i - \frac{1}{\Delta x} \left(\left(q_{i+1/2}^- \right)^3 \Phi(1,1) \boldsymbol{S}_i - \left(q_{i-1/2}^+ \right)^3 \Phi(-1,-1) \boldsymbol{S}_i \right) \\
&\quad + \frac{1}{\Delta x} \left(\hat{q}_{i+1/2}^3 \Phi(1,-1) \boldsymbol{S}_{i+1} - \hat{q}_{i-1/2}^3 \Phi(-1,-1) \boldsymbol{S}_i \right) \\
\boldsymbol{U}_i &= \frac{1}{\Delta x} \left(B_i - \left(q_{i+1/2}^- \right)^3 \Phi(1,1) + \left(\left(q_{i-1/2}^+ \right)^3 - \hat{q}_{i-1/2}^3 \right) \Phi(-1,-1) \right) \boldsymbol{S}_i \\
&\quad + \frac{1}{\Delta x} \left(\hat{q}_{i+1/2}^3 \Phi(1,-1) \boldsymbol{S}_{i+1} \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{Q}_i^k &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(U_i^l \phi_\xi^l(\xi) \right) \phi^k(\xi) \, \mathrm{d}\xi - \frac{1}{\Delta x} \left(\phi^k(1) \hat{U}_{i+1/2} - \phi^k(-1) \hat{U}_{i-1/2} \right) \\
\dot{Q}_i^k &= \frac{1}{\Delta x} (A \boldsymbol{U}_i)_k - \frac{1}{\Delta x} \left(\phi^k(1) \boldsymbol{\phi}^T(1) \boldsymbol{U}_i - \phi^k(-1) \boldsymbol{\phi}^T(1) \boldsymbol{U}_{i-1} \right) \\
\dot{\boldsymbol{Q}}_i^k &= \frac{1}{\Delta x} (A - \Phi(1,1)) \boldsymbol{U}_i + \frac{1}{\Delta x} \Phi(-1,1) \boldsymbol{U}_{i-1}
\end{aligned}
$$