Caleb Logemann, James Rossmanith

Introduction

Derivation

Method

Convection

Diffusion

Numerical Results

Travelling Waves

Conclusion

References

Discontinuous Galerkin Method for Solving Thin Film Equations

Caleb Logemann James Rossmanith

Mathematics Department,
Iowa State University

logemann@iastate.edu

May 10, 2019

Overview

1 Introduction

2 Derivation

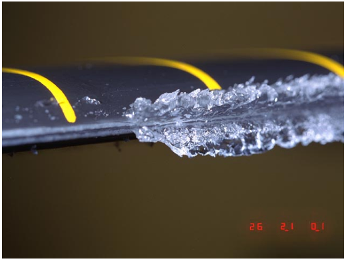
3 Method
- Convection
- Diffusion

4 Numerical Results
- Travelling Waves

5 Conclusion

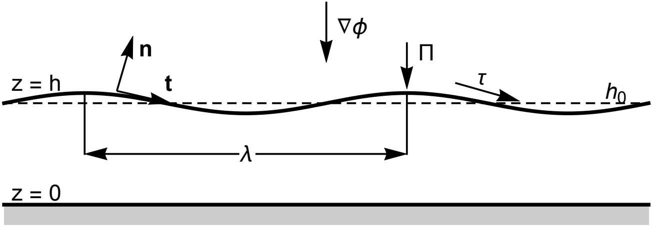
Motivation

- Aircraft Icing
- Runback



- Industrial Coating

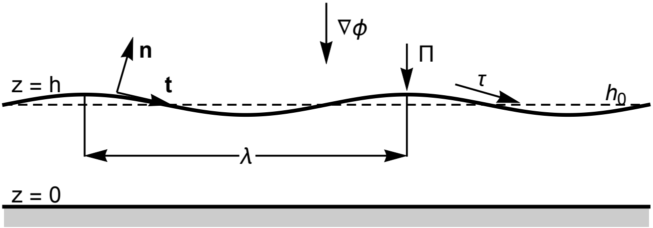
Model Equations



- Incompressible Navier-Stokes Equation

$$
\begin{aligned}
u_x + w_z &= 0 \\
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
w = 0, u &= 0 && \text{at } z = 0 \\
w &= h_t + uh_x && \text{at } z = h \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial \sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } z = h
\end{aligned}
$$

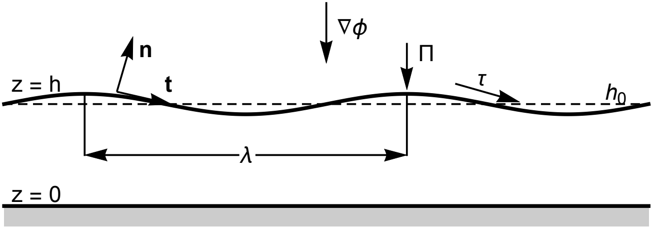
Nondimensionalization



$$\varepsilon = \frac{h_0}{\lambda} \ll 1 \qquad Z = \frac{z}{h_0} \qquad X = \frac{\varepsilon x}{h_0}$$

$$U = \frac{u}{U_0} \qquad W = \frac{w}{\varepsilon U_0} \qquad T = \frac{\varepsilon U_0 t}{h_0}$$

Nondimensionalization



$$U_X + W_Z = 0$$

$$\varepsilon Re(U_T + UU_X + WU_Z) = -P_X + U_{ZZ} + \varepsilon^2 U_{XX} - \Phi_X$$

$$\varepsilon^3 Re(W_T + WW_X + WW_Z) = -P_Z + \varepsilon^2\left(W_{ZZ} + \varepsilon^2 W_{XX}\right) - \Phi_Z$$

$$W = 0, U = 0 \qquad \text{at } Z = 0$$

$$W = H_T + UH_X \qquad \text{at } Z = H$$

$$U_Z + \varepsilon^2 W_X - 4\varepsilon^2 H_X U_X = \tau + \Sigma_X \qquad \text{at } Z = H$$

$$-P - \Pi + \varepsilon^2 U_X\left(\varepsilon^2 H_X^2 - 1\right) = \varepsilon^2 H_X\left(U_Z + \varepsilon^2 W_X\right) + C^{-1}\varepsilon^3 H_{XX} \qquad \text{at } Z = H$$

Take $\varepsilon \to 0$,

$$\begin{aligned} U_X + W_Z &= 0 \\ U_{ZZ} &= P_X + \Phi_X \\ 0 &= -P_Z - \Phi_Z \end{aligned}$$

$$\begin{aligned} W &= 0 \qquad \text{at } Z = 0 \\ U &= 0 \end{aligned}$$

$$\begin{aligned} W &= H_T + UH_X \qquad \text{at } Z = H \\ U_Z &= \tau_0 + \Sigma_X \\ -\Pi_0 - P &= \bar{C}^{-1} H_{XX} \end{aligned}$$

Integrate over Z and simplify

$$0 = H_T + \left(\int_0^H U \,\mathrm{d}Z \right)_X$$

$$P + \Phi = \Phi|_{Z=H} - C^{-1} H_{XX} - \Pi$$

$$U = (\tau + \Sigma_X) Z - (P_X + \Phi_X)\left(HZ - \frac{1}{2} Z^2 \right)$$

$$0 = H_T + \left((\tau + \Sigma_X) \frac{1}{2} H^2 - (P_X + \Phi_X) \frac{1}{3} H^3 \right)_X$$

$$P_X + \Phi_X = \left(\Phi|_{Z=H} - \Pi \right)_X - C^{-1} H_{XXX}$$

$$H_T + \left(\frac{1}{2} (\tau + \Sigma_X) H^2 - \frac{1}{3} \left(\Phi|_{Z=H} - \Pi \right)_X H^3 \right)_X = -\frac{1}{3} C^{-1} \left(H^3 H_{XXX} \right)_X$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

  Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x + \hat{f} v \Big|_{\partial\Omega} = 0$$

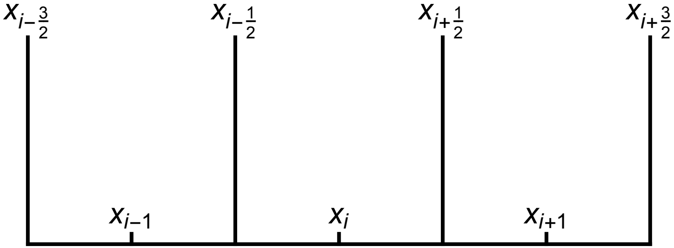
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \,\mathrm{d}x = \int_{I_j} f(Q) v_x \,\mathrm{d}x \\ - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

  for all $v \in V_h$
- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^{\star} &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^{\star}) + \frac{1}{2}\Delta t L(q^{\star}) \end{aligned}$$

Diffusion

- Diffusion Equation

$$q_t = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Linearize operator at $t = t^n$, let $f(x) = q^3(t = t^n, x)$

$$q_t = -(f(x) q_{xxx})_x \qquad (0, T) \times \Omega$$

Finite Difference Approach

- Let cell centers, x_i, form finite difference grid.
- Finite difference space, $\mathbb{R}^N$.
- $Q_{DG} \in V_h \to Q_{FD} \in \mathbb{R}^N$

$$(Q_{FD})_i = \frac{1}{h} \int_{K_i} Q_{DG} \,\mathrm{d}x$$

- $Q_{FD} \in \mathbb{R}^N \to Q_{DG} \in V_h$

$$Q_{DG}|_K \in P^1(K)$$

$$\frac{1}{h} \int_{K_i} Q_{DG} \,\mathrm{d}x = (Q_{FD})_i$$

$$\partial_x Q_{DG}|_{K_i} = \frac{(Q_{FD})_{i+1} - (Q_{FD})_{i-1}}{2h}$$

Finite Difference Approximation

- First derivative approximation

$$\left(-(f(x)q_{xxx})_x\right)_i \approx -\frac{f_{i+1/2}(q_{xxx})_{i+1/2} - f_{i-1/2}(q_{xxx})_{i-1/2}}{h}$$

- Third derivative approximation

$$(q_{xxx})_{i+1/2} \approx \frac{-Q_{i-1} + 3Q_i - 3Q_{i+1} + Q_{i+2}}{h^3}$$

- Value of Q^3 at boundary

$$f_{i+1/2} = q_{i+1/2}^3 = \left(\frac{Q_i + Q_{i+1}}{2}\right)^3$$

- Full operator

$$L(q^{n+1}) = L\left(f = \left(q^{n+1}\right)^3, q^{n+1}\right) = Aq^{n+1}$$

Implicit L-Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$q^\star = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^\star))$$
$$3q^{n+1} = 4q^\star - q^n + \Delta t L(q^{n+1})$$

Nonlinear Solvers

- Picard Iteration

$$q_0^{n+1} = q^n$$

$$q_{m+1}^{n+1} = q^n + \Delta t L\left(f = \left(q_m^{n+1}\right)^3, q_{m+1}^{n+1}\right)$$

$$q_{m+1}^{\star} = q^n + \frac{1}{4}\Delta t\left(L(q^n) + L\left(f = (q_m^{\star})^3, q_{m+1}^{\star}\right)\right)$$
$$3q_{m+1}^{n+1} = 4q^{\star} - q^n + \Delta t L\left(f = \left(q_m^{n+1}\right)^3, q_{m+1}^{n+1}\right)$$

Manufactured Solution

$$
\begin{aligned}
q_t + \left(q^2 - q^3\right)_x &= -\left(q^3 q_{xxx}\right)_x + s(x, t) \\
q_t + \left(q^2 - q^3\right)_x &= s(x, t) \\
q_t &= -\left(q^3 q_{xxx}\right)_x \\
q(x, t) &= 0.1 * \sin(2\pi(x - t)) + 0.15
\end{aligned}
$$

Backward Euler				
	1 Iteration		2 Iterations	
N	error	order	error	order
25	0.1529	—	0.0776	—
50	0.05334	1.52	0.0370	1.06
100	0.02374	1.16	0.0177	1.06
200	0.01186	1.00	0.0091	0.95

Manufactured Solution

$$
\begin{aligned}
q_t + \left(q^2 - q^3\right)_x &= -\left(q^3 q_{xxx}\right)_x + s(x, t) \\
q_t + \left(q^2 - q^3\right)_x &= s(x, t) \\
q_t &= -\left(q^3 q_{xxx}\right)_x \\
q(x, t) &= 0.1 * \sin(2\pi(x - t)) + 0.15
\end{aligned}
$$

2nd Order IRK						
	1 Iteration		2 Iterations		3 Iterations	
N	error	order	error	order	error	order
25	0.03449	—	0.02890	—	0.03103	—
50	0.01061	1.70	0.00875	1.72	0.00910	1.77
100	0.00330	1.68	0.00197	2.14	0.00202	2.17
200	0.00143	1.20	0.00051	1.96	0.00051	1.98

Manufactured Solution

$$
\begin{aligned}
q_t + \left(q^2 - q^3\right)_x &= -\left(q^3 q_{xxx}\right)_x + s(x, t) \\
q_t + \left(q^2 - q^3\right)_x &= s(x, t) \\
q_t &= -\left(q^3 q_{xxx}\right)_x \\
q(x, t) &= \frac{2}{10} e^{-10\left(x - t - \frac{3}{2}\right)^2} + \frac{1}{10}
\end{aligned}
$$

2nd Order IRK				
	1 Iteration		2 Iterations	
N	error	order	error	order
50	0.05609	—	0.3808	—
100	0.04178	0.42	0.2335	0.7
200	0.01182	1.82	0.0429	2.44
400	0.00612	0.94	0.0104	2.04
800			0.0026	2.03

Hyperbolic Wave Structure

- Conservation Law

$$q_t + f(q)_x = 0$$

- Riemann Problem Initial Data

$$q(x, 0) = \begin{cases} q_l & x < d \\ q_r & x > d \end{cases}$$

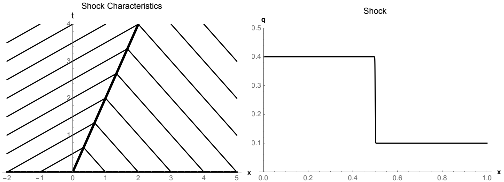
- Rankine-Hugoniot Condition

$$s = \frac{f(q_l) - f(q_r)}{q_l - q_r}$$

Convex Flux Function

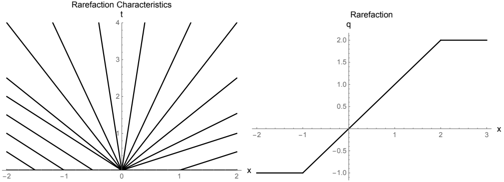
- Shock Wave

$$f'(q_l) > s > f'(q_r)$$



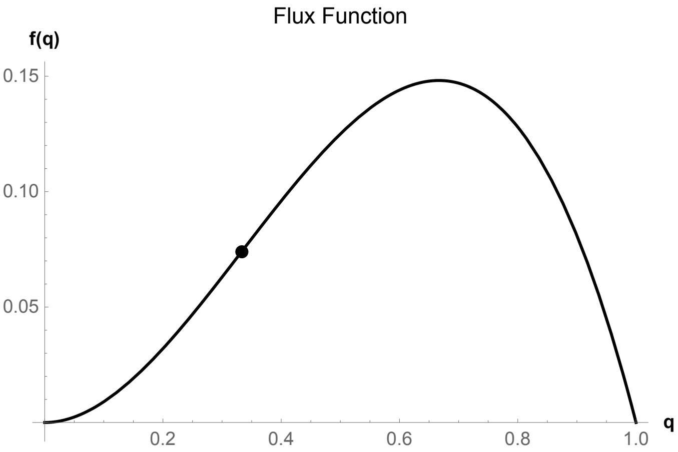
- Rarefaction

$$f'(q_l) < s < f'(q_r)$$



Nonconvex Flux Function

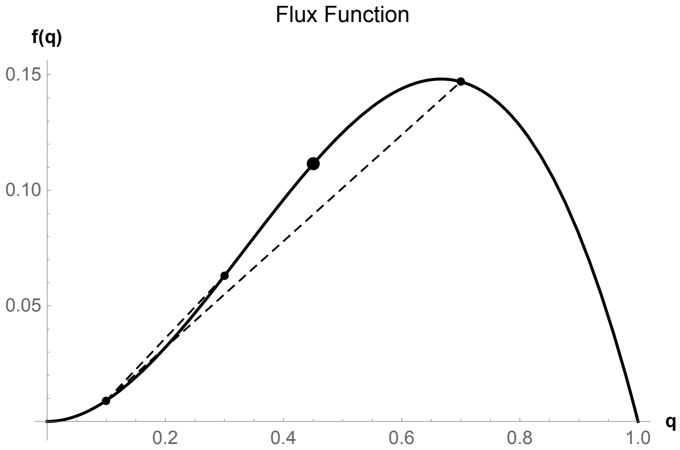
$$q_t + \left(q^2 - q^3\right)_x = 0$$



Nonconvex Flux Function

$$f'(q_b) = s$$

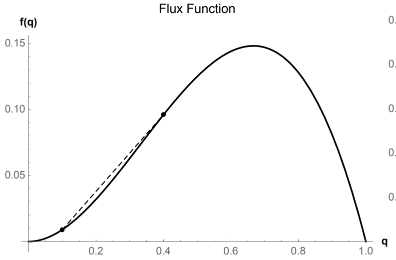
$$q_b = (1 - q_r)/2$$

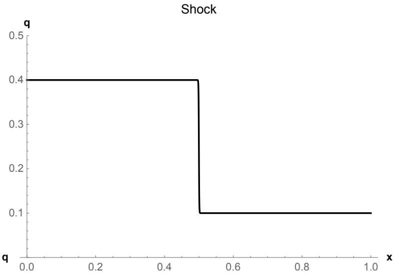


Compressive Shock

$$q_l < q_b$$

$$q(x,t) = \begin{cases} q_l & x \leq st \\ q_r & x > st \end{cases}$$

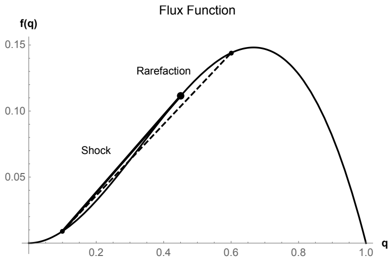


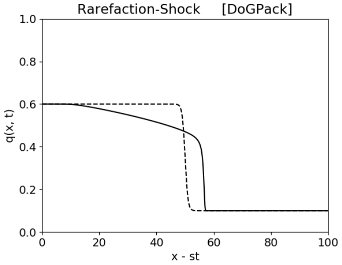


Rarefaction-Compressive Shock

$$q_l > q_b$$

$$q(x,t) = \begin{cases} q_l & x < f'(q_l)t \\ h_r(x) & f'(q_l)t < x < f'(q_b)t \\ q_r & x > f'(q_b)t \end{cases}$$

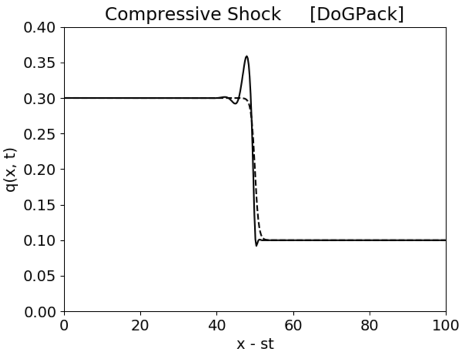




Wave Structure with Nonlinear Hyper Diffusion

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

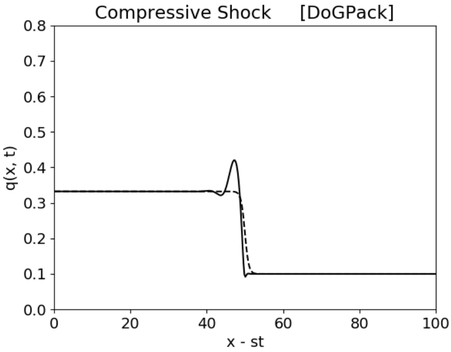
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



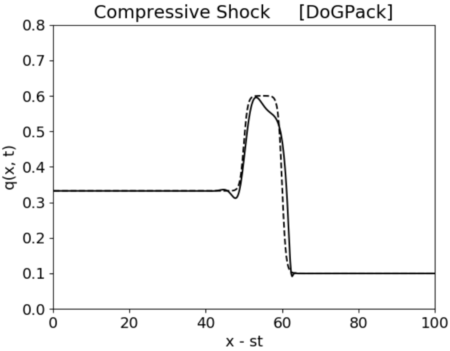
Wave Structure with Nonlinear Hyper Diffusion

Caleb Logemann, James Rossmanith

Introduction
Derivation
Method
Convection
Diffusion
Numerical Results
Travelling Waves
Conclusion
References

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

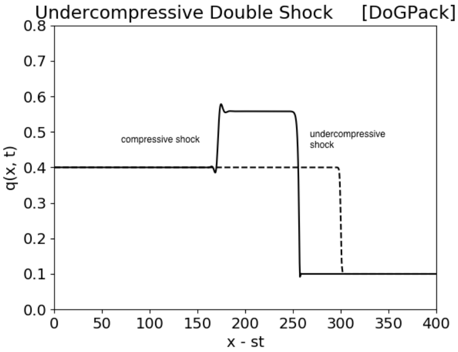
$$q(x,0) = \begin{cases} \frac{q_m - q_l}{2}\tanh(x-50) + \frac{q_m+q_l}{2} & x < 55 \\ -\frac{q_m - q_r}{2}\tanh(x-60) + \frac{q_m+q_r}{2} + q_r & x > 55 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.4$$

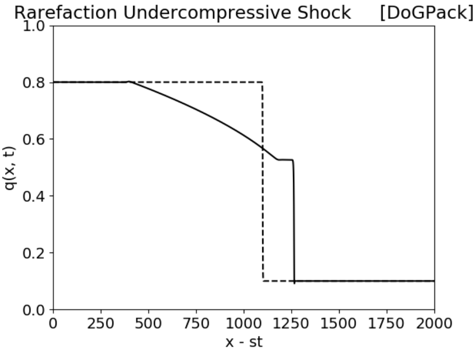
$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x, 0) = (-\tanh(x - 1100) + 1)\frac{q_l - q_r}{2} + q_r$$



Conclusion

Observations

- Nonlinear Hyper Diffusion has subtle instabilities

Future Work

- Higher Order Convergence
    - Runge Kutta IMEX
    - Local Discontinuous Galerkin Method
    - Hybridized Discontinuous Galerkin Method

Bibliography I

[1] Andrea L Bertozzi, Andreas Münch, and Michael Shearer. "Undercompressive shocks in thin film flows". In: *Physica D: Nonlinear Phenomena* 134.4 (1999), pp. 431–464.

[2] Y. Ha, Y.-J. Kim, and T.G. Myers. "On the numerical solution of a driven thin film equation". In: *J. Comp. Phys.* 227.15 (2008), pp. 7246–7263.

[3] T.G. Myers and J.P.F. Charpin. "A mathematical model for atmospheric ice accretion and water flow on a cold surface". In: *Int. J. Heat and Mass Transfer* 47.25 (2004), pp. 5483–5500.

[4] Tim G Myers. "Thin films with high surface tension". In: *SIAM review* 40.3 (1998), pp. 441–462.

[5] NASA. URL: http://icebox.grc.nasa.gov/gallery/images/C95_03918.html.

Bibliography II

[6] Alexander Oron, Stephen H Davis, and S George Bankoff. "Long-scale evolution of thin liquid films". In: *Reviews of modern physics* 69.3 (1997), p. 931.

[7] J.A. Rossmanith. DoGPack. Available from http://www.dogpack-code.org/.