Derivation of Shallow Water Equations

We begin by considering the Navier-Stokes equations,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\boldsymbol{u}_t + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \boldsymbol{g}, \tag{2}$$

where $\boldsymbol{u} = [u, v, w]^T$ is the vector of velocities, p is the pressure, ρ is the constant density, σ is the deviatoric stress tensor, and $\boldsymbol{g}$ is the gravitational force vector. We also have two boundaries, the bottom topography $h_b(t, x, y)$, and the free surface $h_s(t, x, y)$. At both of these boundaries the kinematic boundary conditions are in effect and can be expressed as

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s) \tag{3}$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b). \tag{4}$$

In practice the bottom topography is unchanging in time, but we express h_b with time dependence to allow for a symmetric representation of the boundary conditions.

1 Dimensional Analysis

Now we consider the characteristic scales of the problem. Let L be the characteristic horizontal length scale, and let H be the characteristic vertical length scale. For this problem we assume that $H << L$ and we denote the ratio of these lengths as $\varepsilon = H/L$. With these characteristic lengths we can scale the length variables to a nondimensional form

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}. \tag{5}$$

Now let U be the characteristic horizontal velocity, then because of the shallowness the characteristic vertical velocity will be εU. Therefore the velocity variables can be scaled as follows,

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}. \tag{6}$$

Now with the characteristic length and velocity, the time scaling can be described as

$$t = \frac{L}{U}\hat{t} \tag{7}$$

The pressure will be scaled by the characteristic height, H, and the stresses will be scaled by a characteristic stress, S. It is assumed that the basal shear stresses, σ_{xz} and σ_{yz} are of larger order than the lateral shear stress, σ_{xy}, and the normal stresses, σ_{xx}, σ_{yy}, and σ_{zz}, so that

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}. \tag{8}$$

$$\begin{aligned}
\nabla \cdot \boldsymbol{u} &= 0 \\
\boldsymbol{u}_t + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \boldsymbol{g}, \\
u_x + v_y + w_z &= 0 \\
u_t + \left(u^2\right)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho}p_x + \frac{1}{\rho}\Big((\sigma_{xx})_x + (\sigma_{xy})_y + (\sigma_{xz})_z\Big) + ge_x \\
v_t + (uv)_x + \left(v^2\right)_y + (vw)_z &= -\frac{1}{\rho}p_y + \frac{1}{\rho}\Big((\sigma_{xy})_x + (\sigma_{yy})_y + (\sigma_{yz})_z\Big) + ge_y \\
w_t + (uw)_x + (vw)_y + \left(w^2\right)_z &= -\frac{1}{\rho}p_z + \frac{1}{\rho}\Big((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\Big) + ge_z
\end{aligned}$$

$$u_x = (U\hat{u})_x = (U\hat{u})_{\hat{x}}\hat{x}_x = \frac{U}{L}\hat{u}_{\hat{x}}$$

$$v_y = (U\hat{v})_{\hat{y}}\hat{y}_y = \frac{U}{L}\hat{v}_{\hat{y}}$$

$$w_z = (\varepsilon U\hat{w})_{\hat{z}}\hat{z}_z = \frac{\varepsilon U}{H}\hat{w}_{\hat{z}} = \frac{U}{L}\hat{w}_{\hat{z}}$$

$$u_x + v_y + w_z = \frac{U}{L}\hat{u}_{\hat{x}} + \frac{U}{L}\hat{v}_{\hat{y}} + \frac{U}{L}\hat{w}_{\hat{z}} = 0$$

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$u_t = (U\hat{u})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{u}_{\hat{t}}$$

$$v_t = (U\hat{v})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{v}_{\hat{t}}$$

$$w_t = (\varepsilon U\hat{w})_{\hat{t}}\hat{t}_t = \varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}}$$

$$\left(u^2\right)_x = \left(U^2\hat{u}^2\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}\left(\hat{u}^2\right)_{\hat{x}}$$

$$(uv)_y = \left(U^2\hat{u}\hat{v}\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}}$$

$$(uw)_z = \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}}$$

$$(uv)_x = \left(U^2\hat{u}\hat{v}\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}}$$

$$\left(v^2\right)_y = \left(U^2\hat{v}^2\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}\left(\hat{v}^2\right)_{\hat{y}}$$

$$(vw)_z = \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}}$$

$$(uw)_x = \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{x}}\hat{x}_x = \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}}$$

$$(vw)_y = \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{y}}\hat{y}_y = \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}}$$

$$\left(w^2\right)_z = \left(\varepsilon^2U^2\hat{w}^2\right)_{\hat{z}}\hat{z}_z = \varepsilon\frac{U^2}{L}\left(\hat{w}^2\right)_{\hat{z}}$$

$$
\begin{aligned}
\frac{1}{\rho}p_x &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{x}}\hat{x}_x = \frac{\varepsilon g L}{L}\hat{p}_{\hat{x}} = \varepsilon g\hat{p}_{\hat{x}} \\
\frac{1}{\rho}p_y &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{y}}\hat{y}_y = \frac{\varepsilon g L}{L}\hat{p}_{\hat{y}} = \varepsilon g\hat{p}_{\hat{y}} \\
\frac{1}{\rho}p_z &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{z}}\hat{z}_z = \frac{\varepsilon g L}{\varepsilon L}\hat{p}_{\hat{z}} = g\hat{p}_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xx})_x &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xx})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xx})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{xy})_y &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{xz})_z &= \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{xz})_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xy})_x &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{yy})_y &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{yy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{yz})_z &= \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xz})_x &= \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{x}}\hat{x}_x = \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{yz})_y &= \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{y}}\hat{y}_y = \frac{S}{\rho L}(\hat{\sigma}_{yz})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{zz})_z &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{zz})_{\hat{z}}\hat{z}_z = \varepsilon\frac{S}{\rho H}(\hat{\sigma}_{zz})_{\hat{z}} \\
\frac{U^2}{L}\hat{u}_{\hat{t}} + \frac{U^2}{L}\left(\hat{u}^2\right)_{\hat{x}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}} + \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}} &= -\varepsilon g\hat{p}_{\hat{x}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{y}} + \frac{S}{\rho H}(\hat{\sigma}_{xz})_{\hat{z}} + ge_x \\
\frac{U^2}{L}\hat{v}_{\hat{t}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}} + \frac{U^2}{L}\left(\hat{v}^2\right)_{\hat{y}} + \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}} &= -\varepsilon g\hat{p}_{\hat{y}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}} + \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}} + ge_y \\
\varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}} + \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}\left(\hat{w}^2\right)_{\hat{z}} &= -g\hat{p}_{\hat{z}} + \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}} + \frac{S}{\rho L}(\hat{\sigma}_{yz})_{\hat{y}} + \varepsilon\frac{S}{\rho H}(\hat{\sigma}_{zz})_{\hat{z}} + ge_z \\
\frac{U^2}{L}\left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\right) &= -\varepsilon g\hat{p}_{\hat{x}} + \frac{S}{\rho H}\left(\varepsilon^2(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\right) + ge_x \\
\frac{U^2}{L}\left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\right) &= -\varepsilon g\hat{p}_{\hat{y}} + \frac{S}{\rho H}\left(\varepsilon^2(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\right) + ge_y \\
\varepsilon\frac{U^2}{L}\left(\hat{w}_{\hat{t}} + (\hat{u}\hat{w})_{\hat{x}} + (\hat{v}\hat{w})_{\hat{x}} + \left(\hat{w}^2\right)_{\hat{z}}\right) &= -g\hat{p}_{\hat{z}} + \varepsilon\frac{S}{\rho H}\left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}}\right) + ge_z
\end{aligned}
$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$
\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon\frac{U^2}{gH}\left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\right) &= -\varepsilon\hat{p}_{\hat{x}} + \frac{S}{\rho g H}\left(\varepsilon^2(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\right) + e_x \\
\varepsilon\frac{U^2}{gH}\left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\right) &= -\varepsilon\hat{p}_{\hat{y}} + \frac{S}{\rho g H}\left(\varepsilon^2(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\right) + e_y \\
\varepsilon^2\frac{U^2}{gH}\left(\hat{w}_{\hat{t}} + (\hat{u}\hat{w})_{\hat{x}} + (\hat{v}\hat{w})_{\hat{x}} + \left(\hat{w}^2\right)_{\hat{z}}\right) &= -\hat{p}_{\hat{z}} + \varepsilon\frac{S}{\rho g H}\left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}}\right) + e_z
\end{aligned}
$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon F^2\Big(\hat{u}_{\hat{t}} + \big(\hat{u}^2\big)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{x}} + G\Big(\varepsilon^2(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\Big) + e_x \\
\varepsilon F^2\Big(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \big(\hat{v}^2\big)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{y}} + G\Big(\varepsilon^2(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\Big) + e_y \\
\varepsilon^2 F^2\big(\hat{w}_{\hat{t}} + (\hat{u}\hat{w})_{\hat{x}} + (\hat{v}\hat{w})_{\hat{x}} + \big(\hat{w}^2\big)_{\hat{z}}\big) &= -\hat{p}_{\hat{z}} + \varepsilon G\Big((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}}\Big) + e_z \\
F = \frac{U}{\sqrt{gH}} \approx 1, &\quad G = \frac{S}{\rho g H} < 1
\end{aligned}$$

Drop terms with ε^2 and εG, giving

$$\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon F^2\Big(\hat{u}_{\hat{t}} + \big(\hat{u}^2\big)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{x}} + G(\hat{\sigma}_{xz})_{\hat{z}} + e_x \\
\varepsilon F^2\Big(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \big(\hat{v}^2\big)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{y}} + G(\hat{\sigma}_{yz})_{\hat{z}} + e_y \\
\hat{p}_{\hat{z}} &= e_z
\end{aligned}$$

where we can solve for the hydrostatic pressure

$$\hat{p}(\hat{t}, \hat{x}, \hat{y}) = \Big(\hat{h}_s(\hat{t}, \hat{x}, \hat{y}) - \hat{z}\Big)e_z$$

Transforming back to dimensional variables for readability gives

$$\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + \big(u^2\big)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x \\
v_t + (uv)_x + \big(v^2\big)_y + (vw)_z &= -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y \\
p(t, x, y, z) &= (h_s(t, x, y) - z)\rho g e_z
\end{aligned}$$

2 Mapping

In order to make this system more accessible we will map the vertical variable z to the normalized variable ζ, through the transformation

$$\zeta(t, x, y, z) = \frac{z - h_b(t, x, y)}{h(t, x, y)},$$

or equivalently

$$z(t, x, y, \zeta) = h(t, x, y)\zeta + h_b(t, x, y)$$

where $h(t, x, y) = h_s(t, x, y) - h_b(t, x, y)$. This transformation maps the vertical variable onto a scale from 0 to 1 everywhere.

Consider a function $\Psi(t, x, y, z)$, then is mapped counterpart $\tilde{\Psi}(t, x, y, \zeta)$ can be described as

$$\tilde{\Psi}(t, x, y, \zeta) = \Psi(t, x, y, z(t, x, y, \zeta)) = \Psi(t, x, y, h(t, x, y)\zeta + h_b(t, x, y)),$$

or equivalently

$$\Psi(t, x, y, z) = \tilde{\Psi}(t, x, y, \zeta(t, x, y, z)) = \tilde{\Psi}\bigg(t, x, y, \frac{z - h_b(t, x, y)}{h(t, x, y)}\bigg).$$

We also need to be able to map derivatives of functions in order to be able to map the differential equations. This can be described

$$
\begin{aligned}
\Psi_z(t,x,y,z) &= \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_z \\
\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\zeta_z(t,x,y,z) \\
\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\frac{1}{h(t,x,y)} \\
h(t,x,y)\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z)) \\
h\Psi_z &= \tilde{\Psi}_\zeta
\end{aligned}
$$

For the other derivatives, the partial derivatives are identical for $s \in \{t,x,y\}$.

$$
\begin{aligned}
\zeta_s(t,x,y,z) &= \left(\frac{z-h_b(t,x,y)}{h(t,x,y)}\right)_s \\
&= -\frac{(z-h_b(t,x,y))h_s(t,x,y)}{h(t,x,y)^2} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \\
&= -\zeta(t,x,y,z)\frac{h_s(t,x,y)}{h(t,x,y)} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \\
&= -\frac{\zeta(t,x,y,z)h_s(t,x,y)+(h_b)_s(t,x,y)}{h(t,x,y)}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_s(t,x,y,z) &= \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_s \\
\Psi_s(t,x,y,z) &= \tilde{\Psi}_s(t,x,y,\zeta(t,x,y,z)) + \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))\zeta_s(t,x,y,z) \\
\Psi_s(t,x,y,z) &= \tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)\left(\frac{\zeta h_s(t,x,y)+(h_b)_s(t,x,y)}{h(t,x,y)}\right) \\
h(t,x,y)\Psi_s(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y)+(h_b)_s(t,x,y)) \\
h(t,x,y)\Psi_s(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y)+(h_b)_s(t,x,y)) \\
h\Psi_s &= h\tilde{\Psi}_s - \tilde{\Psi}_\zeta(\zeta h_s + (h_b)_s) \\
h\Psi_s &= h\tilde{\Psi}_s + h_s\tilde{\Psi} - h_s\tilde{\Psi} - \tilde{\Psi}_\zeta(\zeta h + h_b)_s \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(h_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(\left((\zeta h + h_b)_\zeta\right)_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(((\zeta h + h_b)_s)_\zeta\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left((\zeta h + h_b)_s\tilde{\Psi}\right)_\zeta
\end{aligned}
$$

Now we can use these differential transformations to map the continuity equation or mass balance equation onto the normalized space.

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
h(u_x + v_y + w_z) &= 0 \\
hu_x + hv_y + hw_z &= 0 \\
(h\tilde{u})_x - ((\zeta h + h_b)_x \tilde{u})_\zeta + (h\tilde{v})_y - \Big((\zeta h + h_b)_y \tilde{v}\Big)_\zeta + (\tilde{w})_\zeta &= 0 \\
(h\tilde{u})_x + (h\tilde{v})_y + \Big(\tilde{w} - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\Big)_\zeta &= 0 \\
-(h\tilde{u})_x - (h\tilde{v})_y + \Big((\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v}\Big)_\zeta &= (\tilde{w})_\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + \int_0^{\zeta'} \Big((\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v}\Big)_\zeta \,\mathrm{d}\zeta &= \int_0^{\zeta'} (\tilde{w})_\zeta \,\mathrm{d}\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + (\zeta' h + h_b)_x \tilde{u}(t,x,y,\zeta') + (\zeta' h + h_b)_y \tilde{v}(t,x,y,\zeta') & \\
-(h_b)_x \tilde{u}(t,x,y,0) - (h_b)_y \tilde{v}(t,x,y,0) &= \tilde{w}(t,x,y,\zeta') - \tilde{w}(t,x,y,0) \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + (\zeta' h + h_b)_x \tilde{u}(t,x,y,\zeta') + (\zeta' h + h_b)_y \tilde{v}(t,x,y,\zeta') &= \tilde{w}(t,x,y,\zeta')
\end{aligned}
$$

$$
\tilde{w}(t,x,y,\zeta') = -\left(h \int_0^{\zeta'} \tilde{u} \,\mathrm{d}\zeta\right)_x - \left(h \int_0^{\zeta'} \tilde{v} \,\mathrm{d}\zeta\right)_y
$$
$$
+(\zeta' h + h_b)_x \tilde{u}(t,x,y,\zeta') + (\zeta' h + h_b)_y \tilde{v}(t,x,y,\zeta')
$$

$$
h_t + (hu_m)_x + (hv_m)_y = 0
$$

Original Kinematic Boundary Conditions

$$
\begin{aligned}
(h_s)_t + [u(t,x,y,h_s), v(t,x,y,h_s)]^T \cdot \nabla h_s &= w(t,x,y,h_s) \\
(h_b)_t + [u(t,x,y,h_b), v(t,x,y,h_b)]^T \cdot \nabla h_b &= w(t,x,y,h_b)
\end{aligned}
$$

$$
\begin{aligned}
(h_s)_t + u(t,x,y,h_s)(h_s)_x + v(t,x,y,h_s)(h_s)_y &= w(t,x,y,h_s) \\
(h_b)_t + u(t,x,y,h_b)(h_b)_x + v(t,x,y,h_b)(h_b)_y &= w(t,x,y,h_b)
\end{aligned}
$$

Mapped Kinematic Boundary Conditions

$$
\begin{aligned}
(h_s)_t + [\tilde{u}(t,x,y,1), \tilde{v}(t,x,y,1)]^T \cdot \nabla h_s &= \tilde{w}(t,x,y,1) \\
(h_b)_t + [\tilde{u}(t,x,y,0), \tilde{v}(t,x,y,0)]^T \cdot \nabla h_b &= \tilde{w}(t,x,y,0)
\end{aligned}
$$

$$
\begin{aligned}
(h_s)_t + \tilde{u}(t,x,y,1)(h_s)_x + \tilde{v}(t,x,y,1)(h_s)_y &= \tilde{w}(t,x,y,1) \\
(h_b)_t + \tilde{u}(t,x,y,0)(h_b)_x + \tilde{v}(t,x,y,0)(h_b)_y &= \tilde{w}(t,x,y,0)
\end{aligned}
$$

$$
\begin{aligned}
u_t + \big(u^2\big)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho} p_x + \\
hu_t + h\big(u^2\big)_x + h(uv)_y + h(uw)_z + \frac{1}{\rho} h p_x &= +\frac{1}{\rho} h(\sigma \\
(h\tilde{u})_t - ((\zeta h + h_b)_t \tilde{u})_\zeta + \big(h\tilde{u}^2\big)_x - \big((\zeta h + h_b)_x \tilde{u}^2\big)_\zeta + (h\tilde{u}\tilde{v})_y - \Big((\zeta h + h_b)_y \tilde{u}\tilde{v}\Big)_\zeta + (\tilde{u}\tilde{w})_\zeta + \frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho}(\tilde{\sigma}_{xz}) \\
(h\tilde{u})_t + \big(h\tilde{u}^2\big)_x + (h\tilde{u}\tilde{v})_y + \Big(\tilde{u}\Big(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\Big)\Big)_\zeta + \frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho}(\tilde{\sigma}_{xz})
\end{aligned}
$$

$$\tilde{p}(t,x,y,\zeta) = h(t,x,y)(1-\zeta)\rho g e_z$$

$$\begin{aligned}
\frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho}\big(h^2(1-\zeta)\rho g e_z\big)_x - \frac{1}{\rho}((\zeta h + h_b)_x h(1-\zeta)\rho g e_z)_\zeta \\
&= \big(h^2(1-\zeta) g e_z\big)_x - ((\zeta h + h_b)_x h(1-\zeta) g e_z)_\zeta \\
&= 2hh_x(1-\zeta)ge_z - ((\zeta h_x + (h_b)_x) h(1-\zeta) g e_z)_\zeta \\
&= 2hh_x(1-\zeta)ge_z - hge_z((\zeta h_x + (h_b)_x)(1-\zeta))_\zeta \\
&= 2hh_x(1-\zeta)ge_z - hge_z(h_x(1-\zeta) - \zeta h_x - (h_b)_x) \\
&= hh_x g e_z + (h_b)_x h g e_z \\
&= \left(\frac{1}{2}h^2 g e_z\right)_x + (h_b)_x h g e_z
\end{aligned}$$

$$\begin{gathered}
(h\tilde{u})_t + \big(h\tilde{u}^2\big)_x + (h\tilde{u}\tilde{v})_y \\
+\Big(\tilde{u}\Big(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\Big)\Big)_\zeta \\
+\left(\frac{1}{2}h^2 g e_z\right)_x + (h_b)_x h g e_z = \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + g h e_x \\
(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}h^2 g e_z\right)_x + (h\tilde{u}\tilde{v})_y \\
+\Big(\tilde{u}\Big(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\Big)\Big)_\zeta \\
= \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + gh(e_x - (h_b)_x e_z)
\end{gathered}$$

$$
\begin{aligned}
&\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y + (\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v} \\
&\quad - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y - (\zeta h + h_b)_t \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y - \zeta h_t \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y + \zeta\Big((hu_m)_x + (hv_m)_y\Big) \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y + \int_0^\zeta \mathrm{d}\zeta' \Big((hu_m)_x + (hv_m)_y\Big) \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta'\right)_y + \left(h \int_0^\zeta u_m\, \mathrm{d}\zeta'\right)_x + \left(h \int_0^\zeta v_m\, \mathrm{d}\zeta'\right)_y \\
&= -\left(h \int_0^\zeta \tilde{u}\, \mathrm{d}\zeta' + h \int_0^\zeta u_m\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\, \mathrm{d}\zeta' + h \int_0^\zeta v_m\, \mathrm{d}\zeta'\right)_y \\
&= -\left(h \int_0^\zeta \tilde{u} - u_m\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v} - v_m\, \mathrm{d}\zeta'\right)_y \\
&= -\left(h \int_0^\zeta \tilde{u}_d\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}_d\, \mathrm{d}\zeta'\right)_y \\
&\tilde{u}_d = \tilde{u} - u_m \quad \tilde{v}_d = \tilde{v} - v_m
\end{aligned}
$$

We will label this term as

$$h\omega = -\left(h \int_0^\zeta \tilde{u}_d\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}_d\, \mathrm{d}\zeta'\right)_y$$

Note that this is the same as defining

$$\omega = \frac{1}{h}\left(-\left(h \int_0^\zeta \tilde{u}_d\, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}_d\, \mathrm{d}\zeta'\right)_y\right)$$

The