Derivation of Shallow Water Moment Equations in Spherical Coordinates

First I will define the transformation from cartesian $\boldsymbol{x} = [x, y, z]$ to spherical coordinates, $\boldsymbol{r} = [r, \theta, \phi]$

$$r = s_1(x, y, z) = \sqrt{x^2 + y^2 + z^2} \tag{1}$$

$$\theta = s_2(x, y, z) = \arctan\left(\frac{y}{x}\right) \tag{2}$$

$$\phi = s_3(x, y, z) = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{3}$$

$$\boldsymbol{s}(\boldsymbol{x}) = \boldsymbol{s}(x, y, z) = [s_1(x, y, z), s_2(x, y, z), s_3(x, y, z)] \tag{4}$$

$$x = c_1(r, \theta, \phi) = r\cos(\theta)\sin(\phi) \tag{5}$$

$$y = c_2(r, \theta, \phi) = r\sin(\theta)\sin(\phi) \tag{6}$$

$$z = c_3(r, \theta, \phi) = r\cos(\phi) \tag{7}$$

$$\boldsymbol{c}(\boldsymbol{r}) = \boldsymbol{c}(r, \theta, \phi) = [c_1(r, \theta, \phi), c_2(r, \theta, \phi), c_3(r, \theta, \phi)] \tag{8}$$

The projections from cartesian to spherical coordinates can be computed using the following unit vectors

$$\hat{r} = \cos(\theta)\sin(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\phi)\hat{z} \tag{9}$$

$$\hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y} \tag{10}$$

$$\hat{\phi} = \cos(\theta)\cos(\phi)\hat{x} + \sin(\theta)\cos(\phi)\hat{y} - \sin(\phi)\hat{z} \tag{11}$$

$$\hat{x} = \cos(\theta)\sin(\phi)\hat{r} - \sin(\theta)\hat{\theta} + \cos(\theta)\cos(\phi)\hat{\phi} \tag{12}$$

$$\hat{y} = \sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\hat{\theta} + \sin(\theta)\cos(\phi)\hat{\phi} \tag{13}$$

$$\hat{z} = \cos(\phi)\hat{r} - \sin(\phi)\hat{\phi} \tag{14}$$

$$\tag{15}$$

The partial derivatives of the transformation to spherical coordinates are needed, when considering differential equations.

$$s_{1,x} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2x = \cos(\theta)\sin(\phi) \tag{16}$$

$$s_{1,y} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2y = \sin(\theta)\sin(\phi) \tag{17}$$

$$s_{1,z} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2z = \cos(\phi) \tag{18}$$

$$\tag{19}$$

$$s_{2,x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) \tag{20}$$

$$= \frac{1}{1+\frac{r^2\sin^2(\theta)\sin^2(\phi)}{r^2\cos^2(\theta)\sin^2(\phi)}}\left(-\frac{r\sin(\theta)\sin(\phi)}{r^2\cos^2(\theta)\sin^2(\phi)}\right) \tag{21}$$

$$= \frac{1}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}}\left(-\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)}\right) \tag{22}$$

$$= -\frac{1}{1+\frac{1-\cos^2(\theta)}{\cos^2(\theta)}}\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)} \tag{23}$$

$$= -\cos^2(\theta)\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)} \tag{24}$$

$$= -\frac{\sin(\theta)}{r\sin(\phi)} \tag{25}$$

$$s_{2,y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) \tag{26}$$

$$= \cos^2(\theta)\frac{1}{r\cos(\theta)\sin(\phi)} \tag{27}$$

$$= \frac{\cos(\theta)}{r\sin(\phi)} \tag{28}$$

$$s_{2,z} = 0 \tag{29}$$

$$\tag{30}$$

$$s_{3,x} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{1}{2}\frac{2x}{z\sqrt{x^2+y^2}} \tag{31}$$

$$= \frac{1}{1+\frac{r^2\cos^2(\theta)\sin^2(\phi)+r^2\sin^2(\theta)\sin^2(\phi)}{r^2\cos^2(\phi)}}\frac{r\cos(\theta)\sin(\phi)}{r\cos(\phi)\sqrt{r^2\cos^2(\theta)\sin^2(\phi)+r^2\sin^2(\theta)\sin^2(\phi)}} \tag{32}$$

$$= \frac{1}{1+\frac{\sin^2(\phi)}{\cos^2(\phi)}}\frac{\cos(\theta)}{r\cos(\phi)} \tag{33}$$

$$= \frac{1}{1+\frac{1-\cos^2(\phi)}{\cos^2(\phi)}}\frac{\cos(\theta)}{r\cos(\phi)} \tag{34}$$

$$= \cos^2(\phi)\frac{\cos(\theta)}{r\cos(\phi)} \tag{35}$$

$$= \frac{\cos(\theta)\cos(\phi)}{r} \tag{36}$$

$$s_{3,y} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{1}{2}\frac{2y}{z\sqrt{x^2+y^2}} \tag{37}$$

$$= \cos^2(\phi)\frac{r\sin(\theta)\sin(\phi)}{r\cos(\phi)r\sin(\phi)} \tag{38}$$

$$= \frac{\sin(\theta)\cos(\phi)}{r} \tag{39}$$

$$s_{3,z} = -\frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{\sqrt{x^2+y^2}}{z^2} \tag{40}$$

$$= -\cos^2(\phi)\frac{r\sin(\phi)}{r^2\cos^2(\phi)} \tag{41}$$

$$= -\frac{\sin(\phi)}{r} \tag{42}$$

$$\tag{43}$$

The derivative with respect to a cartesian coordinate, can be expressed in terms of the spherical coordinate derivatives using the chain rule. Consider a function of spherical coordinates, $f(r,\theta,\phi)$, then

$$\frac{\partial}{\partial t}(f(\boldsymbol{s}(\boldsymbol{x}))) = \frac{\partial f}{\partial r}\frac{\partial s_1}{\partial t} + \frac{\partial f}{\partial \theta}\frac{\partial s_2}{\partial t} + \frac{\partial f}{\partial \phi}\frac{\partial s_3}{\partial t} \tag{44}$$

where $t = x, y, \text{or} z$

Now let's consider the Navier Stokes equation The velocities in cartesian coordinates are given by $[u, v, w] = \boldsymbol{u}$, then the cartesian Navier Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{45}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^2\right) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + g_x \tag{46}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}\left(v^2\right) + \frac{\partial}{\partial z}(vw) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + g_y \tag{47}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}\left(w^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + g_z \tag{48}$$

$$\tag{49}$$

In spherical coordinates we represent the velocities as $[u_r, u_\theta, u_\phi] = \boldsymbol{u}_r$. Using the unit vectors we can write these velocities in terms of the cartesian velocities.

$$u_r = \cos(\theta)\sin(\phi)u + \sin(\theta)\sin(\phi)v + \cos(\phi)w \tag{50}$$

$$u_\theta = -\sin(\theta)u + \cos(\theta)w \tag{51}$$

$$u_\phi = \cos(\theta)\cos(\phi)u + \sin(\theta)\cos(\phi)v - \sin(\phi)w \tag{52}$$

We can also write the cartesian velocities in terms of the spherical velocities.

$$u = \cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi \tag{53}$$

$$v = \sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi \tag{54}$$

$$w = \cos(\phi)u_r - \sin(\phi)u_\phi \tag{55}$$

Now we can express the continuity equation in spherical coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{56}$$

$$\frac{\partial}{\partial x}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi) \tag{57}$$

$$+\frac{\partial}{\partial y}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi) \tag{58}$$

$$+\frac{\partial}{\partial z}(\cos(\phi)u_r - \sin(\phi)u_\phi) = 0 \tag{59}$$

$$\tag{60}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial x} \tag{61}$$
$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial x} \tag{62}$$
$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial x} \tag{63}$$
$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial y} \tag{64}$$
$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial y} \tag{65}$$
$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial y} \tag{66}$$
$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_1}{\partial z} \tag{67}$$
$$+\frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_2}{\partial z} \tag{68}$$
$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_3}{\partial z} = 0 \tag{69}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \tag{70}$$
$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \tag{71}$$
$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \tag{72}$$
$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \tag{73}$$
$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \tag{74}$$
$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \tag{75}$$
$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \tag{76}$$
$$+\frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)0 \tag{77}$$
$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \tag{78}$$
$$\tag{79}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \tag{80}$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \tag{81}$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \tag{82}$$

$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \tag{83}$$

$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \tag{84}$$

$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \tag{85}$$

$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \tag{86}$$

$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \tag{87}$$

$$\tag{88}$$

Now I will consider the terms of each derivative of each velocity individually,

$\frac{\partial u_r}{\partial r}$

$$\cos^2(\theta)\sin^2(\phi)\frac{\partial u_r}{\partial r} + \sin^2(\theta)\sin^2(\phi)\frac{\partial u_r}{\partial r} + \cos^2(\phi)\frac{\partial u_r}{\partial r} = \sin^2(\phi)\frac{\partial u_r}{\partial r} + \cos^2(\phi)\frac{\partial u_r}{\partial r} = \frac{\partial u_r}{\partial r} \tag{89}$$

$\frac{\partial u_\theta}{\partial r}$

$$-\sin(\theta)\cos(\theta)\sin(\phi)\frac{\partial u_\theta}{\partial r} + \cos(\theta)\sin(\theta)\sin(\phi)\frac{\partial u_\theta}{\partial r} = 0 \tag{90}$$

$\frac{\partial u_\phi}{\partial r}$

$$\cos^2(\theta)\cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} + \sin^2(\theta)\cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} - \sin(\phi)\cos(\phi)\frac{\partial u_\phi}{\partial r} \tag{91}$$

$$= \cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} - \sin(\phi)\cos(\phi)\frac{\partial u_\phi}{\partial r} = 0 \tag{92}$$

$\frac{\partial u_r}{\partial \theta}$

$$-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}(\cos(\theta)u_r) + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}(\sin(\theta)u_r) \tag{93}$$

$$= -\frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial u_r}{\partial \theta} + \frac{\sin^2(\theta)}{r}u_r + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial u_r}{\partial \theta} + \frac{\cos^2(\theta)}{r}u_r = \frac{u_r}{r} \tag{94}$$

$\frac{\partial u_\theta}{\partial \theta}$

$$\frac{\sin(\theta)}{r\sin(\phi)}\frac{\partial}{\partial \theta}(\sin(\theta)u_\theta) + \frac{\cos(\theta)}{r\sin(\phi)}\frac{\partial}{\partial \theta}(\cos(\theta)u_\theta) = \tag{95}$$

$$\frac{\sin(\theta)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\theta}{\partial \theta} + \cos(\theta)u_\theta\right) + \frac{\cos(\theta)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\theta}{\partial \theta} - \sin(\theta)u_\theta\right) = \tag{96}$$

$$\frac{\sin^2(\theta)}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{\cos^2(\theta)}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} \tag{97}$$

$\frac{\partial u_\phi}{\partial \theta}$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\cos(\theta)u_\phi)+\frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\sin(\theta)u_\phi)= \tag{98}$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\phi}{\partial\theta}-\sin(\theta)u_\phi\right)+\frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\phi}{\partial\theta}+\cos(\theta)u_\phi\right)= \tag{99}$$

$$\frac{\sin^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi+\frac{\cos^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi=\frac{\cos(\phi)}{r\sin(\phi)}u_\phi \tag{100}$$

$\frac{\partial u_r}{\partial \phi}$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r)+\frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r)-\frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r)= \tag{101}$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r)-\frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r)= \tag{102}$$

$$\frac{\cos(\phi)}{r}\left(\sin(\phi)\frac{\partial u_r}{\partial\phi}+\cos(\phi)u_r\right)-\frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial u_r}{\partial\phi}-\sin(\phi)u_r\right)= \tag{103}$$

$$\frac{\cos^2(\phi)}{r}u_r+\frac{\sin^2(\phi)}{r}u_r=\frac{1}{r}u_r \tag{104}$$

$\frac{\partial u_\theta}{\partial \phi}$

$$-\frac{\cos(\theta)\sin(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi}+\frac{\sin(\theta)\cos(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi}=0 \tag{105}$$

$\frac{\partial u_\phi}{\partial \phi}$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi)+\frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi)+\frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)= \tag{106}$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi)+\frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)= \tag{107}$$

$$\frac{\cos(\phi)}{r}\left(\cos(\phi)\frac{\partial u_\phi}{\partial\phi}-\sin(\phi)u_\phi\right)+\frac{\sin(\phi)}{r}\left(\sin(\phi)\frac{\partial u_\phi}{\partial\phi}+\cos(\phi)u_\phi\right)= \tag{108}$$

$$\frac{\cos^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{\sin^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi}=\frac{1}{r}\frac{\partial u_\phi}{\partial\phi} \tag{109}$$

The simplified continuity equation is thus

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{\cos(\phi)}{r\sin(\phi)}u_\phi+\frac{1}{r}u_r+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}=0 \tag{110}$$

$$\frac{\partial u_r}{\partial r}+2\frac{u_r}{r}+\frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{\cos(\phi)}{r\sin(\phi)}u_\phi=0 \tag{111}$$

Using the product rule this can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{112}$$

$$\tag{113}$$

Shallow Water in Spherical Coordinates R radius, (θ, ϕ), longitude and latitude

$$h_t + \frac{1}{R\cos(\phi)}(hu_\theta)_\theta + \frac{1}{R\cos(\phi)}(hu_\phi\cos(\phi))_\phi = 0 \tag{114}$$

$$(hu_\theta)_t + \frac{1}{R\cos(\phi)}\left(hu_\theta^2 + \frac{1}{2}gh^2\right)_\theta + \frac{1}{R}(hu_\theta u_\phi) - 2\frac{hu_\theta u_\phi}{R}\tan(\phi) = 0 \tag{115}$$

$$(hu_\phi)_\phi + \frac{1}{R\cos(\phi)}(hu_\theta u_\phi)_\theta + \frac{1}{R}\left(hu_\phi^2 + \frac{1}{2}gh^2\right)_\phi + \frac{hu_\theta^2 - hu_\phi^2}{R}\tan(\phi) = 0 \tag{116}$$

Navier Stokes Equations in Spherical Coordinates r radius, (θ, ϕ) azimuth, and polar angle, $\theta = \arctan(y/x), \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)u_\phi) = 0 \tag{117}$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r \tag{118}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi\cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial \theta} + g_\theta \tag{119}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} - \frac{u_\theta^2\cot(\phi)}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + g_\phi \tag{120}$$

Kinematic Boundary Condition, $h_s(t, \theta, \phi)$, $h_b(t, \theta, \phi)$ $u_{\theta/\phi/r}(t, \theta, \phi, r)$

$$\frac{\partial h_s}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_s)}{r\sin(\phi)}\frac{\partial h_s}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_s)}{r}\frac{\partial h_s}{\partial \phi} = u_r(t, \theta, \phi, h_s) \tag{121}$$

$$\frac{\partial h_b}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_b)}{r\sin(\phi)}\frac{\partial h_b}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_b)}{r}\frac{\partial h_b}{\partial \phi} = u_r(t, \theta, \phi, h_b) \tag{122}$$

$$\tag{123}$$

Dimensional Analysis

$$r = R\hat{r} \quad h = H\hat{h} \quad \frac{H}{R} = \epsilon$$

$$u_\theta = U\hat{u}_\theta \quad u_\phi = U\hat{u}_\phi \quad u_r = U_r\hat{u}_r$$

$$t = T\hat{t} = \frac{R}{U}\hat{t} \quad p = \rho g H\hat{p}$$

$$\frac{1}{R^2\hat{r}^2}\frac{1}{R}\frac{\partial}{\partial \hat{r}}\left(R^2\hat{r}^2\epsilon U\hat{u}_r\right) + \frac{1}{R\hat{r}\sin(\phi)}\frac{\partial U\hat{u}_\theta}{\partial \theta} + \frac{1}{R\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)U\hat{u}_\phi) = 0 \tag{124}$$

$$\frac{U_r}{T}\frac{\partial \hat{u}_r}{\partial \hat{t}} + U_r\hat{u}_r\frac{U_r}{R}\frac{\partial \hat{u}_r}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U_r\frac{\partial \hat{u}_r}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U_r\frac{\partial \hat{u}_r}{\partial \phi} - \frac{U^2}{R}\frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\frac{1}{\rho}\rho g H\frac{1}{R}\frac{\partial \hat{p}}{\partial \hat{r}} + g_r \tag{125}$$

$$\frac{U}{T}\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + U_r\hat{u}_r\frac{U}{R}\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U\frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{U_r\hat{u}_r U\hat{u}_\theta}{R\hat{r}} + \frac{U^2\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{R\hat{r}} = -\frac{1}{\rho R\hat{r}\sin(\phi)}\rho g H\frac{\partial \hat{p}}{\partial \theta} + g_\theta \tag{126}$$

$$\frac{U}{T}\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + U_r\hat{u}_r\frac{U}{R}\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U\frac{\partial \hat{u}_\phi}{\partial \phi} + \frac{U_r U\hat{u}_r\hat{u}_\phi}{R\hat{r}} - \frac{U^2\hat{u}_\theta^2\cot(\phi)}{R\hat{r}} = -\frac{1}{\rho R\hat{r}}\rho g H\frac{\partial \hat{p}}{\partial \phi} + g_\phi \tag{127}$$

$T = \frac{R}{U}$, $U_r = \epsilon U$

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2 \hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{128}$$

$$\frac{U^2}{R}\left(\epsilon\frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2\hat{u}_r\frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon\frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_r}{\partial \theta} + \epsilon\frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r}\right) = -g\frac{H}{R}\frac{\partial \hat{p}}{\partial \hat{r}} + ge_r \tag{129}$$

$$\frac{U^2}{R}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{gH}{R\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + ge_\theta \tag{130}$$

$$\frac{U^2}{R}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{gH}{R\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + ge_\phi \tag{131}$$

$R = \frac{H}{\epsilon}$

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2 \hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{132}$$

$$\epsilon\frac{U^2}{gH}\left(\epsilon\frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2\hat{u}_r\frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon\frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_r}{\partial \theta} + \epsilon\frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r}\right) = -\epsilon\frac{\partial \hat{p}}{\partial \hat{r}} + e_r \tag{133}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + e_\theta \tag{134}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + e_\phi \tag{135}$$

Drop ϵ^2 terms

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2 \hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{136}$$

$$-\epsilon\frac{U^2}{gH}\frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\epsilon\frac{\partial \hat{p}}{\partial \hat{r}} + e_r \tag{137}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + e_\theta \tag{138}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + e_\phi \tag{139}$$

Mapping

$$\rho = \frac{r - h_b(t,\theta,\phi)}{h(t,\theta,\phi)} \tag{140}$$

$$h(t,\theta,\phi) = h_s(t,\theta,\phi) - h_b(t,\theta,\phi) \tag{141}$$

$$r = \rho h(t,\theta,\phi) + h_b(t,\theta,\phi) \tag{142}$$

$$\tilde{\psi}(t,\theta,\phi,\rho) = \psi(t,\theta,\phi,\rho h(t,\theta,\phi) + h_b(t,\theta,\phi)) \tag{143}$$

$$\psi(t,\theta,\phi,r) = \tilde{\psi}(t,\theta,\phi,\frac{r - h_b(t,\theta,\phi)}{h(t,\theta,\phi)}) \tag{144}$$

$$h\frac{\partial \psi}{\partial s} = \frac{\partial}{\partial s}\left(h\tilde{\psi}\right) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial s}(\rho h + h_b)\tilde{\psi}\right) \quad s \in \{t,\theta,\phi\} \tag{145}$$

$$h\frac{\partial \psi}{\partial r} = \frac{\partial \tilde{\psi}}{\partial \rho} \tag{146}$$

Mapping of Mass Balance

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)u_\phi) = 0$$

Multiply by h

$$\frac{h}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{h}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{h}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0$$

Transform from r to ρ

$$\frac{1}{(h\rho+h_b)^2}\frac{\partial}{\partial\rho}\Big((h\rho+h_b)^2\tilde{u}_r\Big)+\frac{1}{(h\rho+h_b)\sin(\phi)}\left(\frac{\partial}{\partial\theta}(hu_\theta)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta\right)\right)$$
$$+\frac{1}{(h\rho+h_b)\sin(\phi)}\left(\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_\phi)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde{u}_\phi\right)\right)=0$$
$$\frac{1}{(h\rho+h_b)\sin(\phi)}\frac{\partial}{\partial\theta}(hu_\theta)+\frac{1}{(h\rho+h_b)\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_\phi)$$
$$-\frac{1}{(h\rho+h_b)\sin(\phi)}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde{u}_\phi\right)=-\frac{1}{(h\rho+h_b)^2}\frac{\partial}{\partial\rho}\Big((h\rho+h_b)^2\tilde{u}_r\Big)$$
$$-\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\theta}(hu_\theta)-\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_\phi)$$
$$+\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde{u}_\phi\right)=\frac{\partial}{\partial\rho}\Big((h\rho+h_b)^2\tilde{u}_r\Big)$$
$$-\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\theta}(hu_\theta)-\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_\phi)$$
$$+\frac{h\rho+h_b}{\sin(\phi)}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde{u}_\phi\right)=\frac{\partial}{\partial\rho}\Big((h\rho+h_b)^2\tilde{u}_r\Big)$$

$$\frac{1}{r_0^2}\frac{\partial}{\partial r}\left(r_0^2 u_r\right)+\frac{1}{r_0\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{147}$$

Mulitply by h

$$h\frac{\partial}{\partial r}(u_r)+\frac{h}{r_0\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{h}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{148}$$

Transform from r to ρ

$$\frac{\partial}{\partial\rho}(\tilde{u}_r)+\frac{1}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{u}_\theta)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta\right)\right) \tag{149}$$
$$+\frac{1}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\theta)\tilde{u}_\phi\right)\right)=0 \tag{150}$$
$$r_0\sin(\phi)\frac{\partial}{\partial\rho}(\tilde{u}_r)=-\frac{\partial}{\partial\theta}(h\tilde{u}_\theta)-\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi) \tag{151}$$
$$+\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\theta)\tilde{u}_\phi\right) \tag{152}$$
$$\int_0^{\rho'} r_0\sin(\phi)\frac{\partial}{\partial\rho}(\tilde{u}_r)\,\mathrm{d}\rho=-\int_0^{\rho'}\frac{\partial}{\partial\theta}(h\tilde{u}_\theta)\,\mathrm{d}\rho-\int_0^{\rho'}\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi)\,\mathrm{d}\rho \tag{153}$$
$$+\int_0^{\rho'}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde{u}_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde{u}_\phi\right)\mathrm{d}\rho \tag{154}$$
$$r_0\sin(\phi)(\tilde{u}_r(t,\theta,\phi,\rho')-\tilde{u}_r(t,\theta,\phi,0))=-\frac{\partial}{\partial\theta}\left(h\int_0^{\rho'}\tilde{u}_\theta\,\mathrm{d}\rho\right)-\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^{\rho'}u_\phi\,\mathrm{d}\rho\right) \tag{155}$$
$$+\frac{\partial}{\partial\theta}(h\rho'+h_b)\tilde{u}_\theta(t,\theta,\phi,\rho')+\frac{\partial}{\partial\phi}(h\rho'+h_b)\sin(\phi)\tilde{u}_\phi(t,\theta,\phi,\rho') \tag{156}$$
$$-\frac{\partial h_b}{\partial\theta}\tilde{u}_\theta(t,\theta,\phi,0)-\frac{\partial h_b}{\partial\phi}\sin(\phi)\tilde{u}_\phi(t,\theta,\phi,0) \tag{157}$$

Let $\rho' = 1$

$$r_0 \sin(\phi)(\tilde{u}_r(t,\theta,\phi,1) - \tilde{u}_r(t,\theta,\phi,0)) = -\frac{\partial}{\partial\theta}(h\tilde{u}_{\theta m}) - \frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_{\phi m}) \tag{158}$$

$$+\frac{\partial h_s}{\partial\theta}\tilde{u}_\theta(t,\theta,\phi,1) + \frac{\partial h_s}{\partial\phi}\sin(\phi)\tilde{u}_\phi(t,\theta,\phi,1) \tag{159}$$

$$-\frac{\partial h_b}{\partial\theta}\tilde{u}_\theta(t,\theta,\phi,0) - \frac{\partial h_b}{\partial\phi}\sin(\phi)\tilde{u}_\phi(t,\theta,\phi,0) \tag{160}$$

$$0 = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\tilde{u}_{\theta m}) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_{\phi m}) \tag{161}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial h_s}{\partial\theta}\tilde{u}_\theta(t,\theta,\phi,1) + \frac{1}{r_0}\frac{\partial h_s}{\partial\phi}\tilde{u}_\phi(t,\theta,\phi,1) - \tilde{u}_r(t,\theta,\phi,1) \tag{162}$$

$$-\frac{1}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta}\tilde{u}_\theta(t,\theta,\phi,0) - \frac{1}{r_0}\frac{\partial h_b}{\partial\phi}\tilde{u}_\phi(t,\theta,\phi,0) + \tilde{u}_r(t,\theta,\phi,0) \tag{163}$$

Using Kinematic Boundary condition with $r_0 = r$

$$\frac{\partial h}{\partial t} + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_{\phi m}) = 0 \tag{164}$$

$$\tag{165}$$

Mapping of Momentum Balance

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial\phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi\cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial\theta} + g_\theta \tag{166}$$

Multiply by h

$$h\frac{\partial u_\theta}{\partial t} + hu_r\frac{\partial u_\theta}{\partial r} + \frac{hu_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{hu_\phi}{r}\frac{\partial u_\theta}{\partial\phi} + \frac{hu_r u_\theta}{r} + \frac{hu_\theta u_\phi\cot(\phi)}{r} = -\frac{h}{\rho r\sin(\phi)}\frac{\partial p}{\partial\theta} + hge_\theta \tag{167}$$

Transform from r to ρ

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta\right) + \tilde{u}_r\frac{\partial h\tilde{u}_\theta}{\partial\rho} \tag{168}$$

$$+\frac{\tilde{u}_\theta}{r\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{169}$$

$$+\frac{\tilde{u}_\phi}{r}\left(\frac{\partial}{\partial\phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{170}$$

$$+\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} = -\frac{1}{\rho_0 r\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{p}) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(\rho h + h_b)\tilde{p}\right)\right) + hge_\theta \tag{171}$$

Transform hydrostatic pressure and simplify terms from momentum equations with pressure

$$p = (h_s(t,\theta,\phi) - r)\rho_0 g e_r \tag{172}$$

$$\tilde{p} = (h_s(t,\theta,\phi) - (h\rho + h_b))\rho_0 g e_r \tag{173}$$

$$= (h_s - h_b + h\rho)\rho_0 g e_r \tag{174}$$

$$= h(1-\rho)\rho_0 g e_r \tag{175}$$

θ momentum equation

$$\frac{\partial}{\partial\theta}(h\tilde{p}) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(\rho h + h_b)\tilde{p}\right) \tag{176}$$

$$= \frac{\partial}{\partial\theta}(hh(1-\rho)\rho_0 g e_r) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(\rho h + h_b)h(1-\rho)\rho_0 g e_r\right) \tag{177}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\theta}\big(h^2(1-\rho)\big) - h\frac{\partial}{\partial\rho}\left(\left(\rho\frac{\partial h}{\partial\theta} + \frac{\partial h_b}{\partial\theta}\right)(1-\rho)\right)\right) \tag{178}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\theta}\big(h^2(1-\rho)\big) - h\left(\frac{\partial h}{\partial\theta}(1-\rho) - \rho\frac{\partial h}{\partial\theta} - \frac{\partial h_b}{\partial\theta}\right)\right) \tag{179}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\theta}\big(h^2(1-\rho)\big) + 2h\frac{\partial h}{\partial\theta}\rho - h\frac{\partial h}{\partial\theta} + h\frac{\partial h_b}{\partial\theta}\right) \tag{180}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\theta}\big(h^2(1-\rho)\big) + \frac{\partial}{\partial\theta}\big(h^2\big)\rho - \frac{1}{2}\frac{\partial}{\partial\theta}\big(h^2\big) + h\frac{\partial h_b}{\partial\theta}\right) \tag{181}$$

$$= \rho_0 g e_r\left(\frac{1}{2}\frac{\partial}{\partial\theta}\big(h^2\big) + h\frac{\partial h_b}{\partial\theta}\right) \tag{182}$$

ϕ momentum equation

$$\frac{\partial}{\partial\phi}(h\tilde{p}) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(\rho h + h_b)\tilde{p}\right) \tag{183}$$

$$= \frac{\partial}{\partial\phi}(hh(1-\rho)\rho_0 g e_r) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(\rho h + h_b)h(1-\rho)\rho_0 g e_r\right) \tag{184}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\phi}\big(h^2(1-\rho)\big) - h\frac{\partial}{\partial\rho}\left(\left(\rho\frac{\partial h}{\partial\theta} + \frac{\partial h_b}{\partial\phi}\right)(1-\rho)\right)\right) \tag{185}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\phi}\big(h^2(1-\rho)\big) - h\left(\frac{\partial h}{\partial\phi}(1-\rho) - \rho\frac{\partial h}{\partial\phi} - \frac{\partial h_b}{\partial\phi}\right)\right) \tag{186}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\phi}\big(h^2(1-\rho)\big) + 2h\frac{\partial h}{\partial\phi}\rho - h\frac{\partial h}{\partial\phi} + h\frac{\partial h_b}{\partial\phi}\right) \tag{187}$$

$$= \rho_0 g e_r\left(\frac{\partial}{\partial\phi}\big(h^2(1-\rho)\big) + \frac{\partial}{\partial\phi}\big(h^2\big)\rho - \frac{1}{2}\frac{\partial}{\partial\phi}\big(h^2\big) + h\frac{\partial h_b}{\partial\phi}\right) \tag{188}$$

$$= \rho_0 g e_r\left(\frac{1}{2}\frac{\partial}{\partial\phi}\big(h^2\big) + h\frac{\partial h_b}{\partial\phi}\right) \tag{189}$$

$$\tag{190}$$

Plug pressure expression into momentum balance equation

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta\right) + \tilde{u}_r\frac{\partial h\tilde{u}_\theta}{\partial \rho} \tag{191}$$

$$+\frac{\tilde{u}_\theta}{r\sin(\phi)}\left(\frac{\partial}{\partial \theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{192}$$

$$+\frac{\tilde{u}_\phi}{r}\left(\frac{\partial}{\partial \phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{193}$$

$$+\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} = -\frac{1}{\rho_0 r\sin(\phi)}\left(\rho_0 g e_r\left(\frac{1}{2}\frac{\partial}{\partial \theta}\left(h^2\right) + h\frac{\partial h_b}{\partial \theta}\right)\right) + hge_\theta \tag{194}$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta\right) + \tilde{u}_r\frac{\partial h\tilde{u}_\theta}{\partial \rho} \tag{195}$$

$$+\frac{\tilde{u}_\theta}{r\sin(\phi)}\left(\frac{\partial}{\partial \theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{196}$$

$$+\frac{\tilde{u}_\phi}{r}\left(\frac{\partial}{\partial \phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{u}_\theta\right)\right) \tag{197}$$

$$+\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} = -\frac{ge_r}{2r\sin(\phi)}\frac{\partial}{\partial \theta}\left(h^2\right) - \frac{hge_r}{r\sin(\phi)}\frac{\partial h_b}{\partial \theta} + hge_\theta \tag{198}$$

$$\tag{199}$$

Rewriting Incompressible Navier Stokes Equations

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)u_\phi) = 0 \tag{200}$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r \tag{201}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi\cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial \theta} + g_\theta \tag{202}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} - \frac{u_\theta^2\cot(\phi)}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + g_\phi \tag{203}$$

Continuity Equation after product rule

$$\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\cot(\phi)}{r}u_\phi = 0 \tag{204}$$

Simplify u_r evolution equation

$$u_r\frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r}\left(u_r^2\right) - u_r\frac{\partial u_r}{\partial r} \tag{205}$$

$$\frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} = \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_r u_\theta) - \frac{u_r}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} \tag{206}$$

$$\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial \phi}(u_r u_\phi) - \frac{u_r}{r}\frac{\partial u_\phi}{\partial \phi} \tag{207}$$

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} \tag{208}$$

$$= \frac{\partial}{\partial r}\left(u_r^2\right) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_r u_\phi) - u_r\left(\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\right) \tag{209}$$

From the continuity equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} = -2\frac{u_r}{r} - \frac{\cot(\phi)}{r}u_\phi \tag{210}$$

So

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} = \frac{\partial}{\partial r}\left(u_r^2\right) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_r u_\phi) + u_r\left(2\frac{u_r}{r} + \frac{\cot(\phi)}{r}u_\phi\right) \tag{211}$$

$$= \frac{\partial}{\partial r}\left(u_r^2\right) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_r u_\phi) + \frac{2u_r^2 + u_r u_\phi \cot(\phi)}{r} \tag{212}$$

Substituting this into the u_r evolution equation gives

$$\frac{\partial u_r}{\partial t} + \frac{\partial}{\partial r}\left(u_r^2\right) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_r u_\phi) + \frac{2u_r^2 + u_r u_\phi \cot(\phi) - u_\theta^2 - u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r \tag{213}$$

$$\tag{214}$$

Simplify u_θ evolution equation

$$u_r \frac{\partial u_\theta}{\partial r} = \frac{\partial}{\partial r}(u_r u_\theta) - u_\theta \frac{\partial u_r}{\partial r} \tag{215}$$

$$\frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}\left(u_\theta^2\right) - \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} \tag{216}$$

$$\frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial \phi}(u_\theta u_\phi) - \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \phi} \tag{217}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} \tag{218}$$

$$= \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}\left(u_\theta^2\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_\theta u_\phi) - u_\theta\left(\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\right) \tag{219}$$

From the Continuity Equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} = -2\frac{u_r}{r} - \frac{\cot(\phi)}{r}u_\phi \tag{220}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} \tag{221}$$

$$= \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}\left(u_\theta^2\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_\theta u_\phi) + u_\theta\left(2\frac{u_r}{r} + \frac{\cot(\phi)}{r}u_\phi\right) \tag{222}$$

Substituting this into u_θ equation gives

$$\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}\left(u_\theta^2\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(u_\theta u_\phi) + 3\frac{u_r u_\theta}{r} + 2\frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial \theta} + g_\theta \tag{223}$$

Simplify the u_ϕ equation

$$u_r \frac{\partial u_\phi}{\partial r} = \frac{\partial}{\partial r}(u_r u_\phi) - u_\phi \frac{\partial u_r}{\partial r} \tag{224}$$

$$\frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} = \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_\theta u_\phi) - \frac{u_\phi}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} \tag{225}$$

$$\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial \phi}\left(u_\phi^2\right) - \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} \tag{226}$$

$$u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} \tag{227}$$

$$= \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(u_\phi^2\right) - u_\phi\left(\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\right) \tag{228}$$

From the Continuity Equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} = -2\frac{u_r}{r} - \frac{\cot(\phi)}{r}u_\phi \tag{229}$$

$$u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} \tag{230}$$

$$= \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(u_\phi^2\right) + u_\phi\left(2\frac{u_r}{r} + \frac{\cot(\phi)}{r}u_\phi\right) \tag{231}$$

Substituting into u_ϕ equation

$$\frac{\partial u_\phi}{\partial t}+\frac{\partial}{\partial r}(u_r u_\phi)+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\theta}(u_\theta u_\phi)+\frac{1}{r}\frac{\partial}{\partial\phi}\left(u_\phi^2\right)+3\frac{u_r u_\phi}{r}+\frac{\left(u_\phi^2-u_\theta^2\right)\cot(\phi)}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\phi}+g_\phi \tag{232}$$

The Incompressible Navier Stokes Equations can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{233}$$

$$\frac{\partial u_r}{\partial t}+\frac{\partial}{\partial r}\left(u_r^2\right)+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\theta}(u_r u_\theta)+\frac{1}{r}\frac{\partial}{\partial\phi}(u_r u_\phi)+\frac{2u_r^2+u_r u_\phi\cot(\phi)-u_\theta^2-u_\phi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+g_r \tag{234}$$

$$\frac{\partial u_\theta}{\partial t}+\frac{\partial}{\partial r}(u_r u_\theta)+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\theta}\left(u_\theta^2\right)+\frac{1}{r}\frac{\partial}{\partial\phi}(u_\theta u_\phi)+3\frac{u_r u_\theta}{r}+2\frac{u_\theta u_\phi\cot(\phi)}{r}=-\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial\theta}+g_\theta \tag{235}$$

$$\frac{\partial u_\phi}{\partial t}+\frac{\partial}{\partial r}(u_r u_\phi)+\frac{1}{r\sin(\phi)}\frac{\partial}{\partial\theta}(u_\theta u_\phi)+\frac{1}{r}\frac{\partial}{\partial\phi}\left(u_\phi^2\right)+3\frac{u_r u_\phi}{r}+\frac{\left(u_\phi^2-u_\theta^2\right)\cot(\phi)}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\phi}+g_\phi \tag{236}$$

Kinematic Boundary Condition

$$\frac{\partial h_s}{\partial t}+\frac{u_\theta(t,\theta,\phi,h_s)}{r\sin(\phi)}\frac{\partial h_s}{\partial\theta}+\frac{u_\phi(t,\theta,\phi,h_s)}{r}\frac{\partial h_s}{\partial\phi}=u_r(t,\theta,\phi,h_s) \tag{237}$$

$$\frac{\partial h_b}{\partial t}+\frac{u_\theta(t,\theta,\phi,h_b)}{r\sin(\phi)}\frac{\partial h_b}{\partial\theta}+\frac{u_\phi(t,\theta,\phi,h_b)}{r}\frac{\partial h_b}{\partial\phi}=u_r(t,\theta,\phi,h_b) \tag{238}$$

Mapped Boundary Condition, assuming $r=r_0$

$$\frac{\partial h_s}{\partial t}+\frac{\tilde u_\theta(t,\theta,\phi,1)}{r_0\sin(\phi)}\frac{\partial h_s}{\partial\theta}+\frac{\tilde u_\phi(t,\theta,\phi,1)}{r_0}\frac{\partial h_s}{\partial\phi}=\tilde u_r(t,\theta,\phi,1) \tag{239}$$

$$\frac{\partial h_b}{\partial t}+\frac{\tilde u_\theta(t,\theta,\phi,0)}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta}+\frac{\tilde u_\phi(t,\theta,\phi,0)}{r_0}\frac{\partial h_b}{\partial\phi}=\tilde u_r(t,\theta,\phi,0) \tag{240}$$

Mapping the Mass Balance Equation

$$\frac{1}{r_0^2}\frac{\partial}{\partial r}\left(r_0^2 u_r\right)+\frac{1}{r_0\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{241}$$

Mulitply by h

$$h\frac{\partial}{\partial r}(u_r)+\frac{h}{r_0\sin(\phi)}\frac{\partial u_\theta}{\partial\theta}+\frac{h}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi)=0 \tag{242}$$

Transform from r to ρ

$$\frac{\partial}{\partial\rho}(\tilde u_r)+\frac{1}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde u_\theta)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde u_\theta\right)\right) \tag{243}$$

$$+\frac{1}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi)-\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\theta)\tilde u_\phi\right)\right)=0 \tag{244}$$

$$\frac{\partial}{\partial\rho}(\tilde u_r)=-\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\tilde u_\theta)-\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi) \tag{245}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde u_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde u_\phi\right) \tag{246}$$

$$\int_0^{\rho'}\frac{\partial}{\partial\rho}(\tilde u_r)\,\mathrm{d}\rho=-\frac{1}{r_0\sin(\phi)}\int_0^{\rho'}\frac{\partial}{\partial\theta}(h\tilde u_\theta)\,\mathrm{d}\rho-\frac{1}{r_0\sin(\phi)}\int_0^{\rho'}\frac{\partial}{\partial\phi}(h\sin(\phi)u_\phi)\,\mathrm{d}\rho \tag{247}$$

$$+\frac{1}{r_0\sin(\phi)}\int_0^{\rho'}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho+h_b)\tilde u_\theta+\frac{\partial}{\partial\phi}(h\rho+h_b)\sin(\phi)\tilde u_\phi\right)\mathrm{d}\rho \tag{248}$$

$$\tilde u_r(t,\theta,\phi,\rho')-\tilde u_r(t,\theta,\phi,0)=-\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^{\rho'}\tilde u_\theta\,\mathrm{d}\rho\right)-\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^{\rho'}u_\phi\,\mathrm{d}\rho\right) \tag{249}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\rho'+h_b)\tilde u_\theta+\frac{1}{r_0}\frac{\partial}{\partial\phi}(h\rho'+h_b)\tilde u_\phi-\frac{1}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta}\tilde u_\theta-\frac{1}{r_0}\frac{\partial h_b}{\partial\phi}\tilde u_\phi \tag{250}$$

Let $\rho' = 1$

$$\tilde{u}_r(t,\theta,\phi,1) - \tilde{u}_r(t,\theta,\phi,0) = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 \tilde{u}_\theta \,\mathrm{d}\rho\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^1 u_\phi \,\mathrm{d}\rho\right) \tag{251}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial h_s}{\partial\theta}\tilde{u}_\theta + \frac{1}{r_0}\frac{\partial h_s}{\partial\phi}\tilde{u}_\phi - \frac{1}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta}\tilde{u}_\theta - \frac{1}{r_0}\frac{\partial h_b}{\partial\phi}\tilde{u}_\phi \tag{252}$$

$$0 = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 \tilde{u}_\theta \,\mathrm{d}\rho\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^1 u_\phi \,\mathrm{d}\rho\right) \tag{253}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial h_s}{\partial\theta}\tilde{u}_\theta + \frac{1}{r_0}\frac{\partial h_s}{\partial\phi}\tilde{u}_\phi - \tilde{u}_r(t,\theta,\phi,1) - \frac{1}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta}\tilde{u}_\theta - \frac{1}{r_0}\frac{\partial h_b}{\partial\phi}\tilde{u}_\phi + \tilde{u}_r(t,\theta,\phi,0) \tag{254}$$

Using Mapped Kinematic Boundary condition with $r_0 = r$

$$0 = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 \tilde{u}_\theta \,\mathrm{d}\rho\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^1 u_\phi \,\mathrm{d}\rho\right) - \frac{\partial h_s}{\partial t} + \frac{\partial h_b}{\partial t} \tag{255}$$

Now we can denote $\tilde{u}_{\theta m} = \int_0^1 \tilde{u}_\theta \,\mathrm{d}\rho$ and $\tilde{u}_{\phi m} = \int_0^1 \tilde{u}_\phi \,\mathrm{d}\rho$, then

$$\frac{\partial h_s}{\partial t} + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}(h\sin(\phi)\tilde{u}_{\phi m}) \tag{256}$$

Also using the mapped boundary conditions we can write an explicit expression for $\tilde{u}_r$

$$\tilde{u}_r(t,\theta,\phi,\rho) = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^\rho \tilde{u}_\theta \,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^\rho u_\phi \,\mathrm{d}\rho'\right) \tag{257}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}(h\rho + h_b)\tilde{u}_\theta + \frac{1}{r_0}\frac{\partial}{\partial\phi}(h\rho + h_b)\tilde{u}_\phi \tag{258}$$

Mapping of Momentum Balance

$$\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial\theta}(u_\theta^2) + \frac{1}{r}\frac{\partial}{\partial\phi}(u_\theta u_\phi) + 3\frac{u_r u_\theta}{r} + 2\frac{u_\theta u_\phi\cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial\theta} + g e_\theta \tag{259}$$

Mulitply by h

$$h\frac{\partial u_\theta}{\partial t} + h\frac{\partial}{\partial r}(u_r u_\theta) + \frac{h}{r\sin(\phi)}\frac{\partial}{\partial\theta}(u_\theta^2) + \frac{h}{r}\frac{\partial}{\partial\phi}(u_\theta u_\phi) + 3\frac{h u_r u_\theta}{r} + 2\frac{h u_\theta u_\phi\cot(\phi)}{r} = -\frac{h}{\rho r\sin(\phi)}\frac{\partial p}{\partial\theta} + g h e_\theta \tag{260}$$

Transform from r to ρ

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\theta\right) + \frac{\partial}{\partial\rho}(\tilde{u}_r\tilde{u}_\theta) + \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{u}_\theta^2) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho + h_b)\tilde{u}_\theta^2\right)\right) \tag{261}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial\phi}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi\right)\right) + 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} \tag{262}$$

$$= -\frac{1}{\rho r\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{p}) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho + h_b)\tilde{p}\right)\right) + g h e_\theta \tag{263}$$

Substitute for transformed pressure

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\theta\right) + \frac{\partial}{\partial\rho}(\tilde{u}_r\tilde{u}_\theta) + \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\tilde{u}_\theta^2) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}(h\rho + h_b)\tilde{u}_\theta^2\right)\right) \tag{264}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial\phi}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\phi}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi\right)\right) + 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} \tag{265}$$

$$= -\frac{1}{\rho r\sin(\phi)}\left(\rho_0 g e_r\left(\frac{1}{2}\frac{\partial}{\partial\theta}(h^2) + h\frac{\partial h_b}{\partial\theta}\right)\right) + g h e_\theta \tag{266}$$

Gather derivatives

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\tilde{u}_\theta^2 + \frac{1}{2}gh^2 e_r\right) + \frac{1}{r}\frac{\partial}{\partial \phi}(h\tilde{u}_\theta\tilde{u}_\phi) \tag{267}$$

$$+\frac{\partial}{\partial \rho}\left(\tilde{u}_r\tilde{u}_\theta - \frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\theta\right) - \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta^2\right) - \frac{1}{r}\frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi\right) \tag{268}$$

$$+3\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r} = -\frac{ghe_r}{r\sin(\phi)}\frac{\partial h_b}{\partial \theta} + ghe_\theta \tag{269}$$

Consider the following separately, and label it as ω

$$\omega = \tilde{u}_r - \frac{\partial}{\partial t}(h\rho + h_b) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi \tag{270}$$

Using expression for $\tilde{u}_r$

$$\omega = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_\theta \,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_\phi \,\mathrm{d}\rho'\right) \tag{271}$$

$$+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta + \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi - \frac{\partial}{\partial t}(h\rho + h_b) \tag{272}$$

$$-\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi \tag{273}$$

$$\omega = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_\theta \,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_\phi \,\mathrm{d}\rho'\right) - \rho\frac{\partial h}{\partial t} \tag{274}$$

Using continuity equation

$$\omega = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_\theta \,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_\phi \,\mathrm{d}\rho'\right) \tag{275}$$

$$+\rho\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}(h\sin(\phi)\tilde{u}_{\phi m})\right) \tag{276}$$

Using the fact that $\rho = \int_0^\rho 1 \,\mathrm{d}\rho'$

$$\omega = \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_{\theta m} - \tilde{u}_\theta \,\mathrm{d}\rho'\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho \tilde{u}_{\phi m} - u_\phi \,\mathrm{d}\rho'\right) \tag{277}$$

Letting $\tilde{u}_{\theta d} = \tilde{u}_\theta - \tilde{u}_{\theta m}$ and $\tilde{u}_{\phi d} = \tilde{u}_\phi - \tilde{u}_{\phi m}$, then

$$\omega = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_{\theta d} \,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho \tilde{u}_{\phi d} \,\mathrm{d}\rho'\right) \tag{278}$$

$$\tag{279}$$

Thus the mapped momentum balance equation is

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\tilde{u}_\theta^2 + \frac{1}{2}gh^2 e_r\right) + \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\tilde{u}_\theta\tilde{u}_\phi) \tag{280}$$

$$+\frac{\partial}{\partial \rho}(\tilde{u}_\theta\omega) + 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r_0} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)}\frac{\partial h_b}{\partial \theta} + ghe_\theta \tag{281}$$

Now we can do the same for the ϕ momentum equation.

$$\frac{\partial u_\phi}{\partial t} + \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(u_\phi^2\right) + 3\frac{u_r u_\phi}{r} + \frac{\left(u_\phi^2 - u_\theta^2\right)\cot(\phi)}{r} = -\frac{1}{\rho_0 r}\frac{\partial p}{\partial \phi} + ge_\phi \tag{282}$$

Multiply by h

$$h\frac{\partial u_\phi}{\partial t} + h\frac{\partial}{\partial r}(u_r u_\phi) + \frac{h}{r\sin(\phi)}\frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{h}{r}\frac{\partial}{\partial \phi}\left(u_\phi^2\right) + 3\frac{hu_r u_\phi}{r} + h\frac{\left(u_\phi^2 - u_\theta^2\right)\cot(\phi)}{r} = -\frac{h}{\rho_0 r}\frac{\partial p}{\partial \phi} + ghe_\phi \tag{283}$$

Transform from r to ρ

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\phi\right) + \frac{\partial}{\partial \rho}(\tilde{u}_r\tilde{u}_\phi) + \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi\right)\right) \tag{284}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial \phi}(h\tilde{u}_\phi^2) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi^2\right)\right) + 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r} + h\frac{\left(\tilde{u}_\phi^2 - \tilde{u}_\theta^2\right)\cot(\phi)}{r} \tag{285}$$

$$= -\frac{1}{\rho_0 r}\left(\frac{\partial}{\partial \phi}(h\tilde{p}) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{p}\right)\right) + ghe_\phi \tag{286}$$

Substitute for transformed pressure

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\phi\right) + \frac{\partial}{\partial \rho}(\tilde{u}_r\tilde{u}_\phi) + \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi\right)\right) \tag{287}$$

$$+\frac{1}{r}\left(\frac{\partial}{\partial \phi}(h\tilde{u}_\phi^2) - \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi^2\right)\right) + 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r} + h\frac{\left(\tilde{u}_\phi^2 - \tilde{u}_\theta^2\right)\cot(\phi)}{r} \tag{288}$$

$$= -\frac{1}{r}\left(ge_r\left(\frac{1}{2}\frac{\partial}{\partial \phi}(h^2) + h\frac{\partial h_b}{\partial \phi}\right)\right) + ghe_\phi \tag{289}$$

Gather derivatives

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0}\frac{\partial}{\partial \phi}\left(h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2\right) \tag{290}$$

$$+\frac{\partial}{\partial \rho}\left(\tilde{u}_\phi\left(\tilde{u}_r - \frac{\partial}{\partial t}(h\rho + h_b) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi\right)\right) \tag{291}$$

$$+3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{\left(\tilde{u}_\phi^2 - \tilde{u}_\theta^2\right)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial \phi} + ghe_\phi \tag{292}$$

Substituting for ω gives

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0}\frac{\partial}{\partial \phi}\left(h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2\right) + \frac{\partial}{\partial \rho}(\tilde{u}_\phi\omega) \tag{293}$$

$$+3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{\left(\tilde{u}_\phi^2 - \tilde{u}_\theta^2\right)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial \phi} + ghe_\phi \tag{294}$$

The complete vertically resolved system is thus

$$\frac{\partial h}{\partial t} + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}(h\sin(\phi)\tilde{u}_{\phi m}) = 0 \tag{295}$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\tilde{u}_\theta^2 + \frac{1}{2}gh^2 e_r\right) + \frac{1}{r_0}\frac{\partial}{\partial \phi}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{\partial}{\partial \rho}(\tilde{u}_\theta\omega) \tag{296}$$

$$+3\frac{h\tilde{u}_r\tilde{u}_\theta}{r_0} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)}\frac{\partial h_b}{\partial \theta} + ghe_\theta \tag{297}$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0}\frac{\partial}{\partial \phi}\left(h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2\right) + \frac{\partial}{\partial \rho}(\tilde{u}_\phi\omega) \tag{298}$$

$$+3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{\left(\tilde{u}_\phi^2 - \tilde{u}_\theta^2\right)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial \phi} + ghe_\phi \tag{299}$$

Where

$$\omega = -\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho \tilde{u}_{\theta d}\,\mathrm{d}\rho'\right) - \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho \tilde{u}_{\phi d}\,\mathrm{d}\rho'\right) \tag{300}$$

$$\tag{301}$$

From now on we will drop the tilde for readability purposes. Now we can express the velocities as polynomial expansions

$$u_\theta(t,\theta,\phi,\rho) = u_{\theta m}(t,\theta,\phi,\rho) + u_{\theta d}(t,\theta,\phi,\rho) = u_{\theta m}(t,\theta,\phi,\rho) + \sum_{j=1}^{N} (\alpha_j(t,\theta,\phi))\psi_j(\rho) \tag{302}$$

$$u_\phi(t,\theta,\phi,\rho) = u_{\phi m}(t,\theta,\phi,\rho) + u_{\phi d}(t,\theta,\phi,\rho) = u_{\phi m}(t,\theta,\phi,\rho) + \sum_{j=1}^{N} (\alpha_j(t,\theta,\phi))\psi_j(\rho) \tag{303}$$

Where

$$\int_0^1 \psi_j(\rho)\psi_j(\rho)\,\mathrm{d}\rho = \frac{1}{2j+1} \qquad \int_0^1 \psi_i(\rho)\psi_j(\rho)\,\mathrm{d}\rho = 0 \text{ for } i \neq j \qquad \psi(0) = 1 \tag{304}$$

Some key quantities are

$$\int_0^1 u_\theta^2\,\mathrm{d}\rho = u_{\theta m}^2 + \sum_{j=1}^{N}\left(\frac{\alpha_j^2}{2j+1}\right) \tag{305}$$

$$\int_0^1 u_\phi^2\,\mathrm{d}\rho = u_{\phi m}^2 + \sum_{j=1}^{N}\left(\frac{\beta_j^2}{2j+1}\right) \tag{306}$$

$$\int_0^1 u_\theta u_\phi\,\mathrm{d}\rho = u_{\theta m}u_{\phi m} + \sum_{j=1}^{N}\left(\frac{\alpha_j\beta_j}{2j+1}\right) \tag{307}$$

Now we depth average

$$\frac{\partial}{\partial t}\left(h\int_0^1 u_\theta\,\mathrm{d}\rho\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 u_\theta^2\,\mathrm{d}\rho + \frac{1}{2}gh^2e_r\right) + \frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\int_0^1 u_\theta u_\phi\,\mathrm{d}\rho\right) + \int_0^1 \frac{\partial}{\partial\rho}(u_\theta\omega)\,\mathrm{d}\rho \tag{308}$$

$$+2\frac{h\int_0^1 u_\theta u_\phi\,\mathrm{d}\rho\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta} + ghe_\theta \tag{309}$$

$$\frac{\partial}{\partial t}(hu_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\left(u_{\theta m}^2 + \sum_{j=1}^{N}\left(\frac{\alpha_j^2}{2j+1}\right)\right) + \frac{1}{2}gh^2e_r\right) + \frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\left(u_{\theta m}u_{\phi m} + \sum_{j=1}^{N}\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\right) \tag{310}$$

$$+ u_\theta\omega\big|_{\rho=0}^{1} + 2\frac{h\left(u_{\theta m}u_{\phi m} + \sum_{j=1}^{N}\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta} + ghe_\theta \tag{311}$$

Since ω vanishes at $\rho = 0$ and $\rho = 1$

$$\frac{\partial}{\partial t}(hu_{\theta m}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\left(u_{\theta m}^2 + \sum_{j=1}^{N}\left(\frac{\alpha_j^2}{2j+1}\right)\right) + \frac{1}{2}gh^2e_r\right) + \frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\left(u_{\theta m}u_{\phi m} + \sum_{j=1}^{N}\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\right) \tag{312}$$

$$+2\frac{h\left(u_{\theta m}u_{\phi m} + \sum_{j=1}^{N}\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)}\frac{\partial h_b}{\partial\theta} + ghe_\theta \tag{313}$$

$$\frac{\partial}{\partial t}\left(h\int_0^1 u_\phi \,\mathrm{d}\rho\right)+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 u_\theta u_\phi \,\mathrm{d}\rho\right)+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\int_0^1 u_\phi^2 \,\mathrm{d}\rho+\frac{1}{2}ge_rh^2\right)+\int_0^1\frac{\partial}{\partial\rho}(u_\phi\omega)\,\mathrm{d}\rho \tag{314}$$

$$+h\frac{\int_0^1 u_\phi^2-u_\theta^2\,\mathrm{d}\rho\cot(\phi)}{r_0}=-\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial\phi}+ghe_\phi \tag{315}$$

$$\frac{\partial}{\partial t}(hu_{\phi m})+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\left(u_{\theta m}u_{\phi m}+\sum_{j=1}^N\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\right)+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\left(u_{\phi m}^2+\sum_{j=1}^N\left(\frac{\beta_j^2}{2j+1}\right)\right)+\frac{1}{2}ge_rh^2\right) \tag{316}$$

$$+u_\phi\omega|_{\rho=0}^1+h\frac{\left(u_{\phi m}^2-u_{\theta m}^2+\sum_{j=1}^N\left(\frac{\beta_j^2-\alpha_j^2}{2j+1}\right)\right)\cot(\phi)}{r_0}=-\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial\phi}+ghe_\phi \tag{317}$$

Since ω vanishes at $\rho=0$ and $\rho=1$

$$\frac{\partial}{\partial t}(hu_{\phi m})+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\left(u_{\theta m}u_{\phi m}+\sum_{j=1}^N\left(\frac{\alpha_j\beta_j}{2j+1}\right)\right)\right)+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\left(u_{\phi m}^2+\sum_{j=1}^N\left(\frac{\beta_j^2}{2j+1}\right)\right)+\frac{1}{2}ge_rh^2\right) \tag{318}$$

$$+h\frac{\left(u_{\phi m}^2-u_{\theta m}^2+\sum_{j=1}^N\left(\frac{\beta_j^2-\alpha_j^2}{2j+1}\right)\right)\cot(\phi)}{r_0}=-\frac{ghe_r}{r_0}\frac{\partial h_b}{\partial\phi}+ghe_\phi \tag{319}$$

In order to find the equations for the higher moments we multiply the conservation of momentum equations by a polynomial basis function and depth average.

$$\frac{\partial}{\partial t}\left(h\int_0^1 u_\theta\psi_i(\rho)\,\mathrm{d}\rho\right)+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 u_\theta^2\psi_i(\rho)\,\mathrm{d}\rho\right)+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\int_0^1 u_\theta u_\phi\psi_i(\rho)\,\mathrm{d}\rho\right) \tag{320}$$

$$+\int_0^1\frac{\partial}{\partial\rho}(u_\theta\omega)\psi_i(\rho)\,\mathrm{d}\rho+2\frac{h\int_0^1 u_\theta u_\phi\psi_i(\rho)\,\mathrm{d}\rho\cot(\phi)}{r_0}=0 \tag{321}$$

$$\int_0^1 u_\theta\psi_i(\rho)\,\mathrm{d}\rho=\frac{\alpha_i}{2i+1} \tag{322}$$

$$\int_0^1 u_\theta^2\psi_i(\rho)\,\mathrm{d}\rho=\int_0^1 u_{\theta m}^2\psi_i(\rho)+2u_{\theta m}\sum_{j=1}^N(\alpha_j\psi_j(\rho))\psi_i(\rho)+\sum_{j=1}^N\left(\sum_{k=1}^N(\alpha_j\alpha_k\psi_j(\rho)\psi_k(\rho))\right)\psi_i(\rho)\,\mathrm{d}\rho \tag{323}$$

$$=2u_{\theta m}\sum_{j=1}^N\left(\alpha_j\int_0^1\psi_i(\rho)\psi_j(\rho)\,\mathrm{d}\rho\right)+\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\alpha_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right) \tag{324}$$

$$=2u_{\theta m}\frac{\alpha_i}{2i+1}+\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\alpha_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right) \tag{325}$$

$$\int_0^1 u_\theta u_\phi\psi_i(\rho)\,\mathrm{d}\rho=\int_0^1 u_{\theta m}u_{\phi m}\psi_i(\rho)\,\mathrm{d}\rho \tag{326}$$

$$+\int_0^1 u_{\theta m}\sum_{j=1}^N(\beta_j\psi_i(\rho)\psi_j(\rho))+u_{\phi m}\sum_{j=1}^N(\alpha_j\psi_i(\rho)\psi_j(\rho))+\sum_{j=1}^N\left(\sum_{k=1}^N(\alpha_j\beta_k\psi_i(\rho)\psi_j(\rho)\psi_k(\rho))\right)\mathrm{d}\rho \tag{327}$$

$$=u_{\theta m}\frac{\beta_i}{2i+1}+u_{\phi m}\frac{\alpha_i}{2i+1}+\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\beta_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right) \tag{328}$$

$$\int_0^1\psi_i(\rho)\frac{\partial}{\partial\rho}(u_\theta\omega)\,\mathrm{d}\rho=\int_0^1\psi_i(\rho)\frac{\partial}{\partial\rho}((u_{\theta m}+u_{\theta d})\omega)\,\mathrm{d}\rho \tag{329}$$

$$=\int_0^1\psi_i(\rho)\frac{\partial}{\partial\rho}(u_{\theta m}\omega)\,\mathrm{d}\rho+\int_0^1\psi_i(\rho)\frac{\partial}{\partial\rho}(u_{\theta d}\omega)\,\mathrm{d}\rho \tag{330}$$

Since $\omega = 0$ at $\rho = 0$ and $\rho = 1$

$$= \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho}(u_{\theta m}\omega)\,\mathrm{d}\rho - \int_0^1 \psi_i'(\rho) u_{\theta d}\omega\,\mathrm{d}\rho \tag{331}$$

$$= \int_0^1 \psi_i(\rho)\frac{\partial}{\partial \rho}\left(u_{\theta m}\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\right)\right)\mathrm{d}\rho \tag{332}$$

$$- \int_0^1 \psi_i'(\rho) u_{\theta d}\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\right)\mathrm{d}\rho \tag{333}$$

$$= \int_0^1 \psi_i(\rho) u_{\theta m}\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\frac{\partial}{\partial \rho}\left(\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right)\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\frac{\partial}{\partial \rho}\left(\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\right)\right)\mathrm{d}\rho \tag{334}$$

$$- \int_0^1 \frac{\psi_i'(\rho) u_{\theta d}}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{\psi_i'(\rho) u_{\theta d}}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\mathrm{d}\rho \tag{335}$$

$$= u_{\theta m}\int_0^1 \left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}(h\psi_i(\rho)u_{\theta d}) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}(h\sin(\phi)\psi_i(\rho)u_{\phi d})\right)\mathrm{d}\rho \tag{336}$$

$$- \int_0^1 \frac{\psi_i'(\rho) u_{\theta d}}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{\psi_i'(\rho) u_{\theta d}}{r_0\sin(\phi)}\frac{\partial}{\partial \phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\mathrm{d}\rho \tag{337}$$

$$= \frac{u_{\theta m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial \theta}\left(h\frac{\alpha_i}{2i+1}\right) + \frac{\partial}{\partial \phi}\left(h\sin(\phi)\frac{\beta_i}{2i+1}\right)\right) \tag{338}$$

$$-\frac{1}{r_0\sin(\phi)}\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\int_0^1\psi_i'(\rho)\psi_j(\rho)\int_0^\rho\psi_k(\rho)\,\mathrm{d}\rho'\,\mathrm{d}\rho\left(\frac{\partial}{\partial \theta}(h\alpha_k) + \frac{\partial}{\partial \phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{339}$$

$$\frac{1}{2i+1}\frac{\partial}{\partial t}(h\alpha_i) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(2hu_{\theta m}\frac{\alpha_i}{2i+1} + h\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\alpha_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right)\right) \tag{340}$$

$$+\frac{1}{r_0}\frac{\partial}{\partial \phi}\left(hu_{\theta m}\frac{\beta_i}{2i+1} + hu_{\phi m}\frac{\alpha_i}{2i+1} + h\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\beta_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right)\right) \tag{341}$$

$$+\frac{u_{\theta m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial \theta}\left(h\frac{\alpha_i}{2i+1}\right) + \frac{\partial}{\partial \phi}\left(h\sin(\phi)\frac{\beta_i}{2i+1}\right)\right) \tag{342}$$

$$-\frac{1}{r_0\sin(\phi)}\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\int_0^1\psi_i'(\rho)\psi_j(\rho)\int_0^\rho\psi_k(\rho)\,\mathrm{d}\rho'\,\mathrm{d}\rho\left(\frac{\partial}{\partial \theta}(h\alpha_k) + \frac{\partial}{\partial \phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{343}$$

$$+\frac{2\cot(\phi)}{r_0}\left(hu_{\theta m}\frac{\beta_i}{2i+1} + hu_{\phi m}\frac{\alpha_i}{2i+1} + h\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j\beta_k\int_0^1\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right)\right) = 0 \tag{344}$$

Or

$$\frac{\partial}{\partial t}(h\alpha_i) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial \theta}\left(2hu_{\theta m}\alpha_i + h\sum_{j=1}^N\left(\sum_{k=1}^N(\alpha_j\alpha_k A_{ijk})\right)\right) \tag{345}$$

$$+\frac{1}{r_0}\frac{\partial}{\partial \phi}\left(hu_{\theta m}\beta_i + hu_{\phi m}\alpha_i + h\sum_{j=1}^N\left(\sum_{k=1}^N(\alpha_j\beta_k A_{ijk})\right)\right) \tag{346}$$

$$+\frac{u_{\theta m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial \theta}(h\alpha_i) + \frac{\partial}{\partial \phi}(h\sin(\phi)\beta_i)\right) \tag{347}$$

$$-\frac{1}{r_0\sin(\phi)}\sum_{j=1}^N\left(\sum_{k=1}^N\left(\alpha_j B_{ijk}\left(\frac{\partial}{\partial \theta}(h\alpha_k) + \frac{\partial}{\partial \phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{348}$$

$$+\frac{2\cot(\phi)}{r_0}\left(hu_{\theta m}\beta_i + hu_{\phi m}\alpha_i + h\sum_{j=1}^N\left(\sum_{k=1}^N(\alpha_j\beta_k A_{ijk})\right)\right) = 0 \tag{349}$$

Where

$$A_{ijk} = (2i+1)\int_0^1 \psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho \tag{350}$$

$$B_{ijk} = (2i+1)\int_0^1 \psi_i'(\rho)\psi_j(\rho)\int_0^\rho \psi_k(\rho)\,\mathrm{d}\rho'\,\mathrm{d}\rho \tag{351}$$

Beta Evolution Equation

$$\frac{\partial}{\partial t}\left(h\int_0^1 u_\phi\psi_i(\rho)\,\mathrm{d}\rho\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^1 u_\theta u_\phi\psi_i(\rho)\,\mathrm{d}\rho\right) + \frac{1}{r_0}\frac{\partial}{\partial\phi}\left(h\int_0^1 u_\phi^2\psi_i(\rho)\,\mathrm{d}\rho\right) + \int_0^1 \frac{\partial}{\partial\rho}(u_\phi\omega)\psi_i(\rho)\,\mathrm{d}\rho \tag{352}$$

$$+h\frac{\int_0^1\left(u_\phi^2 - u_\theta^2\right)\psi_i(\rho)\,\mathrm{d}\rho\cot(\phi)}{r_0} = 0 \tag{353}$$

$$\int_0^1 u_\phi\psi_i(\rho)\,\mathrm{d}\rho = \frac{\beta_i}{2i+1} \tag{354}$$

$$\int_0^1 u_\phi^2\psi_i(\rho)\,\mathrm{d}\rho = \int_0^1 u_{\phi m}^2\psi_i(\rho) + 2u_{\phi m}\sum_{j=1}^N\left(\beta_j\psi_i(\rho)\psi_j(\rho)\right) + \sum_{j=1}^N\left(\sum_{k=1}^N\left(\beta_j\beta_k\psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\right)\right)\mathrm{d}\rho \tag{355}$$

$$= 2u_{\phi m}\frac{\beta_i}{2i+1} + \sum_{j=1}^N\left(\sum_{k=1}^N\left(\beta_j\beta_k\int_0^1 \psi_i(\rho)\psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho\right)\right) \tag{356}$$

$$= 2u_{\phi m}\frac{\beta_i}{2i+1} + \frac{1}{2i+1}\sum_{j=1}^N\left(\sum_{k=1}^N\left(\beta_j\beta_k A_{ijk}\right)\right) \tag{357}$$

$$\int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}(u_\phi\omega)\,\mathrm{d}\rho = \int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}((u_{\phi m}+u_{\phi d})\omega)\,\mathrm{d}\rho \tag{358}$$

$$= u_{\phi m}\int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}(\omega)\,\mathrm{d}\rho + \int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}(u_{\phi d}\omega)\,\mathrm{d}\rho \tag{359}$$

$$= u_{\phi m}\int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}(\omega)\,\mathrm{d}\rho - \int_0^1 \psi_i'(\rho)u_{\phi d}\omega\,\mathrm{d}\rho \tag{360}$$

$$= -u_{\phi m}\int_0^1 \psi_i(\rho)\frac{\partial}{\partial\rho}\left(\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\right)\right)\mathrm{d}\rho \tag{361}$$

$$+\int_0^1 \psi_i'(\rho)u_{\phi d}\left(\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(h\int_0^\rho u_{\theta d}\,\mathrm{d}\rho'\right) + \frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\phi}\left(h\sin(\phi)\int_0^\rho u_{\phi d}\,\mathrm{d}\rho'\right)\right)\mathrm{d}\rho \tag{362}$$

$$= -\frac{u_{\phi m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}\left(h\frac{\alpha_i}{2i+1}\right) + \frac{\partial}{\partial\phi}\left(h\sin(\phi)\frac{\beta_i}{2i+1}\right)\right) \tag{363}$$

$$+\frac{1}{r_0\sin(\phi)}\sum_{j=1}^N\left(\sum_{k=1}^N\left(\beta_j\int_0^1 \psi_i'(\rho)\psi_j(\rho)\int_0^\rho \psi_k(\rho)\,\mathrm{d}\rho'\,\mathrm{d}\rho\left(\frac{\partial}{\partial\theta}(h\alpha_k) + \frac{\partial}{\partial\phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{364}$$

$$= -\frac{u_{\phi m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}\left(h\frac{\alpha_i}{2i+1}\right) + \frac{\partial}{\partial\phi}\left(h\sin(\phi)\frac{\beta_i}{2i+1}\right)\right) \tag{365}$$

$$+\frac{1}{r_0\sin(\phi)(2i+1)}\sum_{j=1}^N\left(\sum_{k=1}^N\left(B_{ijk}\beta_j\left(\frac{\partial}{\partial\theta}(h\alpha_k) + \frac{\partial}{\partial\phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{366}$$

$$\frac{\partial}{\partial t}\left(h\frac{\beta_i}{2i+1}\right)+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(hu_{\theta m}\frac{\beta_i}{2i+1}+hu_{\phi m}\frac{\alpha_i}{2i+1}+\frac{1}{2i+1}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(h\alpha_j\beta_k A_{ijk})\right)\right) \tag{367}$$

$$+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(2hu_{\phi m}\frac{\beta_i}{2i+1}+\frac{1}{2i+1}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(h\beta_j\beta_k A_{ijk})\right)\right) \tag{368}$$

$$-\frac{u_{\phi m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}\left(h\frac{\alpha_i}{2i+1}\right)+\frac{\partial}{\partial\phi}\left(h\sin(\phi)\frac{\beta_i}{2i+1}\right)\right) \tag{369}$$

$$+\frac{1}{r_0\sin(\phi)(2i+1)}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}\left(B_{ijk}\beta_j\left(\frac{\partial}{\partial\theta}(h\alpha_k)+\frac{\partial}{\partial\phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{370}$$

$$+\frac{h\cot(\phi)}{r_0}\left(2u_{\phi m}\frac{\beta_i}{2i+1}-2u_{\theta m}\frac{\alpha_i}{2i+1}+\frac{1}{2i+1}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}(\beta_j\beta_k-\alpha_j\alpha_k))\right)\right)=0 \tag{371}$$

$$\frac{\partial}{\partial t}(h\beta_i)+\frac{1}{r_0\sin(\phi)}\frac{\partial}{\partial\theta}\left(hu_{\theta m}\beta_i+hu_{\phi m}\alpha_i+\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(h\alpha_j\beta_k A_{ijk})\right)\right) \tag{372}$$

$$+\frac{1}{r_0}\frac{\partial}{\partial\phi}\left(2hu_{\phi m}\beta_i+\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(h\beta_j\beta_k A_{ijk})\right)\right) \tag{373}$$

$$-\frac{u_{\phi m}}{r_0\sin(\phi)}\left(\frac{\partial}{\partial\theta}(h\alpha_i)+\frac{\partial}{\partial\phi}(h\sin(\phi)\beta_i)\right) \tag{374}$$

$$+\frac{1}{r_0\sin(\phi)}\sum_{j=1}^{N}\left(\sum_{k=1}^{N}\left(B_{ijk}\beta_j\left(\frac{\partial}{\partial\theta}(h\alpha_k)+\frac{\partial}{\partial\phi}(h\sin(\phi)\beta_k)\right)\right)\right) \tag{375}$$

$$+\frac{h\cot(\phi)}{r_0}\left(2u_{\phi m}\beta_i-2u_{\theta m}\alpha_i+\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}(\beta_j\beta_k-\alpha_j\alpha_k))\right)\right)=0 \tag{376}$$