Derivation of Shallow Water Linearized Moment Equations

Shallow Water Moment Equations

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y$$

$$= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x$$

$$= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big)$$

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k\right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_y$$

$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j\right)$$

$$(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y$$

$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right)$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i \phi_j \phi_k \,\mathrm{d}\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i' \left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right) \phi_k \,\mathrm{d}\zeta$$

$$C_{ij} = \int_0^1 \phi_i' \phi_j' \,\mathrm{d}\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

To get to the Shallow Water Linearized Moment Equations, we assume that $\alpha_i = O(\varepsilon)$ and $\beta_i = O(\varepsilon)$ are drop all terms of $O(\varepsilon^2)$ in the moment equations. The momentum equations remain the same even though they contain some of these terms.

$$
\begin{gathered}
h_t + (hu_m)_x + (hv_m)_y = 0 \\
(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y \\
= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x) \\
(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x \\
= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big) \\
(h\alpha_i)_t + (2hu_m\alpha_i)_x + (hu_m\beta_i + hv_m\alpha_i)_y = u_m D_i - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j\right) \\
(h\beta_i)_t + (hu_m\beta_i + hv_m\alpha_i)_x + (2hv_m\beta_i)_y = v_m D_i - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right) \\
C_{ij} = \int_0^1 \phi_i' \phi_j' \, \mathrm{d}\zeta \\
D_i = (h\alpha_i)_x + (h\beta_i)_y
\end{gathered}
$$

We can write down the shallow water linearized moments equations in the form

$$
\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{1}
$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$
\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{2}
$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $\boldsymbol{n} = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$
n_1(\boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q})) + n_2(\boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})).
$$

If this matrix is diagonalizable with real eigenvalues for all directions $\boldsymbol{n}$, then this system is considered hyperbolic.

Flux Functions

$$
\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2} e_z g h^2 + hu^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j^2\right) \\ huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\ 2hu\alpha_1 \\ \beta_1 hu + \alpha_1 hv \\ \vdots \\ 2hu\alpha_N \\ hu\beta_N + hv\alpha_N \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\ \frac{1}{2} e_z g h^2 + hv^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\beta_j^2\right) \\ hu\beta_1 + hv\alpha_1 \\ 2hv\beta_1 \\ \vdots \\ hu\beta_N + hv\alpha_N \\ 2hv\beta_N \end{pmatrix}
$$

Nonconservative Matrices

$$
g_1(\boldsymbol{q}) = \begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
 & 0 & 0 & & & & & & \\
 & & 0 & u & & & & & \\
 & & & v & 0 & & & & \\
 & & & & 0 & u & & & \\
 & & & & & v & 0 & & \\
 & & & & & & 0 & \ddots & \\
 & & & & & & & \ddots &
\end{pmatrix}, \quad
g_2(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 & & & & & & & \\
 & 0 & 0 & & & & & & \\
 & & 0 & 0 & & & & & \\
 & & & 0 & u & & & & \\
 & & & & v & 0 & & & \\
 & & & & & 0 & u & & \\
 & & & & & & v & 0 & \\
 & & & & & & & 0 & \ddots \\
 & & & & & & & & \ddots
\end{pmatrix}
$$

Flux Jacobians

$$
\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z g h - u^2 - \sum_{i=1}^{N} \left(\frac{1}{2*i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N} \left(\frac{1}{2N+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & 2u & & & & \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & v & u & & & \\
\vdots & \vdots & \vdots & & \ddots & \ddots & & \\
-2u\alpha_N & 2\alpha_N & 0 & & & 0 & 2u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u
\end{pmatrix}
$$

$$
\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z g h - v^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & u & & & \\
-2v\beta_1 & 0 & 2\beta_1 & & 2v & 0 & & \\
\vdots & \vdots & \vdots & & & \ddots & \ddots & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u \\
-2u\beta_N & 0 & 2\beta_N & & & & & 2v
\end{pmatrix}
$$

Quasilinear Matrices, $Q_x = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), Q_y = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$
Q_x = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z g h - u^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & u & & & & \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & & u & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
-2u\alpha_N & 2\alpha_N & 0 & & & & u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & & u
\end{pmatrix}
$$

$$
Q_y = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z g h - v^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & & & & \\
-2v\beta_1 & 0 & 2\beta_1 & & v & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & \\
-2u\beta_N & 0 & 2\beta_N & & & & & v
\end{pmatrix}
$$

Quasilinear Eigenvalues The wavespeeds of this system in direction $\boldsymbol{n} = [n_1, n_2]$ are given by the eigenvalues of the matrix

$$n_1 Q_x + n_2 Q_y.$$

If all of the eigenvalues are real with a full set of eigenvectors, this this system is hyperbolic.

Convenient constants

$$\begin{aligned}
d_0^1 &= n_1\left(e_z g h - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)\\
d_0^2 &= n_1\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2\left(e_z g h - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right)\\
d_i^1 &= n_1(-2u\alpha_i) + n_2(-u\beta_i - \alpha_i v)\\
d_i^2 &= n_1(-u\beta_i - v\alpha_i) + n_2(-2v\beta_i)\\
b_i^1 &= n_1\frac{2}{2i+1}\alpha_i + n_2\frac{1}{2i+1}\beta_i\\
b_i^2 &= n_2\frac{1}{2i+1}\alpha_i\\
b_i^3 &= n_1\frac{1}{2i+1}\beta_i\\
b_i^4 &= n_1\frac{1}{2i+1}\alpha_i + n_2\frac{2}{2i+1}\beta_i\\
c_i^1 &= n_1 2\alpha_i + n_2\beta_i\\
c_i^2 &= n_1\beta_i\\
c_i^3 &= n_2\alpha_i\\
c_i^4 &= n_1\alpha_i + n_2 2\beta_i
\end{aligned}$$

Determinant of Block Matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|\left|A - BD^{-1}C\right|$$

$$\begin{aligned}
\det(n_1 Q_x + n_2 Q_y - \lambda I) &= \begin{vmatrix}
-\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0\\
d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2\\
d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4\\
d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\
d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
d_N^1 & c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\
d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde{\lambda}
\end{vmatrix}\\
&= |D|\left|A - BD^{-1}C\right|
\end{aligned}$$

$$A = \begin{pmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} \end{pmatrix} \tag{3}$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \end{pmatrix} \tag{4}$$

$$C = \begin{pmatrix} d_1^1 & c_1^1 & c_1^3 \\ d_1^2 & c_1^2 & c_1^4 \\ \vdots & \vdots & \vdots \\ d_N^1 & c_N^1 & c_N^3 \\ d_N^2 & c_N^2 & c_N^4 \end{pmatrix} \tag{5}$$

$$D = \begin{pmatrix} -\tilde{\lambda} & & & & \\ & -\tilde{\lambda} & & & \\ & & \ddots & & \\ & & & -\tilde{\lambda} & \\ & & & & -\tilde{\lambda} \end{pmatrix} \tag{6}$$

$$|D| = \tilde{\lambda}^{2N}$$

$$BD^{-1}C = \frac{1}{-\tilde{\lambda}} \begin{pmatrix} 0 & 0 & 0 \\ s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{pmatrix}$$

$$s_1 = \sum_{i=1}^{N} \left(b_i^1 d_i^1 + b_i^2 d_i^2 \right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1} (4\alpha_i u n_1 + \beta_i u n_2 + 3\alpha_i v n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_2 = \sum_{i=1}^{N} \left(b_i^1 c_i^1 + b_i^2 c_i^2 \right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_3 = \sum_{i=1}^{N} \left(b_i^1 c_i^3 + b_i^2 c_i^4 \right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_4 = \sum_{i=1}^{N} \left(b_i^3 d_i^1 + b_i^4 d_i^2 \right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1} (3\beta_i u n_1 + \alpha_i v n_1 + 4\beta_i v n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_5 = \sum_{i=1}^{N} \left(b_i^3 c_i^1 + b_i^4 c_i^2 \right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_6 = \sum_{i=1}^{N} \left(b_i^3 c_i^3 + b_i^4 c_i^4 \right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$A - BD^{-1}C = \frac{1}{\tilde{\lambda}} \begin{pmatrix} -\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2 & \tilde{\lambda}n_1 & \tilde{\lambda}n_2 \\ \tilde{\lambda}d_0^1 + s_1 & \tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2 & \tilde{\lambda}un_2 + s_3 \\ \tilde{\lambda}d_0^2 + s_4 & \tilde{\lambda}vn_1 + s_5 & \tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6 \end{pmatrix}$$

$$\begin{aligned}
\det\big(\tilde{\lambda}\big(A - BD^{-1}C\big)\big) &= \big(-\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2\big)\big(\big(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\big)\big(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\big) - \big(\tilde{\lambda}vn_1 + s_5\big)\big(\tilde{\lambda}un_2 + s_3\big)\big) \\
&\quad -\tilde{\lambda}n_1\big(\big(\tilde{\lambda}d_0^1 + s_1\big)\big(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\big) - \big(\tilde{\lambda}d_0^2 + s_4\big)\big(\tilde{\lambda}un_2 + s_3\big)\big) \\
&\quad +\tilde{\lambda}n_2\big(\big(\tilde{\lambda}d_0^1 + s_1\big)\big(\tilde{\lambda}vn_1 + s_5\big) - \big(\tilde{\lambda}d_0^2 + s_4\big)\big(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\big)\big)
\end{aligned}$$

$$\begin{aligned}
&= \big(-\tilde{\lambda}^2 - (un_1 + vn_2)\tilde{\lambda}\big)\big(\tilde{\lambda}^4 + (-un_1 - vn_2)\tilde{\lambda}^3 + (-s_2 - s_6)\tilde{\lambda}^2 + (un_1 s_6 + vn_2 s_2 - vn_1 s_3 - un_2 s_5)\tilde{\lambda} + s_2 s_6 - s_3 s_5\big) \\
&\quad -\tilde{\lambda}n_1\big(-d_0^1\tilde{\lambda}^3 + \big(d_0^1 vn_2 - s_1 - d_0^2 un_2\big)\tilde{\lambda}^2 + \big(d_0^1 s_6 + vn_2 s_1 - d_0^2 s_3 - un_2 s_4\big)\tilde{\lambda} + s_1 s_6 - s_3 s_4\big) \\
&\quad +\tilde{\lambda}n_2\big(d_0^2\tilde{\lambda}^3 + \big(d_0^1 vn_1 - d_0^2 un_1 + s_4\big)\tilde{\lambda}^2 + \big(d_0^1 s_5 + vn_1 s_1 - d_0^2 s_2 - un_1 s_4\big)\tilde{\lambda} + s_1 s_5 - s_2 s_4\big)
\end{aligned}$$

$$= -\tilde{\lambda}^6 + \Big(s_2 + s_6 + (un_1 + vn_2)^2\Big)\tilde{\lambda}^4 + (vn_1 s_3 + un_2 s_5 + un_1 s_2 + vn_2 s_6)\tilde{\lambda}^3$$

$$+\big(s_3 s_5 - s_2 s_6 + \big(u^2 n_1 n_2 + uvn_2^2\big)s_5 + \big(uvn_1^2 + v^2 n_1 n_2\big)s_3 - \big(u^2 n_1^2 + uvn_1 n_2\big)s_6 - \big(uvn_1 n_2 + v^2 n_2^2\big)s_2\big)\tilde{\lambda}^2 + (s_3 s_5 - s_2 s_6)(un_1 +$$

$$+d_0^1 n_1\tilde{\lambda}^4 + \big(-d_0^1 vn_1 n_2 + s_1 n_1 + d_0^2 un_1 n_2\big)\tilde{\lambda}^3 + \big(-d_0^1 n_1 s_6 - vn_1 n_2 s_1 + d_0^2 n_1 s_3 + un_1 n_2 s_4\big)\tilde{\lambda}^2 + (s_3 s_4 - s_1 s_6)n_1\tilde{\lambda}$$

$$+d_0^2 n_2\tilde{\lambda}^4 + \big(d_0^1 vn_1 n_2 - d_0^2 un_1 n_2 + n_2 s_4\big)\tilde{\lambda}^3 + \big(d_0^1 n_2 s_5 + vn_1 n_2 s_1 - d_0^2 n_2 s_2 - un_1 n_2 s_4\big)\tilde{\lambda}^2 + (s_1 s_5 - s_2 s_4)n_2\tilde{\lambda}$$

$$= -\tilde{\lambda}^6 + \Big(s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1 n_1 + d_0^2 n_2\Big)\tilde{\lambda}^4 + (vn_1 s_3 + un_2 s_5 + un_1 s_2 + vn_2 s_6 + s_1 n_1 + n_2 s_4)\tilde{\lambda}^3$$

$$+\big(s_3 s_5 - s_2 s_6 + \big(u^2 n_1 n_2 + uvn_2^2 + d_0^1 n_2\big)s_5 + \big(uvn_1^2 + v^2 n_1 n_2 + d_0^2 n_1\big)s_3 - \big(u^2 n_1^2 + uvn_1 n_2 + d_0^1 n_1\big)s_6 - \big(uvn_1 n_2 + v^2 n_2^2 + d_0^2 n_2$$

$$+((s_3 s_5 - s_2 s_6)(un_1 + vn_2) + (s_3 s_4 - s_1 s_6)n_1 + (s_1 s_5 - s_2 s_4)n_2)\tilde{\lambda}$$

Look at coefficients of polynomial in $\tilde{\lambda}$.

$$\begin{aligned}
& s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1 n_1 + d_0^2 n_2 \\
&= \sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2)\right) + \sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2)\right) + (un_1 + vn_2)^2 \\
&\quad + n_1^2\left(e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_1 n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) \\
&\quad + n_1 n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2^2\left(e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right) \\
&= \sum_{i=1}^{N}\left(\frac{5}{2i+1}(\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2)\right) + e_z gh\big(n_1^2 + n_2^2\big) - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(\alpha_i^2 n_1^2 + 2\alpha_i\beta_i n_1 n_2 + \beta_i^2 n_2^2\big)\right) \\
&= \sum_{i=1}^{N}\left(\frac{4}{2i+1}(\alpha_i n_1 + \beta_i n_2)^2\right) + e_z gh\big(n_1^2 + n_2^2\big)
\end{aligned}$$

$$
\begin{aligned}
& vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4 \\
&= \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i v n_1 n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&+ \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i u n_1 n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&+ \sum_{i=1}^{N} \left(\frac{1}{2i+1} \big(4\alpha_i u n_1^2 + \beta_i u n_1 n_2\big) (\alpha_i n_1 + \beta_i n_2) \right) \\
&+ \sum_{i=1}^{N} \left(\frac{1}{2i+1} \big(\alpha_i v n_1 n_2 + 4\beta_i v n_2^2\big) (\alpha_i n_1 + \beta_i n_2) \right) \\
&+ \sum_{i=1}^{N} \left(-\frac{1}{2i+1} \big(4\alpha_i u n_1^2 + \beta_i u n_1 n_2 + 3\alpha_i v n_1 n_2\big) (\alpha_i n_1 + \beta_i n_2) \right) \\
&+ \sum_{i=1}^{N} \left(-\frac{1}{2i+1} \big(3\beta_i u n_1 n_2 + \alpha_i v n_1 n_2 + 4\beta_i v n_2^2\big) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= \sum_{i=1}^{N} \left(\frac{1}{2i+1} (3\alpha_i v n_1 n_2 + 3\beta_i u n_1 n_2 - 3\alpha_i v n_1 n_2 - 3\beta_i u n_1 n_2)(\alpha_i n_1 + \beta_i n_2) \right) \\
&= 0
\end{aligned}
$$

$$
a_i = \alpha_i n_1 + \beta_i n_2
$$

$$
\begin{aligned}
& s_3 s_5 - s_2 s_6 \\
&= \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 a_i \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 a_i \right) - \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2) a_i \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2) a_i \right) \\
&= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{3}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right) - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2) a_i a_j \right) \right) \\
&= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} (9\alpha_i \beta_j n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2)) a_i a_j \right) \right) \\
&= \sum_{i=1}^{N} \left(\frac{1}{(2i+1)^2} (9\alpha_i \beta_i n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + 4\beta_i n_2)) a_i a_i \right) \\
&+ \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} \left(\frac{1}{2i+1} \frac{1}{2j+1} (9\alpha_i \beta_j n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2) + 9\alpha_j \beta_i n_1 n_2 - (4\alpha_j n_1 + \beta_j n_2)(\alpha_i n_1 + 4\beta_i n_2)) a_i a_j \right) \right) \\
&= \sum_{i=1}^{N} \left(-\left(\frac{2}{2i+1} \right)^2 (\alpha_i n_1 + \beta_i n_2)^2 a_i a_i \right) + \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} \left(-8 \frac{1}{2i+1} \frac{1}{2j+1} (\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right) \\
&= -\sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{2}{2i+1} \frac{2}{2j+1} a_i^2 a_j^2 \right) \right) \\
&= -\left(\sum_{i=1}^{N} \left(\frac{2}{2i+1} a_i^2 \right) \right)^2
\end{aligned}
$$

$$\big(u^2 n_1 n_2 + uv n_2^2 + d_0^1 n_2\big) s_5$$

$$= \left(n_2 n_1 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_2^2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= \left(e_z g h n_1 n_2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_5 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right)$$

$$\big(uv n_1^2 + v^2 n_1 n_2 + d_0^2 n_1\big) s_3$$

$$= \left(-n_1^2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_1 n_2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_3 - \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_3 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_j \beta_i n_1 n_2 a_i a_j \right) \right)$$

$$\big(u^2 n_1^2 + uv n_1 n_2 + d_0^1 n_1\big) s_6$$

$$= \left(n_1^2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_1 n_2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1^2 s_6 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1^2 s_6 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} \alpha_i n_1 (\alpha_j n_1 + 4\beta_j n_2) a_i a_j \right) \right)$$

$$\big(uv n_1 n_2 + v^2 n_2^2 + d_0^2 n_2\big) s_2$$

$$= \left(n_2 n_1 \left(- \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2^2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_2^2 s_2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_2^2 s_2 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} \beta_i n_2 (4\alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right)$$

$$
\begin{aligned}
&s_3s_5 - s_2s_6 + \big(u^2n_1n_2 + uvn_2^2 + d_0^1n_2\big)s_5 + \big(uvn_1^2 + v^2n_1n_2 + d_0^2n_1\big)s_3 \\
&\quad -\big(u^2n_1^2 + uvn_1n_2 + d_0^1n_1\big)s_6 - \big(uvn_1n_2 + v^2n_2^2 + d_0^2n_2\big)s_2 \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zghn_1n_2s_5 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right) \\
&\quad + e_zghn_1n_2s_3 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_j\beta_in_1n_2a_ia_j\right)\right) \\
&\quad - e_zghn_1^2s_6 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_in_1(\alpha_jn_1 + 4\beta_jn_2)a_ia_j\right)\right) \\
&\quad - e_zghn_2^2s_2 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\beta_in_2(4\alpha_jn_1 + \beta_jn_2)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad -6\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_i\beta_jn_1n_2a_ia_j\right)\right) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\big(\alpha_i\alpha_jn_1^2 + 8\alpha_i\beta_jn_1n_2 + \beta_i\beta_jn_2^2\big)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\big(\alpha_i\alpha_jn_1^2 + 2\alpha_i\beta_jn_1n_2 + \beta_i\beta_jn_2^2\big)a_ia_j\right)\right) \\
&= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big) \\
&\quad + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}a_i^2a_j^2\right)\right) \\
&= -3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_zgh\big(n_1n_2(s_3+s_5) - n_1^2s_6 - n_2^2s_2\big)
\end{aligned}
$$

$$
\begin{aligned}
&n_1n_2(s_3+s_5)-n_1^2s_6-n_2^2s_2\\
&=n_1n_2\sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_i n_2 a_i\right)\\
&+n_1n_2\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_i n_1 a_i\right)\\
&-n_1^2\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_i n_1+4\beta_i n_2)a_i\right)\\
&-n_2^2\sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_i n_1+\beta_i n_2)a_i\right)\\
&=\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(3\alpha_i n_1n_2^2+3\beta_i n_1^2n_2-\big(\alpha_i n_1^3+4\beta_i n_1^2n_2\big)-\big(4\alpha_i n_1n_2^2+\beta_i n_2^3\big)\big)a_i\right)\\
&=-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(\alpha_i n_1^3+\beta_i n_1^2n_2+\alpha_i n_1n_2^2+\beta_i n_2^3\big)a_i\right)\\
&=-\sum_{i=1}^{N}\left(\frac{1}{2i+1}\big(n_1^2a_i+n_2^2a_i\big)a_i\right)\\
&=-\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2+e_zgh\big(n_1n_2(s_3+s_5)-n_1^2s_6-n_2^2s_2\big)\\
&=-3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2-e_zgh\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&(s_3s_5-s_2s_6)(un_1+vn_2)+(s_3s_4-s_1s_6)n_1+(s_1s_5-s_2s_4)n_2\\
&=-4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2(un_1+vn_2)+4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2un_1+4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2vn_2\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
\det\big(\tilde\lambda\big(A-BD^{-1}C\big)\big)&=-\tilde\lambda^6+\left(4\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)+e_zgh\big(n_1^2+n_2^2\big)\right)\tilde\lambda^4\\
&-\left(3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2+e_zgh\big(n_1^2+n_2^2\big)\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)\tilde\lambda^2\\
\det\big(A-BD^{-1}C\big)&=-\frac{1}{\tilde\lambda}\left(\tilde\lambda^2-\left(3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)+e_zgh\big(n_1^2+n_2^2\big)\right)\right)\left(\tilde\lambda^2-\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)
\end{aligned}
$$

Quasilinear Eigenvectors