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# Discontinuous Galerkin Method for solving thin film equations

Caleb Logemann

Mathematics Department, Iowa State University Iogemann@iastate.edu

September 30, 2017

# Overview

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Convection Hyper-Diffusion

- 1 Introduction
- 2 Convection
- 3 Hyper-Diffusion
- 4 Operator Splitting
- 5 Conclusion

### Motivation

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#### Introduction

CONVECTION

Operator Splitting

- Aircraft Icing
- Runback





■ Industrial Coating

# Model Equations

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$$\begin{aligned} \rho_t + (\rho u)_x &= 0\\ (\rho u)_t + (\rho u^2 + p)_x &= \frac{4}{3Re} u_{xx}\\ E_t + (u(E+p))_x &= \frac{1}{Re} \left(\frac{2}{3} (u^2)_{xx} + \frac{\gamma}{(\gamma - 1)Pr} \left(\frac{p}{\rho}\right)_{xx}\right) \end{aligned}$$

- Asymptotic Limit,  $\rho << L$
- Thin-Film Equation 1D with q as fluid height.

$$q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3q_{xxx}\right)_x$$

### Current Model

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Simplified Expression

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

$$q_t + (q^2 - q^3)_x = 0$$
  
 $q_t + (q^3 q_{xxx})_x = 0$ 

### Introduction to Discontinuous Galerkin

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#### Introduction

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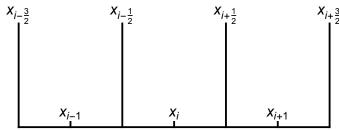
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■ Let  $\mathcal{T}_h$  partition the domain,  $\Omega = [a, b]$ 

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $K_i \in \mathcal{T}_h = [x_{i-1/2}, x_{i+1/2}]$
- $h = x_{i+1/2} x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$ .



# **Function Spaces**

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$$lacksquare$$
  $P^k(K)$  - polynomials of degree less than or equal to  $k$  on  $K\in\mathcal{T}_h$ 

$$V_h = \left\{ v \in L^2(\Omega) : \left. v \right|_K \in P^k(K), \quad \forall K \in \mathcal{T}_h \right\}$$

#### **Numerical Solutions**

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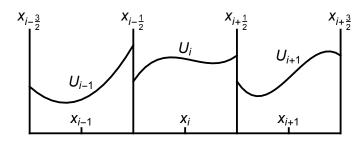
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Operator

- Let  $\{\phi^k(\xi)\}$  be the Legendre polynomials.
- Solution of order *M* on each cell

$$\left.q\right|_{x\in V_i}pprox Q_i=\sum_{k=1}^MQ_i^k\phi^k(\xi)$$



### Convection

Convection

Convection Equation

$$q_t + \frac{2}{\Delta x} f(q)_{\xi} = 0$$
$$f(q) = q^2 - q^3$$

Weak Form

$$\int_{-1}^{1} \left( q_t \phi(\xi) + \frac{2}{\Delta x} f(q) \xi \phi(\xi) \right) d\xi = 0$$

Runge-Kutta Discontinuous Galerkin

$$\dot{Q}_i^\ell = rac{1}{\Delta x} \int_{-1}^1 f(Q_i) \phi_{\xi}^\ell \,\mathrm{d}\xi - rac{1}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2})$$

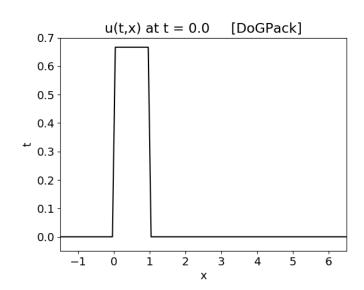
Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(Q_{i+1}(-1)) + f(Q_i(1))}{2} \phi^{\ell}(1)$$

 Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

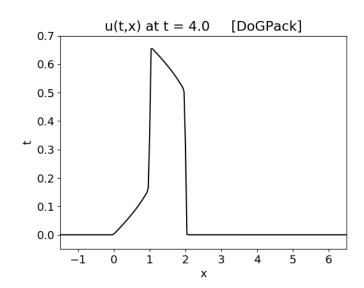
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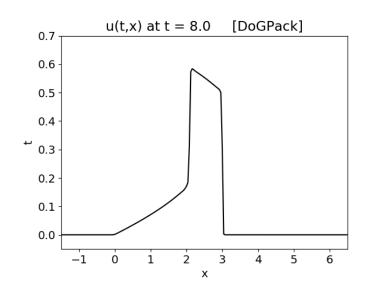
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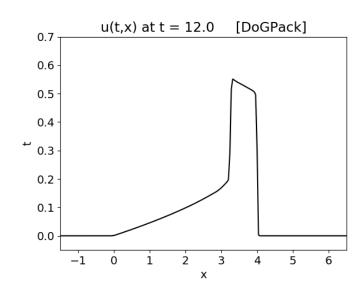
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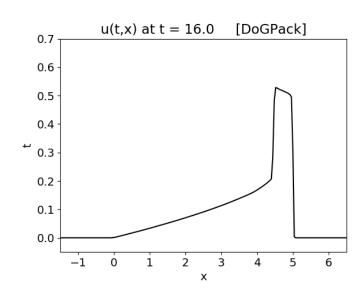
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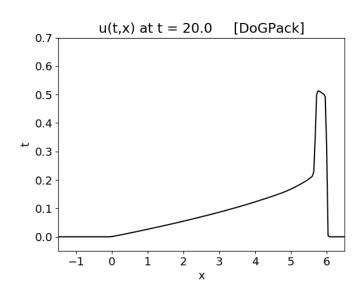
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Convection



# Hyper-Diffusion

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Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4} \left( u^3 u_{\xi\xi\xi} \right)_{\xi} = 0$$

Local Discontinuous Galerkin (LDG)

$$q = \frac{2}{\Delta x} u_{\xi}$$

$$r = \frac{2}{\Delta x} q_{\xi}$$

$$s = \frac{2}{\Delta x} u^{3} r_{\xi}$$

$$u_{t} = -\frac{2}{\Delta x} s_{\xi}$$

### Local Discontinuous Galerkin

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Hyper-Diffusion

$$\eta(\xi) = (U_{i}^{n})^{3} 
Q_{i}^{\ell} = -\frac{1}{\Delta x} \left( \int_{-1}^{1} U_{i} \phi_{\xi}^{\ell} d\xi - \mathcal{F}(U)_{i+1/2}^{\ell} + \mathcal{F}(U)_{i-1/2}^{\ell} \right) 
R_{i}^{\ell} = -\frac{1}{\Delta x} \left( \int_{-1}^{1} Q_{i} \phi_{\xi}^{\ell} d\xi - \mathcal{F}(Q)_{i+1/2}^{\ell} + \mathcal{F}(Q)_{i-1/2}^{\ell} \right) 
S_{i}^{\ell} = \frac{1}{\Delta x} \left( \int_{-1}^{1} (R_{i})_{\xi} \eta(\xi) \phi^{\ell} d\xi \right) 
+ \frac{1}{\Delta x} \left( \mathcal{F}(\eta)_{i+1/2} \mathcal{F}(R)_{i+1/2}^{\ell} - \mathcal{F}(\eta)_{i-1/2} \mathcal{F}(R)_{i-1/2}^{\ell} \right) 
\dot{U}_{i}^{\ell} = \frac{1}{\Delta x} \left( \int_{-1}^{1} S_{i} \phi_{\xi}^{\ell} d\xi - \mathcal{F}(S)_{i+1/2}^{\ell} + \mathcal{F}(S)_{i-1/2}^{\ell} \right)$$

### Local Discontinuous Galerkin

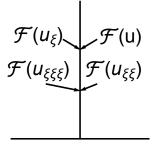
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$$\begin{split} \mathcal{F}(\eta)_{i+1/2} &= \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1)) \\ \mathcal{F}(\eta)_{i-1/2} &= \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1)) \\ \mathcal{F}(*)_{i+1/2}^{\ell} &= \phi^{\ell}(1)_{i+1/2} \end{split}$$



#### Local Discontinuous Galerkin

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■ Explicit SSP Runge Kutta

- Severe time step restriction
- $\Delta t \sim \Delta x^4$
- $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
- $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
  - Linear System Solver
  - Stabilized BiConjugate Gradient
  - MultiGrid Solver

# **Operator Splitting**

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Operator Splitting

- Strang Splitting
  - 1 time step
    - lacksquare 1/2 time step for convection
    - 1 time step for hyper-diffusion
    - 1/2 time step for convection
  - Second order splitting

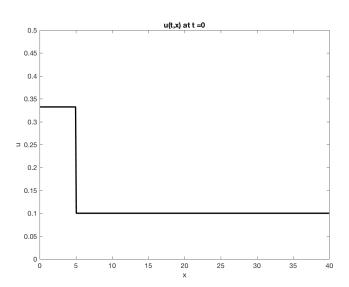
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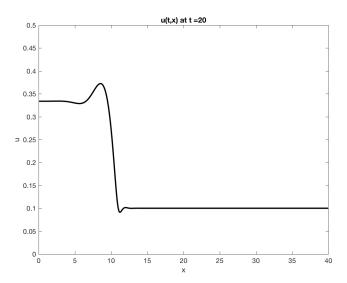
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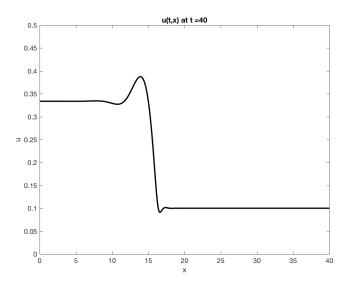
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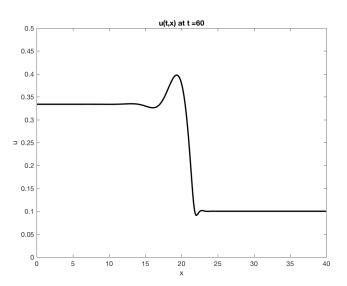
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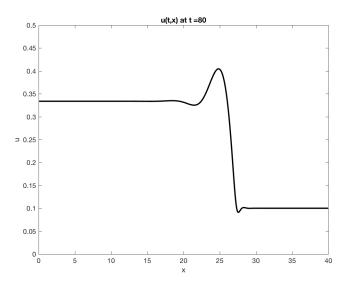
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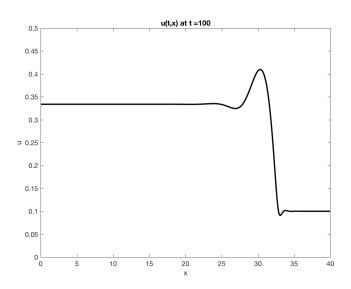


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### Future Work

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- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

### Conclusion

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#### Thanks

- James Rossmanith
- Alric Rothmayer
- Questions?

# **Bibliography**

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Operator

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