Local Discontinuous Galerkin Method for the Diffusion Equation

We would like to solve the 1D diffusion equation with a Discontinuous Galerkin Method. The 1D diffusion equation is given as

$$u_t = u_{xx}.$$

If we were to naively apply DG methods, we could discretize the domain and consider piecewise polynomial approximation. We would then multiply by a test function and integrate by parts.

$$\int_{I_j} u_t v \, \mathrm{d}x = \int_{I_j} u_{xx} v \, \mathrm{d}x$$

$$\int_{I_j} u_t v \, \mathrm{d}x = \left(\left(\hat{u}_x v^- \right)_{j+1/2} - \left(\hat{u}_x v^+ \right)_{j-1/2} \right) - \int_{I_j} u_x v_x \, \mathrm{d}x$$

and we use the average numerical flux

$$\left(\hat{u}_x\right)_{j+1/2} = \frac{(u_x^-)_{j+1/2} + (u_x^+)_{j+1/2}}{2}$$

This method is convergent and stable but it converges to the wrong solution.

Local Discontinuous Galerkin Method The Local Discontinuous Galerkin method proposes a different approach. First rewrite the diffusion equation as a system of first order equations.

$$u_t = q_x$$
$$q = u_x$$

The LDG method becomes the process of finding $u_h, q_h \in V_h$ in the DG solution space, such that for all test functions $v_h, w_h \in V_h$ and for all j the following equations are satisfied

$$\int_{I_j} (u_h)_t v_h \, \mathrm{d}x = \int_{I_j} (q_h)_x v_h \, \mathrm{d}x$$
$$\int_{I_j} q_h w_h \, \mathrm{d}x = \int_{I_j} (u_h)_x w_h \, \mathrm{d}x$$

After integrating by parts, these equations are

$$\int_{I_j} (u_h)_t v_h \, \mathrm{d}x = \left(\left(\hat{q}_h v_h^- \right)_{j+1/2} - \left(\hat{q}_h v_h^+ \right)_{j-1/2} \right) - \int_{I_j} q_h (v_h)_x \, \mathrm{d}x$$
$$\int_{I_j} q_h w_h \, \mathrm{d}x = \left(\left(\hat{u}_h w_h^- \right)_{j+1/2} - \left(\hat{u}_j w_h^+ \right)_{j-1/2} \right) - \int_{I_j} u_h (w_h)_x \, \mathrm{d}x$$

A common choice of numerical fluxes are the so-called alternating fluxes.

$$\hat{q}_h = q_h^+$$
$$\hat{u}_h = u_h^-$$

Implementation If we consider a single cell I_j, do a linear transformation from $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$, and consider specifically the Legendre polynomial basis $\left\{\phi^k(\xi)\right\}$ with the following orthogonality property

$$\frac{1}{2} \int_{-1}^{1} \phi^j(\xi) \phi^k(\xi) \, \mathrm{d}\xi = \delta_{jk}$$

we can form a more concrete LDG method for implementing. The linear transformation can be expressed as

$$x = \frac{\Delta x}{2} \xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

or

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

After this tranformation the diffusion equation become

$$u_t = \frac{4}{\Delta x^2} u_{\xi\xi}$$

on the cell I_j. We can then write this as the following system of first order equations.

$$\begin{aligned} u_t &= \frac{2}{\Delta x} q_\xi \\ q &= \frac{2}{\Delta x} u_\xi \end{aligned}$$

With the Legendre basis, the numerical solution on I_j can be written as

$$\begin{aligned} u \approx u_h &= \sum_{k=1}^{M} \left(U_k \phi^k(\xi)\right) \\ q \approx q_h &= \sum_{k=1}^{M} \left(Q_k \phi^k(\xi)\right) \end{aligned}$$

Now plugging these into the system and multiplying by a Legendre basis and integrating gives.

$$\begin{aligned} q_h &= \frac{2}{\Delta x}(u_h)_\xi \\ \frac{1}{2}\int_{-1}^{1} q_h \phi^l \, \mathrm{d}\xi &= \frac{1}{\Delta x}\int_{-1}^{1} (u_h)_\xi \phi^l \, \mathrm{d}\xi \\ Q_l &= -\frac{1}{\Delta x}\int_{-1}^{1} u_h \phi^l_\xi \, \mathrm{d}\xi + \frac{1}{\Delta x}\left(u^-_{j+1/2}\phi^l(1) - u^-_{j-1/2}\phi^l(-1)\right) \\ (u_h)_t &= \frac{2}{\Delta x}(q_h)_\xi \\ \frac{1}{2}\int_{-1}^{1} (u_h)_t \phi^l \, \mathrm{d}\xi &= \frac{1}{\Delta x}\int_{-1}^{1} (q_h)_\xi \phi^l \, \mathrm{d}\xi \\ \dot{U}_l &= -\frac{1}{\Delta x}\int_{-1}^{1} q_h \phi^l_\xi \, \mathrm{d}\xi + \frac{1}{\Delta x}\left(q^+_{j+1/2}\phi^l(1) - q^+_{j-1/2}\phi^l(-1)\right) \end{aligned}$$

Now this is a system of ODEs, there are $M \times N$ ODEs if M is the spacial order and N is the number of cells.

Proving Stability In order to prove that this method is L^2 stable consider we sum both of the integral equations from before.

$$\begin{aligned} \int_{I_j} (u_h)_t v_h \, \mathrm{d}x + \int_{I_j} q_h w_h \, \mathrm{d}x &= \left(\left(q^+_h v^-_h\right)_{j+1/2} - \left(q^+_h v^+_h\right)_{j-1/2}\right) \\ +\left(\left(u^-_h w^-_h\right)_{j+1/2} - \left(u^-_j w^+_h\right)_{j-1/2}\right) &- \int_{I_j} q_h (v_h)_x \, \mathrm{d}x - \int_{I_j} u_h (w_h)_x \, \mathrm{d}x \end{aligned}$$

Consider using $v_h = u_h$ and $w_h = q_h$.

$$\begin{aligned} \int_{I_j} (u_h)_t u_h \, \mathrm{d}x + \int_{I_j} q_h q_h \, \mathrm{d}x &= \left(\left(q^+_h u^-_h\right)_{j+1/2} - \left(q^+_h u^+_h\right)_{j-1/2}\right) \\ +\left(\left(u^-_h q^-_h\right)_{j+1/2} - \left(u^-_h q^+_h\right)_{j-1/2}\right) &- \int_{I_j} q_h (u_h)_x \, \mathrm{d}x - \int_{I_j} u_h (q_h)_x \, \mathrm{d}x \end{aligned}$$

Consider the following shorthand notation

$$B_j = \int_{I_j} (u_h)_t u_h \,\mathrm{d}x + \int_{I_j} q_h q_h \,\mathrm{d}x$$

$$B_j = \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(q_h^+ u_h^+\right)_{j-1/2}\right) + \left(\left(u_h^- q_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) - \int_{I_j} q_h (u_h)_x \,\mathrm{d}x - \int_{I_j} u_h (q_h)_x \,\mathrm{d}x$$

This can be simplified in several ways. First simplify the left hand side.

$$\begin{aligned}
B_j &= \int_{I_j} (u_h)_t u_h \,\mathrm{d}x + \int_{I_j} q_h q_h \,\mathrm{d}x \\
B_j &= \frac{1}{2}\int_{I_j} \frac{\mathrm{d}}{\mathrm{d}t}\left(u_h^2\right) \mathrm{d}x + \int_{I_j} q_h^2 \,\mathrm{d}x \\
B_j &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_j} u_h^2 \,\mathrm{d}x + \int_{I_j} q_h^2 \,\mathrm{d}x \\
B_j &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L^2(I_j)} + \|q_h\|^2_{L^2(I_j)}
\end{aligned}$$

Second the right hand side can be simplified.

$$\begin{aligned}
\int_{I_j} q_h (u_h)_x \,\mathrm{d}x + \int_{I_j} u_h (q_h)_x \,\mathrm{d}x &= \int_{I_j} q_h (u_h)_x + u_h (q_h)_x \,\mathrm{d}x \\
&= \int_{I_j} (q_h u_h)_x \,\mathrm{d}x \\
&= (q_h^- u_h^-)_{j+1/2} - (q_h^+ u_h^+)_{j-1/2}
\end{aligned}$$

Now

$$B_j = \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(q_h^+ u_h^+\right)_{j-1/2}\right) + \left(\left(u_h^- q_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) - \left((q_h^- u_h^-)_{j+1/2} - (q_h^+ u_h^+)_{j-1/2}\right)$$

$$B_j = \left(q_h^+ u_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}$$

Assuming periodic boundary conditions, and summing B_j over all cells

$$\begin{aligned}
\sum_{j=1}^{N} (B_j) &= \sum_{j=1}^{N} \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) \\
&= -\left(u_h^- q_h^+\right)_{1/2} + \sum_{k=1}^{N}\left(\left(q_h^+ u_h^-\right)_{k+1/2} - \left(u_h^- q_h^+\right)_{k+1/2}\right) + \left(q_h^+ u_h^-\right)_{N+1/2} \\
&= 0
\end{aligned}$$

This shows that

$$\begin{aligned}
\sum_{j=1}^{N} (B_j) &= \sum_{j=1}^{N}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L^2(I_j)} + \|q_h\|^2_{L^2(I_j)}\right) \\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L^2} &+ \|q_h\|^2_{L^2} = 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L^2} &\le 0
\end{aligned}$$