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introductio

Hyper Diffusio

Operator

Conclusion

Discontinuous Galerkin Method for solving a Thin-Film Equation

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Overview

- 1 Introduction
- 2 Convection
- 3 Hyper-Diffusion
- 4 Operator Splitting
- 5 Conclusion

Motivation

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Aircraft Icing







- Industrial Coating
- Paint Drying

Model Equations

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Navier-Stokes Equation

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x =$$
$$E_t + (u(E + p))_x =$$

- Asymptotic Limit, $\rho << L$
- Thin-Film Equation 1D with *u* as fluid height.

$$u_t + (f(x,t)u^2 - g(x,t)u^3)_x = (h(x,t)u^3u_{xxx})_x$$

Current Model

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Simplified Expression

$$u_t + \left(u^2 - u^3\right)_x = -\left(u^3 u_{xxx}\right)_x$$

$$u_t + (u^2 - u^3)_x = 0$$

$$u_t + (u^3 u_{xxx})_x = 0$$

Introduction to Discontinuous Galerkin

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Introduction

Convection

yper-Diffusion

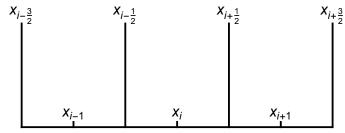
Operator Splitting

Conclusion

■ Partition the domain, [a, b] as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $V_i = [x_{i-1/2}, x_{i+1/2}]$
- $\Delta x = x_{i+1/2} x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$.



Numerical Solutions

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Hyper-Diffusion

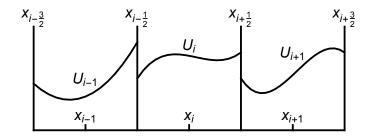
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• Use canonical variable $\xi \in [-1,1]$

- Let $\{\phi^k(\xi)\}$ be the Legendre polynomials.
- Solution of order *M* on each cell

$$u|_{x\in V_i} \approx U_i = \sum_{k=1}^M U_i^k \phi^k(\xi)$$



Convection

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Convection

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Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_{\xi} = 0$$
$$f(u) = u^2 - u^3$$

Weak Form

$$\int_{-1}^{1} \left(u_t \phi(\xi) + \frac{2}{\Delta x} f(u)_{\xi} \phi(\xi) \right) d\xi = 0$$

■ Runge-Kutta Discontinuous Galerkin

$$\dot{U}_i^\ell = rac{1}{\Delta x} \int_{-1}^1 f(U_i) \phi_\xi^\ell \,\mathrm{d}\xi - rac{1}{\Delta x} ig(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}ig)$$

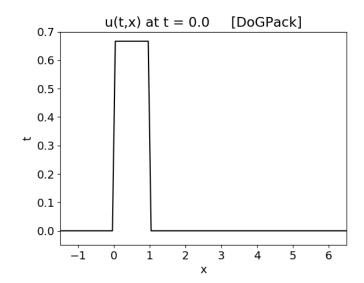
Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^{\ell}(1)$$

 Solve this system of ODEs with any Total Variation Diminishing (TVD) Runge-Kutta Method.

Convection



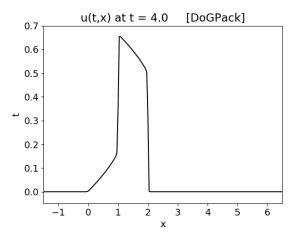


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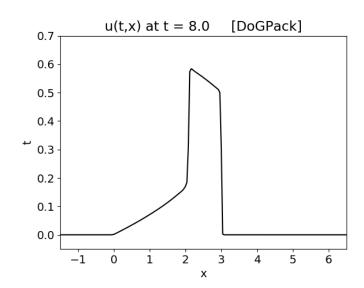
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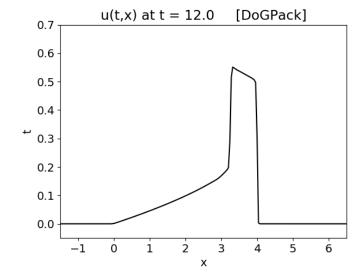
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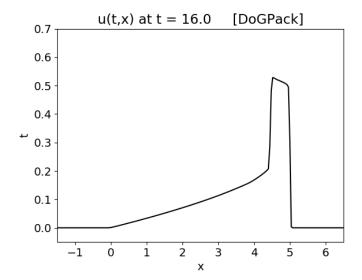
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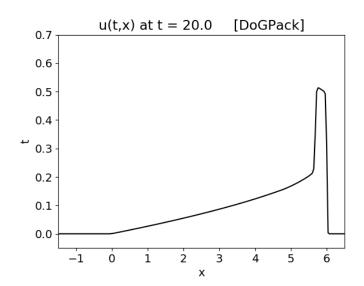
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Hyper-Diffusion

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■ Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4} \left(u^3 u_{\xi\xi\xi} \right)_{\xi} = 0$$

Local Discontinuous Galerkin

$$q = \frac{2}{\Delta x} u_{\xi}$$

$$r = \frac{2}{\Delta x} s_{\xi}$$

$$s = \frac{2}{\Delta x} u^{3} r_{\xi}$$

$$u = -\frac{2}{\Delta x} s_{\xi}$$

Local Discontinuous Galerkin

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$$\begin{aligned} Q_{i}^{\ell} &= -\frac{1}{\Delta x} \left(\int_{-1}^{1} U_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(U)_{i+1/2}^{\ell} + \mathcal{F}(U)_{i-1/2}^{\ell} \right) \\ R_{i}^{\ell} &= -\frac{1}{\Delta x} \left(\int_{-1}^{1} Q_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(Q)_{i+1/2}^{\ell} + \mathcal{F}(Q)_{i-1/2}^{\ell} \right) \\ \eta(\xi) &= (U_{i}^{n})^{3} \\ S_{i}^{\ell} &= \frac{1}{\Delta x} \left(\int_{-1}^{1} (R_{i})_{\xi} \eta(\xi) \phi^{\ell} \, \mathrm{d}\xi \right) \\ &+ \frac{1}{\Delta x} \left(\mathcal{F}(\eta)_{i+1/2} \mathcal{F}(S)_{i+1/2}^{\ell} - \mathcal{F}(\eta)_{i-1/2} \mathcal{F}(S)_{i-1/2}^{\ell} \right) \\ \dot{U}_{i}^{\ell} &= \frac{1}{\Delta x} \left(\int_{-1}^{1} S_{i} \phi_{\xi}^{\ell} \, \mathrm{d}\xi - \mathcal{F}(S)_{i+1/2}^{\ell} + \mathcal{F}(S)_{i-1/2}^{\ell} \right) \end{aligned}$$

Local Discontinuous Galerkin

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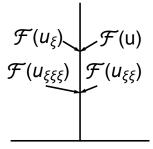
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$$\begin{split} \mathcal{F}(\eta)_{i+1/2} &= \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1)) \\ \mathcal{F}(\eta)_{i-1/2} &= \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1)) \\ \mathcal{F}(*)_{i+1/2}^{\ell} &= \phi^{\ell}(1) *_{i+1/2} \end{split}$$



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■ Explicit TVD Runge Kutta

- Severe time step restriction
- $\Delta t \sim \Delta x^4$
- $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
- $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit TVD Runge Kutta
 - Linear System Solver
 - Stabilized BiConjugate Gradient
 - MultiGrid Solver

Multigrid Solver

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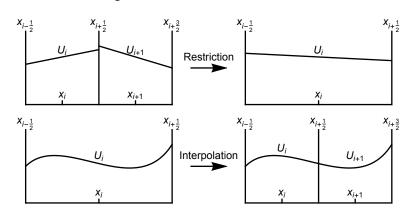
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Reference:

Relaxation e.g. Jacobi Relaxation



Multigrid Solver

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Convection

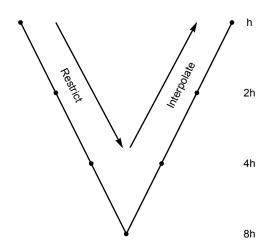
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V-Cycle



Multigrid Solver

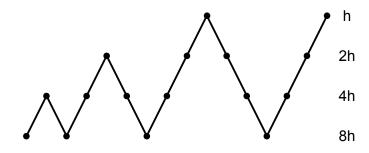
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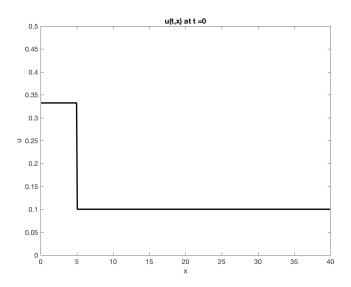
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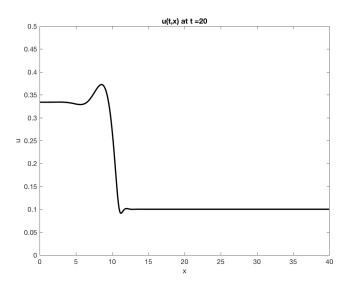
Hyper Diffusio

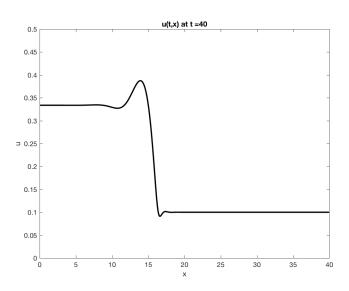
Operator Splitting

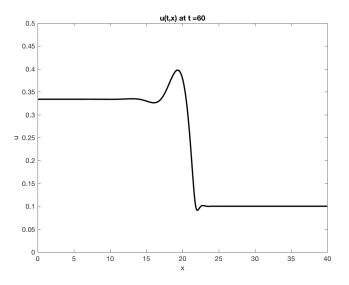
Conclusio

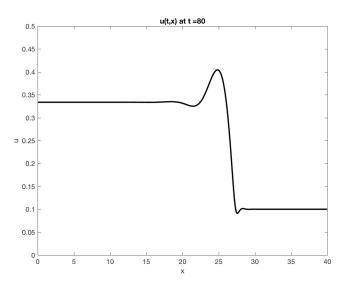
- Strang Splitting
 - 1 time step
 - \blacksquare 1/2 time step for convection
 - 1 time step for hyper-diffusion
 - 1/2 time step for convection
 - Second order splitting

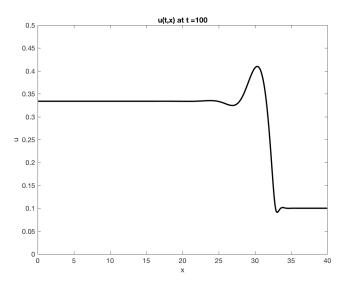












Future Work

Conclusion

- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

Conclusion

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Conclusion

- James Rossmanith
- Alric Rothmayer
- Questions?

Bibliography

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References

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