

Derivation of Shallow Water Equations

We begin by considering the Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}, \quad (2)$$

where $\mathbf{u} = [u, v, w]^T$ is the vector of velocities, p is the pressure, ρ is the constant density, σ is the deviatoric stress tensor, and \mathbf{g} is the gravitational force vector. We also have two boundaries, the bottom topography $h_b(t, x, y)$, and the free surface $h_s(t, x, y)$. At both of these boundaries the kinematic boundary conditions are in effect and can be expressed as

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s) \quad (3)$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b). \quad (4)$$

In practice the bottom topography is unchanging in time, but we express h_b with time dependence to allow for a symmetric representation of the boundary conditions.

1 Dimensional Analysis

Now we consider the characteristic scales of the problem. Let L be the characteristic horizontal length scale, and let H be the characteristic vertical length scale. For this problem we assume that $H \ll L$ and we denote the ratio of these lengths as $\varepsilon = H/L$. With these characteristic lengths we can scale the length variables to a nondimensional form

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}. \quad (5)$$

Now let U be the characteristic horizontal velocity, then because of the shallowness the characteristic vertical velocity will be εU . Therefore the velocity variables can be scaled as follows,

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}. \quad (6)$$

Now with the characteristic length and velocity, the time scaling can be described as

$$t = \frac{L}{U}\hat{t} \quad (7)$$

The pressure will be scaled by the characteristic height, H , and the stresses will be scaled by a characteristic stress, S . It is assumed that the basal shear stresses, σ_{xz} and σ_{yz} are of larger order than the lateral shear stress, σ_{xy} , and the normal stresses, σ_{xx} , σ_{yy} , and σ_{zz} , so that

$$p = \rho g H \hat{p}, \quad \sigma_{xz/yz} = S \hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S \hat{\sigma}_{xx/xy/yy/zz}. \quad (8)$$

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}, \quad u_x + v_y + w_z = 0 \\ u_t + (u^2)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho}p_x + \frac{1}{\rho}\left((\sigma_{xx})_x + (\sigma_{xy})_y + (\sigma_{xz})_z\right) + ge_x \\ v_t + (uv)_x + (v^2)_y + (vw)_z &= -\frac{1}{\rho}p_y + \frac{1}{\rho}\left((\sigma_{xy})_x + (\sigma_{yy})_y + (\sigma_{yz})_z\right) + ge_y \\ w_t + (uw)_x + (vw)_y + (w^2)_z &= -\frac{1}{\rho}p_z + \frac{1}{\rho}\left((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\right) + ge_z \end{aligned}$$

$$\begin{aligned}
u_x &= (U\hat{u})_x = (U\hat{u})_{\hat{x}}\hat{x}_x = \frac{U}{L}\hat{u}_{\hat{x}} \\
v_y &= (U\hat{v})_{\hat{y}}\hat{y}_y = \frac{U}{L}\hat{v}_{\hat{y}} \\
w_z &= (\varepsilon U\hat{w})_{\hat{z}}\hat{z}_z = \frac{\varepsilon U}{H}\hat{w}_{\hat{z}} = \frac{U}{L}\hat{w}_{\hat{z}} \\
u_x + v_y + w_z &= \frac{U}{L}\hat{u}_{\hat{x}} + \frac{U}{L}\hat{v}_{\hat{y}} + \frac{U}{L}\hat{w}_{\hat{z}} = 0 \\
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
u_t &= (U\hat{u})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{u}_{\hat{t}} \\
v_t &= (U\hat{v})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{v}_{\hat{t}} \\
w_t &= (\varepsilon U\hat{w})_{\hat{t}}\hat{t}_t = \varepsilon \frac{U^2}{L}\hat{w}_{\hat{t}} \\
(u^2)_x &= (U^2\hat{u}^2)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}(\hat{u}^2)_{\hat{x}} \\
(uv)_y &= (U^2\hat{u}\hat{v})_{\hat{y}}\hat{y}_y = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}} \\
(uw)_z &= (\varepsilon U^2\hat{u}\hat{w})_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}} \\
(uv)_x &= (U^2\hat{u}\hat{v})_{\hat{x}}\hat{x}_x = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}} \\
(v^2)_y &= (U^2\hat{v}^2)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}(\hat{v}^2)_{\hat{y}} \\
(vw)_z &= (\varepsilon U^2\hat{v}\hat{w})_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}} \\
(uw)_x &= (\varepsilon U^2\hat{u}\hat{w})_{\hat{x}}\hat{x}_x = \varepsilon \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}} \\
(vw)_y &= (\varepsilon U^2\hat{v}\hat{w})_{\hat{y}}\hat{y}_y = \varepsilon \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{y}} \\
(w^2)_z &= (\varepsilon^2 U^2\hat{w}^2)_{\hat{z}}\hat{z}_z = \varepsilon \frac{U^2}{L}(\hat{w}^2)_{\hat{z}}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\rho}p_x &= \frac{1}{\rho}(\rho g H \hat{p})_{\hat{x}} \hat{x}_x = \frac{\varepsilon g L}{L} \hat{p}_{\hat{x}} = \varepsilon g \hat{p}_{\hat{x}} \\
\frac{1}{\rho}p_y &= \frac{1}{\rho}(\rho g H \hat{p})_{\hat{y}} \hat{y}_y = \frac{\varepsilon g L}{L} \hat{p}_{\hat{y}} = \varepsilon g \hat{p}_{\hat{y}} \\
\frac{1}{\rho}p_z &= \frac{1}{\rho}(\rho g H \hat{p})_{\hat{z}} \hat{z}_z = \frac{\varepsilon g L}{\varepsilon L} \hat{p}_{\hat{z}} = g \hat{p}_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xx})_x &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{xx})_{\hat{x}} \hat{x}_x = \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xx})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{xy})_y &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{xy})_{\hat{y}} \hat{y}_y = \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xy})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{xz})_z &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{xz})_{\hat{z}} \hat{z}_z = \frac{S}{\rho H} (\hat{\sigma}_{xz})_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xy})_x &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{xy})_{\hat{x}} \hat{x}_x = \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xy})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{yy})_y &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{yy})_{\hat{y}} \hat{y}_y = \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{yy})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{yz})_z &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{yz})_{\hat{z}} \hat{z}_z = \frac{S}{\rho H} (\hat{\sigma}_{yz})_{\hat{z}} \\
\frac{1}{\rho}(\sigma_{xz})_x &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{xz})_{\hat{x}} \hat{x}_x = \frac{S}{\rho L} (\hat{\sigma}_{xz})_{\hat{x}} \\
\frac{1}{\rho}(\sigma_{yz})_y &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{yz})_{\hat{y}} \hat{y}_y = \frac{S}{\rho L} (\hat{\sigma}_{yz})_{\hat{y}} \\
\frac{1}{\rho}(\sigma_{zz})_z &= \frac{1}{\rho}(\varepsilon S \hat{\sigma}_{zz})_{\hat{z}} \hat{z}_z = \varepsilon \frac{S}{\rho H} (\hat{\sigma}_{zz})_{\hat{z}} \\
\frac{U^2}{L} \hat{u}_{\hat{t}} + \frac{U^2}{L} (\hat{u}^2)_{\hat{x}} + \frac{U^2}{L} (\hat{u} \hat{v})_{\hat{y}} + \frac{U^2}{L} (\hat{u} \hat{w})_{\hat{z}} &= -\varepsilon g \hat{p}_{\hat{x}} + \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xy})_{\hat{y}} + \frac{S}{\rho H} (\hat{\sigma}_{xz})_{\hat{z}} + g e_x \\
\frac{U^2}{L} \hat{v}_{\hat{t}} + \frac{U^2}{L} (\hat{u} \hat{v})_{\hat{x}} + \frac{U^2}{L} (\hat{v}^2)_{\hat{y}} + \frac{U^2}{L} (\hat{v} \hat{w})_{\hat{z}} &= -\varepsilon g \hat{p}_{\hat{y}} + \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon \frac{S}{\rho L} (\hat{\sigma}_{yy})_{\hat{y}} + \frac{S}{\rho H} (\hat{\sigma}_{yz})_{\hat{z}} + g e_y \\
\varepsilon \frac{U^2}{L} \hat{w}_{\hat{t}} + \varepsilon \frac{U^2}{L} (\hat{u} \hat{w})_{\hat{x}} + \varepsilon \frac{U^2}{L} (\hat{v} \hat{w})_{\hat{y}} + \varepsilon \frac{U^2}{L} (\hat{w}^2)_{\hat{z}} &= -g \hat{p}_{\hat{z}} + \frac{S}{\rho L} (\hat{\sigma}_{xz})_{\hat{x}} + \frac{S}{\rho L} (\hat{\sigma}_{yz})_{\hat{y}} + \varepsilon \frac{S}{\rho H} (\hat{\sigma}_{zz})_{\hat{z}} + g e_z \\
\frac{U^2}{L} \left(\hat{u}_{\hat{t}} + (\hat{u}^2)_{\hat{x}} + (\hat{u} \hat{v})_{\hat{y}} + (\hat{u} \hat{w})_{\hat{z}} \right) &= -\varepsilon g \hat{p}_{\hat{x}} + \frac{S}{\rho H} \left(\varepsilon^2 (\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}} \right) + g e_x \\
\frac{U^2}{L} \left(\hat{v}_{\hat{t}} + (\hat{u} \hat{v})_{\hat{x}} + (\hat{v}^2)_{\hat{y}} + (\hat{v} \hat{w})_{\hat{z}} \right) &= -\varepsilon g \hat{p}_{\hat{y}} + \frac{S}{\rho H} \left(\varepsilon^2 (\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}} \right) + g e_y \\
\varepsilon \frac{U^2}{L} \left(\hat{w}_{\hat{t}} + (\hat{u} \hat{w})_{\hat{x}} + (\hat{v} \hat{w})_{\hat{y}} + (\hat{w}^2)_{\hat{z}} \right) &= -g \hat{p}_{\hat{z}} + \varepsilon \frac{S}{\rho H} \left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}} \right) + g e_z
\end{aligned}$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon \frac{U^2}{gH} \left(\hat{u}_{\hat{t}} + (\hat{u}^2)_{\hat{x}} + (\hat{u} \hat{v})_{\hat{y}} + (\hat{u} \hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{x}} + \frac{S}{\rho g H} \left(\varepsilon^2 (\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}} \right) + e_x \\
\varepsilon \frac{U^2}{gH} \left(\hat{v}_{\hat{t}} + (\hat{u} \hat{v})_{\hat{x}} + (\hat{v}^2)_{\hat{y}} + (\hat{v} \hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{y}} + \frac{S}{\rho g H} \left(\varepsilon^2 (\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}} \right) + e_y \\
\varepsilon^2 \frac{U^2}{gH} \left(\hat{w}_{\hat{t}} + (\hat{u} \hat{w})_{\hat{x}} + (\hat{v} \hat{w})_{\hat{y}} + (\hat{w}^2)_{\hat{z}} \right) &= -\hat{p}_{\hat{z}} + \varepsilon \frac{S}{\rho g H} \left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}} \right) + e_z
\end{aligned}$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon F^2 \left(\hat{u}_{\hat{t}} + (\hat{u}^2)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{x}} + G \left(\varepsilon^2 (\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}} \right) + e_x \\
\varepsilon F^2 \left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + (\hat{v}^2)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{y}} + G \left(\varepsilon^2 (\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}} \right) + e_y \\
\varepsilon^2 F^2 \left(\hat{w}_{\hat{t}} + (\hat{u}\hat{w})_{\hat{x}} + (\hat{v}\hat{w})_{\hat{y}} + (\hat{w}^2)_{\hat{z}} \right) &= -\hat{p}_{\hat{z}} + \varepsilon G \left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}} \right) + e_z \\
F &= \frac{U}{\sqrt{gH}} \approx 1, \quad G = \frac{S}{\rho g H} < 1
\end{aligned}$$

Drop terms with ε^2 and εG , giving

$$\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon F^2 \left(\hat{u}_{\hat{t}} + (\hat{u}^2)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{x}} + G (\hat{\sigma}_{xz})_{\hat{z}} + e_x \\
\varepsilon F^2 \left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + (\hat{v}^2)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}} \right) &= -\varepsilon \hat{p}_{\hat{y}} + G (\hat{\sigma}_{yz})_{\hat{z}} + e_y \\
\hat{p}_{\hat{z}} &= e_z
\end{aligned}$$

where we can solve for the hydrostatic pressure

$$\hat{p}(\hat{t}, \hat{x}, \hat{y}) = \left(\hat{h}_s(\hat{t}, \hat{x}, \hat{y}) - \hat{z} \right) e_z$$

Transforming back to dimensional variables for readability gives

$$\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + (u^2)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho} p_x + \frac{1}{\rho} (\sigma_{xz})_z + g e_x \\
v_t + (uv)_x + (v^2)_y + (vw)_z &= -\frac{1}{\rho} p_y + \frac{1}{\rho} (\sigma_{yz})_z + g e_y \\
p(t, x, y, z) &= (h_s(t, x, y) - z) \rho g e_z
\end{aligned}$$

2 Mapping

In order to make this system more accessible we will map the vertical variable z to the normalized variable ζ , through the transformation

$$\zeta(t, x, y, z) = \frac{z - h_b(t, x, y)}{h(t, x, y)},$$

or equivalently

$$z(t, x, y, \zeta) = h(t, x, y) \zeta + h_b(t, x, y)$$

where $h(t, x, y) = h_s(t, x, y) - h_b(t, x, y)$. This transformation maps the vertical variable onto a scale from 0 to 1 everywhere.

Consider a function $\Psi(t, x, y, z)$, then is mapped counterpart $\tilde{\Psi}(t, x, y, \zeta)$ can be described as

$$\tilde{\Psi}(t, x, y, \zeta) = \Psi(t, x, y, z(t, x, y, \zeta)) = \Psi(t, x, y, h(t, x, y) \zeta + h_b(t, x, y)),$$

or equivalently

$$\Psi(t, x, y, z) = \tilde{\Psi}(t, x, y, \zeta(t, x, y, z)) = \tilde{\Psi}\left(t, x, y, \frac{z - h_b(t, x, y)}{h(t, x, y)}\right).$$

We also need to be able to map derivatives of functions in order to be able to map the differential equations. This

can be described

$$\begin{aligned}
\Psi_z(t, x, y, z) &= (\tilde{\Psi}(t, x, y, \zeta(t, x, y, z)))_z \\
\Psi_z(t, x, y, z) &= \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z))\zeta_z(t, x, y, z) \\
\Psi_z(t, x, y, z) &= \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z)) \frac{1}{h(t, x, y)} \\
h(t, x, y)\Psi_z(t, x, y, z) &= \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z)) \\
h\Psi_z &= \tilde{\Psi}_\zeta
\end{aligned}$$

For the other derivatives, the partial derivatives are identical for $s \in \{t, x, y\}$.

$$\begin{aligned}
\zeta_s(t, x, y, z) &= \left(\frac{z - h_b(t, x, y)}{h(t, x, y)} \right)_s \\
&= -\frac{(z - h_b(t, x, y))h_s(t, x, y)}{h(t, x, y)^2} - \frac{(h_b)_s(t, x, y)}{h(t, x, y)} \\
&= -\zeta(t, x, y, z) \frac{h_s(t, x, y)}{h(t, x, y)} - \frac{(h_b)_s(t, x, y)}{h(t, x, y)} \\
&= -\frac{\zeta(t, x, y, z)h_s(t, x, y) + (h_b)_s(t, x, y)}{h(t, x, y)}
\end{aligned}$$

$$\begin{aligned}
\Psi_s(t, x, y, z) &= (\tilde{\Psi}(t, x, y, \zeta(t, x, y, z)))_s \\
\Psi_s(t, x, y, z) &= \tilde{\Psi}_s(t, x, y, \zeta(t, x, y, z)) + \tilde{\Psi}_\zeta(t, x, y, \zeta(t, x, y, z))\zeta_s(t, x, y, z) \\
\Psi_s(t, x, y, z) &= \tilde{\Psi}_s(t, x, y, \zeta) - \tilde{\Psi}_\zeta(t, x, y, \zeta) \left(\frac{\zeta h_s(t, x, y) + (h_b)_s(t, x, y)}{h(t, x, y)} \right) \\
h(t, x, y)\Psi_s(t, x, y, z) &= h(t, x, y)\tilde{\Psi}_s(t, x, y, \zeta) - \tilde{\Psi}_\zeta(t, x, y, \zeta)(\zeta h_s(t, x, y) + (h_b)_s(t, x, y)) \\
h(t, x, y)\Psi_s(t, x, y, z) &= h(t, x, y)\tilde{\Psi}_s(t, x, y, \zeta) - \tilde{\Psi}_\zeta(t, x, y, \zeta)(\zeta h_s(t, x, y) + (h_b)_s(t, x, y)) \\
h\Psi_s &= h\tilde{\Psi}_s - \tilde{\Psi}_\zeta(\zeta h_s + (h_b)_s) \\
h\Psi_s &= h\tilde{\Psi}_s + h_s\tilde{\Psi} - h_s\tilde{\Psi} - \tilde{\Psi}_\zeta(\zeta h + h_b)_s \\
h\Psi_s &= (h\tilde{\Psi})_s - (h_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s) \\
h\Psi_s &= (h\tilde{\Psi})_s - \left(((\zeta h + h_b)_\zeta)_s \tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s \right) \\
h\Psi_s &= (h\tilde{\Psi})_s - \left(((\zeta h + h_b)_s)_\zeta \tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s \right) \\
h\Psi_s &= (h\tilde{\Psi})_s - ((\zeta h + h_b)_s)_\zeta \tilde{\Psi}
\end{aligned}$$

Now we can use these differential transformations to map the continuity equation or mass balance equation onto

the normalized space.

$$\begin{aligned}
u_x + v_y + w_z &= 0 \\
h(u_x + v_y + w_z) &= 0 \\
hu_x + hv_y + hw_z &= 0 \\
(h\tilde{u})_x - ((\zeta h + h_b)_x \tilde{u})_\zeta + (h\tilde{v})_y - ((\zeta h + h_b)_y \tilde{v})_\zeta + (\tilde{w})_\zeta &= 0 \\
(h\tilde{u})_x + (h\tilde{v})_y + \left(\tilde{w} - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \right)_\zeta &= 0 \\
-(h\tilde{u})_x - (h\tilde{v})_y + \left((\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v} \right)_\zeta &= (\tilde{w})_\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x d\zeta - \int_0^{\zeta'} (h\tilde{v})_y d\zeta + \int_0^{\zeta'} \left((\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v} \right)_\zeta d\zeta &= \int_0^{\zeta'} (\tilde{w})_\zeta d\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x d\zeta - \int_0^{\zeta'} (h\tilde{v})_y d\zeta + (\zeta' h + h_b)_x \tilde{u}(t, x, y, \zeta') + (\zeta' h + h_b)_y \tilde{v}(t, x, y, \zeta') \\
-(h_b)_x \tilde{u}(t, x, y, 0) - (h_b)_y \tilde{v}(t, x, y, 0) &= \tilde{w}(t, x, y, \zeta') - \tilde{w}(t, x, y, 0) \\
-\int_0^{\zeta'} (h\tilde{u})_x d\zeta - \int_0^{\zeta'} (h\tilde{v})_y d\zeta + (\zeta' h + h_b)_x \tilde{u}(t, x, y, \zeta') + (\zeta' h + h_b)_y \tilde{v}(t, x, y, \zeta') &= \tilde{w}(t, x, y, \zeta')
\end{aligned}$$

$$\begin{aligned}
\tilde{w}(t, x, y, \zeta') &= - \left(h \int_0^{\zeta'} \tilde{u} d\zeta \right)_x - \left(h \int_0^{\zeta'} \tilde{v} d\zeta \right)_y \\
&+ (\zeta' h + h_b)_x \tilde{u}(t, x, y, \zeta') + (\zeta' h + h_b)_y \tilde{v}(t, x, y, \zeta')
\end{aligned}$$

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

Original Kinematic Boundary Conditions

$$\begin{aligned}
(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s &= w(t, x, y, h_s) \\
(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b &= w(t, x, y, h_b) \\
(h_s)_t + u(t, x, y, h_s)(h_s)_x + v(t, x, y, h_s)(h_s)_y &= w(t, x, y, h_s) \\
(h_b)_t + u(t, x, y, h_b)(h_b)_x + v(t, x, y, h_b)(h_b)_y &= w(t, x, y, h_b)
\end{aligned}$$

Mapped Kinematic Boundary Conditions

$$\begin{aligned}
(h_s)_t + [\tilde{u}(t, x, y, 1), \tilde{v}(t, x, y, 1)]^T \cdot \nabla h_s &= \tilde{w}(t, x, y, 1) \\
(h_b)_t + [\tilde{u}(t, x, y, 0), \tilde{v}(t, x, y, 0)]^T \cdot \nabla h_b &= \tilde{w}(t, x, y, 0) \\
(h_s)_t + \tilde{u}(t, x, y, 1)(h_s)_x + \tilde{v}(t, x, y, 1)(h_s)_y &= \tilde{w}(t, x, y, 1) \\
(h_b)_t + \tilde{u}(t, x, y, 0)(h_b)_x + \tilde{v}(t, x, y, 0)(h_b)_y &= \tilde{w}(t, x, y, 0)
\end{aligned}$$

$$\begin{aligned}
u_t + (u^2)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho} p_x + \frac{1}{\rho} h(\sigma_{xx}) \\
hu_t + h(u^2)_x + h(uv)_y + h(uw)_z + \frac{1}{\rho} hp_x &= +\frac{1}{\rho} h(\sigma_{xx}) \\
(h\tilde{u})_t - ((\zeta h + h_b)_t \tilde{u})_\zeta + (h\tilde{u}^2)_x - ((\zeta h + h_b)_x \tilde{u}^2)_\zeta + (h\tilde{u}\tilde{v})_y - ((\zeta h + h_b)_y \tilde{u}\tilde{v})_\zeta + (\tilde{u}\tilde{w})_\zeta + \frac{1}{\rho} (h\tilde{p})_x - \frac{1}{\rho} ((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho} (\tilde{\sigma}_{xx}) \\
(h\tilde{u})_t + (h\tilde{u}^2)_x + (h\tilde{u}\tilde{v})_y + \left(\tilde{u} \left(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \right) \right)_\zeta + \frac{1}{\rho} (h\tilde{p})_x - \frac{1}{\rho} ((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho} (\tilde{\sigma}_{xx})
\end{aligned}$$

$$\tilde{p}(t, x, y, \zeta) = h(t, x, y)(1 - \zeta)\rho g e_z$$

$$\begin{aligned} \frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta &= \frac{1}{\rho}(h^2(1 - \zeta)\rho g e_z)_x - \frac{1}{\rho}((\zeta h + h_b)_x h(1 - \zeta)\rho g e_z)_\zeta \\ &= (h^2(1 - \zeta)g e_z)_x - ((\zeta h + h_b)_x h(1 - \zeta)g e_z)_\zeta \\ &= 2h h_x(1 - \zeta)g e_z - ((\zeta h_x + (h_b)_x)h(1 - \zeta)g e_z)_\zeta \\ &= 2h h_x(1 - \zeta)g e_z - h g e_z((\zeta h_x + (h_b)_x)(1 - \zeta))_\zeta \\ &= 2h h_x(1 - \zeta)g e_z - h g e_z(h_x(1 - \zeta) - \zeta h_x - (h_b)_x) \\ &= h h_x g e_z + (h_b)_x h g e_z \\ &= \left(\frac{1}{2}h^2 g e_z\right)_x + (h_b)_x h g e_z \end{aligned}$$

$$\begin{aligned} &(h\tilde{u})_t + (h\tilde{u}^2)_x + (h\tilde{u}\tilde{v})_y \\ &+ \left(\tilde{u}\left(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\right)\right)_\zeta \\ &+ \left(\frac{1}{2}h^2 g e_z\right)_x + (h_b)_x h g e_z = \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + g h e_x \\ &(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}h^2 g e_z\right)_x + (h\tilde{u}\tilde{v})_y \\ &+ \left(\tilde{u}\left(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\right)\right)_\zeta \\ &= \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + g h(e_x - (h_b)_x e_z) \end{aligned}$$

$$\begin{aligned}
& \tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y + (\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v} \\
&\quad - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v} \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y - (\zeta h + h_b)_t \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y - \zeta h_t \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y + \zeta \left((hu_m)_x + (hv_m)_y \right) \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y + \int_0^\zeta d\zeta' \left((hu_m)_x + (hv_m)_y \right) \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' \right)_y + \left(h \int_0^\zeta u_m d\zeta' \right)_x + \left(h \int_0^\zeta v_m d\zeta' \right)_y \\
&= - \left(h \int_0^\zeta \tilde{u} d\zeta' + h \int_0^\zeta u_m d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} d\zeta' + h \int_0^\zeta v_m d\zeta' \right)_y \\
&= - \left(h \int_0^\zeta \tilde{u} - u_m d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v} - v_m d\zeta' \right)_y \\
&= - \left(h \int_0^\zeta \tilde{u}_d d\zeta' \right)_x - \left(h \int_0^\zeta \tilde{v}_d d\zeta' \right)_y \\
&\tilde{u}_d = \tilde{u} - u_m \quad \tilde{v}_d = \tilde{v} - v_m
\end{aligned}$$