A Discontinuous Galerkin method applied to nonlinear parabolic equations

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Abstract. Semi-discrete and a family of discrete time locally conservative Discontinuous Galerkin procedures are formulated for approximations to nonlinear parabolic equations. For the continuous time approximations *a priori* $L^\infty(L^2)$ and $L^2(H^1)$ estimates are derived and similarly, $l^\infty(L^2)$ and $l^2(H^1)$ for the discrete time schemes. Spatial rates in H^1 and time truncation errors in L^2 are optimal.

1 Introduction

Over the last two decades there has been a collection of papers devoted to the use of approximation spaces with weak continuity for finite element approximations to elliptic and parabolic problems. The motivation for developing these methods was the flexibility afforded by local approximation spaces. These approaches allow meshes which are more general in their construction and degree of nonuniformity both in time and space than is permitted by the more conventional finite element methods. In general numerical methods defined for discontinuous spaces have less numerical diffusion/dispersion and provide more accurate local approximations for problems with rough solutions. Another advantage that has recently become apparent is the application of domain decomposition algorithms for the discrete solution.

Discontinuous Galerkin methods using interior penalties for elliptic and parabolic equations were first introduced by Douglas, Dupont and Wheeler [11],[4] and Arnold [1] in the seventies. These approaches generalize a method by Nitsche [6] for treating Dirichlet boundary condition by the introduction of penalty terms on the boundary of the domain. Applications of these methods to flow in porous media were presented by Douglas, Wheeler, Darlow, Kendall and Ewing in [5],[2]. These methods frequently referred to as interior penalty Galerkin schemes are not locally mass conservative.

A new type of discontinuous Galerkin method for diffusion problems was introduced and analyzed by Oden, Babuška and Baumann [7]. It was shown that the discontinuous Galerkin method was elementwise conservative. Also, *a priori* error estimates were proven for one-dimensional problems and for polynomials of at least order three. Numerical experiments in higher dimension showed the robustness of the method. The authors [8],[9] have derived *a priori* and *a posteriori* error estimates in higher dimensions. In this paper, a discontinuous Galerkin formulation for nonlinear parabolic equations is introduced and analyzed.

This paper consists of three additional sections. In §2 and §3, notation and problem definition and formulation of the discontinuous Galerkin method are described. In §4 and §5, the proofs of the error estimates in the continuous and discrete time setting are respectively given. Conclusions are described in the last section.

2 Model problem

Consider the nonlinear parabolic partial differential equation

$$u_t - \nabla \cdot a(x,u)\nabla u = f(x,u), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

with the boundary condition

$$a(x,u)\nabla u \cdot \nu = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{2}$$

and the initial condition

$$u(x,0) = \psi(x), \quad x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^d$. Since many of the proofs are highly technical, we shall restrict our attention to $d = 1, 2$. We make several assumptions:

- For $(x,p) \in \Omega \times R$, $\exists\, \gamma$ and γ^* s.t. $0 < \gamma \leq a(x,p) \leq \gamma^*$.
- a, f are uniformly Lipschitz continuous with respect to their second variable.
- $u \in C^2(\Omega \times [0,T])$ is a unique solution to (1), (2), and (3), and $u \in L^2([0,T], H^s(\Omega))$, $u_t \in L^2([0,T], H^{s-1}(\Omega))$ for $s \geq 2$.
- ∇u is bounded in $L^\infty(\Omega \times (0,T])$.

(Here for X normed space and n positive integer, $L^n([0,T],X) = \{f : \int_0^T \|f\|_X^n(t)dt < \infty\}$.)

3 Definitions and the Discontinuous Galerkin procedure

Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a subdivision of Ω, where E_j is a triangle or a quadrilateral. Let $h_j = \mathrm{diam}(E_j)$ and $h = \max\{h_j,\ j = 1 \ldots N_h\}$. We denote the edges of the elements by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_{h+1}}, \ldots, e_{M_h}\}$ where $e_k \subset \Omega, 1 \leq k \leq P_h$,and $e_k \subset \partial\Omega,\ P_h + 1 \leq k \leq M_h$. With each edge e_k , we associate a unit normal vector ν_k. For $k > P_h$, ν_k is taken to be the unit outward vector normal to $\partial\Omega$.
For $s \geq 0$, let

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1 \ldots N_h\}.$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h),\ s > \frac{1}{2}$. Let $1 \leq k \leq P_h$. For $e_k = \partial E_i \cap \partial E_j$ with ν_k exterior to E_i, set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

The L^2 inner product is denoted by $(\cdot,\cdot)$. The usual Sobolev norm on $E \subset I\!R^d$ and for m positive integer, is denoted by $\|\cdot\|_{m,E}$. We define the following broken norms:

$$|||\phi|||_m^2 = \sum_{j=1}^{N_h} \|\phi\|_{m,E_j}^2,$$

$$|||\phi|||^2_{L^2((\alpha,\beta);H^m)} = \int_\alpha^\beta |||\phi(.,t)|||_m^2 dt, \quad |||\phi|||_{L^\infty((\alpha,\beta);H^m)} = \sup_{t\in(\alpha,\beta)} |||\phi(.,t)|||_m.$$

Let r be a positive integer. The finite element subspace is taken to be

$$\mathcal{D}_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j),$$

where $P_r(E_j)$ denotes the set of polynomials of (total) degree less than or equal to r on E_j, even if E_j is a quadrilateral. We introduce the interior penalty term

$$J_0^\sigma(\phi,\psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [\phi][\psi],$$

where $|e_k|$ denotes the length of e_k and σ_k is a real nonnegative number associated to the interior edge e_k.

A proof of the following lemma can be found in [8].

Lemma 1. *Let $u \in H^s(\Omega)$, for $s \geq 2$ and let $r \geq 2$. Let $\bar{a}$ be a positive constant. There is $\hat{u} \in \mathcal{D}_r(\mathcal{E}_h)$ interpolant of p satisfying*

$$\int_{e_k} \{\bar{a}\nabla(\hat{u}-u)\cdot\nu_k\} = 0, \quad \forall k = 1,\ldots,P_h \tag{4}$$

$$\|\hat{u}-u\|_{\infty,E_j} \leq C\frac{h^\mu}{r^{s-1}}\|u\|_{s,E_j}, \quad \forall E_j, \tag{5}$$

$$\|\nabla(\hat{u}-u)\|_{0,E_j} \leq C\frac{h^{\mu-1}}{r^{s-1}}\|u\|_{s,E_j}, \quad \forall E_j, \tag{6}$$

$$\|\nabla^2(\hat{u}-u)\|_{0,E_j} \leq C\frac{h^{\mu-2}}{r^{s-2}}\|u\|_{s,E_j}, \quad \forall E_j, \tag{7}$$

$$\|\hat{u}-u\|_{0,E_j} \leq C\frac{h^\mu}{r^{s-1}}\|u\|_{s,E_j}, \quad \forall E_j, \tag{8}$$

where $\mu = \min(r+1,s)$. Morevoer, for $e_k = \partial E^1 \cap \partial E^2$,

$$\|\nabla\hat{u}\|_{\infty,e_k} \leq C\|\nabla u\|_{\infty,E^1\cup E^2} \tag{9}$$

The Discontinuous Galerkin approximation $U(\cdot,t) \in \mathcal{D}_r(\mathcal{E}_h)$ to the solution u of (1), (2), and (3) is defined by

$$
\begin{aligned}
(\frac{\partial U}{\partial t}, v) + \sum_{j=1}^{N_h} \int_{E_j} a(U)\nabla U \nabla v - \sum_{k=1}^{P_h} \int_{e_k} \{a(U)\nabla U \cdot \nu_k\}[v] \\
+ \sum_{k=1}^{P_h} \int_{e_k} \{a(U)\nabla v \cdot \nu_k\}[U] + J_0^\sigma(U,v) = \int_\Omega f(U)v,\ t>0,\ v \in \mathcal{D}_r(\mathcal{E}_h), \quad (10) \\
U(\cdot,0) = \psi, \quad (11)
\end{aligned}
$$

where we have assumed for simplicity that $\psi \in \mathcal{D}_r(\mathcal{E}_h)$. We note that if $\{v_i\}_{i=1}^M$ is a basis of $\mathcal{D}_r(\mathcal{E}_h)$ and if we write

$$
U(x,t) = \sum_{i=1}^{M} \xi_i(t) v_i(x),
$$

then (10) and (11) reduces to an initial value problem for the system of nonlinear ordinary differential equations

$$
\begin{aligned}
G\xi^{'}(t) &= -B(\xi)\xi + F(\xi) \\
\xi(0) &= b.
\end{aligned}
$$

The matrix G is block diagonal symmetric positive definite. Since a and f are Lipschitz continuous, it follows from the theory of ordinary differential equations that $\xi(t)$ exists and is unique for $t > 0$.

4 Continuous in time *a priori* error estimate

In this section, we demonstrate optimal $L^2(H^1)$ rates of convergence for continuous in time Discontinuous Galerkin approximations of at least quadratic order.

Theorem 2. *Let $s \geq 2$. There exists a constant C^* independent of h and r such that,*

$$
\begin{aligned}
|\!|\!| U-u |\!|\!|^2_{L^\infty((0,T);L^2)} + |\!|\!| U-u |\!|\!|^2_{L^2((0,T);H^1)} \leq C^* \frac{h^{2\mu-2}}{r^{2s-4}} |\!|\!| u |\!|\!|^2_{L^2((0,T),H^s)} + \\
C^* \frac{h^{2\mu-2}}{r^{2s-4}} |\!|\!| u_t |\!|\!|^2_{L^2((0,T),H^{s-1})},
\end{aligned}
$$

where $\mu = \min(r+1,s)$, $\quad r \geq 2$, and $\sigma_k \geq 0$ if $a = a(x)$ and $\sigma_k > 0$ if $a = a(x,u)$.

Proof. It is clear that if u is a solution of (1), (2) and (3), then u satisfies the formulation:

$$(\frac{\partial u}{\partial t}, v) + \int_{\Omega} a(u)\nabla u \nabla v - \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla u \cdot \nu_k\}[v]$$
$$+ \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla v \cdot \nu_k\}[u] + J_0^{\sigma}(u, v) = \int_{\Omega} f(u)v, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

We obtain:

$$(\frac{\partial \hat{u}}{\partial t}, v) + \int_{\Omega} a(U)\nabla \hat{u} \nabla v - \sum_{k=1}^{P_h} \int_{e_k} \{a(U)\nabla \hat{u} \cdot \nu_k\}[v]$$
$$+ \sum_{k=1}^{P_h} \int_{e_k} \{a(U)\nabla v \cdot \nu_k\}[\hat{u}] + J_0^{\sigma}(\hat{u}, v) = \int_{\Omega} \frac{\partial(\hat{u} - u)}{\partial t} v + \int_{\Omega} f(u)v$$
$$+ \int_{\Omega} a(u)\nabla(\hat{u} - u)\nabla v - \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla(\hat{u} - u) \cdot \nu_k\}[v]$$
$$+ \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla v \cdot \nu_k\}[\hat{u} - u] + J_0^{\sigma}(\hat{u} - u, v) + \int_{\Omega} (a(U) - a(u))\nabla \hat{u} \nabla v$$
$$- \sum_{k=1}^{P_h} \int_{e_k} \{(a(U) - a(u))\nabla \hat{u} \cdot \nu_k\}[v] + \sum_{k=1}^{P_h} \int_{e_k} \{(a(U) - a(u))\nabla v \cdot \nu_k\}[\hat{u}]. \quad (12)$$

Subtract (12) from (10), denote $U - \hat{u} = \xi$, $\hat{u} - u = \chi$, and choose $v = \xi$:

$$(\frac{\partial \xi}{\partial t}, \xi) + \sum_{j=1}^{N_h} \int_{E_j} a(U)\nabla \xi \nabla \xi + J_0^{\sigma}(\xi, \xi) = -\int_{\Omega} \frac{\partial \chi}{\partial t} \xi + \int_{\Omega} (f(U) - f(u))\xi$$
$$- \int_{\Omega} a(u)\nabla \chi \nabla \xi + \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla \chi \cdot \nu_k\}[\xi] - \sum_{k=1}^{P_h} \int_{e_k} \{a(u)\nabla \xi \cdot \nu_k\}[\chi]$$
$$+ J_0^{\sigma}(\chi, \xi) + T_1 - T_2 + T_3, \quad (13)$$

where

$$T_1 = \sum_{j=1}^{N_h} \int_{E_j} (a(u) - a(U))\nabla \hat{u} \nabla \xi,$$
$$T_2 = \sum_{k=1}^{P_h} \int_{e_k} \{(a(u) - a(U))\nabla \hat{u} \cdot \nu_k\}[\xi],$$
$$T_3 = \sum_{k=1}^{P_h} \int_{e_k} \{(a(u) - a(U))\nabla \xi \cdot \nu_k\}[\hat{u}].$$

We now derive bounds for all the terms on the right-hand side of (13). The constants C_i are generic constants that vary but do not depend on h or r.

$$
\begin{aligned}
|\int_\Omega \frac{\partial \chi}{\partial t}\xi| &\leq C_1(|||\chi_t|||_0^2 + |||\xi|||_0^2),\\
|\int_\Omega (f(U)-f(u))\xi| &\leq C_2(|||\chi|||_0^2 + |||\xi|||_0^2),\\
|\sum_{j=1}^{N_h}\int_{E_j} a(u)\nabla\chi\nabla\xi| &\leq C_3|||\nabla\chi|||_0|||\nabla\xi|||_0,\\
&\leq \frac{C_3}{\epsilon_1}|||\nabla\chi|||_0^2 + \epsilon_1|||\nabla\xi|||_0^2.
\end{aligned}
$$

To bound the terms involving integrals on the interior edges, we first look at the integral on a given edge e_k, and we assume that $e_k = \partial E^1 \cap \partial E^2$. We denote $E^{12} \equiv E^1 \cup E^2$. Define $\bar{a}$ piecewise constant on each element E_j such that $\bar{a}|_{E_j} = \frac{1}{|E_j|}\int_{E_j} a(u)$.

$$
|\int_{e_k}\{a(u)\nabla\chi\cdot\nu_k\}[\xi]| \leq |\int_{e_k}\{\bar{a}\nabla\chi\cdot\nu_k\}[\xi]| + |\int_{e_k}\{(a(u)-\bar{a})\nabla\chi\cdot\nu_k\}[\xi]|.
$$

Define the constant c_k associated to each interior edge as follows

$$
c_k = \frac{1}{|e_k|}\int_{e_k}[\xi].
$$

By Lemma 1, we see that

$$
\begin{aligned}
|\int_{e_k}\{\bar{a}\nabla\chi\cdot\nu_k\}[\xi]| &= |\int_{e_k}\{\bar{a}\nabla\chi\cdot\nu_k\}([\xi]-c_k)|,\\
&\leq ||\{\bar{a}\nabla\chi\cdot\nu_k\}||_{0,e_k}||[\xi]-c_k||_{0,e_k},\\
&\leq C_4||\nabla\xi||_{0,E^{12}}(||\nabla\chi||_{0,E^{12}} + h||\nabla^2\chi||_{0,E^{12}}).
\end{aligned}
$$

Now,

$$
|\int_{e_k}\{(a(u)-\bar{a})\nabla\chi\cdot\nu_k\}[\xi]| \leq C_4||a(u)-\bar{a}||_{\infty,E^{12}}||\{\nabla\chi\cdot\nu_k\}||_{0,e_k}||[\xi]||_{0,e_k}.
$$

But,

$$
\begin{aligned}
||a(u)-\bar{a}||_{\infty,E^{12}} &\leq C_4 h,\\
||\{\nabla\chi\cdot\nu_k\}||_{0,e_k} &\leq C_4 h^{-\frac{1}{2}}(||\nabla\chi||_{0,E^{12}} + h||\nabla^2\chi||_{0,E^{12}}),\\
||[\xi]||_{0,e_k} &\leq C_4 h^{-\frac{1}{2}}||\xi||_{0,E^{12}}.
\end{aligned}
$$

So,

$$
|\int_{e_k}\{(a(u)-\bar{a})\nabla\chi\cdot\nu_k\}[\xi]| \leq C_4||\xi||_{0,E^{12}}(||\nabla\chi||_{0,E^{12}} + h||\nabla^2\chi||_{0,E^{12}}).
$$

Summing on k, we have

$$|\sum_{k=1}^{P_h}\int_{e_k}\{a(u)\nabla\chi\cdot\nu_k\}[\xi]| \le \epsilon_2 |||\xi|||_0^2 + \epsilon_2|||\nabla\xi|||_0^2 + C_4(|||\nabla\chi|||_0 + h|||\nabla^2\chi|||_0)^2.$$

Similarly, we note that

$$\begin{aligned}|\int_{e_k}\{a(u)\nabla\xi\cdot\nu_k\}[\chi]| &\le C_5||\{a(u)\nabla\xi\cdot\nu_k\}||_{0,e_k}||[\chi]||_{0,e_k},\\ &\le C_5||\nabla\xi||_{0,E^{12}}(h^{-1}||\chi||_{0,E^{12}} + ||\nabla\chi||_{0,E^{12}}).\end{aligned}$$

Thus, by summing on k we obtain

$$|\sum_{k=1}^{P_h}\int_{e_k}\{a(u)\nabla\xi\cdot\nu_k\}[\chi]| \le \epsilon_3|||\nabla\xi|||_0^2 + \frac{C_5}{\epsilon_3}(h^{-1}|||\chi|||_0 + |||\nabla\chi|||_0)^2.$$

For the linear case $a = a(x)$, the following four terms (penalty term and T_1, T_2, T_3) do not appear in (13). For the nonlinear case, we observe that

$$\begin{aligned}|\sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}\int_{e_k}[\chi][\xi]| &\le \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|}||[\chi]||_{0,e_k}||[\xi]||_{0,e_k},\\ &\le \epsilon_4 J_0^\sigma(\xi,\xi) + \frac{C_6}{\epsilon_4}\sum_{k=1}^{P_h}\frac{1}{|e_k|}||[\chi]||_{0,e_k}^2,\\ &\le \epsilon_4 J_0^\sigma(\xi,\xi) + \frac{C_6}{\epsilon_4}\sum_{k=1}^{P_h}\frac{1}{|e_k|}(h^{-1}||\chi||_{0,E^{12}}^2 + h||\nabla\chi||_{0,E^{12}}^2),\\ &\le \epsilon_4 J_0^\sigma(\xi,\xi) + \frac{C_6}{\epsilon_4}(h^{-2}|||\chi|||_0^2 + |||\nabla\chi|||_0^2).\end{aligned}$$

In addition, by estimate (9) in Lemma 1,we see that

$$\begin{aligned}|T_1| &\le C_7\sum_{j=1}^{N_h}\int_{E_j}|u-U||\nabla\hat{u}\cdot\nabla\xi|,\\ &\le C_7||\nabla\hat{u}||_\infty|||u-U|||_0|||\nabla\xi|||_0,\\ &\le \frac{C_7}{\epsilon_5}(|||\chi|||_0^2 + |||\xi|||_0^2) + \epsilon_5|||\nabla\xi|||_0^2.\end{aligned}$$

Similarly, we observe that

$$\begin{aligned}|\int_{e_k}\{(a(u)-a(U))\nabla\hat{u}\cdot\nu_k\}[\xi]| &\le C_8(||\nabla u||_{\infty,E^{12}})||\{u-U\}||_{0,e_k}||[\xi]||_{0,e_k},\\ &\le \epsilon_6\frac{\sigma_k}{|e_k|}||[\xi]||_{0,e_k}^2 + C_8|e_k|h^{-1}||\xi||_{0,E^{12}}^2\\ &\quad + C_8|e_k|(h^{-1}||\chi||_{0,E^{12}}^2 + h||\nabla\chi||_{0,E^{12}}^2).\end{aligned}$$

Summing on $k,\ k=1,\dots,P_h$, we have

$$|T_2| \le \epsilon_6 J_0^\sigma(\xi,\xi) + C_8(\|\chi\|_0^2 + h^2\|\nabla\chi\|_0^2 + \|\xi\|_0^2).$$

Similarly, we have by (8) in Lemma 1

$$|\int_{e_k}\{(a(u)-a(U))\nabla\xi\cdot\nu_k\}[\chi]| \le \frac{C_9}{\epsilon_7}||\nabla u||_{\infty,E^{12}}(||\chi||_{0,E^{12}} + h||\chi||_{0,E^{12}})$$
$$+\epsilon_7||\nabla\xi||_{0,E^{12}} + \frac{C_9}{\epsilon_7}||\nabla u||_{\infty,E^{12}}||\xi||_{0,E^{12}}.$$

Summing on $k,\ k=1,\dots,P_h$, we have

$$|T_3| \le \epsilon_7\|\nabla\xi\|_0^2 + C_9(\|\chi\|_0^2 + h^2\|\nabla\chi\|_0^2 + \|\xi\|_0^2).$$

Combining the above bounds for the right-hand side and choosing the ϵ_i small enough, we have

$$\frac{1}{2}\frac{\partial\|\xi\|_0^2}{\partial t} + \frac{\gamma}{2}\|\nabla\xi\|_0^2 + \frac{1}{2}J_0^\sigma(\xi,\xi) \le (\hat{C}_1 + \frac{\hat{C}_2}{h^2})\|\chi\|_0^2 + \hat{C}_3\|\nabla\chi\|_0^2$$
$$+\hat{C}_4h^2\|\nabla^2\chi\|_0^2 + \hat{C}_5\|\xi\|_0^2 + \hat{C}_6\|\chi_t\|_0.$$

where $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4$ and $\hat{C}_5$ are constants independent of h and r.

Now, we integrate with respect to time between 0 and τ and obtain:

$$\|\xi\|_0^2(\tau) + \frac{\gamma}{2}\int_0^\tau \|\nabla\xi\|_0^2(t)dt + \frac{1}{2}\int_0^\tau J_0^\sigma(\xi,\xi) \le \|\xi\|_0^2(0) + \hat{C}_5\int_0^\tau \|\xi\|_0^2$$
$$+(\hat{C}_1 + \frac{\hat{C}_2}{h^2})\int_0^\tau \|\chi\|_0^2$$
$$+\hat{C}_3\int_0^\tau \|\nabla\chi\|_0^2$$
$$+\hat{C}_4h^2\int_0^\tau \|\nabla^2\chi\|_0^2$$
$$+\hat{C}_6\int_0^\tau \|\chi_t\|_0^2.$$

Using Gronwall's inequality and the approximation results, we obtain

$$\|\xi\|_0^2(\tau) + \frac{\gamma}{2}\int_0^\tau \|\nabla\xi\|_0^2(t)dt + \frac{1}{2}\int_0^\tau J_0^\sigma(\xi,\xi) \le C\frac{h^{2\mu-2}}{r^{2s-4}}\|u\|_{L^2((0,\tau),H^s)}^2 +$$
$$+C\frac{h^{2\mu-2}}{r^{2s-4}}\|u_t\|_{L^2((0,\tau),H^{s-1})}^2.$$

The result follows by triangle inequality. □

5 Discrete in time Discontinuous Galerkin procedures

Let $\Delta t = T/N$ where N is a positive integer and let $t_j = j\Delta t$. We use the following notation:

$$
\begin{aligned}
g_j &= g(x,t_j), \quad 0 \le j \le N,\\
g_{j,\theta} &= \frac{1}{2}(1+\theta)g_{j+1} + \frac{1}{2}(1-\theta)g_j, \quad 0 \le j \le N-1,
\end{aligned}
$$

where $\theta \in [0,1]$. Define the norms:

$$
\|g\|_{l^\infty(L^2)} = \max_{j=0,\dots,N} \|g_j\|_0, \quad \|g\|_{l^2(H^1)} = (\sum_{j=0}^{N-1} \|\nabla g_{j,\theta}\|_0^2)^{\frac{1}{2}}.
$$

Consider the following discrete Discontinuous Galerkin procedure:
Let $\{U_j\}_{j=0}^N$ be a sequence in $\mathcal{D}_r(\mathcal{E}_h)$ that satisfies:

$$
\begin{aligned}
&\int_\Omega \frac{U_{j+1}-U_j}{\Delta t} v + \int_\Omega a(U_{j,\theta})\nabla U_{j,\theta}\nabla v\\
&-\sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla U_{j,\theta}\cdot\nu_k\}[v] + \sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla v\cdot\nu_k\}[U_{j,\theta}]\\
&= \int_\Omega f(x,U_{j,\theta})v + J_0^\sigma(U_{j,\theta},v), \quad t>0, \quad v\in\mathcal{D}_r(\mathcal{E}_h),
\end{aligned}
\tag{14}
$$

$$
U_0 = \psi, \tag{15}
$$

where $\theta \in [0,1]$. If $\theta = 0$, (14) yields the Crank-Nicolson Discontinuous Galerkin approximation; for $\theta = 1$, (14) is a backward difference Discontinuous Galerkin approximation.

We remark that (14) and (15) have solutions (possibly non unique) if Δt is sufficiently small [3] [10].

We have the following result on the interpolant $\hat{u}$ of u in Lemma 1. In particular, $\frac{\partial \hat{u}}{\partial t}$ is the interpolant of $\frac{\partial u}{\partial t}$.

Lemma 3.

$$
\frac{\hat{u}_{j+1}-\hat{u}_j}{\Delta t} = \hat{u}_t(x,t_{j,\theta}) + \Delta t\rho_{j,\theta}, \quad \forall x\in\Omega, \tag{16}
$$

where

$$
\|\rho_{j,\theta}\|_0 \le C_1\|u_{tt}\|_{L^\infty((t_j,t_{j+1});H^1)}
$$

In the particular case $\theta = 0$, we also have

$$
\|\rho_{j,0}\|_0 \le \Delta t C_2\|u_{ttt}\|_{L^\infty((t_j,t_{j+1});H^1)}
$$

C_1 and C_2 are two constants independent of $u, \hat{u}$ and r.

Proof. The Taylor expansions around $t_{j,\theta}$ yield

$$\hat{u}_{j+1} = \hat{u}_{j,\theta} + \frac{1-\theta}{2}\Delta t \hat{u}_t(t_{j,\theta}) + \frac{1}{2}(\frac{1-\theta}{2})^2 \Delta t^2 \hat{u}_{tt}(t_{j,\theta}) + \frac{1}{6}(\frac{1-\theta}{2})^3 \Delta t^3 \hat{u}_{ttt}(t^*), \tag{17}$$

$$\hat{u}_{j} = \hat{u}_{j,\theta} - \frac{1+\theta}{2}\Delta t \hat{u}_t(t_{j,\theta}) + \frac{1}{2}(\frac{1+\theta}{2})^2 \Delta t^2 \hat{u}_{tt}(t_{j,\theta}) - \frac{1}{6}(\frac{1+\theta}{2})^3 \Delta t^3 \hat{u}_{ttt}(t^{**}). \tag{18}$$

Subtracting (18) from (17)

$$\hat{u}_{j+1} - \hat{u}_j = \Delta t \hat{u}_t(t_{j,\theta}) + \Delta t \rho_{j,\theta},$$

where

$$\rho_{j,\theta} = \frac{1}{2}((\frac{1-\theta}{2})^2 - (\frac{1+\theta}{2})^2)\hat{u}_{tt}(t_{j,\theta}) + \frac{1}{6}(\frac{1-\theta}{2})^3 \Delta t \hat{u}_{ttt}(t^*) + \frac{1}{6}(\frac{1+\theta}{2})^3 \Delta t \hat{u}_{ttt}(t^{**}).$$

Clearly, for $\theta \in (0,1]$, we have

$$|||\rho_{j,\theta}|||_0 \leq C(\theta) |||\hat{u}_{tt}|||_{L^\infty((t_j,t_{j+1});L^2)}.$$

Given t, we have

$$\begin{aligned} |||\hat{u}_{tt}|||_0 &\leq |||\hat{u}_{tt} - u_{tt}|||_0 + |||u_{tt}|||_0 \\ &\leq Ch|||u_{tt}|||_1 + |||u_{tt}|||_0. \end{aligned}$$

Thus, for $\theta \in (0,1]$, we have

$$|||\rho_{j,\theta}|||_0 \leq C(h+1)|||u_{tt}|||_{L^\infty((t_j,t_{j+1});H^1)}.$$

For $\theta = 0$, we have in a similar fashion

$$\begin{aligned} |||\rho_{j,\theta}|||_0 &\leq C\Delta t |||\hat{u}_{ttt}|||_{L^\infty((t_j,t_{j+1});H^1)} \\ &\leq C\Delta t(h+1)|||u_{ttt}|||_{L^\infty((t_j,t_{j+1});H^1)}. \end{aligned}$$ □

Theorem 4. *Assume*

- $u_{tt} \in L_\infty([0,T]; H^1(\Omega))$;
- *For* $\theta = 0$, $u_{ttt} \in L_\infty([0,T]; H^1(\Omega))$;

Let $U_j, 0 \leq j \leq N$, *be defined by (14) and (15) for* $\theta \in (0,1]$. *Then, if* Δt *is sufficiently small, there exist* C^* *and* $\hat{C}$ *independent of* h *and* r *such that for*

$\mu = \min(r+1, s)$ *and* $\sigma_k \geq 0$ *if* $a = a(x)$ *and* $\sigma_k > 0$ *if* $a = a(x,u)$,

$$\|U - u\|_{l^\infty(L^2)}^2 + \Delta t\gamma \|U - u\|_{l^2(H^1)}^2 \leq C^* \frac{h^{2\mu-2}}{r^{2s-4}} \Delta t \sum_{j=0}^{N} \|u_j\|_{H^s}^2 + \hat{C}\Delta t^2 \sum_{j=0}^{N-1} \Delta t \|u_{tt}\|_{L^2((t_j,t_{j+1});H^1)}^2.$$

For $\theta = 0$, *we have*

$$\|U - u\|_{l^\infty(L^2)}^2 + \Delta t\gamma \|U - u\|_{l^2(H^1)}^2 \leq C^* \frac{h^{2\mu-2}}{r^{2s-4}} \Delta t \sum_{j=0}^{N} \|u_j\|_{H^s}^2 + \hat{C}\Delta t^4 \sum_{j=0}^{N-1} \Delta t \|u_{ttt}\|_{L^\infty((t_j,t_{j+1});H^1)}^2.$$

Proof. We see that for $t = t_{j,\theta}, 0 \leq j \leq N-1$ and $v \in \mathcal{D}_r(\mathcal{E}_h)$,

$$\begin{aligned}
&(\frac{\hat{u}_{j+1} - \hat{u}j}{\Delta t}, v) + \int_\Omega a(U_{j,\theta})\nabla \hat{u}_{j,\theta}\nabla v - \sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla \hat{u}_{j,\theta}\cdot \nu_k\}[v] \\
&+ \sum_{k=1}^{P_h}\int_{e_k}\{a(U_{j,\theta})\nabla v\cdot\nu_k\}[\hat{u}_{j,\theta}] = \int_\Omega f(x,u_{j,\theta})v + \int_\Omega \Delta t\rho_{j,\theta}v \\
&+ \int_\Omega a(u_{j,\theta})\nabla(\hat{u}_{j,\theta} - u_{j,\theta})\nabla v + J_0^\sigma(\hat{u}_{j,\theta} - u_{j,\theta}, v) \\
&- \sum_{k=1}^{P_h}\int_{e_k}\{a(u_{j,\theta})\nabla(\hat{u}_{j,\theta} - u_{j,\theta})\cdot\nu_k\}[v] \\
&+ \sum_{k=1}^{P_h}\int_{e_k}\{a(u_{j,\theta})\nabla v\cdot\nu_k\}[\hat{u}_{j,\theta} - u_{j,\theta}] \\
&+ \int_\Omega (a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\hat{u}_{j,\theta}\nabla v \\
&- \sum_{k=1}^{P_h}\int_{e_k}\{(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla\hat{u}_{j,\theta}\cdot\nu_k\}[v] \\
&+ \sum_{k=1}^{P_h}\int_{e_k}\{(a(U_{j,\theta}) - a(u_{j,\theta}))\nabla v\cdot\nu_k\}[\hat{u}_{j,\theta}].
\end{aligned} \tag{19}$$

Subtracting (19) from (14), denoting $\xi_{j,\theta} = U_{j,\theta} - \hat{u}_{j,\theta}, \chi_{j,\theta} = \hat{u}_{j,\theta} - u_{j,\theta}$ and choosing $v = \xi_{j,\theta}$:

$$\begin{aligned}
(\frac{\xi_{j+1} - \xi j}{\Delta t}, \xi_{j,\theta}) + \int_\Omega a(U_{j,\theta}) \nabla \xi_{j,\theta} \cdot \nabla \xi_{j,\theta} + J_0^\sigma(\xi_{j,\theta}, \xi_{j,\theta}) = \\
- \int_\Omega a(u_{j,\theta}) \nabla \chi_{j,\theta} \nabla \xi_{j,\theta} + \int_\Omega (f(x, U_{j,\theta}) - f(x, u_{j,\theta})) \xi_{j,\theta} \\
+ \sum_{k=1}^{P_h} \int_{e_k} \{a(u_{j,\theta}) \nabla \chi_{j,\theta} \cdot \nu_k\}[\xi_{j,\theta}] - \sum_{k=1}^{P_h} \int_{e_k} \{a(u_{j,\theta}) \nabla \xi_{j,\theta} \cdot \nu_k\}[\chi_{j,\theta}] \\
- \int_\Omega \Delta t \rho_{j,\theta} \xi_{j,\theta} + J_0^\sigma(\chi_{j,\theta}, \xi_{j,\theta}) + \int_\Omega (a(u_{j,\theta}) - a(U_{j,\theta})) \nabla \hat{u}_{j,\theta} \nabla \xi_{j,\theta} \\
- \sum_{k=1}^{P_h} \int_{e_k} \{(a(u_{j,\theta}) - a(U_{j,\theta})) \nabla \hat{u}_{j,\theta} \cdot \nu_k\}[\xi_{j,\theta}] \\
+ \sum_{k=1}^{P_h} \int_{e_k} \{(a(u_{j,\theta}) - a(U_{j,\theta})) \nabla \xi_{j,\theta} \cdot \nu_k\}[\hat{u}_{j,\theta}].
\end{aligned}$$

It is easy to show that we have:

$$\frac{1}{2\Delta t}(\|\xi_{j+1}\|_0^2 - \|\xi_j\|_0^2) \leq (\frac{\xi_{j+1} - \xi_j}{\Delta t}, \xi_{j,\theta}).$$

By using similar arguments as in the time-continuous case, we have

$$\begin{aligned}
\frac{1}{2\Delta t}(\|\xi_{j+1}\|_0^2 - \|\xi_j\|_0^2) + \frac{\gamma}{2}\|\nabla \xi_{j,\theta}\|_0^2 \leq (C_1 + \frac{C_2}{h^2})\|\chi_{j,\theta}\|_0^2 + C_3\|\nabla \chi_{j,\theta}\|_0^2 \\
+ C_4 h^2 \|\nabla^2 \chi_{j,\theta}\|_0^2 + C_5\|\xi_{j,\theta}\|_0^2 \\
+ C_6\|\chi_{t_{j,\theta}}\|_0^2 + C_7 \Delta t^2 \|\rho_{j,\theta}\|_0^2.
\end{aligned}$$

After some manipulation, we get

$$\begin{aligned}
\frac{1}{2\Delta t}(\|\xi_{j+1}\|_0^2 - \|\xi_j\|_0^2) + \frac{\gamma}{2}\|\nabla \xi_{j,\theta}\|_0^2 \leq (C_1 + \frac{C_2}{h^2})(\|\chi_{j+1}\|_0^2 + \|\chi\|_0^2) \\
+ C_3(\|\nabla \chi_{j+1}\|_0^2 + \|\nabla \chi_j\|_0^2) \\
+ C_4 h^2(\|\nabla^2 \chi_{j+1}\|_0^2 + \|\nabla^2 \chi_j\|_0^2) \\
+ C_5(\|\xi_{j+1}\|_0^2 + \|\xi_j\|_0^2) + C_7 \Delta t^2 \|\rho_{j,\theta}\|_0^2 \\
+ C_6(\|(\chi_t)_{j+1}\|_0^2 + \|(\chi_t)_j\|_0^2).
\end{aligned}$$

Multiplying by $2\Delta t$ and then summing for $j = 0, \ldots, N-1$, we obtain

$$\begin{aligned}
\|\xi_N\|_0^2 - \|\xi_0\|_0^2 + \Delta t \gamma \sum_{j=0}^{N-1} \|\nabla \xi_{j,\theta}\|_0^2 \leq C \Delta t \sum_{j=0}^{N} [(C_1 + \frac{C_2}{h^2})\|\chi_j\|_0^2 + C_6\|(\chi_t)_j\|_0^2 \\
+ C_3\|\nabla \chi_j\|_0^2 + C_4 h^2 \|\nabla^2 \chi_j\|_0^2] \\
+ C_5 \Delta t \sum_{j=0}^{N} \|\xi_j\|_0^2 + C_7 \Delta t^3 \sum_{j=0}^{N} \|\rho_{j,\theta}\|_0^2.
\end{aligned}$$

If Δt is sufficiently small we obtain, by Gronwall's lemma,

$$\|\xi_N\|_0^2 + \Delta t\gamma \sum_{j=0}^{N-1} \|\nabla \xi_{j,\theta}\|_0^2 \le C\|\xi_0\|_0^2 + C_7 \Delta t^3 \sum_{j=0}^{N-1} \|\rho_{j,\theta}\|_0^2$$
$$+ C\Delta t \sum_{j=0}^{N} [(C_1 + \frac{C_2}{h^2})\|\chi_j\|_0^2 + C_6 \|(\chi_t)_j\|_0^2$$
$$+ C_3 \|\nabla \chi_j\|_0^2 + C_4 h^2 \|\nabla^2 \chi_j\|_0^2].$$

Using approximation properties and the choice of the initial condition, we get for $\theta \in (0,1]$:

$$\|\xi_N\|_0^2 + \Delta t\gamma \sum_{j=0}^{N-1} \|\nabla \xi_{j,\theta}\|_0^2 \le C \frac{h^{2\mu-2}}{r^{2s-4}} \sum_{j=0}^{N} \Delta t (\|u_j\|_{H^s}^2 + \|(u_t)_j\|_{H^{s-1}}^2)$$
$$+ \hat{C} \Delta t^2 \sum_{j=0}^{N-1} \Delta t \|u_{tt}\|_{L^\infty((t_j,t_{j+1});H^1)}^2.$$

For $\theta = 0$, we have

$$\|\xi_N\|_0^2 + 2\Delta t\gamma \sum_{j=0}^{N-1} \|\nabla \xi_{j,\theta}\|_0^2 \le C \frac{h^{2\mu-2}}{r^{2s-4}} \sum_{j=0}^{N} \Delta t (\|u_j\|_{H^s}^2 + \|(u_t)_j\|_{H^{s-1}}^2)$$
$$+ \hat{C} \Delta t^4 \sum_{j=0}^{N-1} \Delta t \|u_{ttt}\|_{L^\infty((t_j,t_{j+1});H^1)}^2.$$

□

6 Conclusion

A continuous time and a family of discrete time Discontinuous Galerkin procedures have been formulated for nonlinear parabolic problems. Optimal rates of convergence in $L^2(H^1)$ for the continuous time method and $l_2(H^1)$ were derived. As far as the authors are aware, these are the first optimal H^1 estimates established for the DG method for parabolic problems. These estimates can also be extended to treat the addition of a constraint on the integral average of the jump on edges. Computational results are under development.

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