

# Derivation of Shallow Water Moment Equations in Spherical Coordinates

First I will define the transformation from cartesian  $\mathbf{x} = [x, y, z]$  to spherical coordinates,  $\mathbf{r} = [r, \theta, \phi]$

$$r = s_1(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\theta = s_2(x, y, z) = \arctan\left(\frac{y}{x}\right) \quad (2)$$

$$\phi = s_3(x, y, z) = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \quad (3)$$

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}(x, y, z) = [s_1(x, y, z), s_2(x, y, z), s_3(x, y, z)] \quad (4)$$

$$x = c_1(r, \theta, \phi) = r \cos(\theta) \sin(\phi) \quad (5)$$

$$y = c_2(r, \theta, \phi) = r \sin(\theta) \sin(\phi) \quad (6)$$

$$z = c_3(r, \theta, \phi) = r \cos(\phi) \quad (7)$$

$$\mathbf{c}(\mathbf{r}) = \mathbf{c}(r, \theta, \phi) = [c_1(r, \theta, \phi), c_2(r, \theta, \phi), c_3(r, \theta, \phi)] \quad (8)$$

The projections from cartesian to spherical coordinates can be computed using the following unit vectors

$$\hat{r} = \cos(\theta) \sin(\phi) \hat{x} + \sin(\theta) \sin(\phi) \hat{y} + \cos(\phi) \hat{z} \quad (9)$$

$$\hat{\theta} = -\sin(\theta) \hat{x} + \cos(\theta) \hat{y} \quad (10)$$

$$\hat{\phi} = \cos(\theta) \cos(\phi) \hat{x} + \sin(\theta) \cos(\phi) \hat{y} - \sin(\phi) \hat{z} \quad (11)$$

$$\hat{x} = \cos(\theta) \sin(\phi) \hat{r} - \sin(\theta) \hat{\theta} + \cos(\theta) \cos(\phi) \hat{\phi} \quad (12)$$

$$\hat{y} = \sin(\theta) \sin(\phi) \hat{r} + \cos(\theta) \hat{\theta} + \sin(\theta) \cos(\phi) \hat{\phi} \quad (13)$$

$$\hat{z} = \cos(\phi) \hat{r} - \sin(\phi) \hat{\phi} \quad (14)$$

$$(15)$$

The partial derivatives of the transformation to spherical coordinates are needed, when considering differential equations.

$$s_{1,x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2x = \cos(\theta) \sin(\phi) \quad (16)$$

$$s_{1,y} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2y = \sin(\theta) \sin(\phi) \quad (17)$$

$$s_{1,z} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} 2z = \cos(\phi) \quad (18)$$

$$(19)$$

$$s_{2,x} = \frac{1}{1 + \frac{y^2}{x^2}} \left( -\frac{y}{x^2} \right) \quad (20)$$

$$= \frac{1}{1 + \frac{r^2 \sin^2(\theta) \sin^2(\phi)}{r^2 \cos^2(\theta) \sin^2(\phi)}} \left( -\frac{r \sin(\theta) \sin(\phi)}{r^2 \cos^2(\theta) \sin^2(\phi)} \right) \quad (21)$$

$$= \frac{1}{1 + \frac{\sin^2(\theta)}{\cos^2(\theta)}} \left( -\frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \right) \quad (22)$$

$$= -\frac{1}{1 + \frac{1-\cos^2(\theta)}{\cos^2(\theta)}} \frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \quad (23)$$

$$= -\cos^2(\theta) \frac{\sin(\theta)}{r \cos^2(\theta) \sin(\phi)} \quad (24)$$

$$= -\frac{\sin(\theta)}{r \sin(\phi)} \quad (25)$$

$$s_{2,y} = \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right) \quad (26)$$

$$= \cos^2(\theta) \frac{1}{r \cos(\theta) \sin(\phi)} \quad (27)$$

$$= \frac{\cos(\theta)}{r \sin(\phi)} \quad (28)$$

$$s_{2,z} = 0 \quad (29)$$

$$(30)$$

$$s_{3,x} = \frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{1}{2} \frac{2x}{z \sqrt{x^2+y^2}} \quad (31)$$

$$= \frac{1}{1 + \frac{r^2 \cos^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \sin^2(\phi)}{r^2 \cos^2(\phi)}} \frac{r \cos(\theta) \sin(\phi)}{r \cos(\phi) \sqrt{r^2 \cos^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \sin^2(\phi)}} \quad (32)$$

$$= \frac{1}{1 + \frac{\sin^2(\phi)}{\cos^2(\phi)}} \frac{\cos(\theta)}{r \cos(\phi)} \quad (33)$$

$$= \frac{1}{1 + \frac{1-\cos^2(\phi)}{\cos^2(\phi)}} \frac{\cos(\theta)}{r \cos(\phi)} \quad (34)$$

$$= \cos^2(\phi) \frac{\cos(\theta)}{r \cos(\phi)} \quad (35)$$

$$= \frac{\cos(\theta) \cos(\phi)}{r} \quad (36)$$

$$s_{3,y} = \frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{1}{2} \frac{2y}{z \sqrt{x^2+y^2}} \quad (37)$$

$$= \cos^2(\phi) \frac{r \sin(\theta) \sin(\phi)}{r \cos(\phi) r \sin(\phi)} \quad (38)$$

$$= \frac{\sin(\theta) \cos(\phi)}{r} \quad (39)$$

$$s_{3,z} = -\frac{1}{1 + \frac{x^2+y^2}{z^2}} \frac{\sqrt{x^2+y^2}}{z^2} \quad (40)$$

$$= -\cos^2(\phi) \frac{r \sin(\phi)}{r^2 \cos^2(\phi)} \quad (41)$$

$$= -\frac{\sin(\phi)}{r} \quad (42)$$

$$(43)$$

The derivative with respect to a cartesian coordinate, can be expressed in terms of the spherical coordinate

derivatives using the chain rule. Consider a function of spherical coordinates,  $f(r, \theta, \phi)$ , then

$$\frac{\partial}{\partial t}(f(\mathbf{s}(\mathbf{x}))) = \frac{\partial f}{\partial r} \frac{\partial s_1}{\partial t} + \frac{\partial f}{\partial \theta} \frac{\partial s_2}{\partial t} + \frac{\partial f}{\partial \phi} \frac{\partial s_3}{\partial t} \quad (44)$$

where  $t = x, y, \text{ or } z$

Now let's consider the Navier Stokes equation. The velocities in cartesian coordinates are given by  $[u, v, w] = \mathbf{u}$ , then the cartesian Navier Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (45)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + g_x \quad (46)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(vw) = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + g_y \quad (47)$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(w^2) = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + g_z \quad (48)$$

$$(49)$$

In spherical coordinates we represent the velocities as  $[u_r, u_\theta, u_\phi] = \mathbf{u}_r$ . Using the unit vectors we can write these velocities in terms of the cartesian velocities.

$$u_r = \cos(\theta) \sin(\phi) u + \sin(\theta) \sin(\phi) v + \cos(\phi) w \quad (50)$$

$$u_\theta = -\sin(\theta) u + \cos(\theta) w \quad (51)$$

$$u_\phi = \cos(\theta) \cos(\phi) u + \sin(\theta) \cos(\phi) v - \sin(\phi) w \quad (52)$$

We can also write the cartesian velocities in terms of the spherical velocities.

$$u = \cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi \quad (53)$$

$$v = \sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi \quad (54)$$

$$w = \cos(\phi) u_r - \sin(\phi) u_\phi \quad (55)$$

Now we can express the continuity equation in spherical coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (56)$$

$$\frac{\partial}{\partial x}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \quad (57)$$

$$+ \frac{\partial}{\partial y}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \quad (58)$$

$$+ \frac{\partial}{\partial z}(\cos(\phi) u_r - \sin(\phi) u_\phi) = 0 \quad (59)$$

$$(60)$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial x} \quad (61)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial x} \quad (62)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial x} \quad (63)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial y} \quad (64)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial y} \quad (65)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial y} \quad (66)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_1}{\partial z} \quad (67)$$

$$+ \frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_2}{\partial z} \quad (68)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_3}{\partial z} = 0 \quad (69)$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \quad (70)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \quad (71)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \quad (72)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \quad (73)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \quad (74)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \quad (75)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \quad (76)$$

$$+ \frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)0 \quad (77)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \quad (78)$$

$$(79)$$

$$\frac{\partial}{\partial r}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \cos(\theta) \sin(\phi) \quad (80)$$

$$\frac{\partial}{\partial \theta}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \frac{-\sin(\theta)}{r \sin(\phi)} \quad (81)$$

$$\frac{\partial}{\partial \phi}(\cos(\theta) \sin(\phi) u_r - \sin(\theta) u_\theta + \cos(\theta) \cos(\phi) u_\phi) \frac{\cos(\theta) \cos(\phi)}{r} \quad (82)$$

$$+ \frac{\partial}{\partial r}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \sin(\theta) \sin(\phi) \quad (83)$$

$$+ \frac{\partial}{\partial \theta}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \frac{\cos(\theta)}{r \sin(\phi)} \quad (84)$$

$$+ \frac{\partial}{\partial \phi}(\sin(\theta) \sin(\phi) u_r + \cos(\theta) u_\theta + \sin(\theta) \cos(\phi) u_\phi) \frac{\sin(\theta) \cos(\phi)}{r} \quad (85)$$

$$+ \frac{\partial}{\partial r}(\cos(\phi) u_r - \sin(\phi) u_\phi) \cos(\phi) \quad (86)$$

$$+ \frac{\partial}{\partial \phi}(\cos(\phi) u_r - \sin(\phi) u_\phi) \frac{-\sin(\phi)}{r} = 0 \quad (87)$$

$$(88)$$

Now I will consider the terms of each derivative of each velocity individually,

$$\frac{\partial u_r}{\partial r}$$

$$\cos^2(\theta) \sin^2(\phi) \frac{\partial u_r}{\partial r} + \sin^2(\theta) \sin^2(\phi) \frac{\partial u_r}{\partial r} + \cos^2(\phi) \frac{\partial u_r}{\partial r} = \sin^2(\phi) \frac{\partial u_r}{\partial r} + \cos^2(\phi) \frac{\partial u_r}{\partial r} = \frac{\partial u_r}{\partial r} \quad (89)$$

$$\frac{\partial u_\theta}{\partial r}$$

$$- \sin(\theta) \cos(\theta) \sin(\phi) \frac{\partial u_\theta}{\partial r} + \cos(\theta) \sin(\theta) \sin(\phi) \frac{\partial u_\theta}{\partial r} = 0 \quad (90)$$

$$\frac{\partial u_\phi}{\partial r}$$

$$\cos^2(\theta) \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} + \sin^2(\theta) \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} - \sin(\phi) \cos(\phi) \frac{\partial u_\phi}{\partial r} \quad (91)$$

$$= \cos(\phi) \sin(\phi) \frac{\partial u_\phi}{\partial r} - \sin(\phi) \cos(\phi) \frac{\partial u_\phi}{\partial r} = 0 \quad (92)$$

$$\frac{\partial u_r}{\partial \theta}$$

$$- \frac{\sin(\theta)}{r} \frac{\partial}{\partial \theta}(\cos(\theta) u_r) + \frac{\cos(\theta)}{r} \frac{\partial}{\partial \theta}(\sin(\theta) u_r) \quad (93)$$

$$= - \frac{\sin(\theta) \cos(\theta)}{r} \frac{\partial u_r}{\partial \theta} + \frac{\sin^2(\theta)}{r} u_r + \frac{\cos(\theta) \sin(\theta)}{r} \frac{\partial u_r}{\partial \theta} + \frac{\cos^2(\theta)}{r} u_r = \frac{u_r}{r} \quad (94)$$

$$\frac{\partial u_\theta}{\partial \theta}$$

$$\frac{\sin(\theta)}{r \sin(\phi)} \frac{\partial}{\partial \theta}(\sin(\theta) u_\theta) + \frac{\cos(\theta)}{r \sin(\phi)} \frac{\partial}{\partial \theta}(\cos(\theta) u_\theta) = \quad (95)$$

$$\frac{\sin(\theta)}{r \sin(\phi)} \left( \sin(\theta) \frac{\partial u_\theta}{\partial \theta} + \cos(\theta) u_\theta \right) + \frac{\cos(\theta)}{r \sin(\phi)} \left( \cos(\theta) \frac{\partial u_\theta}{\partial \theta} - \sin(\theta) u_\theta \right) = \quad (96)$$

$$\frac{\sin^2(\theta)}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{\cos^2(\theta)}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} = \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} \quad (97)$$

$$\frac{\partial u_\phi}{\partial \theta}$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\cos(\theta)u_\phi) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\sin(\theta)u_\phi) = \quad (98)$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\phi}{\partial\theta} - \sin(\theta)u_\phi\right) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\phi}{\partial\theta} + \cos(\theta)u_\phi\right) = \quad (99)$$

$$\frac{\sin^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{\cos^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi = \frac{\cos(\phi)}{r\sin(\phi)}u_\phi \quad (100)$$

$$\frac{\partial u_r}{\partial \phi}$$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \quad (101)$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \quad (102)$$

$$\frac{\cos(\phi)}{r}\left(\sin(\phi)\frac{\partial u_r}{\partial\phi} + \cos(\phi)u_r\right) - \frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial u_r}{\partial\phi} - \sin(\phi)u_r\right) = \quad (103)$$

$$\frac{\cos^2(\phi)}{r}u_r + \frac{\sin^2(\phi)}{r}u_r = \frac{1}{r}u_r \quad (104)$$

$$\frac{\partial u_\theta}{\partial \phi}$$

$$-\frac{\cos(\theta)\sin(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} + \frac{\sin(\theta)\cos(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} = 0 \quad (105)$$

$$\frac{\partial u_\phi}{\partial \phi}$$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \quad (106)$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \quad (107)$$

$$\frac{\cos(\phi)}{r}\left(\cos(\phi)\frac{\partial u_\phi}{\partial\phi} - \sin(\phi)u_\phi\right) + \frac{\sin(\phi)}{r}\left(\sin(\phi)\frac{\partial u_\phi}{\partial\phi} + \cos(\phi)u_\phi\right) = \quad (108)$$

$$\frac{\cos^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\sin^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} \quad (109)$$

The simplified continuity equation is thus

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} = 0 \quad (110)$$

$$\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi = 0 \quad (111)$$

Using the product rule this can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = 0 \quad (112)$$

$$(113)$$

Shallow Water in Spherical Coordinates R radius,  $(\theta, \phi)$ , longitude and latitude

$$h_t + \frac{1}{R \cos(\phi)} (hu_\theta)_\theta + \frac{1}{R \cos(\phi)} (hu_\phi \cos(\phi))_\phi = 0 \quad (114)$$

$$(hu_\theta)_t + \frac{1}{R \cos(\phi)} \left( hu_\theta^2 + \frac{1}{2} gh^2 \right)_\theta + \frac{1}{R} (hu_\theta u_\phi) - 2 \frac{hu_\theta u_\phi}{R} \tan(\phi) = 0 \quad (115)$$

$$(hu_\phi)_\phi + \frac{1}{R \cos(\phi)} (hu_\theta u_\phi)_\theta + \frac{1}{R} \left( hu_\phi^2 + \frac{1}{2} gh^2 \right)_\phi + \frac{hu_\theta^2 - hu_\phi^2}{R} \tan(\phi) = 0 \quad (116)$$

Navier Stokes Equations in Spherical Coordinates r radius,  $(\theta, \phi)$  azimuth, and polar angle,  $\theta = \arctan(y/x)$ ,  $\phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) u_\phi) = 0 \quad (117)$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r \quad (118)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g_\theta \quad (119)$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} - \frac{u_\theta^2 \cot(\phi)}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + g_\phi \quad (120)$$

Kinematic Boundary Condition,  $h_s(t, \theta, \phi)$ ,  $h_b(t, \theta, \phi)$   $u_{\theta/\phi/r}(t, \theta, \phi, r)$

$$\frac{\partial h_s}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_s)}{r \sin(\phi)} \frac{\partial h_s}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_s)}{r} \frac{\partial h_s}{\partial \phi} = u_r(t, \theta, \phi, h_s) \quad (121)$$

$$\frac{\partial h_b}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_b)}{r \sin(\phi)} \frac{\partial h_b}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_b)}{r} \frac{\partial h_b}{\partial \phi} = u_r(t, \theta, \phi, h_b) \quad (122)$$

$$(123)$$

Dimensional Analysis

$$r = R\hat{r} \quad h = H\hat{h} \quad \frac{H}{R} = \epsilon$$

$$u_\theta = U\hat{u}_\theta \quad u_\phi = U\hat{u}_\phi \quad u_r = U_r\hat{u}_r$$

$$t = T\hat{t} = \frac{R}{U}\hat{t} \quad p = \rho g H \hat{p}$$

$$\frac{1}{R^2 \hat{r}^2} \frac{1}{R} \frac{\partial}{\partial \hat{r}} (R^2 \hat{r}^2 \epsilon U \hat{u}_r) + \frac{1}{R \hat{r} \sin(\phi)} \frac{\partial U \hat{u}_\theta}{\partial \theta} + \frac{1}{R \hat{r} \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) U \hat{u}_\phi) = 0 \quad (124)$$

$$\frac{U_r}{T} \frac{\partial \hat{u}_r}{\partial \hat{t}} + U_r \hat{u}_r \frac{U_r}{R} \frac{\partial \hat{u}_r}{\partial \hat{r}} + \frac{U \hat{u}_\theta}{R \hat{r} \sin(\phi)} U_r \frac{\partial \hat{u}_r}{\partial \theta} + \frac{U \hat{u}_\phi}{R \hat{r}} U_r \frac{\partial \hat{u}_r}{\partial \phi} - \frac{U^2}{R} \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\frac{1}{\rho} \rho g H \frac{1}{R} \frac{\partial \hat{p}}{\partial \hat{r}} + g_r \quad (125)$$

$$\frac{U}{T} \frac{\partial \hat{u}_\theta}{\partial \hat{t}} + U_r \hat{u}_r \frac{U}{R} \frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{U \hat{u}_\theta}{R \hat{r} \sin(\phi)} U \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{U \hat{u}_\phi}{R \hat{r}} U \frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{U_r \hat{u}_r U \hat{u}_\theta}{R \hat{r}} + \frac{U^2 \hat{u}_\theta \hat{u}_\phi \cot(\phi)}{R \hat{r}} = -\frac{1}{\rho R \hat{r} \sin(\phi)} \rho g H \frac{\partial \hat{p}}{\partial \theta} + g_\theta \quad (126)$$

$$\frac{U}{T} \frac{\partial \hat{u}_\phi}{\partial \hat{t}} + U_r \hat{u}_r \frac{U}{R} \frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{U \hat{u}_\theta}{R \hat{r} \sin(\phi)} U \frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{U \hat{u}_\phi}{R \hat{r}} U \frac{\partial \hat{u}_\phi}{\partial \phi} + \frac{U_r U \hat{u}_r \hat{u}_\phi}{R \hat{r}} - \frac{U^2 \hat{u}_\theta^2 \cot(\phi)}{R \hat{r}} = -\frac{1}{\rho R \hat{r}} \rho g H \frac{\partial \hat{p}}{\partial \phi} + g_\phi \quad (127)$$

$$T = \frac{R}{U}, U_r = \epsilon U$$

$$\frac{\epsilon}{\hat{r}^2} \frac{\partial}{\partial \hat{r}} (\hat{r}^2 \hat{u}_r) + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) \hat{u}_\phi) = 0 \quad (128)$$

$$\frac{U^2}{R} \left( \epsilon \frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2 \hat{u}_r \frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_r}{\partial \theta} + \epsilon \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} \right) = -g \frac{H}{R} \frac{\partial \hat{p}}{\partial \hat{r}} + g e_r \quad (129)$$

$$\frac{U^2}{R} \left( \frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon \hat{u}_r \frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon \frac{\hat{u}_r \hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta \hat{u}_\phi \cot(\phi)}{\hat{r}} \right) = -\frac{gH}{R \hat{r} \sin(\phi)} \frac{\partial \hat{p}}{\partial \theta} + g e_\theta \quad (130)$$

$$\frac{U^2}{R} \left( \frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon \hat{u}_r \frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon \frac{\hat{u}_r \hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2 \cot(\phi)}{\hat{r}} \right) = -\frac{gH}{R \hat{r}} \frac{\partial \hat{p}}{\partial \phi} + g e_\phi \quad (131)$$

$$R = \frac{H}{\epsilon}$$

$$\frac{\epsilon}{\hat{r}^2} \frac{\partial}{\partial \hat{r}} (\hat{r}^2 \hat{u}_r) + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) \hat{u}_\phi) = 0 \quad (132)$$

$$\epsilon \frac{U^2}{gH} \left( \epsilon \frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2 \hat{u}_r \frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_r}{\partial \theta} + \epsilon \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} \right) = -\epsilon \frac{\partial \hat{p}}{\partial \hat{r}} + e_r \quad (133)$$

$$\epsilon \frac{U^2}{gH} \left( \frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon \hat{u}_r \frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon \frac{\hat{u}_r \hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta \hat{u}_\phi \cot(\phi)}{\hat{r}} \right) = -\frac{\epsilon}{\hat{r} \sin(\phi)} \frac{\partial \hat{p}}{\partial \theta} + e_\theta \quad (134)$$

$$\epsilon \frac{U^2}{gH} \left( \frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon \hat{u}_r \frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon \frac{\hat{u}_r \hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2 \cot(\phi)}{\hat{r}} \right) = -\frac{\epsilon}{\hat{r}} \frac{\partial \hat{p}}{\partial \phi} + e_\phi \quad (135)$$

Drop  $\epsilon^2$  terms

$$\frac{\epsilon}{\hat{r}^2} \frac{\partial}{\partial \hat{r}} (\hat{r}^2 \hat{u}_r) + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r} \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) \hat{u}_\phi) = 0 \quad (136)$$

$$-\epsilon \frac{U^2}{gH} \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\epsilon \frac{\partial \hat{p}}{\partial \hat{r}} + e_r \quad (137)$$

$$\epsilon \frac{U^2}{gH} \left( \frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{\hat{u}_\theta \hat{u}_\phi \cot(\phi)}{\hat{r}} \right) = -\frac{\epsilon}{\hat{r} \sin(\phi)} \frac{\partial \hat{p}}{\partial \theta} + e_\theta \quad (138)$$

$$\epsilon \frac{U^2}{gH} \left( \frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r} \sin(\phi)} \frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}} \frac{\partial \hat{u}_\phi}{\partial \phi} - \frac{\hat{u}_\theta^2 \cot(\phi)}{\hat{r}} \right) = -\frac{\epsilon}{\hat{r}} \frac{\partial \hat{p}}{\partial \phi} + e_\phi \quad (139)$$

Mapping

$$\rho = \frac{r - h_b(t, \theta, \phi)}{h(t, \theta, \phi)} \quad (140)$$

$$h(t, \theta, \phi) = h_s(t, \theta, \phi) - h_b(t, \theta, \phi) \quad (141)$$

$$r = \rho h(t, \theta, \phi) + h_b(t, \theta, \phi) \quad (142)$$

$$\tilde{\psi}(t, \theta, \phi, \rho) = \psi(t, \theta, \phi, \rho h(t, \theta, \phi) + h_b(t, \theta, \phi)) \quad (143)$$

$$\psi(t, \theta, \phi, r) = \tilde{\psi}(t, \theta, \phi, \frac{r - h_b(t, \theta, \phi)}{h(t, \theta, \phi)}) \quad (144)$$

$$h \frac{\partial \psi}{\partial s} = \frac{\partial}{\partial s} (h \tilde{\psi}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial s} (\rho h + h_b) \tilde{\psi} \right) \quad s \in \{t, \theta, \phi\} \quad (145)$$

$$h \frac{\partial \psi}{\partial r} = \frac{\partial \tilde{\psi}}{\partial \rho} \quad (146)$$

Mapping of Mass Balance

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) u_\phi) = 0$$



Multiply by h

$$\frac{h}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{h}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{h}{r \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) u_\phi) = 0$$

Transform from r to  $\rho$

$$\begin{aligned} & \frac{1}{(h\rho + h_b)^2} \frac{\partial}{\partial \rho} \left( (h\rho + h_b)^2 \tilde{u}_r \right) + \frac{1}{(h\rho + h_b) \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h u_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta \right) \right) \\ & + \frac{1}{(h\rho + h_b) \sin(\phi)} \left( \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) \right) = 0 \\ & \frac{1}{(h\rho + h_b) \sin(\phi)} \frac{\partial}{\partial \theta} (h u_\theta) + \frac{1}{(h\rho + h_b) \sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_\phi) \\ & - \frac{1}{(h\rho + h_b) \sin(\phi)} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) = - \frac{1}{(h\rho + h_b)^2} \frac{\partial}{\partial \rho} \left( (h\rho + h_b)^2 \tilde{u}_r \right) \\ & - \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \theta} (h u_\theta) - \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_\phi) \\ & + \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) = \frac{\partial}{\partial \rho} \left( (h\rho + h_b)^2 \tilde{u}_r \right) \\ & - \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \theta} (h u_\theta) - \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_\phi) \\ & + \frac{h\rho + h_b}{\sin(\phi)} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) = \frac{\partial}{\partial \rho} \left( (h\rho + h_b)^2 \tilde{u}_r \right) \end{aligned}$$

$$\frac{1}{r_0^2} \frac{\partial}{\partial r} (r_0^2 u_r) + \frac{1}{r_0 \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) u_\phi) = 0 \quad (147)$$

Multiply by h

$$h \frac{\partial}{\partial r} (u_r) + \frac{h}{r_0 \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{h}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} (\sin(\phi) u_\phi) = 0 \quad (148)$$

Transform from r to  $\rho$

$$\frac{\partial}{\partial \rho} (\tilde{u}_r) + \frac{1}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta \right) \right) \quad (149)$$

$$+ \frac{1}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \phi} (h \sin(\phi) u_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) \right) = 0 \quad (150)$$

$$r_0 \sin(\phi) \frac{\partial}{\partial \rho} (\tilde{u}_r) = - \frac{\partial}{\partial \theta} (h \tilde{u}_\theta) - \frac{\partial}{\partial \phi} (h \sin(\phi) u_\phi) \quad (151)$$

$$+ \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) \quad (152)$$

$$\int_0^{\rho'} r_0 \sin(\phi) \frac{\partial}{\partial \rho} (\tilde{u}_r) d\rho = - \int_0^{\rho'} \frac{\partial}{\partial \theta} (h \tilde{u}_\theta) d\rho - \int_0^{\rho'} \frac{\partial}{\partial \phi} (h \sin(\phi) u_\phi) d\rho \quad (153)$$

$$+ \int_0^{\rho'} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi} (h\rho + h_b) \sin(\phi) \tilde{u}_\phi \right) d\rho \quad (154)$$

$$r_0 \sin(\phi) (\tilde{u}_r(t, \theta, \phi, \rho') - \tilde{u}_r(t, \theta, \phi, 0)) = - \frac{\partial}{\partial \theta} \left( h \int_0^{\rho'} \tilde{u}_\theta d\rho \right) - \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^{\rho'} u_\phi d\rho \right) \quad (155)$$

$$+ \frac{\partial}{\partial \theta} (h\rho' + h_b) \tilde{u}_\theta(t, \theta, \phi, \rho') + \frac{\partial}{\partial \phi} (h\rho' + h_b) \sin(\phi) \tilde{u}_\phi(t, \theta, \phi, \rho') \quad (156)$$

$$- \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta(t, \theta, \phi, 0) - \frac{\partial h_b}{\partial \phi} \sin(\phi) \tilde{u}_\phi(t, \theta, \phi, 0) \quad (157)$$

Let  $\rho' = 1$

$$r_0 \sin(\phi)(\tilde{u}_r(t, \theta, \phi, 1) - \tilde{u}_r(t, \theta, \phi, 0)) = -\frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) - \frac{\partial}{\partial \phi}(h \sin(\phi)\tilde{u}_{\phi m}) \quad (158)$$

$$+ \frac{\partial h_s}{\partial \theta} \tilde{u}_\theta(t, \theta, \phi, 1) + \frac{\partial h_s}{\partial \phi} \sin(\phi) \tilde{u}_\phi(t, \theta, \phi, 1) \quad (159)$$

$$- \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta(t, \theta, \phi, 0) - \frac{\partial h_b}{\partial \phi} \sin(\phi) \tilde{u}_\phi(t, \theta, \phi, 0) \quad (160)$$

$$0 = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi}(h \sin(\phi)\tilde{u}_{\phi m}) \quad (161)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial h_s}{\partial \theta} \tilde{u}_\theta(t, \theta, \phi, 1) + \frac{1}{r_0} \frac{\partial h_s}{\partial \phi} \tilde{u}_\phi(t, \theta, \phi, 1) - \tilde{u}_r(t, \theta, \phi, 1) \quad (162)$$

$$- \frac{1}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta(t, \theta, \phi, 0) - \frac{1}{r_0} \frac{\partial h_b}{\partial \phi} \tilde{u}_\phi(t, \theta, \phi, 0) + \tilde{u}_r(t, \theta, \phi, 0) \quad (163)$$

Using Kinematic Boundary condition with  $r_0 = r$

$$\frac{\partial h}{\partial t} + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi}(h \sin(\phi)\tilde{u}_{\phi m}) = 0 \quad (164)$$

$$(165)$$

Mapping of Momentum Balance

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g_\theta \quad (166)$$

Multiply by h

$$h \frac{\partial u_\theta}{\partial t} + h u_r \frac{\partial u_\theta}{\partial r} + \frac{h u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{h u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} + \frac{h u_r u_\theta}{r} + \frac{h u_\theta u_\phi \cot(\phi)}{r} = -\frac{h}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + h g_\theta \quad (167)$$

Transform from r to  $\rho$

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta \right) + \tilde{u}_r \frac{\partial h\tilde{u}_\theta}{\partial \rho} \quad (168)$$

$$+ \frac{\tilde{u}_\theta}{r \sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (169)$$

$$+ \frac{\tilde{u}_\phi}{r} \left( \frac{\partial}{\partial \phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (170)$$

$$+ \frac{h\tilde{u}_r \tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta \tilde{u}_\phi \cot(\phi)}{r} = -\frac{1}{\rho_0 r \sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{p}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{p} \right) \right) + h g_\theta \quad (171)$$

Transform hydrostatic pressure and simplify terms from momentum equations with pressure

$$p = (h_s(t, \theta, \phi) - r)\rho_0 g_e r \quad (172)$$

$$\tilde{p} = (h_s(t, \theta, \phi) - (h\rho + h_b))\rho_0 g_e r \quad (173)$$

$$= (h_s - h_b + h\rho)\rho_0 g_e r \quad (174)$$

$$= h(1 - \rho)\rho_0 g_e r \quad (175)$$

$\theta$  momentum equation

$$\frac{\partial}{\partial \theta}(h\tilde{p}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{p} \right) \quad (176)$$

$$= \frac{\partial}{\partial \theta}(hh(1-\rho)\rho_0 g e_r) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)h(1-\rho)\rho_0 g e_r \right) \quad (177)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \theta}(h^2(1-\rho)) - h \frac{\partial}{\partial \rho} \left( \left( \rho \frac{\partial h}{\partial \theta} + \frac{\partial h_b}{\partial \theta} \right) (1-\rho) \right) \right) \quad (178)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \theta}(h^2(1-\rho)) - h \left( \frac{\partial h}{\partial \theta}(1-\rho) - \rho \frac{\partial h}{\partial \theta} - \frac{\partial h_b}{\partial \theta} \right) \right) \quad (179)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \theta}(h^2(1-\rho)) + 2h \frac{\partial h}{\partial \theta} \rho - h \frac{\partial h}{\partial \theta} + h \frac{\partial h_b}{\partial \theta} \right) \quad (180)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \theta}(h^2(1-\rho)) + \frac{\partial}{\partial \theta}(h^2)\rho - \frac{1}{2} \frac{\partial}{\partial \theta}(h^2) + h \frac{\partial h_b}{\partial \theta} \right) \quad (181)$$

$$= \rho_0 g e_r \left( \frac{1}{2} \frac{\partial}{\partial \theta}(h^2) + h \frac{\partial h_b}{\partial \theta} \right) \quad (182)$$

$\phi$  momentum equation

$$\frac{\partial}{\partial \phi}(h\tilde{p}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{p} \right) \quad (183)$$

$$= \frac{\partial}{\partial \phi}(hh(1-\rho)\rho_0 g e_r) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(\rho h + h_b)h(1-\rho)\rho_0 g e_r \right) \quad (184)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \phi}(h^2(1-\rho)) - h \frac{\partial}{\partial \rho} \left( \left( \rho \frac{\partial h}{\partial \phi} + \frac{\partial h_b}{\partial \phi} \right) (1-\rho) \right) \right) \quad (185)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \phi}(h^2(1-\rho)) - h \left( \frac{\partial h}{\partial \phi}(1-\rho) - \rho \frac{\partial h}{\partial \phi} - \frac{\partial h_b}{\partial \phi} \right) \right) \quad (186)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \phi}(h^2(1-\rho)) + 2h \frac{\partial h}{\partial \phi} \rho - h \frac{\partial h}{\partial \phi} + h \frac{\partial h_b}{\partial \phi} \right) \quad (187)$$

$$= \rho_0 g e_r \left( \frac{\partial}{\partial \phi}(h^2(1-\rho)) + \frac{\partial}{\partial \phi}(h^2)\rho - \frac{1}{2} \frac{\partial}{\partial \phi}(h^2) + h \frac{\partial h_b}{\partial \phi} \right) \quad (188)$$

$$= \rho_0 g e_r \left( \frac{1}{2} \frac{\partial}{\partial \phi}(h^2) + h \frac{\partial h_b}{\partial \phi} \right) \quad (189)$$

$$(190)$$

Plug pressure expression into momentum balance equation

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta \right) + \tilde{u}_r \frac{\partial h\tilde{u}_\theta}{\partial \rho} \quad (191)$$

$$+ \frac{\tilde{u}_\theta}{r \sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (192)$$

$$+ \frac{\tilde{u}_\phi}{r} \left( \frac{\partial}{\partial \phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (193)$$

$$+ \frac{h\tilde{u}_r\tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta\tilde{u}_\phi \cot(\phi)}{r} = -\frac{1}{\rho_0 r \sin(\phi)} \left( \rho_0 g e_r \left( \frac{1}{2} \frac{\partial}{\partial \theta}(h^2) + h \frac{\partial h_b}{\partial \theta} \right) \right) + h g e_\theta \quad (194)$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t}(\rho h + h_b)\tilde{u}_\theta \right) + \tilde{u}_r \frac{\partial h\tilde{u}_\theta}{\partial \rho} \quad (195)$$

$$+ \frac{\tilde{u}_\theta}{r \sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (196)$$

$$+ \frac{\tilde{u}_\phi}{r} \left( \frac{\partial}{\partial \phi}(h\tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(\rho h + h_b)\tilde{u}_\theta \right) \right) \quad (197)$$

$$+ \frac{h\tilde{u}_r\tilde{u}_\theta}{r} + \frac{h\tilde{u}_\theta\tilde{u}_\phi \cot(\phi)}{r} = -\frac{g e_r}{2r \sin(\phi)} \frac{\partial}{\partial \theta}(h^2) - \frac{h g e_r}{r \sin(\phi)} \frac{\partial h_b}{\partial \theta} + h g e_\theta \quad (198)$$

$$(199)$$

Rewriting Incompressible Navier Stokes Equations

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \phi}(\sin(\phi) u_\phi) = 0 \quad (200)$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r \quad (201)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g_\theta \quad (202)$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} - \frac{u_\theta^2 \cot(\phi)}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + g_\phi \quad (203)$$

Continuity Equation after product rule

$$\frac{\partial u_r}{\partial r} + 2 \frac{u_r}{r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\cot(\phi)}{r} u_\phi = 0 \quad (204)$$

Simplify  $u_r$  evolution equation

$$u_r \frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r}(u_r^2) - u_r \frac{\partial u_r}{\partial r} \quad (205)$$

$$\frac{u_\theta}{r \sin(\phi)} \frac{\partial u_r}{\partial \theta} = \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) - \frac{u_r}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} \quad (206)$$

$$\frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} = \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) - \frac{u_r}{r} \frac{\partial u_\phi}{\partial \phi} \quad (207)$$

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} \quad (208)$$

$$= \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) - u_r \left( \frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} \right) \quad (209)$$

From the continuity equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} = -2 \frac{u_r}{r} - \frac{\cot(\phi)}{r} u_\phi \quad (210)$$

So

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} = \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) + u_r \left( 2 \frac{u_r}{r} + \frac{\cot(\phi)}{r} u_\phi \right) \quad (211)$$

$$= \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) + \frac{2u_r^2 + u_r u_\phi \cot(\phi)}{r} \quad (212)$$

Substituting this into the  $u_r$  evolution equation gives

$$\frac{\partial u_r}{\partial t} + \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) + \frac{2u_r^2 + u_r u_\phi \cot(\phi) - u_\theta^2 - u_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r \quad (213)$$

$$(214)$$

Simplify  $u_\theta$  evolution equation

$$u_r \frac{\partial u_\theta}{\partial r} = \frac{\partial}{\partial r}(u_r u_\theta) - u_\theta \frac{\partial u_r}{\partial r} \quad (215)$$

$$\frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} = \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta^2) - \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} \quad (216)$$

$$\frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} = \frac{1}{r} \frac{\partial}{\partial \phi}(u_\theta u_\phi) - \frac{u_\theta}{r} \frac{\partial u_\phi}{\partial \phi} \quad (217)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} \quad (218)$$

$$= \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta^2) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\theta u_\phi) - u_\theta \left( \frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} \right) \quad (219)$$

From the Continuity Equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} = -2 \frac{u_r}{r} - \frac{\cot(\phi)}{r} u_\phi \quad (220)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\theta}{\partial \phi} \quad (221)$$

$$= \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta^2) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\theta u_\phi) + u_\theta \left( 2 \frac{u_r}{r} + \frac{\cot(\phi)}{r} u_\phi \right) \quad (222)$$

Substituting this into  $u_\theta$  equation gives

$$\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta^2) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\theta u_\phi) + 3 \frac{u_r u_\theta}{r} + 2 \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g_\theta \quad (223)$$

Simplify the  $u_\phi$  equation

$$u_r \frac{\partial u_\phi}{\partial r} = \frac{\partial}{\partial r}(u_r u_\phi) - u_\phi \frac{\partial u_r}{\partial r} \quad (224)$$

$$\frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\phi}{\partial \theta} = \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta u_\phi) - \frac{u_\phi}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} \quad (225)$$

$$\frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} = \frac{1}{r} \frac{\partial}{\partial \phi}(u_\phi^2) - \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} \quad (226)$$

$$u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} \quad (227)$$

$$= \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\phi^2) - u_\phi \left( \frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} \right) \quad (228)$$

From the Continuity Equation

$$\frac{\partial u_r}{\partial r} + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} = -2 \frac{u_r}{r} - \frac{\cot(\phi)}{r} u_\phi \quad (229)$$

$$u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r \sin(\phi)} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} \quad (230)$$

$$= \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\phi^2) + u_\phi \left( 2 \frac{u_r}{r} + \frac{\cot(\phi)}{r} u_\phi \right) \quad (231)$$

Substituting into  $u_\phi$  equation

$$\frac{\partial u_\phi}{\partial t} + \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\phi^2) + 3 \frac{u_r u_\phi}{r} + \frac{(u_\phi^2 - u_\theta^2) \cot(\phi)}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + g_\phi \quad (232)$$

The Incompressible Navier Stokes Equations can be rewritten as

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \phi}(\sin(\phi) u_\phi) = 0 \quad (233)$$

$$\frac{\partial u_r}{\partial t} + \frac{\partial}{\partial r}(u_r^2) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_r u_\phi) + \frac{2u_r^2 + u_r u_\phi \cot(\phi) - u_\theta^2 - u_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + g_r \quad (234)$$

$$\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial r}(u_r u_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta^2) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\theta u_\phi) + 3 \frac{u_r u_\theta}{r} + 2 \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g_\theta \quad (235)$$

$$\frac{\partial u_\phi}{\partial t} + \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta}(u_\theta u_\phi) + \frac{1}{r} \frac{\partial}{\partial \phi}(u_\phi^2) + 3 \frac{u_r u_\phi}{r} + \frac{(u_\phi^2 - u_\theta^2) \cot(\phi)}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + g_\phi \quad (236)$$

Kinematic Boundary Condition

$$\frac{\partial h_s}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_s)}{r \sin(\phi)} \frac{\partial h_s}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_s)}{r} \frac{\partial h_s}{\partial \phi} = u_r(t, \theta, \phi, h_s) \quad (237)$$

$$\frac{\partial h_b}{\partial t} + \frac{u_\theta(t, \theta, \phi, h_b)}{r \sin(\phi)} \frac{\partial h_b}{\partial \theta} + \frac{u_\phi(t, \theta, \phi, h_b)}{r} \frac{\partial h_b}{\partial \phi} = u_r(t, \theta, \phi, h_b) \quad (238)$$

Mapped Boundary Condition, assuming  $r = r_0$

$$\frac{\partial h_s}{\partial t} + \frac{\tilde{u}_\theta(t, \theta, \phi, 1)}{r_0 \sin(\phi)} \frac{\partial h_s}{\partial \theta} + \frac{\tilde{u}_\phi(t, \theta, \phi, 1)}{r_0} \frac{\partial h_s}{\partial \phi} = \tilde{u}_r(t, \theta, \phi, 1) \quad (239)$$

$$\frac{\partial h_b}{\partial t} + \frac{\tilde{u}_\theta(t, \theta, \phi, 0)}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + \frac{\tilde{u}_\phi(t, \theta, \phi, 0)}{r_0} \frac{\partial h_b}{\partial \phi} = \tilde{u}_r(t, \theta, \phi, 0) \quad (240)$$

Mapping the Mass Balance Equation

$$\frac{1}{r_0^2} \frac{\partial}{\partial r}(r_0^2 u_r) + \frac{1}{r_0 \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi}(\sin(\phi) u_\phi) = 0 \quad (241)$$

Multiply by h

$$h \frac{\partial}{\partial r}(u_r) + \frac{h}{r_0 \sin(\phi)} \frac{\partial u_\theta}{\partial \theta} + \frac{h}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi}(\sin(\phi) u_\phi) = 0 \quad (242)$$

Transform from r to  $\rho$

$$\frac{\partial}{\partial \rho}(\tilde{u}_r) + \frac{1}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta}(h \tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h \rho + h_b) \tilde{u}_\theta \right) \right) \quad (243)$$

$$+ \frac{1}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \phi}(h \sin(\phi) u_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(h \rho + h_b) \sin(\phi) \tilde{u}_\phi \right) \right) = 0 \quad (244)$$

$$\frac{\partial}{\partial \rho}(\tilde{u}_r) = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta}(h \tilde{u}_\theta) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi}(h \sin(\phi) u_\phi) \quad (245)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h \rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi}(h \rho + h_b) \sin(\phi) \tilde{u}_\phi \right) \quad (246)$$

$$\int_0^{\rho'} \frac{\partial}{\partial \rho}(\tilde{u}_r) d\rho = -\frac{1}{r_0 \sin(\phi)} \int_0^{\rho'} \frac{\partial}{\partial \theta}(h \tilde{u}_\theta) d\rho - \frac{1}{r_0 \sin(\phi)} \int_0^{\rho'} \frac{\partial}{\partial \phi}(h \sin(\phi) u_\phi) d\rho \quad (247)$$

$$+ \frac{1}{r_0 \sin(\phi)} \int_0^{\rho'} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h \rho + h_b) \tilde{u}_\theta + \frac{\partial}{\partial \phi}(h \rho + h_b) \sin(\phi) \tilde{u}_\phi \right) d\rho \quad (248)$$

$$\tilde{u}_r(t, \theta, \phi, \rho') - \tilde{u}_r(t, \theta, \phi, 0) = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^{\rho'} \tilde{u}_\theta d\rho \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^{\rho'} u_\phi d\rho \right) \quad (249)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta}(h \rho' + h_b) \tilde{u}_\theta + \frac{1}{r_0} \frac{\partial}{\partial \phi}(h \rho' + h_b) \tilde{u}_\phi - \frac{1}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta - \frac{1}{r_0} \frac{\partial h_b}{\partial \phi} \tilde{u}_\phi \quad (250)$$

Let  $\rho' = 1$

$$\tilde{u}_r(t, \theta, \phi, 1) - \tilde{u}_r(t, \theta, \phi, 0) = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 \tilde{u}_\theta d\rho \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^1 u_\phi d\rho \right) \quad (251)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial h_s}{\partial \theta} \tilde{u}_\theta + \frac{1}{r_0} \frac{\partial h_s}{\partial \phi} \tilde{u}_\phi - \frac{1}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta - \frac{1}{r_0} \frac{\partial h_b}{\partial \phi} \tilde{u}_\phi \quad (252)$$

$$0 = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 \tilde{u}_\theta d\rho \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^1 u_\phi d\rho \right) \quad (253)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial h_s}{\partial \theta} \tilde{u}_\theta + \frac{1}{r_0} \frac{\partial h_s}{\partial \phi} \tilde{u}_\phi - \tilde{u}_r(t, \theta, \phi, 1) - \frac{1}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} \tilde{u}_\theta - \frac{1}{r_0} \frac{\partial h_b}{\partial \phi} \tilde{u}_\phi + \tilde{u}_r(t, \theta, \phi, 0) \quad (254)$$

Using Mapped Kinematic Boundary condition with  $r_0 = r$

$$0 = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 \tilde{u}_\theta d\rho \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^1 u_\phi d\rho \right) - \frac{\partial h_s}{\partial t} + \frac{\partial h_b}{\partial t} \quad (255)$$

Now we can denote  $\tilde{u}_{\theta m} = \int_0^1 \tilde{u}_\theta d\rho$  and  $\tilde{u}_{\phi m} = \int_0^1 \tilde{u}_\phi d\rho$ , then

$$\frac{\partial h_s}{\partial t} + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \tilde{u}_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_{\phi m}) \quad (256)$$

Also using the mapped boundary conditions we can write an explicit expression for  $\tilde{u}_r$

$$\tilde{u}_r(t, \theta, \phi, \rho) = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_\theta d\rho' \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_\phi d\rho' \right) \quad (257)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h\rho + h_b) \tilde{u}_\phi \quad (258)$$

Mapping of Momentum Balance

$$\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial r} (u_r u_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta} (u_\theta^2) + \frac{1}{r} \frac{\partial}{\partial \phi} (u_\theta u_\phi) + 3 \frac{u_r u_\theta}{r} + 2 \frac{u_\theta u_\phi \cot(\phi)}{r} = -\frac{1}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g e_\theta \quad (259)$$

Multitply by h

$$h \frac{\partial u_\theta}{\partial t} + h \frac{\partial}{\partial r} (u_r u_\theta) + \frac{h}{r \sin(\phi)} \frac{\partial}{\partial \theta} (u_\theta^2) + \frac{h}{r} \frac{\partial}{\partial \phi} (u_\theta u_\phi) + 3 \frac{h u_r u_\theta}{r} + 2 \frac{h u_\theta u_\phi \cot(\phi)}{r} = -\frac{h}{\rho r \sin(\phi)} \frac{\partial p}{\partial \theta} + g h e_\theta \quad (260)$$

Transform from r to  $\rho$

$$\frac{\partial}{\partial t} (h \tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t} (h\rho + h_b) \tilde{u}_\theta \right) + \frac{\partial}{\partial \rho} (\tilde{u}_r \tilde{u}_\theta) + \frac{1}{r \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \tilde{u}_\theta^2) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta^2 \right) \right) \quad (261)$$

$$+ \frac{1}{r} \left( \frac{\partial}{\partial \phi} (h \tilde{u}_\theta \tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi} (h\rho + h_b) \tilde{u}_\theta \tilde{u}_\phi \right) \right) + 3 \frac{h \tilde{u}_r \tilde{u}_\theta}{r} + 2 \frac{h \tilde{u}_\theta \tilde{u}_\phi \cot(\phi)}{r} \quad (262)$$

$$= -\frac{1}{\rho r \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \tilde{p}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{p} \right) \right) + g h e_\theta \quad (263)$$

Substitute for transformed pressure

$$\frac{\partial}{\partial t} (h \tilde{u}_\theta) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t} (h\rho + h_b) \tilde{u}_\theta \right) + \frac{\partial}{\partial \rho} (\tilde{u}_r \tilde{u}_\theta) + \frac{1}{r \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \tilde{u}_\theta^2) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta} (h\rho + h_b) \tilde{u}_\theta^2 \right) \right) \quad (264)$$

$$+ \frac{1}{r} \left( \frac{\partial}{\partial \phi} (h \tilde{u}_\theta \tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi} (h\rho + h_b) \tilde{u}_\theta \tilde{u}_\phi \right) \right) + 3 \frac{h \tilde{u}_r \tilde{u}_\theta}{r} + 2 \frac{h \tilde{u}_\theta \tilde{u}_\phi \cot(\phi)}{r} \quad (265)$$

$$= -\frac{1}{\rho r \sin(\phi)} \left( \rho_0 g e_r \left( \frac{1}{2} \frac{\partial}{\partial \theta} (h^2) + h \frac{\partial h_b}{\partial \theta} \right) \right) + g h e_\theta \quad (266)$$

Gather derivatives

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta} \left( h\tilde{u}_\theta^2 + \frac{1}{2}gh^2e_r \right) + \frac{1}{r} \frac{\partial}{\partial \phi} (h\tilde{u}_\theta\tilde{u}_\phi) \quad (267)$$

$$+ \frac{\partial}{\partial \rho} \left( \tilde{u}_r\tilde{u}_\theta - \frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\theta \right) - \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta^2 \right) - \frac{1}{r} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi \right) \quad (268)$$

$$+ 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi \cot(\phi)}{r} = -\frac{ghe_r}{r \sin(\phi)} \frac{\partial h_b}{\partial \theta} + ghe_\theta \quad (269)$$

Consider the following separately, and label it as  $\omega$

$$\omega = \tilde{u}_r - \frac{\partial}{\partial t}(h\rho + h_b) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0} \frac{\partial}{\partial \phi} (h\rho + h_b)\tilde{u}_\phi \quad (270)$$

Using expression for  $\tilde{u}_r$

$$\omega = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_\theta d\rho' \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_\phi d\rho' \right) \quad (271)$$

$$+ \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\rho + h_b)\tilde{u}_\theta + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h\rho + h_b)\tilde{u}_\phi - \frac{\partial}{\partial t}(h\rho + h_b) \quad (272)$$

$$- \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0} \frac{\partial}{\partial \phi} (h\rho + h_b)\tilde{u}_\phi \quad (273)$$

$$\omega = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_\theta d\rho' \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_\phi d\rho' \right) - \rho \frac{\partial h}{\partial t} \quad (274)$$

Using continuity equation

$$\omega = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_\theta d\rho' \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_\phi d\rho' \right) \quad (275)$$

$$+ \rho \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\tilde{u}_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \tilde{u}_{\phi m}) \right) \quad (276)$$

Using the fact that  $\rho = \int_0^\rho 1 d\rho'$

$$\omega = \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_{\theta m} - \tilde{u}_\theta d\rho' \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho \tilde{u}_{\phi m} - u_\phi d\rho' \right) \quad (277)$$

Letting  $\tilde{u}_{\theta d} = \tilde{u}_\theta - \tilde{u}_{\theta m}$  and  $\tilde{u}_{\phi d} = \tilde{u}_\phi - \tilde{u}_{\phi m}$ , then

$$\omega = -\frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_{\theta d} d\rho' \right) - \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho \tilde{u}_{\phi d} d\rho' \right) \quad (278)$$

$$(279)$$

Thus the mapped momentum balance equation is

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h\tilde{u}_\theta^2 + \frac{1}{2}gh^2e_r \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h\tilde{u}_\theta\tilde{u}_\phi) \quad (280)$$

$$+ \frac{\partial}{\partial \rho} (\tilde{u}_\theta\omega) + 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r_0} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi \cot(\phi)}{r_0} = -\frac{ghe_r}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + ghe_\theta \quad (281)$$

Now we can do the same for the  $\phi$  momentum equation.

$$\frac{\partial u_\phi}{\partial t} + \frac{\partial}{\partial r}(u_r u_\phi) + \frac{1}{r \sin(\phi)} \frac{\partial}{\partial \theta} (u_\theta u_\phi) + \frac{1}{r} \frac{\partial}{\partial \phi} (u_\phi^2) + 3\frac{u_r u_\phi}{r} + \frac{(u_\phi^2 - u_\theta^2) \cot(\phi)}{r} = -\frac{1}{\rho_0 r} \frac{\partial p}{\partial \phi} + ge_\phi \quad (282)$$

Multiply by  $h$

$$h \frac{\partial u_\phi}{\partial t} + h \frac{\partial}{\partial r}(u_r u_\phi) + \frac{h}{r \sin(\phi)} \frac{\partial}{\partial \theta} (u_\theta u_\phi) + \frac{h}{r} \frac{\partial}{\partial \phi} (u_\phi^2) + 3\frac{h u_r u_\phi}{r} + h \frac{(u_\phi^2 - u_\theta^2) \cot(\phi)}{r} = -\frac{h}{\rho_0 r} \frac{\partial p}{\partial \phi} + ghe_\phi \quad (283)$$



Transform from  $r$  to  $\rho$

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\phi \right) + \frac{\partial}{\partial \rho}(\tilde{u}_r\tilde{u}_\phi) + \frac{1}{r\sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi \right) \right) \quad (284)$$

$$+ \frac{1}{r} \left( \frac{\partial}{\partial \phi}(h\tilde{u}_\phi^2) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi^2 \right) \right) + 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r} + h\frac{(\tilde{u}_\phi^2 - \tilde{u}_\theta^2)\cot(\phi)}{r} \quad (285)$$

$$= -\frac{1}{\rho_0 r} \left( \frac{\partial}{\partial \phi}(h\tilde{p}) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{p} \right) \right) + ghe_\phi \quad (286)$$

Substitute for transformed pressure

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial t}(h\rho + h_b)\tilde{u}_\phi \right) + \frac{\partial}{\partial \rho}(\tilde{u}_r\tilde{u}_\phi) + \frac{1}{r\sin(\phi)} \left( \frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta\tilde{u}_\phi \right) \right) \quad (287)$$

$$+ \frac{1}{r} \left( \frac{\partial}{\partial \phi}(h\tilde{u}_\phi^2) - \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi^2 \right) \right) + 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r} + h\frac{(\tilde{u}_\phi^2 - \tilde{u}_\theta^2)\cot(\phi)}{r} \quad (288)$$

$$= -\frac{1}{r} \left( ge_r \left( \frac{1}{2} \frac{\partial}{\partial \phi}(h^2) + h \frac{\partial h_b}{\partial \phi} \right) \right) + ghe_\phi \quad (289)$$

Gather derivatives

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2 \right) \quad (290)$$

$$+ \frac{\partial}{\partial \rho} \left( \tilde{u}_\phi \left( \tilde{u}_r - \frac{\partial}{\partial t}(h\rho + h_b) - \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta}(h\rho + h_b)\tilde{u}_\theta - \frac{1}{r_0} \frac{\partial}{\partial \phi}(h\rho + h_b)\tilde{u}_\phi \right) \right) \quad (291)$$

$$+ 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{(\tilde{u}_\phi^2 - \tilde{u}_\theta^2)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0} \frac{\partial h_b}{\partial \phi} + ghe_\phi \quad (292)$$

Substituting for  $\omega$  gives

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2 \right) + \frac{\partial}{\partial \rho}(\tilde{u}_\phi\omega) \quad (293)$$

$$+ 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{(\tilde{u}_\phi^2 - \tilde{u}_\theta^2)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0} \frac{\partial h_b}{\partial \phi} + ghe_\phi \quad (294)$$

The complete vertically resolved system is thus

$$\frac{\partial h}{\partial t} + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_{\theta m}) + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \phi}(h\sin(\phi)\tilde{u}_{\phi m}) = 0 \quad (295)$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\theta) + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta} \left( h\tilde{u}_\theta^2 + \frac{1}{2}gh^2e_r \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{\partial}{\partial \rho}(\tilde{u}_\theta\omega) \quad (296)$$

$$+ 3\frac{h\tilde{u}_r\tilde{u}_\theta}{r_0} + 2\frac{h\tilde{u}_\theta\tilde{u}_\phi\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0\sin(\phi)} \frac{\partial h_b}{\partial \theta} + ghe_\theta \quad (297)$$

$$\frac{\partial}{\partial t}(h\tilde{u}_\phi) + \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta}(h\tilde{u}_\theta\tilde{u}_\phi) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h\tilde{u}_\phi^2 + \frac{1}{2}ge_r h^2 \right) + \frac{\partial}{\partial \rho}(\tilde{u}_\phi\omega) \quad (298)$$

$$+ 3\frac{h\tilde{u}_r\tilde{u}_\phi}{r_0} + h\frac{(\tilde{u}_\phi^2 - \tilde{u}_\theta^2)\cot(\phi)}{r_0} = -\frac{ghe_r}{r_0} \frac{\partial h_b}{\partial \phi} + ghe_\phi \quad (299)$$

Where

$$\omega = -\frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho \tilde{u}_{\theta d} d\rho' \right) - \frac{1}{r_0\sin(\phi)} \frac{\partial}{\partial \phi} \left( h\sin(\phi) \int_0^\rho \tilde{u}_{\phi d} d\rho' \right) \quad (300)$$

$$(301)$$

From now on we will drop the tilde for readability purposes. Now we can express the velocities as polynomial expansions

$$u_\theta(t, \theta, \phi, \rho) = u_{\theta m}(t, \theta, \phi, \rho) + u_{\theta d}(t, \theta, \phi, \rho) = u_{\theta m}(t, \theta, \phi, \rho) + \sum_{j=1}^N (\alpha_j(t, \theta, \phi)) \psi_j(\rho) \quad (302)$$

$$u_\phi(t, \theta, \phi, \rho) = u_{\phi m}(t, \theta, \phi, \rho) + u_{\phi d}(t, \theta, \phi, \rho) = u_{\phi m}(t, \theta, \phi, \rho) + \sum_{j=1}^N (\alpha_j(t, \theta, \phi)) \psi_j(\rho) \quad (303)$$

Where

$$\int_0^1 \psi_j(\rho) \psi_j(\rho) d\rho = \frac{1}{2j+1} \quad \int_0^1 \psi_i(\rho) \psi_j(\rho) d\rho = 0 \text{ for } i \neq j \quad \psi(0) = 1 \quad (304)$$

Some key quantities are

$$\int_0^1 u_\theta^2 d\rho = u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\alpha_j^2}{2j+1} \right) \quad (305)$$

$$\int_0^1 u_\phi^2 d\rho = u_{\phi m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2}{2j+1} \right) \quad (306)$$

$$\int_0^1 u_\theta u_\phi d\rho = u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \quad (307)$$

Now we depth average

$$\frac{\partial}{\partial t} \left( h \int_0^1 u_\theta d\rho \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 u_\theta^2 d\rho + \frac{1}{2} g h^2 e_r \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \int_0^1 u_\theta u_\phi d\rho \right) + \int_0^1 \frac{\partial}{\partial \rho} (u_\theta \omega) d\rho \quad (308)$$

$$+ 2 \frac{h \int_0^1 u_\theta u_\phi d\rho \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + g h e_\theta \quad (309)$$

$$\frac{\partial}{\partial t} (h u_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\alpha_j^2}{2j+1} \right) \right) + \frac{1}{2} g h^2 e_r \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) \quad (310)$$

$$+ u_\theta \omega|_{\rho=0} + 2 \frac{h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + g h e_\theta \quad (311)$$

Since  $\omega$  vanishes at  $\rho = 0$  and  $\rho = 1$

$$\frac{\partial}{\partial t} (h u_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\alpha_j^2}{2j+1} \right) \right) + \frac{1}{2} g h^2 e_r \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) \quad (312)$$

$$+ 2 \frac{h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + g h e_\theta \quad (313)$$

$$\frac{\partial}{\partial t} \left( h \int_0^1 u_\phi \, d\rho \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 u_\theta u_\phi \, d\rho \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \int_0^1 u_\phi^2 \, d\rho + \frac{1}{2} g e_r h^2 \right) + \int_0^1 \frac{\partial}{\partial \rho} (u_\phi \omega) \, d\rho \quad (314)$$

$$+ h \frac{\int_0^1 u_\phi^2 - u_\theta^2 \, d\rho \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0} \frac{\partial h_b}{\partial \phi} + g h e_\phi \quad (315)$$

$$\frac{\partial}{\partial t} (h u_{\phi m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\phi m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2}{2j+1} \right) \right) + \frac{1}{2} g e_r h^2 \right) \quad (316)$$

$$+ u_\phi \omega|_{\rho=0} + h \frac{\left( u_{\phi m}^2 - u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2 - \alpha_j^2}{2j+1} \right) \right) \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0} \frac{\partial h_b}{\partial \phi} + g h e_\phi \quad (317)$$

Since  $\omega$  vanishes at  $\rho = 0$  and  $\rho = 1$

$$\frac{\partial}{\partial t} (h u_{\phi m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\phi m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2}{2j+1} \right) \right) + \frac{1}{2} g e_r h^2 \right) \quad (318)$$

$$+ h \frac{\left( u_{\phi m}^2 - u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2 - \alpha_j^2}{2j+1} \right) \right) \cot(\phi)}{r_0} = - \frac{g h e_r}{r_0} \frac{\partial h_b}{\partial \phi} + g h e_\phi \quad (319)$$

In order to find the equations for the higher moments we multiply the conservation of momentum equations by a polynomial basis function and depth average.

$$\frac{\partial}{\partial t} \left( h \int_0^1 u_\theta \psi_i(\rho) \, d\rho \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 u_\theta^2 \psi_i(\rho) \, d\rho \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \int_0^1 u_\theta u_\phi \psi_i(\rho) \, d\rho \right) \quad (320)$$

$$+ \int_0^1 \frac{\partial}{\partial \rho} (u_\theta \omega) \psi_i(\rho) \, d\rho + 2 \frac{h \int_0^1 u_\theta u_\phi \psi_i(\rho) \, d\rho \cot(\phi)}{r_0} = 0 \quad (321)$$

$$\int_0^1 u_\theta \psi_i(\rho) \, d\rho = \frac{\alpha_i}{2i+1} \quad (322)$$

$$\int_0^1 u_\theta^2 \psi_i(\rho) \, d\rho = \int_0^1 u_{\theta m}^2 \psi_i(\rho) \, d\rho + 2 u_{\theta m} \sum_{j=1}^N (\alpha_j \psi_j(\rho)) \psi_i(\rho) + \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \alpha_k \psi_j(\rho) \psi_k(\rho)) \right) \psi_i(\rho) \, d\rho \quad (323)$$

$$= 2 u_{\theta m} \sum_{j=1}^N \left( \alpha_j \int_0^1 \psi_i(\rho) \psi_j(\rho) \, d\rho \right) + \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \alpha_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) \, d\rho \right) \right) \quad (324)$$

$$= 2 u_{\theta m} \frac{\alpha_i}{2i+1} + \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \alpha_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) \, d\rho \right) \right) \quad (325)$$

$$\int_0^1 u_\theta u_\phi \psi_i(\rho) \, d\rho = \int_0^1 u_{\theta m} u_{\phi m} \psi_i(\rho) \, d\rho \quad (326)$$

$$+ \int_0^1 u_{\theta m} \sum_{j=1}^N (\beta_j \psi_i(\rho) \psi_j(\rho)) + u_{\phi m} \sum_{j=1}^N (\alpha_j \psi_i(\rho) \psi_j(\rho)) + \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \beta_k \psi_i(\rho) \psi_j(\rho) \psi_k(\rho)) \right) \, d\rho \quad (327)$$

$$= u_{\theta m} \frac{\beta_i}{2i+1} + u_{\phi m} \frac{\alpha_i}{2i+1} + \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \beta_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) \, d\rho \right) \right) \quad (328)$$

$$\int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_\theta \omega) \, d\rho = \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} ((u_{\theta m} + u_{\theta d}) \omega) \, d\rho \quad (329)$$

$$= \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_{\theta m} \omega) \, d\rho + \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_{\theta d} \omega) \, d\rho \quad (330)$$

Since  $\omega = 0$  at  $\rho = 0$  and  $\rho = 1$

$$= \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_{\theta m} \omega) d\rho - \int_0^1 \psi'_i(\rho) u_{\theta d} \omega d\rho \quad (331)$$

$$= - \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} \left( u_{\theta m} \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) \right) \right) d\rho \quad (332)$$

$$+ \int_0^1 \psi'_i(\rho) u_{\theta d} \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) \right) d\rho \quad (333)$$

$$= - \int_0^1 \psi_i(\rho) u_{\theta m} \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \frac{\partial}{\partial \rho} \left( \int_0^\rho u_{\theta d} d\rho' \right) \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\partial}{\partial \rho} \left( \int_0^\rho u_{\phi d} d\rho' \right) \right) \right) d\rho \quad (334)$$

$$+ \int_0^1 \frac{\psi'_i(\rho) u_{\theta d}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{\psi'_i(\rho) u_{\theta d}}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) d\rho \quad (335)$$

$$= -u_{\theta m} \int_0^1 \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \psi_i(\rho) u_{\theta d}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} (h \sin(\phi) \psi_i(\rho) u_{\phi d}) \right) d\rho \quad (336)$$

$$+ \int_0^1 \frac{\psi'_i(\rho) u_{\theta d}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{\psi'_i(\rho) u_{\theta d}}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) d\rho \quad (337)$$

$$= -\frac{u_{\theta m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} \left( h \frac{\alpha_i}{2i+1} \right) + \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\beta_i}{2i+1} \right) \right) \quad (338)$$

$$+ \frac{1}{r_0 \sin(\phi)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \int_0^1 \psi'_i(\rho) \psi_j(\rho) \int_0^\rho \psi_k(\rho) d\rho' d\rho \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (339)$$

$$\frac{1}{2i+1} \frac{\partial}{\partial t} (h \alpha_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( 2h u_{\theta m} \frac{\alpha_i}{2i+1} + h \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \alpha_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) d\rho \right) \right) \right) \quad (340)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h u_{\theta m} \frac{\beta_i}{2i+1} + h u_{\phi m} \frac{\alpha_i}{2i+1} + h \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \beta_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) d\rho \right) \right) \right) \quad (341)$$

$$- \frac{u_{\theta m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} \left( h \frac{\alpha_i}{2i+1} \right) + \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\beta_i}{2i+1} \right) \right) \quad (342)$$

$$+ \frac{1}{r_0 \sin(\phi)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \int_0^1 \psi'_i(\rho) \psi_j(\rho) \int_0^\rho \psi_k(\rho) d\rho' d\rho \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (343)$$

$$+ \frac{2 \cot(\phi)}{r_0} \left( h u_{\theta m} \frac{\beta_i}{2i+1} + h u_{\phi m} \frac{\alpha_i}{2i+1} + h \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j \beta_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) d\rho \right) \right) \right) = 0 \quad (344)$$

Or

$$\frac{\partial}{\partial t} (h \alpha_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( 2h u_{\theta m} \alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \alpha_k A_{ijk}) \right) \right) \quad (345)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \beta_k A_{ijk}) \right) \right) \quad (346)$$

$$- \frac{u_{\theta m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \alpha_i) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_i) \right) \quad (347)$$

$$+ \frac{1}{r_0 \sin(\phi)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j B_{ijk} \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (348)$$

$$+ \frac{2 \cot(\phi)}{r_0} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \beta_k A_{ijk}) \right) \right) = 0 \quad (349)$$

Or

$$\frac{\partial}{\partial t}(h\alpha_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( 2hu_{\theta m}\alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \alpha_k A_{ijk}) \right) \right) \quad (350)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( hu_{\theta m}\beta_i + hu_{\phi m}\alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \beta_k A_{ijk}) \right) \right) \quad (351)$$

$$- \frac{u_{\theta m}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h\alpha_i) - \frac{u_{\theta m}}{r_0} \frac{\partial}{\partial \phi} (h\beta_i) + \frac{1}{r_0} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j B_{ijk} \left( \frac{1}{\sin(\phi)} \frac{\partial}{\partial \theta} (h\alpha_k) + \frac{\partial}{\partial \phi} (h\beta_k) \right) \right) \right) \quad (352)$$

$$= -\frac{h \cot(\phi)}{r_0} \left( u_{\theta m}\beta_i + 2u_{\phi m}\alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N ((2A_{ijk} + B_{ijk})\alpha_j \beta_k) \right) \right) \quad (353)$$

Where

$$A_{ijk} = (2i+1) \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) d\rho \quad (354)$$

$$B_{ijk} = (2i+1) \int_0^1 \psi'_i(\rho) \psi_j(\rho) \int_0^\rho \psi_k(\rho) d\rho' d\rho \quad (355)$$

Beta Evolution Equation

$$\frac{\partial}{\partial t} \left( h \int_0^1 u_\phi \psi_i(\rho) d\rho \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^1 u_\theta u_\phi \psi_i(\rho) d\rho \right) + \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \int_0^1 u_\phi^2 \psi_i(\rho) d\rho \right) + \int_0^1 \frac{\partial}{\partial \rho} (u_\phi \omega) \psi_i(\rho) d\rho \quad (356)$$

$$+ h \frac{\int_0^1 (u_\phi^2 - u_\theta^2) \psi_i(\rho) d\rho \cot(\phi)}{r_0} = 0 \quad (357)$$

$$\int_0^1 u_\phi \psi_i(\rho) d\rho = \frac{\beta_i}{2i+1} \quad (358)$$

$$\int_0^1 u_\phi^2 \psi_i(\rho) d\rho = \int_0^1 u_{\phi m}^2 \psi_i(\rho) + 2u_{\phi m} \sum_{j=1}^N (\beta_j \psi_i(\rho) \psi_j(\rho)) + \sum_{j=1}^N \left( \sum_{k=1}^N (\beta_j \beta_k \psi_i(\rho) \psi_j(\rho) \psi_k(\rho)) \right) d\rho \quad (359)$$

$$= 2u_{\phi m} \frac{\beta_i}{2i+1} + \sum_{j=1}^N \left( \sum_{k=1}^N \left( \beta_j \beta_k \int_0^1 \psi_i(\rho) \psi_j(\rho) \psi_k(\rho) d\rho \right) \right) \quad (360)$$

$$= 2u_{\phi m} \frac{\beta_i}{2i+1} + \frac{1}{2i+1} \sum_{j=1}^N \left( \sum_{k=1}^N (\beta_j \beta_k A_{ijk}) \right) \quad (361)$$

$$\int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_\phi \omega) d\rho = \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} ((u_{\phi m} + u_{\phi d}) \omega) d\rho \quad (362)$$

$$= u_{\phi m} \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (\omega) d\rho + \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (u_{\phi d} \omega) d\rho \quad (363)$$

$$= u_{\phi m} \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} (\omega) d\rho - \int_0^1 \psi'_i(\rho) u_{\phi d} \omega d\rho \quad (364)$$

$$= -u_{\phi m} \int_0^1 \psi_i(\rho) \frac{\partial}{\partial \rho} \left( \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) \right) \right) d\rho \quad (365)$$

$$+ \int_0^1 \psi'_i(\rho) u_{\phi d} \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \int_0^\rho u_{\theta d} d\rho' \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \phi} \left( h \sin(\phi) \int_0^\rho u_{\phi d} d\rho' \right) \right) d\rho \quad (366)$$

$$= -\frac{u_{\phi m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} \left( h \frac{\alpha_i}{2i+1} \right) + \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\beta_i}{2i+1} \right) \right) \quad (367)$$

$$+ \frac{1}{r_0 \sin(\phi)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \beta_j \int_0^1 \psi'_i(\rho) \psi_j(\rho) \int_0^\rho \psi_k(\rho) d\rho' d\rho \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (368)$$

$$= -\frac{u_{\phi m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} \left( h \frac{\alpha_i}{2i+1} \right) + \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\beta_i}{2i+1} \right) \right) \quad (369)$$

$$+ \frac{1}{r_0 \sin(\phi)(2i+1)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( B_{ijk} \beta_j \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (370)$$

$$\frac{\partial}{\partial t} \left( h \frac{\beta_i}{2i+1} \right) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h u_{\theta m} \frac{\beta_i}{2i+1} + h u_{\phi m} \frac{\alpha_i}{2i+1} + \frac{1}{2i+1} \sum_{j=1}^N \left( \sum_{k=1}^N (h \alpha_j \beta_k A_{ijk}) \right) \right) \quad (371)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( 2 h u_{\phi m} \frac{\beta_i}{2i+1} + \frac{1}{2i+1} \sum_{j=1}^N \left( \sum_{k=1}^N (h \beta_j \beta_k A_{ijk}) \right) \right) \quad (372)$$

$$- \frac{u_{\phi m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} \left( h \frac{\alpha_i}{2i+1} \right) + \frac{\partial}{\partial \phi} \left( h \sin(\phi) \frac{\beta_i}{2i+1} \right) \right) \quad (373)$$

$$+ \frac{1}{r_0 \sin(\phi) (2i+1)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( B_{ijk} \beta_j \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (374)$$

$$+ \frac{h \cot(\phi)}{r_0} \left( 2 u_{\phi m} \frac{\beta_i}{2i+1} - 2 u_{\theta m} \frac{\alpha_i}{2i+1} + \frac{1}{2i+1} \sum_{j=1}^N \left( \sum_{k=1}^N (A_{ijk} (\beta_j \beta_k - \alpha_j \alpha_k)) \right) \right) = 0 \quad (375)$$

$$\frac{\partial}{\partial t} (h \beta_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \alpha_j \beta_k A_{ijk}) \right) \right) \quad (376)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( 2 h u_{\phi m} \beta_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \beta_j \beta_k A_{ijk}) \right) \right) \quad (377)$$

$$- \frac{u_{\phi m}}{r_0 \sin(\phi)} \left( \frac{\partial}{\partial \theta} (h \alpha_i) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_i) \right) \quad (378)$$

$$+ \frac{1}{r_0 \sin(\phi)} \sum_{j=1}^N \left( \sum_{k=1}^N \left( B_{ijk} \beta_j \left( \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \sin(\phi) \beta_k) \right) \right) \right) \quad (379)$$

$$+ \frac{h \cot(\phi)}{r_0} \left( 2 u_{\phi m} \beta_i - 2 u_{\theta m} \alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N (A_{ijk} (\beta_j \beta_k - \alpha_j \alpha_k)) \right) \right) = 0 \quad (380)$$

$$\frac{\partial}{\partial t} (h \beta_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \alpha_j \beta_k A_{ijk}) \right) \right) \quad (381)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( 2 h u_{\phi m} \beta_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \beta_j \beta_k A_{ijk}) \right) \right) \quad (382)$$

$$- \frac{u_{\phi m}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_i) - \frac{u_{\phi m}}{r_0} \frac{\partial}{\partial \phi} (h \beta_i) + \sum_{j=1}^N \left( \sum_{k=1}^N \left( B_{ijk} \beta_j \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h \beta_k) \right) \right) \right) \quad (383)$$

$$= \frac{h \cot(\phi)}{r_0} \left( 2 u_{\theta m} \alpha_i - u_{\phi m} \beta_i + \sum_{j=1}^N \left( \sum_{k=1}^N (A_{ijk} (\alpha_j \alpha_k - \beta_j \beta_k) - B_{ijk} \beta_j \beta_k) \right) \right) \quad (384)$$

$$(385)$$

$$\frac{\partial h}{\partial t} + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h u_{\theta m}) + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h u_{\phi m}) = -\frac{h u_{\phi m} \cot(\phi)}{r_0} \quad (386)$$

$$\frac{\partial}{\partial t} (h u_{\theta m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\alpha_j^2}{2j+1} \right) \right) + \frac{1}{2} g h^2 e_r \right) \quad (387)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) \quad (388)$$

$$= -\frac{2h \cot(\phi)}{r_0} \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) - \frac{g h e_r}{r_0 \sin(\phi)} \frac{\partial h_b}{\partial \theta} + g h e_{\theta} \quad (389)$$

$$\frac{\partial}{\partial t} (h u_{\phi m}) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h \left( u_{\theta m} u_{\phi m} + \sum_{j=1}^N \left( \frac{\alpha_j \beta_j}{2j+1} \right) \right) \right) \quad (390)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h \left( u_{\phi m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2}{2j+1} \right) \right) + \frac{1}{2} g e_r h^2 \right) \quad (391)$$

$$= -\frac{h \cot(\phi)}{r_0} \left( u_{\phi m}^2 - u_{\theta m}^2 + \sum_{j=1}^N \left( \frac{\beta_j^2 - \alpha_j^2}{2j+1} \right) \right) - \frac{g h e_r}{r_0} \frac{\partial h_b}{\partial \phi} + g h e_{\phi} \quad (392)$$

$$\frac{\partial}{\partial t} (h \alpha_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( 2h u_{\theta m} \alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \alpha_k A_{ijk}) \right) \right) \quad (393)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + h \sum_{j=1}^N \left( \sum_{k=1}^N (\alpha_j \beta_k A_{ijk}) \right) \right) \quad (394)$$

$$- \frac{u_{\theta m}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_i) - \frac{u_{\theta m}}{r_0} \frac{\partial}{\partial \phi} (h \beta_i) + \frac{1}{r_0} \sum_{j=1}^N \left( \sum_{k=1}^N \left( \alpha_j B_{ijk} \left( \frac{1}{\sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{\partial}{\partial \phi} (h \beta_k) \right) \right) \right) \quad (395)$$

$$= -\frac{h \cot(\phi)}{r_0} \left( u_{\theta m} \beta_i + 2u_{\phi m} \alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N ((2A_{ijk} + B_{ijk}) \alpha_j \beta_k) \right) \right) \quad (396)$$

$$\frac{\partial}{\partial t} (h \beta_i) + \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} \left( h u_{\theta m} \beta_i + h u_{\phi m} \alpha_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \alpha_j \beta_k A_{ijk}) \right) \right) \quad (397)$$

$$+ \frac{1}{r_0} \frac{\partial}{\partial \phi} \left( 2h u_{\phi m} \beta_i + \sum_{j=1}^N \left( \sum_{k=1}^N (h \beta_j \beta_k A_{ijk}) \right) \right) \quad (398)$$

$$- \frac{u_{\phi m}}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_i) - \frac{u_{\phi m}}{r_0} \frac{\partial}{\partial \phi} (h \beta_i) + \sum_{j=1}^N \left( \sum_{k=1}^N \left( B_{ijk} \beta_j \left( \frac{1}{r_0 \sin(\phi)} \frac{\partial}{\partial \theta} (h \alpha_k) + \frac{1}{r_0} \frac{\partial}{\partial \phi} (h \beta_k) \right) \right) \right) \quad (399)$$

$$= \frac{h \cot(\phi)}{r_0} \left( 2u_{\theta m} \alpha_i - u_{\phi m} \beta_i + \sum_{j=1}^N \left( \sum_{k=1}^N (A_{ijk} (\alpha_j \alpha_k - \beta_j \beta_k) - B_{ijk} \beta_j \beta_k) \right) \right) \quad (400)$$