Local Discontinuous Galerkin Method for Solving Thin Film Equations

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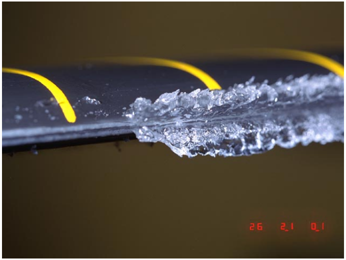
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating

Model Equations

- Navier-Stokes Equation

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$\partial_t h_s + (u, v)^T \cdot \nabla h_s = w$$

$$\partial_t h_b + (u, v)^T \cdot \nabla h_b = w$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation - 1D with q as fluid height.

$$q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x$$

Method Overview

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Runge Kutta Implicit Explicit (IMEX)

$$q_t = F(q) + G(q)$$

  - F evaluated explicitly
  - G solved implicitly

$$F(q) = -\left(q^2 - q^3\right)_x$$
$$G(q) = \left(q^3 q_{xxx}\right)_x$$

Convection

- Convection Equation

$$F(q) = f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} \left(F(q)v - f(q)v_x\right) \mathrm{d}x + \hat{f}v\Big|_{\partial\Omega} = 0$$

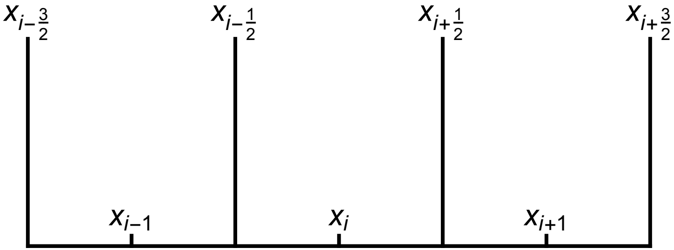
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{v \in L^1(\Omega) : v\big|_{I_j} \in P^k(I_j)\right\}$

$$\int_{I_j} F(Q) v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2})\right)$$

  for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{\left|f'(q)\right|\right\}\left(Q^-_{j+1/2} - Q^+_{j+1/2}\right)$$

Diffusion

- Diffusion Equation

$$G(q) = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Local Discontinuous Galerkin

$$\begin{aligned} r &= q_x \\ s &= r_x \\ u &= s_x \\ G(q) &= \left(q^3 u\right)_x \end{aligned}$$

Local Discontinuous Galerkin

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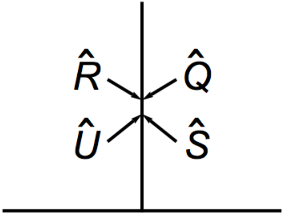
Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t,\cdot), R, S, U \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$
\begin{aligned}
\int_{I_j} R v \, \mathrm{d}x &= -\int_{I_j} Q v_x \, \mathrm{d}x + \left(\hat{Q}_{j+1/2} v_{j+1/2}^- - \hat{Q}_{j-1/2} v_{j-1/2}^+ \right) \\
\int_{I_j} S w \, \mathrm{d}x &= -\int_{I_j} R w_x \, \mathrm{d}x + \left(\hat{R}_{j+1/2} w_{j+1/2}^- - \hat{R}_{j-1/2} w_{j-1/2}^+ \right) \\
\int_{I_j} U y \, \mathrm{d}x &= -\int_{I_j} S y_x \, \mathrm{d}x + \left(\hat{S}_{j+1/2} y_{j+1/2}^- - \hat{S}_{j-1/2} y_{j-1/2}^+ \right) \\
\int_{I_j} G(Q) z \, \mathrm{d}x &= -\int_{I_j} Q^3 U z_x \, \mathrm{d}x + \left(\hat{U}_{j+1/2} z_{j+1/2}^- - \hat{U}_{j-1/2} z_{j-1/2}^+ \right)
\end{aligned}
$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$
$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$
$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$
$$\hat{U}_{j+1/2} = \left(Q^3 U\right)^{-}_{j+1/2}$$



IMEX Runge Kutta

- IMEX scheme

$$q^{n+1} = q^n + \Delta t \sum_{i=1}^{s} \left(b_i' F(t_i, u_i)\right) + \Delta t \sum_{i=1}^{s} \left(b_i G(t_i, u_i)\right)$$

$$u_i = q^n + \Delta t \sum_{j=1}^{i-1} \left(a_{ij}' F(t_j, u_j)\right) + \Delta t \sum_{j=1}^{i} \left(a_{ij} G(t_j, u_j)\right)$$

$$t_i = t^n + c_i \Delta t$$

- Double Butcher Tableaus

$$\begin{array}{c|c} c' & a' \\ \hline & b'^T \end{array} \quad \begin{array}{c|c} c & a \\ \hline & b^T \end{array}$$

- 1st Order - L-Stable SSP

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

- 2nd Order - SSP

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

- 3rd Order - L-Stable SSP

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|cccc}
\alpha & \alpha & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 \\
\frac{1}{2} & \beta & \eta & \zeta & \alpha \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

$$
\begin{aligned}
\alpha &= 0.24169426078821 \\
\beta &= 0.06042356519705 \\
\eta &= 0.1291528696059 \\
\zeta &= \frac{1}{2} - \beta - \eta - \alpha
\end{aligned}
$$

Nonlinear Solvers

- Nonlinear System

$$u_i - a_{ii}\Delta t G(u_i) = b$$

- Picard Iteration

$$\tilde{G}(q, u) = (q^3 u_{xxx})_x$$

$$u_0 = q^n \qquad u_i^0 = u_{i-1}$$
$$u_i^j - a_{ii}\Delta t \tilde{G}(u_i^{j-1}, u_i^j) = b$$

- Number of picard iterations equals order in time and space

Manufactured Solution

$$q_t + (q^2 - q^3)_x = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = 0.1 * \sin(2\pi(x - t)) + 0.15$$

1st Order IMEX		
N	error	order
50	0.0278	-
100	0.0144	0.955
200	0.0072	0.988

Manufactured Solution

$$q_t + (q^2 - q^3)_x = -\left(q^3 q_{xxx}\right)_x + s(x,t)$$
$$q(x,t) = 0.1 * \sin(2\pi(x-t)) + 0.15$$

2nd Order IMEX		
N	error	order
20	0.00265	-
40	0.000689	1.94
80	0.000184	1.91

Manufactured Solution

$$q_t + (q^2 - q^3)_x = -\left(q^3 q_{xxx}\right)_x + s(x,t)$$
$$q(x,t) = 0.1 * \sin(2\pi(x-t)) + 0.15$$

3rd Order IMEX		
N	error	order
20	6.57×10^{-5}	-
40	8.35×10^{-6}	2.97
80	1.07×10^{-6}	2.96

Observations

- CFL Restrictions
- 1st Order - 0.15
- 2nd Order - 0.05
- 3rd Order - 0.01
- Linearized Problem

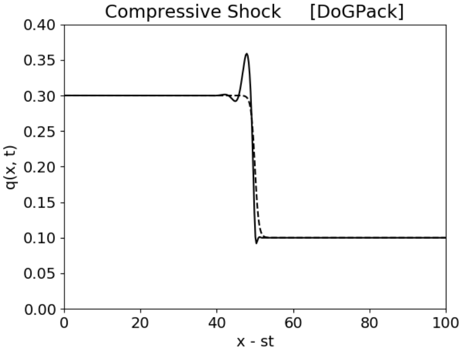
$$q_t + (q^2 - q^3)_x = -(f(x,t)q_x xx)_x + s(x,t)$$
$$f(x,t) = (\tilde{q}(x,t))^3$$

- Same CFL restrictions

Wave Structure with Nonlinear Hyper Diffusion

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

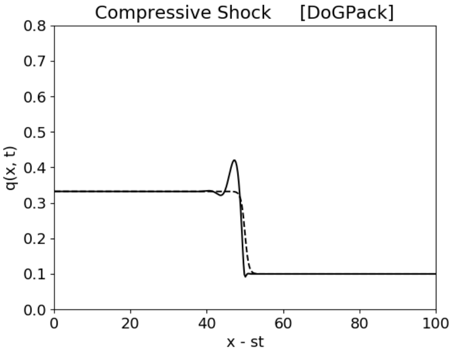
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

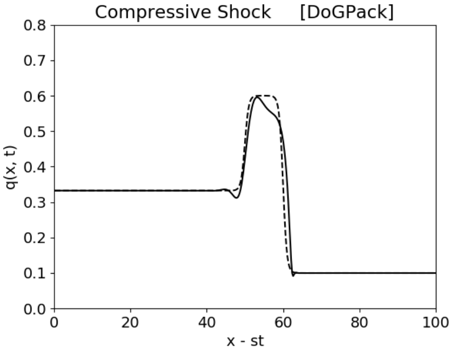
$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

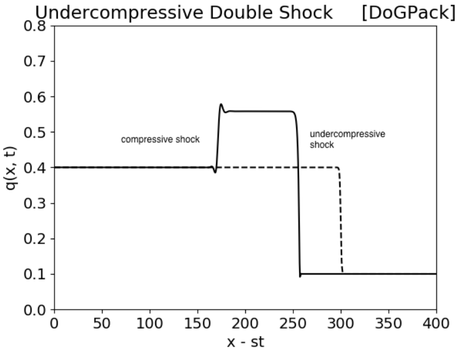
$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2}\tanh(x - 50) + \frac{q_m + q_l}{2} & x < 55 \\ -\frac{q_m - q_r}{2}\tanh(x - 60) + \frac{q_m + q_r}{2} + q_r & x > 55 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.4$$

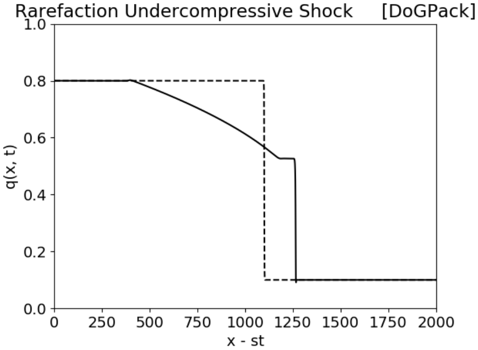
$$q(x,0) = (-\tanh(x-300)+1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x, 0) = (-\tanh(x - 1100) + 1)\frac{q_l - q_r}{2} + q_r$$



Conclusion

Observations

- Expensive computations
- Nonlinear Hyper Diffusion has subtle instabilities

Future Work

- Hybridized Discontinuous Galerkin Method

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