Discontinuous Galerkin Method for Solving Thin Film Equations

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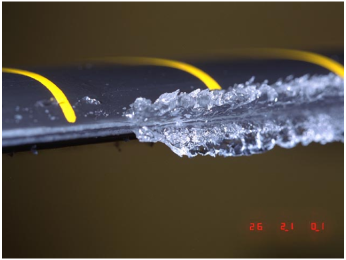
Overview

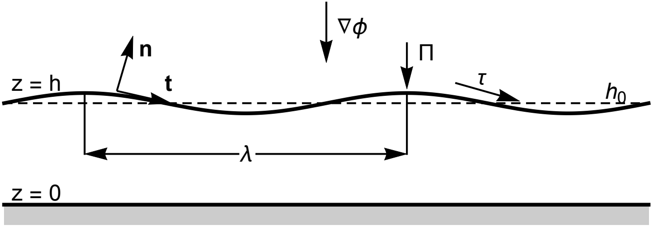
Motivation

- Aircraft Icing
- Runback



- Industrial Coating

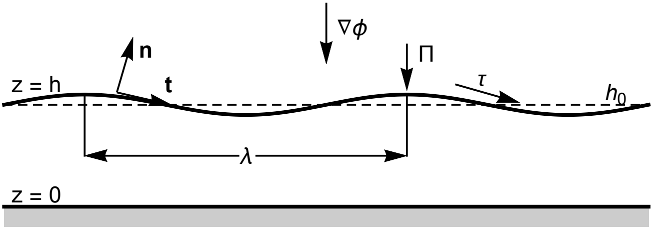
Model Equations



- Incompressible Navier-Stokes Equation

$$
\begin{aligned}
u_x + w_z &= 0 \\
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
w = 0, u &= 0 && \text{at } z = 0 \\
w &= h_t + uh_x && \text{at } z = h \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } z = h
\end{aligned}
$$

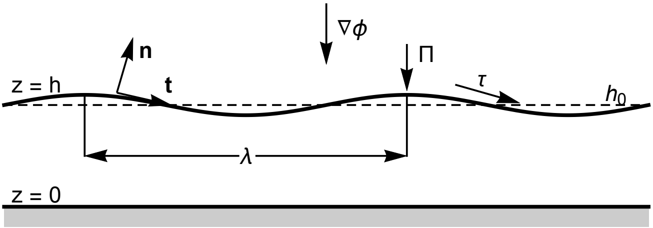
Nondimensionalization



$$\varepsilon = \frac{h_0}{\lambda} \ll 1 \qquad Z = \frac{z}{h_0} \qquad X = \frac{\varepsilon x}{h_0}$$

$$U = \frac{u}{U_0} \qquad W = \frac{w}{\varepsilon U_0} \qquad T = \frac{\varepsilon U_0 t}{h_0}$$

Nondimensionalization



$$
\begin{aligned}
U_X + W_Z &= 0 \\
\varepsilon Re(U_T + UU_X + WU_Z) &= -P_X + U_{ZZ} + \varepsilon^2 U_{XX} - \Phi_X \\
\varepsilon^3 Re(W_T + WW_X + WW_Z) &= -P_Z + \varepsilon^2\left(W_{ZZ} + \varepsilon^2 W_{XX}\right) - \Phi_Z \\
W = 0, U &= 0 && \text{at } Z = 0 \\
W &= H_T + UH_X && \text{at } Z = H \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial \sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } Z = H
\end{aligned}
$$

$$
\begin{aligned}
U_X + W_Z &= 0 \\
U_{ZZ} &= P_X + \Phi_X \\
0 &= -P_Z - \Phi_Z
\end{aligned}
$$

$$
\begin{aligned}
W &= 0 \qquad \text{at } Z = 0 \\
U &= 0
\end{aligned}
$$

$$
\begin{aligned}
W &= H_T + UH_X \qquad \text{at } Z = H \\
U_Z &= \tau_0 + \Sigma_X \\
-\Pi_0 - P &= C^{-1} H_{XX}
\end{aligned}
$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

  Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x + \hat{f} v \Big|_{\partial\Omega} = 0$$

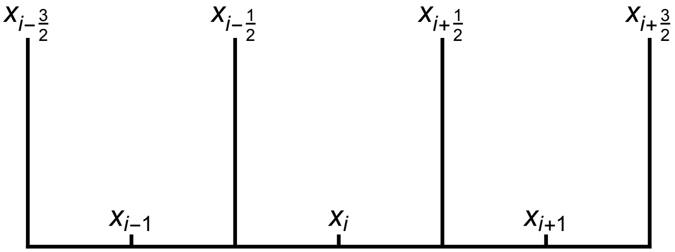
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t,x)$ such that for each time $t \in (0, T)$, $Q(t,\cdot) \in V_h = \left\{ v \in L^1(\Omega) : v\big|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \,\mathrm{d}x = \int_{I_j} f(Q) v_x \,\mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2})\right)$$

  for all $v \in V_h$
- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2}\max_q\left\{\left|f'(q)\right|\right\}\left(Q^-_{j+1/2} - Q^+_{j+1/2}\right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^\star &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^\star) + \frac{1}{2}\Delta t L(q^\star) \end{aligned}$$

Diffusion

- Diffusion Equation

$$q_t = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Linearize operator at $t = t^n$, let $f(x) = q^3(t = t^n, x)$

$$q_t = -(f(x) q_{xxx})_x \qquad (0, T) \times \Omega$$

Local Discontinuous Galerkin

Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t,\cdot), R, S, U \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Rv \, \mathrm{d}x = -\int_{I_j} Q v_x \, \mathrm{d}x + \left(\hat{Q}_{j+1/2} v^-_{j+1/2} - \hat{Q}_{j-1/2} v^+_{j-1/2} \right)$$

$$\int_{I_j} Sw \, \mathrm{d}x = -\int_{I_j} R w_x \, \mathrm{d}x + \left(\hat{R}_{j+1/2} w^-_{j+1/2} - \hat{R}_{j-1/2} w^+_{j-1/2} \right)$$

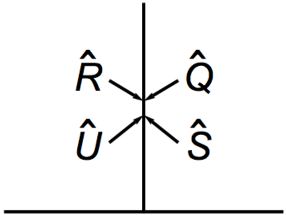
$$\begin{aligned}\int_{I_j} Uy \, \mathrm{d}x = \int_{I_j} S_x f y \, \mathrm{d}x &- \left(S^-_{j+1/2} f^-_{j+1/2} y^-_{j+1/2} - S^+_{j-1/2} f^+_{j-1/2} y^+_{j-1/2} \right) \\ &+ \left(\hat{S}_{j+1/2} \hat{f}_{j+1/2} y^-_{j+1/2} - \hat{S}_{j-1/2} \hat{f}_{j-1/2} y^+_{j-1/2} \right)\end{aligned}$$

$$\int_{I_j} Q_t z \, \mathrm{d}x = -\int_{I_j} U z_x \, \mathrm{d}x + \left(\hat{U}_{j+1/2} z^-_{j+1/2} - \hat{U}_{j-1/2} z^+_{j-1/2} \right)$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$\hat{f}_{j+1/2} = \frac{1}{2}\left(f^{+}_{j+1/2} + f^{-}_{j+1/2}\right)$$
$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$
$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$
$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$
$$\hat{U}_{j+1/2} = U^{-}_{j+1/2}$$



LDG Complications

- Explicit time step scales with h^4
- Implicit System is difficult to solve efficiently
  - GMRES iterations scale with size of system
  - Preconditioned GMRES

$$P = A_0^{-1}$$
$$PAx = Pb$$

  - Geometric Multigrid fails to converge

Finite Difference Approach

- Let cell centers, x_i, form finite difference grid.
- Finite difference space, $\mathbb{R}^N$.
- $Q_{DG} \in V_h \to Q_{FD} \in \mathbb{R}^N$

$$(Q_{FD})_i = \frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x$$

- $Q_{FD} \in \mathbb{R}^N \to Q_{DG} \in V_h$

$$Q_{DG}|_K \in P^1(K)$$

$$\frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x = (Q_{FD})_i$$

$$\partial_x Q_{DG}|_{K_i} = \frac{(Q_{FD})_{i+1} - (Q_{FD})_{i-1}}{2h}$$

Finite Difference Approximation

- First derivative approximation

$$(-(f(x)q_{xxx})_x)_i \approx -\frac{q^3_{i+1/2}(q_{xxx})_{i+1/2} - q^3_{i-1/2}(q_{xxx})_{i-1/2}}{h}$$

- Third derivative approximation

$$(q_{xxx})_{i+1/2} \approx \frac{-Q_{i-1} + 3Q_i - 3Q_{i+1} + Q_{i+2}}{h^3}$$

- Value of Q^3 at boundary

$$q^3_{i+1/2} = \left(\frac{Q_i + Q_{i+1}}{2}\right)^3$$

Implicit L-Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$q^{\star} = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^{\star}))$$
$$3q^{n+1} = 4q^{\star} - q^n + \Delta t L(q^{n+1})$$

Nonlinear Solvers

- Picard Iteration

$$L(q) = A(f \approx q^3)q$$
$$q_0^{n+1} = q^n$$

$$q_{m+1}^{n+1} = q^n + \Delta t A(q_m^{n+1})q_{m+1}^{n+1}$$

$$q_{m+1}^{\star} = q^n + \frac{1}{4}\Delta t\big(L(q^n) + A(q_m^{\star})q_{m+1}^{\star}\big)$$
$$3q_{m+1}^{n+1} = 4q^{\star} - q^n + \Delta t A(q_m^{n+1})q_{m+1}^{n+1}$$

- Newton's Method

$$q_{m+1}^{n+1} = q_m^{n+1} - J(q_m^{n+1})^{-1}F(q_m^{n+1})$$

$$F(q) = q - q^n - \Delta t L(q)$$
$$J(q) = I - \Delta t L'(q)$$

Manufactured Solution

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x,t)$$
$$q(x,t) = 0.1 * \sin(2\pi(x-t)) + 0.15$$

Backward Euler				
	1 Iteration		2 Iterations	
N	error	order	error	order
100	0.0131	—	0.0053	—
200	0.0064	1.0264	0.0026	1.0466
400	0.0033	0.96	0.0013	0.9704
800	0.0016	1.0069	0.0007	1.0134

Manufactured Solution

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = 0.1 * \sin(2\pi(x - t)) + 0.15$$

2nd Order IRK						
	1 Iteration		2 Iterations		3 Iterations	
N	error	order	error	order	error	order
50	0.0075	—	0.00047	—	0.0004901	—
100	0.0041	0.8601	0.00012	1.9844	0.0001209	2.0194
200	0.0020	1.0391	0.0000312	1.9451	0.0000305	1.9887
400	0.0010	0.9652	0.0000082	1.9244	0.0000078	1.9641

Manufactured Solution

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = \frac{2}{10} e^{-10t} e^{-300\left(x - \frac{1}{2}\right)^2} + \frac{1}{10}$$

Backward Euler				
	1 Iteration		2 Iterations	
N	error	order	error	order
100	0.0097	—	0.0933	—
200	0.0050	0.95	0.0421	1.1494
400	0.0027	0.87	3.756	-6.48
800	33.21	-13.5	16.51	-2.14

Manufactured Solution with Newton's Method

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$

$$q(x, t) = \frac{2}{10} e^{-10t} e^{-300\left(x - \frac{1}{2}\right)^2} + \frac{1}{10}$$

Backward Euler		
N	error	order
50	0.0280	—
100	0.0153	0.8765
200	0.0080	0.9249
400	5.5e75	-258

Hyperbolic Wave Structure

- Conservation Law

$$q_t + f(q)_x = 0$$

- Riemann Problem Initial Data

$$q(x,0) = \begin{cases} q_l & x < d \\ q_r & x > d \end{cases}$$

- Rankine-Hugoniot Condition

$$s = \frac{f(q_l) - f(q_r)}{q_l - q_r}$$

Convex Flux Function

- Shock Wave

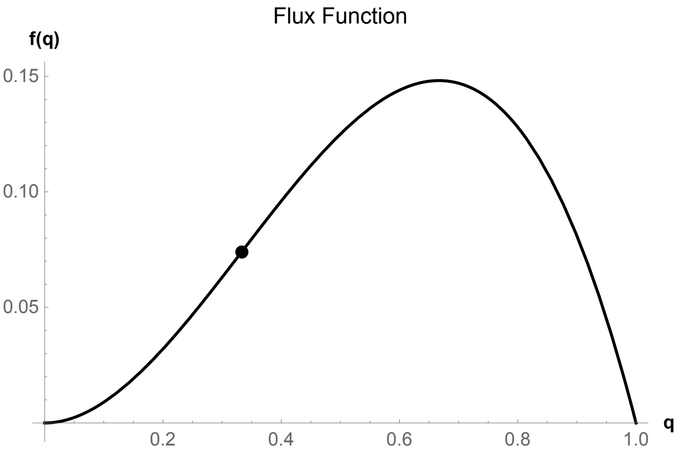
$$f'(q_l) > s > f'(q_r)$$

- Rarefaction

$$f'(q_l) < s < f'(q_r)$$

Nonconvex Flux Function

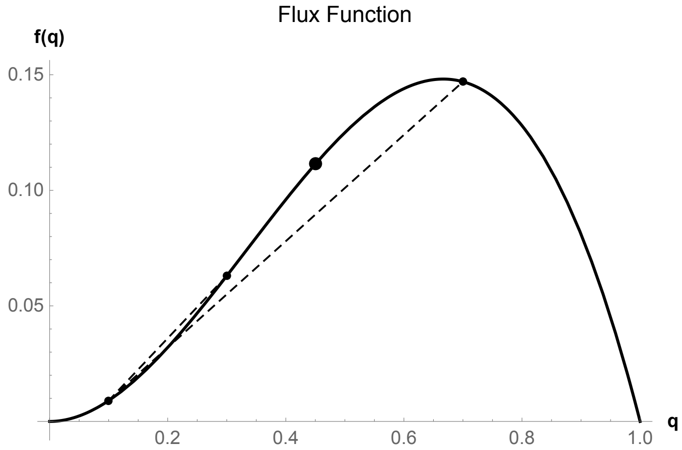
$$q_t + \left(q^2 - q^3\right)_x = 0$$



Nonconvex Flux Function

$$f'(q_b) = s$$

$$q_b = (1 - q_r)/2$$



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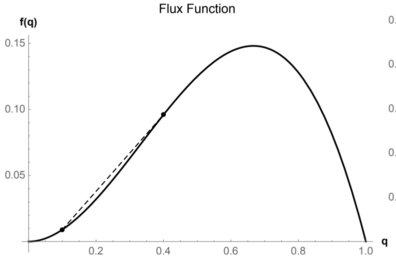
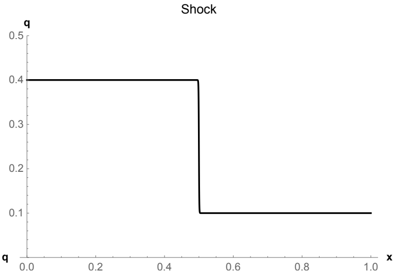
Compressive Shock

$$q_l < q_b$$

$$q(x,t) = \begin{cases} q_l & x \leq st \\ q_r & x > st \end{cases}$$

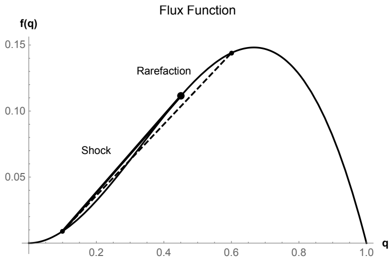


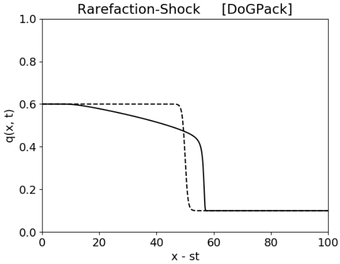


Rarefaction-Compressive Shock

$$q_l > q_b$$

$$q(x,t) = \begin{cases} q_l & x < f'(q_l)t \\ h_r(x) & f'(q_l)t < x < f'(q_b)t \\ q_r & x > f'(q_b)t \end{cases}$$

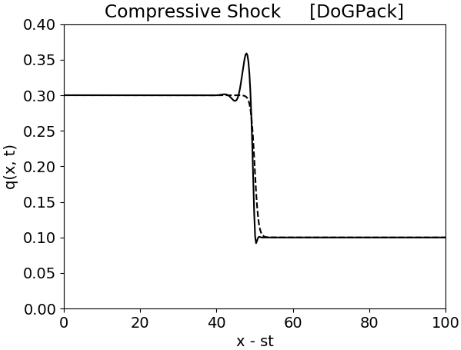




Wave Structure with Nonlinear Hyper Diffusion

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

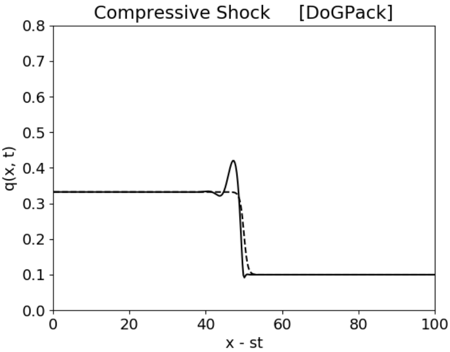
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

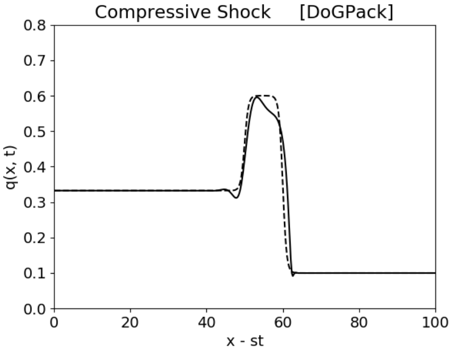
$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

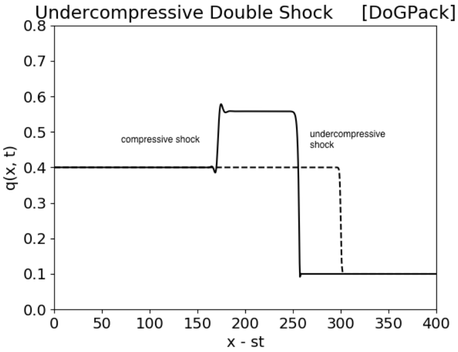
$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2} \tanh(x - 50) + \frac{q_m + q_l}{2} & x < 55 \\ -\frac{q_m - q_r}{2} \tanh(x - 60) + \frac{q_m + q_r}{2} + q_r & x > 55 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.4$$

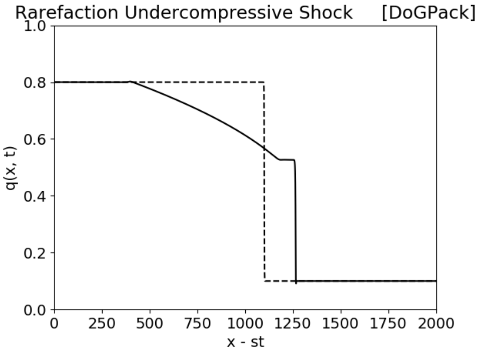
$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x, 0) = (-\tanh(x - 1100) + 1)\frac{q_l - q_r}{2} + q_r$$



Conclusion

Observations

- Nonlinear Hyper Diffusion has subtle instabilities

Future Work

- Higher Order Convergence
  - Runge Kutta IMEX
  - Local Discontinuous Galerkin Method
  - Hybridized Discontinuous Galerkin Method

Bibliography

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