Hybridized Discontinuous Galerkin Method

These notes are intended to give background on Hyridized Discontinuous Galerkin methods and then explore how to apply this method to Thin Film Equations

1 Introduction/Main Idea

To start we will consider Poisson's equation with Dirichlet boundary conditions

$$-\Delta u = f \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega$$

We will consider this in mixed form by introducing the auxilliary variable $\boldsymbol{q} = -\nabla u$, then the equation becomes

$$\boldsymbol{q} + \nabla u = 0 \qquad \text{in } \Omega$$
$$\nabla \cdot \boldsymbol{q} = f \qquad \text{in } \Omega$$
$$u = g \qquad \text{on } \partial\Omega$$

This problem has an exact solution that can be found analytically assuming some nice properties.

I will introduce a triangulation of Ω, $\mathcal{T}_h$, and reformulate the problem on this triangulation that will give the same exact solution.

1.1 Notation

First some notation,

$$\partial\mathcal{T}_h = \{\partial K : K \in \mathcal{T}_h\}$$
$$F = \partial K \cap \partial\Omega \text{ for } K \in \mathcal{T}_h$$
$$F = \partial K^+ \cap \partial K^- \text{ for } K^+, K^- \in \mathcal{T}_h$$

Let ε_h be the set of all faces, F, and ε_h^0 be interior faces, and ε_h^∂ be boundary faces.

Let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ be the outward unit normals of ∂K^+ and ∂K^- respectively, and $(\boldsymbol{q}^\pm, u^\pm)$ be the interior values of $(\boldsymbol{q}, u)$ on F for $K^\pm$. Define

$$[\![\boldsymbol{q} \cdot n]\!] = \boldsymbol{q}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}^- \cdot \boldsymbol{n}^-$$
$$[\![u\boldsymbol{n}]\!] = u^+\boldsymbol{n}^+ + u^-\boldsymbol{n}^-$$
$$\{\boldsymbol{q}\} = \frac{\boldsymbol{q}^+ + \boldsymbol{q}^-}{2}$$
$$\{u\} = \frac{u^+ + u^-}{2}$$

1.2 Reformulation

Now we can reformulate the original Poisson's problem on $\mathcal{T}_h$ as a local problem for each K

$$\boldsymbol{q} + \nabla u = 0$$
$$\nabla \cdot \boldsymbol{q} = f$$

a transmission condition on each interior face, $F \in \varepsilon_h^0$

$$[\![u\boldsymbol{n}]\!] = \boldsymbol{0}$$
$$[\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] = 0$$

and the boundary condition on each boundary face, $F \in \varepsilon_h^\partial$

$$u = g$$

This problem is equivalent to the original problem on Ω. The $(\boldsymbol{q}, u)$ that satisfies this problem also solve the original problem.

We would like to be able to solve the local problem locally, but this requires boundary conditions on each element K for the local problem to be solved. Therefore consider the local problem

$$\boldsymbol{q} + \nabla u = 0 \quad \text{in } K$$
$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } K$$
$$u = \hat{u} \quad \text{on } \partial K$$

We have introduced another unknown $\hat{u}$ on each interior face $F \in \varepsilon_h^0$. This unknown automatically makes us satisfy $[\![u\boldsymbol{n}]\!] = \boldsymbol{0}$, so the transmission condition becomes

$$[\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] = 0 \quad \text{on } F \in \varepsilon_h^0$$

and we still have the boundary condition

$$u = g \quad \text{on } F \in \varepsilon_h^\partial$$

Now solving for $(\boldsymbol{q}, u, \hat{u})$ will give the same solution as the original problem, however $\boldsymbol{q}$ and u can be solved locally and only $\hat{u}$ needs to be solved globally.

1.3 General Algorithm

Here is the outline for solving for $\boldsymbol{q}$, u, and $\hat{u}$. First split the local problem in two, so that one part depends on f and the other part depends on $\hat{u}$, that is let $\boldsymbol{q} = \boldsymbol{Q}_f + \boldsymbol{Q}_{\hat{u}}$ and $u = U_f + U_{\hat{u}}$, where

$$\boldsymbol{Q}_f + \nabla U_f = 0 \quad \text{in } K$$
$$\nabla \cdot \boldsymbol{Q}_f = f \quad \text{in } K$$
$$U_f = 0 \quad \text{on } \partial K$$

and

$$\boldsymbol{Q}_{\hat{u}} + \nabla U_{\hat{u}} = 0 \quad \text{in } K$$
$$\nabla \cdot \boldsymbol{Q}_{\hat{u}} = 0 \quad \text{in } K$$
$$U_{\hat{u}} = \hat{u} \quad \text{on } \partial K$$

Now the transmission condition becomes

$$[\![\boldsymbol{Q}_{\hat{u}}]\!] = -[\![\boldsymbol{Q}_f]\!]$$

First solve for $\boldsymbol{Q}_f$ exactly, then solve for $\boldsymbol{Q}_{\hat{u}}$ in terms of $\hat{u}$. Now the transmission condition gives a global linear algebra problem

$$[\![\boldsymbol{Q}_{\hat{u}}]\!] = -[\![\boldsymbol{Q}_f]\!]$$

since $[\![\boldsymbol{Q}_{\hat{u}}]\!]$ is a linear system in $\hat{u}$ and $-[\![\boldsymbol{Q}_f]\!]$ is known.

After this linear algebra problem is solved, the values of $U_{\hat{u}}$ and $\boldsymbol{Q}_{\hat{u}}$ can be found/reconstructed locally. The full solution is then $\boldsymbol{q} = \boldsymbol{Q}_f + \boldsymbol{Q}_{\hat{u}}$ and $u = U_f + U_{\hat{u}}$.

1.4 1D example

In 1D, the problem becomes

$$Q_f + U_f' = 0 \quad \text{in } K$$
$$Q_f' = f \quad \text{in } K$$
$$U_f = 0 \quad \text{on } \partial K$$

and

$$Q_{\hat{u}} + U'_{\hat{u}} = 0 \quad \text{in } K$$
$$Q'_{\hat{u}} = 0 \quad \text{in } K$$
$$U_{\hat{u}} = \hat{u} \quad \text{on } \partial K$$

and

$$Q_{\hat{u}}(x^-_{j+1/2}) - Q_{\hat{u}}(x^+_{j+1/2}) = -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2})$$

where the uniform mesh is given by x_i at cell centers, $x_{i+1/2}$ at cell interfaces, and spacing h.

Solving the $\hat{u}$ system we see that

$$Q'_{\hat{u}} = 0$$
$$Q_{\hat{u}} = c$$

and

$$Q_{\hat{u}} + U'_{\hat{u}} = 0$$
$$U'_{\hat{u}} = -c$$
$$U_{\hat{u}} = -cx + b$$

with the boundary conditions, we know $U_{\hat{u}}$ is a line from $\hat{u}_{j-1/2}$ to $\hat{u}_{j+1/2}$, and $Q_{\hat{u}}$ is the opposite of the slope of this line.

$$U_{\hat{u}} = \frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h}(x - x_{j-1/2}) - \hat{u}_{j-1/2}$$
$$Q_{\hat{u}} = -\frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h}$$

Now we can form the linear system given by the transmission condition

$$Q_{\hat{u}}(x^-_{j+1/2}) - Q_{\hat{u}}(x^+_{j+1/2}) = -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2})$$
$$-\frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h} + \frac{\hat{u}_{j+3/2} - \hat{u}_{j+1/2}}{h} = -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2})$$
$$\frac{\hat{u}_{j-1/2} - 2\hat{u}_{j+1/2} + \hat{u}_{j+3/2}}{h} = -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2})$$

After solving this linear system we already have expressions for $Q_{\hat{u}}$ and $U_{\hat{u}}$ in terms of $\hat{u}$.

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