Discontinuous Galerkin Method for solving thin film equations

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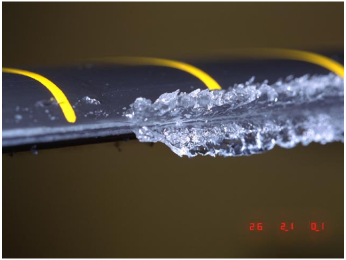
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating

Model Equations

- Navier-Stokes Equation

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + \left(\rho u^2 + p\right)_x &= \frac{4}{3Re} u_{xx} \\
E_t + (u(E + p))_x &= \frac{1}{Re}\left(\frac{2}{3}\left(u^2\right)_{xx} + \frac{\gamma}{(\gamma - 1)Pr}\left(\frac{p}{\rho}\right)_{xx}\right)
\end{aligned}
$$

- Asymptotic Limit, $\rho << L$
- Thin-Film Equation - 1D with q as fluid height.

$$
q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x
$$

Current Model

- Simplified Expression

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

- Operator Splitting

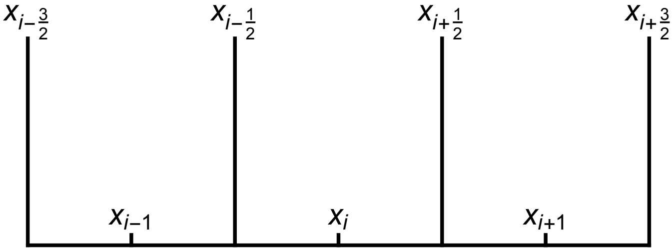
$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 q_{xxx}\right)_x &= 0 \end{aligned}$$

Introduction to Discontinuous Galerkin

- Let $\mathcal{T}_h$ partition the domain, $\Omega = [a, b]$

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $K_i \in \mathcal{T}_h = \left[x_{i-1/2}, x_{i+1/2}\right]$
- $h = x_{i+1/2} - x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$.



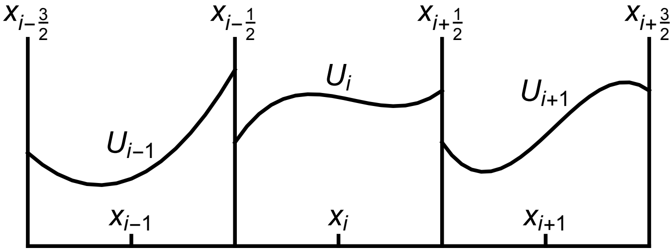
Function Spaces

- $P^k(K)$ - polynomials of degree less than or equal to k on $K \in \mathcal{T}_h$
- $$V_h = \left\{ v \in L^2(\Omega) : v|_K \in P^k(K), \quad \forall K \in \mathcal{T}_h \right\}$$

Numerical Solutions

- Let $\left\{\phi^k(\xi)\right\}$ be the Legendre polynomials.
- Solution of order M on each cell

$$q|_{x \in V_i} \approx Q_i = \sum_{k=1}^{M} Q_i^k \phi^k(\xi)$$



Convection

- Convection Equation

$$q_t + \frac{2}{\Delta x} f(q)_\xi = 0$$
$$f(q) = q^2 - q^3$$

- Weak Form

$$\int_{-1}^{1} \left(q_t \phi(\xi) + \frac{2}{\Delta x} f(q)_\xi \phi(\xi) \right) \mathrm{d}\xi = 0$$

- Runge-Kutta Discontinuous Galerkin

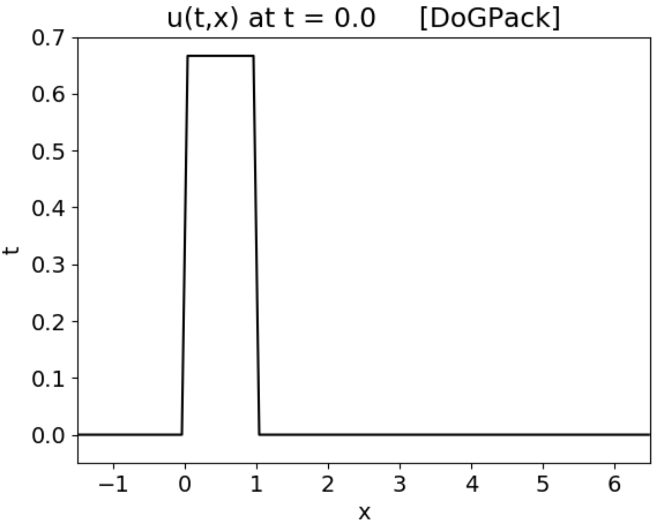
$$\dot{Q}_i^\ell = \frac{1}{\Delta x} \int_{-1}^{1} f(Q_i) \phi_\xi^\ell \, \mathrm{d}\xi - \frac{1}{\Delta x} \left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2} \right)$$

- Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(Q_{i+1}(-1)) + f(Q_i(1))}{2} \phi^\ell(1)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Numerical Example - Square Wave



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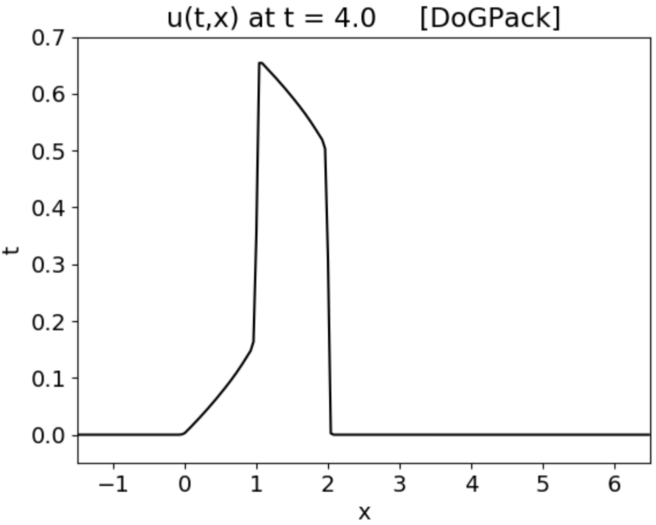

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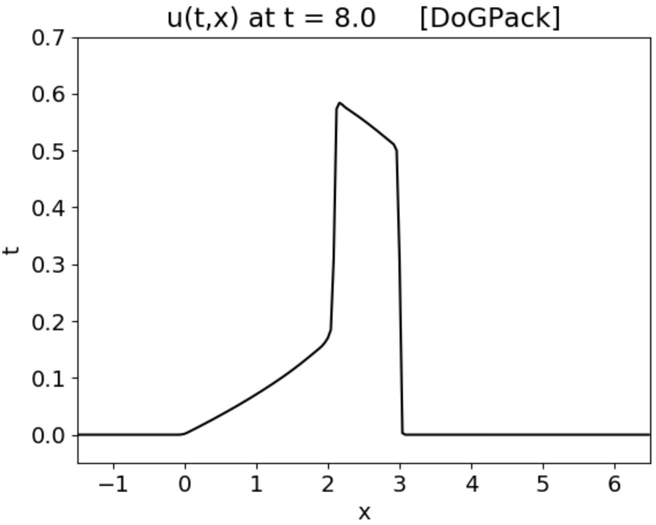

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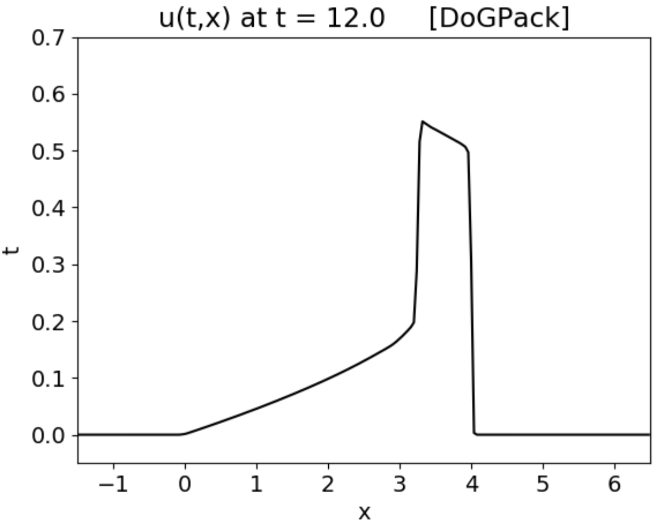

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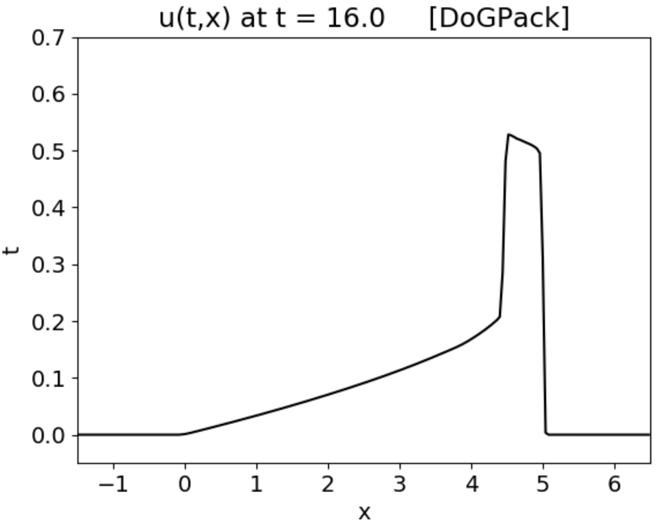

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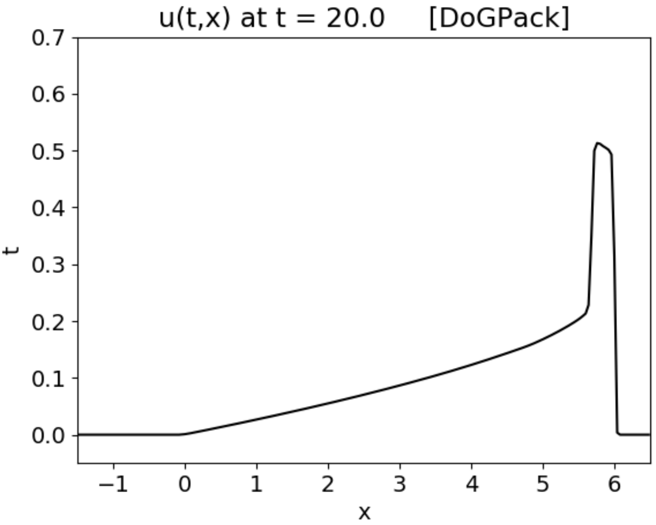


Hyper-Diffusion

- Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4}\left(u^3 u_{\xi\xi\xi}\right)_\xi = 0$$

- Local Discontinuous Galerkin (LDG)

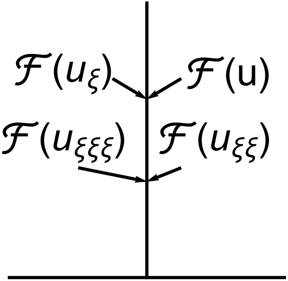
$$\begin{aligned} q &= \frac{2}{\Delta x} u_\xi \\ r &= \frac{2}{\Delta x} q_\xi \\ s &= \frac{2}{\Delta x} u^3 r_\xi \\ u_t &= -\frac{2}{\Delta x} s_\xi \end{aligned}$$

Local Discontinuous Galerkin

$$
\begin{aligned}
\eta(\xi) &= \left(U_i^n\right)^3 \\
Q_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} U_i \phi_\xi^\ell \, \mathrm{d}\xi - \mathcal{F}(U)_{i+1/2}^\ell + \mathcal{F}(U)_{i-1/2}^\ell\right) \\
R_i^\ell &= -\frac{1}{\Delta x}\left(\int_{-1}^{1} Q_i \phi_\xi^\ell \, \mathrm{d}\xi - \mathcal{F}(Q)_{i+1/2}^\ell + \mathcal{F}(Q)_{i-1/2}^\ell\right) \\
S_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} (R_i)_\xi \eta(\xi) \phi^\ell \, \mathrm{d}\xi\right) \\
&\quad + \frac{1}{\Delta x}\left(\mathcal{F}(\eta)_{i+1/2}\mathcal{F}(R)_{i+1/2}^\ell - \mathcal{F}(\eta)_{i-1/2}\mathcal{F}(R)_{i-1/2}^\ell\right) \\
\dot{U}_i^\ell &= \frac{1}{\Delta x}\left(\int_{-1}^{1} S_i \phi_\xi^\ell \, \mathrm{d}\xi - \mathcal{F}(S)_{i+1/2}^\ell + \mathcal{F}(S)_{i-1/2}^\ell\right)
\end{aligned}
$$

Local Discontinuous Galerkin

$$\mathcal{F}(\eta)_{i+1/2} = \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1))$$
$$\mathcal{F}(\eta)_{i-1/2} = \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1))$$
$$\mathcal{F}(*)^{\ell}_{i+1/2} = \phi^{\ell}(1) *_{i+1/2}$$



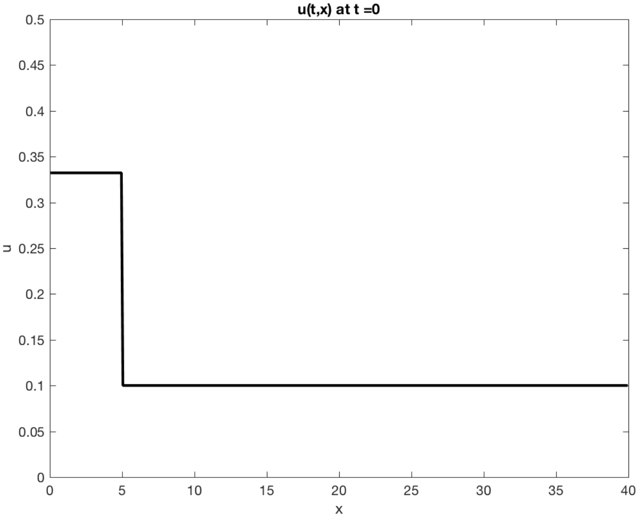
Local Discontinuous Galerkin

- Explicit SSP Runge Kutta
  - Severe time step restriction
  - $\Delta t \sim \Delta x^4$
  - $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
  - $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
  - Linear System Solver
  - Stabilized BiConjugate Gradient
  - MultiGrid Solver

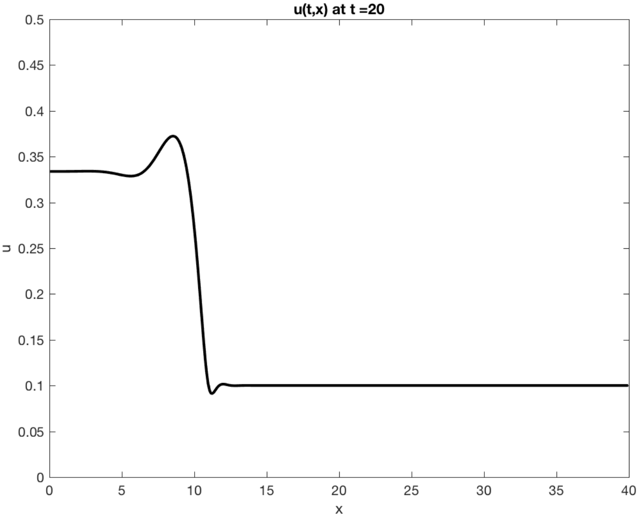
Operator Splitting

- Strang Splitting
  - 1 time step
    - 1/2 time step for convection
    - 1 time step for hyper-diffusion
    - 1/2 time step for convection
  - Second order splitting

Numerical Results - Riemann Problem



Numerical Results - Riemann Problem



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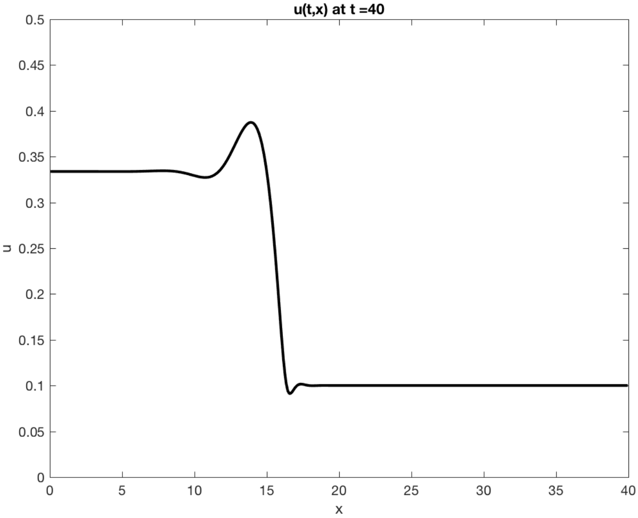

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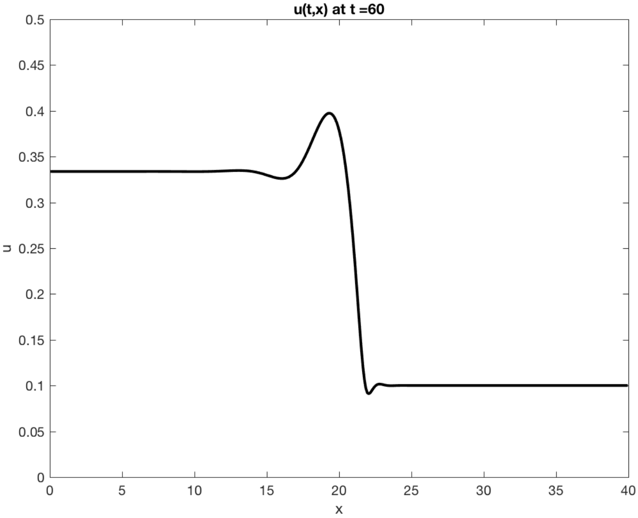

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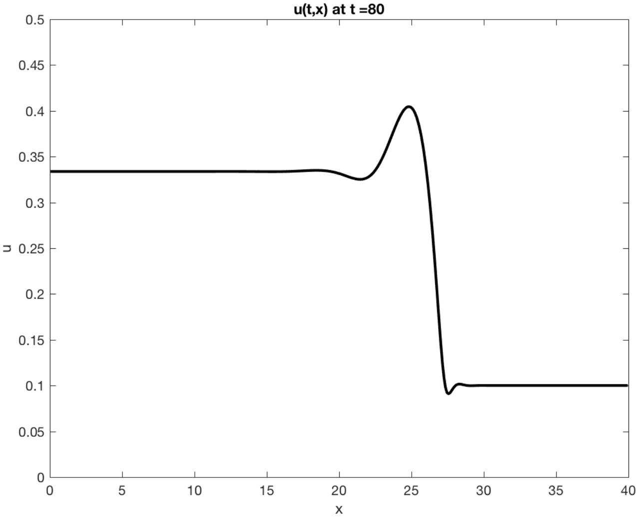

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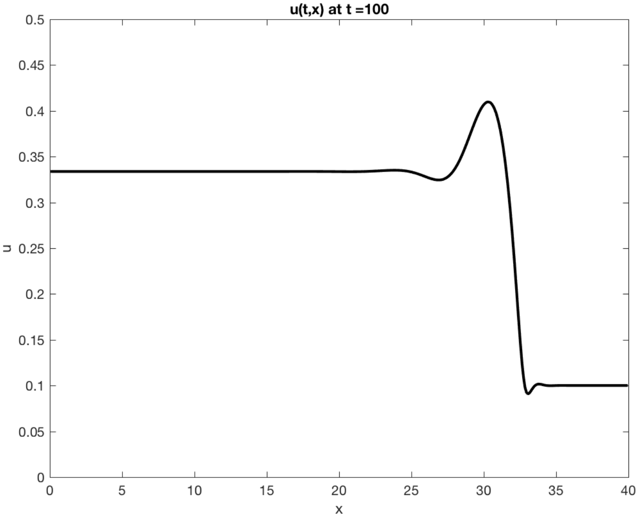

Future Work

- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

Conclusion

- Thanks
  - James Rossmanith
  - Alric Rothmayer
- Questions?

Bibliography

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