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# Discontinuous Galerkin Method for solving a Thin-Film Equation

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# Overview

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# Motivation

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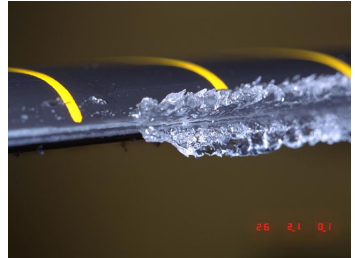
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- Aircraft Icing
- Runback



- Industrial Coating
- Paint Drying

# Model Equations

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## ■ Navier-Stokes Equation

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = \frac{4}{3Re} u_{xx}$$

$$E_t + (u(E + p))_x = \frac{1}{Re} \left( \frac{2}{3} (u^2)_{xx} + \frac{\gamma}{(\gamma - 1)Pr} \left( \frac{p}{\rho} \right)_{xx} \right)$$

## ■ Asymptotic Limit, $\rho \ll L$

## ■ Thin-Film Equation - 1D with $u$ as fluid height.

$$u_t + (f(x, t)u^2 - g(x, t)u^3)_x = -(h(x, t)u^3 u_{xxx})_x$$

# Current Model

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## ■ Simplified Expression

$$u_t + (u^2 - u^3)_x = -(u^3 u_{xxx})_x$$

## ■ Operator Splitting

$$u_t + (u^2 - u^3)_x = 0$$

$$u_t + (u^3 u_{xxx})_x = 0$$

# Introduction to Discontinuous Galerkin

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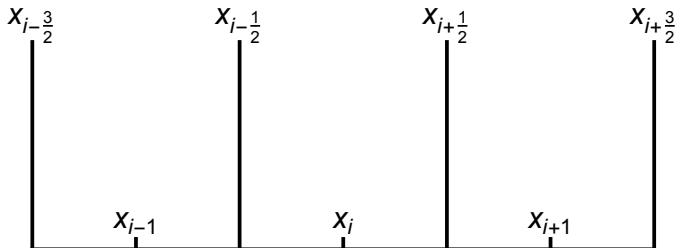
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- Partition the domain,  $[a, b]$  as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $V_i = [x_{i-1/2}, x_{i+1/2}]$
- $\Delta x = x_{i+1/2} - x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$ .



# Numerical Solutions

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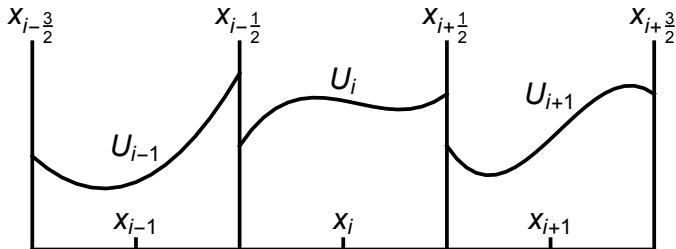
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- Use canonical variable  $\xi \in [-1, 1]$
- Let  $\{\phi^k(\xi)\}$  be the Legendre polynomials.
- Solution of order  $M$  on each cell

$$u|_{x \in V_i} \approx U_i = \sum_{k=1}^M U_i^k \phi^k(\xi)$$



# Convection

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## ■ Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_\xi = 0$$
$$f(u) = u^2 - u^3$$

## ■ Weak Form

$$\int_{-1}^1 \left( u_t \phi(\xi) + \frac{2}{\Delta x} f(u)_\xi \phi(\xi) \right) d\xi = 0$$

## ■ Runge-Kutta Discontinuous Galerkin

$$\dot{U}_i^\ell = \frac{1}{\Delta x} \int_{-1}^1 f(U_i) \phi_\xi^\ell d\xi - \frac{1}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2})$$

## ■ Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^\ell(1)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.



# Numerical Example - Square Wave

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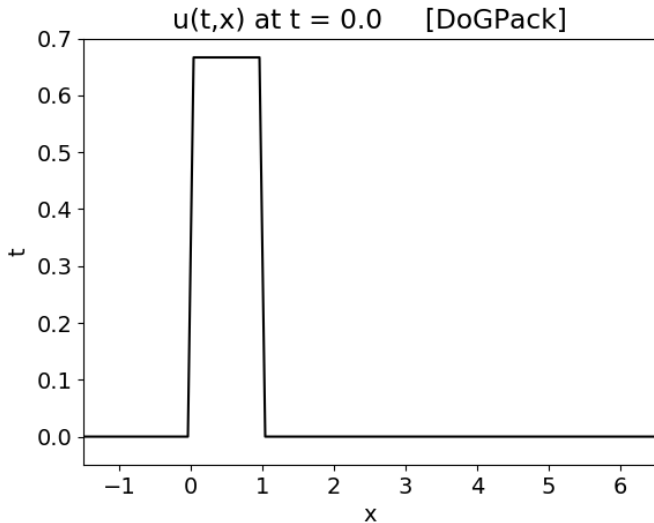
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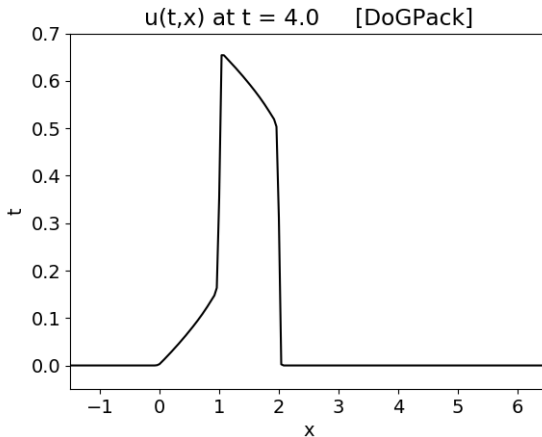
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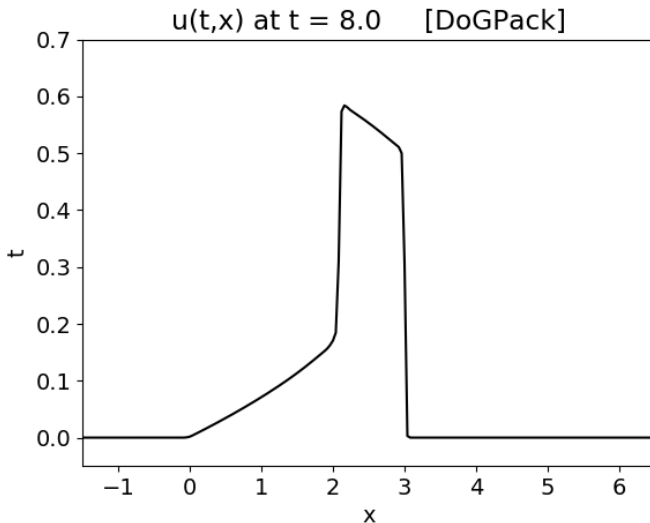
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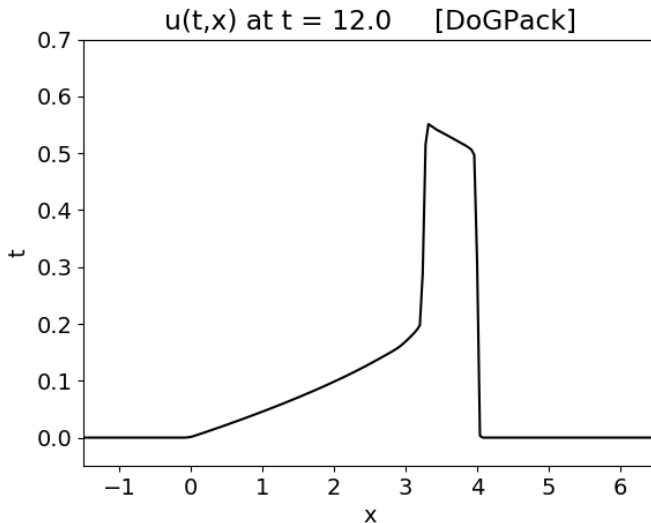
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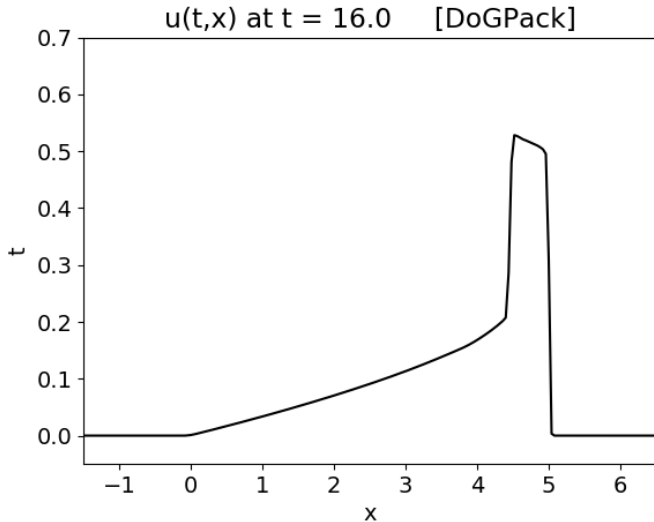
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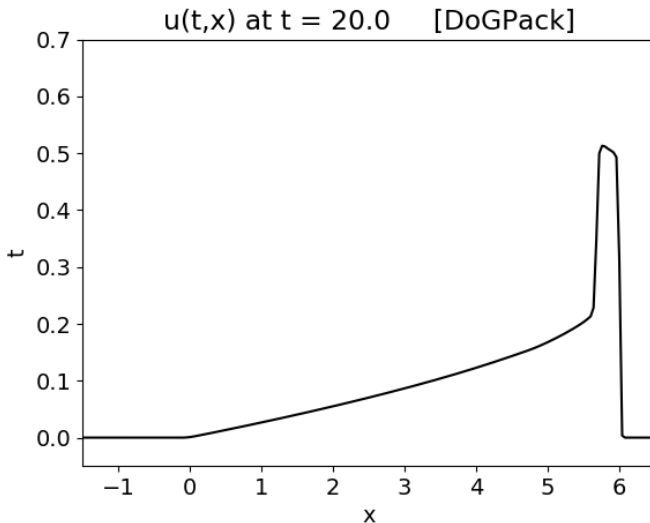
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# Hyper-Diffusion

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## ■ Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4} (u^3 u_{\xi\xi\xi})_{\xi} = 0$$

## ■ Local Discontinuous Galerkin

$$q = \frac{2}{\Delta x} u_{\xi}$$

$$r = \frac{2}{\Delta x} q_{\xi}$$

$$s = \frac{2}{\Delta x} u^3 r_{\xi}$$

$$u_t = -\frac{2}{\Delta x} s_{\xi}$$

# Local Discontinuous Galerkin

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$$\eta(\xi) = (U_i^n)^3$$

$$Q_i^\ell = -\frac{1}{\Delta x} \left( \int_{-1}^1 U_i \phi_\xi^\ell d\xi - \mathcal{F}(U)_{i+1/2}^\ell + \mathcal{F}(U)_{i-1/2}^\ell \right)$$

$$R_i^\ell = -\frac{1}{\Delta x} \left( \int_{-1}^1 Q_i \phi_\xi^\ell d\xi - \mathcal{F}(Q)_{i+1/2}^\ell + \mathcal{F}(Q)_{i-1/2}^\ell \right)$$

$$S_i^\ell = \frac{1}{\Delta x} \left( \int_{-1}^1 (R_i)_\xi \eta(\xi) \phi^\ell d\xi \right) \\ + \frac{1}{\Delta x} \left( \mathcal{F}(\eta)_{i+1/2} \mathcal{F}(S)_{i+1/2}^\ell - \mathcal{F}(\eta)_{i-1/2} \mathcal{F}(S)_{i-1/2}^\ell \right)$$

$$\dot{U}_i^\ell = \frac{1}{\Delta x} \left( \int_{-1}^1 S_i \phi_\xi^\ell d\xi - \mathcal{F}(S)_{i+1/2}^\ell + \mathcal{F}(S)_{i-1/2}^\ell \right)$$



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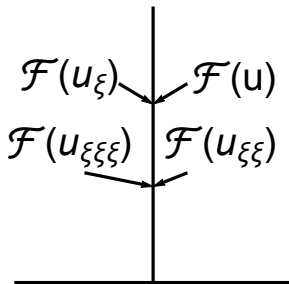
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$$\mathcal{F}(\eta)_{i+1/2} = \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1))$$

$$\mathcal{F}(\eta)_{i-1/2} = \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1))$$

$$\mathcal{F}(\ast)_{i+1/2}^\ell = \phi^\ell(1) \ast_{i+1/2}$$



# Local Discontinuous Galerkin

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- Explicit SSP Runge Kutta
  - Severe time step restriction
  - $\Delta t \sim \Delta x^4$
  - $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
  - $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
  - Linear System Solver
  - Stabilized BiConjugate Gradient
  - MultiGrid Solver

# Multigrid Solver

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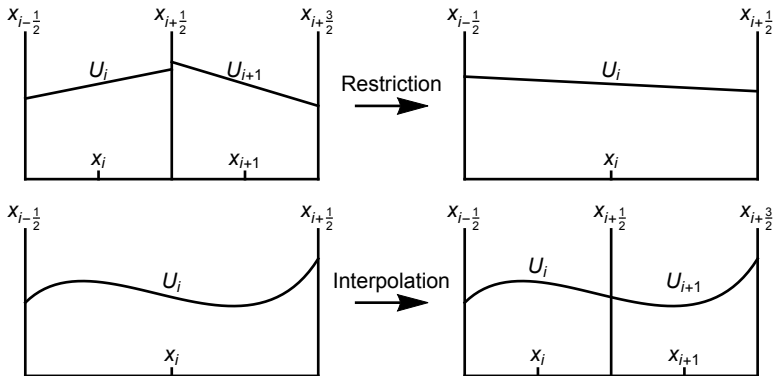
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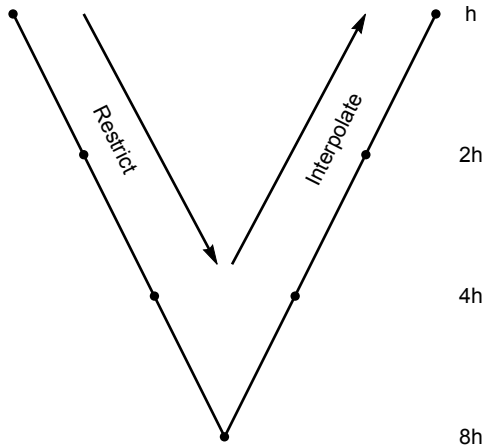
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## ■ Relaxation e.g. Jacobi Relaxation



# Multigrid Solver

## V-Cycle



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# Multigrid Solver

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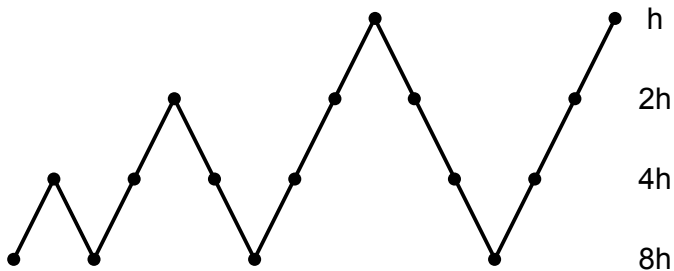
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# Operator Splitting

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- Strang Splitting
  - 1 time step
    - $1/2$  time step for convection
    - 1 time step for hyper-diffusion
    - $1/2$  time step for convection
  - Second order splitting

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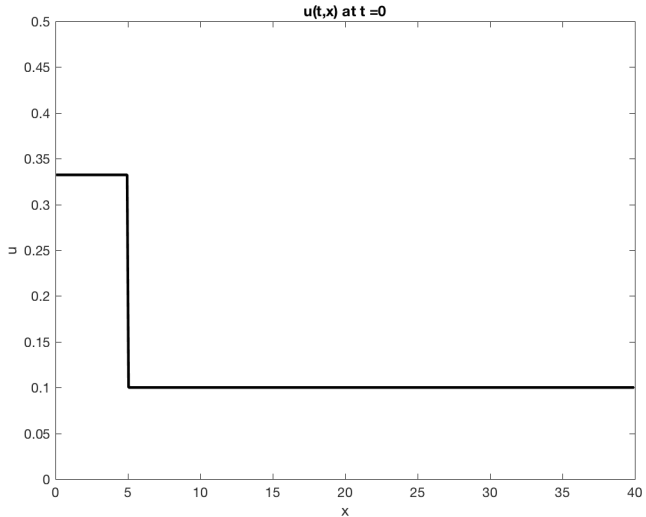
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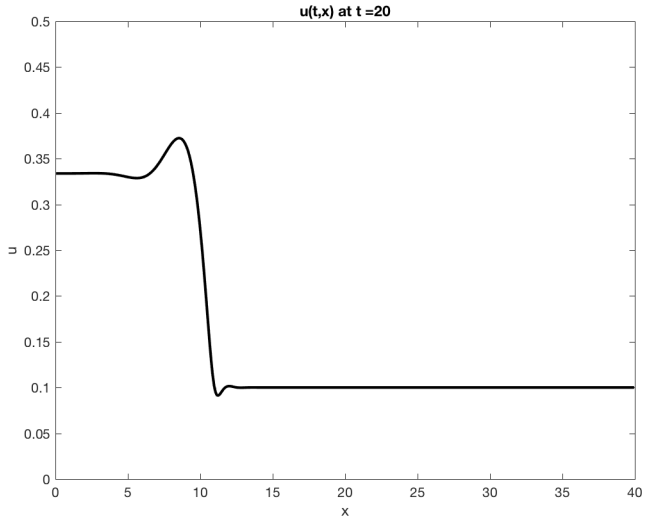
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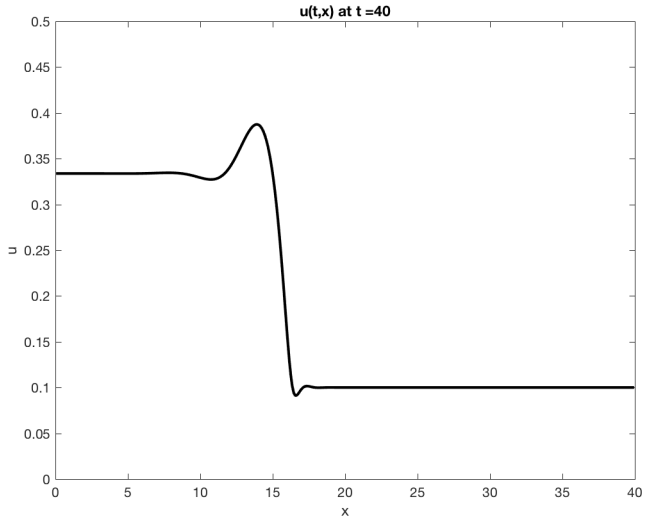
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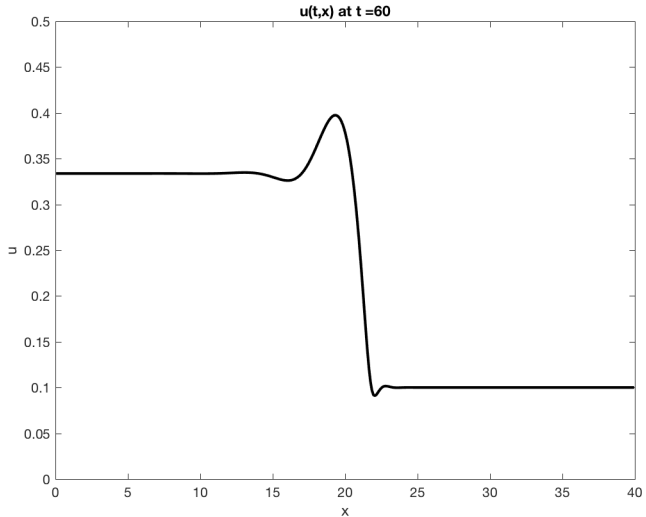
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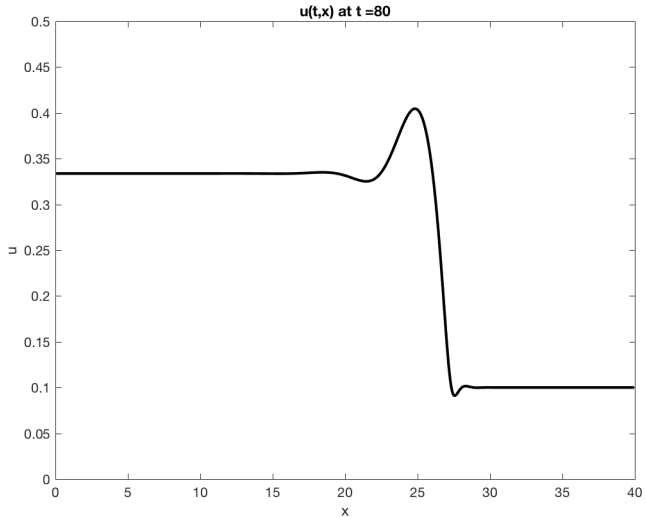
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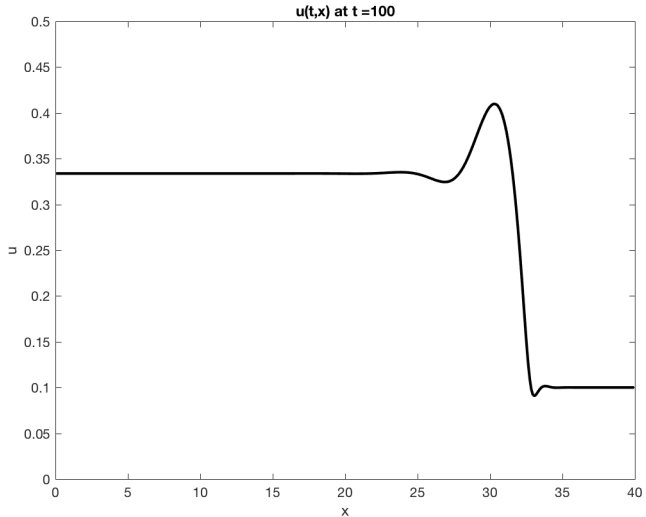
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# Future Work

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- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

# Conclusion

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References

- Thanks
  - James Rossmannith
  - Alric Rothmayer
- Questions?

# Bibliography

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References

- [1] Y. Ha, Y.-J. Kim, and T.G. Myers. “On the numerical solution of a driven thin film equation”. In: *J. Comp. Phys.* 227.15 (2008), pp. 7246–7263.
- [2] NASA. URL: [http://icebox.grc.nasa.gov/gallery/images/C95\\_03918.html](http://icebox.grc.nasa.gov/gallery/images/C95_03918.html).
- [3] J.A. Rossmanith. DOGPack. Available from <http://www.dogpack-code.org/>.