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Hybridizable Discontinuous Galerkin Methods for Timoshenko Beams

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Abstract In this paper, we introduce a new class of discontinuous Galerkin methods for Timoshenko beams. The main feature of these methods is that they can be implemented in an efficient way through a hybridization procedure which reduces the globally coupled unknowns to approximations to the displacement and bending moment at the element boundaries. After displaying the methods, we obtain conditions under which they are well defined. We then compare these new methods with the already existing discontinuous Galerkin methods for Timoshenko beams. Finally, we display extensive numerical results to ascertain the influence of the stabilization parameters on the accuracy of the approximation. In particular, we find specific choices for which all the variables, namely, the displacement, the rotation, the bending moment and the shear force converge with the optimal order of $k+1$ when each of their approximations are taken to be piecewise polynomial of degree $k \geq 0$.

Keywords Discontinuous Galerkin methods · Timoshenko beams · Hybridization

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1 Introduction

In this paper, we introduce a new class of discontinuous Galerkin methods for Timoshenko beams called hybridizable discontinuous Galerkin (HDG) methods. The dimensionless form of the Timoshenko beam model [9], is given by the differential equations, see [4],

$$w' = \theta - d^2 \frac{T}{GA}, \qquad \theta' = \frac{M}{EI}, \qquad M' = T, \qquad T' = q, \tag{1.1}$$

in $\Omega := (0, 1)$. Here, the unknowns are the transverse displacement w, the rotation of the transverse cross-section of the beam θ, the bending moment M, and the shear force T. The material and geometrical properties of the beam are characterized by the shear modulus G, the cross-section area A, the Young modulus E, and the moment of inertia I. The remaining data of the problem are the transverse load, q, and the boundary conditions which we take to be

$$w = w_D, \qquad \theta = \theta_N \quad \text{on } \partial\Omega = \{0, 1\}. \tag{1.2}$$

The parameter $0 < d < 1$ represents the thickness of the beam. The case $d \ll 1$ is of particular interest from a numerical point of view since we intend to devise methods whose convergence properties remain invariant as d goes to zero. We will also consider the limiting case in which $d = 0$. Note that this corresponds to the so called Euler-Bernoulli beam model, and is a one-dimensional form of the biharmonic problem. The Timoshenko beam model on the other hand is a one-dimensional form of the Reissner-Mindlin plate model.

To put our results in perspective, let us briefly review the work done on the subject. In [3], DG methods for the Timoshenko beams were introduced and sufficient conditions ensuring the existence and uniqueness of their approximate solutions were proved. Preliminary numerical experiments were obtained which indicated that, when polynomials of degree k are used, the optimal order of convergence of $k + 1$ is achieved for all variables. Later, in [2], the fact that all the numerical traces of the h-version of the DG method superconverge with order $2k + 1$ was uncovered and a local post-processing resulting in a uniformly accurate solution of order $2k + 1$ was devised and numerically tested. These results hold uniformly with respect to the thickness of the beam. In [4], these results were put in firm mathematical ground.

On the other hand, the HDG methods were introduced in [7] in the framework of second-order elliptic problems. The main feature of these methods is that their approximate solutions can be expressed in an element-by-element fashion in terms of an approximate trace satisfying a global weak formulation. Since the associated matrix is symmetric and positive definite, these methods can be efficiently implemented. In [5], the single-face HDG method (SFH) for second order elliptic problems was introduced. It was proved that by using polynomials of degree $k \geq 0$ for both the potential as well as the flux, the order of convergence in L^2 of both unknowns is $k + 1$. Later it was shown [8] that many other DG methods, including a wide class of HDG methods, have these optimal convergence properties as well.

In [6], the SFH method was applied to the biharmonic problem $\Delta^2 u = f$ with the hope that all variables would converge optimally. However, for this HDG method, optimal convergence orders were obtained only for u and ∇u. The approximations to Δu and $\nabla \Delta u$ where proven to converge with order $k + \frac{1}{2}$ and $k - \frac{1}{2}$, respectively, and numerical experiments confirming the sharpness of these results were obtained. As a first step for uncovering HDG methods with optimal orders of convergence for all the variables, not only for the biharmonic but also for plates and shells, we introduce here a wide class of HDG methods for the Timoshenko beam.

The rest of the paper is organized as follows. In Sect. 2 we describe the HDG methods, and in Sect. 3 we state conditions under which they are well defined. In Sect. 4, we state a theorem showing that they can be implemented efficiently through an hybridization procedure. In Sect. 5, we compare them with the already existing DG methods. In Sect. 6, we display extensive numerical results to ascertain the influence of the stabilization parameters on the accuracy of their approximations. We end with some concluding remarks in Sect. 7.

2 The HDG Methods

Let us describe the HDG methods under consideration. We begin by introducing our notation. Set

$$\Omega_h := \{I_j = (x_{j-1}, x_j) : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1\}.$$

We associate to this partition of the domain Ω the set of its nodes,

$$\mathscr{E}_h := \{x_0, x_1, \ldots, x_N\},$$

and the set of its interior nodes $\mathscr{E}_h^\circ := \mathscr{E}_h \backslash \partial\Omega$. Moreover, for each mesh element $K \in \Omega_h$, we denote its length by h_K and set

$$h := \max_{K\in\Omega_h} \{h_K\}.$$

Next, given a polynomial degree $k \geq 0$ and an element $K \in \Omega_h$, we define $\mathcal{P}^k(K)$ as the set of polynomials of degree less than or equal to k on K. The space of piecewise polynomials on Ω is defined accordingly as

$$V_h^k := \{v : \Omega_h \mapsto \mathbb{R} : \, v|_K \in \mathcal{P}^k(K) \text{ for all } K \in \Omega_h\}.$$

We also set

$$L_0^2(\mathscr{E}_h) := \{\mathsf{w} \in L^2(\mathscr{E}_h) : \mathsf{w} = 0 \text{ on } \partial\Omega\}.$$

Here, $L^2(\mathscr{E}_h)$ is simply the set of vectors $\mathbf{v} = (v_1, v_2, \ldots, v_{N+1}) \in \mathbb{R}^{N+1}$ such that $\sum_{i=1}^{N+1} v_i^2 < \infty$, i.e. that $v_i < \infty$ for all $i = 1, 2, \ldots, N+1$.

We are now ready to define the HDG methods. The HDG methods seek an approximation $(T_h, M_h, \theta_h, w_h, \widehat{M}_h, \widehat{w}_h)$ to the exact solution $(T, M, \theta, w, M|_{\mathscr{E}_h}, w|_{\mathscr{E}_h})$, in the finite dimensional space $[V_h^k]^4 \times L^2(\mathscr{E}_h) \times L^2(\mathscr{E}_h)$ determined by requiring that

$$-(w_h, v_1')_{\Omega_h} + \langle \widehat{w}_h, v_1\, n\rangle_{\partial\Omega_h} = (\theta_h, v_1)_{\Omega_h} - d^2\,(T_h/GA, v_1)_{\Omega_h}\,, \tag{2.1a}$$

$$-(\theta_h, v_2')_{\Omega_h} + \langle \widehat{\theta}_h, v_2\, n\rangle_{\partial\Omega_h} = (M_h/EI, v_2)_{\Omega_h}\,, \tag{2.1b}$$

$$-(M_h, v_3')_{\Omega_h} + \langle \widehat{M}_h, v_3\, n\rangle_{\partial\Omega_h} = (T_h, v_3)_{\Omega_h}, \tag{2.1c}$$

$$-(T_h, v_4')_{\Omega_h} + \langle \widehat{T}_h, v_4\, n\rangle_{\partial\Omega_h} = (q, v_4)_{\Omega_h}, \tag{2.1d}$$

$$\langle \widehat{\theta}_h, \mathsf{m}\, n\rangle_{\partial\Omega_h} = \langle \theta_N, \mathsf{m}\, n\rangle_{\partial\Omega}, \tag{2.1e}$$

$$\langle \widehat{T}_h, \mathsf{w}\, n\rangle_{\partial\Omega_h} = 0, \tag{2.1f}$$

hold for all $(v_1, v_2, v_3, v_4, \mathsf{m}, \mathsf{w}) \in [V_h^k]^4 \times L^2(\mathscr{E}_h) \times L_0^2(\mathscr{E}_h)$. Here, the outward unit normal vectors are $n(x^{\mp}) := \pm 1$ for $x \in \mathscr{E}_h$. The "volume" inner product is defined as

$$(u, v)_{\Omega_h} := \sum_{j=1}^{N} (u, v)_{I_j} \quad \text{where } (u, v)_{I_j} := \int_{I_j} u(x) v(x)\, dx,$$

and the boundary inner product is defined as $\langle u, v\, n\rangle_{\partial\Omega_h} := \sum_{j=1}^{N} \langle u, v\, n\rangle_{\partial I_j}$ where $\langle u, v\rangle_{\partial I_j} := u(x_j^-) v(x_j^-) + u(x_{j-1}^+) v(x_{j-1}^+)$, and $\varphi(x^{\pm}) := \lim_{\epsilon \downarrow 0} \varphi(x \pm \epsilon)$ for $x \in \mathscr{E}_h$.

Note that the boundary condition (1.2) on θ is imposed by (2.1e). The boundary condition (1.2) on w is imposed as follows:

$$\widehat{w}_h = w_D \quad \text{on } \partial\Omega. \tag{2.2a}$$

To complete the definition of the HDG method, we need to express the numerical traces $\widehat{T}_h$ and $\widehat{\theta}_h$ in terms of the unknowns:

$$\widehat{\theta}_h = \theta_h - \alpha_\theta (M_h - \widehat{M}_h) n - \tau (w_h - \widehat{w}_h) n, \tag{2.2b}$$

$$\widehat{T}_h = T_h - \tau (M_h - \widehat{M}_h) n + \alpha_T (w_h - \widehat{w}_h) n, \tag{2.2c}$$

where α_θ, α_T, and τ are non-negative functions defined on $\partial\Omega_h$. They have to be suitably defined to guarantee the existence and uniqueness of the approximate solution.

3 Existence and Uniqueness of the HDG Solution

In this section we provide sufficient conditions under which the HDG method introduced in the previous section defines a unique solution. As is usual for DG methods, the existence and uniqueness of the approximation depends strongly on the definition of the numerical traces (2.2), and hence on the parameters α_θ, α_T, and τ. We state our existence and uniqueness results in the following two theorems.

Theorem 3.1 *Consider the HDG method defined by the weak formulation* (2.1), *and the formulas* (2.2) *for the numerical traces. Suppose that the parameter d is strictly positive, and that the stabilization parameters α_θ, α_T, and τ are non-negative. Then the method has a unique solution in the following cases*:

(1) $\alpha_T, \alpha_\theta > 0$ *on* $\partial\Omega_h$.
(2) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $\alpha_\theta > 0$ *on at least one point of* $\partial\Omega_h$
(3) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $\tau > 0$ *on at least one point of* $\partial\Omega_h$
(4) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $k \geq 1$.

Next, we state an analogous result for the $d = 0$ case.

Theorem 3.2 *Consider the HDG method defined by the weak formulation* (2.1), *and the formulas* (2.2) *for the numerical traces. Suppose that the parameter d is equal to zero, namely, we consider HDG approximations to the Euler-Bernoulli beam model. Suppose* *further that the stabilization parameters α_θ, α_T, and τ are non-negative. Then the method has a unique solution in the following cases.*

(1) $\alpha_T, \alpha_\theta > 0$ *on* $\partial\Omega_h$.
(2) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $\alpha_\theta > 0$ *on* $\partial\Omega$.
(3) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $k \geq 1$, *and* $\tau > 0$ *on at least one point of* $\partial\Omega_h$.
(4) α_T *or* $\tau > 0$ *on at least one point of* ∂I_j *for all* j, *and* $k = 0$, *and* $\tau(x_0^+) > 0$, $\tau(x_1^-) = 0$, $\tau(x_{N-1}^+) = 0$, $\tau(x_N^-) > 0$.

Note that it is possible to take $\tau = 0$ on $\partial\Omega_h$. It is also possible to take $\alpha_T = \alpha_\theta = 0$ on $\partial\Omega_h$. Furthermore, we have existence and uniqueness if $\alpha_T = \alpha_\theta = 0$ on $\partial\Omega_h$, and $\tau > 0$ at only one node of each element. This corresponds to the single-face hybridizable (SFH) method considered in [6] for biharmonic problems. Note that, for the SFH methods it is required that $\tau > 0$ on $\partial\Omega$. For this one dimensional case, we were able to relax this condition for $k \geq 1$, as can be seen from Case (3) of Theorem 3.2.

To retain the clarity of the presentation, we give a proof of Theorems 3.1 and 3.2 in Appendix A.

4 Characterization of the Approximate Solution

In this section, we show that the *only* globally coupled unknowns of the HDG method defined by the weak formulation (2.1), and the formulas (2.2) for the numerical traces are the approximations at the nodes to the displacement and bending moment given by the numerical traces $\widehat{w}_h$ and $\widehat{M}_h$, respectively. We also show that the remaining components of the approximate solution can be expressed solely in terms of element-by-element-defined operators acting on $\widehat{w}_h$ and $\widehat{M}_h$. To do this, we follow the framework provided in [7].

4.1 The Local Solvers

We begin by introducing the above-mentioned locally defined operators which we call the *local solvers* associated with the method. The first local solver is defined on the element $K \in \Omega_h$ as the mapping $\omega \in L^2(\partial K) \mapsto (\mathcal{T}\omega, \mathcal{M}\omega, \Theta\omega, \mathcal{W}\omega) \in [\mathcal{P}^k(K)]^4$ where

$$-(\mathcal{W}\omega, v_1')_K + \langle \omega, v_1\, n\rangle_{\partial K} = (\Theta\omega, v_1)_K - d^2\,(\mathcal{T}\omega/GA, v_1)_K\,, \tag{4.1a}$$

$$-(\Theta\omega, v_2')_K + \langle \widehat{\Theta}\omega, v_2\, n\rangle_{\partial K} = (\mathcal{M}\omega/EI, v_2)_K\,, \tag{4.1b}$$

$$-(\mathcal{M}\omega, v_3')_K = (\mathcal{T}\omega, v_3)_K, \tag{4.1c}$$

$$-(\mathcal{T}\omega, v_4')_K + \langle \widehat{\mathcal{T}}\omega, v_4\, n\rangle_{\partial K} = 0 \tag{4.1d}$$

for all $v_i \in \mathcal{P}^k(K)$ for $i = 1, 2, 3, 4$. Here,

$$\widehat{\Theta}\omega = \Theta\omega - \alpha_\theta \mathcal{M}\omega\, n - \tau(\mathcal{W}\omega - \omega)\, n, \tag{4.2a}$$

$$\widehat{\mathcal{T}}\omega = \mathcal{T}\omega - \tau\mathcal{M}\omega\, n + \alpha_T(\mathcal{W}\omega - \omega)\, n. \tag{4.2b}$$

The second local solver is defined on the element $K \in \Omega_h$ as the mapping $\mu \in L^2(\partial K) \mapsto (\mathcal{T}\mu, \mathcal{M}\mu, \Theta\mu, \mathcal{W}\mu) \in [\mathcal{P}^k(K)]^4$ where

$$
\begin{aligned}
-(\mathcal{W}\mu, v_1')_K &= (\Theta\mu, v_1)_K - d^2\,(\mathcal{T}\mu/GA, v_1)_K\,, &\text{(4.3a)}\\
-(\Theta\mu, v_2')_K + \langle\widehat{\Theta}\mu, v_2\,n\rangle_{\partial K} &= (\mathcal{M}\mu/EI, v_2)_K\,, &\text{(4.3b)}\\
-(\mathcal{M}\mu, v_3')_K + \langle\mu, v_3\,n\rangle_{\partial K} &= (\mathcal{T}\mu, v_3)_K, &\text{(4.3c)}\\
-(\mathcal{T}\mu, v_4')_K + \langle\widehat{\mathcal{T}}\mu, v_4\,n\rangle_{\partial K} &= 0 &\text{(4.3d)}
\end{aligned}
$$

for all $v_i \in \mathcal{P}^k(K)$ for $i = 1, 2, 3, 4$. Here,

$$
\begin{aligned}
\widehat{\Theta}\mu &= \Theta\mu - \alpha_\theta(\mathcal{M}\mu - \mu)\,n - \tau\mathcal{W}\mu\,n, &\text{(4.4a)}\\
\widehat{\mathcal{T}}\mu &= \mathcal{T}\mu - \tau(\mathcal{M}\mu - \mu)\,n + \alpha_T\mathcal{W}\mu\,n. &\text{(4.4b)}
\end{aligned}
$$

The third local solver is defined on the element $K \in \Omega_h$ as the mapping $q \in L^2(K) \mapsto (\mathcal{T}q, \mathcal{M}q, \Theta q, \mathcal{W}q) \in [\mathcal{P}^k(K)]^4$ where

$$
\begin{aligned}
-(\mathcal{W}q, v_1')_K &= (\Theta q, v_1)_K - d^2\,(\mathcal{T}q/GA, v_1)_K\,, &\text{(4.5a)}\\
-(\Theta q, v_2')_K + \langle\widehat{\Theta}q, v_2\,n\rangle_{\partial K} &= (\mathcal{M}q/EI, v_2)_K\,, &\text{(4.5b)}\\
-(\mathcal{M}q, v_3')_K &= (\mathcal{T}q, v_3)_K, &\text{(4.5c)}\\
-(\mathcal{T}q, v_4')_K + \langle\widehat{\mathcal{T}}q, v_4\,n\rangle_{\partial K} &= (q, v_4)_K &\text{(4.5d)}
\end{aligned}
$$

for all $v_i \in \mathcal{P}^k(K)$ for $i = 1, 2, 3, 4$. Here,

$$
\begin{aligned}
\widehat{\Theta}q &= \Theta q - \alpha_\theta\mathcal{M}q\,n - \tau\mathcal{W}q\,n, &\text{(4.6a)}\\
\widehat{\mathcal{T}}q &= \mathcal{T}q - \tau\mathcal{M}q\,n + \alpha_T\mathcal{W}q\,n. &\text{(4.6b)}
\end{aligned}
$$

4.2 The Characterization of the HDG Solution for $k \geq 1$

The function w_D, as well as any other function defined only on $\partial\Omega = \{0, 1\}$ is extended to $\mathcal{E}_h$ by zero. We also set

$$
\omega_h := \begin{cases} \widehat{w}_h & \text{on } \partial\Omega_h\backslash\partial\Omega \\ 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7a}
$$

so that we have that $\widehat{w}_h = \omega_h + w_D$ where $\omega_h \in L^2_0(\mathcal{E}_h)$. Also, to simplify the notation we write

$$
\mu_h := \widehat{M}_h \quad \text{on } \partial\Omega_h. \tag{4.7b}
$$

We can now state a characterization of the approximate solution in terms of the local solvers.

Theorem 4.1 *Suppose that the conditions of Theorem 3.1 (Theorem 3.2, if $d = 0$) are satisfied. Suppose also that $k \geq 1$. Then the approximate solution $(T_h, M_h, \theta_h, w_h, \mu_h, \omega_h) \in [V_h^k]^4 \times L^2(\mathcal{E}_h) \times L^2_0(\mathcal{E}_h)$ given by the HDG method can be expressed in terms of the* *local solvers as*

$$T_h = \mathcal{T}\omega_h + \mathcal{T}w_D + \mathcal{T}\mu_h + \mathcal{T}q, \tag{4.8a}$$

$$M_h = \mathcal{M}\omega_h + \mathcal{M}w_D + \mathcal{M}\mu_h + \mathcal{M}q, \tag{4.8b}$$

$$\theta_h = \Theta\omega_h + \Theta w_D + \Theta\mu_h + \Theta q, \tag{4.8c}$$

$$w_h = \mathcal{W}\omega_h + \mathcal{W}w_D + \mathcal{W}\mu_h + \mathcal{W}q, \tag{4.8d}$$

where $(\mu_h, \omega_h) \in L^2(\mathscr{E}_h) \times L^2_0(\mathscr{E}_h)$ *satisfies*

$$a_h(\mu_h, \mathsf{m}) + b_h(\omega_h, \mathsf{m}) = \ell_h^{\Theta}(\mathsf{m}), \tag{4.9a}$$

$$b_h(\mathsf{w}, \mu_h) = \ell_h^{\mathcal{T}}(\mathsf{w}) \tag{4.9b}$$

for all $\mathsf{m} \in L^2(\mathscr{E}_h)$ *and* $\mathsf{w} \in L^2_0(\mathscr{E}_h)$. *Here*

$$\begin{aligned}
a_h(\mu, \mathsf{m}) &= (\mathcal{M}\mu/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2(\mathcal{T}\mu/GA, \mathcal{T}\mathsf{m})_{\Omega_h} \\
&\qquad + \langle 1, \alpha_\theta(\mu - \mathcal{M}\mu)(\mathsf{m} - \mathcal{M}\mathsf{m})\rangle_{\partial\Omega_h} + \langle 1, \alpha_T \mathcal{W}\mu\mathcal{W}\mathsf{m}\rangle_{\partial\Omega_h}, \\
b_h(\omega, \mathsf{m}) &= (\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h}, \\
\ell_h^{\Theta}(\mathsf{m}) &= -(q, \mathcal{W}\mathsf{m})_{\Omega_h} - (\Theta w_D, \mathcal{T}\mathsf{m})_{\Omega_h} + \langle \theta_N, \mathsf{m}\, n\rangle_{\partial\Omega}, \\
\ell_h^{\mathcal{T}}(\mathsf{w}) &= -(q, \mathcal{W}\mathsf{w})_{\Omega_h}.
\end{aligned}$$

Next, let us briefly discuss this result whose detailed proof can be found in Appendix B. Note that the system of equations for the numerical traces μ_h and ω_h is of the form

$$\begin{bmatrix} \mathsf{A}_h & \mathsf{B}_h \\ \mathsf{B}_h^t & 0 \end{bmatrix} \begin{bmatrix} [\mu_h] \\ [\omega_h] \end{bmatrix} = \begin{bmatrix} \mathsf{b}_{h,1} \\ \mathsf{b}_{h,2} \end{bmatrix}. \tag{4.10}$$

This is a reflection of the fact that μ_h approximates the bending moment M, that ω_h approximates the displacement w, and that we can write our original problem (1.1) as

$$\begin{bmatrix} \frac{I}{EI} - \frac{d}{dx}\left(\frac{d^2}{GA}\frac{d}{dx}\right) & -\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} M \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -q \end{bmatrix}.$$

Let us illustrate this in the case in which q, GA, and EI are constants, and we take a uniform grid of meshsize $h := 1/N$ *and* a polynomial degree $k \geq 3$. In this case, we obtain that

$$\mathsf{A}_h = \frac{1}{6EI}\begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & \ddots & & \\ & & 1 & \ddots & 1 & \\ & & & \ddots & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} - \frac{d^2}{h^2\,GA}\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & 1 & \ddots & 1 & \\ & & & \ddots & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}$$

and

$$\mathsf{B}_h = -\frac{1}{h^2}\begin{pmatrix} 1 & & & & \\ -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & 1 & \ddots & 1 \\ & & & \ddots & -2 \\ & & & & 1 \end{pmatrix}$$

where A_h is a square matrix of order $N+1$ and B_h an $(N+1)\times(N-1)$ matrix. Moreover,

$$\mathsf{b}_{1,h} = \begin{pmatrix} -w_D(x_0) \\ w_D(x_0) \\ 0 \\ \vdots \\ 0 \\ w_D(x_N) \\ -w_D(x_N) \end{pmatrix} + hq\begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{pmatrix} + h\begin{pmatrix} \theta_D(x_0) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \theta_D(x_N) \end{pmatrix}, \qquad \mathsf{b}_{2,h} = -q\begin{pmatrix} 1/2 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1/2 \end{pmatrix}$$

where $\mathsf{b}_{1,h}$ and $\mathsf{b}_{2,h}$ are $1\times(N+1)$ and $1\times(N-1)$ matrices, respectively.

We immediately recognize that the matrix A_h is a finite difference approximation of the operator $\frac{I}{EI} - \frac{d}{dx}(\frac{d^2}{GA}\frac{d}{dx})$ and that the matrix B_h is a finite difference approximation of $-\frac{d^2}{dx^2}$. Similarly, we see that $\mathsf{b}_{1,h}$ is an approximation of 0 and $\mathsf{b}_{2,h}$ an approximation of $-q$. Finally, note that all of the above matrices are *independent* of the stabilization functions τ, α_θ and α_T as well as of the polynomial degree k. This is a reflection of the fact that the approximation μ_h and ω_h is in fact *exact*. In other words, in this case, (4.10) is nothing but the matrix equation determining the values of the exact bending moment and displacement at the nodes.

4.3 Characterization of the HDG Solution for $k=0$

To end this section, we would like to note that the above characterization theorem is valid only for $k\ge 1$. This is not because the approach to obtain this result is different for $k=0$, but only because the local solvers take a form which does not allow for some simplifications that take place in the case $k\ge 1$. For $k=0$, the characterization of the approximation is given in the next result.

Theorem 4.2 *Suppose that the conditions of Theorem 3.1 (Theorem 3.2, if $d=0$) are satisfied. Suppose also that $k=0$. Then the approximate solution $(T_h, M_h, \theta_h, w_h, \mu_h, \omega_h) \in [V_h^0]^4 \times L^2(\mathscr{E}_h) \times L_0^2(\mathscr{E}_h)$ given by the HDG method can be expressed in terms of the local solvers as*

$$\begin{aligned} T_h &= \mathcal{T}\omega_h + \mathcal{T}w_D + \mathcal{T}\mu_h + \mathcal{T}q, \\ M_h &= \mathcal{M}\omega_h + \mathcal{M}w_D + \mathcal{M}\mu_h + \mathcal{M}q, \\ \theta_h &= \Theta\omega_h + \Theta w_D + \Theta\mu_h + \Theta q, \\ w_h &= \mathcal{W}\omega_h + \mathcal{W}w_D + \mathcal{W}\mu_h + \mathcal{W}q, \end{aligned}$$

where $(\mu_h, \omega_h) \in L^2(\mathscr{E}_h) \times L^2_0(\mathscr{E}_h)$ *satisfies*

$$a_h^\Theta(\mu_h, \mathsf{m}) + b_h^\Theta(\omega_h, \mathsf{m}) = \ell_h^\Theta(\mathsf{m}), \tag{4.11a}$$

$$a_h^\mathcal{T}(\mu_h, \mathsf{w}) + b_h^\mathcal{T}(\omega_h, \mathsf{w}) = \ell_h^\mathcal{T}(\mathsf{w}) \tag{4.11b}$$

for all $\mathsf{m} \in L^2(\mathscr{E}_h)$ *and* $\mathsf{w} \in L^2_0(\mathscr{E}_h)$. *Here*

$$\begin{aligned}
a_h^\Theta(\mu, \mathsf{m}) &= (\mathcal{M}\mu/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2(\mathcal{T}\mu/GA, \mathcal{T}\mathsf{m})_{\Omega_h} \\
&\quad + \langle 1, \alpha_T \mathcal{W}\mu\mathcal{W}\mathsf{m}\rangle_{\partial\Omega_h} + \langle 1, \alpha_\theta(\mu - \mathcal{M}\mu)(\mathsf{m} - \mathcal{M}\mathsf{m})\rangle_{\partial\Omega_h}, \\
b_h^\Theta(\omega, \mathsf{m}) &= (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + \langle \omega, (\mathcal{T}\mathsf{m})n\rangle_{\partial\Omega_h} \\
&\quad + \langle 1, \alpha_T \mathcal{W}\omega\mathcal{W}\mathsf{m}\rangle_{\partial\Omega_h} + \langle 1, (\tau\omega - \alpha_\theta\mathcal{M}\omega)(\mathsf{m} - \mathcal{M}\mathsf{m})\rangle_{\partial\Omega_h}, \\
a_h^\mathcal{T}(\mu, \mathsf{w}) &= -(\mathcal{M}\mu/EI, \mathcal{M}\mathsf{w})_{\Omega_h} + \langle \mu, (\Theta\mathsf{w})n\rangle_{\partial\Omega_h} \\
&\quad - \langle 1, \alpha_\theta\mathcal{M}\mu\mathcal{M}\mathsf{w}\rangle_{\partial\Omega_h} + \langle 1, (\tau\mu + \alpha_T\mathcal{W}\mu)(\mathsf{w} - \mathcal{W}\mathsf{w})\rangle_{\partial\Omega_h}, \\
b_h^\mathcal{T}(\omega, \mathsf{w}) &= -(\mathcal{M}\omega/EI, \mathcal{M}\mathsf{w})_{\Omega_h} - \langle 1, \alpha_\theta\mathcal{M}\omega\mathcal{M}\mathsf{w}\rangle_{\partial\Omega_h} \\
&\quad - \langle 1, \alpha_T(\omega - \mathcal{W}\omega)(\mathsf{w} - \mathcal{W}\mathsf{w})\rangle_{\partial\Omega_h},
\end{aligned}$$

and

$$\begin{aligned}
\ell_h^\Theta(\mathsf{m}) &= -(q, \mathcal{W}\mathsf{m})_{\Omega_h} - \langle w_D, (\widehat{\mathcal{T}}\mathsf{m})n\rangle_{\partial\Omega} + \langle \theta_N, \mathsf{m}\,n\rangle_{\partial\Omega}, \\
\ell_h^\mathcal{T}(\mathsf{w}) &= -(q, \mathcal{W}\mathsf{w})_{\Omega_h} - \langle w_D, (\widehat{\mathcal{T}}\mathsf{w})n\rangle_{\partial\Omega}.
\end{aligned}$$

The proof of this result easily follows from Lemma B.2 in Appendix B.

5 Relationship Between the HDG Methods and Other DG Methods

In this section, we investigate the relationship between the HDG methods introduced in this paper with the DG methods for Timoshenko beams introduced in [3], and analyzed in [4] and [1].

These methods were defined by the weak formulation, (2.1a)–(2.1d). The conservativity conditions (2.1e) and (2.1f) were not taken into account explicitly, because they were automatically satisfied by the definition of the numerical traces which we give next. For an interior node $x \in \mathscr{E}_h^\circ$ they are defined by

$$\widehat{w}_h = \{\!\{ w_h \}\!\} + C_{11}[\![w_h]\!] + C_{12}[\![\theta_h]\!] + C_{13}[\![M_h]\!] + C_{14}[\![T_h]\!], \tag{5.1a}$$

$$\widehat{\theta}_h = \{\!\{\theta_h\}\!\} + C_{21}[\![w_h]\!] + C_{22}[\![\theta_h]\!] + C_{23}[\![M_h]\!] + C_{24}[\![T_h]\!], \tag{5.1b}$$

$$\widehat{M}_h = \{\!\{M_h\}\!\} + C_{31}[\![w_h]\!] + C_{32}[\![\theta_h]\!] + C_{33}[\![M_h]\!] + C_{34}[\![T_h]\!], \tag{5.1c}$$

$$\widehat{T}_h = \{\!\{T_h\}\!\} + C_{41}[\![w_h]\!] + C_{42}[\![\theta_h]\!] + C_{43}[\![M_h]\!] + C_{44}[\![T_h]\!], \tag{5.1d}$$

where the *average*, $\{\!\{\cdot\}\!\}$, and the *jump*, $[\![\cdot]\!]$, operators are defined by

$$\{\!\{\varphi\}\!\}(x) := \frac{1}{2}(\varphi(x^+) + \varphi(x^-)), \qquad [\![\varphi]\!](x) := \varphi(x^-) - \varphi(x^+).$$

At $x = 0$, the numerical traces of the DG methods are defined by

$$\widehat{w}_h(0) = w_D(0), \tag{5.2a}$$

$$\widehat{\theta}_h(0) = \theta_N(0), \tag{5.2b}$$

$$\widehat{M}_h(0) = M_h(0^+) + C_{31}(0)(w_D(0) - w_h(0^+)) + C_{32}(0)(\theta_N(0) - \theta_h(0^+)), \tag{5.2c}$$

$$\widehat{T}_h(0) = T_h(0^+) + C_{41}(0)(w_D(0) - w_h(0^+)) + C_{42}(0)(\theta_N(0) - \theta_h(0^+)). \tag{5.2d}$$

And at $x = 1$,

$$\widehat{w}_h(1) = w_D(1), \tag{5.3a}$$

$$\widehat{\theta}_h(1) = \theta_N(1), \tag{5.3b}$$

$$\widehat{M}_h(1) = M_h(1^-) + C_{31}(1)(w_h(1^-) - w_D(1)) + C_{32}(1)(\theta_h(1^-) - \theta_N(1)), \tag{5.3c}$$

$$\widehat{T}_h(1) = T_h(1^-) + C_{41}(1)(w_h(1^-) - w_D(1)) + C_{42}(1)(\theta_h(1^-) - \theta_N(1)). \tag{5.3d}$$

The following theorem provides a recipe for choosing the coefficients C_{ij}, $1 \le i, j \le 4$, so that the solution provided by an HDG method is identical to the one provided by the previously defined DG methods. We emphasize the fact we are interested in such an identification only for theoretical purposes because the HDG methods are computationally much more efficient than the DG methods as we have shown in Sect. 4.

Let us introduce some notation that will be used in the next theorem. For any two functions φ and α which are possibly double valued on $\mathscr{E}_h^\circ$ we define

$$[\![\varphi]\!]_\alpha = \varphi^-\alpha^+ - \varphi^+\alpha^-.$$

Let us also note that

$$[\![\varphi n]\!] = (\varphi n)^- - (\varphi n)^+ = \varphi^- n^- - \varphi^+ n^+ = \varphi^- + \varphi^+,$$

and, similarly, that

$$[\![\varphi n]\!]_\alpha = \varphi^-\alpha^+ + \varphi^+\alpha^-.$$

We are now ready to state our result.

Theorem 5.1 *Let $(T_h^H, M_h^H, \theta_h^H, w_h^H, \widehat{M}_h, \widehat{w}_h)$ be the solution of the HDG method defined by the weak formulation (2.1) where the numerical traces $\widehat{w}_h, \widehat{\theta}_h$ and $\widehat{T}_h$ satisfy (2.2). Suppose that $(T_h, M_h, \theta_h, w_h)$ is the solution of the DG method defined by the weak formulation (2.1a)–(2.1d), and the formulas (5.1)–(5.3) for the numerical traces. Then*

$$(T_h^H, M_h^H, \theta_h^H, w_h^H) = (T_h, M_h, \theta_h, w_h)$$

provided that

$$\begin{aligned} C_{11} &= ([\![\tau n]\!][\![\tau]\!] + [\![\alpha_\theta n]\!][\![\alpha_T]\!])/2D, \\ C_{12} &= -[\![\tau n]\!]/D, \\ C_{13} &= -[\![\tau]\!]_{\alpha_\theta}/D, \\ C_{14} &= [\![\alpha_\theta n]\!]/D, \end{aligned} \tag{5.4}$$

$$\begin{aligned}
C_{21} &= -([\![\alpha_\theta\alpha_T]\!]_\tau + [\![\tau^2 n]\!]_\tau)/D, \\
C_{22} &= -([\![\tau n]\!][\![\tau]\!] + [\![\alpha_T n]\!][\![\alpha_\theta]\!])/2D, \\
C_{23} &= -([\![\alpha_\theta n]\!]_{\tau^2} + [\![\alpha_\theta\alpha_T n]\!]_{\alpha_\theta})/D, \\
C_{24} &= -C_{13},
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
C_{31} &= [\![\tau]\!]_{\alpha_T}/D, \\
C_{32} &= -[\![\alpha_T n]\!]/D \\
C_{33} &= -C_{22}, \\
C_{34} &= C_{12},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
C_{41} &= ([\![\alpha_T n]\!]_{\tau^2} + [\![\alpha_T\alpha_\theta n]\!]_{\alpha_T})/D, \\
C_{42} &= -C_{31}, \\
C_{43} &= C_{21}, \\
C_{44} &= -C_{11},
\end{aligned} \tag{5.7}$$

at the interior nodes $x \in \mathscr{E}_h^\circ$, *and*

$$\begin{aligned}
C_{31} &= \tau n/\alpha_\theta, \\
C_{32} &= -1/\alpha_\theta, \\
C_{41} &= (\tau^2 + \alpha_T\alpha_\theta)/\alpha_\theta, \\
C_{42} &= -C_{31},
\end{aligned} \tag{5.8}$$

at $\partial\Omega$. *Here*

$$D := [\![\tau n]\!]^2 + [\![\alpha_T n]\!][\![\alpha_\theta n]\!],$$

and we assume that $D > 0$ *on* $\mathscr{E}_h^\circ$, *and* $\alpha_\theta > 0$ *on* $\partial\Omega$.

A detailed proof of this theorem can be found in Appendix C. Next, we discuss three applications of this result.

• An optimally convergent DG method was studied by Celiker *et al.* in [4]. This method is obtained by setting

$$C_{11} = C_{22} = -C_{33} = -C_{44} = 1/2,$$

at all interior nodes,

$$-C_{32}(1) = C_{41}(1) = \frac{1}{h_N},$$

and all of the remaining coefficients to zero. It is not difficult to see that it is impossible to express this DG method as an HDG method. In other words, it is impossible to pick the parameters τ, α_T, and α_θ in (2.2) so that the resulting numerical traces coincide with the numerical traces of the DG method above. Let us see why this is so. For this DG method

$C_{12}(x)=0$ at all interior nodes $x\in\mathcal{E}_h^\circ$. By Theorem 5.1 $C_{12}=[\![\tau n]\!]/D=(\tau^-+\tau^+)/D$. This implies that $\tau^-+\tau^+=0$, and hence that $\tau^-=\tau^+=0$. Then

$$D=[\![\tau n]\!]^2+[\![\alpha_T n]\!][\![\alpha_\theta n]\!]=[\![\alpha_T n]\!][\![\alpha_\theta n]\!]=(\alpha_T^-+\alpha_T^+)(\alpha_\theta^-+\alpha_\theta^+),$$

and since we have assumed $D>0$ at the interior nodes, we must have that

$$\alpha_T^- \text{ or } \alpha_T^+>0 \quad \text{and} \quad \alpha_\theta^- \text{ or } \alpha_\theta^+>0, \tag{5.9}$$

at all interior nodes. On the other hand, $C_{14}=0$ implies by Theorem 5.1 that $[\![\alpha_\theta n]\!]/D=0$, and hence $\alpha_\theta^-+\alpha_\theta^+=0$. But this is possible only if $\alpha_\theta^-=\alpha_\theta^+=0$, which contradicts the second part of (5.9).

• Another family of DG method studied in Celiker [1] is obtained as follows. Let c be an arbitrary positive number independent of the mesh-size h, and suppose that

$$C_{14}(x)=-C_{23}(x)=-C_{32}(x)=C_{41}(x)=\mathsf{c} \quad \forall x\in\mathcal{E}_h. \tag{5.10}$$

Suppose further that

$$\begin{aligned}(C_{ii}(x)-1/2)^2\le\mathsf{c} \qquad &\text{for } i=1,2,3,4,\\ C_{12}^2(x),\ C_{13}^2(x),\ C_{21}^2(x),\ C_{24}^2(x),\ C_{31}^2(x),\ C_{34}^2(x),\ C_{42}^2(x),\ &C_{43}^2\le\mathsf{c},\end{aligned} \tag{5.11}$$

for all $x\in\mathcal{E}_h$. It was shown in [1] that these methods converge optimally in $L^2(\Omega_h)$-norm to the exact solution for all the unknowns. By a straightforward computation we can show that these DG methods can be expressed as HDG methods if we take

$$\mathsf{c}=\frac{1}{2},$$

and

$$\tau=0, \qquad \alpha_T=\alpha_\theta=1$$

at all interior nodes, and

$$\tau=0, \qquad \alpha_T=\frac{1}{2}, \qquad \alpha_\theta=2$$

at the boundary nodes. It is easy to see that with this choice of the parameters the conditions of Theorem 5.1 are satisfied, and hence the resulting DG method is identical to the HDG method.

• The one-dimensional version of the SFH method for the biharmonic [6] takes $\alpha_T=\alpha_\theta=0$ on $\partial\Omega_h$, and $\tau>0$ at one end and zero at the other end of every subinterval. It is also required that for the subintervals which intersect $\partial\Omega$ we should take $\tau>0$. We can thus take

$$\begin{aligned}\tau(x_0^+)>0, \qquad \tau(x_j^+)=0 \quad &\text{for } j=1,2,\dots,N-1,\\ \tau(x_1^-)=0, \qquad \tau(x_j^-)>0 \quad &\text{for } j=2,3,\dots,N.\end{aligned} \tag{5.12}$$

In particular, $\tau(x_1^-)=\tau(x_1^+)=0$ and as a consequence, $D(x_1)=[\![\tau n]\!]^2(x_1)=0$. Therefore, by Theorem 5.1, we see that it is impossible to identify this SFH method with a previously introduced DG method. A similar argument can be used to show that this is true for any other SFH method.

6 Numerical Results

In this section, we display numerical results showing the performance of the numerical methods considered in this paper. We solve (1.1) with

$$q(x) = e^x, \qquad (EI)(x) = e^x, \qquad (GA)(x) = e^{-x},$$

together with the boundary conditions

$$w(0) = w(1) = \theta(0) = \theta(1) = 0.$$

To study the effect of the penalization parameters τ, α_T, and α_θ on the performance of the method, we display results for a variety of choices of these parameters. Also, to find out if the HDG method is free from shear locking, the thickness of the beam, d, is taken be 10^{-2} and then decreased down to 10^{-8}.

We display our numerical results in Tables 1 through 10. In Table 1, we display results corresponding to the HDG method obtained by setting $\alpha_T = \alpha_\theta = \tau = 1$ on $\partial\Omega_h$. Therein, the column labeled k denotes the polynomial degree used for the approximation, and "mesh $= i$" means we employed a uniform mesh with 2^i elements. We also display numerical orders of convergence which are computed as follows. Let $e_u(i)$ denote the error where a mesh with 2^i elements has been employed to obtain the HDG solution. Then the order of convergence, r_i, at level i is defined as

$$r_i := \frac{\log(\frac{e_u(i-1)}{e_u(i)})}{\log 2}.$$

The first part of the table shows the numerical results for the problem (1.1) with $d = 10^{-2}$, and the second part for $d = 10^{-8}$. We see that the method converges optimally for all the variables. We also notice that the convergence is independent of the parameter d. This indicates that the method is free from shear locking.

In Table 2, we carry out a similar history of convergence study for the SFH method (5.12). We see that the method converges optimally for all the variables, and the convergence remains invariant as we decrease d. It is worth noting that we do not see the suboptimal convergence of the same method that was proved, and observed numerically, for the biharmonic problems in two-space dimensions [6].

In order to be able to display an extensive set of numerical results, in Tables 3 through 10 we report only a summary of what we have observed as a result of our study of the history of convergence. In these tables the column k again shows the polynomial degree of the approximation, and the columns $\|e_u\|_{L^2(\Omega)}$, for $u = T, M, \theta, w$ shows the order of convergence we have observed as a result of the numerical experiments whose details we suppress. The dash "$-$" means that the method failed to converge for that particular combination of the polynomial degree and the choice of the stabilization parameters.

From Table 3 we see that whenever $\alpha_T \equiv 0$, $\alpha_\theta \equiv 0$, and τ identically equal to a constant on $\partial\Omega_h$, the only choice which converges optimally for all the variables is obtained by setting $\tau \equiv O(1)$. Table 5 shows that in an analogous situation we have a little more flexibility if we employ an SFH method. That is, if we set $\alpha_T \equiv 0$, $\alpha_\theta \equiv 0$, and $\tau^- \equiv 0$ then we observe optimal convergence for $\tau^+ = 1/h^\mu$ if $\mu = 0, 1, 2$.

In Table 4 we consider the other extreme, namely, we take $\tau \equiv 0$, and $\alpha_T = \alpha_\theta \equiv \nu$ on $\partial\Omega_h$ for some constant $\nu \geq 0$. Once again, we see that the only choice to get optimal convergence for all the unknowns is obtained by setting $\nu = O(1)$.

Table 1 $\tau \equiv 1, \alpha_\theta \equiv 1, \alpha_T \equiv 1$

k	Mesh	$\|e_T\|_{L^2(\Omega)}$ Error	Order	$\|e_M\|_{L^2(\Omega)}$ Error	Order	$\|e_\theta\|_{L^2(\Omega)}$ Error	Order	$\|e_W\|_{L^2(\Omega)}$ Error	Order
				$d = 10^{-2}$					
0	5	1.37E-02	0.82	7.34E-03	0.86	7.41E-03	0.94	1.61E-02	0.99
	6	7.45E-03	0.88	3.89E-03	0.91	3.78E-03	0.97	8.05E-03	1.00
	7	3.90E-03	0.93	2.01E-03	0.95	1.91E-03	0.99	4.03E-03	1.00
	8	2.00E-03	0.96	1.03E-03	0.97	9.60E-04	0.99	2.02E-03	1.00
1	3	1.32E-03	1.98	6.63E-04	1.91	1.27E-03	1.92	6.69E-04	2.01
	4	3.33E-04	1.99	1.70E-04	1.96	3.27E-04	1.96	1.67E-04	2.00
	5	8.36E-05	1.99	4.31E-05	1.98	8.29E-05	1.98	4.16E-05	2.00
	6	2.09E-05	2.00	1.08E-05	1.99	2.09E-05	1.99	1.04E-05	2.00
2	3	1.27E-05	2.98	9.93E-06	2.96	1.33E-05	2.98	4.57E-06	2.99
	4	1.60E-06	2.99	1.26E-06	2.98	1.68E-06	2.99	5.73E-07	3.00
	5	2.00E-07	2.99	1.58E-07	2.99	2.10E-07	3.00	7.16E-08	3.00
	6	2.51E-08	3.00	1.98E-08	3.00	2.63E-08	3.00	8.94E-09	3.00
3	3	6.79E-08	3.95	3.87E-08	3.96	1.49E-07	3.96	1.04E-07	3.97
	4	4.29E-09	3.98	2.44E-09	3.99	9.46E-09	3.98	6.55E-09	3.99
	5	2.69E-10	3.99	1.53E-10	4.00	5.95E-10	3.99	4.11E-10	3.99
	6	1.69E-11	4.00	9.55E-12	4.00	3.73E-11	4.00	2.57E-11	4.00
				$d = 10^{-8}$					
0	5	1.39E-02	0.81	7.40E-03	0.86	7.42E-03	0.94	1.61E-02	0.99
	6	7.55E-03	0.88	3.93E-03	0.91	3.78E-03	0.97	8.05E-03	1.00
	7	3.96E-03	0.93	2.03E-03	0.95	1.91E-03	0.99	4.03E-03	1.00
	8	2.03E-03	0.96	1.04E-03	0.97	9.61E-04	0.99	2.02E-03	1.00
1	3	1.32E-03	1.98	6.64E-04	1.91	1.28E-03	1.92	6.69E-04	2.01
	4	3.33E-04	1.99	1.70E-04	1.96	3.28E-04	1.96	1.67E-04	2.00
	5	8.35E-05	1.99	4.31E-05	1.98	8.30E-05	1.98	4.16E-05	2.00
	6	2.09E-05	2.00	1.08E-05	1.99	2.09E-05	1.99	1.04E-05	2.00
2	3	1.27E-05	2.98	9.92E-06	2.96	1.34E-05	2.98	4.57E-06	2.99
	4	1.60E-06	2.99	1.26E-06	2.98	1.68E-06	2.99	5.73E-07	3.00
	5	2.00E-07	2.99	1.58E-07	2.99	2.11E-07	3.00	7.16E-08	3.00
	6	2.51E-08	3.00	1.98E-08	3.00	2.64E-08	3.00	8.95E-09	3.00
3	3	6.78E-08	3.95	3.87E-08	3.96	1.49E-07	3.96	1.04E-07	3.97
	4	4.29E-09	3.98	2.43E-09	3.99	9.46E-09	3.98	6.55E-09	3.99
	5	2.69E-10	3.99	1.53E-10	4.00	5.95E-10	3.99	4.11E-10	3.99
	6	1.68E-11	4.00	9.54E-12	4.00	3.73E-11	4.00	2.57E-11	4.00

In Table 6 we carry out a more extensive study of what we have done for Table 1. We take $\tau \equiv \alpha_\theta \equiv \alpha_T \equiv \nu$ for $\nu = h^2, h, 1, h^{-1}, h^{-2}$. We see that the only choice among these options to get optimal convergence for all the unknowns is the one we considered in Table 1,

Table 2 $\tau^{+} \equiv 1, \tau^{-} \equiv 0, \alpha_\theta \equiv 0, \alpha_T \equiv 0$

k	Mesh	$\|e_T\|_{L^2(\Omega)}$		$\|e_M\|_{L^2(\Omega)}$		$\|e_\theta\|_{L^2(\Omega)}$		$\|e_w\|_{L^2(\Omega)}$	
		Error	Order	Error	Order	Error	Order	Error	Order
				$d = 10^{-2}$					
0	5	7.09E-02	0.86	4.95E-03	1.27	4.95E-04	0.96	1.17E-03	1.09
	6	3.72E-02	0.93	1.98E-03	1.33	2.70E-04	0.88	5.75E-04	1.02
	7	1.91E-02	0.97	7.99E-04	1.31	1.43E-04	0.92	2.87E-04	1.00
	8	9.64E-03	0.98	3.37E-04	1.24	7.36E-05	0.95	1.44E-04	1.00
1	3	1.94E-03	2.55	3.52E-03	1.78	2.33E-04	2.04	4.66E-04	2.10
	4	3.84E-04	2.34	9.43E-04	1.90	5.74E-05	2.02	1.14E-04	2.03
	5	8.75E-05	2.13	2.44E-04	1.95	1.43E-05	2.00	2.83E-05	2.01
	6	2.13E-05	2.04	6.20E-05	1.98	3.58E-06	2.00	7.07E-06	2.00
2	3	1.31E-05	3.00	3.42E-05	2.81	8.41E-06	2.91	1.59E-05	2.92
	4	1.64E-06	3.00	4.57E-06	2.90	1.08E-06	2.96	2.03E-06	2.97
	5	2.04E-07	3.00	5.90E-07	2.95	1.37E-07	2.98	2.57E-07	2.98
	6	2.55E-08	3.00	7.50E-08	2.98	1.73E-08	2.99	3.23E-08	2.99
3	3	9.87E-08	3.99	2.58E-07	3.81	1.24E-07	3.98	1.89E-07	4.04
	4	6.16E-09	4.00	1.72E-08	3.90	7.81E-09	3.99	1.17E-08	4.02
	5	3.84E-10	4.00	1.11E-09	3.95	4.89E-10	4.00	7.24E-10	4.01
	6	2.40E-11	4.00	7.07E-11	3.98	3.06E-11	4.00	4.51E-11	4.00
				$d = 10^{-8}$					
0	5	7.11E-02	0.86	4.95E-03	1.27	4.98E-04	0.95	1.16E-03	1.09
	6	3.73E-02	0.93	1.97E-03	1.33	2.72E-04	0.87	5.74E-04	1.02
	7	1.91E-02	0.97	7.96E-04	1.31	1.44E-04	0.92	2.87E-04	1.00
	8	9.67E-03	0.98	3.36E-04	1.25	7.43E-05	0.95	1.43E-04	1.00
1	3	1.94E-03	2.55	3.52E-03	1.78	2.33E-04	2.04	4.66E-04	2.09
	4	3.85E-04	2.34	9.43E-04	1.90	5.73E-05	2.02	1.14E-04	2.03
	5	8.75E-05	2.14	2.44E-04	1.95	1.43E-05	2.00	2.83E-05	2.01
	6	2.13E-05	2.04	6.20E-05	1.98	3.57E-06	2.00	7.06E-06	2.00
2	3	1.31E-05	3.00	3.42E-05	2.81	8.40E-06	2.91	1.59E-05	2.92
	4	1.64E-06	3.00	4.57E-06	2.90	1.08E-06	2.96	2.03E-06	2.97
	5	2.04E-07	3.00	5.90E-07	2.95	1.37E-07	2.98	2.57E-07	2.98
	6	2.55E-08	3.00	7.50E-08	2.98	1.73E-08	2.99	3.23E-08	2.99
3	3	9.87E-08	3.99	2.58E-07	3.81	1.24E-07	3.98	1.89E-07	4.04
	4	6.16E-09	4.00	1.72E-08	3.90	7.80E-09	3.99	1.16E-08	4.02
	5	3.84E-10	4.00	1.11E-09	3.95	4.89E-10	4.00	7.23E-10	4.01
	6	2.40E-11	4.00	7.07E-11	3.98	3.06E-11	4.00	4.51E-11	4.00

namely, take $\nu = 1$. All the other options result in suboptimal convergence for either M and w, or for T and θ.

In all of the Tables 1 through 6 we have considered cases where $\alpha_T \equiv \alpha_\theta$. Therefore, in Tables 7 to 10 we consider possible cases in which these two parameters take different

Table 3 $\alpha_\theta \equiv 0, \alpha_T \equiv 0$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_{\mathcal{W}}\|_{L^2(\Omega)}$
		$\tau \equiv h^2$		
0	–	–	–	–
1	k	–	k	–
2	$k+1$	$k-1$	k	–
3	$k+1$	$k-1$	k	$k-2$
		$\tau \equiv h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	k
2	$k+1$	k	$k+1$	k
3	$k+1$	k	$k+1$	k
		$\tau \equiv 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\tau \equiv 1/h$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$
		$\tau \equiv 1/h^2$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$

Table 4 $\tau \equiv 0$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_{\mathcal{W}}\|_{L^2(\Omega)}$
		$\alpha_\theta \equiv \alpha_T \equiv h^2$		
0	–	–	–	–
1	$k+1$	k	$k+1$	–
2	$k+1$	k	$k+1$	$k-1$
3	$k+1$	k	$k+1$	$k-1$
		$\alpha_\theta \equiv \alpha_T \equiv h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	k
2	$k+1$	k	$k+1$	k
3	$k+1$	k	$k+1$	k
		$\alpha_\theta \equiv \alpha_T \equiv 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_\theta \equiv \alpha_T \equiv 1/h$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$
		$\alpha_\theta \equiv \alpha_T \equiv 1/h^2$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$

Table 5 $\tau^- \equiv 0, \alpha_\theta \equiv 0, \alpha_T \equiv 0$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_\mathcal{W}\|_{L^2(\Omega)}$
		$\tau^+ \equiv h^2$		
0	–	–	–	–
1	k	–	k	–
2	$k+1$	$k-1$	k	–
3	$k+1$	$k-1$	k	$k-2$
		$\tau^+ \equiv h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	k
2	$k+1$	k	$k+1$	k
3	$k+1$	k	$k+1$	k
		$\tau^+ \equiv 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\tau^+ \equiv 1/h$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\tau^+ \equiv 1/h^2$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$

Table 6 $\tau \equiv \alpha_\theta \equiv \alpha_T \equiv \nu$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_\mathcal{W}\|_{L^2(\Omega)}$
		$\nu = h^2$		
0	–	–	–	–
1	$k+1$	k	$k+1$	–
2	$k+1$	k	$k+1$	$k-1$
3	$k+1$	k	$k+1$	$k-1$
		$\nu = h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	k
2	$k+1$	k	$k+1$	k
3	$k+1$	k	$k+1$	k
		$\nu = 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\nu = 1/h$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$
		$\nu = 1/h^2$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$

Table 7 $\alpha_\theta \equiv 1, \alpha_T \equiv 0$

k	$\lVert e_T \rVert_{L^2(\Omega)}$	$\lVert e_M \rVert_{L^2(\Omega)}$	$\lVert e_\theta \rVert_{L^2(\Omega)}$	$\lVert e_{\mathcal{W}} \rVert_{L^2(\Omega)}$
		$\tau = h^2$		
0	–	–	–	–
1	k	–	k	–
2	$k+1$	$k-1$	k	–
3	$k+1$	$k-1$	k	–
		$\tau = h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	–
2	$k+1$	k	$k+1$	$k-1$
3	$k+1$	k	$k+1$	$k-1$
		$\tau = 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\tau = 1/h$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$
		$\tau = 1/h^2$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$

Table 8 $\alpha_\theta \equiv 0, \alpha_T \equiv 1$

k	$\lVert e_T \rVert_{L^2(\Omega)}$	$\lVert e_M \rVert_{L^2(\Omega)}$	$\lVert e_\theta \rVert_{L^2(\Omega)}$	$\lVert e_{\mathcal{W}} \rVert_{L^2(\Omega)}$
		$\tau = h^2$		
0	–	–	–	–
1	$k+1$	k	$k+1$	$k+1$
2	$k+1$	k	$k+1$	$k+1$
3	$k+1$	k	$k+1$	$k+1$
		$\tau = h$		
0	–	–	–	–
1	$k+1$	k	$k+1$	$k+1$
2	$k+1$	k	$k+1$	$k+1$
3	$k+1$	k	$k+1$	$k+1$
		$\tau = 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\tau = 1/h$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$
		$\tau = 1/h^2$		
0	–	–	–	–
1	k	$k+1$	k	$k+1$
2	k	$k+1$	k	$k+1$
3	k	$k+1$	k	$k+1$

Table 9 $\alpha_T \equiv 0, \tau \equiv 1$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_{\mathcal{W}}\|_{L^2(\Omega)}$
		$\alpha_\theta = h^2$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_\theta = h$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_\theta = 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_\theta = 1/h$		
0	–	–	–	–
1	$k+1$	$k+1$	k	k
2	$k+1$	$k+1$	k	k
3	$k+1$	$k+1$	k	k
		$\alpha_\theta = 1/h^2$		
0	–	–	–	–
1	$k+1$	$k+1$	–	–
2	$k+1$	$k+1$	$k-1$	$k-1$
3	$k+1$	$k+1$	$k-1$	$k-1$

Table 10 $\alpha_\theta \equiv 0, \tau \equiv 1$

k	$\|e_T\|_{L^2(\Omega)}$	$\|e_M\|_{L^2(\Omega)}$	$\|e_\theta\|_{L^2(\Omega)}$	$\|e_{\mathcal{W}}\|_{L^2(\Omega)}$
		$\alpha_T = h^2$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_T = h$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_T = 1$		
0	$k+1$	$k+1$	$k+1$	$k+1$
1	$k+1$	$k+1$	$k+1$	$k+1$
2	$k+1$	$k+1$	$k+1$	$k+1$
3	$k+1$	$k+1$	$k+1$	$k+1$
		$\alpha_T = 1/h$		
0	–	–	–	–
1	k	k	$k+1$	$k+1$
2	k	k	$k+1$	$k+1$
3	k	k	$k+1$	$k+1$
		$\alpha_T = 1/h^2$		
0	–	–	–	–
1	–	k	k	k
2	$k-1$	k	$k+1$	$k+1$
3	$k-1$	k	$k+1$	$k+1$

values. We observe similar convergence behavior to what we already observed in the cases we considered in Tables 1 through 6. An interesting observation that might be worth noting that we did not observe any convergence at all for w_h if $\alpha_\theta \equiv 1$, $\alpha_T \equiv 0$, and $\tau = h^2$, see Table 7. Another observation we would like to emphasize is that the choice $\alpha_T \equiv 0$, $\tau \equiv 1$ converges optimally for $\alpha_\theta = h^\mu$ with $\mu = 0, 1, 2$. We have a similar behavior if we keep τ the same but switch the roles of α_T and α_θ. These conclusions are based on Tables 9 and 10.

7 Concluding Remarks

We have introduced a new class of discontinuous Galerkin methods for Timoshenko beams called HDG methods and shown that they can be efficiently implemented through an hybridization procedure. Our numerical results indicate that it is possible to devise optimally convergent HDG methods which are free from shear locking for a variety of choices of the stabilization parameters. A rigorous error analysis of these HDG methods constitutes the subject of a forthcoming paper.

Appendix A: Proof of the Existence and Uniqueness Results

In this appendix we give proofs of the existence and uniqueness theorems stated in Sect. 3. The proofs are based on the following technical lemmas.

Lemma A.1 *Let $(T_h, M_h, \theta_h, w_h, \widehat{M}_h, \widehat{w}_h)$ be the HDG solution defined by the weak formulation* (2.1), *and the formulas* (2.2) *for the numerical traces. Then we have the following identities*:

$$\begin{aligned}
&(M_h/EI, M_h)_{\Omega_h} + d^2\,(T_h/GA, T_h)_{\Omega_h}\\
&\qquad - \langle \widehat{\theta}_h - \theta_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h - T_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h}\\
&\quad = (q, w_h)_{\Omega_h} + \langle \widehat{\theta}_h n, \widehat{M}_h\rangle_{\partial\Omega} - \langle \widehat{T}_h n, \widehat{w}_h\rangle_{\partial\Omega},
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&(\theta_h, \theta_h)_{\Omega_h} - \langle \widehat{\theta}_h - \theta_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h}\\
&\quad = d^2\,(T_h/GA, \theta_h)_{\Omega_h} - (M_h/EI, w_h)_{\Omega_h} + \langle \widehat{\theta}_h n, \widehat{w}_h\rangle_{\partial\Omega},
\end{aligned} \tag{A.2}$$

$$(T_h, T_h)_{\Omega_h} - \langle \widehat{T}_h - T_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h} = -(q, M_h)_{\Omega_h} + \langle \widehat{T}_h n, \widehat{M}_h\rangle_{\partial\Omega}. \tag{A.3}$$

Lemma A.2 *With the same notation as in Lemma A.1 the following identities hold true*

$$\begin{aligned}
&-\langle \widehat{\theta}_h - \theta_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h - T_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h}\\
&\quad = \langle \alpha_\theta, (M_h - \widehat{M}_h)^2\rangle_{\partial\Omega_h} + \langle \alpha_T, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h},
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
&-\langle \widehat{\theta}_h - \theta_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h}\\
&\quad = \langle \tau, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h} + \langle \alpha_\theta, (M_h - \widehat{M}_h)(w_h - \widehat{w}_h)\rangle_{\partial\Omega_h},
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&-\langle \widehat{T}_h - T_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h}\\
&\quad = \langle \tau, (M_h - \widehat{M}_h)^2\rangle_{\partial\Omega_h} - \langle \alpha_T, (M_h - \widehat{M}_h)(w_h - \widehat{w}_h)\rangle_{\partial\Omega_h}.
\end{aligned} \tag{A.6}$$

The proofs of Lemmas A.1 and A.2 are delayed until the end of this section.

Before starting to prove Theorems 3.1 and 3.2 we state and prove an auxiliary lemma in which we collect some intermediate results.

Lemma A.3 *Consider the HDG method defined by the weak formulation* (2.1), *and the formulas* (2.2) *for the numerical traces. Suppose that the data of the problem is given by*

$$q = 0 \quad \text{in } \Omega, \qquad w_D = \theta_N = 0 \quad \text{on } \partial\Omega \tag{A.7}$$

and that the stabilization parameters α_θ, α_T, *and* τ *are non-negative. Then we have*

$$d\,T_h = 0 \quad \text{in } \Omega_h, \tag{A.8a}$$

$$M_h = 0 \quad \text{in } \Omega_h, \tag{A.8b}$$

$$\theta_h = 0 \quad \text{in } \Omega_h, \tag{A.8c}$$

$$\alpha_\theta \widehat{M}_h = 0 \quad \text{on } \partial\Omega_h, \tag{A.8d}$$

$$\alpha_T w_h = 0 \quad \text{on } \partial\Omega_h, \tag{A.8e}$$

$$\tau w_h = 0 \quad \text{on } \partial\Omega_h, \tag{A.8f}$$

$$\widehat{w}_h = 0 \quad \text{on } \mathscr{E}_h. \tag{A.8g}$$

Furthermore, let $L_\ell(\cdot)$ *be the Legendre polynomial of order* ℓ, *and set*

$$\varphi_j^\ell(x) = L_\ell\left(\frac{2x - x_j - x_{j-1}}{h_j}\right) \quad \text{for } j = 1, 2, \dots, N.$$

Then, there exist constants a_j *such that*

$$w_h(x) = a_j \varphi_j^k(x) \quad \text{on each element } I_j \in \Omega_h. \tag{A.9}$$

Proof Inserting (A.4) into (A.1), and using (A.7), we obtain

$$\begin{aligned}(M_h/EI, M_h)_{\Omega_h} &+ d^2\,(T_h/GA, T_h)_{\Omega_h}\\ &+ \langle \alpha_\theta, (M_h - \widehat{M}_h)^2\rangle_{\partial\Omega_h} + \langle \alpha_T, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h}\\ &= 0.\end{aligned}$$

Since, $\alpha_T, \alpha_\theta \geq 0$ on $\partial\Omega_h$, we immediately obtain (A.8a), (A.8b), (A.8d), and

$$\alpha_T(w_h - \widehat{w}_h) = 0 \quad \text{on } \partial\Omega_h. \tag{A.10}$$

Then, inserting (A.5) into (A.2) we get,

$$(\theta_h, \theta_h)_{\Omega_h} + \langle \tau, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h} = 0.$$

Since, $\tau \geq 0$ on $\partial\Omega_h$ this implies (A.8c), and

$$\tau(w_h - \widehat{w}_h) = 0 \quad \text{on } \partial\Omega_h. \tag{A.11}$$

By (A.8a) and (A.8c), (2.1a) reduces to,

$$-(w_h, v')_{\Omega_h} + \langle \widehat{w}_h, vn\rangle_{\partial\Omega_h} = 0 \quad \text{for all } v \in V_h^k. \tag{A.12}$$

Thus, taking $v = \chi_{I_j}$, the characteristic function of I_j, and varying $j = 1, 2, \dots, N$ we see that $\widehat{w}_h$ is identically constant on $\mathscr{E}_h$. Since $\widehat{w}_h|_{\partial\Omega} = w_D = 0$, this constant is zero, which proves (A.8g). Hence, (A.10) implies (A.8e), and (A.11) implies (A.8f).

It remains to prove (A.9). Clearly, (A.8g) further reduces (A.12) to

$$-(w_h, v')_{\Omega_h} = 0 \quad \text{for all } v \in V_h^k.$$

Restricting this equation to a single but arbitrary element I_j by taking a test function v with support on that element we get

$$-(w_h, v')_{I_j} = 0 \quad \text{for all } v \in P^k(I_j). \tag{A.13}$$

Let

$$w_h(x) = \sum_{\ell=0}^{k} a_j^\ell \varphi_j^\ell(x) \quad \text{for some constants } a_j^\ell.$$

If $k = 0$ then (A.9) is trivially true, so we assume that $k \geq 1$. Taking v in (A.13) such that $v' = \varphi_j^\ell$, we see that $a_j^\ell = 0$ for $\ell = 0, 1, \dots, k-1$, and (A.9) follows. □

We are now ready to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1 Due to the linearity of the problem, it is enough show that the only solution to (2.1) with data given by (A.7) is

$$T_h = M_h = \theta_h = w_h = 0 \quad \text{in } \Omega_h, \qquad \omega_h = 0 \quad \text{on } \mathscr{E}_h^\circ, \qquad \mu_h = 0 \quad \text{on } \mathscr{E}_h.$$

In Lemma A.3 we have already proved that $M_h = \theta_h = 0$ in Ω_h. Since $d > 0$, (A.8a) implies $T_h = 0$ in Ω_h. Also since $\omega_h = \widehat{w}_h$ on $\mathscr{E}_h^\circ$, (A.8g) implies that $\omega_h = 0$ on $\mathscr{E}_h^\circ$. Thus it remains to prove that

$$w_h = 0, \quad \text{in } \Omega_h, \tag{A.14a}$$

$$\mu_h = 0, \quad \text{on } \mathscr{E}_h. \tag{A.14b}$$

We proceed case by case.

Case (1): Suppose that $\alpha_T > 0$ on $\partial\Omega_h$. Then by (A.8e) we see that $w_h = 0$ on $\partial\Omega_h$. Since $\varphi_j^k|_{\partial I_j} \neq 0$, (A.9) implies that $a_j = 0$ for all $j = 1, 2, \dots, N$. Hence $w_h = 0$ on Ω_h. Similarly, suppose that $\alpha_\theta > 0$ on $\partial\Omega_h$. Then (A.8d) implies $\widehat{M}_h = 0$ on $\mathscr{E}_h$. Since $\widehat{M}_h = \mu_h$ by (4.7b) we reach at (A.14b).

Case (2): Suppose that α_T or $\tau > 0$ on at least one point of ∂I_j for all j. Then by (A.8e) and (A.8f) $w_h = 0$ on $\partial\Omega_h$. Proceeding as in Case (1) we get (A.14a). Since $T_h = M_h = 0$ in Ω_h, (2.1c) reduces to

$$\langle \widehat{M}_h, vn\rangle_{\partial\Omega_h} = 0 \quad \text{for all } v \in V_h^k.$$

Thus,

$$\widehat{M}_h = \begin{cases} 0 & \text{if } k \geq 1, \\ \text{constant} & \text{if } k = 0. \end{cases}$$

For the case $k = 0$, if $\alpha_\theta > 0$ for at least one point of $\partial\Omega_h$, then (A.8d) implies (A.14b).

Case (3): The proof of (A.14a) was given in the proof of Case (2). To prove (A.14b) suppose that $\tau > 0$ on at least one point of $\partial\Omega_h$, say $\tau(x_i^-) > 0$. Note that $T_h = 0$ on Ω_h implies that $\widehat{T}_h$ is constant on $\mathscr{E}_h$ by (2.1d). In particular, $\widehat{T}_h(x_{i-1}^+) = \widehat{T}_h(x_i^-)$. Recall that by (A.8b), (A.8g), and (A.14a), the definition of $\widehat{T}_h$ reduces to

$$\widehat{T}_h = \tau\widehat{M}_h n.$$

Thus,

$$-\tau(x_{i-1}^+)\widehat{M}_h(x_{i-1}) = \tau(x_i^-)\widehat{M}_h(x_i).$$

We know that $\widehat{M}_h =$ constant on $\mathscr{E}_h$. If this constant is not zero then the above equation implies that

$$\tau(x_i^-) = -\tau(x_{i-1}^+)$$

which is impossible since $\tau(x_i^-) > 0$ and $-\tau(x_{i-1}^+) \le 0$. Hence, this constant must be zero, i.e. $\widehat{M}_h = 0$ on $\mathscr{E}_h$, and hence (A.14b) holds.

Case (4): This was already proved in the proof of Case (2). □

Proof of Theorem 3.2 We proceed as we did in the proof of Theorem 3.1. So, we assume (A.7), and deduce from Lemma A.3 that $M_h = \theta_h = 0$ in Ω_h, and that $\widehat{w}_h = 0$ on $\mathscr{E}_h$. However, since $d = 0$ we can not deduce $T_h = 0$. Hence, we need to prove

$$T_h = 0, \quad \text{in } \Omega_h, \tag{A.15a}$$

$$w_h = 0, \quad \text{in } \Omega_h, \tag{A.15b}$$

$$\mu_h = 0, \quad \text{on } \mathscr{E}_h, \tag{A.15c}$$

for each case below.

Case (1): The proof that $w_h = 0$ in Ω_h, and that $\mu_h = 0$ on $\mathscr{E}_h$ follow as in the proof of Case (1) of Theorem 3.1. Then (A.3) yields (A.15a).

Case (2): The proof that $w_h = 0$ in Ω_h follows as in Case (1). By (A.3) and (A.6) we have

$$(T_h, T_h)_{\Omega_h} + \langle\tau, (\widehat{M}_h)^2\rangle_{\partial\Omega_h} = \langle\widehat{T}_h n, \widehat{M}_h\rangle_{\partial\Omega}. \tag{A.16}$$

Since $\alpha_\theta > 0$ on $\partial\Omega$, (A.8d) implies that $\widehat{M}_h = 0$ on $\partial\Omega$. Then, the above equation yields (A.15a) because $\tau \ge 0$ on $\partial\Omega_h$. Consequently, if $k \ge 1$ then (2.1c) implies that $\widehat{M}_h = 0$ on $\partial\Omega_h$. If $k = 0$, then we deduce from (2.1c) that $\widehat{M}_h$ is constant on $\partial\Omega_h$, but since $\alpha_\theta > 0$ on $\partial\Omega$, (A.8d) shows that this constant is zero.

Case (3): As in Cases (1) and (2) above, (A.15b) follows from the assumption that α_T or $\tau > 0$ on at least one point of ∂I_j for all j. We know by (2.1d) with $q = 0$ that $\widehat{T}_h$ is a constant on $\partial\Omega_h$. Then, for any $v \in V_h^k$, upon integration by parts we get that $(\widehat{T}_h, v')_{\Omega_h} = \langle\widehat{T}_h, vn\rangle_{\partial\Omega_h}$. Inserting this into (2.1d) we obtain

$$(T_h - \widehat{T}_h, v')_{\Omega_h} = 0 \quad \text{for all } v \in V_h^k.$$

Now, assuming that $k \ge 1$, and taking $v \in V_h^k$ such that v' is equal to the constant $\widehat{T}_h$ we see that

$$(T_h - \widehat{T}_h, \widehat{T}_h)_{\Omega_h} = 0. \tag{A.17}$$

Taking $v_3 = \widehat{T}_h$ in (2.1c) we obtain

$$(T_h, \widehat{T}_h)_{\Omega_h} = \langle \widehat{M}_h, \widehat{T}_h n\rangle_{\partial\Omega_h}. \tag{A.18}$$

Since, $M_h = w_h = 0$ in Ω_h, and $\widehat{w}_h = 0$ on $\mathscr{E}_h$, the definition of the numerical trace $\widehat{T}_h$, (2.2c), reduces to $\widehat{T}_h = T_h + \tau \widehat{M}_h n$. Inserting this into (A.18) implies

$$\langle \widehat{M}_h, T_h n\rangle_{\partial\Omega_h} + \langle \tau, (\widehat{M}_h)^2\rangle_{\partial\Omega_h} = (T_h, \widehat{T}_h)_{\Omega_h}. \tag{A.19}$$

Taking $v_3 = T_h$ in (2.1c) gives $\langle \widehat{M}_h, T_h n\rangle_{\partial\Omega_h} = (T_h, T_h)_{\Omega_h}$. Thus, (A.19) can be written as

$$(T_h, T_h - \widehat{T}_h)_{\Omega_h} + \langle \tau, (\widehat{M}_h)^2\rangle_{\partial\Omega_h} = 0.$$

Then, by (A.17) we get that

$$(T_h - \widehat{T}_h, T_h - \widehat{T}_h)_{\Omega_h} + \langle \tau, \widehat{M}_h^2\rangle_{\partial\Omega_h} = 0.$$

Thus,

$$\tau \widehat{M}_h = 0 \quad \text{on } \partial\Omega_h, \quad \text{and} \quad T_h = \widehat{T}_h = \text{constant} \quad \text{on } \Omega_h.$$

If $\tau > 0$ at some point of $\partial\Omega_h$, say x_i^+, then $\widehat{M}_h(x_i) = 0$. Then taking v_3 in (2.1c) as the linear function with support (x_{i-1}, x_i) such that $v_3(x_i^-) = 1$ and $v_3(x_{i-1}^+) = 0$, we see that

$$T_h(1, v_3)_{I_j} = \widehat{M}_h(x_i)v_3(x_i^-) - \widehat{M}_h(x_{i-1})v_3(x_{i-1}^+) = \widehat{M}_h(x_i) = 0$$

where we have used the fact that T_h is constant on Ω_h. Thus, $T_h = 0$ on Ω_h. Using this in (2.1c) implies $\langle \widehat{M}_h, vn\rangle_{\partial\Omega_h} = 0$, for all $v \in V_h^k$, and since $k \geq 1$, we see that $\widehat{M}_h = 0$ on $\mathscr{E}_h$.

Case (4): We have already proved (A.15b). Integrating by parts in (2.1d) gives

$$-\langle T_h, vn\rangle_{\partial\Omega_h} + (T_h', v)_{\Omega_h} + \langle \widehat{T}_h, vn\rangle_{\partial\Omega_h} = 0.$$

Suppose that $k = 0$, then $T_h' = 0$ and hence

$$\langle \widehat{T}_h - T_h, vn\rangle_{\partial\Omega_h} = 0 \quad \text{for all } v \in V_h^0.$$

By (2.2c), $\widehat{T}_h = T_h + \tau \widehat{M}_h n$, and so

$$\langle \tau \widehat{M}_h n, vn\rangle_{\partial\Omega_h} = 0 \quad \text{for all } v \in V_h^0. \tag{A.20}$$

Taking $v = \chi_{I_1}$, we see that

$$\tau(x_1^-)\widehat{M}_h(x_1) - \tau(x_0^+)\widehat{M}_h(x_0) = 0.$$

Now, if we take

$$\tau(x_1^-) = 0 \quad \text{and} \quad \tau(x_0^+) > 0$$

we see that $\widehat{M}_h(x_0) = 0$. Similarly, taking $v = \chi_{I_N}$ in (A.20), we get that

$$\tau(x_N^-)\widehat{M}_h(x_N) - \tau(x_{N-1}^+)\widehat{M}_h(x_{N-1}) = 0.$$

Thus taking

$$\tau(x_{N-1}^+) = 0 \quad \text{and} \quad \tau(x_N^-) > 0$$

we get that $\widehat{M}_h(x_N) = 0$. Thus $\widehat{M}_h = 0$ on $\partial\Omega$. Inserting this into (A.16) we see that

$$(T_h, T_h)_{\Omega_h} + \langle \tau, (\widehat{M}_h)^2 \rangle_{\partial\Omega_h \setminus \partial\Omega} = 0.$$

Since $\tau \geq 0$, we immediately deduce that $T_h = 0$ in Ω_h. Using this in (2.1c) we obtain

$$\langle \widehat{M}_h, vn \rangle_{\partial\Omega_h} = 0 \quad \text{for all } v \in V_h^0$$

and hence $\widehat{M}_h$ is identically constant on $\mathscr{E}_h$. Since $\widehat{M}_h = 0$ on $\partial\Omega$ we deduce that $\widehat{M}_h = 0$ on $\mathscr{E}_h$. This finishes the proof. □

It remains to prove Lemmas A.1 and A.2.

Proof of Lemma A.1 To prove the first identity we begin with taking $v_1 = -T_h$ in (2.1a) and $v_2 = M_h$ in (2.1b), and adding the resulting equations to obtain

$$E = -(\theta_h, M_h')_{\Omega_h} + \langle \widehat{\theta}_h, M_h n \rangle_{\partial\Omega_h} + (w_h, T_h')_{\Omega_h} - \langle \widehat{w}_h, T_h n \rangle_{\partial\Omega_h} + (\theta_h, T_h)_{\Omega_h}$$

where

$$E := (M_h/EI, M_h)_{\Omega_h} + d^2\, (T_h/GA, T_h)_{\Omega_h}\,.$$

Integrating by parts on the term $(w_h, T_h')_{\Omega_h}$, and using (2.1d) with $v_4 = w_h$, we can rewrite the last equation as

$$\begin{aligned} &E - \langle \widehat{\theta}_h, M_h n \rangle_{\partial\Omega_h} + \langle \widehat{T}_h, w_h n \rangle_{\partial\Omega_h} - \langle T_h, w_h n \rangle_{\partial\Omega_h} \\ &\quad = -(\theta_h, M_h')_{\Omega_h} + (q, w_h)_{\Omega_h} - \langle \widehat{w}_h, T_h n \rangle_{\partial\Omega_h} + (\theta_h, T_h)_{\Omega_h}. \end{aligned}$$

Integrating by parts on the term $(\theta_h, M_h')_{\Omega_h}$, and using (2.1c) with $v_3 = \theta_h$, the last equation can be written as

$$\begin{aligned} &E - \langle (\widehat{\theta}_h - \theta_h), M_h n \rangle_{\partial\Omega_h} + \langle (\widehat{T}_h - T_h), w_h n \rangle_{\partial\Omega_h} \\ &\quad = (q, w_h)_{\Omega_h} + \langle \widehat{M}_h, \theta_h n \rangle_{\partial\Omega_h} - \langle \widehat{w}_h, T_h n \rangle_{\partial\Omega_h}. \end{aligned}$$

Adding to both sides the quantity

$$\langle \widehat{\theta}_h - \theta_h, \widehat{M}_h n \rangle_{\partial\Omega_h} - \langle \widehat{T}_h - T_h, \widehat{w}_h n \rangle_{\partial\Omega_h}$$

we get

$$\begin{aligned} &E - \langle (\widehat{\theta}_h - \theta_h), (M_h - \widehat{M}_h) n \rangle_{\partial\Omega_h} + \langle (\widehat{T}_h - T_h), (w_h - \widehat{w}_h) n \rangle_{\partial\Omega_h} \\ &\quad = (q, w_h)_{\Omega_h} + \langle \widehat{\theta}_h n, \widehat{M}_h \rangle_{\partial\Omega_h} - \langle \widehat{T}_h n, \widehat{w}_h \rangle_{\partial\Omega_h} \\ &\quad = (q, w_h)_{\Omega_h} + \langle \widehat{\theta}_h n, \widehat{M}_h \rangle_{\partial\Omega} - \langle \widehat{T}_h n, \widehat{w}_h \rangle_{\partial\Omega}. \end{aligned}$$

In the last step we used the fact that

$$\langle \widehat{\theta}_h n, \widehat{M}_h \rangle_{\partial\Omega_h} = \langle \widehat{\theta}_h n, \widehat{M}_h \rangle_{\partial\Omega} \quad \text{and} \quad \langle \widehat{T}_h n, \widehat{w}_h \rangle_{\partial\Omega_h} = \langle \widehat{T}_h n, \widehat{w}_h \rangle_{\partial\Omega}$$

since $[\![\widehat{\theta}_h]\!] = [\![\widehat{T}_h]\!] = 0$ on $\mathscr{E}_h^\circ$ by (2.1e) and (2.1f), respectively. This finishes the proof of the first identity, (A.1), in Lemma A.1.

To prove the second identity we begin with taking $v_1 = \theta_h$ in (2.1a) to obtain

$$d^2(T_h/GA, \theta_h)_{\Omega_h} = (\theta_h, \theta_h)_{\Omega_h} + (w_h, \theta_h')_{\Omega_h} - \langle \widehat{w}_h, \theta_h n\rangle_{\partial\Omega_h}.$$

Taking $v_2 = w_h$ in (2.1b) implies

$$-(M_h/EI, w_h)_{\Omega_h} = (\theta_h, w_h')_{\Omega_h} - \langle \widehat{\theta}_h, w_h n\rangle_{\partial\Omega_h}.$$

Adding these two equations we get

$$F = (\theta_h, \theta_h)_{\Omega_h} + (w_h, \theta_h')_{\Omega_h} - \langle \widehat{w}_h, \theta_h n\rangle_{\partial\Omega_h} + (\theta_h, w_h')_{\Omega_h} - \langle \widehat{\theta}_h, w_h n\rangle_{\partial\Omega_h}$$

where

$$F := d^2(T_h/GA, \theta_h)_{\Omega_h} - (M_h/EI, w_h)_{\Omega_h}.$$

Integrating by parts on the term $(w_h, \theta_h')_{\Omega_h}$ we get

$$\begin{aligned}
F &= (\theta_h, \theta_h)_{\Omega_h} + \langle w_h, \theta_h n\rangle_{\partial\Omega_h} - \langle \widehat{w}_h, \theta_h n\rangle_{\partial\Omega_h} - \langle \widehat{\theta}_h, w_h n\rangle_{\partial\Omega_h} \\
&= (\theta_h, \theta_h)_{\Omega_h} + \langle \theta_h, (w_h - \widehat{w}_h) n\rangle_{\partial\Omega_h} - \langle \widehat{\theta}_h, w_h n\rangle_{\partial\Omega_h} \\
&= (\theta_h, \theta_h)_{\Omega_h} - \langle (\widehat{\theta}_h - \theta_h), (w_h - \widehat{w}_h) n\rangle_{\partial\Omega_h} - \langle \widehat{\theta}_h, \widehat{w}_h n\rangle_{\partial\Omega_h} \\
&= (\theta_h, \theta_h)_{\Omega_h} - \langle (\widehat{\theta}_h - \theta_h), (w_h - \widehat{w}_h) n\rangle_{\partial\Omega_h} - \langle \widehat{\theta}_h, \widehat{w}_h n\rangle_{\partial\Omega}.
\end{aligned}$$

Thus,

$$(\theta_h, \theta_h)_{\Omega_h} - \langle (\widehat{\theta}_h - \theta_h), (w_h - \widehat{w}_h) n\rangle_{\partial\Omega_h} = F + \langle \widehat{\theta}_h, \widehat{w}_h n\rangle_{\partial\Omega}$$

which is (A.2).

To prove the third identity we take $v_3 = T_h$ in (2.1c), $v_4 = M_h$ in (2.1d), and add the resulting equations to obtain

$$\begin{aligned}
&(T_h, T_h)_{\Omega_h} + (q, M_h)_{\Omega_h} \\
&\quad = -(M_h, T_h')_{\Omega_h} - (T_h, M_h')_{\Omega_h} + \langle \widehat{M}_h, T_h n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h, M_h n\rangle_{\partial\Omega_h}.
\end{aligned}$$

Integrating by parts on the term $(M_h, T_h')_{\Omega_h}$, and carrying out some simple algebraic manipulations we get

$$\begin{aligned}
(T_h, T_h)_{\Omega_h} &= -(q, M_h)_{\Omega_h} - \langle T_h, M_h n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h, M_h n\rangle_{\partial\Omega_h} + \langle \widehat{M}_h, T_h n\rangle_{\partial\Omega_h} \\
&= -(q, M_h)_{\Omega_h} + \langle (\widehat{T}_h - T_h), (M_h - \widehat{M}_h) n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h n, \widehat{M}_h\rangle_{\partial\Omega_h} \\
&= -(q, M_h)_{\Omega_h} + \langle (\widehat{T}_h - T_h), (M_h - \widehat{M}_h) n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h n, \widehat{M}_h\rangle_{\partial\Omega},
\end{aligned}$$

where in the last step we have used (2.1f). Hence

$$(T_h, T_h)_{\Omega_h} - \langle (\widehat{T}_h - T_h), (M_h - \widehat{M}_h) n\rangle_{\partial\Omega_h} = -(q, M_h)_{\Omega_h} + \langle \widehat{T}_h n, \widehat{M}_h\rangle_{\partial\Omega}$$

which is the desired identity, (A.3).

This finishes the proof of Lemma A.1. □

Proof of Lemma A.2 These identities follow simply from the definition of the numerical traces $\widehat{\theta}_h$ and $\widehat{T}_h$ given by (2.2b) and (2.2c). Indeed, (2.2c) implies

$$\langle \widehat{T}_h - T_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h} = \langle -\tau, (M_h - \widehat{M}_h)(w_h - \widehat{w}_h)\rangle_{\partial\Omega_h} + \langle \alpha_T, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h}.$$

Similarly, (2.2b) gives

$$-\langle \widehat{\theta}_h - \theta_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h} = \langle \tau, (w_h - \widehat{w}_h)(M_h - \widehat{M}_h)\rangle_{\partial\Omega_h} + \langle \alpha_\theta, (M_h - \widehat{M}_h)^2\rangle_{\partial\Omega_h}.$$

Adding these equations together we get

$$\begin{aligned} &-\langle \widehat{\theta}_h - \theta_h, (M_h - \widehat{M}_h)n\rangle_{\partial\Omega_h} + \langle \widehat{T}_h - T_h, (w_h - \widehat{w}_h)n\rangle_{\partial\Omega_h} \\ &\quad = \langle \alpha_\theta, (M_h - \widehat{M}_h)^2\rangle_{\partial\Omega_h} + \langle \alpha_T, (w_h - \widehat{w}_h)^2\rangle_{\partial\Omega_h} \end{aligned}$$

which is (A.4). Similarly, using (2.2b) we get (A.5), and using (2.2c) we get (A.6).
This finishes the proof of Lemma A.2. □

Appendix B: Proof of the Characterization Result of Theorem 4.1

In this appendix we prove the characterization result of Theorem 4.1. The proof will be based on the following auxiliary lemmas. Our first auxiliary result is a simple implication of the definition of the local solvers. Indeed, adding each of the equations (4.1), (4.3), and (4.5) over $K \in \Omega_h$, we immediately obtain the following lemma.

Lemma B.1 *For any $(\omega, \mu) \in [L^2(\mathscr{E}_h)]^2$, and $(v_1, v_2, v_3, v_4) \in [V_h^k]^4$, we have*

$$-(\mathcal{W}\omega, v_1')_{\Omega_h} + \langle \omega, v_1 n\rangle_{\partial\Omega_h} = (\Theta\omega, v_1)_{\Omega_h} - d^2(\mathcal{T}\omega/GA, v_1)_{\Omega_h}, \tag{B.1a}$$

$$-(\Theta\omega, v_2')_{\Omega_h} + \langle 1, (\widehat{\Theta}\omega) v_2 n\rangle_{\partial\Omega_h} = (\mathcal{M}\omega/EI, v_2)_{\Omega_h}, \tag{B.1b}$$

$$-(\mathcal{M}\omega, v_3')_{\Omega_h} = (\mathcal{T}\omega, v_3)_{\Omega_h}, \tag{B.1c}$$

$$-(\mathcal{T}\omega, v_4')_{\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\omega) v_4 n\rangle_{\partial\Omega_h} = 0; \tag{B.1d}$$

$$-(\mathcal{W}\mu, v_1')_{\Omega_h} = (\Theta\mu, v_1)_{\Omega_h} - d^2(\mathcal{T}\mu/GA, v_1)_{\Omega_h}, \tag{B.2a}$$

$$-(\Theta\mu, v_2')_{\Omega_h} + \langle 1, (\widehat{\Theta}\mu) v_2 n\rangle_{\partial\Omega_h} = (\mathcal{M}\mu/EI, v_2)_{\Omega_h}, \tag{B.2b}$$

$$-(\mathcal{M}\mu, v_3')_{\Omega_h} + \langle \mu, v_3 n\rangle_{\partial\Omega_h} = (\mathcal{T}\mu, v_3)_{\Omega_h}, \tag{B.2c}$$

$$-(\mathcal{T}\mu, v_4')_{\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\mu) v_4 n\rangle_{\partial\Omega_h} = 0; \tag{B.2d}$$

and that

$$-(\mathcal{W}q, v_1')_{\Omega_h} = (\Theta q, v_1)_{\Omega_h} - d^2(\mathcal{T}q/GA, v_1)_{\Omega_h}, \tag{B.3a}$$

$$-(\Theta q, v_2')_{\Omega_h} + \langle 1, (\widehat{\Theta}q) v_2 n\rangle_{\partial\Omega_h} = (\mathcal{M}q/EI, v_2)_{\Omega_h}, \tag{B.3b}$$

$$-(\mathcal{M}q, v_3')_{\Omega_h} = (\mathcal{T}q, v_3)_{\Omega_h}, \tag{B.3c}$$

$$-(\mathcal{T}q, v_4')_{\Omega_h} + \langle 1, (\widehat{\mathcal{T}}q) v_4 n\rangle_{\partial\Omega_h} = (q, v_4)_{\Omega_h}. \tag{B.3d}$$

The following lemma contains some elementary identities.

Lemma B.2 (Elementary identities) *For any* ω, μ, m *in* $L^2(\mathscr{E}_h)$, *and any* w *in* $L^2_0(\mathscr{E}_h)$ *we have*

$$\begin{aligned}
\langle \widehat{\Theta}\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} = {}& (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2\,(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h} \\
&+ \langle \omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} + \langle 1, (\widehat{\Theta}\omega - \Theta\omega)(\mathsf{m} - \mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}\mathsf{m} - \mathcal{T}\mathsf{m})(\mathcal{W}\omega)\, n\rangle_{\partial\Omega_h}, &\text{(B.4a)} \\
\langle \widehat{\Theta}w_D, \mathsf{m}\, n\rangle_{\partial\Omega_h} = {}& \langle w_D, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} - \langle 1, (\widehat{\mathcal{T}}w_D - \mathcal{T}w_D)(\mathcal{W}\mathsf{m})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}\mathsf{m} - \Theta\mathsf{m})(\mathcal{M}w_D)\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}\mathsf{m} - \mathcal{T}\mathsf{m})(\mathcal{W}w_D)\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}w_D - \Theta w_D)(\mathsf{m} - \mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h}, &\text{(B.4b)} \\
\langle \widehat{\Theta}\mu, \mathsf{m}\, n\rangle_{\partial\Omega_h} = {}& (\mathcal{M}\mu/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2\,(\mathcal{T}\mu/GA, \mathcal{T}\mathsf{m})_{\Omega_h} \\
&+ \langle 1, \alpha_\theta(\mu - \mathcal{M}\mu)(\mathsf{m} - \mathcal{M}\mathsf{m})\rangle_{\partial\Omega_h} \\
&+ \langle 1, \alpha_T \mathcal{W}\mu\mathcal{W}\mathsf{m}\rangle_{\partial\Omega_h}, &\text{(B.4c)} \\
\langle \widehat{\Theta}q, \mathsf{m}\, n\rangle_{\partial\Omega_h} = {}& (q, \mathcal{W}\mathsf{m})_{\Omega_h} - \langle 1, (\widehat{\mathcal{T}}q - \mathcal{T}q)(\mathcal{W}\mathsf{m})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}\mathsf{m} - \Theta\mathsf{m})(\mathcal{M}q)\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}\mathsf{m} - \mathcal{T}\mathsf{m})(\mathcal{W}q)\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}q - \Theta q)(\mathsf{m} - \mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h}, &\text{(B.4d)}
\end{aligned}$$

and that

$$\begin{aligned}
\langle \widehat{\mathcal{T}}\omega, \mathsf{w}\, n\rangle_{\partial\Omega_h} = {}& -(\mathcal{M}\omega/EI, \mathcal{M}\mathsf{w})_{\Omega_h} - d^2\,(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{w})_{\Omega_h} \\
&- \langle 1, \alpha_T(\omega - \mathcal{W}\omega)(\mathsf{w} - \mathcal{W}\mathsf{w})\rangle_{\partial\Omega_h} \\
&- \langle 1, \alpha_\theta \mathcal{M}\omega\mathcal{M}\mathsf{w}\rangle_{\partial\Omega_h}, &\text{(B.5a)} \\
\langle \widehat{\mathcal{T}}w_D, \mathsf{w}\, n\rangle_{\partial\Omega_h} = {}& \langle w_D, (\mathcal{T}\mathsf{w})\, n\rangle_{\partial\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\mathsf{w} - \mathcal{T}\mathsf{w})(\mathcal{W}w_D)\, n\rangle_{\partial\Omega_h} \\
&- \langle 1, (\widehat{\Theta}w_D - \Theta w_D)(\mathcal{M}\mathsf{w})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}w_D - \mathcal{T}w_D)(\mathsf{w} - \mathcal{W}\mathsf{w})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}\mathsf{w} - \Theta\mathsf{w})(\mathcal{M}w_D)\, n\rangle_{\partial\Omega_h}, &\text{(B.5b)} \\
\langle \widehat{\mathcal{T}}\mu, \mathsf{w}\, n\rangle_{\partial\Omega_h} = {}& -(\mathcal{M}\mu/EI, \mathcal{M}\mathsf{w})_{\Omega_h} - d^2\,(\mathcal{T}\mu/GA, \mathcal{T}\mathsf{w})_{\Omega_h} \\
&+ \langle \mu, (\Theta\mathsf{w})\, n\rangle_{\partial\Omega_h} + \langle 1, (\widehat{\Theta}\mathsf{w} - \Theta\mathsf{w})(\mathcal{M}\mu)\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}\mu - \mathcal{T}\mu)(\mathsf{w} - \mathcal{W}\mathsf{w})\, n\rangle_{\partial\Omega_h}, &\text{(B.5c)} \\
\langle \widehat{\mathcal{T}}q, \mathsf{w}\, n\rangle_{\partial\Omega_h} = {}& (q, \mathcal{W}\mathsf{w})_{\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\mathsf{w} - \mathcal{T}\mathsf{w})(\mathcal{W}q)\, n\rangle_{\partial\Omega_h} \\
&- \langle 1, (\widehat{\Theta}q - \Theta q)(\mathcal{M}\mathsf{w})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\mathcal{T}}q - \mathcal{T}q)(\mathsf{w} - \mathcal{W}\mathsf{w})\, n\rangle_{\partial\Omega_h} \\
&+ \langle 1, (\widehat{\Theta}\mathsf{w} - \Theta\mathsf{w})(\mathcal{M}q)\, n\rangle_{\partial\Omega_h}. &\text{(B.5d)}
\end{aligned}$$

We leave the proof of this lemma to the end of this appendix.

The next lemma provides the explicit solution of the local solver defined by (4.1).

Lemma B.3 *Suppose that the conditions of one of the cases in Theorem 3.1 (Theorem 3.2, if $d = 0$) are satisfied. Let $k \geq 1$, and consider an arbitrary element $K = (x_L, x_R)$. Then the solution of the local solver (4.1) is given by*

$$(\mathcal{W}\omega)(x) = \frac{\omega_R - \omega_L}{h_K}(x - x_L) + \omega_L, \tag{B.6a}$$

$$(\Theta\omega)(x) = \frac{\omega_R - \omega_L}{h_K}, \tag{B.6b}$$

$$(\mathcal{M}\omega)(x) = 0, \tag{B.6c}$$

$$(\mathcal{T}\omega)(x) = 0, \tag{B.6d}$$

for any $x \in K$, where $\omega_L = \omega(x_L)$, $\omega_R = \omega(x_R)$, and $h_K = x_R - x_L$.

Proof The fact that the local solver (4.1) with the numerical traces (4.2) defines a unique solution is a direct corollary of Theorem 3.1 (Theorem 3.2, if $d = 0$). The result now follows directly from substituting the solution given by (B.6) into (4.1), and verifying that it satisfies the equations. □

A direct implication of (B.6a) is that,

$$\mathsf{w} - \mathcal{W}\mathsf{w} = 0 \quad \text{on } \partial\Omega_h. \tag{B.7}$$

Hence, by the definition of the numerical traces, (4.2), and (B.6c) and (B.6d), we have

$$\widehat{\Theta}\omega - \Theta\omega = 0, \quad \text{on } \partial\Omega_h, \tag{B.8a}$$

$$\widehat{\mathcal{T}}\omega = 0 \quad \text{on } \partial\Omega_h. \tag{B.8b}$$

Using the results of Lemmas B.3 and B.1 in the identities given by Lemma B.2 we get our next auxiliary lemma.

Lemma B.4 *Suppose that $k \geq 1$, then for any ω, μ, m in $L^2(\mathcal{E}_h)$, and any w in $L^2_0(\mathcal{E}_h)$ we have*

(i) $\langle \widehat{\Theta}\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} = (\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h}$,
(ii) $\langle \widehat{\Theta}q, \mathsf{m}\, n\rangle_{\partial\Omega_h} = (q, \mathcal{W}\mathsf{m})_{\Omega_h}$,
(iii) $\langle \widehat{\mathcal{T}}\omega, \mathsf{w}\, n\rangle_{\partial\Omega_h} = 0$,
(iv) $\langle \widehat{\mathcal{T}}\mu, \mathsf{w}\, n\rangle_{\partial\Omega_h} = (\Theta\mathsf{w}, \mathcal{T}\mu)_{\Omega_h}$,
(v) $\langle \widehat{\mathcal{T}}q, \mathsf{w}\, n\rangle_{\partial\Omega_h} = (q, \mathcal{W}\mathsf{w})_{\Omega_h}$.

We leave the proof of this lemma to the end of this section.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 Using the identity (i) in Lemma B.4 with $\omega = \omega_h$ we get

$$\langle \widehat{\Theta}\omega_h, \mathsf{m}\, n\rangle_{\partial\Omega_h} = (\Theta\omega_h, \mathcal{T}\mathsf{m}) = b_h(\omega_h, \mathsf{m}). \tag{B.9}$$

Using the same identity with $\omega = w_D$ we get

$$\langle \widehat{\Theta} w_D, \mathsf{m}\, n\rangle_{\partial\Omega_h} = (\Theta w_D, \mathcal{T}\mathsf{m}). \tag{B.10}$$

Taking $\mu = \mu_h$ in (B.4c) gives

$$\langle \widehat{\Theta}\mu_h, \mathsf{m}\, n\rangle_{\partial\Omega_h} = a_h(\omega_h, \mathsf{m}). \tag{B.11}$$

Combining (B.9), (B.10), (B.11), and the identity (ii) of Lemma B.4 yields

$$\langle(\widehat{\Theta}\omega_h + \widehat{\Theta} w_D + \widehat{\Theta}\mu_h + \widehat{\Theta} q), \mathsf{m}\, n\rangle_{\partial\Omega_h} = a_h(\mu_h, \mathsf{m}) + b_h(\omega_h, \mathsf{m}) - \ell_h^{\Theta}(\mathsf{m}) + \langle\theta_N, \mathsf{m}\, n\rangle_{\partial\Omega}.$$

Hence, (2.1e) implies (4.9a).

Similarly, using the identities (iii), (iv), and (v) of Lemma B.4 we get

$$\begin{aligned}
\langle \widehat{\mathcal{T}}\omega_h, \mathsf{w}\, n\rangle_{\partial\Omega_h} &= 0,\\
\langle \widehat{\mathcal{T}} w_D, \mathsf{w}\, n\rangle_{\partial\Omega_h} &= 0,\\
\langle \widehat{\mathcal{T}}\mu_h, \mathsf{w}\, n\rangle_{\partial\Omega_h} &= (\Theta\mathsf{w}, \mathcal{T}\mu_h)_{\Omega_h} = b_h(\mathsf{w}, \mu_h),\\
\langle \widehat{\mathcal{T}} q, \mathsf{w}\, n\rangle_{\partial\Omega_h} &= (q, \mathcal{W}\mathsf{w})_{\Omega_h}.
\end{aligned}$$

Hence,

$$\langle(\widehat{\mathcal{T}}\omega_h + \widehat{\mathcal{T}} w_D + \widehat{\mathcal{T}}\mu_h + \widehat{\mathcal{T}} q), \mathsf{w}\, n\rangle_{\partial\Omega_h} = b_h(\mathsf{w}, \omega_h) - \ell_h^{\mathcal{T}}(\mathsf{w}).$$

The result, (4.9b), now follows from (2.1f). This completes the proof of the theorem. □

It remains to prove Lemma B.2 and Lemma B.4.

Proof of Lemma B.2 We only give a proof of the identities (B.4a) and (B.4b), the proofs of the remaining ones are very similar. To prove (B.4a) we begin by writing

$$\langle \widehat{\Theta}\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} = \langle \widehat{\Theta}\omega - \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} + \langle \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h}. \tag{B.12}$$

Taking $\mu = \mathsf{m}$ and $v_3 = \Theta\omega$ in (B.2c) we get

$$\langle \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} = (\mathcal{M}\mathsf{m}, (\Theta\omega)')_{\Omega_h} + (\mathcal{T}\mathsf{m}, \Theta\omega)_{\Omega_h}. \tag{B.13}$$

By a simple integration by parts

$$(\mathcal{M}\mathsf{m}, (\Theta\omega)')_{\Omega_h} = \langle 1, (\mathcal{M}\mathsf{m})(\Theta\omega)\, n\rangle_{\partial\Omega_h} - ((\mathcal{M}\mathsf{m})', \Theta\omega)_{\Omega_h}. \tag{B.14}$$

Using (B.1a) with $v_1 = \mathcal{T}\mathsf{m}$ yields

$$(\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h} = -(\mathcal{W}\omega, (\mathcal{T}\mathsf{m})')_{\Omega_h} + \langle\omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} + d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}. \tag{B.15}$$

Inserting (B.14) and (B.15) into (B.13) we obtain

$$\begin{aligned}
\langle \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} = {} & \langle 1, (\mathcal{M}\mathsf{m})(\Theta\omega)\, n\rangle_{\partial\Omega_h} - ((\mathcal{M}\mathsf{m})', \Theta\omega)_{\Omega_h}\\
& - (\mathcal{W}\omega, (\mathcal{T}\mathsf{m})')_{\Omega_h} + \langle\omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h}\\
& + d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}.
\end{aligned} \tag{B.16}$$

Integrating by parts on the term $(\mathcal{W}\omega, (\mathcal{T}\mathsf{m})')_{\Omega_h}$, and taking $v_2 = \mathcal{M}\mathsf{m}$ in (B.1b) we obtain

$$-(\mathcal{W}\omega, (\mathcal{T}\mathsf{m})')_{\Omega_h} = -\langle 1, (\mathcal{W}\omega)(\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} + ((\mathcal{W}\omega)', \mathcal{T}\mathsf{m})_{\Omega_h},$$
$$-((\mathcal{M}\mathsf{m})', \Theta\omega)_{\Omega_h} = -\langle 1, (\widehat{\Theta}\omega)(\mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h} + (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h}.$$

Inserting these into (B.16) implies

$$\begin{aligned}\langle \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} &= (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}\\ &\quad + \langle \omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} - \langle 1, (\mathcal{W}\omega)(\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h}\\ &\quad + ((\mathcal{W}\omega)', \mathcal{T}\mathsf{m})_{\Omega_h} - \langle 1, (\widehat{\Theta}\omega - \Theta\omega)(\mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h}.\end{aligned}$$

Taking $\mu = \mathsf{m}$ and $v_4 = \mathcal{W}\omega$ in (B.2d), and inserting the resulting identity into the last equation above we get that

$$\begin{aligned}\langle \Theta\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} &= (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h} + d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}\\ &\quad + \langle \omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} - \langle 1, (\widehat{\Theta}\omega - \Theta\omega)(\mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h}\\ &\quad + \langle 1, (\widehat{\mathcal{T}}\mathsf{m} - \mathcal{T}\mathsf{m})(\mathcal{W}\omega)\, n\rangle_{\partial\Omega_h}.\end{aligned}$$

Combining this with (B.12) we reach at (B.4a).

To prove (B.4b) we need to further work on the first two terms on the right-hand side of (B.4a). To do that, we start with taking $\mu = \mathsf{m}$ and $v_1 = \mathcal{T}\omega$ in (B.2a) to obtain

$$d^2(\mathcal{T}\mathsf{m}/GA, \mathcal{T}\omega)_{\Omega_h} = (\Theta\mathsf{m}, \mathcal{T}\omega)_{\Omega_h} + (\mathcal{W}\mathsf{m}, (\mathcal{T}\omega)')_{\Omega_h}. \tag{B.17}$$

Integrating by parts we obtain

$$(\mathcal{W}\mathsf{m}, (\mathcal{T}\omega)')_{\Omega_h} = \langle 1, (\mathcal{W}\mathsf{m})(\mathcal{T}\omega)\, n\rangle_{\partial\Omega_h} - ((\mathcal{W}\mathsf{m})', \mathcal{T}\omega)_{\Omega_h}. \tag{B.18}$$

Taking $v_3 = \Theta\mathsf{m}$ in (B.1c) gives

$$(\Theta\mathsf{m}, \mathcal{T}\omega)_{\Omega_h} = -(\mathcal{M}\omega, (\Theta\mathsf{m})')_{\Omega_h}. \tag{B.19}$$

Using (B.18) and (B.19) in (B.17) we get

$$d^2(\mathcal{T}\mathsf{m}/GA, \mathcal{T}\omega)_{\Omega_h} = -(\mathcal{M}\omega, (\Theta\mathsf{m})')_{\Omega_h} + \langle 1, (\mathcal{W}\mathsf{m})(\mathcal{T}\omega)\, n\rangle_{\partial\Omega_h} - ((\mathcal{W}\mathsf{m})', \mathcal{T}\omega)_{\Omega_h}. \tag{B.20}$$

Integrating by parts on the term $(\mathcal{M}\omega, (\Theta\mathsf{m})')_{\Omega_h}$ and taking $v_4 = \mathcal{W}\mathsf{m}$ in (B.1d) we obtain

$$-(\mathcal{M}\omega, (\Theta\mathsf{m})')_{\Omega_h} = -\langle 1, (\mathcal{M}\omega)(\Theta\mathsf{m})\, n\rangle_{\partial\Omega_h} + ((\mathcal{M}\omega)', \Theta\mathsf{m})_{\Omega_h}, \tag{B.21}$$

and

$$-((\mathcal{W}\mathsf{m})', \mathcal{T}\omega)_{\Omega_h} = -\langle 1, (\widehat{\mathcal{T}}\omega)(\mathcal{W}\mathsf{m})\, n\rangle_{\partial\Omega_h}. \tag{B.22}$$

Inserting (B.21) and (B.22) into (B.20) implies

$$\begin{aligned}d^2(\mathcal{T}\mathsf{m}/GA, \mathcal{T}\omega)_{\Omega_h} &= -\langle 1, (\mathcal{M}\omega)(\Theta\mathsf{m})\, n\rangle_{\partial\Omega_h} + ((\mathcal{M}\omega)', \Theta\mathsf{m})_{\Omega_h}\\ &\quad - \langle 1, (\widehat{\mathcal{T}}\omega - \mathcal{T}\omega)(\mathcal{W}\mathsf{m})\, n\rangle_{\partial\Omega_h}.\end{aligned} \tag{B.23}$$

Taking $\mu = \mathsf{m}$ and $v_2 = \mathcal{M}\omega$ in (B.2b) we get

$$((\mathcal{M}\omega)', \Theta\mathsf{m})_{\Omega_h} = \langle 1, (\widehat{\Theta}\mathsf{m})(\mathcal{M}\omega)\, n\rangle_{\partial\Omega_h} - (\mathcal{M}\mathsf{m}/EI, \mathcal{M}\omega)_{\Omega_h}.$$

Inserting this into (B.23) we get

$$\begin{aligned}&(\mathcal{M}\mathsf{m}/EI, \mathcal{M}\omega)_{\Omega_h} + d^2(\mathcal{T}\mathsf{m}/GA, \mathcal{T}\omega)_{\Omega_h}\\ &\quad = -\langle 1, (\widehat{\mathcal{T}}\omega - \mathcal{T}\omega)(\mathcal{W}\mathsf{m})\, n\rangle_{\partial\Omega_h} + \langle 1, (\widehat{\Theta}\mathsf{m} - \Theta\mathsf{m})(\mathcal{M}\omega)\, n\rangle_{\partial\Omega_h}.\end{aligned}$$

Combining this with (B.4a), and setting $\omega = w_D$ in the resulting expression, we finally obtain (B.4b). □

Proof of Lemma B.4 (i) Taking $v_1 = \mathcal{T}\mathsf{m}$ in (B.1), we obtain

$$-(\mathcal{W}\omega, (\mathcal{T}\mathsf{m})')_{\Omega_h} + \langle \omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} = (\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h} - d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}.$$

Similarly, taking $\mu = \mathsf{m}$ and $v_4 = \mathcal{W}\omega$ in (B.2), we get

$$-(\mathcal{T}\mathsf{m}, (\mathcal{W}\omega)')_{\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\mathsf{m})(\mathcal{W}\omega)\, n\rangle_{\partial\Omega_h} = 0.$$

Combining these two equations, and carrying out some simple algebraic manipulations, we obtain

$$\langle \omega, (\mathcal{T}\mathsf{m})\, n\rangle_{\partial\Omega_h} + \langle 1, (\widehat{\mathcal{T}}\mathsf{m} - \mathcal{T}\mathsf{m})(\mathcal{W}\omega)\, n\rangle_{\partial\Omega_h} = (\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h} - d^2(\mathcal{T}\omega/GA, \mathcal{T}\mathsf{m})_{\Omega_h}.$$

Using this in (B.4a) we get

$$\begin{aligned}\langle \widehat{\Theta}\omega, \mathsf{m}\, n\rangle_{\partial\Omega_h} &= (\Theta\omega, \mathcal{T}\mathsf{m})_{\Omega_h} + (\mathcal{M}\omega/EI, \mathcal{M}\mathsf{m})_{\Omega_h}\\ &\quad + \langle 1, (\widehat{\Theta}\omega - \Theta\omega)(\mathsf{m} - \mathcal{M}\mathsf{m})\, n\rangle_{\partial\Omega_h}.\end{aligned}$$

The identity (i) now follows from (B.6c) and (B.8a).

(ii) This follows from inserting the definition of the numerical traces, (4.4), and (4.6), into the elementary identity (B.4d), and carrying out some simplifications.

(iii) This follows trivially from (B.8b).

(iv) Using (B.6c), (B.6d), (B.7), and (B.8), in (B.5c) we get

$$\langle \widehat{\mathcal{T}}\mu, \mathsf{w}\, n\rangle_{\partial\Omega_h} = \langle \mu, (\Theta\mathsf{w}) n\rangle_{\partial\Omega_h}.$$

Taking $v_3 = \Theta\mathsf{w}$ in (B.2c) we get

$$-(\mathcal{M}\mu, (\Theta\mathsf{w})')_{\Omega_h} + \langle \mu, (\Theta\mathsf{w})\, n\rangle_{\partial\Omega_h} = (\mathcal{T}\mu, \Theta\mathsf{w})_{\Omega_h}.$$

The identity (iv) now follows from the last two equations once we note that $(\Theta\mathsf{w})' = 0$ by (B.6b).

(v) This is a simple result of inserting (B.6c), (B.6d), (B.7), and (B.8) into (B.5d). □

Appendix C: Proof of Theorem 5.1

In this appendix we prove Theorem 5.1. The key to the proof is the fact that it is possible to compute the numerical traces $\widehat{M}_h$ and $\widehat{w}_h$ of the HDG method explicitly in terms of the averages and the jumps of the approximate solution $(T_h, M_h, \theta_h, w_h)$. We state and prove this result next.

In this appendix, we will make a slight abuse of notation and denote both the DG solution and the HDG solution as $(T_h, M_h, \theta_h, w_h)$, i.e. we will drop the superscript "H" in the HDG solution. We can do this because the solutions will be identical *a posteriori*.

Proposition C.1 *Consider the HDG method defined by the weak formulation* (2.1) *and the formulas* (2.2) *for the numerical traces. Then at every interior node $x \in \mathcal{E}_h^\circ$*

$$\begin{aligned}\widehat{M}_h = \{\!\!\{M_h\}\!\!\} + \frac{1}{2D}\Big[&(\llbracket \tau n\rrbracket \llbracket \tau\rrbracket + \llbracket \alpha_T n\rrbracket \llbracket \alpha_\theta\rrbracket)\llbracket M_h\rrbracket \\ &+ 2\llbracket \tau\rrbracket_{\alpha_T}\llbracket w_h\rrbracket - 2\llbracket \alpha_T n\rrbracket\llbracket \theta_h\rrbracket - 2\llbracket \tau n\rrbracket\llbracket T_h\rrbracket\Big],\end{aligned} \tag{C.1}$$

and

$$\begin{aligned}\widehat{w}_h = \{\!\!\{w_h\}\!\!\} + \frac{1}{2D}\Big[&(\llbracket \tau n\rrbracket \llbracket \tau\rrbracket + \llbracket \alpha_\theta n\rrbracket \llbracket \alpha_T\rrbracket)\llbracket w_h\rrbracket \\ &- 2\llbracket \tau\rrbracket_{\alpha_\theta}\llbracket M_h\rrbracket + 2\llbracket \alpha_\theta n\rrbracket\llbracket T_h\rrbracket - 2\llbracket \tau n\rrbracket\llbracket \theta_h\rrbracket\Big].\end{aligned} \tag{C.2}$$

Before proving this proposition, let us collect some elementary algebraic identities.

Lemma C.2 *For functions κ, α, and φ which are possibly double valued on $\mathcal{E}_h^\circ$ we have*

$$\llbracket \kappa n\rrbracket \llbracket \alpha\varphi n\rrbracket - \llbracket \alpha n\rrbracket\llbracket \kappa\varphi n\rrbracket = -\llbracket \varphi\rrbracket\llbracket \kappa\rrbracket_\alpha, \tag{C.3a}$$

$$\llbracket \alpha\varphi n\rrbracket = \llbracket \alpha n\rrbracket\{\!\!\{\varphi\}\!\!\} + \frac{1}{2}\llbracket \alpha\rrbracket\llbracket \varphi\rrbracket. \tag{C.3b}$$

Proof The first identity is proved as follows

$$\begin{aligned}\llbracket \kappa n\rrbracket \llbracket \alpha\varphi n\rrbracket - \llbracket \alpha n\rrbracket\llbracket \tau\varphi n\rrbracket &= (\kappa^+ + \kappa^-)(\alpha^+\varphi^+ + \alpha^-\varphi^-) \\ &\quad - (\alpha^+ + \alpha^-)(\kappa^+\varphi^+ + \kappa^-\varphi^-) \\ &= \varphi^+(\kappa^+\alpha^+ + \kappa^-\alpha^+ - \alpha^+\kappa^+ - \alpha^-\kappa^+) \\ &\quad - \varphi^-(-\kappa^+\alpha^- - \kappa^-\alpha^- + \alpha^+\kappa^- + \alpha^-\kappa^-) \\ &= (\varphi^+ - \varphi^-)(\alpha^+\kappa^- - \alpha^-\kappa^+) \\ &= -\llbracket \varphi\rrbracket \cdot \llbracket \kappa\rrbracket_\alpha.\end{aligned}$$

The second identity can be proved by expanding the right-hand side and showing that it is equal to the left-hand side after simplifications. □

Proof of Proposition C.1 By the conservativity conditions (2.1e) and (2.1f), we have that $\llbracket \widehat{T}_h\rrbracket = \llbracket \widehat{\theta}_h\rrbracket = 0$ at every interior node. Hence, by the definition of the numerical traces, (2.2b) and (2.2c), we have

$$0 = [\![T_h]\!] - [\![\tau M_h n]\!] + [\![\tau n]\!] \widehat{M}_h + [\![\alpha_T w_h n]\!] - [\![\alpha_T n]\!] \widehat{w}_h,$$
$$0 = [\![\theta_h]\!] - [\![\alpha_\theta M_h n]\!] + [\![\alpha_\theta n]\!] \widehat{M}_h - [\![\tau w_h n]\!] + [\![\tau n]\!] \widehat{w}_h$$

which is a simple linear system for $\widehat{M}_h$ and $\widehat{w}_h$. Inverting this system yields

$$\begin{aligned}\widehat{M}_h = \frac{1}{D}\Big[&[\![\tau n]\!][\![\tau M_h n]\!] + [\![\alpha_T n]\!][\![\alpha_\theta M_h n]\!]\\ &- [\![\tau n]\!][\![\alpha_T w_h n]\!] + [\![\alpha_T n]\!][\![\tau w_h n]\!]\\ &- [\![\alpha_T n]\!][\![\theta_h]\!] - [\![\tau n]\!][\![T_h]\!]\Big],\end{aligned} \tag{C.4}$$

$$\begin{aligned}\widehat{w}_h = \frac{1}{D}\Big[&[\![\tau n]\!][\![\tau w_h n]\!] + [\![\alpha_\theta n]\!][\![\alpha_T w_h n]\!]\\ &+ [\![\tau n]\!][\![\alpha_\theta M_h n]\!] - [\![\alpha_\theta n]\!][\![\tau M_h n]\!]\\ &- [\![\tau n]\!][\![\theta_h]\!] + [\![\alpha_\theta n]\!][\![T_h]\!]\Big].\end{aligned} \tag{C.5}$$

Using the identity (C.3a) with $\kappa = \tau$, $\alpha = \alpha_T$, and $\varphi = w_h$, we obtain

$$-[\![\tau n]\!][\![\alpha_T w_h n]\!] + [\![\alpha_T n]\!][\![\tau w_h n]\!] = [\![w_h]\!] \cdot [\![\tau]\!]_{\alpha_T}.$$

Using the same identity with $\kappa = \tau$, $\alpha = \alpha_\theta$, and $\varphi = M_h$, we get

$$[\![\tau n]\!][\![\alpha_\theta M_h n]\!] - [\![\alpha_\theta n]\!][\![\tau M_h n]\!] = -[\![M_h]\!] \cdot [\![\tau]\!]_{\alpha_\theta}.$$

Inserting these into (C.4) and (C.5) we get

$$\begin{aligned}\widehat{M}_h = \frac{1}{D}\Big[&[\![\tau n]\!][\![\tau M_h n]\!] + [\![\alpha_T n]\!][\![\alpha_\theta M_h n]\!]\\ &+ [\![\tau]\!]_{\alpha_T} - [\![\alpha_T n]\!][\![\theta_h]\!] - [\![\tau n]\!][\![T_h]\!]\Big],\end{aligned} \tag{C.6}$$

$$\begin{aligned}\widehat{w}_h = \frac{1}{D}\Big[&[\![\tau n]\!][\![\tau w_h n]\!] + [\![\alpha_\theta n]\!][\![\alpha_T w_h n]\!]\\ &- [\![\tau]\!]_{\alpha_\theta} - [\![\tau n]\!][\![\theta_h]\!] + [\![\alpha_\theta n]\!][\![T_h]\!]\Big].\end{aligned} \tag{C.7}$$

Using the identity (C.3b) with $\alpha = \tau$ and $\varphi = M_h$ we get

$$[\![\tau M_h n]\!] = [\![\tau n]\!]\{\!\{M_h\}\!\} + \frac{1}{2}[\![\tau]\!][\![M_h]\!].$$

Similarly, $\alpha = \alpha_\theta$ and $\varphi = M_h$ we get

$$[\![\alpha_\theta M_h n]\!] = [\![\alpha_\theta n]\!]\{\!\{M_h\}\!\} + \frac{1}{2}[\![\alpha_\theta]\!][\![M_h]\!].$$

Hence,

$$[\![\tau n]\!][\![\tau M_h n]\!] + [\![\alpha_T n]\!][\![\alpha_\theta M_h n]\!]$$

$$= \{\!\!\{M_h\}\!\!\}(\llbracket \tau n\rrbracket^2 + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta n\rrbracket) + \frac{1}{2}\left(\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta\rrbracket\right)\llbracket M_h\rrbracket.$$

Upon dividing both sides by $D = \llbracket \tau n\rrbracket^2 + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta n\rrbracket$ we get

$$\frac{\llbracket \tau n\rrbracket\llbracket \tau M_h n\rrbracket + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta M_h n\rrbracket}{D} = \{\!\!\{M_h\}\!\!\} + \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta\rrbracket}{2D}\llbracket M_h\rrbracket, \tag{C.8}$$

and similarly,

$$\frac{\llbracket \tau n\rrbracket\llbracket \tau w_h n\rrbracket + \llbracket \alpha_\theta n\rrbracket\llbracket \alpha_T w_h n\rrbracket}{D} = \{\!\!\{w_h\}\!\!\} + \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_\theta n\rrbracket\llbracket \alpha_T\rrbracket}{2D}\llbracket w_h\rrbracket. \tag{C.9}$$

Inserting (C.8) and (C.9) into (C.6) and (C.7), respectively, we reach at (C.1) and (C.2). □

Proof of Theorem 5.1 Since both the HDG method and the DG method are defined by the same weak formulation, their solutions are identical if and only if their numerical traces, $(\widehat{T}_h, \widehat{M}_h, \widehat{\theta}_h, \widehat{w}_h)$, are identical at every node of the mesh. We will prove that this is the case if we pick the coefficients C_{ij} as prescribed in the theorem. We proceed in five steps, in the first four steps we give the proof for the interior nodes, namely, we will derive the identities (5.4)–(5.7), and in the last one we give a proof for the boundary nodes, namely, the identity (5.8).

Step 1: Proof of (5.4).

Inserting (5.4) into (5.1a), and comparing the resulting expression with (C.2) shows that the numerical trace $\widehat{w}_h$ of the DG method and that of the HDG method are identical.

Step 2: Proof of (5.5).

Inserting the expressions for $\widehat{M}_h$ and $\widehat{w}_h$ provided, respectively, by (C.1) and (C.2) into (2.2b) we obtain

$$\begin{aligned}
\widehat{\theta}_h^+ = \theta_h^+ + \alpha_\theta^+\Bigg[&-\frac{1}{2}\llbracket M_h\rrbracket - \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta\rrbracket}{2D}\llbracket M_h\rrbracket \\
&- \frac{\llbracket \tau\rrbracket_{\alpha_T}}{D}\llbracket w_h\rrbracket + \frac{\llbracket \alpha_T n\rrbracket}{D}\llbracket \theta_h\rrbracket + \frac{\llbracket \tau n\rrbracket}{D}\llbracket T_h\rrbracket\Bigg] \\
+ \tau^+\Bigg[&-\frac{1}{2}\llbracket w_h\rrbracket - \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_\theta n\rrbracket\llbracket \alpha_T\rrbracket}{2D}\llbracket w_h\rrbracket \\
&+ \frac{\llbracket \tau\rrbracket_{\alpha_\theta}}{D}\llbracket M_h\rrbracket - \frac{\llbracket \alpha_\theta n\rrbracket}{D}\llbracket T_h\rrbracket + \frac{\llbracket \tau n\rrbracket}{D}\llbracket \theta_h\rrbracket\Bigg].
\end{aligned}$$

After a recollection of terms we can rewrite this equation as

$$\begin{aligned}
\widehat{\theta}_h^+ = &-\left(\frac{\alpha_\theta^+\llbracket \tau\rrbracket_{\alpha_T}}{D} + \tau^+\left(\frac{1}{2} + \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_\theta n\rrbracket\llbracket \alpha_T\rrbracket}{2D}\right)\right)\llbracket w_h\rrbracket \\
&+ \theta_h^+ + \frac{\alpha_\theta^+\llbracket \alpha_T n\rrbracket + \tau^+\llbracket \tau n\rrbracket}{D}\llbracket \theta_h\rrbracket \\
&- \alpha_\theta^+\left(\frac{1}{2} + \frac{\llbracket \tau n\rrbracket\llbracket \tau\rrbracket + \llbracket \alpha_T n\rrbracket\llbracket \alpha_\theta\rrbracket}{2D} - \frac{\tau^+\llbracket \tau\rrbracket_{\alpha_\theta}}{D}\right)\llbracket M_h\rrbracket
\end{aligned}$$

$$+\frac{\alpha_\theta^+[\![\tau n]\!]-\tau^+[\![\alpha_\theta n]\!]}{D}[\![T_h]\!].$$

Comparing this with (5.1b), and carrying out some simple algebraic manipulations we see that the two numerical traces are identical if we pick C_{2j} as prescribed by (5.5).

Step 3: Proof of (5.6).

Inserting (5.6) into (5.1c), and comparing the resulting expression with (C.1) shows that the numerical trace $\widehat{M}_h$ of the DG method and that of the HDG method are identical.

Step 4: Proof of (5.7).

We proceed as we did in the proof of Step 2. Inserting the expressions for $\widehat{M}_h$ and $\widehat{w}_h$ provided, respectively, by (C.1) and (C.2) into (2.2c), and recollecting terms, we obtain

$$\begin{aligned}\widehat{T}_h^+ = &-\left(\frac{\tau^+[\![\tau]\!]_{\alpha_T}}{D}-\alpha_T^+\left(\frac{1}{2}+\frac{[\![\tau n]\!][\![\tau]\!]+[\![\alpha_\theta n]\!][\![\alpha_T]\!]}{2D}\right)\right)[\![w_h]\!]\\&+\frac{\tau^+[\![\alpha_T n]\!]-\alpha_T^+[\![\tau n]\!]}{D}[\![\theta_h]\!]\\&-\left(\frac{\alpha_T^+[\![\tau]\!]_{\alpha_\theta}}{D}+\tau^+\left(\frac{1}{2}+\frac{[\![\tau n]\!][\![\tau]\!]+[\![\alpha_T n]\!][\![\alpha_\theta]\!]}{2D}\right)\right)[\![M_h]\!]\\&+T_h^+ +\frac{\tau^+[\![\tau n]\!]+\alpha_T^+[\![\alpha_\theta n]\!]}{D}[\![T_h]\!].\end{aligned}$$

Comparing this with (5.1d), and carrying out some simple algebraic manipulations we see that the two numerical traces are identical if we pick C_{4j} as prescribed by (5.7).

Step 5: Proof of (5.8).

We only prove this for $x=0$, the proof for $x=1$ is similar.

The conservativity condition (2.1e) implies $\widehat{\theta}_h(0)=\theta_N(0)$. Thus, the numerical trace (2.2b) at $x=0$ can be written as

$$\theta_N(0)=\theta_h(0^+)+\alpha_\theta(0^+)(M_h(0^+)-\widehat{M}_h(0))+\tau(0^+)(w_h(0^+)-w_D(0)).$$

Using the assumption $\alpha_\theta(0^+)>0$ we get

$$\widehat{M}_h(0)=M_h(0^+)+\frac{1}{\alpha_\theta(0^+)}\Big[-\tau(0^+)(w_D(0)-w_h(0^+))-(\theta_N(0)-\theta_h(0^+))\Big]. \tag{C.10}$$

Comparing this with (5.2c) we see that the numerical trace $\widehat{M}_h(0)$ of the DG method is identical to that of the HDG method if we set $C_{31}(0)$ and $C_{32}(0)$ as prescribed by (5.8).

Similarly, using (C.10) in (2.2c) we get

$$\begin{aligned}\widehat{T}_h(0^+)=T_h(0^+)&+\frac{\tau^2(0^+)+\alpha_T(0^+)\alpha_\theta(0^+)}{\alpha_\theta(0^+)}(w_D(0)-w_h(0^+))\\&+\frac{\tau(0^+)}{\alpha_\theta(0^+)}(\theta_N(0)-\theta_h(0^+)).\end{aligned}$$

Comparing this with (5.2d) we see that the numerical trace $\widehat{T}_h(0)$ of the DG method is identical to that of the HDG method if we set $C_{41}(0)$ and $C_{42}(0)$ as prescribed by (5.8).

This completes the proof of Theorem 5.1. □

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