Discontinuous Galerkin Method for Solving Thin Film Equations

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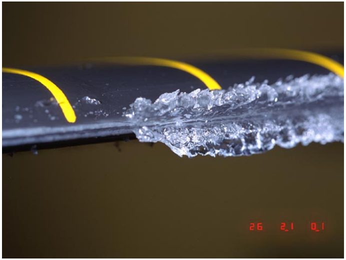
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating

Model Equations

- Navier-Stokes Equation

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$\partial_t h_s + (u, v)^T \cdot \nabla h_s = w$$

$$\partial_t h_b + (u, v)^T \cdot \nabla h_b = w$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation - 1D with q as fluid height.

$$q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

$\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form

  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x = 0$$

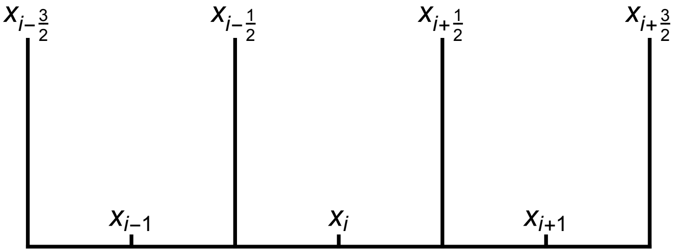
  for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2} \left(f\left(Q_{j+1/2}^-\right) + f\left(Q_{j+1/2}^+\right) \right) + \max_q \left\{ \left| f'(q) \right| \right\} \left(Q_{j+1/2}^- - Q_{j+1/2}^+ \right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^{\star} &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^{\star}) + \frac{1}{2}\Delta t L(q^{\star}) \end{aligned}$$

Diffusion

- Diffusion Equation

$$q_t = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Linearize operator at $t = t^n$, let $f(x) = q^3(t = t^n, x)$

$$q_t = -(f(x) q_{xxx})_x \qquad (0, T) \times \Omega$$

Finite Difference Approach

- Let cell centers, x_i form finite difference grid.
- Finite difference space, $\mathbb{R}^N$.
- $Q_{DG} \in V_h \to Q_{FD} \in \mathbb{R}^N$

$$(Q_{FD})_i = \frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x$$

- $Q_{FD} \in \mathbb{R}^N \to Q_{DG} \in V_h$

$$Q_{DG}|_K \in P^1(K)$$

$$\frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x = (Q_{FD})_i$$

$$\partial_x Q_{DG}|_{K_i} = \frac{(Q_{FD})_{i+1} - (Q_{FD})_{i-1}}{2h}$$

Finite Difference Approximation

$$(q_{xxx})_{i+1/2} \approx \frac{-Q_{i-1} + 3Q_i - 3Q_{i+1} + Q_{i+2}}{h^3}$$

$$F_{i+1/2} = \frac{f(x_{i+1/2}^{+}) + f(x_{i+1/2}^{-})}{2}$$

$$(-(f(x)q_{xxx})_x)_i \approx -\frac{F_{i+1/2}(q_{xxx})_{i+1/2} - F_{i-1/2}(q_{xxx})_{i-1/2}}{h}$$

Implicit L-Stable Runge Kutta

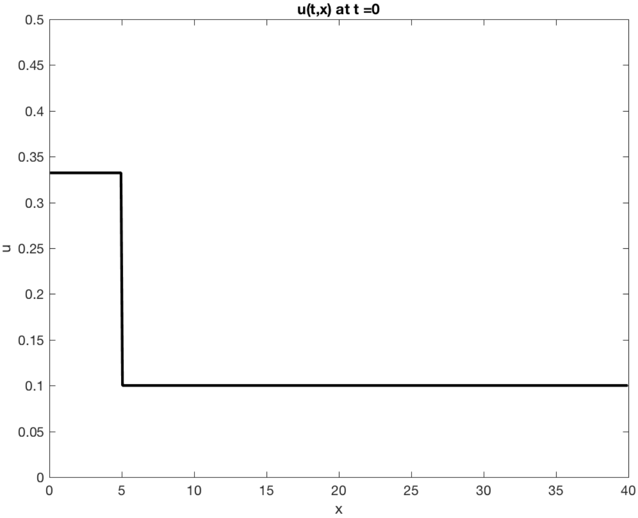
- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

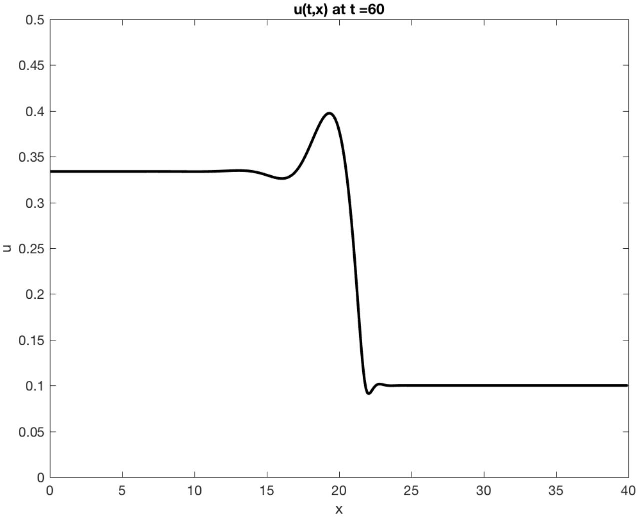
- 2nd Order

$$q^\star = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^\star))$$
$$3q^{n+1} = 4q^\star - q^n + \Delta t L(q^{n+1})$$

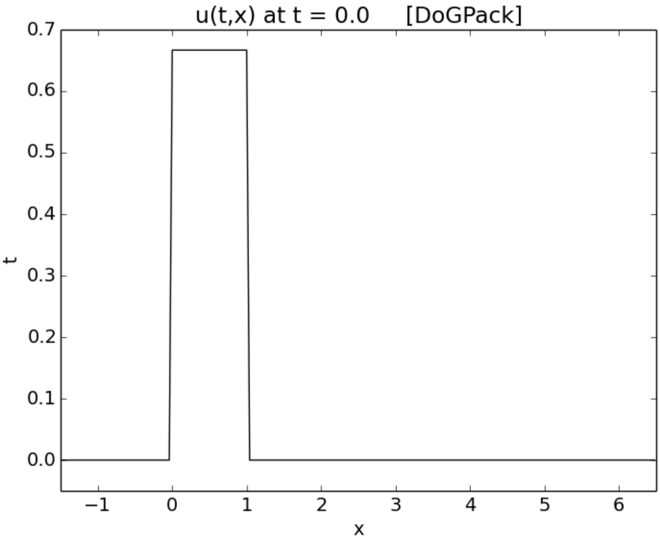
Riemann Problem



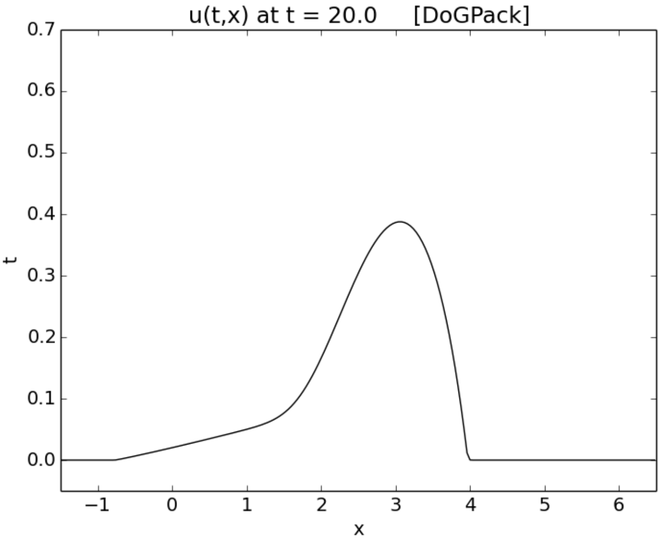
Riemann Problem



Square Wave



Square Wave



Future Work

- Show second order convergence
- Runge Kutta IMEX
- Space and time dependent coefficients

Bibliography

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