Discontinuous Galerkin Method for Solving Thin Film and Shallow Water Equations

Caleb Logemann James Rossmanith

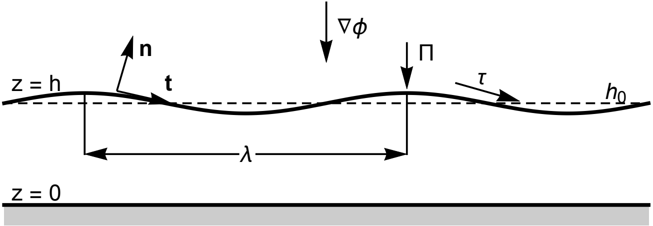
Mathematics Department,
Iowa State University

logemann@iastate.edu

May 13, 2019

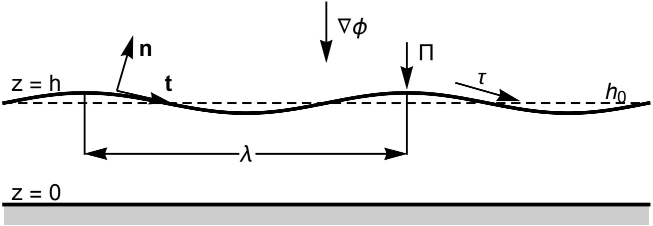
Overview

Model Equations



- Incompressible Navier-Stokes Equation

$$
\begin{aligned}
u_x + w_z &= 0 \\
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
w = 0, u &= 0 && \text{at } z = 0 \\
w &= h_t + uh_x && \text{at } z = h \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } z = h
\end{aligned}
$$



Nondimensionalize, integrate over Z, and simplify, gives

$$H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}\left(\Phi|_{Z=H} - \Pi\right)_X H^3\right)_X = -\frac{1}{3}\bar{C}^{-1}\left(H^3 H_{XXX}\right)_X$$

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

Method Overview

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Runge Kutta Implicit Explicit (IMEX)

$$q_t = F(q) + G(q)$$

  - F evaluated explicitly
  - G solved implicitly

$$F(q) = -\left(q^2 - q^3\right)_x$$
$$G(q) = \left(q^3 q_{xxx}\right)_x$$

Convection

- Convection Equation

$$F(q) = f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_\Omega (F(q)v - f(q)v_x)\,\mathrm{d}x + \hat{f}\,v\Big|_{\partial\Omega} = 0$$

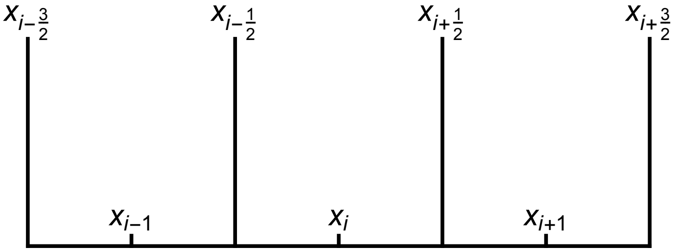
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2}+x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} F(Q) v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

  for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

Diffusion

- Diffusion Equation

$$G(q) = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Local Discontinuous Galerkin

$$\begin{aligned} r &= q_x \\ s &= r_x \\ u &= s_x \\ G(q) &= \left(q^3 u\right)_x \end{aligned}$$

Local Discontinuous Galerkin

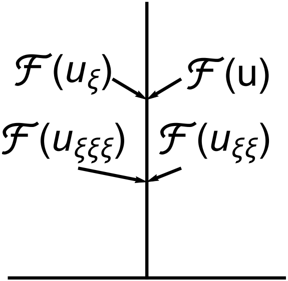
Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t,\cdot), R, S, U \in V_h = \left\{v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j)\right\}$

$$\int_{I_j} Rv \,\mathrm{d}x = -\int_{I_j} Qv_x \,\mathrm{d}x + \left(\hat{Q}_{j+1/2} v_{j+1/2}^{-} - \hat{Q}_{j-1/2} v_{j-1/2}^{+}\right)$$
$$\int_{I_j} Sw \,\mathrm{d}x = -\int_{I_j} Rw_x \,\mathrm{d}x + \left(\hat{R}_{j+1/2} w_{j+1/2}^{-} - \hat{R}_{j-1/2} w_{j-1/2}^{+}\right)$$
$$\int_{I_j} Uy \,\mathrm{d}x = -\int_{I_j} Sy_x \,\mathrm{d}x + \left(\hat{S}_{j+1/2} y_{j+1/2}^{-} - \hat{S}_{j-1/2} y_{j-1/2}^{+}\right)$$
$$\int_{I_j} G(Q)z \,\mathrm{d}x = -\int_{I_j} Q^3 U z_x \,\mathrm{d}x + \left(\hat{U}_{j+1/2} z_{j+1/2}^{-} - \hat{U}_{j-1/2} z_{j-1/2}^{+}\right)$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$
$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$
$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$
$$\hat{U}_{j+1/2} = \left(Q^3 U\right)^{-}_{j+1/2}$$



IMEX Runge Kutta

- IMEX scheme

$$q^{n+1} = q^n + \Delta t \sum_{i=1}^{s} (b'_i F(t_i, u_i)) + \Delta t \sum_{i=1}^{s} (b_i G(t_i, u_i))$$

$$u_i = q^n + \Delta t \sum_{j=1}^{i-1} (a'_{ij} F(t_j, u_j)) + \Delta t \sum_{j=1}^{i} (a_{ij} G(t_j, u_j))$$

$$t_i = t^n + c_i \Delta t$$

- Double Butcher Tableaus

$$\begin{array}{c|c} c' & a' \\ \hline & b'^T \end{array} \quad \begin{array}{c|c} c & a \\ \hline & b^T \end{array}$$

- 1st Order — L-Stable SSP

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

- 2nd Order — SSP

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

- 3rd Order — L-Stable SSP

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|cccc}
\alpha & \alpha & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 \\
\frac{1}{2} & \beta & \eta & \zeta & \alpha \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

$$\alpha = 0.24169426078821$$

$$\beta = 0.06042356519705$$

$$\eta = 0.1291528696059$$

$$\zeta = \frac{1}{2} - \beta - \eta - \alpha$$

Nonlinear Solvers

- Nonlinear System

$$u_i - a_{ii}\Delta t G(u_i) = b$$

- Picard Iteration

$$\tilde{G}(q, u) = \left(q^3 u_{xxx}\right)_x$$

$$u_0 = q^n \qquad u_i^0 = u_{i-1}$$
$$u_i^j - a_{ii}\Delta t \tilde{G}(u_i^{j-1}, u_i^j) = b$$

- Number of picard iterations equals order in time and space

Manufactured Solution

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x + s$$

$$s = \hat{q}_t + \left(\hat{q}^2 - \hat{q}^3\right)_x + \left(\hat{q}^3 \hat{q}_{xxx}\right)_x$$

$$\hat{q} = 0.1 \times \sin(2\pi/20.0 \times (x - t)) + 0.15 \quad \text{for } (x, t) \in [0, 40] \times [0, 5.0]$$

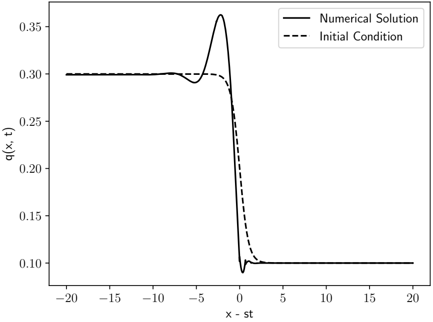
	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	0.136	—	7.33×10^{-3}	—	5.29×10^{-4}	—
40	0.0719	0.92	1.99×10^{-3}	1.88	5.38×10^{-5}	3.30
80	0.0378	0.93	5.60×10^{-4}	1.83	7.47×10^{-6}	2.85
160	0.0191	0.99	1.56×10^{-4}	1.85	9.97×10^{-7}	2.91
320	0.00961	0.99	3.98×10^{-5}	1.97	1.26×10^{-7}	2.98
640	0.00483	0.99	1.00×10^{-5}	1.99	1.58×10^{-8}	3.00
1280	0.00242	1.00	2.50×10^{-6}	2.00	1.98×10^{-9}	3.00

Table: Convergence table with a constant, linear, quadratic polynomial bases. CFL $= 0.9, 0.2, 0.1$ respectively.

Wave Structure with Nonlinear Hyper Diffusion

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

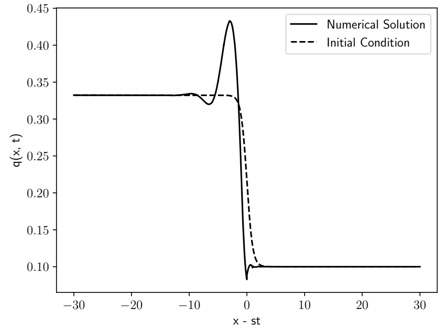
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

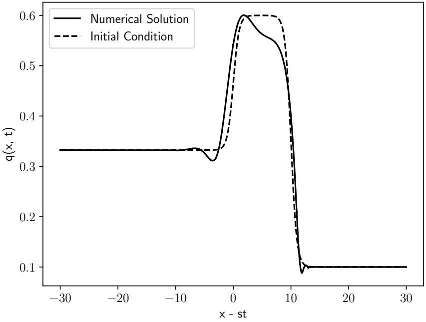
$$q(x,0) = (-\tanh(x) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

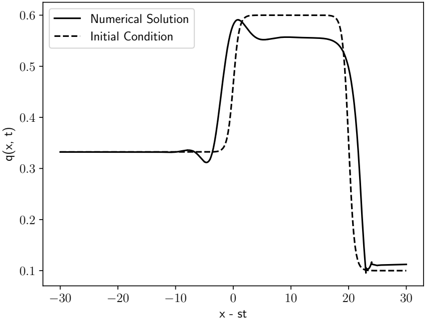
$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2} \tanh(x) + \frac{q_m + q_l}{2} & x < 5 \\ -\frac{q_m - q_r}{2} \tanh(x - 10) + \frac{q_m + q_r}{2} + q_r & x > 5 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2}\tanh(x) + \frac{q_m + q_l}{2} & x < 10 \\ -\frac{q_m - q_r}{2}\tanh(x - 20) + \frac{q_m + q_r}{2} + q_r & x > 10 \end{cases}$$



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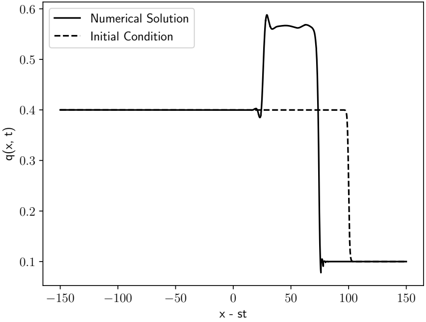
Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.4$$

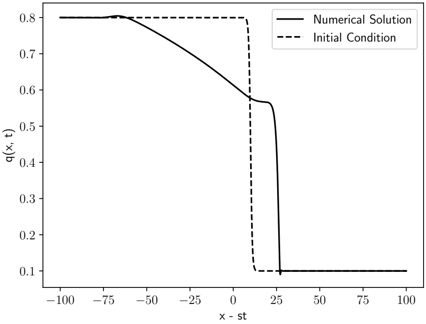
$$q(x, 0) = (-\tanh(x - 100) + 1)\frac{q_l - q_r}{2} + q_r$$



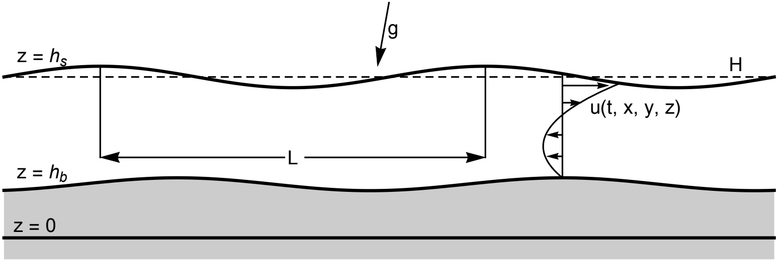
Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x, 0) = (-\tanh(x - 10) + 1)\frac{q_l - q_r}{2} + q_r$$



Generalized Shallow Water



$$\div \mathbf{u} = 0$$

$$\mathbf{u}_t + \div * \mathbf{u}\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho} \div \sigma + \mathbf{g}$$

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s)$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b)$$

Nondimensionalization

Characteristic Lengths

$$\varepsilon = \frac{H}{L}, \quad x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}$$

Characteristic Velocities

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}$$

Characteristic Time

$$t = \frac{L}{U}\hat{t}$$

Characteristic Stresses

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}$$

Nondimensionalization

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$\varepsilon F^2 \left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + \left(\hat{u}\hat{v}\right)_{\hat{y}} + \left(\hat{u}\hat{w}\right)_{\hat{z}} \right) = -\varepsilon \hat{p}_{\hat{x}}$$

$$+ G\left(\varepsilon^2 (\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}} \right) + e_x$$

$$\varepsilon F^2 \left(\hat{v}_{\hat{t}} + \left(\hat{u}\hat{v}\right)_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + \left(\hat{v}\hat{w}\right)_{\hat{z}} \right) = -\varepsilon \hat{p}_{\hat{y}}$$

$$+ G\left(\varepsilon^2 (\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2 (\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}} \right) + e_y$$

$$\varepsilon^2 F^2 \left(\hat{w}_{\hat{t}} + \left(\hat{u}\hat{w}\right)_{\hat{x}} + \left(\hat{v}\hat{w}\right)_{\hat{x}} + \left(\hat{w}^2\right)_{\hat{z}} \right) = -\hat{p}_{\hat{z}}$$

$$+ \varepsilon G\left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}} \right) + e_z$$

$$F = \frac{U}{\sqrt{gH}} \approx 1, \quad G = \frac{S}{\rho g H} < 1$$

Nondimensionalization

Drop ε^2 and εG terms

$$
\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0 \\
\varepsilon F^2\left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + \left(\hat{u}\hat{v}\right)_{\hat{y}} + \left(\hat{u}\hat{w}\right)_{\hat{z}}\right) &= -\varepsilon\hat{p}_{\hat{x}} + G\left(\hat{\sigma}_{xz}\right)_{\hat{z}} + e_x \\
\varepsilon F^2\left(\hat{v}_{\hat{t}} + \left(\hat{u}\hat{v}\right)_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + \left(\hat{v}\hat{w}\right)_{\hat{z}}\right) &= -\varepsilon\hat{p}_{\hat{y}} + G\left(\hat{\sigma}_{yz}\right)_{\hat{z}} + e_y \\
\hat{p}_{\hat{z}} &= e_z
\end{aligned}
$$

Solving for the hydrostatic pressure

$$
\hat{p}(\hat{t}, \hat{x}, \hat{y}) = \left(\hat{h}_s(\hat{t}, \hat{x}, \hat{y}) - \hat{z}\right)e_z
$$

Nondimensionalization

Dimensional Variables

$$u_x + v_y + w_z = 0$$

$$u_t + \left(u^2\right)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x$$

$$v_t + (uv)_x + \left(v^2\right)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y$$

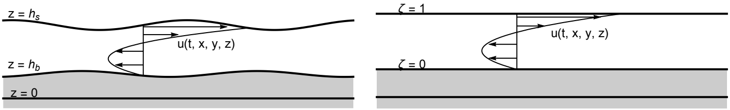
$$p(t, x, y, z) = (h_s(t, x, y) - z)\rho ge_z$$

Kinematic Boundary Conditions

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s)$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b)$$

Mapping



Transform from $z \to \zeta$, by

$$\zeta = \frac{z - h_b(t, x, y)}{h(t, x, y)},$$

or equivalently

$$z = h(t, x, y)\zeta + h_b(t, x, y)$$

where $h(t, x, y) = h_s(t, x, y) - h_b(t, x, y)$.

$$\tilde{\Psi}(t, x, y, \zeta) = \Psi(t, x, y, h(t, x, y)\zeta + h_b(t, x, y))$$

Mapping Continuity Equation

$$u_x + v_y + w_z = 0$$

Map to new space

$$(h\tilde{u})_x - ((\zeta h + h_b)_x \tilde{u})_\zeta + (h\tilde{v})_y - \left((\zeta h + h_b)_y \tilde{v}\right)_\zeta + (\tilde{w})_\zeta = 0$$

Solve for vertical velocity, w,

$$\tilde{w}(t, x, y, \zeta') = -\left(h \int_0^{\zeta'} \tilde{u} \, \mathrm{d}\zeta\right)_x - \left(h \int_0^{\zeta'} \tilde{v} \, \mathrm{d}\zeta\right)_y$$
$$+(\zeta' h + h_b)_x \tilde{u}(t, x, y, \zeta') + (\zeta' h + h_b)_y \tilde{v}(t, x, y, \zeta')$$

Depth averaged equation

$$h_t + \left(h \int_0^1 \tilde{u} \, \mathrm{d}\zeta\right)_x + \left(h \int_0^1 \tilde{v} \, \mathrm{d}\zeta\right)_y = 0$$

Let u_m and v_m denote the mean velocity

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

Mapping Momentum Equations

$$u_t + \left(u^2\right)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x$$

$$v_t + (uv)_x + \left(v^2\right)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y$$

$$(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}ge_z h^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz}\right)_\zeta = gh\left(e_x - e_z(h_b)_x\right)$$

$$(h\tilde{v})_t + (h\tilde{u}\tilde{v})_x + \left(h\tilde{v}^2 + \frac{1}{2}ge_z h^2\right)_y + \left(h\tilde{v}\omega - \frac{1}{\rho}\tilde{\sigma}_{yz}\right)_\zeta = gh\left(e_x - e_z(h_b)_y\right)$$

where

$$\omega = \frac{1}{h}\left(-\left(h\int_0^\zeta \tilde{u} - u_m \,\mathrm{d}\zeta'\right)_x - \left(h\int_0^\zeta \tilde{v} - v_m \,\mathrm{d}\zeta'\right)_y\right)$$

Mapped Reference System

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}ge_z h^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz}\right)_\zeta = gh\big(e_x - e_z(h_b)_x\big)$$

$$(h\tilde{v})_t + (h\tilde{u}\tilde{v})_x + \left(h\tilde{v}^2 + \frac{1}{2}ge_z h^2\right)_y + \left(h\tilde{v}\omega - \frac{1}{\rho}\tilde{\sigma}_{yz}\right)_\zeta = gh\big(e_x - e_z(h_b)_y\big)$$

$$\omega = \frac{1}{h}\left(-\left(h\int_0^\zeta \tilde{u}_d \,\mathrm{d}\zeta'\right)_x - \left(h\int_0^\zeta \tilde{v}_d \,\mathrm{d}\zeta'\right)_y\right)$$

with

$$\tilde{u}_d = \tilde{u} - u_m \quad \tilde{v}_d = \tilde{v} - v_m$$

Newtonian Flow

Newtonian Stree Tensor

$$\sigma_{xz} = \mu u_z \quad \sigma_{yz} = \mu v_z$$

Kinematic Viscosity

$$\nu = \frac{\mu}{\rho}$$

Mapped stress tensor

$$\frac{1}{\rho}\tilde{\sigma}_{xz} = \frac{\nu}{h}\tilde{u}_\zeta \quad \frac{1}{\rho}\tilde{\sigma}_{yz} = \frac{\nu}{h}\tilde{v}_\zeta$$

Boundary Conditions

Stree Free Condition at surface

$$u_z\big|_{z=h_s} = v_z\big|_{z=h_s} = 0$$

Mixed Slip Condition at bottom topography

$$u - \frac{\lambda}{\mu}\sigma_{xz}\bigg|_{z=h_b} = v - \frac{\lambda}{\mu}\sigma_{yz}\bigg|_{z=h_b} = 0$$

Mapped with Newtonian Stress

$$\tilde{u}_\zeta\big|_{\zeta=1} = \tilde{v}_\zeta\big|_{\zeta=1} = 0$$

and

$$\tilde{u} - \frac{\lambda}{h}\tilde{u}_\zeta\bigg|_{\zeta=0} = \tilde{v} - \frac{\lambda}{h}\tilde{v}_\zeta\bigg|_{\zeta=0} = 0$$

Moment Closure

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Depth Averaged Momentum Equations

$$
\begin{aligned}
&(hu_m)_t + \left(h\int_0^1 \tilde{u}^2 \, \mathrm{d}\zeta + \frac{1}{2} g e_z h^2\right)_x + \left(h\int_0^1 \tilde{u}\tilde{v} \, \mathrm{d}\zeta\right)_y \\
&\quad + \frac{\nu}{\lambda} \, u|_{\zeta=0} = hg(e_x - e_z(h_b)_x) \\
&(hv_m)_t + \left(h\int_0^1 \tilde{u}\tilde{v} \, \mathrm{d}\zeta\right)_y + \left(h\int_0^1 \tilde{v}^2 \, \mathrm{d}\zeta + \frac{1}{2} g e_z h^2\right)_y \\
&\quad + \frac{\nu}{\lambda} \, v|_{\zeta=0} = hg\left(e_x - e_z(h_b)_y\right)
\end{aligned}
$$

Polynomial Ansatz

$$\begin{aligned}
\tilde{u}(t,x,y,\zeta) &= u_m(t,x,y) + u_d(t,x,y,\zeta) \\
&= u_m(t,x,y) + \sum_{j=1}^{N} \left(\alpha_j(t,x,y)\phi_j(\zeta)\right) \\
\tilde{v}(t,x,y,\zeta) &= v_m(t,x,y) + v_d(t,x,y,\zeta) \\
&= v_m(t,x,y) + \sum_{j=1}^{N} \left(\beta_j(t,x,y)\phi_j(\zeta)\right)
\end{aligned}$$

Orthogonality Condition

$$\int_0^1 \phi_j(\zeta)\phi_i(\zeta)\,\mathrm{d}\zeta = 0 \quad \text{for } j \neq i$$

$$\phi_0(\zeta) = 1, \quad \phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$$

Constant Moments

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y = -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg\left(e_x - e_z (h_b)_x\right)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x = -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\left(e_y - e_z (h_b)_y\right)$$

Higher Order Moments

Moment Equation

$$\int_0^1 \phi_i \left((h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2} g e_z h^2 \right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz} \right)_\zeta \right) \mathrm{d}\zeta$$
$$= \int_0^1 \phi_i (gh(e_x - e_z (h_b)_x)) \, \mathrm{d}\zeta$$

Simplified gives

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k \right)_x$$
$$+ \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k \right)_y$$
$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j \right)$$

Higher Order Moments

$$(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y$$

$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \, \mathrm{d}\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^{\zeta} \phi_j \, \mathrm{d}\hat{\zeta}\right)\phi_k \, \mathrm{d}\zeta$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \, \mathrm{d}\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

Example Systems

1D model with h_b constant, $e_x = e_y = 0$, and $e_z = 1$
Constant System

$$\begin{bmatrix} h \\ hu_m \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{bmatrix} = -\frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m \end{bmatrix}$$

Linear System, $s = \alpha_1$

$$\begin{bmatrix} h \\ hu_m \\ hs \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 \\ 2hu_m s \end{bmatrix} = Q\begin{bmatrix} h \\ hu_m \\ hs \end{bmatrix}_x - P$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{bmatrix} \quad P = \frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m + s \\ 3(u_m + s + 4\frac{\lambda}{h}s) \end{bmatrix}$$

Nonconservative Flux

$$\int Q\mathbf{q}_x \phi \, \mathrm{d}x$$

$$\begin{bmatrix} Q_x^1 \\ Q_x^2 \\ Q_x^3 \\ Q_x^4 \\ Q_x^5 \end{bmatrix} = \frac{1}{2\Delta x} \begin{bmatrix} \Delta Q^1 - 2\sqrt{5}\Delta Q^3 + 78\Delta Q^5 \\ \Delta Q^2 - \frac{10}{3}\sqrt{3}\sqrt{7}\Delta Q^4 \\ \Delta Q^3 - 14\sqrt{5}\Delta Q^5 \\ \Delta Q^4 \\ \Delta Q^5 \end{bmatrix}$$

Higer Momemt Equations

Quadratic Vertical Profile

$$\begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 + \frac{1}{5}h\kappa^2 \\ 2hus + \frac{4}{5}hs\kappa \\ 2hu\kappa + \frac{2}{3}hs^2 + \frac{2}{7}h\kappa^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_x - P$$

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