Derivation of Shallow Water Equations

We begin by considering the Navier-Stokes equations,

$$\div \boldsymbol{u} = 0 \tag{1}$$

$$\boldsymbol{u}_t + \div * \boldsymbol{u}\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\div \sigma + \boldsymbol{g}, \tag{2}$$

j where $\boldsymbol{u} = [u, v, w]^T$ is the vector of velocities, p is the pressure, ρ is the constant density, σ is the deviatoric stress tensor, and $\boldsymbol{g}$ is the gravitational force vector. We also have two boundaries, the bottom topography $h_b(t, x, y)$, and the free surface $h_s(t, x, y)$. At both of these boundaries the kinimatic boundary conditions are in effect and can be expressed as

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s) \tag{3}$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b). \tag{4}$$

In practice the bottom topography is unchanging in time, but we express h_b with time dependence to allow for a symmetric representation of the boundary conditions.

1   Dimensional Analysis

Now we consider the characteristic scales of the problem. Let L be the characteristic horizontal length scale, and let H be the characteristic vertical length scale. For this problem we assume that $H << L$ and we denote the ratio of these lengths as $\varepsilon = H/L$. With these characteristic lengths we can scale the length variables to a nondimensional form

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}. \tag{5}$$

Now let U be the characteristic horizontal velocity, then because of the shallowness the characteristic vertical velocity will be εU. Therefore the velocity variables can be scaled as follows,

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}. \tag{6}$$

Now with the characteristic length and velocity, the time scaling can be described as

$$t = \frac{L}{U}\hat{t} \tag{7}$$

The pressure will be scaled by the characteristic height, H, and the stresses will be scaled by a characteristic stress, S. It is assumed that the basal shear stresses, σ_{xz} and σ_{yz} are of larger order than the lateral shear stress, σ_{xy}, and the normal stresses, σ_{xx}, σ_{yy}, and σ_{zz}, so that

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}. \tag{8}$$

$$\div \boldsymbol{u} = 0$$

$$\boldsymbol{u}_t + \div * \boldsymbol{u}\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\div \sigma + \boldsymbol{g}, u_x + v_y + w_z \qquad\qquad = 0$$

$$u_t + \left(u^2\right)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}\left((\sigma_{xx})_x + (\sigma_{xy})_y + (\sigma_{xz})_z\right) + ge_x$$

$$v_t + (uv)_x + \left(v^2\right)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}\left((\sigma_{xy})_x + (\sigma_{yy})_y + (\sigma_{yz})_z\right) + ge_y$$

$$w_t + (uw)_x + (vw)_y + \left(w^2\right)_z = -\frac{1}{\rho}p_z + \frac{1}{\rho}\left((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\right) + ge_z$$

$$u_x = (U\hat{u})_x = (U\hat{u})_{\hat{x}}\hat{x}_x = \frac{U}{L}\hat{u}_{\hat{x}}$$

$$v_y = (U\hat{v})_{\hat{y}}\hat{y}_y = \frac{U}{L}\hat{v}_{\hat{y}}$$

$$w_z = (\varepsilon U\hat{w})_{\hat{z}}\hat{z}_z = \frac{\varepsilon U}{H}\hat{w}_{\hat{z}} = \frac{U}{L}\hat{w}_{\hat{z}}$$

$$u_x + v_y + w_z = \frac{U}{L}\hat{u}_{\hat{x}} + \frac{U}{L}\hat{v}_{\hat{y}} + \frac{U}{L}\hat{w}_{\hat{z}} = 0$$

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$u_t = (U\hat{u})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{u}_{\hat{t}}$$

$$v_t = (U\hat{v})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{v}_{\hat{t}}$$

$$w_t = (\varepsilon U\hat{w})_{\hat{t}}\hat{t}_t = \varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}}$$

$$\left(u^2\right)_x = \left(U^2\hat{u}^2\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}\left(\hat{u}^2\right)_{\hat{x}}$$

$$(uv)_y = \left(U^2\hat{u}\hat{v}\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}}$$

$$(uw)_z = \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}}$$

$$(uv)_x = \left(U^2\hat{u}\hat{v}\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}}$$

$$\left(v^2\right)_y = \left(U^2\hat{v}^2\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}\left(\hat{v}^2\right)_{\hat{y}}$$

$$(vw)_z = \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}}$$

$$(uw)_x = \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{x}}\hat{x}_x = \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}}$$

$$(vw)_y = \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{y}}\hat{y}_y = \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}}$$

$$\left(w^2\right)_z = \left(\varepsilon^2 U^2\hat{w}^2\right)_{\hat{z}}\hat{z}_z = \varepsilon\frac{U^2}{L}\left(\hat{w}^2\right)_{\hat{z}}$$

$$\frac{1}{\rho}p_x = \frac{1}{\rho}(\rho g H\hat{p})_{\hat{x}}\hat{x}_x = \frac{\varepsilon g L}{L}\hat{p}_{\hat{x}} = \varepsilon g\hat{p}_{\hat{x}}$$

$$\frac{1}{\rho}p_y = \frac{1}{\rho}(\rho g H\hat{p})_{\hat{y}}\hat{y}_y = \frac{\varepsilon g L}{L}\hat{p}_{\hat{y}} = \varepsilon g\hat{p}_{\hat{y}}$$

$$\frac{1}{\rho}p_z = \frac{1}{\rho}(\rho g H\hat{p})_{\hat{z}}\hat{z}_z = \frac{\varepsilon g L}{\varepsilon L}\hat{p}_{\hat{z}} = g\hat{p}_{\hat{z}}$$

$$\frac{1}{\rho}(\sigma_{xx})_x = \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xx})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xx})_{\hat{x}}$$

$$\frac{1}{\rho}(\sigma_{xy})_y = \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{y}}$$

$$\frac{1}{\rho}(\sigma_{xz})_z = \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{xz})_{\hat{z}}$$

$$\frac{1}{\rho}(\sigma_{xy})_x = \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}}$$

$$\frac{1}{\rho}(\sigma_{yy})_y = \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{yy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}}$$

$$\frac{1}{\rho}(\sigma_{yz})_z = \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}}$$

$$\frac{1}{\rho}(\sigma_{xz})_x = \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{x}}\hat{x}_x = \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}}$$

$$\frac{1}{\rho}(\sigma_{yz})_y = \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{y}}\hat{y}_y = \frac{S}{\rho L}(\hat{\sigma}_{yz})_{\hat{y}}$$

$$\frac{1}{\rho}(\sigma_{zz})_z = \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{zz})_{\hat{z}}\hat{z}_z = \varepsilon\frac{S}{\rho H}(\hat{\sigma}_{zz})_{\hat{z}}\frac{U^2}{L}\hat{u}_{\hat{t}} + \frac{U^2}{L}(\hat{u}^2)_{\hat{x}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}} + \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}} \quad = -\varepsilon g\hat{p}_{\hat{x}}$$

$$\frac{U^2}{L}\hat{v}_{\hat{t}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}} + \frac{U^2}{L}(\hat{v}^2)_{\hat{y}} + \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}} = -\varepsilon g\hat{p}_{\hat{y}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}} + \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}} + ge_y$$

$$\varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}} + \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}(\hat{w}^2)_{\hat{z}} = -g\hat{p}_{\hat{z}} + \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}}\Big((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\Big) + ge_z$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$\frac{U^2}{gL}\Big(\hat{u}_{\hat{t}} + (\hat{u}^2)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) = -\varepsilon\hat{p}_{\hat{x}} + \frac{S}{\rho gL}\Big(\varepsilon(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\Big) + e_x$$

$$\frac{U^2}{gL}\Big(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + (\hat{v}^2)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\Big) = -\varepsilon\hat{p}_{\hat{y}} + \frac{S}{\rho gL}\Big(\varepsilon(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\Big) + e_y$$

$$u_x + v_y + w_z = 0$$

$$u_t + (u^2)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x$$

$$v_t + (uv)_x + (v^2)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y$$

2 Mapping

In order to make this system more accessible we will map the vertical variable z to the normalized variable ζ, through the transformation

$$\zeta(t,x,y,z) = \frac{z - h_b(t,x,y)}{h(t,x,y)},$$

or equivalently
$$z(t, x, y, \zeta) = h(t, x, y)\zeta + h_b(t, x, y)$$
where $h(t, x, y) = h_s(t, x, y) - h_b(t, x, y)$. This transformation maps the vertical variable onto a scale from 0 to 1 everywhere.

Consider a function $\Psi(t, x, y, z)$, then is mapped counterpart $\tilde{\Psi}(t, x, y, \zeta)$ can be described as

$$\tilde{\Psi}(t, x, y, \zeta) = \Psi(t, x, y, z(t, x, y, \zeta)) = \Psi(t, x, y, h(t, x, y)\zeta + h_b(t, x, y)),$$

or equivalently

$$\Psi(t, x, y, z) = \tilde{\Psi}(t, x, y, \zeta(t, x, y, z)) = \tilde{\Psi}\left(t, x, y, \frac{z - h_b(t, x, y)}{h(t, x, y)}\right).$$

We also need to be able to map derivatives of functions in order to be able to map the differential equations. This can be described

$$\Psi_z(t, x, y, z) = \left(\tilde{\Psi}(t, x, y, \zeta(t, x, y, z))\right)_z$$
$$\Psi_z(t, x, y, z) = \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z))\zeta_z(t, x, y, z)$$
$$\Psi_z(t, x, y, z) = \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z))\frac{1}{h(t, x, y)}$$
$$h(t, x, y)\Psi_z(t, x, y, z) = \tilde{\Psi}_\zeta(t, z, y, \zeta(t, x, y, z))$$
$$h\Psi_z = \tilde{\Psi}_\zeta$$

$$\Psi_t(t, x, y, z) = \left(\tilde{\Psi}(t, x, y, \zeta(t, x, y, z))\right)_t$$
$$\Psi_t(t, x, y, z) = \tilde{\Psi}_t(t, x, y, \zeta(t, x, y, z)) + \tilde{\Psi}_\zeta(t, x, y, \zeta(t, x, y, z))\zeta_t(t, x, y, z)$$