Nonconservative Products

1 Definition

Consider the nonconservative product

$$g(\boldsymbol{q})\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x},$$

where $g(\boldsymbol{q}) : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ is continuous, but $\boldsymbol{q}$ is possibly discontinuous. In this case, the product is traditionally not well-defined at the discontinuities of $\boldsymbol{q}$. In order to define this product for discontinuous functions, $\boldsymbol{q}$, it is possible to regularize $\boldsymbol{q}$ with a path ϕ at discontinuities according to the theory laid out by Dal Maso, Le Floch, and Murat. To this end consider Lipschitz continuous paths, $\boldsymbol{\psi} : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, that satisfy the following properties.

1. $\forall \boldsymbol{q}_L, \boldsymbol{q}_R \in \mathbb{R}^p$, $\boldsymbol{\psi}(0, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{q}_L$ and $\boldsymbol{\psi}(1, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{q}_R$
2. $\exists k > 0$, $\forall \boldsymbol{q}_L, \boldsymbol{q}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, $\left|\frac{\partial \boldsymbol{\psi}}{\partial s}(s, \boldsymbol{q}_L, \boldsymbol{q}_R)\right| \leq k|\boldsymbol{q}_L - \boldsymbol{q}_R|$ elementwise
3. $\exists k > 0$, $\forall \boldsymbol{q}_L, \boldsymbol{q}_R, \boldsymbol{u}_L, \boldsymbol{u}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, elementwise

$$\left|\frac{\partial \boldsymbol{\psi}}{\partial s}(s, \boldsymbol{q}_L, \boldsymbol{q}_R) - \frac{\partial \boldsymbol{\psi}}{\partial s}(s, \boldsymbol{u}_L, \boldsymbol{u}_R)\right| \leq k(|\boldsymbol{q}_L - \boldsymbol{u}_L| + |\boldsymbol{q}_R - \boldsymbol{u}_R|)$$

Once we have these paths, $\boldsymbol{\psi}$, we can define the nonconservative product.

Let $\boldsymbol{q} : [a,b] \to \mathbb{R}^p$ be a function of bounded variation, let $g : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ be a continuous function, and let $\boldsymbol{\psi}$ satisfy the properties given above. Then there exists a unique real-valued bounded Borel measure μ on $[a,b]$ characterized by the two following properties.

1. If q is continuous on a Borel set $B \subset [a,b]$, then

$$\mu(B) = \int_B g(\boldsymbol{q})\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\,\mathrm{d}x$$

2. If q is discontinuous at a point $x_0 \in [a,b]$, then

$$\mu(x_0) = \int_0^1 g(\psi(s; q(x_0^-), q(x_0^+)))\frac{\partial \psi}{\partial s}(s; q(x_0^-), q(x_0^+))\,\mathrm{d}s$$

By definition, this measure μ is the nonconservative product $g(q)\frac{\mathrm{d}q}{\mathrm{d}x}$ and will be denoted by

$$\mu = \left[g(q)\frac{\mathrm{d}q}{\mathrm{d}x}\right]_\psi$$

In higher dimensions the paths, $\boldsymbol{\psi}$ must also have the property that

4. $\boldsymbol{\psi}(s, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{\psi}(1 - s, \boldsymbol{q}_L, \boldsymbol{q}_R)$

Then the following Thereom can be given in spacetime Let $\boldsymbol{q} : \Omega \to \mathbb{R}^m$ be a bounded function of bounded variation defined on an open subset Ω of $\mathbb{R}^{n+1}$ and $\boldsymbol{t} : \mathbb{R}^m \to \mathbb{R}^m$ be a locally bounded Borel function. Then there existsa unique family of real-valued bounded Borel measures μ_i on Ω, $i = 1, 2, \ldots, m$ such that

1. if B is a continuous Borel subset of Ω, then

$$\mu_i(B) = \int_B t_{ik}(q)\boldsymbol{q}_{x_k}\,\mathrm{d}\lambda$$

where λ is the Borel measure;

2. if B is a discontinuous subset of Ω of approximate jump, then

$$\mu_i(B) = \int_B \int_0^1 t_{ik}(\boldsymbol{\psi}(s, \boldsymbol{q}^L, \boldsymbol{q}^R)) \frac{\partial \boldsymbol{\psi}}{\partial s}(s, \boldsymbol{q}^L, \boldsymbol{q}^R) \, \mathrm{d}s \boldsymbol{n}_k^L \, \mathrm{d}H^n$$

with $\boldsymbol{q}^L$ and $\boldsymbol{q}^R$ the left and right traces at the discontinuity, where H^n is the n-dimensional Hausdorf measure and where we choose $\boldsymbol{n}^L$ the outward normal with respect to the left state,

3. if B is an irregular Borel subset of Ω, then $\mu_i(B) = 0$

This is given in Rhebergen without proof, but I haven't found any outside original references for this Theorem. Mostly this reflects the one dimensional theorem, but I don't understand the appearance of the outward facing normal. It appears to have been an arbitrary choice between $\boldsymbol{n}^L$ and $\boldsymbol{n}^R$. Also I am not sure why it is necessary at all.

2 Weak Solutions

A function $\boldsymbol{q}$ of bounded variation is a weak solution to

$$\boldsymbol{q}_t + g(\boldsymbol{q})\boldsymbol{q}_x = 0 \tag{1}$$

if

$$\boldsymbol{q}_t + [g(\boldsymbol{q})\boldsymbol{q}_x]_\phi = 0 \tag{2}$$

as a bounded Borel measure on $\mathbb{R} \times \mathbb{R}_+$. This is equivalent to finding $\boldsymbol{q}$ that satisfies,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} v_t(t,x)\boldsymbol{q}(t,x) \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}_+} \int_{\mathbb{R}} v(t,\cdot)[g(\boldsymbol{q}(t,\cdot))\boldsymbol{q}_x(t,\cdot)]_\psi \, \mathrm{d}t = \mathbf{0} \tag{3}$$

for all functions $v \in C_0^\infty(\mathbb{R}_t \times \mathbb{R})$.

3 DG Weak Formulation

3.1 Rhebergen Weak Formulation

Find $\boldsymbol{q} \in V_h$ such that for all $\boldsymbol{v} \in V_h$,

$$\sum_j \left(\int_{K_j} \boldsymbol{v}^T \boldsymbol{q}_t - \boldsymbol{v}_x^T \boldsymbol{f}(\boldsymbol{q}) + \boldsymbol{v}^T g(\boldsymbol{q})\boldsymbol{q}_x \, \mathrm{d}x \right) + \sum_S \left(\int_S \left(\boldsymbol{v}^L - \boldsymbol{v}^R\right)^T \hat{\boldsymbol{P}}^{nc} \, \mathrm{d}S \right) \tag{4}$$

$$+ \sum_S \left(\int_S \frac{1}{2}\left(\boldsymbol{v}^R + \boldsymbol{v}^L\right)^T \int_0^1 g\big(\boldsymbol{\psi}(\tau, \boldsymbol{q}^L, \boldsymbol{q}^R)\big) \frac{\partial \boldsymbol{\psi}}{\partial \tau}(\tau, \boldsymbol{q}^L, \boldsymbol{q}^R) \, \mathrm{d}\tau \, \mathrm{d}S \right) \tag{5}$$

where $\hat{\boldsymbol{P}}^{nc}$ is the nonconservative numerical flux, if symmetrical wave speeds are assumed, then the Rusanov or Local Lax Friedrichs flux can be used, otherwise the nonconservative product will affect the numerical flux.

3.2 Pure Nonconservative DG Formulation

Consider the 1D nonconservative equation shown below,

$$\boldsymbol{q}_t + g(\boldsymbol{q})\boldsymbol{q}_x = \mathbf{0} \qquad x \in [a,b], 0 < t < T$$

Let $\{K_j\}$ be a mesh of the domain $[a,b]$. Also denote the DG space as

$$V_h = \left\{ v \in L^1([a,b]) \middle| \, v|_{K_j} \in \mathbb{P}^M(K_j) \right\}$$

Now the semi discrete DG formulation for this problem becomes finding $\boldsymbol{q}_h \in V_h$ for all $v_h \in V_h$ that satisfies

$$\int_a^b v_h \boldsymbol{q}_{h,t} \, \mathrm{d}x + \int_a^b v_h [g(\boldsymbol{q}_h)\boldsymbol{q}_{h,x}]_\psi \, \mathrm{d}x = \mathbf{0} \tag{6}$$

$$\sum_j \left(\int_{K_j} v_h \boldsymbol{q}_{h,t} \, \mathrm{d}x \right) + \sum_j \left(\int_{K_j} v_h g(\boldsymbol{q}_h)\boldsymbol{q}_{h,x} \, \mathrm{d}x \right) + \sum_I \left(\hat{v}_h \int_0^1 g(\psi(s, \boldsymbol{q}_h^L, \boldsymbol{q}_h^R)) \frac{\partial \psi}{\partial s}(s, \boldsymbol{q}_h^L, \boldsymbol{q}_h^R) \, \mathrm{d}s \right) = 0 \tag{7}$$