

Derivation of Thin Film Equations

In this model we consider a thin film of liquid on a flat surface with a free interface. This liquid is driven by gravity, shear and normal forces on the surface, and surface tension (see Figure 1). We begin by considering the two dimensional incompressible Navier-Stokes equations, which have the form,

$$u_x + w_z = 0 \quad (1)$$

$$\rho(u_t + uu_x + ww_z) = -p_x + \mu\Delta u - \phi_x \quad (2)$$

$$\rho(w_t + uw_x + ww_z) = -p_z + \mu\Delta w - \phi_z \quad (3)$$

where ρ is the density, u is the horizontal velocity, w is the vertical velocity, p is the pressure, and ϕ is the force of gravity. Equation (1) is the incompressibility condition and also represents conservation of mass. Equations (2) and (3) represent the conservation of momentum in the x and z directions respectively. We take a no penetration and no slip boundary condition at the lower boundary and the kinematic boundary condition at the upper boundary. These boundary conditions can be expressed as follows,

$$w = 0, u = 0 \quad \text{at } z = 0 \quad (4)$$

$$w = h_t + uh_x \quad \text{at } z = h \quad (5)$$

where h is the height of the liquid. We can also describe the stress tensor, \mathbf{T} , at the free surface, $z = h$, as

$$\mathbf{T} \cdot \mathbf{n} = (-\kappa\sigma + \Pi_0)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau_0\right)\mathbf{t} \quad \text{at } z = h$$

where κ is the mean curvature, σ is the surface tension, and Π_0 and τ_0 are the normal and tangential components of the forcing respectively.

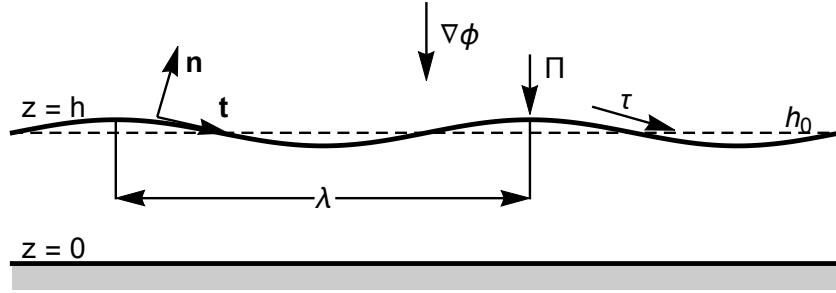


Figure 1: A diagram of the situation in question.

These equations completely describe the fluid, but we are now going to make a lubrication approximation, through a scaling argument. For the lubrication approximation we are going to assume that the average height of the liquid, h_0 , is much smaller than the characteristic wavelength of the liquid, λ . We will denote the ratio of these two lengths as ε , that is

$$\varepsilon = \frac{h_0}{\lambda} \ll 1. \quad (6)$$

Now we nondimensionalize the rest of the variables with respect to this ratio. We denote the nondimensional variables as the uppercase variables. As we stated earlier the characteristic height is h_0 and the characteristic length is $\lambda = h_0/\varepsilon$, so the nondimensional length variables are

$$Z = \frac{z}{h_0}, \quad X = \frac{\varepsilon x}{h_0}, \quad H = \frac{h}{h_0}. \quad (7)$$

Let U_0 be the characteristic horizontal velocity, then

$$U = \frac{u}{U_0}, \quad W = \frac{w}{\varepsilon U_0} \quad (8)$$

where the nondimensional vertical velocity, W , follows from the continuity equation, equation (1). It follows that time will be scaled by λ/U_0 , so nondimensional time is

$$T = \frac{\varepsilon U_0 t}{h_0}. \quad (9)$$

Finally we assume that the flow is locally parallel or equivalently $p_x \sim \mu u_{zz}$. This gives us the proper scaling for the pressure, gravity, and surface stresses,

$$P = \frac{\varepsilon h_0}{\mu U_0} p, \quad \Phi = \frac{\varepsilon h_0}{\mu U_0} \phi, \quad \Pi = \frac{\varepsilon h_0}{\mu U_0} \Pi_0, \quad \tau = \frac{h_0}{\mu U_0} \tau_0. \quad (10)$$

Lastly we can nondimensionalize the surface tension as

$$\Sigma = \frac{\varepsilon \sigma}{\mu U_0} \quad (11)$$

The following give some intermediate steps in how to nondimensionalize variables that include derivatives.

$$\begin{aligned} u_x &= U_0 U_X \frac{\partial X}{\partial x} = \frac{\varepsilon U_0}{h_0} U_X \\ w_z &= \varepsilon U_0 W_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon U_0}{h_0} W_Z \\ u_t &= U_0 U_T \frac{\partial T}{\partial t} = \frac{\varepsilon U_0^2}{h_0} U_T \\ uu_x &= U_0 U U_0 U_X \frac{\partial X}{\partial x} = \frac{\varepsilon U_0^2}{h_0} U U_X \\ wu_z &= \varepsilon U_0 W U_0 U_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon U_0^2}{h_0} W U_Z \\ w_t &= \varepsilon U_0 W_T \frac{\partial T}{\partial t} = \frac{\varepsilon^2 U_0^2}{h_0} W_T \\ uw_x &= U_0 U \varepsilon U_0 W_X \frac{\partial X}{\partial x} = \frac{\varepsilon^2 U_0^2}{h_0} U W_X \\ ww_z &= \varepsilon U_0 W \varepsilon U_0 W_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon^2 U_0^2}{h_0} W W_Z \\ p_x &= \frac{\mu U_0}{\varepsilon h_0} P_X \frac{\partial X}{\partial x} = \frac{\mu U_0}{h_0^2} P_X \\ p_z &= \frac{\mu U_0}{\varepsilon h_0} P_Z \frac{\partial Z}{\partial z} = \frac{\mu U_0}{\varepsilon h_0^2} P_Z \\ u_{xx} &= U_0 U_{XX} \left(\frac{\partial X}{\partial x} \right)^2 = \frac{\varepsilon^2 U_0}{h_0^2} U_{XX} \\ u_{zz} &= U_0 U_{ZZ} \left(\frac{\partial Z}{\partial z} \right)^2 = \frac{U_0}{h_0^2} U_{ZZ} \\ w_{xx} &= \varepsilon U_0 W_{XX} \left(\frac{\partial X}{\partial x} \right)^2 = \frac{\varepsilon^3 U_0}{h_0^2} W_{XX} \\ w_{zz} &= \varepsilon U_0 W_{ZZ} \left(\frac{\partial Z}{\partial z} \right)^2 = \frac{\varepsilon U_0}{h_0^2} W_{ZZ} \\ \phi_x &= \frac{\mu U_0}{\varepsilon h_0} \Phi_X \frac{\partial X}{\partial x} = \frac{\mu U_0}{h_0^2} \Phi_X \\ \phi_z &= \frac{\mu U_0}{\varepsilon h_0} \Phi_Z \frac{\partial Z}{\partial z} = \frac{\mu U_0}{\varepsilon h_0^2} \Phi_Z \\ h_t &= h_0 H_T \frac{\partial T}{\partial t} = \varepsilon U_0 H_T \\ uh_x &= U_0 U h_0 H_X \frac{\partial X}{\partial x} = \varepsilon U_0 U H_X \end{aligned}$$

Substituting these nondimensional variables into the continuity equation (1), gives

$$\begin{aligned} u_x + w_z &= 0 \\ \frac{\varepsilon U_0}{h_0} U_X + \frac{\varepsilon U_0}{h_0} W_Z &= 0 \\ U_X + W_Z &= 0 \end{aligned}$$

This computation also justifies the scaling of w as the nondimensional variables should also conserve mass.

We can nondimensionalize the conservation of momentum equations as follows,

$$\begin{aligned}\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\ \rho\left(\frac{\varepsilon U_0^2}{h_0}U_T + \frac{\varepsilon U_0^2}{h_0}UU_X + \frac{\varepsilon U_0^2}{h_0}WU_Z\right) &= -\frac{\mu U_0}{h_0^2}P_X + \mu\left(\frac{\varepsilon^2 U_0}{h_0^2}U_{XX} + \frac{U_0}{h_0^2}U_{ZZ}\right) - \frac{\mu U_0}{h_0^2}\Phi_X \\ \frac{\varepsilon U_0^2 \rho}{h_0}(U_T + UU_X + WU_Z) &= \frac{\mu U_0}{h_0^2}(-P_X + (\varepsilon^2 U_{XX} + U_{ZZ}) - \Phi_X) \\ \frac{\varepsilon U_0 \rho h_0}{\mu}(U_T + UU_X + WU_Z) &= (-P_X + (\varepsilon^2 U_{XX} + U_{ZZ}) - \Phi_X)\end{aligned}$$

and

$$\begin{aligned}\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\ \rho\left(\frac{\varepsilon^2 U_0^2}{h_0}W_T + \frac{\varepsilon^2 U_0^2}{h_0}UW_X + \frac{\varepsilon^2 U_0^2}{h_0}WW_Z\right) &= -\frac{\mu U_0}{\varepsilon h_0^2}P_Z + \mu\left(\frac{\varepsilon^3 U_0}{h_0^2}W_{XX} + \frac{\varepsilon U_0}{h_0^2}W_{ZZ}\right) - \frac{\mu U_0}{\varepsilon h_0^2}\Phi_Z \\ \frac{\varepsilon^2 \rho U_0^2}{h_0}(W_T + UW_X + WW_Z) &= \frac{\mu U_0}{\varepsilon h_0^2}(-P_Z + (\varepsilon^4 W_{XX} + \varepsilon^2 W_{ZZ}) - \Phi_Z) \\ \varepsilon^3 \frac{\rho U_0 h_0}{\mu}(W_T + UW_X + WW_Z) &= (-P_Z + \varepsilon^2(\varepsilon^2 W_{XX} + W_{ZZ}) - \Phi_Z)\end{aligned}$$

The boundary conditions are nondimensionalized as below,

$$\begin{aligned}w = 0, u = 0 & \quad \text{at } z = 0 \\ \varepsilon U_0 W = 0, U_0 U = 0 & \quad \text{at } Z = 0 \\ W = 0, U = 0 & \quad \text{at } Z = 0\end{aligned}$$

with

$$\begin{aligned}w = h_t + uh_x & \quad \text{at } z = h \\ \varepsilon U_0 W = \varepsilon U_0 H_T + \varepsilon U_0 U H_x & \quad \text{at } Z = H \\ W = H_T + U H_x & \quad \text{at } Z = H\end{aligned}$$

In order to nondimensionalize the stress tensor at the free surface, we consider the normal and tangential components of $\mathbf{T} \cdot \mathbf{n}$ separately. The normal and tangential vector at the surface can be expressed in terms of the free surface as

$$\mathbf{n} = \frac{\langle -h_x, 1 \rangle}{(1 + h_x^2)^{1/2}} \quad \mathbf{t} = \frac{\langle 1, h_x \rangle}{(1 + h_x^2)^{1/2}} \quad (12)$$

and the mean curvature, κ , can be expressed in terms of h as

$$\kappa = -\frac{h_{xx}}{(1 + h_x^2)^{3/2}}. \quad (13)$$

We would now like to write the following vector equation in terms of its normal and tangential components.

$$\mathbf{T} \cdot \mathbf{n} = (-\kappa\sigma + \Pi_0)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau_0\right)\mathbf{t} \quad \text{at } z = h$$

Note that we are using a Newtonian stress tensor which is given by

$$\mathbf{T} = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + \mu \begin{bmatrix} 2u_x & u_z + w_x \\ u_z + w_x & 2w_z \end{bmatrix} \quad (14)$$

The normal component of this vector equation is given by

$$\langle \mathbf{T} \cdot \mathbf{n}, \mathbf{n} \rangle = (-\kappa\sigma + \Pi_0) \quad (15)$$

First we will simplify the left hand side.

$$\begin{aligned}
a &= (1 + h_x^2)^{1/2} \\
\langle \mathbf{T} \cdot \mathbf{n}, \mathbf{n} \rangle &= \frac{1}{a^2} \begin{bmatrix} -h_x & 1 \end{bmatrix} \begin{bmatrix} -p + 2\mu u_x & \mu(u_z + w_x) \\ \mu(u_z + w_x) & -p + 2\mu w_z \end{bmatrix} \begin{bmatrix} -h_x \\ 1 \end{bmatrix} \\
&= \frac{1}{a^2} \begin{bmatrix} -h_x & 1 \end{bmatrix} \begin{bmatrix} -h_x(-p + 2\mu u_x) + \mu(u_z + w_x) \\ -h_x\mu(u_z + w_x) + -p + 2\mu w_z \end{bmatrix} \\
&= \frac{1}{a^2} (h_x^2(-p + 2\mu u_x) - \mu h_x(u_z + w_x) - \mu h_x(u_z + w_x) + -p + 2\mu w_z) \\
&= \frac{1}{a^2} (h_x^2(-p + 2\mu u_x) - 2\mu h_x(u_z + w_x) - p + 2\mu w_z) \\
&= \frac{1}{a^2} ((1 + h_x^2)(-p) + 2\mu(h_x^2 u_x + w_z) - 2\mu h_x(u_z + w_x)) \\
&= -p + \frac{2\mu}{a^2} ((h_x^2 u_x + w_z) - h_x(u_z + w_x))
\end{aligned}$$

Next simplify the right hand side.

$$(-\kappa\sigma + \Pi_0) = \frac{h_{xx}}{(1 + h_x^2)^{3/2}} \sigma + \Pi_0$$

This gives the following scalar equation,

$$-p - \Pi_0 + \frac{2\mu}{1 + h_x^2} ((h_x^2 u_x + w_z) - h_x(u_z + w_x)) = \frac{h_{xx}}{(1 + h_x^2)^{3/2}} \sigma. \quad (16)$$

The tangential component is given by

$$\langle \mathbf{T} \cdot \mathbf{n}, \mathbf{t} \rangle = \left(\frac{\partial \sigma}{\partial s} + \tau_0 \right) \quad (17)$$

First simplify the left hand side,

$$\begin{aligned}
\langle \mathbf{T} \cdot \mathbf{n}, \mathbf{n} \rangle &= \frac{1}{a^2} \begin{bmatrix} 1 & h_x \end{bmatrix} \begin{bmatrix} -p + 2\mu u_x & \mu(u_z + w_x) \\ \mu(u_z + w_x) & -p + 2\mu w_z \end{bmatrix} \begin{bmatrix} -h_x \\ 1 \end{bmatrix} \\
&= \frac{1}{a^2} \begin{bmatrix} 1 & h_x \end{bmatrix} \begin{bmatrix} -h_x(-p + 2\mu u_x) + \mu(u_z + w_x) \\ -h_x\mu(u_z + w_x) + -p + 2\mu w_z \end{bmatrix} \\
&= \frac{1}{a^2} (-h_x(-p + 2\mu u_x) + \mu(u_z + w_x) + -h_x^2\mu(u_z + w_x) + h_x(-p + 2\mu w_z)) \\
&= \frac{1}{a^2} (-h_x(2\mu u_x) + \mu(u_z + w_x) + -h_x^2\mu(u_z + w_x) + h_x(2\mu w_z)) \\
&= \frac{\mu}{a^2} (2h_x(w_z - u_x) + (1 - h_x^2)(u_z + w_x))
\end{aligned}$$

Next simplify the right hand side

$$\begin{aligned}
\left(\frac{\partial \sigma}{\partial s} + \tau_0 \right) &= \frac{\partial \sigma}{\partial x} \frac{\partial x}{\partial s} + \tau_0 \\
&= \frac{\partial \sigma}{\partial x} \frac{1}{(1 + h_x^2)^{1/2}} + \tau_0
\end{aligned}$$

This gives the following scalar equation.

$$\mu(2h_x(w_z - u_x) + (1 - h_x^2)(u_z + w_x)) = \frac{\partial \sigma}{\partial x} (1 + h_x^2)^{1/2} + \tau_0 (1 + h_x^2) \quad (18)$$

Lastly we will nondimensionalize these two equations.

$$\begin{aligned}
& -p - \pi + \frac{2\mu}{1 + h_x^2} ((h_x^2 u_x + w_z) - h_x(u_z + w_x)) = \frac{h_{xx}}{(1 + h_x^2)^{3/2}} \sigma \\
& -\frac{\mu U_0}{\varepsilon h_0} P - \frac{\mu U_0}{\varepsilon h_0} \Pi + \frac{2\mu}{1 + \varepsilon^2 H_X^2} \left(\left(\varepsilon^2 H_X^2 \frac{\varepsilon U_0}{h_0} U_X + \frac{\varepsilon U_0}{h_0} W_Z \right) - \varepsilon H_X \left(\frac{U_0}{h_0} U_Z + \frac{\varepsilon^2 U_0}{h_0} W_X \right) \right) \\
& = \frac{\varepsilon^2}{h_0} \frac{H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \frac{\mu U_0}{\varepsilon} \Sigma \\
& \left(-\frac{\mu U_0}{\varepsilon h_0} P - \frac{\mu U_0}{\varepsilon h_0} \Pi + \frac{\varepsilon \mu U_0}{h_0} \frac{2}{1 + \varepsilon^2 H_X^2} ((\varepsilon^2 H_X^2 U_X + W_Z) - H_X(U_Z + \varepsilon^2 W_X)) \right) \\
& = \frac{\varepsilon \mu U_0}{h_0} \frac{H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \Sigma \\
& \left(-\frac{\mu U_0}{\varepsilon h_0} P - \frac{\mu U_0}{\varepsilon h_0} \Pi + \frac{\varepsilon \mu U_0}{h_0} \frac{2}{1 + \varepsilon^2 H_X^2} ((\varepsilon^2 H_X^2 U_X + W_Z) - H_X(U_Z + \varepsilon^2 W_X)) \right) \\
& = \frac{\varepsilon \mu U_0}{h_0} \frac{H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \Sigma \\
& \frac{\mu U_0}{\varepsilon h_0} \left(-P - \Pi + \frac{2\varepsilon^2}{1 + \varepsilon^2 H_X^2} ((\varepsilon^2 H_X^2 U_X + W_Z) - H_X(U_Z + \varepsilon^2 W_X)) \right) = \frac{\mu U_0}{\varepsilon h_0} \frac{\varepsilon^2 H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \Sigma \\
& \left(-P - \Pi + \frac{2\varepsilon^2}{1 + \varepsilon^2 H_X^2} ((\varepsilon^2 H_X^2 U_X + W_Z) - H_X(U_Z + \varepsilon^2 W_X)) \right) = \frac{\varepsilon^2 H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \Sigma
\end{aligned}$$

and

$$\begin{aligned}
& \mu(2h_x(w_z - u_x) + (1 - h_x^2)(u_z + w_x)) = \frac{\partial \sigma}{\partial x} (1 + h_x^2)^{1/2} + \tau_0(1 + h_x^2) \\
& \mu \left(2\varepsilon H_X \left(\frac{\varepsilon U_0}{h_0} W_Z - \frac{\varepsilon U_0}{h_0} U_X \right) + (1 - \varepsilon^2 H_X^2) \left(\frac{U_0}{h_0} U_Z + \frac{\varepsilon^2 U_0}{h_0} W_X \right) \right) \\
& = \frac{\mu U_0}{h_0} \Sigma_X (1 + \varepsilon^2 H_X^2)^{1/2} + \frac{\mu U_0}{h_0} \tau (1 + \varepsilon^2 H_X^2) \\
& \frac{\mu U_0}{h_0} (2\varepsilon^2 H_X (W_Z - U_X) + (1 - \varepsilon^2 H_X^2) (U_Z + \varepsilon^2 W_X)) \\
& = \frac{\mu U_0}{h_0} (\Sigma_X (1 + \varepsilon^2 H_X^2)^{1/2} + \tau (1 + \varepsilon^2 H_X^2)) \\
& 2\varepsilon^2 H_X (W_Z - U_X) + (1 - \varepsilon^2 H_X^2) (U_Z + \varepsilon^2 W_X) \\
& = \Sigma_X (1 + \varepsilon^2 H_X^2)^{1/2} + \tau (1 + \varepsilon^2 H_X^2)
\end{aligned}$$

The full nondimensional equations are thus

$$U_X + W_Z = 0 \quad (19)$$

$$\frac{\varepsilon U_0 \rho h_0}{\mu} (U_T + U U_X + W U_Z) = -P_X + (\varepsilon^2 U_{XX} + U_{ZZ}) - \Phi_X \quad (20)$$

$$\varepsilon^3 \frac{\rho U_0 h_0}{\mu} (W_T + U W_X + W W_Z) = -P_Z + \varepsilon^2 (\varepsilon^2 W_{XX} + W_{ZZ}) - \Phi_Z \quad (21)$$

at $Z = 0$

$$W = 0, \quad U = 0 \quad (22)$$

and at $Z = H$

$$W = H_T + U H_x \quad (23)$$

$$-P - \Pi + \frac{2\varepsilon^2}{1 + \varepsilon^2 H_X^2} ((\varepsilon^2 H_X^2 U_X + W_Z) - H_X(U_Z + \varepsilon^2 W_X)) = \frac{\varepsilon^2 H_{XX}}{(1 + \varepsilon^2 H_X^2)^{3/2}} \Sigma \quad (24)$$

$$2\varepsilon^2 H_X (W_Z - U_X) + (1 - \varepsilon^2 H_X^2) (U_Z + \varepsilon^2 W_X) = \Sigma_X (1 + \varepsilon^2 H_X^2)^{1/2} + \tau (1 + \varepsilon^2 H_X^2) \quad (25)$$

We can now let $\varepsilon \rightarrow 0$ which results in

$$U_X + W_Z = 0 \quad (26)$$

$$P_X + \Phi_X = U_{ZZ} \quad (27)$$

$$P_Z + \Phi_Z = 0 \quad (28)$$

at $Z = 0$,

$$W = 0, \quad U = 0 \quad (29)$$

and at $Z = H$

$$W = H_T + UH_x \quad (31)$$

$$-P - \Pi = \bar{\Sigma}H_{XX} \quad (32)$$

$$U_Z = \Sigma_X + \tau \quad (33)$$

Note that we assume that the surface tension is large, so that $\bar{\Sigma} = \varepsilon^2 \Sigma = O(1)$. This is important in order to keep surface tension effects in the final equation.

Next we integrate the continuity equation over Z .

$$\begin{aligned} \int_0^H U_X + W_Z \, dZ &= 0 \\ \int_0^H U_X \, dZ + W|_{Z=0}^H &= 0 \\ \int_0^H U_X \, dZ + H_T + UH_X &= 0 \\ H_T + \int_0^H U_X \, dZ + UH_X &= 0 \\ H_T + \frac{\partial}{\partial X} \left(\int_0^H U \, dZ \right) &= 0 \end{aligned}$$

Using the boundary conditions we can solve for an expression of U as follows

$$\begin{aligned} \int_Z^H U_{ZZ} \, dZ &= \int_Z^H P_X + \Phi_X \, dZ \\ U_Z|_Z^H &= (P_X + \Phi_X)(H - Z) \\ (\tau + \Sigma_X) - U_Z &= (P_X + \Phi_X)(H - Z) \\ \int_0^Z (\tau + \Sigma_X) - U_Z \, dZ &= \int_0^Z (P_X + \Phi_X)(H - Z) \, dZ \\ (\tau + \Sigma_X)Z - U|_{Z=0}^Z &= (P_X + \Phi_X) \left(HZ - \frac{1}{2}Z^2 \right) \\ (\tau + \Sigma_X)Z - U &= (P_X + \Phi_X) \left(HZ - \frac{1}{2}Z^2 \right) \\ U &= (\tau + \Sigma_X)Z + (P_X + \Phi_X) \left(\frac{1}{2}Z^2 - HZ \right) \end{aligned}$$

The boundary conditions also give an expression for $P + \Phi$,

$$\begin{aligned} P_Z + \Phi_Z &= 0 \\ \int_Z^H P_Z + \Phi_Z \, dZ &= 0 \\ P|_{Z=H} - P + \Phi|_{Z=H} - \Phi &= 0 \\ -\Pi - \bar{\Sigma}H_{XX} - P + \Phi|_{Z=H} - \Phi &= 0 \\ P + \Phi &= \Phi|_{Z=H} - \Pi - \bar{\Sigma}H_{XX} \end{aligned}$$

Plugging both of these into the integrated continuity equation gives,

$$\begin{aligned}
H_T + \frac{\partial}{\partial X} \left(\int_0^H U \, dZ \right) &= 0 \\
H_T + \frac{\partial}{\partial X} \left(\int_0^H (\tau + \Sigma_X) Z + (P_X + \Phi_X) \left(\frac{1}{2} Z^2 - HZ \right) dZ \right) &= 0 \\
H_T + \left(\frac{1}{2} (\tau + \Sigma_X) H^2 + (P_X + \Phi_X) \left(\frac{1}{6} H^3 - \frac{1}{2} H^3 \right) \right)_X &= 0 \\
H_T + \left(\frac{1}{2} (\tau + \Sigma_X) H^2 - \frac{1}{3} (P + \Phi)_X H^3 \right)_X &= 0 \\
H_T + \left(\frac{1}{2} (\tau + \Sigma_X) H^2 - \frac{1}{3} (\Phi|_{Z=H} - \Pi - \bar{\Sigma} H_{XX})_X H^3 \right)_X &= 0 \\
H_T + \left(\frac{1}{2} (\tau + \Sigma_X) H^2 - \frac{1}{3} (\Phi|_{Z=H} - \Pi)_X H^3 \right)_X &= -(\bar{\Sigma} H^3 H_{XXX})_X
\end{aligned}$$

This is our final Thin Film equation and taking all of the constants to be one gives

$$H_T + (H^2 - H^3)_X = -(H^3 H_{XXX})_X \quad (34)$$