Discontinuous Galerkin Method for Solving Thin Film and Shallow Water Equations

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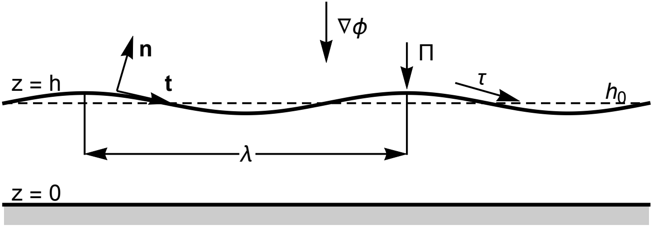
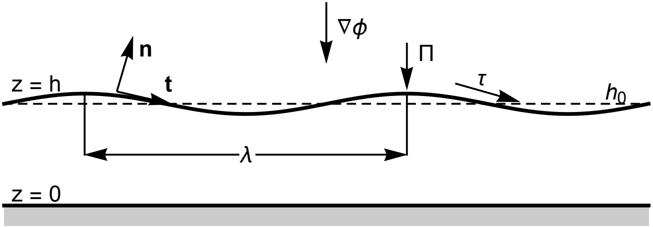
Overview

Model Equations



- Incompressible Navier-Stokes Equation

$$
\begin{aligned}
u_x + w_z &= 0 \\
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
w = 0, u &= 0 && \text{at } z = 0 \\
w &= h_t + uh_x && \text{at } z = h \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } z = h
\end{aligned}
$$



Nondimensionalize, integrate over Z, and simplify, gives

$$H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}\left(\Phi|_{Z=H} - \Pi\right)_X H^3\right)_X = -\frac{1}{3}\bar{C}^{-1}\left(H^3 H_{XXX}\right)_X$$

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

Method Overview

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Runge Kutta Implicit Explicit (IMEX)

$$q_t = F(q) + G(q)$$

  - F evaluated explicitly
  - G solved implicitly

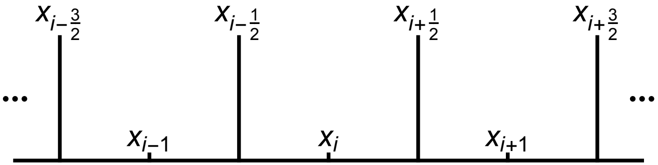
$$F(q) = -\left(q^2 - q^3\right)_x$$
$$G(q) = \left(q^3 q_{xxx}\right)_x$$

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$
- $\Delta x_j = x_{j+1/2} - x_{j-1/2}$
- $\Delta x_j = \Delta x$ for all j.



Convection

- Convection Equation

$$F(q) = f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (F(q)v - f(q)v_x) \, \mathrm{d}x + \hat{f} v \Big|_{\partial\Omega} = 0$$

  for all test functions v

Runge Kutta Discontinuous Galerkin

- Find $Q(t,x)$ such that for each time $t \in (0, T)$, $Q(t,\cdot) \in V_h = \left\{ v \in L^1(\Omega) : v\big|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} F(Q) v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

  for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

Diffusion

- Diffusion Equation

$$G(q) = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Local Discontinuous Galerkin

$$\begin{aligned} r &= q_x \\ s &= r_x \\ u &= s_x \\ G(q) &= \left(q^3 u\right)_x \end{aligned}$$

Local Discontinuous Galerkin

Find $Q(t, x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t, \cdot), R, S, U \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Rv \, \mathrm{d}x = -\int_{I_j} Qv_x \, \mathrm{d}x + \left(\hat{Q}_{j+1/2} v_{j+1/2}^{-} - \hat{Q}_{j-1/2} v_{j-1/2}^{+} \right)$$

$$\int_{I_j} Sw \, \mathrm{d}x = -\int_{I_j} Rw_x \, \mathrm{d}x + \left(\hat{R}_{j+1/2} w_{j+1/2}^{-} - \hat{R}_{j-1/2} w_{j-1/2}^{+} \right)$$

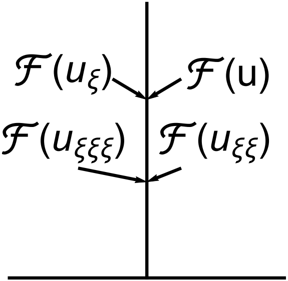
$$\int_{I_j} Uy \, \mathrm{d}x = -\int_{I_j} Sy_x \, \mathrm{d}x + \left(\hat{S}_{j+1/2} y_{j+1/2}^{-} - \hat{S}_{j-1/2} y_{j-1/2}^{+} \right)$$

$$\int_{I_j} G(Q)z \, \mathrm{d}x = -\int_{I_j} Q^3 U z_x \, \mathrm{d}x + \left(\hat{U}_{j+1/2} z_{j+1/2}^{-} - \hat{U}_{j-1/2} z_{j-1/2}^{+} \right)$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$
\begin{aligned}
\hat{Q}_{j+1/2} &= Q^{+}_{j+1/2} \\
\hat{R}_{j+1/2} &= R^{-}_{j+1/2} \\
\hat{S}_{j+1/2} &= S^{+}_{j+1/2} \\
\hat{U}_{j+1/2} &= \left(Q^3 U\right)^{-}_{j+1/2}
\end{aligned}
$$



IMEX Runge Kutta

- IMEX scheme

$$
\begin{aligned}
q^{n+1} &= q^n + \Delta t \sum_{i=1}^{s} \left(b_i' F(t_i, u_i)\right) + \Delta t \sum_{i=1}^{s} \left(b_i G(t_i, u_i)\right) \\
u_i &= q^n + \Delta t \sum_{j=1}^{i-1} \left(a_{ij}' F(t_j, u_j)\right) + \Delta t \sum_{j=1}^{i} \left(a_{ij} G(t_j, u_j)\right) \\
t_i &= t^n + c_i \Delta t
\end{aligned}
$$

- Double Butcher Tableaus

$$
\begin{array}{c|c}
c' & a' \\
\hline
 & b'^T
\end{array}
\quad
\begin{array}{c|c}
c & a \\
\hline
 & b^T
\end{array}
$$

- 1st Order — L-Stable SSP

$$
\begin{array}{c|c}
0 & 0 \\
\hline
 & 1
\end{array}
\qquad
\begin{array}{c|c}
1 & 1 \\
\hline
 & 1
\end{array}
$$

- 2nd Order — SSP

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
\hline
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
\qquad
\begin{array}{c|ccc}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} \\
\hline
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

- 3rd Order — L-Stable SSP

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|cccc}
\alpha & \alpha & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 \\
\frac{1}{2} & \beta & \eta & \zeta & \alpha \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

$$
\begin{aligned}
\alpha &= 0.24169426078821 \\
\beta &= 0.06042356519705 \\
\eta &= 0.1291528696059 \\
\zeta &= \frac{1}{2} - \beta - \eta - \alpha
\end{aligned}
$$

Nonlinear Solvers

- Nonlinear System

$$u_i - a_{ii}\Delta t G(u_i) = b$$

- Picard Iteration

$$\tilde{G}(q, u) = \left(q^3 u_{xxx}\right)_x$$

$$u_0 = q^n \qquad u_i^0 = u_{i-1}$$
$$u_i^j - a_{ii}\Delta t \tilde{G}(u_i^{j-1}, u_i^j) = b$$

Manufactured Solution

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x + s$$
$$s = \hat{q}_t + \left(\hat{q}^2 - \hat{q}^3\right)_x + \left(\hat{q}^3 \hat{q}_{xxx}\right)_x$$
$$\hat{q} = 0.1 \times \sin(2\pi/20.0 \times (x - t)) + 0.15 \quad \text{for } (x, t) \in [0, 40] \times [0, 5.0]$$

	1st Order		2nd Order		3rd Order	
n	error	order	error	order	error	order
20	0.136	—	7.33×10^{-3}	—	5.29×10^{-4}	—
40	0.0719	0.92	1.99×10^{-3}	1.88	5.38×10^{-5}	3.30
80	0.0378	0.93	5.60×10^{-4}	1.83	7.47×10^{-6}	2.85
160	0.0191	0.99	1.56×10^{-4}	1.85	9.97×10^{-7}	2.91
320	0.00961	0.99	3.98×10^{-5}	1.97	1.26×10^{-7}	2.98
640	0.00483	0.99	1.00×10^{-5}	1.99	1.58×10^{-8}	3.00
1280	0.00242	1.00	2.50×10^{-6}	2.00	1.98×10^{-9}	3.00

Table: Convergence table with a constant, linear, quadratic polynomial bases. CFL = 0.9, 0.2, 0.1 respectively.

Wave Structure with Nonlinear Hyper Diffusion

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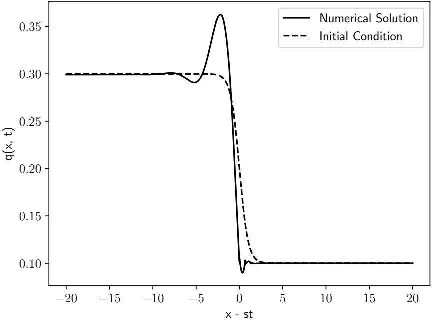
$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

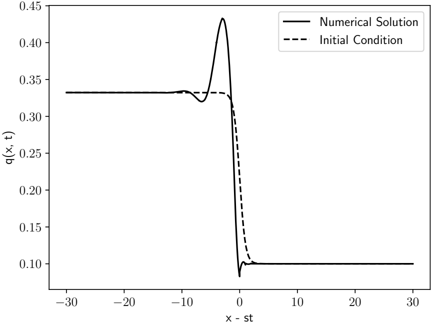
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

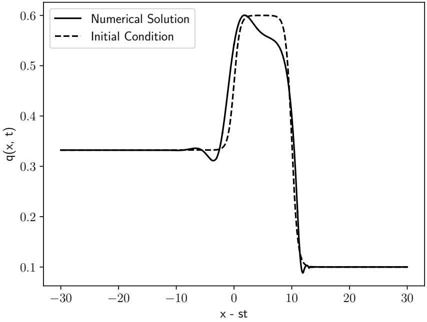
$$q(x, 0) = (-\tanh(x) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

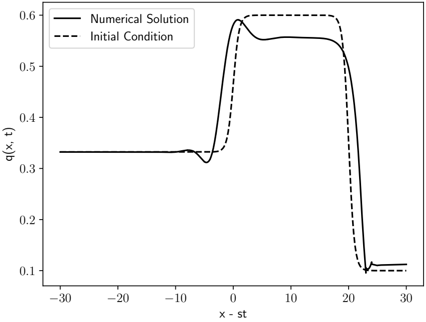
$$q(x,0) = \begin{cases} \frac{q_m - q_l}{2}\tanh(x) + \frac{q_m + q_l}{2} & x < 5 \\ -\frac{q_m - q_r}{2}\tanh(x - 10) + \frac{q_m + q_r}{2} + q_r & x > 5 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

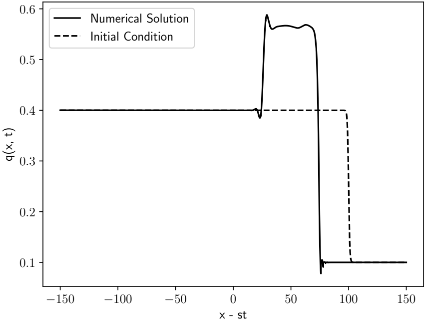
$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2}\tanh(x) + \frac{q_m + q_l}{2} & x < 10 \\ -\frac{q_m - q_r}{2}\tanh(x - 20) + \frac{q_m + q_r}{2} + q_r & x > 10 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.4$$

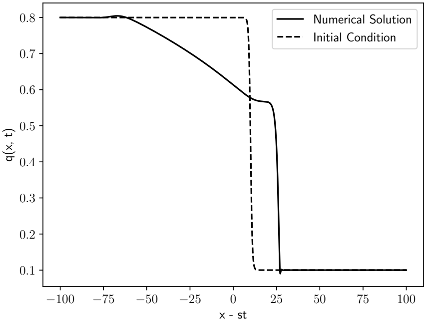
$$q(x, 0) = (-\tanh(x - 100) + 1)\frac{q_l - q_r}{2} + q_r$$



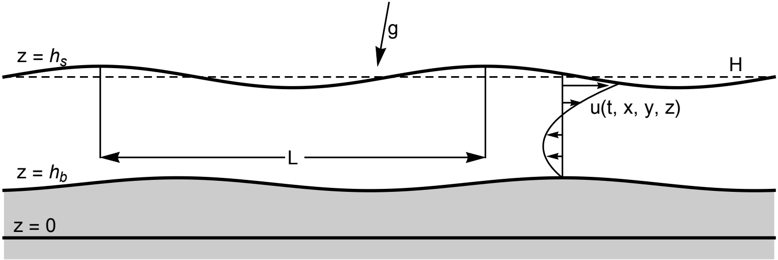
Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x,0) = (-\tanh(x - 10) + 1)\frac{q_l - q_r}{2} + q_r$$



Generalized Shallow Water



$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s)$$
$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b)$$

Nondimensionalization

Characteristic Lengths

$$\varepsilon = \frac{H}{L}, \quad x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}$$

Characteristic Velocities

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}$$

Characteristic Time

$$t = \frac{L}{U}\hat{t}$$

Characteristic Stresses

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}$$

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Nondimensionalization

Nondimensional Equations

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$\varepsilon F^2\left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + \left(\hat{u}\hat{v}\right)_{\hat{y}} + \left(\hat{u}\hat{w}\right)_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{x}}$$

$$+G\left(\varepsilon^2(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\right) + e_x$$

$$\varepsilon F^2\left(\hat{v}_{\hat{t}} + \left(\hat{u}\hat{v}\right)_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + \left(\hat{v}\hat{w}\right)_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{y}}$$

$$+G\left(\varepsilon^2(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\right) + e_y$$

$$\varepsilon^2 F^2\left(\hat{w}_{\hat{t}} + \left(\hat{u}\hat{w}\right)_{\hat{x}} + \left(\hat{v}\hat{w}\right)_{\hat{x}} + \left(\hat{w}^2\right)_{\hat{z}}\right) = -\hat{p}_{\hat{z}}$$

$$+\varepsilon G\left((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}}\right) + e_z$$

$$F = \frac{U}{\sqrt{gH}} \approx 1, \quad G = \frac{S}{\rho g H} < 1$$

Nondimensionalization

Drop ε^2 and εG terms

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0$$

$$\varepsilon F^2\left(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + \left(\hat{u}\hat{v}\right)_{\hat{y}} + \left(\hat{u}\hat{w}\right)_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{x}} + G\left(\hat{\sigma}_{xz}\right)_{\hat{z}} + e_x$$

$$\varepsilon F^2\left(\hat{v}_{\hat{t}} + \left(\hat{u}\hat{v}\right)_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + \left(\hat{v}\hat{w}\right)_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{y}} + G\left(\hat{\sigma}_{yz}\right)_{\hat{z}} + e_y$$

$$\hat{p}_{\hat{z}} = e_z$$

Solving for the hydrostatic pressure

$$\hat{p}(\hat{t}, \hat{x}, \hat{y}) = \left(\hat{h}_s(\hat{t}, \hat{x}, \hat{y}) - \hat{z}\right)e_z$$

Nondimensionalization

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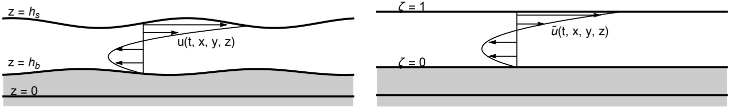
Dimensional Variables

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + \left(u^2\right)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho} p_x + \frac{1}{\rho}(\sigma_{xz})_z + g e_x \\
v_t + (uv)_x + \left(v^2\right)_y + (vw)_z &= -\frac{1}{\rho} p_y + \frac{1}{\rho}(\sigma_{yz})_z + g e_y \\
p(t, x, y, z) &= (h_s(t, x, y) - z)\rho g e_z
\end{aligned}
$$

Kinematic Boundary Conditions

$$
\begin{aligned}
(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s &= w(t, x, y, h_s) \\
(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b &= w(t, x, y, h_b)
\end{aligned}
$$

Mapping



Transform from $z \to \zeta$, by

$$\zeta = \frac{z - h_b(t,x,y)}{h(t,x,y)},$$

or equivalently

$$z = h(t,x,y)\zeta + h_b(t,x,y)$$

where $h(t,x,y) = h_s(t,x,y) - h_b(t,x,y)$.

$$\tilde{\Psi}(t,x,y,\zeta) = \Psi(t,x,y,h(t,x,y)\zeta + h_b(t,x,y))$$

Mapping Continuity Equation

$$u_x + v_y + w_z = 0$$

Map to new space

$$(h\tilde{u})_x - ((\zeta h + h_b)_x \tilde{u})_\zeta + (h\tilde{v})_y - \left((\zeta h + h_b)_y \tilde{v}\right)_\zeta + (\tilde{w})_\zeta = 0$$

Solve for vertical velocity, w,

$$\tilde{w}(t, x, y, \zeta) = -\left(h \int_0^\zeta \tilde{u} \, \mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v} \, \mathrm{d}\zeta'\right)_y$$
$$+(\zeta h + h_b)_x \tilde{u}(t, x, y, \zeta) + (\zeta h + h_b)_y \tilde{v}(t, x, y, \zeta)$$

Depth averaged equation

$$h_t + \left(h \int_0^1 \tilde{u} \, \mathrm{d}\zeta\right)_x + \left(h \int_0^1 \tilde{v} \, \mathrm{d}\zeta\right)_y = 0$$

Let u_m and v_m denote the mean velocity

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

Mapping Momentum Equations

$$u_t + \left(u^2\right)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x$$
$$v_t + (uv)_x + \left(v^2\right)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y$$

$$(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}ge_z h^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz}\right)_\zeta = gh\left(e_x - e_z(h_b)_x\right)$$
$$(h\tilde{v})_t + (h\tilde{u}\tilde{v})_x + \left(h\tilde{v}^2 + \frac{1}{2}ge_z h^2\right)_y + \left(h\tilde{v}\omega - \frac{1}{\rho}\tilde{\sigma}_{yz}\right)_\zeta = gh\left(e_x - e_z(h_b)_y\right)$$

where

$$\omega = \frac{1}{h}\left(-\left(h\int_0^\zeta \tilde{u} - u_m \,\mathrm{d}\zeta'\right)_x - \left(h\int_0^\zeta \tilde{v} - v_m \,\mathrm{d}\zeta'\right)_y\right)$$

Mapped Reference System

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2} g e_z h^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz}\right)_\zeta = gh\big(e_x - e_z (h_b)_x\big)$$

$$(h\tilde{v})_t + (h\tilde{u}\tilde{v})_x + \left(h\tilde{v}^2 + \frac{1}{2} g e_z h^2\right)_y + \left(h\tilde{v}\omega - \frac{1}{\rho}\tilde{\sigma}_{yz}\right)_\zeta = gh\big(e_x - e_z (h_b)_y\big)$$

$$\omega = \frac{1}{h}\left(-\left(h\int_0^\zeta \tilde{u}_d \,\mathrm{d}\zeta'\right)_x - \left(h\int_0^\zeta \tilde{v}_d \,\mathrm{d}\zeta'\right)_y\right)$$

with

$$\tilde{u}_d = \tilde{u} - u_m \quad \tilde{v}_d = \tilde{v} - v_m$$

Newtonian Flow

Newtonian Stree Tensor

$$\sigma_{xz} = \mu u_z \quad \sigma_{yz} = \mu v_z$$

Kinematic Viscosity

$$\nu = \frac{\mu}{\rho}$$

Mapped stress tensor

$$\frac{1}{\rho}\tilde{\sigma}_{xz} = \frac{\nu}{h}\tilde{u}_\zeta \quad \frac{1}{\rho}\tilde{\sigma}_{yz} = \frac{\nu}{h}\tilde{v}_\zeta$$

Boundary Conditions

Stree Free Condition at surface

$$u_z\big|_{z=h_s} = v_z\big|_{z=h_s} = 0$$

Mixed Slip Condition at bottom topography

$$u - \frac{\lambda}{\mu}\sigma_{xz}\bigg|_{z=h_b} = v - \frac{\lambda}{\mu}\sigma_{yz}\bigg|_{z=h_b} = 0$$

Mapped with Newtonian Stress

$$\tilde{u}_\zeta\big|_{\zeta=1} = \tilde{v}_\zeta\big|_{\zeta=1} = 0$$

and

$$\tilde{u} - \frac{\lambda}{h}\tilde{u}_\zeta\bigg|_{\zeta=0} = \tilde{v} - \frac{\lambda}{h}\tilde{v}_\zeta\bigg|_{\zeta=0} = 0$$

Mapped Reference System

$$
\begin{aligned}
h_t + (hu_m)_x + (hv_m)_y &= 0 \\
(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2} g e_z h^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{\nu}{h}\tilde{u}_\zeta\right)_\zeta &= gh\left(e_x - e_z (h_b)_x\right) \\
(h\tilde{v})_t + (h\tilde{u}\tilde{v})_x + \left(h\tilde{v}^2 + \frac{1}{2} g e_z h^2\right)_y + \left(h\tilde{v}\omega - \frac{\nu}{h}\tilde{v}_\zeta\right)_\zeta &= gh\left(e_x - e_z (h_b)_y\right)
\end{aligned}
$$

Boundary Conditions

$$
\tilde{u}_\zeta\big|_{\zeta=1} = \tilde{v}_\zeta\big|_{\zeta=1} = 0
$$

and

$$
\tilde{u} - \frac{\lambda}{h}\tilde{u}_\zeta\bigg|_{\zeta=0} = \tilde{v} - \frac{\lambda}{h}\tilde{v}_\zeta\bigg|_{\zeta=0} = 0
$$

Moment Closure

Depth Averaged Momentum Equations

$$
\begin{gathered}
(hu_m)_t + \left(h\int_0^1 \tilde{u}^2 \,\mathrm{d}\zeta + \frac{1}{2} g e_z h^2\right)_x + \left(h\int_0^1 \tilde{u}\tilde{v}\,\mathrm{d}\zeta\right)_y \\
+\frac{\nu}{\lambda}\left. u\right|_{\zeta=0} = hg(e_x - e_z(h_b)_x) \\
(hv_m)_t + \left(h\int_0^1 \tilde{u}\tilde{v}\,\mathrm{d}\zeta\right)_y + \left(h\int_0^1 \tilde{v}^2\,\mathrm{d}\zeta + \frac{1}{2} g e_z h^2\right)_y \\
+\frac{\nu}{\lambda}\left. v\right|_{\zeta=0} = hg\left(e_x - e_z(h_b)_y\right)
\end{gathered}
$$

Polynomial Ansatz

$$
\begin{aligned}
\tilde{u}(t, x, y, \zeta) &= u_m(t, x, y) + u_d(t, x, y, \zeta) \\
&= u_m(t, x, y) + \sum_{j=1}^{N} \left(\alpha_j(t, x, y)\phi_j(\zeta)\right) \\
\tilde{v}(t, x, y, \zeta) &= v_m(t, x, y) + v_d(t, x, y, \zeta) \\
&= v_m(t, x, y) + \sum_{j=1}^{N} \left(\beta_j(t, x, y)\phi_j(\zeta)\right)
\end{aligned}
$$

Orthogonality Condition

$$
\int_0^1 \phi_j(\zeta)\phi_i(\zeta)\,\mathrm{d}\zeta = 0 \quad \text{for } j \neq i
$$

$$
\phi_0(\zeta) = 1, \quad \phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2
$$

Constant Moments

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y = -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg\left(e_x - e_z (h_b)_x\right)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x = -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\left(e_y - e_z (h_b)_y\right)$$

Higher Order Moments

Moment Equation

$$\int_0^1 \phi_i \left((h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2} g e_z h^2 \right)_x + (h\tilde{u}\tilde{v})_y + \left(h\tilde{u}\omega - \frac{1}{\rho}\tilde{\sigma}_{xz} \right)_\zeta \right) \mathrm{d}\zeta$$

$$= \int_0^1 \phi_i (gh(e_x - e_z (h_b)_x)) \, \mathrm{d}\zeta$$

Simplified gives

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k \right)_x$$

$$+ \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k \right)_y$$

$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \left(1 + \frac{\lambda}{h} C_{ij} \right) \alpha_j \right)$$

Higher Order Moments

$$
\begin{aligned}
&(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y \\
&= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right)
\end{aligned}
$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \,\mathrm{d}\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right)\phi_k \,\mathrm{d}\zeta$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \,\mathrm{d}\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

Example Systems

1D model with h_b constant, $e_x = e_y = 0$, and $e_z = 1$
Constant System

$$\begin{bmatrix} h \\ hu_m \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = -\frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m \end{bmatrix}$$

Linear System, $\tilde{u} = u_m + s\phi_1$, $s = \alpha_1$

$$\begin{bmatrix} h \\ hu_m \\ hs \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 \\ 2hu_m s \end{bmatrix}_x = Q\begin{bmatrix} h \\ hu_m \\ hs \end{bmatrix}_x - P$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{bmatrix} \quad P = \frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m + s \\ 3(u_m + s + 4\frac{\lambda}{h}s) \end{bmatrix}$$

Numerical Methods

Model Equation

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = Q(\mathbf{q})\mathbf{q}_x - \mathbf{P}(\mathbf{q}) \quad \text{for } (x, t) \in [a, b] \times [0, T]$$

Weak Form, find $\mathbf{q}$ such that

$$\int_a^b \mathbf{q}_t v \, \mathrm{d}x + \int_a^b \mathbf{f}(\mathbf{q})_x v \, \mathrm{d}x = \int_a^b Q(\mathbf{q})\mathbf{q}_x v \, \mathrm{d}x - \int_a^b \mathbf{P}(\mathbf{q}) v \, \mathrm{d}x$$

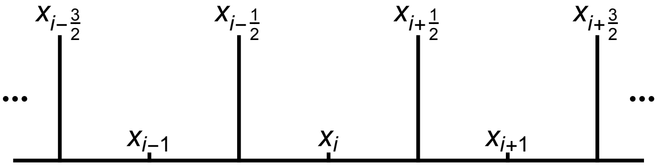
for all $v \in L^2([a, b] \times [0, T])$

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$
- $\Delta x_j = x_{j+1/2} - x_{j-1/2}$
- $\Delta x_j = \Delta x$ for all j.



Discontinuous Galerkin Space

Finite Dimensional DG Space

$$V^k = \left\{ v \in L^2([a,b]) \middle| v|_{I_j} \in P^k(I_j) \right\}$$

Basis for V^k

$$\left\{\phi_j^\ell\right\} \text{ where } \phi_j^\ell(x)\Big|_{I_j} = \phi^\ell(\xi_j(x)) \text{ and } \phi_j^\ell(x)\Big|_{\bar{I_j}} = 0$$

for $j = 1, \dots, N$ and $\ell = 1, \dots k$.
Legendre Polynomials

$$\phi^k \in P^k([-1,1]) \text{ with } \frac{1}{2}\int_{-1}^{1} \phi^k(\xi)\phi^\ell(\xi) \, \mathrm{d}\xi = \delta_{k\ell}$$

and

$$\xi_j(x) = \frac{2}{\Delta x_j}(x - x_j)$$

Numerical Methods

Find $\mathbf{q}_h \in V^k$ such that

$$\begin{aligned}\int_{I_j} (\mathbf{q}_h)_t \phi_j^\ell(x) \,\mathrm{d}x = \int_{I_j} &\mathbf{f}(\mathbf{q}_h)_x \phi_j^\ell \,\mathrm{d}x \\ &- F_{j+1/2}\phi_j^\ell(x_{j+1/2}) + F_{j-1/2}\phi_j^\ell(x_{j-1/2}) \\ &+ \int_{I_j} Q(\mathbf{q}_h)(\mathbf{q}_h)_x \phi_j^\ell \,\mathrm{d}x - \int_{I_j} \mathbf{P}(\mathbf{q}_h)\phi_j^\ell \,\mathrm{d}x\end{aligned}$$

for all ϕ_j^l.

Local Lax-Friedrichs Flux

$$\mathbf{q}_h^+ = \lim_{x \to x_{j+1/2}^+} (\mathbf{q}_h(x))$$

$$\mathbf{q}_h^- = \lim_{x \to x_{j+1/2}^-} (\mathbf{q}_h(x))$$

$$\lambda = \max_{q \in [\mathbf{q}_h^-, \mathbf{q}_h^+]} \{\rho(\mathbf{f}'(\mathbf{q}) - Q(\mathbf{q}))\}$$

$$F_{j+1/2} = \frac{1}{2}\left(\mathbf{f}\left(\mathbf{q}_h^+\right) + \mathbf{f}\left(\mathbf{q}_h^-\right)\right) - \frac{1}{2}\lambda\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right)$$

Nonconservative Flux

Need to evaluate

$$\int^{I_j} Q\mathbf{q}_x \phi_j^\ell \,\mathrm{d}x$$

$$\mathbf{q}|_{I_j} = \sum_{\ell=1}^{k} \left(Q_j^\ell \phi_j^\ell(x)\right), \quad \mathbf{q}_x|_{I_j} = \sum_{\ell=1}^{k} \left(Q_x^\ell \phi_j^\ell(x)\right)$$

where

$$\begin{bmatrix} Q_x^1 \\ Q_x^2 \\ Q_x^3 \\ Q_x^4 \\ Q_x^5 \end{bmatrix} = \frac{1}{2\Delta x} \begin{bmatrix} \Delta Q^1 - 2\sqrt{5}\Delta Q^3 + 78\Delta Q^5 \\ \Delta Q^2 - \frac{10}{3}\sqrt{3}\sqrt{7}\Delta Q^4 \\ \Delta Q^3 - 14\sqrt{5}\Delta Q^5 \\ \Delta Q^4 \\ \Delta Q^5 \end{bmatrix}$$

$$\Delta Q^\ell = Q_{i+1}^\ell - Q_{i-1}^\ell$$

Inviscid Example

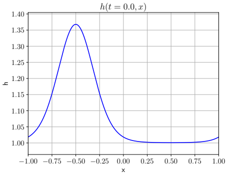
$$x \in [-1, 1] \quad t \in [0, 2.0]$$

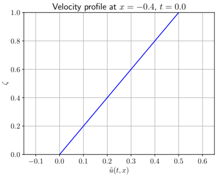
$$h(t = 0, x) = 1 + e^{3\cos(\pi(x+0.5))-4}$$

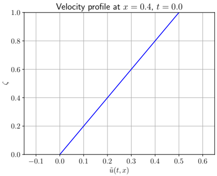
$$\tilde{u}(t = 0, x) = \begin{cases} 0.25 & \text{constant} \\ 0.5\zeta & \text{linear} \end{cases}$$

$$u_m = 0.25$$

$$s = -0.25$$

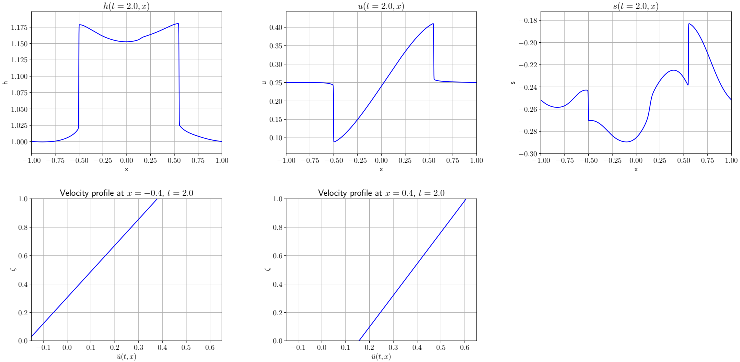






Caleb Logemann,
James
Rossmanith

Inviscid Example



Higher Moment Equations

1 dimensional with h_b constant, $e_x = e_y = 0$, and $e_z = 1$ Quadratic Vertical Profile, $\tilde{u} = u_m + s\phi_1 + \kappa\phi_2$

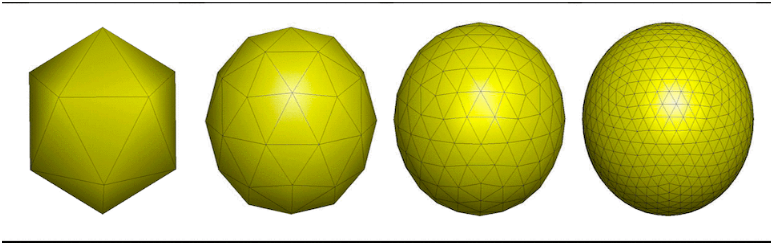
$$\begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 + \frac{1}{5}h\kappa^2 \\ 2hus + \frac{4}{5}hs\kappa \\ 2hu\kappa + \frac{2}{3}hs^2 + \frac{2}{7}h\kappa^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_x - P$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\kappa}{5} & \frac{s}{5} \\ 0 & 0 & s & u_m + \frac{\kappa}{7} \end{bmatrix} \quad P = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u_m + s + \kappa \\ 3\left(u_m + s + \kappa + 4\frac{\lambda}{h}s\right) \\ 5\left(u_m + s + \kappa + 12\frac{\lambda}{h}\kappa\right) \end{bmatrix}$$

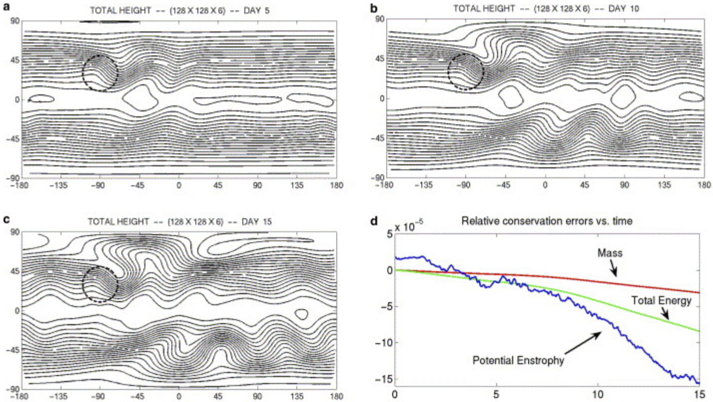
Future Work

- Higher Order Numerical Methods
- Slope Limiters
- Two Dimensional Meshes
- Icosahedral Spherical Mesh
- Positivity Preserving Limiters

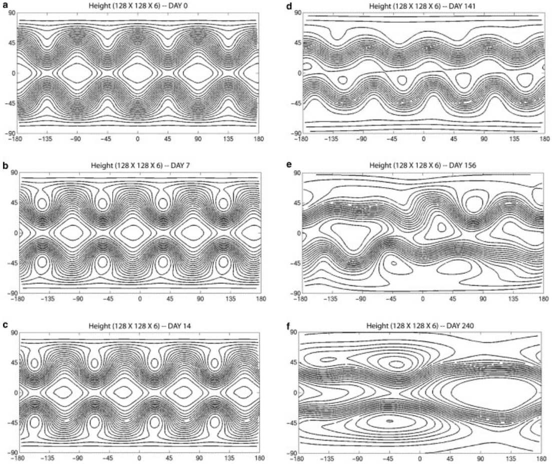
Icosahedral Mesh



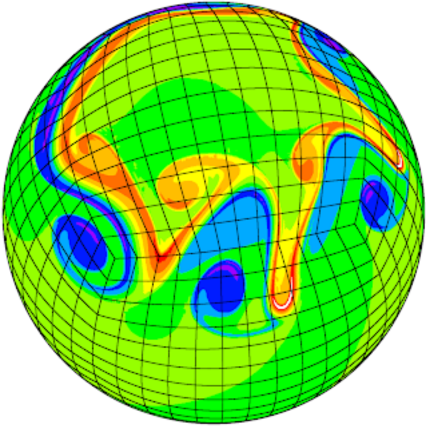
Spherical Test Cases



Spherical Test Cases



Spherical Test Cases



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