

Discontinuous Galerkin Methods for Shallow Fluid Models

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Overview

1 Discontinuous Galerkin Methods

2 Thin Film Model

3 Shallow Water Moment Model

Hyperbolic Conservation Laws

$$\mathbf{q}_t(\mathbf{x}, t) + \nabla \cdot \mathbf{f}(\mathbf{q}, \mathbf{x}, t) = \mathbf{0} \quad \text{for } (\mathbf{x}, t) \in \Omega \times [0, T]$$

Weak Solutions Find $\mathbf{q} \in L^2(\Omega \times [0, T])$, such that

$$\int_0^T \int_{\Omega} \mathbf{q}(\mathbf{x}, t) v_t(\mathbf{x}, t) + \mathbf{f}(\mathbf{q}, \mathbf{x}, t) \cdot \nabla v(\mathbf{x}, t) d\mathbf{x} dt = \mathbf{0}$$

for all $v \in C_0^\infty(\Omega \times [0, T])$

Discontinuous Galerkin Methods

Mesh

$$\Omega \approx \Omega_h = \cup_{i=1}^N K_i$$

Discontinuous Galerkin Space

$$V_h^k = \{v \in L^2(\Omega) : v|_{K_i} \in P^k(K_i)\}$$

or approximate solution of the form

$$\mathbf{q}_h|_{K_i} = \sum_{j=1}^{N_p} \left(\mathbf{q}_i^j(t) \phi_i^j(\mathbf{x}) \right)$$

Discontinuous Galerkin Methods

Semi-Discrete Weak Formulation

$$\int_{K_i} \mathbf{q}_{h,t} \phi_i^j - \mathbf{f}(\mathbf{q}_h) \cdot \nabla \phi_i^j \, d\mathbf{x} = - \int_{\partial K_i} \mathbf{n} \cdot \mathbf{f}^* \phi_i^j \, d\mathbf{x}$$

for all elements K_i and all basis functions ϕ_i^j .

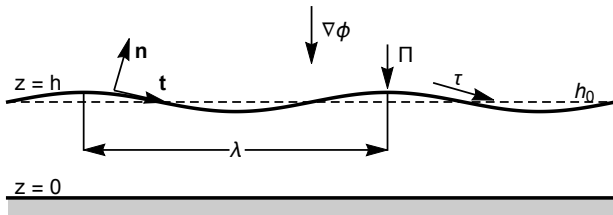
Numerical Flux or Approximate Riemann Solver

For example, Local Lax-Friedrichs Flux

$$\mathbf{f}^* = \frac{1}{2} (\mathbf{f}(\mathbf{q}_h^L) + \mathbf{f}(\mathbf{q}_h^R)) + \lambda (\mathbf{q}_h^L - \mathbf{q}_h^R) \mathbf{n}$$

where λ is the maximum wavespeed at the interface.

Model Equations



■ Incompressible Navier-Stokes Equation

$$u_x + w_z = 0$$

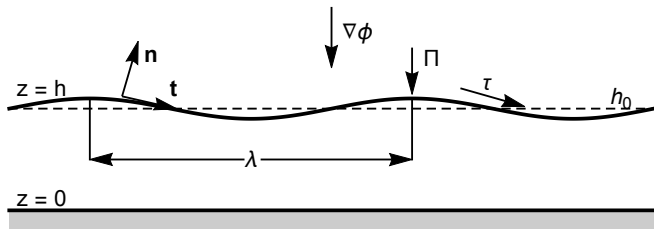
$$\rho(u_t + uu_x + ww_z) = -p_x + \mu\Delta u - \phi_x$$

$$\rho(w_t + uw_x + ww_z) = -p_z + \mu\Delta w - \phi_z$$

$$w = 0, u = 0 \quad \text{at } z = 0$$

$$w = h_t + uh_x \quad \text{at } z = h$$

$$\mathbf{T} \cdot \mathbf{n} = (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau\right)\mathbf{t} \quad \text{at } z = h$$



Nondimensionalize, integrate over Z , and simplify, gives

$$H_T + \left(\frac{1}{2}(\tau + \Sigma_x)H^2 - \frac{1}{3}(\Phi|_{z=H} - \Pi)_x H^3 \right)_x = -\frac{1}{3}\bar{C}^{-1}(H^3 H_{xxx})_x$$

$$q_t + (q^2 - q^3)_x = -(q^3 q_{xxx})_x$$

Method Overview

- Simplified Model

$$q_t + (q^2 - q^3)_x = -(q^3 q_{xxx})_x \quad (0, T) \times \Omega$$

- Runge Kutta Implicit Explicit (IMEX)

$$q_t = F(q) + G(q)$$

- F evaluated explicitly
- G solved implicitly

$$F(q) = -(q^2 - q^3)_x$$

$$G(q) = (q^3 q_{xxx})_x$$

Diffusion

■ Diffusion Equation

$$G(q) = -(q^3 q_{xxx})_x \quad (0, T) \times \Omega$$

■ Local Discontinuous Galerkin

$$r = q_x$$

$$s = r_x$$

$$u = s_x$$

$$G(q) = -(q^3 u)_x$$

Local Discontinuous Galerkin (Cockburn and Shu [2])

Find $q_h(t, x), r_h(x), s_h(x), u_h(x) \in V_h$ such that for all $t \in (0, T)$

$$\begin{aligned} \int_{K_i} r_h \phi^j + q_h \phi_x^j \, d\mathbf{x} &= \int_{\partial K_i} n \cdot \mathbf{q}^* \phi^j \, d\mathbf{x} \\ \int_{K_i} s_h \phi^j + s_h \phi_x^j \, d\mathbf{x} &= \int_{\partial K_i} n \cdot \mathbf{r}^* \phi^j \, d\mathbf{x} \\ \int_{K_i} u_h \phi^j + s_h \phi_x^j \, d\mathbf{x} &= \int_{\partial K_i} n \cdot \mathbf{s}^* \phi^j \, d\mathbf{x} \\ \int_{K_i} G(q_h) \phi^j - q_h^3 u_h \phi_x^j \, d\mathbf{x} &= - \int_{\partial K_i} n \cdot \mathbf{u}^* \phi^j \, d\mathbf{x} \end{aligned}$$

for all K_i and all $\phi^j \in V_h$.

Numerical Fluxes

Alternating Fluxes

$$q^* = \sum_{j=1}^{N_p} \left(q_h^j \phi^j(\mathbf{x}^L) \right)$$

$$r^* = \sum_{j=1}^{N_p} \left(r_h^j \phi^j(\mathbf{x}^R) \right)$$

$$s^* = \sum_{j=1}^{N_p} \left(s_h^j \phi^j(\mathbf{x}^L) \right)$$

$$u^* = \left(\sum_{j=1}^{N_p} \left(q_h^j \phi^j(\mathbf{x}^R) \right) \right)^3 \sum_{j=1}^{N_p} \left(u_h^j \phi^j(\mathbf{x}^R) \right)$$

Implicit Explicit Runge Kutta

■ IMEX scheme

$$\begin{aligned}
 q^{n+1} &= q^n + \Delta t \sum_{i=1}^s (b'_i F(t_i, u_i)) + \Delta t \sum_{i=1}^s (b_i G(t_i, u_i)) \\
 u_i &= q^n + \Delta t \sum_{j=1}^{i-1} (a'_{ij} F(t_j, u_j)) + \Delta t \sum_{j=1}^i (a_{ij} G(t_j, u_j)) \\
 t_i &= t^n + c_i \Delta t
 \end{aligned}$$

■ Double Butcher Tableaus

$$\begin{array}{c|c} c' & a' \\ \hline & b'^T \end{array} \quad \begin{array}{c|c} c & a \\ \hline & b^T \end{array}$$

SSP Runge Kutta IMEX Schemes (Pareschi and Russo [14])

■ 1st Order — L-Stable SSP

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \quad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

■ 2nd Order — SSP

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \quad \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

SSP Runge Kutta IMEX Schemes (Pareschi and Russo [13])

■ 3rd Order — L-Stable SSP

0		0	0	0	0
0		0	0	0	0
1		0	1	0	0
$\frac{1}{2}$		0	$\frac{1}{4}$	$\frac{1}{4}$	0
<hr/>					
		0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

α		α	0	0	0
0		$-\alpha$	α	0	0
1		0	$1 - \alpha$	α	0
$\frac{1}{2}$		β	η	ζ	α
<hr/>					
		0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

$$\alpha = 0.24169426078821$$

$$\beta = 0.06042356519705$$

$$\eta = 0.1291528696059$$

$$\zeta = \frac{1}{2} - \beta - \eta - \alpha$$

Nonlinear Solvers

■ Nonlinear System

$$u_i - a_{ii}\Delta t G(u_i) = b$$

■ Picard Iteration

$$\tilde{G}(q, u) = (q^3 u_{xxx})_x$$

$$\begin{aligned} u_0 &= q^n & u_i^0 &= u_{i-1} \\ u_i^j - a_{ii}\Delta t \tilde{G}(u_i^{j-1}, u_i^j) &= b \end{aligned}$$

Manufactured Solution

$$q_t + (q^2 - q^3)_x = -(q^3 q_{xxx})_x + s$$

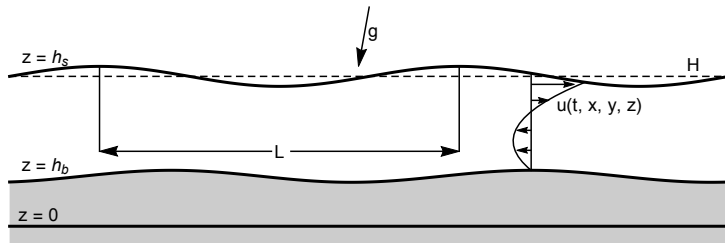
$$s = \hat{q}_t + (\hat{q}^2 - \hat{q}^3)_x + (\hat{q}^3 \hat{q}_{xxx})_x$$

$$\hat{q} = 0.1 \times \sin(2\pi/20.0 \times (x - t)) + 0.15 \quad \text{for } (x, t) \in [0, 40] \times [0, 5.0]$$

1st Order			2nd Order		3rd Order	
n	error	order	error	order	error	order
20	0.136	—	7.33×10^{-3}	—	5.29×10^{-4}	—
40	0.0719	0.92	1.99×10^{-3}	1.88	5.38×10^{-5}	3.30
80	0.0378	0.93	5.60×10^{-4}	1.83	7.47×10^{-6}	2.85
160	0.0191	0.99	1.56×10^{-4}	1.85	9.97×10^{-7}	2.91
320	0.00961	0.99	3.98×10^{-5}	1.97	1.26×10^{-7}	2.98
640	0.00483	0.99	1.00×10^{-5}	1.99	1.58×10^{-8}	3.00
1280	0.00242	1.00	2.50×10^{-6}	2.00	1.98×10^{-9}	3.00

Table: Convergence table with a constant, linear, quadratic polynomial bases.
CFL = 0.9, 0.2, 0.1 respectively.

Generalized Shallow Water, (Kowalski and Torrilhon [6])



Incompressible Navier Stokes Equations with a free surface

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s)$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b)$$

Polynomial Ansatz

$$\begin{aligned}\tilde{u}(t, x, y, \zeta) &= u_m(t, x, y) + u_d(t, x, y, \zeta) \\ &= u_m(t, x, y) + \sum_{j=1}^N (\alpha_j(t, x, y) \phi_j(\zeta)) \\ \tilde{v}(t, x, y, \zeta) &= v_m(t, x, y) + v_d(t, x, y, \zeta) \\ &= v_m(t, x, y) + \sum_{j=1}^N (\beta_j(t, x, y) \phi_j(\zeta))\end{aligned}$$

Orthogonality Condition

$$\int_0^1 \phi_j(\zeta) \phi_i(\zeta) d\zeta = 0 \quad \text{for } j \neq i$$

$$\phi_0(\zeta) = 1, \quad \phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2$$

Constant Moments

Continuity Equation

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

Conservation of Momentum Equations

$$\begin{aligned} & (hu_m)_t + \left(h \left(u_m^2 + \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_x \\ & + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_y = -\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \alpha_j \right) + hg(e_x - e_z(h_b)_x) \\ & (hv_m)_t + \left(h \left(v_m^2 + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_y \\ & + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_x = -\frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \beta_j \right) + hg(e_y - e_z(h_b)_y) \end{aligned}$$

Higher Order Moments

$$\begin{aligned}
 & (h\alpha_i)_t + \left(2hu_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\alpha_k \right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\beta_k \right)_y \\
 & = u_mD_i - \sum_{j,k=1}^N B_{ijk}D_j\alpha_k - (2i+1)\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \alpha_j \right) \\
 & (h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\beta_k \right)_x + \left(2hv_m\beta_i + h \sum_{j,k=1}^N A_{ijk}\beta_j\beta_k \right)_y \\
 & = v_mD_i - \sum_{j,k=1}^N B_{ijk}D_j\beta_k - (2i+1)\frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \beta_j \right)
 \end{aligned}$$

Example Systems

1D model with h_b constant, $e_x = e_y = 0$, and $e_z = 1$

Constant System

$$\begin{bmatrix} h \\ hu_m \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = -\frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u_m \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u_m \pm \sqrt{gh}$

Linear System, $\tilde{u} = u_m + \alpha_1 \phi_1$

$$\begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{bmatrix} \quad \mathbf{s} = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{bmatrix}$$

Quasilinear Matrix Eigenvalues, $u_m \pm \sqrt{gh + \alpha_1^2}$, u_m

Example Systems

1 dimensional with h_b constant, $e_x = e_y = 0$, and $e_z = 1$

Quadratic Vertical Profile, $\tilde{u} = u + \alpha_1 \phi_1 + \alpha_2 \phi_2$

$$\begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u + \frac{\alpha_2}{7} \end{bmatrix}, P = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u + \alpha_1 + \alpha_2 \\ 3(u + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1) \\ 5(u + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2) \end{bmatrix}$$

Quasilinear Matrix Eigenvalues, $u \pm c\sqrt{gh}$

Nonconservative Products, (Dal Maso, Lefloch, and Murat [4])

Model Equation

$$\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + g_i(\mathbf{q})\mathbf{q}_{x_i} = \mathbf{s}(\mathbf{q}) \quad \text{for } (\mathbf{x}, t) \in \Omega \times [0, T]$$

Traditionally searching for weak solutions, find \mathbf{q} such that

$$\int_0^T \int_{\Omega} (\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + g_i(\mathbf{q})\mathbf{q}_{x_i}) v \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega} \mathbf{s}(\mathbf{q}) v \, d\mathbf{x} \, dt$$

for all $v \in C_0^1(\Omega \times [0, T])$

Regularization Paths

Consider Lipschitz continuous paths, $\psi : [0, 1] \times \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^p$, that satisfy the following properties.

- 1 $\forall \mathbf{q}_L, \mathbf{q}_R \in \mathbb{R}^p$, $\psi(0, \mathbf{q}_L, \mathbf{q}_R) = \mathbf{q}_L$ and $\psi(1, \mathbf{q}_L, \mathbf{q}_R) = \mathbf{q}_R$
- 2 $\exists k > 0$, $\forall \mathbf{q}_L, \mathbf{q}_R \in \mathbb{R}^p$, $\forall s \in [0, 1]$, $\left| \frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R) \right| \leq k |\mathbf{q}_L - \mathbf{q}_R|$
elementwise
- 3 $\exists k > 0$, $\forall \mathbf{q}_L, \mathbf{q}_R, \mathbf{u}_L, \mathbf{u}_R \in \mathbb{R}^p$, $\forall s \in [0, 1]$, elementwise

$$\left| \frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R) - \frac{\partial \psi}{\partial s}(s, \mathbf{u}_L, \mathbf{u}_R) \right| \leq k(|\mathbf{q}_L - \mathbf{u}_L| + |\mathbf{q}_R - \mathbf{u}_R|)$$

Let $u = u_0 + H(x - x_0)(u_1 - u_0)$, then regularize

$$u^\varepsilon(x) = \begin{cases} u_0 & x < x_0 - \varepsilon \\ \psi\left(\frac{x - x_0 + \varepsilon}{2\varepsilon}, u_0, u_1\right) & x_0 - \varepsilon < x < x_0 + \varepsilon \\ u_1 & x > x_0 + \varepsilon \end{cases}$$

Nonconservative Product Definition

Let $\mathbf{q} \in BV(\Omega \rightarrow \mathbb{R}^p)$ and $g \in C^1(\mathbb{R}^p \rightarrow \mathbb{R}^p \times \mathbb{R}^p)$, then μ is a Borel measure.

- 1 If \mathbf{q} is continuous on a Borel set $B \subset \Omega$, then

$$\mu(B) = \int_B g(\mathbf{q}) \frac{d\mathbf{q}}{dx} dx$$

- 2 If \mathbf{q} is discontinuous at a point $x_0 \in \Omega$, then

$$\mu(x_0) = \int_0^1 g(\psi(s; \mathbf{q}(x_0^-), \mathbf{q}(x_0^+))) \frac{\partial \psi}{\partial s}(s; \mathbf{q}(x_0^-), \mathbf{q}(x_0^+)) ds$$

Define

$$\mu = \left[g(\mathbf{q}) \frac{d\mathbf{q}}{dx} \right]_{\psi}$$

Nonconservative Products

If there exists $\mathbf{f}(\mathbf{q})$ such that $\mathbf{f}'(\mathbf{q}) = g(\mathbf{q})$, then

$$[g(\mathbf{q})\mathbf{q}_x]_{\psi} = \mathbf{f}(\mathbf{q})_x$$

or

$$\int_0^1 \mathbf{f}'(\psi(s, \mathbf{q}_L, \mathbf{q}_R)) \frac{\partial \psi}{\partial s}(s, \mathbf{q}_L, \mathbf{q}_R) ds = \mathbf{f}(\mathbf{q}_L) - \mathbf{f}(\mathbf{q}_R)$$

Find weak solution \mathbf{q} such that

$$\begin{aligned} & \int_0^T \int_{\Omega} \mathbf{q} v_t \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega} \mathbf{f}(\mathbf{q}) \nabla \cdot \mathbf{v} \, d\mathbf{x} \, dt \\ & + \int_0^T \int_{\Omega} [g_i(\mathbf{q})\mathbf{q}_{x_i}]_{\psi} v \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega} \mathbf{s}(\mathbf{q}) v \, d\mathbf{x} \, dt \end{aligned}$$

for all $v \in C_0^1(\Omega \times [0, T])$.

Nonconservative DG Formulation

Semi Discrete formulation

find $\mathbf{q} \in V_h = \left\{ v \in L^1(\Omega) \mid v|_{K_j} \in \mathbb{P}^M(K_j) \right\}$ such that

$$\int_{\Omega} v_h \mathbf{q}_t \, dx + \int_{\Omega} v_h \nabla \cdot \mathbf{f}(\mathbf{q}) \, dx + \int_{\Omega} v_h [g_i(\mathbf{q}) \mathbf{q}_{x_i}]_{,\psi} = \int_{\Omega} v_h \mathbf{s}(\mathbf{q}) \, dx$$

or

$$\begin{aligned} & \sum_j \left(\int_{K_j} v_h \mathbf{q}_t \, dx \right) - \sum_j \left(\int_{K_j} \nabla \cdot v_h \mathbf{f}(\mathbf{q}) \, dx \right) \\ & + \sum_l \left((v_h^L - v_h^R) \hat{\mathbf{f}} \right) + \sum_j \left(\int_{K_j} v_h g_i(\mathbf{q}) \mathbf{q}_{x_i} \, dx \right) \\ & + \sum_l \left(\int_l \hat{v}_h \left(\int_0^1 g(\psi(s, \mathbf{q}_h^L, \mathbf{q}_h^R)) \psi_s(s, \mathbf{q}_h^L, \mathbf{q}_h^R) \, ds \mathbf{n} \right) dl \right) = \int_{\Omega} v_h \mathbf{s}(\mathbf{q}) \, dx \end{aligned}$$

for all $v_h \in V_h$.

Nonconservative DG Formulation, (Rhebergen, Bokhove, and Vegt [15])

Test Function Flux,

$$\hat{v}_h = \frac{1}{2}(v_h^+ + v_h^-)$$

consistent with conservative DG formulation when $\mathbf{h}'(\mathbf{q}) = g(\mathbf{q})$.

Local Lax-Friedrichs Numerical Flux

$$\lambda = \max_{q \in [\mathbf{q}_h^-, \mathbf{q}_h^+]} \{\rho(\mathbf{f}'(\mathbf{q}) + g(\mathbf{q}))\}$$
$$\hat{\mathbf{f}} = \frac{1}{2}(\mathbf{f}(\mathbf{q}_h^+) + \mathbf{f}(\mathbf{q}_h^-)) - \frac{1}{2}\lambda(\mathbf{q}_h^+ - \mathbf{q}_h^-)$$

Manufactured Solution

Shallow Water Equations, constant vertical velocity profile

1st Order			2nd Order		3rd Order	
n	error	order	error	order	error	order
20	0.226	—	8.57×10^{-3}	—	1.67×10^{-4}	—
40	0.117	0.96	2.17×10^{-3}	1.98	2.07×10^{-5}	3.02
80	0.058	1.00	5.40×10^{-4}	2.01	2.57×10^{-6}	3.01
160	0.028	1.06	1.35×10^{-4}	2.00	3.21×10^{-7}	3.00
320	0.014	0.99	3.37×10^{-5}	2.00	4.01×10^{-8}	3.00

4th Order			5th Order	
n	error	order	error	order
20	3.172×10^{-6}	—	7.606×10^{-8}	0.00
40	1.982×10^{-7}	4.00	2.380×10^{-9}	5.00
80	1.240×10^{-8}	4.00	7.713×10^{-11}	4.95
160	7.755×10^{-10}	4.00	4.035×10^{-11}	0.93
320	4.849×10^{-11}	4.00	8.085×10^{-11}	-1.00

Manufactured Solution

One moment, linear vertical velocity profile

n	1st Order		2nd Order		3rd Order	
	error	order	error	order	error	order
20	2.53×10^{-1}	—	9.97×10^{-3}	—	1.71×10^{-3}	—
40	1.30×10^{-1}	0.96	2.52×10^{-3}	1.98	3.85×10^{-4}	2.15
80	6.47×10^{-2}	1.00	6.28×10^{-4}	2.00	6.13×10^{-5}	2.65
160	3.13×10^{-2}	1.05	1.57×10^{-4}	2.00	9.09×10^{-6}	2.75
320	1.58×10^{-2}	0.99	3.92×10^{-5}	2.00	1.73×10^{-6}	2.39

n	4th Order		5th Order	
	error	order	error	order
20	1.14×10^{-4}	—	2.68×10^{-5}	—
40	1.74×10^{-5}	2.72	8.01×10^{-7}	5.06
80	7.50×10^{-7}	4.53	1.53×10^{-8}	5.71
160	1.25×10^{-7}	2.59	4.04×10^{-10}	5.25
320	8.79×10^{-9}	3.83	8.40×10^{-11}	2.27

Manufactured Solution

Two moments, quadratic vertical velocity profile

1st Order			2nd Order		3rd Order	
n	error	order	error	order	error	order
20	2.778×10^{-1}	—	1.141×10^{-2}	—	5.350×10^{-3}	—
40	1.424×10^{-1}	0.96	2.884×10^{-3}	1.98	6.466×10^{-4}	3.05
80	7.121×10^{-2}	1.00	7.191×10^{-4}	2.00	7.836×10^{-5}	3.04
160	3.454×10^{-2}	1.04	1.797×10^{-4}	2.00	1.270×10^{-5}	2.63
320	1.740×10^{-2}	0.99	4.493×10^{-5}	2.00	2.546×10^{-6}	2.32

4th Order			5th Order	
n	error	order	error	order
20	3.688×10^{-4}	—	5.194×10^{-5}	—
40	2.461×10^{-5}	3.91	1.121×10^{-6}	5.53
80	1.403×10^{-6}	4.13	1.934×10^{-8}	5.86
160	1.144×10^{-7}	3.62	5.859×10^{-10}	5.04
320	1.092×10^{-8}	3.39	8.791×10^{-11}	2.74

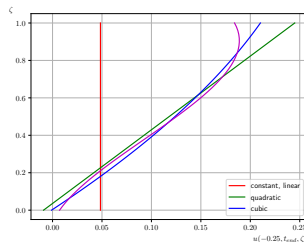
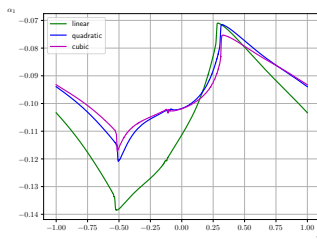
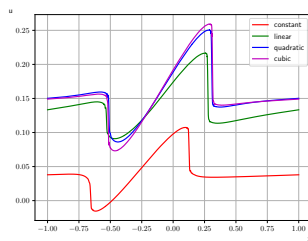
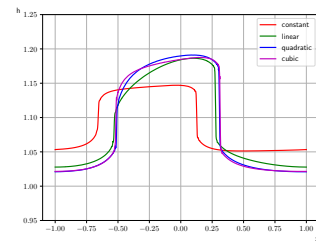
Manufactured Solution

Three moments, cubic vertical velocity profile

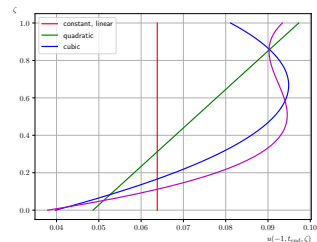
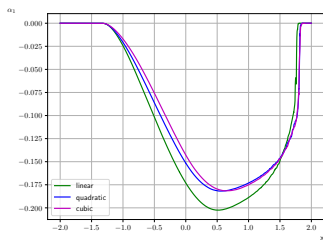
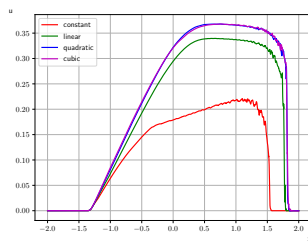
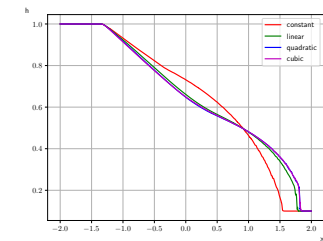
n	1st Order		2nd Order		3rd Order	
	error	order	error	order	error	order
20	3.024×10^{-1}	—	1.300×10^{-2}	—	7.015×10^{-3}	—
40	1.556×10^{-1}	0.96	3.283×10^{-3}	1.99	6.992×10^{-4}	3.33
80	7.808×10^{-2}	0.99	8.188×10^{-4}	2.00	1.183×10^{-4}	2.56
160	3.802×10^{-2}	1.04	2.046×10^{-4}	2.00	2.545×10^{-5}	2.22
320	1.916×10^{-2}	0.99	5.117×10^{-5}	2.00	5.110×10^{-6}	2.32

n	4th Order		5th Order	
	error	order	error	order
20	3.167×10^{-4}	—	5.571×10^{-5}	—
40	2.384×10^{-5}	3.73	1.099×10^{-6}	5.66
80	2.509×10^{-6}	3.25	2.639×10^{-8}	5.38
160	3.168×10^{-7}	2.99	1.371×10^{-9}	4.27
320	4.675×10^{-8}	2.76	1.171×10^{-10}	3.55

Effect of Higher Moments



Effect of Higher Moments



Conclusions

Results

- Discontinuous Galerkin Method for Generalized Shallow Water Equations
- High Order Method
- Properly Discretized Nonconservative Product

Future Work

- Two Dimensional Meshes
- Icosahedral Spherical Mesh
- Shallow Water test cases on the sphere

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