Discontinuous Galerkin Method for Solving Thin Film Equations

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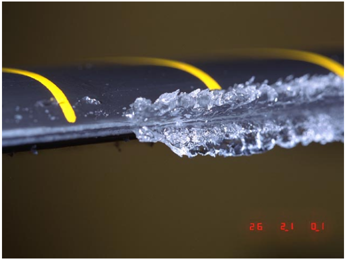
Overview

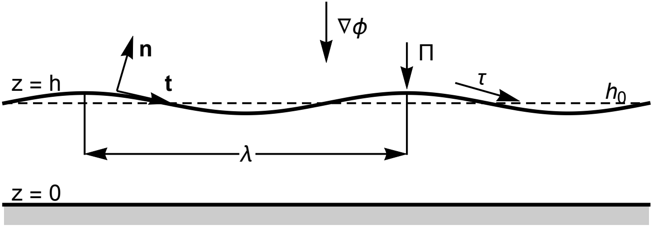
Motivation

- Aircraft Icing
- Runback



- Industrial Coating

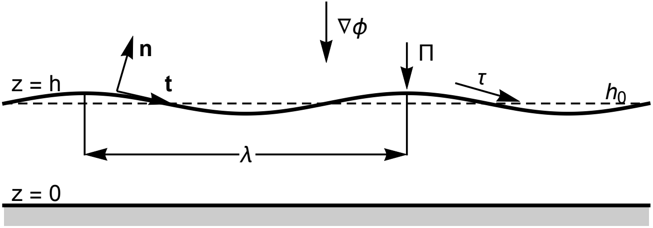
Model Equations



- Incompressible Navier-Stokes Equation

$$
\begin{aligned}
u_x + w_z &= 0 \\
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
w = 0, u &= 0 && \text{at } z = 0 \\
w &= h_t + uh_x && \text{at } z = h \\
\mathbf{T}\cdot\mathbf{n} &= (-\kappa\sigma + \Pi)\mathbf{n} + \left(\frac{\partial\sigma}{\partial s} + \tau\right)\mathbf{t} && \text{at } z = h
\end{aligned}
$$

Nondimensionalization

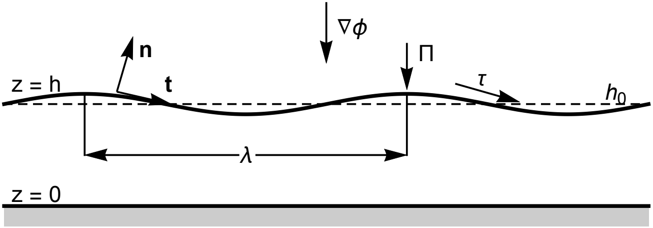


$$\varepsilon = \frac{h_0}{\lambda} \ll 1 \qquad Z = \frac{z}{h_0} \qquad X = \frac{\varepsilon x}{h_0}$$

$$U = \frac{u}{U_0} \qquad W = \frac{w}{\varepsilon U_0} \qquad T = \frac{\varepsilon U_0 t}{h_0}$$

$$Re = \frac{U_0 h_0 \rho}{\mu} \qquad C = \frac{U_0 \mu}{\sigma}$$

Nondimensionalization



$$
\begin{aligned}
U_X + W_Z &= 0 \\
\varepsilon Re(U_T + UU_X + WU_Z) &= -P_X + U_{ZZ} + \varepsilon^2 U_{XX} - \Phi_X \\
\varepsilon^3 Re(W_T + WW_X + WW_Z) &= -P_Z + \varepsilon^2\left(W_{ZZ} + \varepsilon^2 W_{XX}\right) - \Phi_Z \\
W &= 0, U = 0 && \text{at } Z = 0 \\
W &= H_T + UH_X && \text{at } Z = H \\
U_Z + \varepsilon^2 W_X - 4\varepsilon^2 H_X U_X &= \tau + \Sigma_X && \text{at } Z = H \\
-P - \Pi + \varepsilon^2 U_X\left(\varepsilon^2 H_X^2 - 1\right) &= \varepsilon^2 H_X\left(U_Z + \varepsilon^2 W_X\right) + C^{-1}\varepsilon^3 H_{XX} && \text{at } Z = H
\end{aligned}
$$

Take $\varepsilon \to 0$,

$$
\begin{aligned}
U_X + W_Z &= 0 \\
U_{ZZ} &= P_X + \Phi_X \\
0 &= -P_Z - \Phi_Z
\end{aligned}
$$

$$
\begin{aligned}
W &= 0 \qquad \text{at } Z = 0 \\
U &= 0
\end{aligned}
$$

$$
\begin{aligned}
W &= H_T + UH_X \qquad \text{at } Z = H \\
U_Z &= \tau_0 + \Sigma_X \\
-\Pi_0 - P &= \bar{C}^{-1} H_{XX}
\end{aligned}
$$

Integrate over Z and simplify

$$0 = H_T + \left(\int_0^H U \,\mathrm{d}Z\right)_X$$

$$P + \Phi = \Phi|_{Z=H} - \bar{C}^{-1} H_{XX} - \Pi$$

$$U = (\tau + \Sigma_X)Z - (P_X + \Phi_X)\left(HZ - \frac{1}{2}Z^2\right)$$

$$0 = H_T + \left((\tau + \Sigma_X)\frac{1}{2}H^2 - (P_X + \Phi_X)\frac{1}{3}H^3\right)_X$$

$$P_X + \Phi_X = (\Phi|_{Z=H} - \Pi)_X - \bar{C}^{-1} H_{XXX}$$

$$H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}\left(\Phi|_{Z=H} - \Pi\right)_X H^3\right)_X = -\frac{1}{3}\bar{C}^{-1}\left(H^3 H_{XXX}\right)_X$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$q_t + \left(q^2 - q^3\right)_x = 0$$
$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

- Strang Splitting

$\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

$\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x + \hat{f} v \Big|_{\partial\Omega} = 0$$

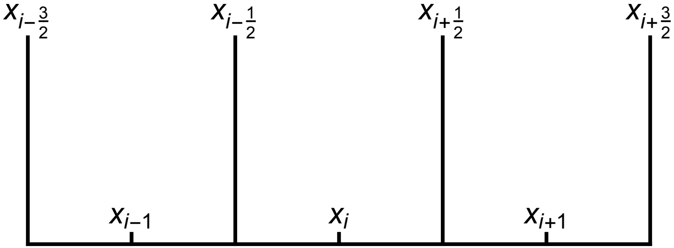
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$, $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \, \mathrm{d}x = \int_{I_j} f(Q) v_x \, \mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

  for all $v \in V_h$

- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^{\star} &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^{\star}) + \frac{1}{2}\Delta t L(q^{\star}) \end{aligned}$$

Finite Difference Approach

- Diffusion Equation

$$q_t = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Let cell centers, x_i, form finite difference grid.
- Finite difference space, $\mathbb{R}^N$.
- $Q_{DG} \in V_h \to Q_{FD} \in \mathbb{R}^N$

$$(Q_{FD})_i = \frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x$$

- $Q_{FD} \in \mathbb{R}^N \to Q_{DG} \in V_h$

$$Q_{DG}|_K \in P^1(K)$$
$$\frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x = (Q_{FD})_i$$
$$\partial_x Q_{DG}|_{K_i} = \frac{(Q_{FD})_{i+1} - (Q_{FD})_{i-1}}{2h}$$

Finite Difference Approximation

- First derivative approximation

$$\left(-(f(x)q_{xxx})_x\right)_i \approx -\frac{f_{i+1/2}(q_{xxx})_{i+1/2} - f_{i-1/2}(q_{xxx})_{i-1/2}}{h}$$

- Third derivative approximation

$$(q_{xxx})_{i+1/2} \approx \frac{-Q_{i-1} + 3Q_i - 3Q_{i+1} + Q_{i+2}}{h^3}$$

- Value of Q^3 at boundary

$$f_{i+1/2} = q^3_{i+1/2} = \left(\frac{Q_i + Q_{i+1}}{2}\right)^3$$

- Full operator

$$L(q) = L(f = q^3, q) = A(f)q$$

Implicit L-Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$q^{\star} = q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^{\star}))$$
$$3q^{n+1} = 4q^{\star} - q^n + \Delta t L(q^{n+1})$$

Nonlinear Solvers

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- Picard Iteration

$$q_0^{n+1} = q^n$$

$$q_{m+1}^{n+1} = q^n + \Delta t L\left(f = \left(q_m^{n+1}\right)^3, q_{m+1}^{n+1}\right)$$

$$q_{m+1}^{\star} = q^n + \frac{1}{4}\Delta t\left(L(q^n) + L\left(f = (q_m^{\star})^3, q_{m+1}^{\star}\right)\right)$$

$$3q_{m+1}^{n+1} = 4q^{\star} - q^n + \Delta t L\left(f = \left(q_m^{n+1}\right)^3, q_{m+1}^{n+1}\right)$$

Manufactured Solution

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q_t + \left(q^2 - q^3\right)_x = s(x, t)$$
$$q_t = -\left(q^3 q_{xxx}\right)_x$$
$$q(x, t) = 0.1 * \sin(2\pi(x - t)) + 0.15$$

1st Order				
	1 Iteration		2 Iterations	
N	error	order	error	order
25	0.1529	—	0.0776	—
50	0.05334	1.52	0.0370	1.06
100	0.02374	1.16	0.0177	1.06
200	0.01186	1.00	0.0091	0.95

Manufactured Solution

$$
\begin{aligned}
q_t + \left(q^2 - q^3\right)_x &= -\left(q^3 q_{xxx}\right)_x + s(x, t) \\
q_t + \left(q^2 - q^3\right)_x &= s(x, t) \\
q_t &= -\left(q^3 q_{xxx}\right)_x \\
q(x, t) &= 0.1 * \sin(2\pi(x - t)) + 0.15
\end{aligned}
$$

2nd Order						
	1 Iteration		2 Iterations		3 Iterations	
N	error	order	error	order	error	order
25	0.03449	—	0.02890	—	0.03103	—
50	0.01061	1.70	0.00875	1.72	0.00910	1.77
100	0.00330	1.68	0.00197	2.14	0.00202	2.17
200	0.00143	1.20	0.00051	1.96	0.00051	1.98

Manufactured Solution

$$
\begin{aligned}
q_t + \left(q^2 - q^3\right)_x &= -\left(q^3 q_{xxx}\right)_x + s(x, t) \\
q_t + \left(q^2 - q^3\right)_x &= s(x, t) \\
q_t &= -\left(q^3 q_{xxx}\right)_x \\
q(x, t) &= \frac{2}{10} e^{-10\left(x - t - \frac{3}{2}\right)^2} + \frac{1}{10}
\end{aligned}
$$

2nd Order				
	1 Iteration		2 Iterations	
N	error	order	error	order
50	0.05609	—	0.3808	—
100	0.04178	0.42	0.2335	0.7
200	0.01182	1.82	0.0429	2.44
400	0.00612	0.94	0.0104	2.04
800	0.00268	1.19	0.0026	2.03

Hyperbolic Wave Structure

- Conservation Law

$$q_t + f(q)_x = 0$$

- Riemann Problem Initial Data

$$q(x,0) = \begin{cases} q_l & x < d \\ q_r & x > d \end{cases}$$

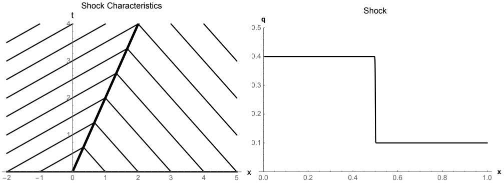
- Rankine-Hugoniot Condition

$$s = \frac{f(q_l) - f(q_r)}{q_l - q_r}$$

Convex Flux Function

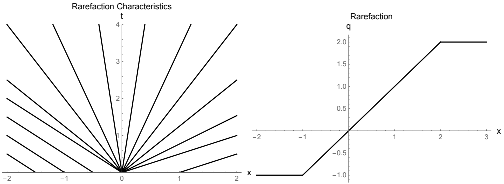
- Shock Wave

$$f'(q_l) > s > f'(q_r)$$



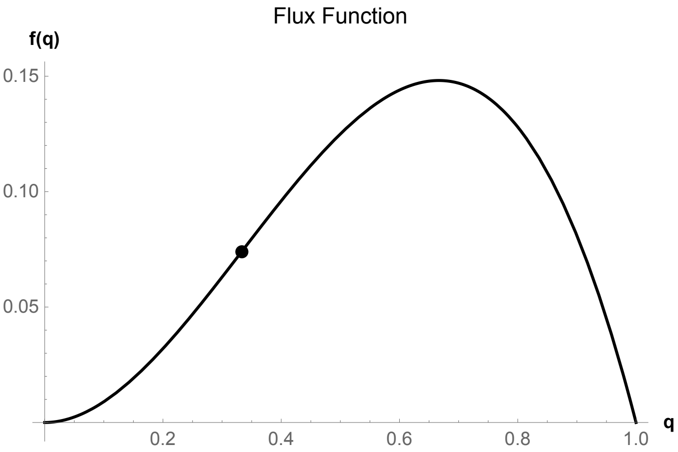
- Rarefaction

$$f'(q_l) < s < f'(q_r)$$



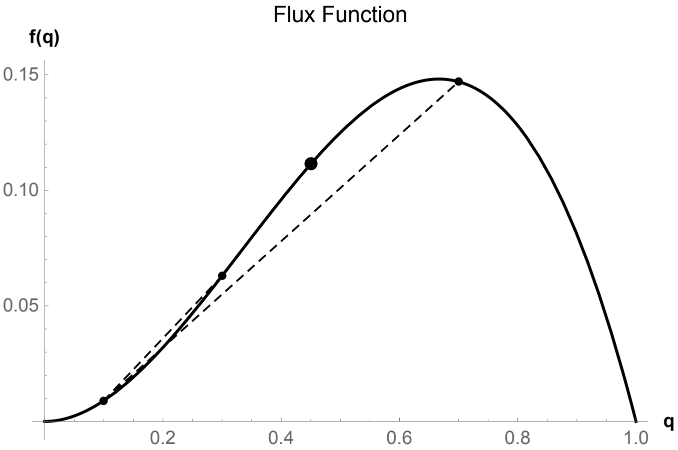
Nonconvex Flux Function

$$q_t + \left(q^2 - q^3\right)_x = 0$$



Nonconvex Flux Function

$$f'(q_b) = s$$
$$q_b = (1 - q_r)/2$$



Compressive Shock

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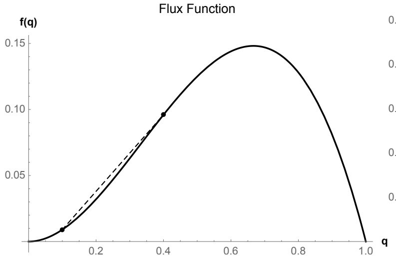
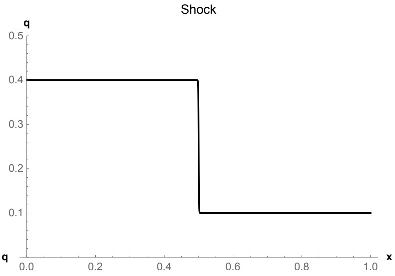
$$q_l < q_b$$

$$q(x,t) = \begin{cases} q_l & x \leq st \\ q_r & x > st \end{cases}$$

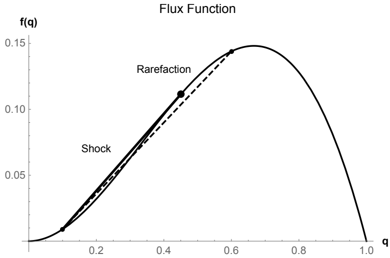


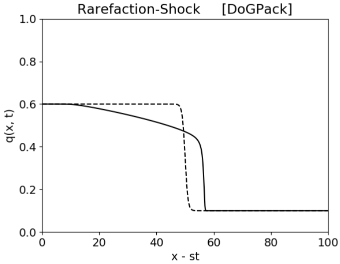


Rarefaction-Compressive Shock

$$q_l > q_b$$

$$q(x,t) = \begin{cases} q_l & x < f'(q_l)t \\ h_r(x) & f'(q_l)t < x < f'(q_b)t \\ q_r & x > f'(q_b)t \end{cases}$$

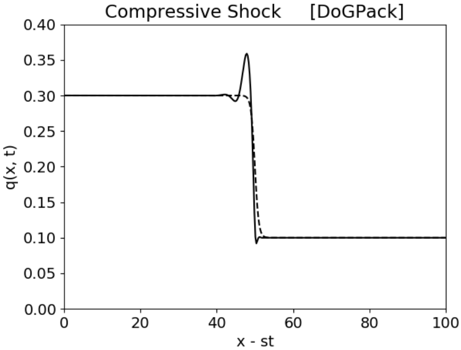




Wave Structure with Nonlinear Hyper Diffusion

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x$$

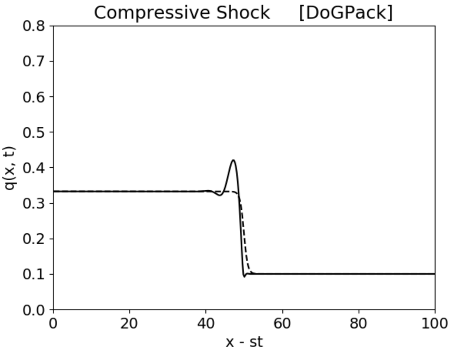
$$q_r = 0.1 \qquad q_l = 0.3$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323$$

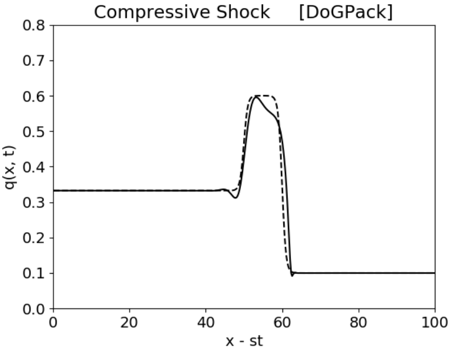
$$q(x, 0) = (-\tanh(x - 50) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.3323 \qquad q_m = 0.6$$

$$q(x, 0) = \begin{cases} \frac{q_m - q_l}{2} \tanh(x - 50) + \frac{q_m + q_l}{2} & x < 55 \\ -\frac{q_m - q_r}{2} \tanh(x - 60) + \frac{q_m + q_r}{2} + q_r & x > 55 \end{cases}$$



Wave Structure with Nonlinear Hyper Diffusion

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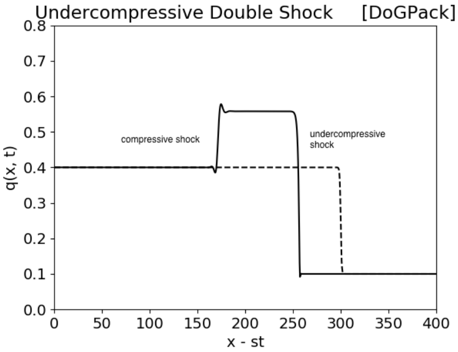
$$q_r = 0.1 \qquad q_l = 0.4$$

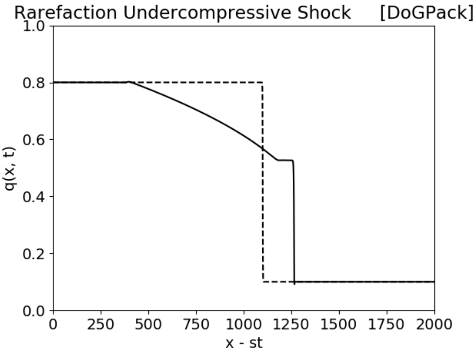
$$q(x,0) = (-\tanh(x - 300) + 1)\frac{q_l - q_r}{2} + q_r$$



Wave Structure with Nonlinear Hyper Diffusion

$$q_r = 0.1 \qquad q_l = 0.8$$

$$q(x, 0) = (-\tanh(x - 1100) + 1)\frac{q_l - q_r}{2} + q_r$$



Conclusion

Observations

- Nonlinear Hyper Diffusion has subtle instabilities

Future Work

- Higher Order Convergence
    - Runge Kutta IMEX
    - Local Discontinuous Galerkin Method
    - Hybridized Discontinuous Galerkin Method

Questions?

Bibliography I

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