Discontinuous Galerkin Method for Solving Thin Film Equations

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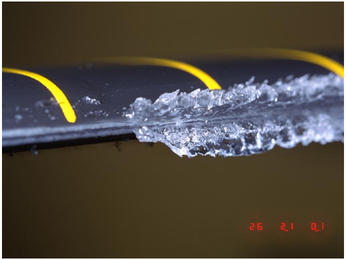
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating

Model Equations

- Navier-Stokes Equation

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$\partial_t h_s + (u, v)^T \cdot \nabla h_s = w$$

$$\partial_t h_b + (u, v)^T \cdot \nabla h_b = w$$

- Lubrication or reduced Reynolds number approximation
- Thin-Film Equation - 1D with q as fluid height.

$$q_t + \left(f(x,t)q^2 - g(x,t)q^3\right)_x = -\left(h(x,t)q^3 q_{xxx}\right)_x$$

Operator Splitting

- Simplified Model

$$q_t + \left(q^2 - q^3\right)_x = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Operator Splitting

$$\begin{aligned} q_t + \left(q^2 - q^3\right)_x &= 0 \\ q_t + \left(q^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

- Strang Splitting
  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

  Δt step of Diffusion

$$q_t + \left(q^3 u_{xxx}\right)_x = 0$$

  $\frac{1}{2}\Delta t$ step of Convection

$$q_t + \left(q^2 - q^3\right)_x = 0$$

Convection

- Convection Equation

$$q_t + f(q)_x = 0 \qquad (0, T) \times \Omega$$
$$f(q) = q^2 - q^3$$

- Weak Form
  Find q such that

$$\int_{\Omega} (q_t v - f(q) v_x) \, \mathrm{d}x + \hat{f} v \Big|_{\partial \Omega} = 0$$

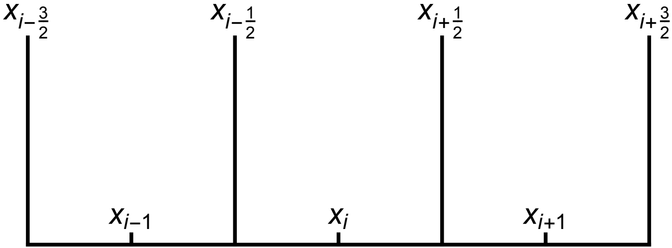
for all test functions v

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$.



Runge Kutta Discontinuous Galerkin

- Find $Q(t, x)$ such that for each time $t \in (0, T)$,
  $Q(t, \cdot) \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Q_t v \,\mathrm{d}x = \int_{I_j} f(Q) v_x \,\mathrm{d}x - \left(\mathcal{F}_{j+1/2} v^-(x_{j+1/2}) - \mathcal{F}_{j-1/2} v^+(x_{j-1/2}) \right)$$

  for all $v \in V_h$
- Rusanov/Local Lax-Friedrichs Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{1}{2}\left(f\left(Q^-_{j+1/2}\right) + f\left(Q^+_{j+1/2}\right)\right) + \frac{1}{2} \max_q \left\{ \left| f'(q) \right| \right\} \left(Q^-_{j+1/2} - Q^+_{j+1/2} \right)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Explicit SSP Runge Kutta Methods

- Forward Euler

$$q^{n+1} = q^n + \Delta t L(q^n)$$

- Second Order

$$\begin{aligned} q^{\star} &= q^n + \Delta t L(q^n) \\ q^{n+1} &= \frac{1}{2}(q^n + q^{\star}) + \frac{1}{2}\Delta t L(q^{\star}) \end{aligned}$$

Diffusion

- Diffusion Equation

$$q_t = -\left(q^3 q_{xxx}\right)_x \qquad (0, T) \times \Omega$$

- Linearize operator at $t = t^n$, let $f(x) = q^3(t = t^n, x)$

$$q_t = -(f(x) q_{xxx})_x \qquad (0, T) \times \Omega$$

Local Discontinuous Galerkin

Find $Q(t,x), R(x), S(x), U(x)$ such that for all $t \in (0, T)$
$Q(t, \cdot), R, S, U \in V_h = \left\{ v \in L^1(\Omega) : v|_{I_j} \in P^k(I_j) \right\}$

$$\int_{I_j} Rv \,\mathrm{d}x = -\int_{I_j} Qv_x \,\mathrm{d}x + \left(\hat{Q}_{j+1/2} v^-_{j+1/2} - \hat{Q}_{j-1/2} v^+_{j-1/2} \right)$$

$$\int_{I_j} Sw \,\mathrm{d}x = -\int_{I_j} Rw_x \,\mathrm{d}x + \left(\hat{R}_{j+1/2} w^-_{j+1/2} - \hat{R}_{j-1/2} w^+_{j-1/2} \right)$$

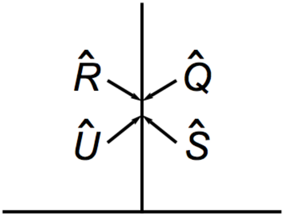
$$\int_{I_j} Uy \,\mathrm{d}x = \int_{I_j} S_x f y \,\mathrm{d}x - \left(S^-_{j+1/2} f^-_{j+1/2} y^-_{j+1/2} - S^+_{j-1/2} f^+_{j-1/2} y^+_{j-1/2} \right) + \left(\hat{S}_{j+1/2} \hat{f}_{j+1/2} y^-_{j+1/2} - \hat{S}_{j-1/2} \hat{f}_{j-1/2} y^+_{j-1/2} \right)$$

$$\int_{I_j} Q_t z \,\mathrm{d}x = -\int_{I_j} Uz_x \,\mathrm{d}x + \left(\hat{U}_{j+1/2} z^-_{j+1/2} - \hat{U}_{j-1/2} z^+_{j-1/2} \right)$$

for all $I_j \in \Omega$ and all $v, w, y, z \in V_h$.

Numerical Fluxes

$$\hat{f}_{j+1/2} = \frac{1}{2}\left(f^{+}_{j+1/2} + f^{-}_{j+1/2}\right)$$
$$\hat{Q}_{j+1/2} = Q^{+}_{j+1/2}$$
$$\hat{R}_{j+1/2} = R^{-}_{j+1/2}$$
$$\hat{S}_{j+1/2} = S^{+}_{j+1/2}$$
$$\hat{U}_{j+1/2} = U^{-}_{j+1/2}$$



LDG Complications

- Explicit time step scales with h^4
- Implicit System is difficult to solve efficiently
  - GMRES iterations scale with size of system
  - Preconditioned GMRES

$$P = A_0^{-1}$$
$$PAx = Pb$$

  - Geometric Multigrid fails to converge

Finite Difference Approach

- Let cell centers, x_i, form finite difference grid.
- Finite difference space, $\mathbb{R}^N$.
- $Q_{DG} \in V_h \to Q_{FD} \in \mathbb{R}^N$

$$\left(Q_{FD}\right)_i = \frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x$$

- $Q_{FD} \in \mathbb{R}^N \to Q_{DG} \in V_h$

$$Q_{DG}\big|_K \in P^1(K)$$

$$\frac{1}{h} \int_{K_i} Q_{DG} \, \mathrm{d}x = \left(Q_{FD}\right)_i$$

$$\partial_x Q_{DG}\big|_{K_i} = \frac{\left(Q_{FD}\right)_{i+1} - \left(Q_{FD}\right)_{i-1}}{2h}$$

Finite Difference Approximation

- First derivative approximation

$$\left(-(f(x)q_{xxx})_x\right)_i \approx -\frac{q^3_{i+1/2}(q_{xxx})_{i+1/2} - q^3_{i-1/2}(q_{xxx})_{i-1/2}}{h}$$

- Third derivative approximation

$$(q_{xxx})_{i+1/2} \approx \frac{-Q_{i-1} + 3Q_i - 3Q_{i+1} + Q_{i+2}}{h^3}$$

- Value of Q^3 at boundary

$$q^3_{i+1/2} = \left(\frac{Q_i + Q_{i+1}}{2}\right)^3$$

Implicit L-Stable Runge Kutta

- Backward Euler

$$q^{n+1} = q^n + \Delta t L(q^{n+1})$$

- 2nd Order

$$\begin{aligned} q^{\star} &= q^n + \frac{1}{4}\Delta t(L(q^n) + L(q^{\star})) \\ 3q^{n+1} &= 4q^{\star} - q^n + \Delta t L(q^{n+1}) \end{aligned}$$

Nonlinear Solvers

- Picard Iteration

$$L(q) = A(f \approx q^3)q$$
$$q_0^{n+1} = q^n$$

$$q_{m+1}^{n+1} = q^n + \Delta t A(q_m^{n+1}) q_{m+1}^{n+1}$$

$$q_{m+1}^{\star} = q^n + \frac{1}{4}\Delta t \big(L(q^n) + A(q_m^{\star}) q_{m+1}^{\star} \big)$$
$$3q_{m+1}^{n+1} = 4q^{\star} - q^n + \Delta t A(q_m^{n+1}) q_{m+1}^{n+1}$$

- Newton's Method

$$q_{m+1}^{n+1} = q_m^{n+1} - J(q_m^{n+1})^{-1} F(q_m^{n+1})$$

$$F(q) = q - q^n - \Delta t L(q)$$
$$J(q) = I - \Delta t L'(q)$$

Manufactured Solution

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x,t)$$
$$q(x,t) = 0.1 * \sin(2\pi(x-t)) + 0.15$$

Backward Euler				
	1 Iteration		2 Iterations	
N	error	order	error	order
100	.0131	-	.0053	-
200	.0064	1.0264	.0026	1.0466
400	.0033	0.96	.0013	0.9704
800	.0016	1.0069	.0007	1.0134

Manufactured Solution

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = 0.1 * \sin(2\pi(x - t)) + 0.15$$

2nd Order IRK						
	1 Iteration		2 Iterations		3 Iterations	
N	error	order	error	order	error	order
50	.0075	-	.00047	-	.0004901	-
100	.0041	0.8601	.00012	1.9844	.0001209	2.0194
200	.0020	1.0391	.0000312	1.9451	.0000305	1.9887
400	.0010	0.9652	.0000082	1.9244	.0000078	1.9641

Manufactured Solution

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$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = \frac{2}{10} e^{-10t} e^{-300\left(x - \frac{1}{2}\right)^2} + \frac{1}{10}$$

Backward Euler				
	1 Iteration		2 Iterations	
N	error	order	error	order
100	.0097	-	.0933	-
200	.0050	0.95	.0421	1.1494
400	.0027	0.87	3.756	-6.48
800	33.21	-13.5	16.51	-2.14

Manufactured Solution - Newton's Method

$$q_t = -\left(q^3 q_{xxx}\right)_x + s(x, t)$$
$$q(x, t) = \frac{2}{10} e^{-10t} e^{-300\left(x - \frac{1}{2}\right)^2} + \frac{1}{10}$$

Backward Euler		
N	error	order
50	0.0280	-
100	0.0153	0.8765
200	0.0080	0.9249
400	5.5e75	-258

Conclusion

Observations

- Expensive computations
- Nonlinear Hyper Diffusion has subtle instabilities

Future Work

- Hybridized Discontinuous Galerkin Method
- Higher Order Convergence
  - Higher order finite difference approximations
  - More accurate transition from finite difference to discontinuous Galerkin
  - Runge Kutta IMEX
- Space and time dependent coefficients

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