Hybridized Discontinuous Galerkin Method

These notes are intended to give background on Hyridized Discontinuous Galerkin methods and then explore how to apply this method to Thin Film Equations

1 Introduction/Main Idea

To start we will consider Poisson's equation with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

We will consider this in mixed form by introducing the auxilliary variable $\boldsymbol{q} = -\nabla u$, then the equation becomes

$$\begin{aligned} \boldsymbol{q} + \nabla u &= 0 \qquad \text{in } \Omega \\ \nabla \cdot \boldsymbol{q} &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

This problem has an exact solution that can be found analytically assuming some nice properties.

I will introduce a triangulation of Ω, $\mathcal{T}_h$, and reformulate the problem on this triangulation that will give the same exact solution.

First some notation,

$$\begin{aligned} \partial\mathcal{T}_h &= \{\partial K : K \in \mathcal{T}_h\} \\ F &= \partial K \cap \partial\Omega \text{ for } K \in \mathcal{T}_h \\ F &= \partial K^+ \cap \partial K^- \text{ for } K^+, K^- \in \mathcal{T}_h \end{aligned}$$

Let ε_h be the set of all faces, F, and ε_h^0 be interior faces, and ε_h^∂ be boundary faces.

Let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ be the outward unit normals of ∂K^+ and ∂K^- respectively, and $(\boldsymbol{q}^\pm, u^\pm)$ be the interior values of $(\boldsymbol{q}, u)$ on F for $K^\pm$. Define

$$\begin{aligned} [\![\boldsymbol{q} \cdot n]\!] &= \boldsymbol{q}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}^- \cdot \boldsymbol{n}^- \\ [\![u\boldsymbol{n}]\!] &= u^+\boldsymbol{n}^+ + u^-\boldsymbol{n}^- \\ \{\boldsymbol{q}\} &= \frac{\boldsymbol{q}^+ + \boldsymbol{q}^-}{2} \\ \{u\} &= \frac{u^+ + u^-}{2} \end{aligned}$$

Now we can reformulate the original Poisson's problem on $\mathcal{T}_h$ as a local problem for each K

$$\begin{aligned} \boldsymbol{q} + \nabla u &= 0 \\ \nabla \cdot \boldsymbol{q} &= f \end{aligned}$$

a transmission condition on each interior face, $F \in \varepsilon_h^0$

$$\begin{aligned} [\![u\boldsymbol{n}]\!] &= \boldsymbol{0} \\ [\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] &= 0 \end{aligned}$$

and the boundary condition on each boundary face, $F \in \varepsilon_h^\partial$

$$u = g$$

This problem is equivalent to the original problem on Ω. The $(\boldsymbol{q}, u)$ that satisfies this problem also solve the original problem.

We would like to be able to solve the local problem locally, but this requires boundary conditions on each element K for the local problem to be solved. Therefore consider the local problem

$$\begin{aligned} \boldsymbol{q} + \nabla u &= 0 \quad \text{in } K \\ \nabla \cdot \boldsymbol{q} &= f \quad \text{in } K \\ u &= \hat{u} \quad \text{on } \partial K \end{aligned}$$

We have introduced another unknown $\hat{u}$ on each interior face $F \in \varepsilon_h^0$. This unknown automatically makes us satisfy $[\![u\boldsymbol{n}]\!] = \boldsymbol{0}$, so the transmission condition becomes

$$[\![\boldsymbol{q} \cdot \boldsymbol{n}]\!] = 0 \qquad \text{on } F \in \varepsilon_h^0$$

and we still have the boundary condition

$$u = g \qquad \text{on } F \in \varepsilon_h^\partial$$

Now solving for $(\boldsymbol{q}, u, \hat{u})$ will give the same solution as the original problem, however $\boldsymbol{q}$ and u can be solved locally and only $\hat{u}$ needs to be solved globally.

Here is the outline for solving for $\boldsymbol{q}$, u, and $\hat{u}$. First split the local problem in two, so that one part depends on f and the other part depends on $\hat{u}$, that is let $\boldsymbol{q} = \boldsymbol{Q}_f + \boldsymbol{Q}_{\hat{u}}$ and $u = U_f + U_{\hat{u}}$, where

$$\begin{aligned} \boldsymbol{Q}_f + \nabla U_f &= 0 \quad \text{in } K \\ \nabla \cdot \boldsymbol{Q}_f &= f \quad \text{in } K \\ U_f &= 0 \quad \text{on } \partial K \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{Q}_{\hat{u}} + \nabla U_{\hat{u}} &= 0 \quad \text{in } K \\ \nabla \cdot \boldsymbol{Q}_{\hat{u}} &= 0 \quad \text{in } K \\ U_{\hat{u}} &= \hat{u} \quad \text{on } \partial K \end{aligned}$$

Now the transmission condition becomes

$$[\![\boldsymbol{Q}_{\hat{u}}]\!] = -[\![\boldsymbol{Q}_f]\!]$$

First solve for $\boldsymbol{Q}_f$ exactly, then solve for $\boldsymbol{Q}_{\hat{u}}$ in terms of $\hat{u}$. Now the transmission condition gives a global linear algebra problem

$$[\![\boldsymbol{Q}_{\hat{u}}]\!] = -[\![\boldsymbol{Q}_f]\!]$$

since $[\![\boldsymbol{Q}_{\hat{u}}]\!]$ is a linear system in $\hat{u}$ and $-[\![\boldsymbol{Q}_f]\!]$ is known.

After this linear algebra problem is solved, the values of $U_{\hat{u}}$ and $\boldsymbol{Q}_{\hat{u}}$ can be found/reconstructed locally. The full solution is then $\boldsymbol{q} = \boldsymbol{Q}_f + \boldsymbol{Q}_{\hat{u}}$ and $u = U_f + U_{\hat{u}}$.

In 1D, the problem becomes

$$\begin{aligned} Q_f + U_f' &= 0 \quad \text{in } K \\ Q_f' &= f \quad \text{in } K \\ U_f &= 0 \quad \text{on } \partial K \end{aligned}$$

and

$$\begin{aligned} Q_{\hat{u}} + U_{\hat{u}}' &= 0 \quad \text{in } K \\ Q_{\hat{u}}' &= 0 \quad \text{in } K \\ U_{\hat{u}} &= \hat{u} \quad \text{on } \partial K \end{aligned}$$

and

$$Q_{\hat{u}}(x_{j+1/2}^-) - Q_{\hat{u}}(x_{j+1/2}^+) = -Q_f(x_{j+1/2}^-) + Q_f(x_{j+1/2}^+$$

where the uniform mesh is given by x_i at cell centers, $x_{i+1/2}$ at cell interfaces, and spacing h.

Solving the $\hat{u}$ system we see that

$$\begin{aligned} Q'_{\hat{u}} &= 0 \\ Q_{\hat{u}} &= c \end{aligned}$$

and

$$\begin{aligned} Q_{\hat{u}} + U'_{\hat{u}} &= 0 \\ U'_{\hat{u}} &= -c \\ U_{\hat{u}} &= -cx + b \end{aligned}$$

with the boundary conditions, we know $U_{\hat{u}}$ is a line from $\hat{u}_{j-1/2}$ to $\hat{u}_{j+1/2}$, and $Q_{\hat{u}}$ is the opposite of the slope of this line.

$$\begin{aligned} U_{\hat{u}} &= \frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h}\left(x - x_{j-1/2}\right) - \hat{u}_{j-1/2} \\ Q_{\hat{u}} &= -\frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h} \end{aligned}$$

Now we can form the linear system given by the transmission condition

$$\begin{aligned} Q_{\hat{u}}(x^-_{j+1/2}) - Q_{\hat{u}}(x^+_{j+1/2}) &= -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2} \\ -\frac{\hat{u}_{j+1/2} - \hat{u}_{j-1/2}}{h} + \frac{\hat{u}_{j+3/2} - \hat{u}_{j+1/2}}{h} &= -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2} \\ \frac{\hat{u}_{j-1/2} - 2\hat{u}_{j+1/2} + \hat{u}_{j+3/2}}{h} &= -Q_f(x^-_{j+1/2}) + Q_f(x^+_{j+1/2} \end{aligned}$$

After solving this linear system we already have expressions for $Q_{\hat{u}}$ and $U_{\hat{u}}$ in terms of $\hat{u}$.