Nonconservative Discontinuous Galerkin Method for Generalized Shallow Water Equations

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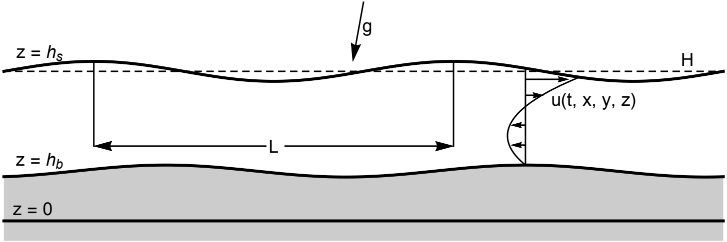
Overview

Generalized Shallow Water



Navier Stokes Equations with a free surface

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \mathbf{g}$$

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s)$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b)$$

Polynomial Ansatz

$$
\begin{aligned}
\tilde{u}(t, x, y, \zeta) &= u_m(t, x, y) + u_d(t, x, y, \zeta) \\
&= u_m(t, x, y) + \sum_{j=1}^{N} \left(\alpha_j(t, x, y)\phi_j(\zeta)\right) \\
\tilde{v}(t, x, y, \zeta) &= v_m(t, x, y) + v_d(t, x, y, \zeta) \\
&= v_m(t, x, y) + \sum_{j=1}^{N} \left(\beta_j(t, x, y)\phi_j(\zeta)\right)
\end{aligned}
$$

Orthogonality Condition

$$
\int_0^1 \phi_j(\zeta)\phi_i(\zeta)\,\mathrm{d}\zeta = 0 \quad \text{for } j \neq i
$$

$$
\phi_0(\zeta) = 1, \quad \phi_1(\zeta) = 1 - 2\zeta, \quad \phi_2(\zeta) = 1 - 6\zeta + 6\zeta^2
$$

Constant Moments

$$h_t + \left(hu_m\right)_x + \left(hv_m\right)_y = 0$$

$$\left(hu_m\right)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y = -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg\left(e_x - e_z\left(h_b\right)_x\right)$$

$$\left(hv_m\right)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y$$

$$+ \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x = -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\left(e_y - e_z\left(h_b\right)_y\right)$$

Higher Order Moments

$$\left(h\alpha_i\right)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k\right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_y$$
$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right)$$

$$\left(h\beta_i\right)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y$$
$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)$$

Example Systems

1D model with h_b constant, $e_x = e_y = 0$, and $e_z = 1$
Constant System

$$\begin{bmatrix} h \\ hu_m \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = -\frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u_m \pm \sqrt{gh}$
Linear System, $\tilde{u} = u_m + \alpha_1\phi_1$

$$\begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_t + \begin{bmatrix} hu_m \\ hu_m^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 \\ 2hu_m\alpha_1 \end{bmatrix}_x = Q\begin{bmatrix} h \\ hu_m \\ h\alpha_1 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u_m \end{bmatrix} \quad \mathbf{s} = \frac{\nu}{\lambda}\begin{bmatrix} 0 \\ u_m + \alpha_1 \\ 3(u_m + \alpha_1 + 4\frac{\lambda}{h}\alpha_1) \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u_m \pm \sqrt{gh + \alpha_1^2}, u_m$

Example Systems

1 dimensional with h_b constant, $e_x = e_y = 0$, and $e_z = 1$ Quadratic Vertical Profile, $\tilde{u} = u + \alpha_1 \phi_1 + \alpha_2 \phi_2$

$$\begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h\alpha_1^2 + \frac{1}{5}h\alpha_2^2 \\ 2hu\alpha_1 + \frac{4}{5}h\alpha_1\alpha_2 \\ 2hu\alpha_2 + \frac{2}{3}h\alpha_1^2 + \frac{2}{7}h\alpha_2^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ h\alpha_1 \\ h\alpha_2 \end{bmatrix}_x - \mathbf{s}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u - \frac{\alpha_2}{5} & \frac{\alpha_1}{5} \\ 0 & 0 & \alpha_1 & u + \frac{\alpha_2}{7} \end{bmatrix}, P = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u + \alpha_1 + \alpha_2 \\ 3\left(u + \alpha_1 + \alpha_2 + 4\frac{\lambda}{h}\alpha_1\right) \\ 5\left(u + \alpha_1 + \alpha_2 + 12\frac{\lambda}{h}\alpha_2\right) \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u \pm c\sqrt{gh}$

$$c^4 - \frac{10\alpha_2}{7}c^3 - \left(1 + \frac{6\alpha_2^2}{35} + \frac{6\alpha_1^2}{5}\right)c^2$$

$$+\left(\frac{22\alpha_2^3}{35} - \frac{6\alpha_2\alpha_1^2}{35} + \frac{10\alpha_2}{7}\right)c - \frac{\alpha_2^4}{35} - \frac{6\alpha_2^2\alpha_1^2}{35} - \frac{3\alpha_2^2}{7} + \frac{\alpha_1^4}{5} + \frac{\alpha_1^2}{5} = 0$$

Nonconservative Products

Model Equation

$$\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + Q_i(\mathbf{q})\mathbf{q}_{x_i} = \mathbf{s}(\mathbf{q}) \quad \text{for } (\mathbf{x}, t) \in \Omega \times [0, T]$$

Traditionally searching for weak solutions, find **q** such that

$$\int_0^T \int_\Omega (\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) + Q_i(\mathbf{q})\mathbf{q}_{x_i}) v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^T \int_\Omega \mathbf{s}(\mathbf{q}) v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for all $v \in C_0^1(\Omega \times [0, T])$

Manufactured Solution

Effect of Higher Moments

Numerical Methods

Model Equation

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = Q(\mathbf{q})\mathbf{q}_x - \mathbf{P}(\mathbf{q}) \quad \text{for } (x, t) \in [a, b] \times [0, T]$$

Weak Form, find $\mathbf{q}$ such that

$$\int_a^b \mathbf{q}_t v \, \mathrm{d}x + \int_a^b \mathbf{f}(\mathbf{q})_x v \, \mathrm{d}x = \int_a^b Q(\mathbf{q})\mathbf{q}_x v \, \mathrm{d}x - \int_a^b \mathbf{P}(\mathbf{q}) v \, \mathrm{d}x$$

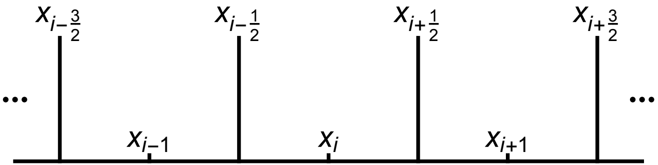
for all $v \in L^2([a, b] \times [0, T])$

Notation

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = b$$

- $I_j = \left[x_{j-1/2}, x_{j+1/2}\right]$
- $x_j = \frac{x_{j+1/2} + x_{j-1/2}}{2}$
- $\Delta x_j = x_{j+1/2} - x_{j-1/2}$
- $\Delta x_j = \Delta x$ for all j.



Discontinuous Galerkin Space

Finite Dimensional DG Space

$$V^k = \left\{ v \in L^2([a,b]) \middle| \, v|_{I_j} \in P^k(I_j) \right\}$$

Basis for V^k

$$\left\{\phi_j^\ell\right\} \text{ where } \phi_j^\ell(x)\Big|_{I_j} = \phi^\ell(\xi_j(x)) \text{ and } \phi_j^\ell(x)\Big|_{\bar{I}_j} = 0$$

for $j = 1, \ldots, N$ and $\ell = 1, \ldots k$.
Legendre Polynomials

$$\phi^k \in P^k([-1,1]) \text{ with } \frac{1}{2}\int_{-1}^{1} \phi^k(\xi)\phi^\ell(\xi)\,\mathrm{d}\xi = \delta_{k\ell}$$

and

$$\xi_j(x) = \frac{2}{\Delta x_j}(x - x_j)$$

Numerical Methods

Find $\mathbf{q}_h \in V^k$ such that

$$\begin{aligned}\int_{I_j} (\mathbf{q}_h)_t \phi_j^\ell(x) \, \mathrm{d}x = &\int_{I_j} \mathbf{f}(\mathbf{q}_h)_x \phi_j^\ell \, \mathrm{d}x \\ &- F_{j+1/2}\phi_j^\ell(x_{j+1/2}) + F_{j-1/2}\phi_j^\ell(x_{j-1/2}) \\ &+ \int_{I_j} Q(\mathbf{q}_h)(\mathbf{q}_h)_x \phi_j^\ell \, \mathrm{d}x - \int_{I_j} \mathbf{P}(\mathbf{q}_h)\phi_j^\ell \, \mathrm{d}x\end{aligned}$$

for all ϕ_j^ℓ.

Local Lax-Friedrichs Flux

$$\begin{aligned}\mathbf{q}_h^+ &= \lim_{x \to x_{j+1/2}^+} (\mathbf{q}_h(x)) \\ \mathbf{q}_h^- &= \lim_{x \to x_{j+1/2}^-} (\mathbf{q}_h(x)) \\ a &= \max_{q \in \left[\mathbf{q}_h^-, \mathbf{q}_h^+\right]} \{\rho(\mathbf{f}'(\mathbf{q}) - Q(\mathbf{q}))\} \\ F_{j+1/2} &= \frac{1}{2}\left(\mathbf{f}\left(\mathbf{q}_h^+\right) + \mathbf{f}\left(\mathbf{q}_h^-\right)\right) - \frac{1}{2}a\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right)\end{aligned}$$

Nonconservative Flux

Need to evaluate

$$\int^{I_j} Q\mathbf{q}_x \phi_j^\ell \, \mathrm{d}x$$

$$\mathbf{q}|_{I_j} = \sum_{\ell=1}^{k} \left(Q_j^\ell \phi_j^\ell(x)\right), \quad \mathbf{q}_x|_{I_j} = \sum_{\ell=1}^{k} \left(Q_x^\ell \phi_j^\ell(x)\right)$$

where

$$\begin{bmatrix} Q_x^1 \\ Q_x^2 \\ Q_x^3 \\ Q_x^4 \\ Q_x^5 \end{bmatrix} = \frac{1}{2\Delta x} \begin{bmatrix} \Delta Q^1 - 2\sqrt{5}\Delta Q^3 + 78\Delta Q^5 \\ \Delta Q^2 - \frac{10}{3}\sqrt{3}\sqrt{7}\Delta Q^4 \\ \Delta Q^3 - 14\sqrt{5}\Delta Q^5 \\ \Delta Q^4 \\ \Delta Q^5 \end{bmatrix}$$

$$\Delta Q^\ell = Q_{i+1}^\ell - Q_{i-1}^\ell$$

Inviscid Example

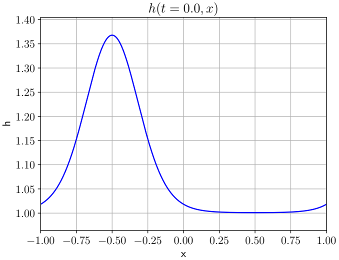
$$x \in [-1, 1] \quad t \in [0, 2.0]$$

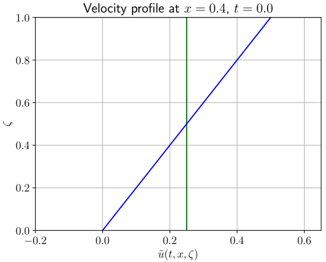
$$h(t = 0, x) = 1 + e^{3\cos(\pi(x+0.5))-4}$$

$$\tilde{u}(t = 0, x, \zeta) = \begin{cases} 0.25 & \text{constant} \\ 0.5\zeta & \text{linear} \end{cases}$$

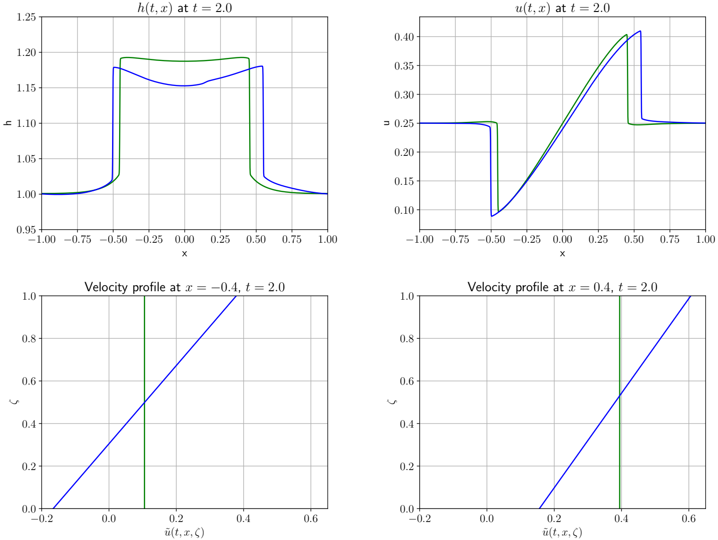
$$u_m = 0.25$$

$$s = -0.25$$





Inviscid Example



Higher Moment Equations

1 dimensional with h_b constant, $e_x = e_y = 0$, and $e_z = 1$ Quadratic Vertical Profile, $\tilde{u} = u_m + s\phi_1 + \kappa\phi_2$

$$\begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 + \frac{1}{5}h\kappa^2 \\ 2hus + \frac{4}{5}hs\kappa \\ 2hu\kappa + \frac{2}{3}hs^2 + \frac{2}{7}h\kappa^2 \end{bmatrix}_x = Q \begin{bmatrix} h \\ hu \\ hs \\ h\kappa \end{bmatrix}_x - P$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_m - \frac{\kappa}{5} & \frac{s}{5} \\ 0 & 0 & s & u_m + \frac{\kappa}{7} \end{bmatrix} \qquad P = \frac{\nu}{\lambda} \begin{bmatrix} 0 \\ u_m + s + \kappa \\ 3\left(u_m + s + \kappa + 4\frac{\lambda}{h}s\right) \\ 5\left(u_m + s + \kappa + 12\frac{\lambda}{h}\kappa\right) \end{bmatrix}$$

Flux Jacobian Eigenvalues, $u_m \pm c\sqrt{gh}$

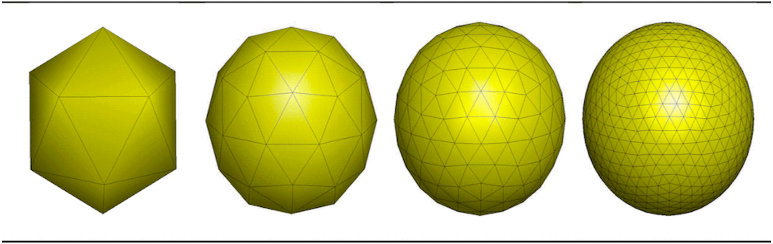
$$c^4 - \frac{10\kappa}{7}c^3 - \left(1 + \frac{6\kappa^2}{35} + \frac{6s^2}{5}\right)c^2$$

$$+\left(\frac{22\kappa^3}{35} - \frac{6\kappa s^2}{35} + \frac{10\kappa}{7}\right)c - \frac{\kappa^4}{35} - \frac{6\kappa^2 s^2}{35} - \frac{3\kappa^2}{7} + \frac{s^4}{5} + \frac{s^2}{5} = 0$$

Future Work

- Higher Order Numerical Methods
- Slope Limiters
- Two Dimensional Meshes
- Icosahedral Spherical Mesh
- Positivity Preserving Limiters

Icosahedral Mesh



Subdivide each edge
Project vertices onto sphere

Caleb Logemann, James Rossmanith

Generalized Shallow Water Equations

Nonconservative Products

Nonconservative DG Formulation

Results

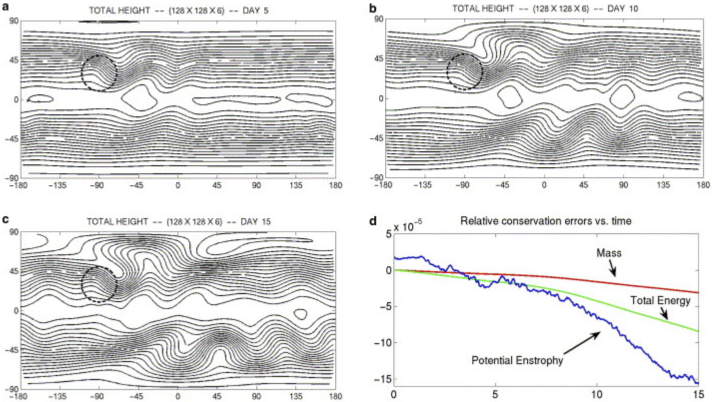
Numerical Methods

Results

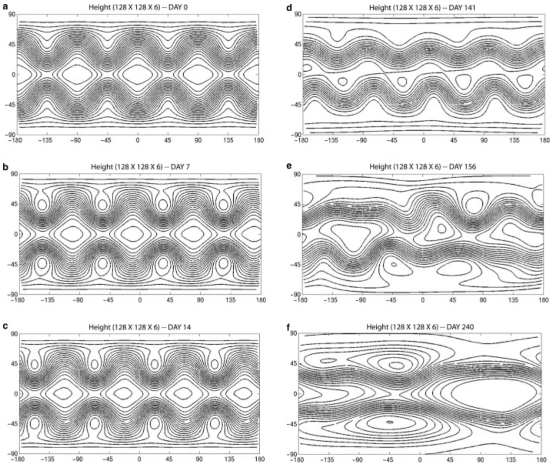
Future Work

References

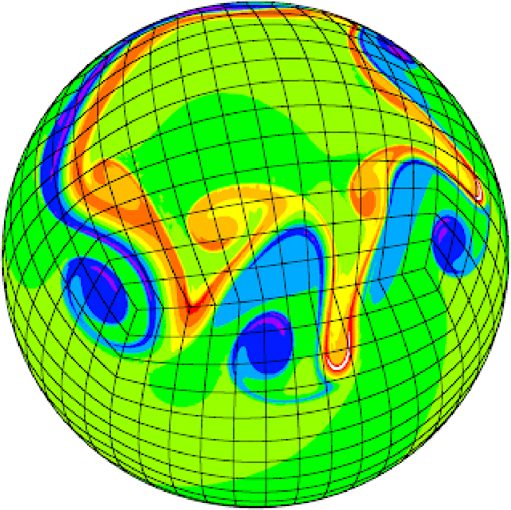
Spherical Test Cases



Spherical Test Cases



Spherical Test Cases



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