Nonconservative discontinuous Galerkin methods for shallow water moment models on the sphere

by

Caleb Logemann

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
James Rossmanith, Major Professor
Hailiang Liu
Songting Luo
Alric Rothmayer
Jue Yan

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation/thesis. The Graduate College will ensure this dissertation/thesis is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University

Ames, Iowa

2021

Contents

List of Tables

List of Figures

ABSTRACT

We present a discontinuous Galerkin method for the generalized shallow water equations first introduced by Kowalski and Torrilhon. These generalized shallow water equations introduce vertical moments into the shallow flow's velocity profile. As a result of these additional moments a nonconservative term appears in these hyperbolic equations. We use the Dal Maso, Le Floch, and Murat theory of nonlinear hyperbolic systems in nonconservative form to correctly discretize this nonconservative term. Using this theory a high order discontinuous Galerkin method is presented for the generalized equations on the sphere. This work also updates some of the standard shallow water tests on the sphere for the generalized shallow water equations.

Chapter1. INTRODUCTION

In this thesis, I present discontinuous Galerkin methods for

Chapter2. The Models

2.1 Shallow Water Moment Models

The shallow water moment equations (SWME) were first introduced by Kowalski and Torrilhon. The goal of this new model is to add vertical resolution to the velocity of the shallow water equations. The standard shallow water equations make several key assumptions. The shallow water equations assume hydrostatic pressure and that the horizontal velocity is constant in the vertical direction. The assumption that the horizontal velocity is constant in the vertical direction is particularly restricting. One common approach to add vertical resolution the the shallow water models is the so-called multilayer shallow water model. The multilayer shallow water model assumes that the horizontal velocity consists of multiple layers of constant velocity. This approach can reflect nature, where the oceans and atmosphere do have multiple layers. However the multilayer model has a significant numerical downside. The multilayer model is not globally hyperbolic, which means that the problem can become ill-posed. When the velocities of the different layers become too different the system is no longer hyperbolic. In this case the fluid should create vortices at the interface between the layers. However the multilayer shallow water model does not allow for these roll-ups and so becomes ill-posed.

Kowalski and Torrilhon have introduced a new approach to adding vertical resolution to the shallow water equations, which has better hyperbolicity properties. The main idea of their approach is to approximate the horizontal velocity as an Ansatz expansion in the vertical direction, that is the velocities can be represented as

$$u(x,y,z,t) = u_m(x,y,t) + \sum_{j=1}^{N} (\alpha_j(x,y,t)\phi_j(z)) \tag{2.1}$$

$$v(x,y,z,t) = v_m(x,y,t) + \sum_{j=1}^{N} (\beta_j(x,y,t)\phi_j(z)), \tag{2.2}$$

where $u_m(x, y, t)$ and $v_m(x, y, t)$ are the mean velocities in the x and y directions respectively. In general the functions ϕ_j can be arbitrary. In fact if ϕ_j are characteristic functions, then the multilayer shallow water model can be derived. However in this work we will assume that ϕ_j are polynomials. This approach maintains computational efficiency compared with fully vertically resolved models.

2.1.1 Derivation

We begin by considering the Navier-Stokes equations,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2.3}$$

$$\boldsymbol{u}_t + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \sigma + \boldsymbol{g}, \tag{2.4}$$

where $\boldsymbol{u} = [u, v, w]^T$ is the vector of velocities, p is the pressure, ρ is the constant density, σ is the deviatoric stress tensor, and $\boldsymbol{g}$ is the gravitational force vector. We also have two boundaries, the bottom topography $h_b(t, x, y)$, and the free surface $h_s(t, x, y)$. At both of these boundaries the kinematic boundary conditions are in effect and can be expressed as

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s) \tag{2.5}$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b). \tag{2.6}$$

In practice the bottom topography is unchanging in time, but we express h_b with time dependence to allow for a symmetric representation of the boundary conditions.

2.1.1.1 Dimensional Analysis

Now we consider the characteristic scales of the problem. Let L be the characteristic horizontal length scale, and let H be the characteristic vertical length scale. For this problem we assume that $H << L$ and we denote the ratio of these lengths as $\varepsilon = H/L$. With these characteristic lengths we can scale the length variables to a nondimensional form

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}. \tag{2.7}$$

Now let U be the characteristic horizontal velocity, then because of the shallowness the characteristic vertical velocity will be εU. Therefore the velocity variables can be scaled as follows,

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}. \tag{2.8}$$

Now with the characteristic length and velocity, the time scaling can be described as

$$t = \frac{L}{U}\hat{t} \tag{2.9}$$

The pressure will be scaled by the characteristic height, H, and the stresses will be scaled by a characteristic stress, S. It is assumed that the basal shear stresses, σ_{xz} and σ_{yz} are of larger order than the lateral shear stress, σ_{xy}, and the normal stresses, σ_{xx}, σ_{yy}, and σ_{zz}, so that

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}. \tag{2.10}$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0 \tag{2.11}$$

$$\varepsilon F^2\Big(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) = -\varepsilon\hat{p}_{\hat{x}} + G\Big(\varepsilon^2(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\Big) + e_x \tag{2.12}$$

$$\varepsilon F^2\left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{y}} + G\Big(\varepsilon^2(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon^2(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\Big) + e_y \tag{2.13}$$

$$\varepsilon^2 F^2\Big(\hat{w}_{\hat{t}} + (\hat{u}\hat{w})_{\hat{x}} + (\hat{v}\hat{w})_{\hat{x}} + \left(\hat{w}^2\right)_{\hat{z}}\Big) = -\hat{p}_{\hat{z}} + \varepsilon G\Big((\hat{\sigma}_{xz})_{\hat{x}} + (\hat{\sigma}_{yz})_{\hat{y}} + (\hat{\sigma}_{zz})_{\hat{z}}\Big) + e_z \tag{2.14}$$

$$F = \frac{U}{\sqrt{gH}} \approx 1, \quad G = \frac{S}{\rho g H} < 1 \tag{2.15}$$

Drop terms with ε^2 and εG, giving

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0 \tag{2.16}$$

$$\varepsilon F^2\Big(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) = -\varepsilon\hat{p}_{\hat{x}} + G(\hat{\sigma}_{xz})_{\hat{z}} + e_x \tag{2.17}$$

$$\varepsilon F^2\left(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\right) = -\varepsilon\hat{p}_{\hat{y}} + G(\hat{\sigma}_{yz})_{\hat{z}} + e_y \tag{2.18}$$

$$\hat{p}_{\hat{z}} = e_z \tag{2.19}$$

where we can solve for the hydrostatic pressure

$$\hat{p}(\hat{t}, \hat{x}, \hat{y}) = \left(\hat{h}_s(\hat{t}, \hat{x}, \hat{y}) - \hat{z}\right)e_z \tag{2.20}$$

For the rest of the derivation we will transform back into dimensional variables for readability purposes.

$$u_x + v_y + w_z = 0 \tag{2.21}$$

$$u_t + \left(u^2\right)_x + (uv)_y + (uw)_z = -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x \tag{2.22}$$

$$v_t + (uv)_x + \left(v^2\right)_y + (vw)_z = -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y \tag{2.23}$$

$$p(t, x, y, z) = (h_s(t, x, y) - z)\rho g e_z \tag{2.24}$$

2.1.1.2 Mapping

In order to make this system more accessible we will map the vertical variable z to the normalized variable ζ, through the transformation

$$\zeta(t, x, y, z) = \frac{z - h_b(t, x, y)}{h(t, x, y)}, \tag{2.25}$$

or equivalently

$$z(t, x, y, \zeta) = h(t, x, y)\zeta + h_b(t, x, y) \tag{2.26}$$

where $h(t, x, y) = h_s(t, x, y) - h_b(t, x, y)$. This transformation maps the vertical variable, z onto $\zeta \in [0, 1]$. In order to transform the partial differential equations we consider a function $\Psi(t, x, y, z)$, then it's mapped counterpart $\tilde{\Psi}(t, x, y, \zeta)$ can be described as

$$\tilde{\Psi}(t, x, y, \zeta) = \Psi(t, x, y, z(t, x, y, \zeta)) = \Psi(t, x, y, h(t, x, y)\zeta + h_b(t, x, y)),$$

or equivalently

$$\Psi(t, x, y, z) = \tilde{\Psi}(t, x, y, \zeta(t, x, y, z)) = \tilde{\Psi}\left(t, x, y, \frac{z - h_b(t, x, y)}{h(t, x, y)}\right).$$

We also need to be able to map derivatives of functions in order to be able to map the differential equations. This can be described

$$\Psi_z(t,x,y,z) = \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_z \tag{2.27}$$

$$\Psi_z(t,x,y,z) = \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\zeta_z(t,x,y,z) \tag{2.28}$$

$$\Psi_z(t,x,y,z) = \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\frac{1}{h(t,x,y)} \tag{2.29}$$

$$h(t,x,y)\Psi_z(t,x,y,z) = \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z)) \tag{2.30}$$

$$h\Psi_z = \tilde{\Psi}_\zeta \tag{2.31}$$

For the other variables, $\{t,x,y\}$, the partial derivatives are identical. Let $s \in \{t,x,y\}$, then

$$\zeta_s(t,x,y,z) = \left(\frac{z - h_b(t,x,y)}{h(t,x,y)}\right)_s \tag{2.32}$$

$$= -\frac{(z - h_b(t,x,y))h_s(t,x,y)}{h(t,x,y)^2} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \tag{2.33}$$

$$= -\zeta(t,x,y,z)\frac{h_s(t,x,y)}{h(t,x,y)} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \tag{2.34}$$

$$= -\frac{\zeta(t,x,y,z)h_s(t,x,y) + (h_b)_s(t,x,y)}{h(t,x,y)} \tag{2.35}$$

and

$$\Psi_s(t,x,y,z) = \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_s \tag{2.36}$$

$$\Psi_s(t,x,y,z) = \tilde{\Psi}_s(t,x,y,\zeta(t,x,y,z)) + \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))\zeta_s(t,x,y,z) \tag{2.37}$$

$$\Psi_s(t,x,y,z) = \tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)\left(\frac{\zeta h_s(t,x,y) + (h_b)_s(t,x,y)}{h(t,x,y)}\right) \tag{2.38}$$

$$h(t,x,y)\Psi_s(t,x,y,z) = h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y) + (h_b)_s(t,x,y)) \tag{2.39}$$

$$h(t,x,y)\Psi_s(t,x,y,z) = h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y) + (h_b)_s(t,x,y)) \tag{2.40}$$

$$h\Psi_s = h\tilde{\Psi}_s - \tilde{\Psi}_\zeta(\zeta h_s + (h_b)_s) \tag{2.41}$$

$$h\Psi_s = h\tilde{\Psi}_s + h_s\tilde{\Psi} - h_s\tilde{\Psi} - \tilde{\Psi}_\zeta(\zeta h + h_b)_s \tag{2.42}$$

$$h\Psi_s = \left(h\tilde{\Psi}\right)_s - \left(h_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \tag{2.43}$$

$$h\Psi_s = \left(h\tilde{\Psi}\right)_s - \left(\left((\zeta h + h_b)_\zeta\right)_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \tag{2.44}$$

$$h\Psi_s = \left(h\tilde{\Psi}\right)_s - \left(((\zeta h + h_b)_s)_\zeta\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \tag{2.45}$$

$$h\Psi_s = \left(h\tilde{\Psi}\right)_s - \left((\zeta h + h_b)_s\tilde{\Psi}\right)_\zeta \tag{2.46}$$

Mapping of the Mass Balance Equation Now we can use these differential transformations to map the continuity equation or mass balance equation onto the normalized space. We begin by multiplying the continuity equation by h

$$h(u_x + v_y + w_z) = 0, \tag{2.47}$$

and then transforming from z to ζ

$$(h\tilde{u})_x + (h\tilde{v})_y + \left(\tilde{w} - (\zeta h + h_b)_x\tilde{u} - (\zeta h + h_b)_y\tilde{v}\right)_\zeta = 0. \tag{2.48}$$

We can then integrate over ζ to find an explicit expression for w the vertical velocity.

$$\begin{gathered}\tilde{w}(t,x,y,\zeta) - \tilde{w}(t,x,y,0) = \\ -\int_0^\zeta (h\tilde{u})_x \,\mathrm{d}\zeta' - \int_0^\zeta (h\tilde{v})_y \,\mathrm{d}\zeta' + (\zeta h + h_b)_x\tilde{u} + (\zeta h + h_b)_y\tilde{v} - (h_b)_x\tilde{u} - (h_b)_y\tilde{v}.\end{gathered} \tag{2.49}$$

This can be simplified using the kinematic boundary condition at the bottom surface, to show that the vertical velocity can be expressed as

$$\tilde{w}(t,x,y,\zeta) = -\int_0^{\zeta} (h\tilde{u})_x \, \mathrm{d}\zeta' - \int_0^{\zeta} (h\tilde{v})_y \, \mathrm{d}\zeta' + (\zeta h + h_b)_x \tilde{u} + (\zeta h + h_b)_y \tilde{v}. \tag{2.50}$$

Lastly by consider the vertical velocity at the free surface and using the kinematic boundary condition at that surface we arrive at the mass conservation equation,

$$h_t + (hu_m)_x + (hv_m)_y = 0, \tag{2.51}$$

where $u_m = \int_0^1 \tilde{u} \, \mathrm{d}\zeta$ and $v_m = \int_0^1 \tilde{v} \, \mathrm{d}\zeta$ are the mean velocities in the x and y directions respectively. This mass conservation equation is identical to the corresponding equation in the standard shallow water equations.

Mapping of the Momentum Equations Next we map the conservation of momentum equations. Again we multiply by h,

$$hu_t + h\left(u^2\right)_x + h(uv)_y + h(uw)_z + \frac{1}{\rho} hp_x = \frac{1}{\rho} h(\sigma_{xz})_z + ghe_x \tag{2.52}$$

and transform from z to ζ,

$$(h\tilde{u})_t + \left(h\tilde{u}^2\right)_x + (h\tilde{u}\tilde{v})_y + \left(\tilde{u}\left(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\right)\right)_\zeta \tag{2.53}$$

$$+\frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta = \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + ghe_x. \tag{2.54}$$

The hydrostatic pressure can be mapped onto ζ as

$$\tilde{p}(t,x,y,\zeta) = h(t,x,y)(1-\zeta)\rho g e_z, \tag{2.55}$$

and then the pressure terms in the momentum equation can be simplified in the following way,

$$\frac{1}{\rho}(h\tilde{p})_x - \frac{1}{\rho}((\zeta h + h_b)_x \tilde{p})_\zeta = \left(\frac{1}{2} h^2 g e_z\right)_x + (h_b)_x h g e_z. \tag{2.56}$$

The resulting momentum balance equation is

$$(h\tilde{u})_t + \left(h\tilde{u}^2 + \frac{1}{2}h^2 g e_z\right)_x + (h\tilde{u}\tilde{v})_y + \left(\tilde{u}\left(\tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}\right)\right)_\zeta \tag{2.57}$$

$$= \frac{1}{\rho}(\tilde{\sigma}_{xz})_\zeta + gh(e_x - (h_b)_x e_z) \tag{2.58}$$

Next we consider the vertical coupling term, ω

$$\omega = \tilde{w} - (\zeta h + h_b)_t - (\zeta h + h_b)_x \tilde{u} - (\zeta h + h_b)_y \tilde{v}, \tag{2.59}$$

Using the expression for vertical velocity in (2.50), we find that,

$$\omega = -\left(h \int_0^\zeta \tilde{u}\,\mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}\,\mathrm{d}\zeta'\right)_y - \zeta h_t, \tag{2.60}$$

and then using (2.51), the vertical coupling becomes

$$\omega = -\left(h \int_0^\zeta \tilde{u}_d\,\mathrm{d}\zeta'\right)_x - \left(h \int_0^\zeta \tilde{v}_d\,\mathrm{d}\zeta'\right)_y, \tag{2.61}$$

where

$$\tilde{u}_d = \tilde{u} - u_m \quad \tilde{v}_d = \tilde{v} - v_m. \tag{2.62}$$

Chapter3. METHODS AND PROCEDURES

This is the opening paragraph to my thesis which explains in general terms the concepts and hypothesis which will be used in my thesis.

With more general information given here than really necessary.

3.1 Introduction

Here initial concepts and conditions are explained and several hypothesis are mentioned in brief.

obvious what I am saying is true.

3.1.1 Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

rest is obvious.

3.1.1.1 Parts of the hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

3.1.2 Second Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

3.1.2.1 Parts of the second hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

3.2 Criteria Review

Here certain criteria are explained thus eventually leading to a foregone conclusion as can be seen in

Chapter4. RESULTS

This is the opening paragraph to my thesis which explains in general terms the concepts and hypothesis which will be used in my thesis.

With more general information given here than really necessary.

4.1 Introduction

Here initial concepts and conditions are explained and several hypothesis are mentioned in brief.

is few and far between.

4.1.1 Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

it is certain that my hypothesis is true.

4.1.1.1 Parts of the hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

4.1.2 Second Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

4.1.2.1 Parts of the second hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

4.2 Criteria Review

Here certain criteria are explained thus eventually leading to a foregone conclusion.

Chapter5. SUMMARY AND DISCUSSION

This is the opening paragraph to my thesis which explains in general terms the concepts and hypothesis which will be used in my thesis.

With more general information given here than really necessary.

5.1 Introduction

Here initial concepts and conditions are explained and several hypothesis are mentioned in brief.

it is certain that my hypothesis is true.

5.1.1 Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

truly obvious what I am saying is true.

5.1.1.1 Parts of the hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

5.1.2 Second Hypothesis

Here one particular hypothesis is explained in depth and is examined in the light of current literature.

5.1.2.1 Parts of the second hypothesis

Here one particular part of the hypothesis that is currently being explained is examined and particular elements of that part are given careful scrutiny.

5.2 Criteria Review

Here certain criteria are explained thus eventually leading to a foregone conclusion.

5.3 Results And Discussion

Here the results can be inserted

BIBLIOGRAPHY

APPENDIX A. ADDITIONAL MATERIAL

This is now the same as any other chapter except that all sectioning levels below the chapter level must begin with the *-form of a sectioning command.

More stuff

Supplemental material.

APPENDIX B. STATISTICAL RESULTS

This is now the same as any other chapter except that all sectioning levels below the chapter level must begin with the *-form of a sectioning command.

Supplemental Statistics

More stuff.