Local Discontinuous Galerkin Method for Thin Film Diffusion

We would like to solve the 1D thin film diffusion equation with a Discontinuous Galerkin Method. The equation is given as

$$q_t = -\left(q^3 q_{xxx}\right)_x.$$

Local Discontinuous Galerkin Method First rewrite the diffusion equation as a system of first order equations.

$$\begin{aligned}
r &= q_x \\
s &= r_x = q_{xx} \\
u &= q^3 s_x = q^3 q_{xxx} \\
q_t &= -u_x = -\left(q^3 q_{xxx}\right)_x
\end{aligned}$$

The LDG method becomes the process of finding $q_h, r_h, s_h, u_h \in V_h$ in the DG solution space, such that for all test functions $v_h, w_h, y_h, z_h \in V_h$ and for all j the following equations are satisfied

$$\begin{aligned}
\int_{I_j} r_h w_h \,\mathrm{d}x &= \int_{I_j} (q_h)_x w_h \,\mathrm{d}x \\
\int_{I_j} s_h y_h \,\mathrm{d}x &= \int_{I_j} (r_h)_x y_h \,\mathrm{d}x \\
\int_{I_j} u_h z_h \,\mathrm{d}x &= \int_{I_j} q_h^3 (s_h)_x z_h \,\mathrm{d}x \\
\int_{I_j} (q_h)_t v_h \,\mathrm{d}x &= -\int_{I_j} (u_h)_x v_h \,\mathrm{d}x
\end{aligned}$$

After integrating by parts, these equations are

$$\begin{aligned}
\int_{I_j} r_h w_h \,\mathrm{d}x &= \left(\left(\hat{q}_h w_h^-\right)_{j+1/2} - \left(\hat{q}_j w_h^+\right)_{j-1/2}\right) - \int_{I_j} q_h (w_h)_x \,\mathrm{d}x \\
\int_{I_j} s_h y_h \,\mathrm{d}x &= \left(\left(\hat{r}_h y_h^-\right)_{j+1/2} - \left(\hat{r}_j y_h^+\right)_{j-1/2}\right) - \int_{I_j} r_h (y_h)_x \,\mathrm{d}x \\
\int_{I_j} (q_h)_t v_h \,\mathrm{d}x &= -\left(\left(\hat{u}_h v_h^-\right)_{j+1/2} - \left(\hat{u}_h v_h^+\right)_{j-1/2}\right) + \int_{I_j} u_h (v_h)_x \,\mathrm{d}x
\end{aligned}$$

The third equation is trickier and requires integrating by parts twice.

$$\begin{aligned}
\int_{I_j} u_h z_h \,\mathrm{d}x &= \int_{I_j} q_h^3 (r_h)_x z_h \,\mathrm{d}x \\
\int_{I_j} u_h z_h \,\mathrm{d}x &= \left(\left(\hat{r}_h \hat{q_h^3} z_h^-\right)_{j+1/2} - \left(\hat{r}_j \hat{q_h^3} z_h^+\right)_{j-1/2}\right) - \int_{I_j} r_h (q_h^3 z_h)_x \,\mathrm{d}x \\
\int_{I_j} u_h z_h \,\mathrm{d}x &= \left(\left(\hat{r}_h \hat{q_h^3} z_h^-\right)_{j+1/2} - \left(\hat{r}_j \hat{q_h^3} z_h^+\right)_{j-1/2}\right) \\
&\quad - \left(\left(\hat{r}_h \left(q_h^-\right)^3 z_h^-\right)_{j+1/2} - \left(\hat{r}_j \left(q_h^+\right)^3 z_h^+\right)_{j-1/2}\right) + \int_{I_j} (r_h)_x q_h^3 z_h \,\mathrm{d}x
\end{aligned}$$

A common choice of numerical fluxes are the so-called alternating fluxes.

$$\begin{aligned}
\hat{u}_h &= u_h^- \\
\hat{q}_h &= q_h^+ \\
\hat{r}_h &= r_h^- \\
\hat{s}_h &= s_h^+
\end{aligned}$$

or

$$\begin{aligned}
\hat{u}_h &= u_h^+ \\
\hat{q}_h &= q_h^- \\
\hat{r}_h &= r_h^+ \\
\hat{s}_h &= s_h^-
\end{aligned}$$

Implementation If we consider a single cell I_j, do a linear transformation from $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$, and consider specifically the Legendre polynomial basis $\left\{\phi^k(\xi)\right\}$ with the following orthogonality property

$$\frac{1}{2}\int_{-1}^{1} \phi^j(\xi)\phi^k(\xi)\,\mathrm{d}\xi = \delta_{jk}$$

we can form a more concrete LDG method for implementing. The linear transformation can be expressed as

$$x = \frac{\Delta x}{2}\xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

or

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

After this tranformation the thin film diffusion equation become

$$u_t = -\frac{16}{\Delta x^4}\left(u^3 u_{\xi\xi\xi}\right)_\xi$$

on the cell I_j. We can then write this as the following system of first order equations.

$$\begin{aligned}
r &= \frac{2}{\Delta x} q_\xi \\
s &= \frac{2}{\Delta x} r_\xi = \frac{4}{\Delta x^2} q_{\xi\xi} \\
u &= \frac{2}{\Delta x} q^3 s_\xi = \frac{8}{\Delta x^3} q^3 q_{\xi\xi\xi} \\
q_t &= -\frac{2}{\Delta x} u_\xi = -\frac{16}{\Delta x^4}\left(q^3 q_{\xi\xi\xi}\right)_\xi
\end{aligned}$$

With the Legendre basis, the numerical solution on I_j can be written as

$$\begin{aligned}
q \approx q_h &= \sum_{k=1}^{M}\left(Q_k \phi^k(\xi)\right) \\
r \approx r_h &= \sum_{k=1}^{M}\left(R_k \phi^k(\xi)\right) \\
s \approx s_h &= \sum_{k=1}^{M}\left(S_k \phi^k(\xi)\right) \\
u \approx u_h &= \sum_{k=1}^{M}\left(U_k \phi^k(\xi)\right)
\end{aligned}$$

Now plugging these into the system and multiplying by a Legendre basis and integrating gives.

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\begin{aligned}
q_h &= \frac{2}{\Delta x}(u_h)_\xi \\
\frac{1}{2}\int_{-1}^{1} q_h \phi^l \, \mathrm{d}\xi &= \frac{1}{\Delta x}\int_{-1}^{1} (u_h)_\xi \phi^l \, \mathrm{d}\xi \\
Q_l &= -\frac{1}{\Delta x}\int_{-1}^{1} u_h \phi^l_\xi \, \mathrm{d}\xi + \frac{1}{\Delta x}\Big(u^-_{j+1/2}\phi^l(1) - u^-_{j-1/2}\phi^l(-1)\Big) \\
(u_h)_t &= \frac{2}{\Delta x}(q_h)_\xi \\
\frac{1}{2}\int_{-1}^{1} (u_h)_t \phi^l \, \mathrm{d}\xi &= \frac{1}{\Delta x}\int_{-1}^{1} (q_h)_\xi \phi^l \, \mathrm{d}\xi \\
\dot{U}_l &= -\frac{1}{\Delta x}\int_{-1}^{1} q_h \phi^l_\xi \, \mathrm{d}\xi + \frac{1}{\Delta x}\Big(q^+_{j+1/2}\phi^l(1) - q^+_{j-1/2}\phi^l(-1)\Big)
\end{aligned}
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Now this is a system of ODEs, there are $M \times N$ ODEs if M is the spacial order and N is the number of cells.