Hyperbolicity of Generalized Shallow Water Equations

The one dimensional first order inviscid generalized shallow water equations in primitive variables are given as

$$\begin{bmatrix} h \\ hu \\ hs \end{bmatrix}_t + \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}hs^2 \\ 2hus \end{bmatrix}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} h \\ hu \\ hs \end{bmatrix}_x, \tag{1}$$

where h is the height of the water, u is the mean horizontal velocity, s is linear scaling of the horizontal velocity, and g is the gravitational constant. Often it may be better to use the conserved variables of mass and momentum instead of height and velocity. Let $q_1 = h$ be the mass and $q_2 = hu$ be the momentum of a cross section of the water, and $q_3 = hs$ be the final conserved variable. In the conserved variables the shallow water equations are

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}_t + \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + \frac{1}{2}gq_1^2 + \frac{1}{3}\frac{q_3^2}{q_1} \\ 2\frac{q_2 q_3}{q_1} \end{bmatrix}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{q_2}{q_1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}_x, \tag{2}$$

or in vector form

$$\boldsymbol{q}_t + \boldsymbol{f}(\boldsymbol{q})_x = Q\boldsymbol{q}_x, \tag{3}$$

where

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \boldsymbol{f}(\boldsymbol{q}) = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + \frac{1}{2}gq_1^2 + \frac{1}{3}\frac{q_3^2}{q_1} \\ 2\frac{q_2 q_3}{q_1} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{q_2}{q_1} \end{bmatrix} \tag{4}$$

For smooth solutions this can be expressed in its quasilinear form,

$$\boldsymbol{q}_t + (\boldsymbol{f}'(\boldsymbol{q}) - Q)\boldsymbol{q}_x = \boldsymbol{0}, \tag{5}$$

where $\boldsymbol{f}'(\boldsymbol{q})$ is the Jacobian matrix. A system of balance laws is said to be hyperbolic if the matrix $A = \boldsymbol{f}'(\boldsymbol{q}) - Q$ is diagonalizable with real eigenvalues. To show that this system is hyperbolic we proceed to find the eigenvalues of the matrix A, where the flux Jacobian is

$$\boldsymbol{f}'(\boldsymbol{q}) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{q_2^2}{q_1^2} + gq_1 - \frac{1}{3}\frac{q_3^2}{q_1^2} & 2\frac{q_2}{q_1} & \frac{2}{3}\frac{q_3}{q_1} \\ -2\frac{q_2 q_3}{q_1} & 2\frac{q_3}{q_1} & 2\frac{q_2}{q_1} \end{bmatrix} \tag{6}$$

or in primitive variables

$$\boldsymbol{f}'(\boldsymbol{q}) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh - \frac{1}{3}s^2 & 2u & \frac{2}{3}s \\ -2us & 2s & 2u \end{bmatrix} \tag{7}$$

So the matrix A is thus

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{q_2^2}{q_1^2} + gq_1 - \frac{1}{3}\frac{q_3^2}{q_1^2} & 2\frac{q_2}{q_1} & \frac{2}{3}\frac{q_3}{q_1} \\ -2\frac{q_2 q_3}{q_1} & 2\frac{q_3}{q_1} & \frac{q_2}{q_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh - \frac{1}{3}s^2 & 2u & \frac{2}{3}s \\ -2us & 2s & u \end{bmatrix}. \tag{8}$$

The eigenvalues, λ of A satisfy $\det(A-\lambda I)=0$. This equation can be solved as follows,

$$\begin{aligned}
\det(A-\lambda I) &= 0\\
\begin{vmatrix} -\lambda & 1 & 0\\ -u^2+gh-\frac{1}{3}s^2 & 2u-\lambda & \frac{2}{3}s\\ -2us & 2s & u-\lambda \end{vmatrix} &= 0\\
\begin{vmatrix} -\lambda & 1 & 0\\ -u^2+gh-\frac{1}{3}s^2 & 2u-\lambda & \frac{2}{3}s\\ -2us & 2s & u-\lambda \end{vmatrix} &= 0\\
-\lambda\left((2u-\lambda)(u-\lambda)-\frac{4}{3}s^2\right)-\left(\left(-u^2+gh-\frac{1}{3}s^2\right)(u-\lambda)+\frac{4}{3}s^2u\right) &= 0\\
(u-\lambda)\lambda(\lambda-2u)+\frac{4}{3}s^2\lambda-\frac{4}{3}s^2u+(u-\lambda)\left(u^2-gh+\frac{1}{3}s^2\right) &= 0\\
(u-\lambda)\left(\lambda(\lambda-2u)-\frac{4}{3}s^2+u^2-gh+\frac{1}{3}s^2\right) &= 0\\
(u-\lambda)\left(\lambda^2-2u\lambda+u^2-gh-s^2\right) &= 0
\end{aligned}$$

Thus the eigenvalues of A are

$$\begin{aligned}
\lambda &= u\\
\lambda &= \frac{2u\pm\sqrt{4u^2-4(u^2-gh-s^2)}}{2}\\
\lambda &= u\pm\sqrt{gh+s^2}
\end{aligned}$$