Derivation of Shallow Water Moment Equations in Spherical Coordinates

First I will define the transformation from cartesian $\boldsymbol{x} = [x, y, z]$ to spherical coordinates, $\boldsymbol{r} = [r, \theta, \phi]$

$$r = s_1(x, y, z) = \sqrt{x^2 + y^2 + z^2} \tag{1}$$
$$\theta = s_2(x, y, z) = \arctan\left(\frac{y}{x}\right) \tag{2}$$
$$\phi = s_3(x, y, z) = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{3}$$
$$\boldsymbol{s}(\boldsymbol{x}) = \boldsymbol{s}(x, y, z) = [s_1(x, y, z), s_2(x, y, z), s_3(x, y, z)] \tag{4}$$
$$x = c_1(r, \theta, \phi) = r\cos(\theta)\sin(\phi) \tag{5}$$
$$y = c_2(r, \theta, \phi) = r\sin(\theta)\sin(\phi) \tag{6}$$
$$z = c_3(r, \theta, \phi) = r\cos(\phi) \tag{7}$$
$$\boldsymbol{c}(\boldsymbol{r}) = \boldsymbol{c}(r, \theta, \phi) = [c_1(r, \theta, \phi), c_2(r, \theta, \phi), c_3(r, \theta, \phi)] \tag{8}$$

The projections from cartesian to spherical coordinates can be computed using the following unit vectors

$$\hat{r} = \cos(\theta)\sin(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\phi)\hat{z} \tag{9}$$
$$\hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y} \tag{10}$$
$$\hat{\phi} = \cos(\theta)\cos(\phi)\hat{x} + \sin(\theta)\cos(\phi)\hat{y} - \sin(\phi)\hat{z} \tag{11}$$
$$\hat{x} = \cos(\theta)\sin(\phi)\hat{r} - \sin(\theta)\hat{\theta} + \cos(\theta)\cos(\phi)\hat{\phi} \tag{12}$$
$$\hat{y} = \sin(\theta)\sin(\phi)\hat{r} + \cos(\theta)\hat{\theta} + \sin(\theta)\cos(\phi)\hat{\phi} \tag{13}$$
$$\hat{z} = \cos(\phi)\hat{r} - \sin(\phi)\hat{\phi} \tag{14}$$
$$\tag{15}$$

The partial derivatives of the transformation to spherical coordinates are needed, when considering differential equations.

$$s_{1,x} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2x = \cos(\theta)\sin(\phi) \tag{16}$$
$$s_{1,y} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2y = \sin(\theta)\sin(\phi) \tag{17}$$
$$s_{1,z} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}2z = \cos(\phi) \tag{18}$$
$$\tag{19}$$

$$s_{2,x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) \tag{20}$$

$$= \frac{1}{1+\frac{r^2\sin^2(\theta)\sin^2(\phi)}{r^2\cos^2(\theta)\sin^2(\phi)}}\left(-\frac{r\sin(\theta)\sin(\phi)}{r^2\cos^2(\theta)\sin^2(\phi)}\right) \tag{21}$$

$$= \frac{1}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}}\left(-\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)}\right) \tag{22}$$

$$= -\frac{1}{1+\frac{1-\cos^2(\theta)}{\cos^2(\theta)}}\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)} \tag{23}$$

$$= -\cos^2(\theta)\frac{\sin(\theta)}{r\cos^2(\theta)\sin(\phi)} \tag{24}$$

$$= -\frac{\sin(\theta)}{r\sin(\phi)} \tag{25}$$

$$s_{2,y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) \tag{26}$$

$$= \cos^2(\theta)\frac{1}{r\cos(\theta)\sin(\phi)} \tag{27}$$

$$= \frac{\cos(\theta)}{r\sin(\phi)} \tag{28}$$

$$s_{2,z} = 0 \tag{29}$$

$$\tag{30}$$

$$s_{3,x} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{1}{2}\frac{2x}{z\sqrt{x^2+y^2}} \tag{31}$$

$$= \frac{1}{1+\frac{r^2\cos^2(\theta)\sin^2(\phi)+r^2\sin^2(\theta)\sin^2(\phi)}{r^2\cos^2(\phi)}}\frac{r\cos(\theta)\sin(\phi)}{r\cos(\phi)\sqrt{r^2\cos^2(\theta)\sin^2(\phi)+r^2\sin^2(\theta)\sin^2(\phi)}} \tag{32}$$

$$= \frac{1}{1+\frac{\sin^2(\phi)}{\cos^2(\phi)}}\frac{\cos(\theta)}{r\cos(\phi)} \tag{33}$$

$$= \frac{1}{1+\frac{1-\cos^2(\phi)}{\cos^2(\phi)}}\frac{\cos(\theta)}{r\cos(\phi)} \tag{34}$$

$$= \cos^2(\phi)\frac{\cos(\theta)}{r\cos(\phi)} \tag{35}$$

$$= \frac{\cos(\theta)\cos(\phi)}{r} \tag{36}$$

$$s_{3,y} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{1}{2}\frac{2y}{z\sqrt{x^2+y^2}} \tag{37}$$

$$= \cos^2(\phi)\frac{r\sin(\theta)\sin(\phi)}{r\cos(\phi)r\sin(\phi)} \tag{38}$$

$$= \frac{\sin(\theta)\cos(\phi)}{r} \tag{39}$$

$$s_{3,z} = -\frac{1}{1+\frac{x^2+y^2}{z^2}}\frac{\sqrt{x^2+y^2}}{z^2} \tag{40}$$

$$= -\cos^2(\phi)\frac{r\sin(\phi)}{r^2\cos^2(\phi)} \tag{41}$$

$$= -\frac{\sin(\phi)}{r} \tag{42}$$

$$\tag{43}$$

The derivative with respect to a cartesian coordinate, can be expressed in terms of the spherical coordinate derivatives using the chain rule. Consider a function of spherical coordinates, $f(r,\theta,\phi)$, then

$$\frac{\partial}{\partial t}(f(\boldsymbol{s}(\boldsymbol{x}))) = \frac{\partial f}{\partial r}\frac{\partial s_1}{\partial t} + \frac{\partial f}{\partial \theta}\frac{\partial s_2}{\partial t} + \frac{\partial f}{\partial \phi}\frac{\partial s_3}{\partial t} \tag{44}$$

where $t = x, y, \mathrm{or} z$

Now let's consider the Navier Stokes equation The velocities in cartesian coordinates are given by $[u, v, w] = \boldsymbol{u}$, then the cartesian Navier Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{45}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^2\right) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + g_x \tag{46}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}\left(v^2\right) + \frac{\partial}{\partial z}(vw) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + g_y \tag{47}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}\left(w^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + g_z \tag{48}$$

$$\tag{49}$$

In spherical coordinates we represent the velocities as $[u_r, u_\theta, u_\phi] = \boldsymbol{u}_r$. Using the unit vectors we can write these velocities in terms of the cartesian velocities.

$$u_r = \cos(\theta)\sin(\phi)u + \sin(\theta)\sin(\phi)v + \cos(\phi)w \tag{50}$$

$$u_\theta = -\sin(\theta)u + \cos(\theta)w \tag{51}$$

$$u_\phi = \cos(\theta)\cos(\phi)u + \sin(\theta)\cos(\phi)v - \sin(\phi)w \tag{52}$$

We can also write the cartesian velocities in terms of the spherical velocities.

$$u = \cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi \tag{53}$$

$$v = \sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi \tag{54}$$

$$w = \cos(\phi)u_r - \sin(\phi)u_\phi \tag{55}$$

Now we can express the continuity equation in spherical coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{56}$$

$$\frac{\partial}{\partial x}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi) \tag{57}$$

$$+\frac{\partial}{\partial y}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi) \tag{58}$$

$$+\frac{\partial}{\partial z}(\cos(\phi)u_r - \sin(\phi)u_\phi) = 0 \tag{59}$$

$$\tag{60}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial x} \tag{61}$$
$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial x} \tag{62}$$
$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial x} \tag{63}$$
$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_1}{\partial y} \tag{64}$$
$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_2}{\partial y} \tag{65}$$
$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\partial s_3}{\partial y} \tag{66}$$
$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_1}{\partial z} \tag{67}$$
$$+\frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_2}{\partial z} \tag{68}$$
$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{\partial s_3}{\partial z} = 0 \tag{69}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \tag{70}$$
$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \tag{71}$$
$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \tag{72}$$
$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \tag{73}$$
$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \tag{74}$$
$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \tag{75}$$
$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \tag{76}$$
$$+\frac{\partial}{\partial \theta}(\cos(\phi)u_r - \sin(\phi)u_\phi)0 \tag{77}$$
$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \tag{78}$$
$$\tag{79}$$

$$\frac{\partial}{\partial r}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\cos(\theta)\sin(\phi) \tag{80}$$

$$\frac{\partial}{\partial \theta}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{-\sin(\theta)}{r\sin(\phi)} \tag{81}$$

$$\frac{\partial}{\partial \phi}(\cos(\theta)\sin(\phi)u_r - \sin(\theta)u_\theta + \cos(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)\cos(\phi)}{r} \tag{82}$$

$$+\frac{\partial}{\partial r}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\sin(\theta)\sin(\phi) \tag{83}$$

$$+\frac{\partial}{\partial \theta}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\cos(\theta)}{r\sin(\phi)} \tag{84}$$

$$+\frac{\partial}{\partial \phi}(\sin(\theta)\sin(\phi)u_r + \cos(\theta)u_\theta + \sin(\theta)\cos(\phi)u_\phi)\frac{\sin(\theta)\cos(\phi)}{r} \tag{85}$$

$$+\frac{\partial}{\partial r}(\cos(\phi)u_r - \sin(\phi)u_\phi)\cos(\phi) \tag{86}$$

$$+\frac{\partial}{\partial \phi}(\cos(\phi)u_r - \sin(\phi)u_\phi)\frac{-\sin(\phi)}{r} = 0 \tag{87}$$

$$\tag{88}$$

Now I will consider the terms of each derivative of each velocity individually,

$\frac{\partial u_r}{\partial r}$

$$\cos^2(\theta)\sin^2(\phi)\frac{\partial u_r}{\partial r} + \sin^2(\theta)\sin^2(\phi)\frac{\partial u_r}{\partial r} + \cos^2(\phi)\frac{\partial u_r}{\partial r} = \sin^2(\phi)\frac{\partial u_r}{\partial r} + \cos^2(\phi)\frac{\partial u_r}{\partial r} = \frac{\partial u_r}{\partial r} \tag{89}$$

$\frac{\partial u_\theta}{\partial r}$

$$-\sin(\theta)\cos(\theta)\sin(\phi)\frac{\partial u_\theta}{\partial r} + \cos(\theta)\sin(\theta)\sin(\phi)\frac{\partial u_\theta}{\partial r} = 0 \tag{90}$$

$\frac{\partial u_\phi}{\partial r}$

$$\cos^2(\theta)\cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} + \sin^2(\theta)\cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} - \sin(\phi)\cos(\phi)\frac{\partial u_\phi}{\partial r} \tag{91}$$

$$= \cos(\phi)\sin(\phi)\frac{\partial u_\phi}{\partial r} - \sin(\phi)\cos(\phi)\frac{\partial u_\phi}{\partial r} = 0 \tag{92}$$

$\frac{\partial u_r}{\partial \theta}$

$$-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}(\cos(\theta)u_r) + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}(\sin(\theta)u_r) \tag{93}$$

$$= -\frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial u_r}{\partial \theta} + \frac{\sin^2(\theta)}{r}u_r + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial u_r}{\partial \theta} + \frac{\cos^2(\theta)}{r}u_r = \frac{u_r}{r} \tag{94}$$

$\frac{\partial u_\theta}{\partial \theta}$

$$\frac{\sin(\theta)}{r\sin(\phi)}\frac{\partial}{\partial \theta}(\sin(\theta)u_\theta) + \frac{\cos(\theta)}{r\sin(\phi)}\frac{\partial}{\partial \theta}(\cos(\theta)u_\theta) = \tag{95}$$

$$\frac{\sin(\theta)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\theta}{\partial \theta} + \cos(\theta)u_\theta\right) + \frac{\cos(\theta)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\theta}{\partial \theta} - \sin(\theta)u_\theta\right) = \tag{96}$$

$$\frac{\sin^2(\theta)}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{\cos^2(\theta)}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} \tag{97}$$

$\dfrac{\partial u_\phi}{\partial \theta}$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\cos(\theta)u_\phi) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\frac{\partial}{\partial\theta}(\sin(\theta)u_\phi) = \tag{98}$$

$$-\frac{\sin(\theta)\cos(\phi)}{r\sin(\phi)}\left(\cos(\theta)\frac{\partial u_\phi}{\partial\theta} - \sin(\theta)u_\phi\right) + \frac{\cos(\theta)\cos(\phi)}{r\sin(\phi)}\left(\sin(\theta)\frac{\partial u_\phi}{\partial\theta} + \cos(\theta)u_\phi\right) = \tag{99}$$

$$\frac{\sin^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{\cos^2(\theta)\cos(\phi)}{r\sin(\phi)}u_\phi = \frac{\cos(\phi)}{r\sin(\phi)}u_\phi \tag{100}$$

$\dfrac{\partial u_r}{\partial \phi}$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \tag{101}$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_r) - \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_r) = \tag{102}$$

$$\frac{\cos(\phi)}{r}\left(\sin(\phi)\frac{\partial u_r}{\partial\phi} + \cos(\phi)u_r\right) - \frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial u_r}{\partial\phi} - \sin(\phi)u_r\right) = \tag{103}$$

$$\frac{\cos^2(\phi)}{r}u_r + \frac{\sin^2(\phi)}{r}u_r = \frac{1}{r}u_r \tag{104}$$

$\dfrac{\partial u_\theta}{\partial \phi}$

$$-\frac{\cos(\theta)\sin(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} + \frac{\sin(\theta)\cos(\theta)\cos(\phi)}{r}\frac{\partial u_\theta}{\partial\phi} = 0 \tag{105}$$

$\dfrac{\partial u_\phi}{\partial \phi}$

$$\frac{\cos^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin^2(\theta)\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \tag{106}$$

$$\frac{\cos(\phi)}{r}\frac{\partial}{\partial\phi}(\cos(\phi)u_\phi) + \frac{\sin(\phi)}{r}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = \tag{107}$$

$$\frac{\cos(\phi)}{r}\left(\cos(\phi)\frac{\partial u_\phi}{\partial\phi} - \sin(\phi)u_\phi\right) + \frac{\sin(\phi)}{r}\left(\sin(\phi)\frac{\partial u_\phi}{\partial\phi} + \cos(\phi)u_\phi\right) = \tag{108}$$

$$\frac{\cos^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\sin^2(\phi)}{r}\frac{\partial u_\phi}{\partial\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} \tag{109}$$

The simplified continuity equation is thus

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} = 0 \tag{110}$$

$$\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\cos(\phi)}{r\sin(\phi)}u_\phi = 0 \tag{111}$$

Using the product rule this can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) = 0 \tag{112}$$

$$\tag{113}$$

Shallow Water in Spherical Coordinates R radius, (θ, ϕ), longitude and latitude

$$h_t + \frac{1}{R\cos(\phi)}(hu_\theta)_\theta + \frac{1}{R\cos(\phi)}(hu_\phi\cos(\phi))_\phi = 0 \tag{114}$$

$$(hu_\theta)_t + \frac{1}{R\cos(\phi)}\left(hu_\theta^2 + \frac{1}{2}gh^2\right)_\theta + \frac{1}{R}(hu_\theta u_\phi) - 2\frac{hu_\theta u_\phi}{R}\tan(\phi) = 0 \tag{115}$$

$$(hu_\phi)_\phi + \frac{1}{R\cos(\phi)}(hu_\theta u_\phi)_\theta + \frac{1}{R}\left(hu_\phi^2 + \frac{1}{2}gh^2\right)_\phi + \frac{hu_\theta^2 - hu_\phi^2}{R}\tan(\phi) = 0 \tag{116}$$

Navier Stokes Equations in Spherical Coordinates r radius, (θ, ϕ) azimuth, and polar angle, $\theta = \arctan(y/x), \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)u_\phi) = 0 \tag{117}$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r \tag{118}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} + \frac{u_\theta u_\phi\cot(\phi)}{r} = -\frac{1}{\rho r\sin(\phi)}\frac{\partial p}{\partial \theta} + g_\theta \tag{119}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r\sin(\phi)}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} - \frac{u_\theta^2\cot(\phi)}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + g_\phi \tag{120}$$

Kinematic Boundary Condition, $h_s(t,\theta,\phi)$, $h_b(t,\theta,\phi)$ $u_{\theta/\phi/r}(t,\theta,\phi,r)$

$$\frac{\partial h_s}{\partial t} + u_\theta(t,\theta,\phi,h_s)\frac{\partial h_s}{\partial \theta} + u_\phi(t,\theta,\phi,h_s)\frac{\partial h_s}{\partial \phi} = u_r(t,\theta,\phi,h_s) \tag{121}$$

$$\frac{\partial h_b}{\partial t} + u_\theta(t,\theta,\phi,h_b)\frac{\partial h_b}{\partial \theta} + u_\phi(t,\theta,\phi,h_b)\frac{\partial h_b}{\partial \phi} = u_r(t,\theta,\phi,h_b) \tag{122}$$

$$\tag{123}$$

Dimensional Analysis

$$r = R\hat{r} \quad h = H\hat{h} \quad \frac{H}{R} = \epsilon$$

$$u_\theta = U\hat{u}_\theta \quad u_\phi = U\hat{u}_\phi \quad u_r = U_r\hat{u}_r$$

$$t = T\hat{t} = \frac{R}{U}\hat{t} \quad p = \rho g H\hat{p}$$

$$\frac{1}{R^2\hat{r}^2}\frac{1}{R}\frac{\partial}{\partial \hat{r}}\left(R^2\hat{r}^2\epsilon U\hat{u}_r\right) + \frac{1}{R\hat{r}\sin(\phi)}\frac{\partial U\hat{u}_\theta}{\partial \theta} + \frac{1}{R\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)U\hat{u}_\phi) = 0 \tag{124}$$

$$\frac{U_r}{T}\frac{\partial \hat{u}_r}{\partial \hat{t}} + U_r\hat{u}_r\frac{U_r}{R}\frac{\partial \hat{u}_r}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U_r\frac{\partial \hat{u}_r}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U_r\frac{\partial \hat{u}_r}{\partial \phi} - \frac{U^2}{R}\frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\frac{1}{\rho}\rho g H\frac{1}{R}\frac{\partial \hat{p}}{\partial \hat{r}} + g_r \tag{125}$$

$$\frac{U}{T}\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + U_r\hat{u}_r\frac{U}{R}\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U\frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{U_r\hat{u}_r U\hat{u}_\theta}{R\hat{r}} + \frac{U^2\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{R\hat{r}} = -\frac{1}{\rho R\hat{r}\sin(\phi)}\rho g H\frac{\partial \hat{p}}{\partial \theta} + g_\theta \tag{126}$$

$$\frac{U}{T}\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + U_r\hat{u}_r\frac{U}{R}\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{U\hat{u}_\theta}{R\hat{r}\sin(\phi)}U\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{U\hat{u}_\phi}{R\hat{r}}U\frac{\partial \hat{u}_\phi}{\partial \phi} + \frac{U_r U\hat{u}_r\hat{u}_\phi}{R\hat{r}} - \frac{U^2\hat{u}_\theta^2\cot(\phi)}{R\hat{r}} = -\frac{1}{\rho R\hat{r}}\rho g H\frac{\partial \hat{p}}{\partial \phi} + g_\phi \tag{127}$$

$T = \frac{R}{U}$, $U_r = \epsilon U$

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2\hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{128}$$

$$\frac{U^2}{R}\left(\epsilon\frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2\hat{u}_r\frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon\frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_r}{\partial \theta} + \epsilon\frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r}\right) = -g\frac{H}{R}\frac{\partial \hat{p}}{\partial \hat{r}} + ge_r \tag{129}$$

$$\frac{U^2}{R}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{gH}{R\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + ge_\theta \tag{130}$$

$$\frac{U^2}{R}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{gH}{R\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + ge_\phi \tag{131}$$

$R = \frac{H}{\epsilon}$

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2\hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{132}$$

$$\epsilon\frac{U^2}{gH}\left(\epsilon\frac{\partial \hat{u}_r}{\partial \hat{t}} + \epsilon^2\hat{u}_r\frac{\partial \hat{u}_r}{\partial \hat{r}} + \epsilon\frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_r}{\partial \theta} + \epsilon\frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_r}{\partial \phi} - \frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r}\right) = -\epsilon\frac{\partial \hat{p}}{\partial \hat{r}} + e_r \tag{133}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\theta}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\theta}{\hat{r}} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + e_\theta \tag{134}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \epsilon\hat{u}_r\frac{\partial \hat{u}_\phi}{\partial \hat{r}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} + \epsilon\frac{\hat{u}_r\hat{u}_\phi}{\hat{r}} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + e_\phi \tag{135}$$

Drop ϵ^2 terms

$$\frac{\epsilon}{\hat{r}^2}\frac{\partial}{\partial \hat{r}}\left(\hat{r}^2\hat{u}_r\right) + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{1}{\hat{r}\sin(\phi)}\frac{\partial}{\partial \phi}(\sin(\phi)\hat{u}_\phi) = 0 \tag{136}$$

$$-\epsilon\frac{U^2}{gH}\frac{\hat{u}_\theta^2 + \hat{u}_\phi^2}{r} = -\epsilon\frac{\partial \hat{p}}{\partial \hat{r}} + e_r \tag{137}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\theta}{\partial \phi} + \frac{\hat{u}_\theta\hat{u}_\phi\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}\sin(\phi)}\frac{\partial \hat{p}}{\partial \theta} + e_\theta \tag{138}$$

$$\epsilon\frac{U^2}{gH}\left(\frac{\partial \hat{u}_\phi}{\partial \hat{t}} + \frac{\hat{u}_\theta}{\hat{r}\sin(\phi)}\frac{\partial \hat{u}_\phi}{\partial \theta} + \frac{\hat{u}_\phi}{\hat{r}}\frac{\partial \hat{u}_\phi}{\partial \phi} - \frac{\hat{u}_\theta^2\cot(\phi)}{\hat{r}}\right) = -\frac{\epsilon}{\hat{r}}\frac{\partial \hat{p}}{\partial \phi} + e_\phi \tag{139}$$