# Derivation of Shallow Water Equations

We begin by considering the Navier-Stokes equations,

$$\div \boldsymbol{u} = 0 \tag{1}$$

$$\boldsymbol{u}_t + \div * \boldsymbol{u}\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho} \div \sigma + \boldsymbol{g}, \tag{2}$$

j where $\boldsymbol{u} = [u, v, w]^T$ is the vector of velocities, $p$ is the pressure, $\rho$ is the constant density, $\sigma$ is the deviatoric stress tensor, and $\boldsymbol{g}$ is the gravitational force vector. We also have two boundaries, the bottom topography $h_b(t, x, y)$, and the free surface $h_s(t, x, y)$. At both of these boundaries the kinimatic boundary conditions are in effect and can be expressed as

$$(h_s)_t + [u(t, x, y, h_s), v(t, x, y, h_s)]^T \cdot \nabla h_s = w(t, x, y, h_s) \tag{3}$$

$$(h_b)_t + [u(t, x, y, h_b), v(t, x, y, h_b)]^T \cdot \nabla h_b = w(t, x, y, h_b). \tag{4}$$

In practice the bottom topography is unchanging in time, but we express $h_b$ with time dependence to allow for a symmetric representation of the boundary conditions.

## 1 Dimensional Analysis

Now we consider the characteristic scales of the problem. Let $L$ be the characteristic horizontal length scale, and let $H$ be the characteristic vertical length scale. For this problem we assume that $H << L$ and we denote the ratio of these lengths as $\varepsilon = H/L$. With these characteristic lengths we can scale the length variables to a nondimensional form

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = H\hat{z}. \tag{5}$$

Now let $U$ be the characteristic horizontal velocity, then because of the shallowness the characteristic vertical velocity will be $\varepsilon U$. Therefore the velocity variables can be scaled as follows,

$$u = U\hat{u}, \quad v = U\hat{v}, \quad w = \varepsilon U\hat{w}. \tag{6}$$

Now with the characteristic length and velocity, the time scaling can be described as

$$t = \frac{L}{U}\hat{t} \tag{7}$$

The pressure will be scaled by the characteristic height, $H$, and the stresses will be scaled by a characteristic stress, $S$. It is assumed that the basal shear stresses, $\sigma_{xz}$ and $\sigma_{yz}$ are of larger order than the lateral shear stress, $\sigma_{xy}$, and the normal stresses, $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{zz}$, so that

$$p = \rho g H\hat{p}, \quad \sigma_{xz/yz} = S\hat{\sigma}_{xz/yz}, \quad \sigma_{xx/xy/yy/zz} = \varepsilon S\hat{\sigma}_{xx/xy/yy/zz}. \tag{8}$$

$$\begin{aligned}
\div \boldsymbol{u} &= 0 \\
\boldsymbol{u}_t + \div * \boldsymbol{u}\boldsymbol{u} &= -\frac{1}{\rho}\nabla p + \frac{1}{\rho} \div \sigma + \boldsymbol{g}, u_x + v_y + w_z && = 0 \\
u_t + \left(u^2\right)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho}p_x + \frac{1}{\rho}\Big((\sigma_{xx})_x + (\sigma_{xy})_y + (\sigma_{xz})_z\Big) + ge_x \\
v_t + (uv)_x + \left(v^2\right)_y + (vw)_z &= -\frac{1}{\rho}p_y + \frac{1}{\rho}\Big((\sigma_{xy})_x + (\sigma_{yy})_y + (\sigma_{yz})_z\Big) + ge_y \\
w_t + (uw)_x + (vw)_y + \left(w^2\right)_z &= -\frac{1}{\rho}p_z + \frac{1}{\rho}\Big((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\Big) + ge_z
\end{aligned}$$

$$
\begin{aligned}
u_x &= (U\hat{u})_x = (U\hat{u})_{\hat{x}}\hat{x}_x = \frac{U}{L}\hat{u}_{\hat{x}} \\
v_y &= (U\hat{v})_{\hat{y}}\hat{y}_y = \frac{U}{L}\hat{v}_{\hat{y}} \\
w_z &= (\varepsilon U\hat{w})_{\hat{z}}\hat{z}_z = \frac{\varepsilon U}{H}\hat{w}_{\hat{z}} = \frac{U}{L}\hat{w}_{\hat{z}}
\end{aligned}
$$

$$
u_x + v_y + w_z = \frac{U}{L}\hat{u}_{\hat{x}} + \frac{U}{L}\hat{v}_{\hat{y}} + \frac{U}{L}\hat{w}_{\hat{z}} = 0
$$

$$
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} = 0
$$

$$
\begin{aligned}
u_t &= (U\hat{u})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{u}_{\hat{t}} \\
v_t &= (U\hat{v})_{\hat{t}}\hat{t}_t = \frac{U^2}{L}\hat{v}_{\hat{t}} \\
w_t &= (\varepsilon U\hat{w})_{\hat{t}}\hat{t}_t = \varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}} \\
\left(u^2\right)_x &= \left(U^2\hat{u}^2\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}\left(\hat{u}^2\right)_{\hat{x}} \\
(uv)_y &= \left(U^2\hat{u}\hat{v}\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}} \\
(uw)_z &= \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}} \\
(uv)_x &= \left(U^2\hat{u}\hat{v}\right)_{\hat{x}}\hat{x}_x = \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}} \\
\left(v^2\right)_y &= \left(U^2\hat{v}^2\right)_{\hat{y}}\hat{y}_y = \frac{U^2}{L}\left(\hat{v}^2\right)_{\hat{y}} \\
(vw)_z &= \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{z}}\hat{z}_z = \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}} \\
(uw)_x &= \left(\varepsilon U^2\hat{u}\hat{w}\right)_{\hat{x}}\hat{x}_x = \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}} \\
(vw)_y &= \left(\varepsilon U^2\hat{v}\hat{w}\right)_{\hat{y}}\hat{y}_y = \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}} \\
\left(w^2\right)_z &= \left(\varepsilon^2U^2\hat{w}^2\right)_{\hat{z}}\hat{z}_z = \varepsilon\frac{U^2}{L}\left(\hat{w}^2\right)_{\hat{z}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\rho}p_x &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{x}}\hat{x}_x = \frac{\varepsilon g L}{L}\hat{p}_{\hat{x}} = \varepsilon g\hat{p}_{\hat{x}}\\
\frac{1}{\rho}p_y &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{y}}\hat{y}_y = \frac{\varepsilon g L}{L}\hat{p}_{\hat{y}} = \varepsilon g\hat{p}_{\hat{y}}\\
\frac{1}{\rho}p_z &= \frac{1}{\rho}(\rho g H\hat{p})_{\hat{z}}\hat{z}_z = \frac{\varepsilon g L}{\varepsilon L}\hat{p}_{\hat{z}} = g\hat{p}_{\hat{z}}\\
\frac{1}{\rho}(\sigma_{xx})_x &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xx})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xx})_{\hat{x}}\\
\frac{1}{\rho}(\sigma_{xy})_y &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{y}}\\
\frac{1}{\rho}(\sigma_{xz})_z &= \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{xz})_{\hat{z}}\\
\frac{1}{\rho}(\sigma_{xy})_x &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{xy})_{\hat{x}}\hat{x}_x = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}}\\
\frac{1}{\rho}(\sigma_{yy})_y &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{yy})_{\hat{y}}\hat{y}_y = \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}}\\
\frac{1}{\rho}(\sigma_{yz})_z &= \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{z}}\hat{z}_z = \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}}\\
\frac{1}{\rho}(\sigma_{xz})_x &= \frac{1}{\rho}(S\hat{\sigma}_{xz})_{\hat{x}}\hat{x}_x = \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}}\\
\frac{1}{\rho}(\sigma_{yz})_y &= \frac{1}{\rho}(S\hat{\sigma}_{yz})_{\hat{y}}\hat{y}_y = \frac{S}{\rho L}(\hat{\sigma}_{yz})_{\hat{y}}\\
\frac{1}{\rho}(\sigma_{zz})_z &= \frac{1}{\rho}(\varepsilon S\hat{\sigma}_{zz})_{\hat{z}}\hat{z}_z = \varepsilon\frac{S}{\rho H}(\hat{\sigma}_{zz})_{\hat{z}}\frac{U^2}{L}\hat{u}_{\hat{t}} + \frac{U^2}{L}\left(\hat{u}^2\right)_{\hat{x}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{y}} + \frac{U^2}{L}(\hat{u}\hat{w})_{\hat{z}} \quad = -\varepsilon g\hat{p}_{\hat{x}}\\
\frac{U^2}{L}\hat{v}_{\hat{t}} + \frac{U^2}{L}(\hat{u}\hat{v})_{\hat{x}} + \frac{U^2}{L}\left(\hat{v}^2\right)_{\hat{y}} + \frac{U^2}{L}(\hat{v}\hat{w})_{\hat{z}} &= -\varepsilon g\hat{p}_{\hat{y}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon\frac{S}{\rho L}(\hat{\sigma}_{yy})_{\hat{y}} + \frac{S}{\rho H}(\hat{\sigma}_{yz})_{\hat{z}} + ge_y\\
\varepsilon\frac{U^2}{L}\hat{w}_{\hat{t}} + \varepsilon\frac{U^2}{L}(\hat{u}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}(\hat{v}\hat{w})_{\hat{x}} + \varepsilon\frac{U^2}{L}\left(\hat{w}^2\right)_{\hat{z}} &= -g\hat{p}_{\hat{z}} + \frac{S}{\rho L}(\hat{\sigma}_{xz})_{\hat{x}}\Big((\sigma_{xz})_x + (\sigma_{yz})_y + (\sigma_{zz})_z\Big) + ge_z
\end{aligned}
$$

Substituting all of these scaled variables into the Navier-Stokes system gives,

$$
\begin{aligned}
\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} + \hat{w}_{\hat{z}} &= 0\\
\frac{U^2}{gL}\Big(\hat{u}_{\hat{t}} + \left(\hat{u}^2\right)_{\hat{x}} + (\hat{u}\hat{v})_{\hat{y}} + (\hat{u}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{x}} + \frac{S}{\rho g L}\Big(\varepsilon(\hat{\sigma}_{xx})_{\hat{x}} + \varepsilon(\hat{\sigma}_{xy})_{\hat{y}} + (\hat{\sigma}_{xz})_{\hat{z}}\Big) + e_x\\
\frac{U^2}{gL}\Big(\hat{v}_{\hat{t}} + (\hat{u}\hat{v})_{\hat{x}} + \left(\hat{v}^2\right)_{\hat{y}} + (\hat{v}\hat{w})_{\hat{z}}\Big) &= -\varepsilon\hat{p}_{\hat{y}} + \frac{S}{\rho g L}\Big(\varepsilon(\hat{\sigma}_{xy})_{\hat{x}} + \varepsilon(\hat{\sigma}_{yy})_{\hat{y}} + (\hat{\sigma}_{yz})_{\hat{z}}\Big) + e_y
\end{aligned}
$$

$$
\begin{aligned}
u_x + v_y + w_z &= 0\\
u_t + \left(u^2\right)_x + (uv)_y + (uw)_z &= -\frac{1}{\rho}p_x + \frac{1}{\rho}(\sigma_{xz})_z + ge_x\\
v_t + (uv)_x + \left(v^2\right)_y + (vw)_z &= -\frac{1}{\rho}p_y + \frac{1}{\rho}(\sigma_{yz})_z + ge_y
\end{aligned}
$$

## 2 Mapping

In order to make this system more accessible we will map the vertical variable $z$ to the normalized variable $\zeta$, through the transformation

$$
\zeta(t,x,y,z) = \frac{z - h_b(t,x,y)}{h(t,x,y)},
$$

or equivalently

$$z(t,x,y,\zeta) = h(t,x,y)\zeta + h_b(t,x,y)$$

where $h(t,x,y) = h_s(t,x,y) - h_b(t,x,y)$. This transformation maps the vertical variable onto a scale from 0 to 1 everywhere.

Consider a function $\Psi(t,x,y,z)$, then is mapped counterpart $\tilde{\Psi}(t,x,y,\zeta)$ can be described as

$$\tilde{\Psi}(t,x,y,\zeta) = \Psi(t,x,y,z(t,x,y,\zeta)) = \Psi(t,x,y,h(t,x,y)\zeta + h_b(t,x,y)),$$

or equivalently

$$\Psi(t,x,y,z) = \tilde{\Psi}(t,x,y,\zeta(t,x,y,z)) = \tilde{\Psi}\left(t,x,y,\frac{z-h_b(t,x,y)}{h(t,x,y)}\right).$$

We also need to be able to map derivatives of functions in order to be able to map the differential equations. This can be described

$$\begin{aligned}
\Psi_z(t,x,y,z) &= \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_z \\
\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\zeta_z(t,x,y,z) \\
\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z))\frac{1}{h(t,x,y)} \\
h(t,x,y)\Psi_z(t,x,y,z) &= \tilde{\Psi}_\zeta(t,z,y,\zeta(t,x,y,z)) \\
h\Psi_z &= \tilde{\Psi}_\zeta
\end{aligned}$$

For the other derivatives, the partial derivatives are identical for $s \in \{t,x,y\}$.

$$\begin{aligned}
\zeta_s(t,x,y,z) &= \left(\frac{z-h_b(t,x,y)}{h(t,x,y)}\right)_s \\
&= -\frac{(z-h_b(t,x,y))h_s(t,x,y)}{h(t,x,y)^2} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \\
&= -\zeta(t,x,y,z)\frac{h_s(t,x,y)}{h(t,x,y)} - \frac{(h_b)_s(t,x,y)}{h(t,x,y)} \\
&= -\frac{\zeta(t,x,y,z)h_s(t,x,y) + (h_b)_s(t,x,y)}{h(t,x,y)}
\end{aligned}$$

$$\begin{aligned}
\Psi_s(t,x,y,z) &= \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_s \\
\Psi_s(t,x,y,z) &= \tilde{\Psi}_s(t,x,y,\zeta(t,x,y,z)) + \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))\zeta_s(t,x,y,z) \\
\Psi_s(t,x,y,z) &= \tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)\left(\frac{\zeta h_s(t,x,y) + (h_b)_s(t,x,y)}{h(t,x,y)}\right) \\
h(t,x,y)\Psi_s(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y) + (h_b)_s(t,x,y)) \\
h(t,x,y)\Psi_s(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_s(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_s(t,x,y) + (h_b)_s(t,x,y)) \\
h\Psi_s &= h\tilde{\Psi}_s - \tilde{\Psi}_\zeta(\zeta h_s + (h_b)_s) \\
h\Psi_s &= h\tilde{\Psi}_s + h_s\tilde{\Psi} - h_s\tilde{\Psi} - \tilde{\Psi}_\zeta(\zeta h + h_b)_s \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(h_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(\left((\zeta h + h_b)_\zeta\right)_s\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left(((\zeta h + h_b)_s)_\zeta\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_s\right) \\
h\Psi_s &= \left(h\tilde{\Psi}\right)_s - \left((\zeta h + h_b)_s\tilde{\Psi}\right)_\zeta
\end{aligned}$$

$$
\begin{aligned}
\Psi_t(t,x,y,z) &= \left(\tilde{\Psi}(t,x,y,\zeta(t,x,y,z))\right)_t \\
\Psi_t(t,x,y,z) &= \tilde{\Psi}_t(t,x,y,\zeta(t,x,y,z)) + \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))\zeta_t(t,x,y,z) \\
\Psi_t(t,x,y,z) &= \tilde{\Psi}_t(t,x,y,\zeta(t,x,y,z)) - \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))\left(\frac{\zeta(t,x,y,z)h_t(t,x,y) + (h_b)_t(t,x,y)}{h(t,x,y)}\right) \\
h(t,x,y)\Psi_t(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_t(t,x,y,\zeta(t,x,y,z)) - \tilde{\Psi}_\zeta(t,x,y,\zeta(t,x,y,z))(\zeta(t,x,y,z)h_t(t,x,y) + (h_b)_t(t,x,y)) \\
h(t,x,y)\Psi_t(t,x,y,z) &= h(t,x,y)\tilde{\Psi}_t(t,x,y,\zeta) - \tilde{\Psi}_\zeta(t,x,y,\zeta)(\zeta h_t(t,x,y) + (h_b)_t(t,x,y)) \\
h\Psi_t &= h\tilde{\Psi}_t - \tilde{\Psi}_\zeta(\zeta h_t + (h_b)_t) \\
h\Psi_t &= h\tilde{\Psi}_t + h_t\tilde{\Psi} - h_t\tilde{\Psi} - \tilde{\Psi}_\zeta(\zeta h + h_b)_t \\
h\Psi_t &= \left(h\tilde{\Psi}\right)_t - \left(h_t\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_t\right) \\
h\Psi_t &= \left(h\tilde{\Psi}\right)_t - \left(\left((\zeta h + h_b)_\zeta\right)_t\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_t\right) \\
h\Psi_t &= \left(h\tilde{\Psi}\right)_t - \left(((\zeta h + h_b)_t)_\zeta\tilde{\Psi} + \tilde{\Psi}_\zeta(\zeta h + h_b)_t\right) \\
h\Psi_t &= \left(h\tilde{\Psi}\right)_t - \left((\zeta h + h_b)_t\tilde{\Psi}\right)_\zeta
\end{aligned}
$$

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
h(u_x + v_y + w_z) &= 0 \\
hu_x + hv_y + hw_z &= 0 \\
(h\tilde{u})_x - ((\zeta h + h_b)_x\tilde{u})_\zeta + (h\tilde{v})_y - \left((\zeta h + h_b)_y\tilde{v}\right)_\zeta + (\tilde{w})_\zeta &= 0 \\
(h\tilde{u})_x + (h\tilde{v})_y + \left(\tilde{w} - (\zeta h + h_b)_x\tilde{u} - (\zeta h + h_b)_y\tilde{v}\right)_\zeta &= 0 \\
-(h\tilde{u})_x - (h\tilde{v})_y + \left((\zeta h + h_b)_x\tilde{u} + (\zeta h + h_b)_y\tilde{v}\right)_\zeta &= (\tilde{w})_\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + \int_0^{\zeta'} \left((\zeta h + h_b)_x\tilde{u} + (\zeta h + h_b)_y\tilde{v}\right)_\zeta \mathrm{d}\zeta &= \int_0^{\zeta'} (\tilde{w})_\zeta \,\mathrm{d}\zeta \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + (\zeta' h + h_b)_x\tilde{u}(t,x,y,\zeta') + (\zeta' h + h_b)_y\tilde{v}(t,x,y,\zeta') & \\
-(h_b)_x\tilde{u}(t,x,y,0) - (h_b)_y\tilde{v}(t,x,y,0) &= \tilde{w}(t,x,y,\zeta') - \tilde{w}(t,x,y,0) \\
-\int_0^{\zeta'} (h\tilde{u})_x \,\mathrm{d}\zeta - \int_0^{\zeta'} (h\tilde{v})_y \,\mathrm{d}\zeta + (\zeta' h + h_b)_x\tilde{u}(t,x,y,\zeta') + (\zeta' h + h_b)_y\tilde{v}(t,x,y,\zeta') &= \tilde{w}(t,x,y,\zeta')
\end{aligned}
$$

Original Kinematic Boundary Conditions

$$
\begin{aligned}
(h_s)_t + [u(t,x,y,h_s), v(t,x,y,h_s)]^T \cdot \nabla h_s &= w(t,x,y,h_s) \\
(h_b)_t + [u(t,x,y,h_b), v(t,x,y,h_b)]^T \cdot \nabla h_b &= w(t,x,y,h_b)
\end{aligned}
$$

Mapped Kinematic Boundary Conditions

$$
\begin{aligned}
(h_s)_t + [\tilde{u}(t,x,y,1), \tilde{v}(t,x,y,1)]^T \cdot \nabla h_s &= \tilde{w}(t,x,y,1) \\
(h_b)_t + [\tilde{u}(t,x,y,0), \tilde{v}(t,x,y,0)]^T \cdot \nabla h_b &= \tilde{w}(t,x,y,0)
\end{aligned}
$$