Generalized Shallow Water Equations

1 Generalized Shallow Water Equations

The generalized shallow water equations describe the movement of fluid when driven by gravity. The height of the fluid is given by h, the velocity profiles u^* and v^* in the x and y directions are approximated with the following Ansatz,

$$u^* = u + \sum_{j=1}^{N} (\alpha_j \phi_j)$$

$$v^* = v + \sum_{j=1}^{N} (\beta_j \phi_j),$$

where ϕ_j are the Legendre polynomials orthogonal on the domain $[0, 1]$, such that $\phi_j(0) = 1$. The first few of these Legendre polynomials are

$$\phi_0(\zeta) = 1, \qquad \phi_1(\zeta) = -2\zeta + 1, \qquad \phi_2(\zeta) = 6\zeta^2 - 6\zeta + 1, \qquad \phi_3(\zeta) = -20\zeta^3 + 30\zeta^2 - 12\zeta + 1.$$

Note that the mean velocities u and v can be expressed as coefficients of the constant moment, ϕ_0. They could be written as α_0 and β_0 respectively, but are given as u and v to match the standard shallow water equations.

The bottom topography is given by h_b, the kinematic viscosity ν, the slip length λ, the gravitational constant g, and the gravity direction $\boldsymbol{e} = [e_x, e_y, e_z]^T$.

The generalized shallow water equations are then given as follows.

$$h_t + (hu)_x + (hv)_x = 0 \tag{1}$$

$$\begin{aligned}(hu)_t + \left(hu^2 + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j^2\right) + \frac{1}{2}ge_zh^2\right)_x + \left(huv + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j\beta_j\right)\right)_y \\ = -\frac{\nu}{\lambda}\left(u + \sum_{j=1}^{N}(\alpha_j)\right) + hge_x - hge_z(h_b)_x\end{aligned} \tag{2}$$

$$\begin{aligned}(hv)_t + \left(huv + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j\beta_j\right)\right)_x + \left(hv^2 + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\beta_j^2\right) + \frac{1}{2}ge_zh^2\right)_y \\ = -\frac{\nu}{\lambda}\left(v + \sum_{j=1}^{N}(\beta_j)\right) + hge_y - hge_z(h_b)_y\end{aligned} \tag{3}$$

$$\begin{aligned}(h\alpha_i)_t + \left(2hu\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\alpha_k)\right)\right)_x + \left(hu\beta_i + hv\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\beta_k)\right)\right)_y \\ = u_mD_i - \sum_{j=1}^{N}\left(D_j\sum_{k=1}^{N}(B_{ijk}\alpha_k)\right) - (2i+1)\frac{\nu}{\lambda}\left(u + \sum_{j=1}^{N}\left(\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right)\right)\end{aligned} \tag{4}$$

$$\begin{aligned}(h\beta_i)_t + \left(hu\beta_i + hv\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\beta_k)\right)\right)_x + \left(2hv\beta_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\beta_j\beta_k)\right)\right)_y \\ = v_mD_i - \sum_{j=1}^{N}\left(D_j\sum_{k=1}^{N}(B_{ijk}\beta_k)\right) - (2i+1)\frac{\nu}{\lambda}\left(v + \sum_{j=1}^{N}\left(\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)\right)\end{aligned} \tag{5}$$

where

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \,\mathrm{d}\zeta \tag{6}$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right)\phi_k \,\mathrm{d}\zeta \tag{7}$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \,\mathrm{d}\zeta \tag{8}$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y \tag{9}$$

2 1D Equations

In one dimension the generalized shallow water equations will have the following form,

$$\boldsymbol{q}_t + \boldsymbol{f}(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{10}$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{11}$$

where the number of components depends on the number of moments in the velocity profiles.

2.1 Zeroth Order

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 \end{pmatrix} \boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu u}{\lambda} \end{pmatrix} \tag{12}$$

2.2 First Order

$$\boldsymbol{f}(\boldsymbol{q}) \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + \frac{1}{3}\,\alpha_1^2 h + hu^2 \\ 2\,\alpha_1 hu \end{pmatrix} \boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{pmatrix}, \tag{13}$$

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1 + u)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\alpha_1 + u)\nu}{\lambda} \end{pmatrix} \tag{14}$$

2.3 Second Order

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 + \frac{1}{15}\,(5\,\alpha_1^2 + 3\,\alpha_2^2)h \\ \frac{4}{5}\,\alpha_1\alpha_2 h + 2\,\alpha_1 hu \\ 2\,\alpha_2 hu + \frac{2}{21}\,(7\,\alpha_1^2 + 3\,\alpha_2^2)h \end{pmatrix}, \tag{15}$$

$$g(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & \frac{1}{5}\,\alpha_1 \\ 0 & 0 & \alpha_1 & \frac{1}{7}\,\alpha_2 + u \end{pmatrix}, \tag{16}$$

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1 + \alpha_2 + u)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\alpha_1 + \alpha_2 + u)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h} + 1)\alpha_2 + \alpha_1 + u)\nu}{\lambda} \end{pmatrix} \tag{17}$$

2.4 Third Order

$$
\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 + \frac{1}{105}\,(35\,\alpha_1^2 + 21\,\alpha_2^2 + 15\,\alpha_3^2)h \\ 2\,\alpha_1 hu + \frac{2}{35}\,(14\,\alpha_1\alpha_2 + 9\,\alpha_2\alpha_3)h \\ 2\,\alpha_2 hu + \frac{2}{21}\,(7\,\alpha_1^2 + 3\,\alpha_2^2 + 9\,\alpha_1\alpha_3 + 2\,\alpha_3^2)h \\ 2\,\alpha_3 hu + \frac{2}{15}\,(9\,\alpha_1\alpha_2 + 4\,\alpha_2\alpha_3)h \end{pmatrix}, \tag{18}
$$

$$
g(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & \frac{1}{5}\,\alpha_1 - \frac{3}{35}\,\alpha_3 & \frac{3}{35}\,\alpha_2 \\ 0 & 0 & \alpha_1 - \frac{3}{7}\,\alpha_3 & \frac{1}{7}\,\alpha_2 + u & \frac{2}{7}\,\alpha_1 + \frac{1}{21}\,\alpha_3 \\ 0 & 0 & \frac{6}{5}\,\alpha_2 & \frac{4}{5}\,\alpha_1 + \frac{2}{15}\,\alpha_3 & \frac{1}{5}\,\alpha_2 + u \end{pmatrix}, \tag{19}
$$

$$
\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x)gh - \frac{\nu(\alpha_1+\alpha_2+\alpha_3+u)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h}+1)\alpha_1+(\frac{4\,\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h}+1)\alpha_2+\alpha_1+\alpha_3+u)\nu}{\lambda} \\ -\frac{7\,((\frac{4\,\lambda}{h}+1)\alpha_1+(\frac{24\,\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \end{pmatrix} \tag{20}
$$

3 2D Equations

In two dimensions the generalized shallow water equations will have the following form,

$$
\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{21}
$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$
\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{22}
$$

where the number of components depends on the number of moments in the velocity profiles.

3.1 Zeroth Order

The zeroth order system is exactly the standard shallow water equations, where only the average velocity is considered. This velocity profiles in this system only consider the constant moment. In this case the nonconservative product disappears and the equation has the following form.

$$
\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = \boldsymbol{p}. \tag{23}
$$

where

$$
\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \qquad \boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 \\ huv \end{pmatrix}, \qquad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv \\ \frac{1}{2}\, e_z g h^2 + hv^2 \end{pmatrix} \tag{24}
$$

$$
\boldsymbol{p} = \begin{pmatrix} 0 \\ -\left(e_z \frac{\partial}{\partial x}(h_b) - e_x\right) gh - \frac{\nu}{\lambda} u \\ -\left(e_z \frac{\partial}{\partial y}(h_b) - e_y\right) gh - \frac{\nu}{\lambda} v \end{pmatrix} \tag{25}
$$

3.2 First Order

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \end{pmatrix}, \quad \boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + \frac{1}{3}\, \alpha_1^2 h + hu^2 \\ \frac{1}{3}\, \alpha_1 \beta_1 h + huv \\ 2\, \alpha_1 hu \\ \beta_1 hu + \alpha_1 hv \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ \frac{1}{3}\, \alpha_1 \beta_1 h + huv \\ \frac{1}{2}\, e_z g h^2 + \frac{1}{3}\, \beta_1^2 h + hv^2 \\ \beta_1 hu + \alpha_1 hv \\ 2\, \beta_1 hv \end{pmatrix}, \tag{26}$$

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & v & 0 \end{pmatrix}, \quad g_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & v \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) gh - \frac{\nu(\alpha_1 + u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y) gh - \frac{\nu(\beta_1 + v)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\alpha_1 + u)\nu}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\beta_1 + v)\nu}{\lambda} \end{pmatrix} \tag{27}$$

3.3 Second Order

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \\ h\alpha_2 \\ h\beta_2 \end{pmatrix}, \tag{28}$$

$$\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 + \frac{1}{15}\,(5\,\alpha_1^2 + 3\,\alpha_2^2) h \\ huv + \frac{1}{15}\,(5\,\alpha_1\beta_1 + 3\,\alpha_2\beta_2) h \\ \frac{4}{5}\,\alpha_1\alpha_2 h + 2\,\alpha_1 hu \\ \beta_1 hu + \alpha_1 hv + \frac{2}{5}\,(\alpha_2\beta_1 + \alpha_1\beta_2) h \\ 2\,\alpha_2 hu + \frac{2}{21}\,(7\,\alpha_1^2 + 3\,\alpha_2^2) h \\ \beta_2 hu + \alpha_2 hv + \frac{2}{21}\,(7\,\alpha_1\beta_1 + 3\,\alpha_2\beta_2) h \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \frac{1}{15}\,(5\,\alpha_1\beta_1 + 3\,\alpha_2\beta_2) h \\ \frac{1}{2}\, e_z g h^2 + hv^2 + \frac{1}{15}\,(5\,\beta_1^2 + 3\,\beta_2^2) h \\ \beta_1 hu + \alpha_1 hv + \frac{2}{5}\,(\alpha_1\beta_1 + \alpha_2\beta_2) h \\ \frac{4}{5}\,\beta_1\beta_2 h + 2\,\beta_1 hv \\ \beta_2 hu + \alpha_2 hv + \frac{2}{21}\,(7\,\alpha_1\beta_1 + 3\,\alpha_2\beta_2) h \\ 2\,\beta_2 hv + \frac{2}{21}\,(7\,\beta_1^2 + 3\,\beta_2^2) h \end{pmatrix}, \tag{29}$$

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & 0 & \frac{1}{5}\,\alpha_1 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\,\beta_2 + v & 0 & \frac{1}{5}\,\beta_1 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 & \frac{1}{7}\,\alpha_2 + u & 0 \\ 0 & 0 & 0 & \beta_1 & 0 & \frac{1}{7}\,\beta_2 + v & 0 \end{pmatrix}, \quad g_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & 0 & \frac{1}{5}\,\alpha_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\,\beta_2 + v & 0 & \frac{1}{5}\,\beta_1 \\ 0 & 0 & 0 & 0 & \alpha_1 & 0 & \frac{1}{7}\,\alpha_2 + u \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & \frac{1}{7}\,\beta_2 + v \end{pmatrix}, \tag{30}$$

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) gh - \frac{\nu(\alpha_1 + \alpha_2 + u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y) gh - \frac{\nu(\beta_1 + \beta_2 + v)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\alpha_1 + \alpha_2 + u)\nu}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h} + 1)\beta_1 + \beta_2 + v)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h} + 1)\alpha_2 + \alpha_1 + u)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h} + 1)\beta_2 + \beta_1 + v)\nu}{\lambda} \end{pmatrix} \tag{31}$$

3.4 Third Order

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \\ h\alpha_2 \\ h\beta_2 \\ h\alpha_3 \\ h\beta_3 \end{pmatrix}, \tag{32}$$

$$\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}\, e_z g h^2 + hu^2 + \frac{1}{105}\,(35\,\alpha_1^2 + 21\,\alpha_2^2 + 15\,\alpha_3^2)h \\ huv + \frac{1}{105}\,(35\,\alpha_1\beta_1 + 21\,\alpha_2\beta_2 + 15\,\alpha_3\beta_3)h \\ 2\,\alpha_1 hu + \frac{2}{35}\,(14\,\alpha_1\alpha_2 + 9\,\alpha_2\alpha_3)h \\ \beta_1 hu + \alpha_1 hv + \frac{1}{35}\,(14\,\alpha_2\beta_1 + 14\,\alpha_1\beta_2 + 9\,\alpha_3\beta_2 + 9\,\alpha_2\beta_3)h \\ 2\,\alpha_2 hu + \frac{2}{21}\,(7\,\alpha_1^2 + 3\,\alpha_2^2 + 9\,\alpha_1\alpha_3 + 2\,\alpha_3^2)h \\ \beta_2 hu + \alpha_2 hv + \frac{1}{21}\,(14\,\alpha_1\beta_1 + 9\,\alpha_3\beta_1 + 6\,\alpha_2\beta_2 + 9\,\alpha_1\beta_3 + 4\,\alpha_3\beta_3)h \\ 2\,\alpha_3 hu + \frac{2}{15}\,(9\,\alpha_1\alpha_2 + 4\,\alpha_2\alpha_3)h \\ \beta_3 hu + \alpha_3 hv + \frac{1}{15}\,(9\,\alpha_2\beta_1 + 9\,\alpha_1\beta_2 + 4\,\alpha_3\beta_2 + 4\,\alpha_2\beta_3)h \end{pmatrix} \tag{33}$$

$$\boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \frac{1}{105}\,(35\,\alpha_1\beta_1 + 21\,\alpha_2\beta_2 + 15\,\alpha_3\beta_3)h \\ \frac{1}{2}\, e_z g h^2 + hv^2 + \frac{1}{105}\,(35\,\beta_1^2 + 21\,\beta_2^2 + 15\,\beta_3^2)h \\ \beta_1 hu + \alpha_1 hv + \frac{1}{35}\,(14\,\alpha_1\beta_1 + 23\,\alpha_2\beta_2 + 9\,\alpha_3\beta_3)h \\ 2\,\beta_1 hv + \frac{2}{35}\,(14\,\beta_1\beta_2 + 9\,\beta_2\beta_3)h \\ \beta_2 hu + \alpha_2 hv + \frac{1}{21}\,(23\,\alpha_1\beta_1 + 6\,\alpha_2\beta_2 + 13\,\alpha_3\beta_3)h \\ 2\,\beta_2 hv + \frac{2}{21}\,(7\,\beta_1^2 + 3\,\beta_2^2 + 9\,\beta_1\beta_3 + 2\,\beta_3^2)h \\ \beta_3 hu + \alpha_3 hv + \frac{1}{15}\,(9\,\alpha_1\beta_1 + 13\,\alpha_2\beta_2 + 4\,\alpha_3\beta_3)h \\ 2\,\beta_3 hv + \frac{2}{15}\,(9\,\beta_1\beta_2 + 4\,\beta_2\beta_3)h \end{pmatrix} \tag{34}$$

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & 0 & \frac{1}{5}\,\alpha_1 - \frac{3}{35}\,\alpha_3 & 0 & \frac{3}{35}\,\alpha_2 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\,\beta_2 + v & 0 & \frac{1}{5}\,\beta_1 - \frac{3}{35}\,\beta_3 & 0 & \frac{3}{35}\,\beta_2 & 0 \\ 0 & 0 & 0 & \alpha_1 - \frac{3}{7}\,\alpha_3 & 0 & \frac{1}{7}\,\alpha_2 + u & 0 & \frac{2}{7}\,\alpha_1 + \frac{1}{21}\,\alpha_3 & 0 \\ 0 & 0 & 0 & \beta_1 - \frac{3}{7}\,\beta_3 & 0 & \frac{1}{7}\,\beta_2 + v & 0 & \frac{2}{7}\,\beta_1 + \frac{1}{21}\,\beta_3 & 0 \\ 0 & 0 & 0 & \frac{6}{5}\,\alpha_2 & 0 & \frac{4}{5}\,\alpha_1 + \frac{2}{15}\,\alpha_3 & 0 & \frac{1}{5}\,\alpha_2 + u & 0 \\ 0 & 0 & 0 & \frac{6}{5}\,\beta_2 & 0 & \frac{4}{5}\,\beta_1 + \frac{2}{15}\,\beta_3 & 0 & \frac{1}{5}\,\beta_2 + v & 0 \end{pmatrix} \tag{35}$$

$$g_2(\boldsymbol{q}) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\,\alpha_2 + u & 0 & \frac{1}{5}\,\alpha_1 - \frac{3}{35}\,\alpha_3 & 0 & \frac{3}{35}\,\alpha_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\,\beta_2 + v & 0 & \frac{1}{5}\,\beta_1 - \frac{3}{35}\,\beta_3 & 0 & \frac{3}{35}\,\beta_2 \\ 0 & 0 & 0 & 0 & \alpha_1 - \frac{3}{7}\,\alpha_3 & 0 & \frac{1}{7}\,\alpha_2 + u & 0 & \frac{2}{7}\,\alpha_1 + \frac{1}{21}\,\alpha_3 \\ 0 & 0 & 0 & 0 & \beta_1 - \frac{3}{7}\,\beta_3 & 0 & \frac{1}{7}\,\beta_2 + v & 0 & \frac{2}{7}\,\beta_1 + \frac{1}{21}\,\beta_3 \\ 0 & 0 & 0 & 0 & \frac{6}{5}\,\alpha_2 & 0 & \frac{4}{5}\,\alpha_1 + \frac{2}{15}\,\alpha_3 & 0 & \frac{1}{5}\,\alpha_2 + u \\ 0 & 0 & 0 & 0 & \frac{6}{5}\,\beta_2 & 0 & \frac{4}{5}\,\beta_1 + \frac{2}{15}\,\beta_3 & 0 & \frac{1}{5}\,\beta_2 + v \end{pmatrix} \tag{36}$$

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x)gh - \frac{\nu(\alpha_1+\alpha_2+\alpha_3+u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y)gh - \frac{\nu(\beta_1+\beta_2+\beta_3+v)}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h}+1)\alpha_1+(\frac{4\,\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \\ -\frac{3\,((\frac{4\,\lambda}{h}+1)\beta_1+(\frac{4\,\lambda}{h}+1)\beta_3+\beta_2+v)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h}+1)\alpha_2+\alpha_1+\alpha_3+u)\nu}{\lambda} \\ -\frac{5\,((\frac{12\,\lambda}{h}+1)\beta_2+\beta_1+\beta_3+v)\nu}{\lambda} \\ -\frac{7\,((\frac{4\,\lambda}{h}+1)\alpha_1+(\frac{24\,\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \\ -\frac{7\,((\frac{4\,\lambda}{h}+1)\beta_1+(\frac{24\,\lambda}{h}+1)\beta_3+\beta_2+v)\nu}{\lambda} \end{pmatrix} \tag{37}$$