

Derivation of Shallow Water Linearized Moment Equations

Shallow Water Moment Equations

$$\begin{aligned}
 & h_t + (hu_m)_x + (hv_m)_y = 0 \\
 & (hu_m)_t + \left(h \left(u_m^2 + \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_x + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_y \\
 & \quad = -\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \alpha_j \right) + hg(e_x - e_z(h_b)_x) \\
 & (hv_m)_t + \left(h \left(v_m^2 + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_y + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_x \\
 & \quad = -\frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \beta_j \right) + hg(e_y - e_z(h_b)_y) \\
 & (h\alpha_i)_t + \left(2hu_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\alpha_k \right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\beta_k \right)_y \\
 & \quad = u_m D_i - \sum_{j,k=1}^N B_{ijk} D_j \alpha_k - (2i+1) \frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \alpha_j \right) \\
 & (h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h \sum_{j,k=1}^N A_{ijk}\alpha_j\beta_k \right)_x + \left(2hv_m\beta_i + h \sum_{j,k=1}^N A_{ijk}\beta_j\beta_k \right)_y \\
 & \quad = v_m D_i - \sum_{j,k=1}^N B_{ijk} D_j \beta_k - (2i+1) \frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \beta_j \right) \\
 & A_{ijk} = (2i+1) \int_0^1 \phi_i \phi_j \phi_k \, d\zeta \\
 & B_{ijk} = (2i+1) \int_0^1 \phi'_i \left(\int_0^\zeta \phi_j \, d\hat{\zeta} \right) \phi_k \, d\zeta \\
 & C_{ij} = \int_0^1 \phi'_i \phi'_j \, d\zeta \\
 & D_i = (h\alpha_i)_x + (h\beta_i)_y
 \end{aligned}$$

To get to the Shallow Water Linearized Moment Equations, we assume that $\alpha_i = O(\varepsilon)$ and $\beta_i = O(\varepsilon)$ are drop all terms of $O(\varepsilon^2)$ in the moment equations. The momentum equations remain the same even though they contain

some of these terms.

$$\begin{aligned}
& h_t + (hu_m)_x + (hv_m)_y = 0 \\
& (hu_m)_t + \left(h \left(u_m^2 + \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_x + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_y \\
& \quad = -\frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \alpha_j \right) + hg(e_x - e_z(h_b)_x) \\
& (hv_m)_t + \left(h \left(v_m^2 + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) + \frac{1}{2} g e_z h^2 \right)_y + \left(h \left(u_m v_m + \sum_{j=1}^N \frac{\alpha_j \beta_j}{2j+1} \right) \right)_x \\
& \quad = -\frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \beta_j \right) + hg(e_y - e_z(h_b)_y) \\
& (h\alpha_i)_t + (2hu_m\alpha_i)_x + (hu_m\beta_i + hv_m\alpha_i)_y = u_m D_i - (2i+1) \frac{\nu}{\lambda} \left(u_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \alpha_j \right) \\
& (h\beta_i)_t + (hu_m\beta_i + hv_m\alpha_i)_x + (2hv_m\beta_i)_y = v_m D_i - (2i+1) \frac{\nu}{\lambda} \left(v_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h} C_{ij} \right) \beta_j \right) \\
& \quad C_{ij} = \int_0^1 \phi'_i \phi'_j d\zeta \\
& \quad D_i = (h\alpha_i)_x + (h\beta_i)_y
\end{aligned}$$

We can write down the shallow water linearized moments equations in the form

$$\mathbf{q}_t + \mathbf{f}_1(\mathbf{q})_x + \mathbf{f}_2(\mathbf{q})_y = g_1(\mathbf{q})\mathbf{q}_x + g_2(\mathbf{q})\mathbf{q}_y + \mathbf{p}. \quad (1)$$

In this case the unknown \mathbf{q} will have the form

$$\mathbf{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \dots]^T, \quad (2)$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $\mathbf{n} = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$n_1(\mathbf{f}'_1(\mathbf{q}) - g_1(\mathbf{q})) + n_2(\mathbf{f}'_2(\mathbf{q}) - g_2(\mathbf{q})).$$

If this matrix is diagonalizable with real eigenvalues for all directions \mathbf{n} , then this system is considered hyperbolic.

Flux Functions

$$\mathbf{f}_1(\mathbf{q}) = \begin{pmatrix} hu \\ \frac{1}{2} e_z g h^2 + hu^2 + \sum_{j=1}^N \left(\frac{1}{2j+1} h \alpha_j^2 \right) \\ huv + \sum_{j=1}^N \left(\frac{1}{2j+1} h \alpha_j \beta_j \right) \\ 2hu\alpha_1 \\ \beta_1 hu + \alpha_1 hv \\ \vdots \\ 2hu\alpha_N \\ hu\beta_N + hv\alpha_N \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{q}) = \begin{pmatrix} hv \\ huv + \sum_{j=1}^N \left(\frac{1}{2j+1} h \alpha_j \beta_j \right) \\ \frac{1}{2} e_z g h^2 + hv^2 + \sum_{j=1}^N \left(\frac{1}{2j+1} h \beta_j^2 \right) \\ hu\beta_1 + hv\alpha_1 \\ 2hv\beta_1 \\ \vdots \\ hu\beta_N + hv\alpha_N \\ 2hv\beta_N \end{pmatrix}$$

Nonconservative Matrices

$$g_1(\mathbf{q}) = \begin{pmatrix} 0 & & & & & & & \\ 0 & 0 & & & & & & \\ & 0 & 0 & & & & & \\ & & 0 & u & & & & \\ & & & v & 0 & & & \\ & & & & 0 & u & & \\ & & & & & v & 0 & \\ & & & & & & 0 & \ddots \\ & & & & & & & \ddots \end{pmatrix}, \quad g_2(\mathbf{q}) = \begin{pmatrix} 0 & 0 & & & & & & \\ & 0 & 0 & & & & & \\ & & 0 & 0 & & & & \\ & & & 0 & u & & & \\ & & & & v & 0 & & \\ & & & & & 0 & u & \\ & & & & & & v & 0 \\ & & & & & & & 0 & \ddots \\ & & & & & & & & \ddots \end{pmatrix}$$

Flux Jacobians

$$\mathbf{f}'_1(\mathbf{q}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ e_z gh - u^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) & 2u & 0 & \frac{2}{3} \alpha_1 & 0 & \cdots & \frac{2}{2N+1} \alpha_N & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2N+1} \alpha_i \beta_i \right) & v & u & \frac{1}{3} \beta_1 & \frac{1}{3} \alpha_1 & \cdots & \frac{1}{2N+1} \beta_N & \frac{1}{2N+1} \alpha_N \\ -2u\alpha_1 & 2\alpha_1 & 0 & 2u & & & & \\ -u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & v & u & & & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & & \\ -2u\alpha_N & 2\alpha_N & 0 & & & 0 & 2u & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u \end{pmatrix}$$

$$\mathbf{f}'_2(\mathbf{q}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) & v & u & \frac{1}{3} \beta_1 & \frac{1}{3} \alpha_1 & \cdots & \frac{1}{2N+1} \beta_N & \frac{1}{2N+1} \alpha_N \\ e_z gh - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) & 0 & 2v & 0 & \frac{2}{3} \beta_1 & \cdots & 0 & \frac{2}{2N+1} \beta_N \\ -u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & u & & & \\ -2v\beta_1 & 0 & 2\beta_1 & & 2v & 0 & & \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u \\ -2u\beta_N & 0 & 2\beta_N & & & & & 2v \end{pmatrix}$$

Quasilinear Matrices, $Q_x = \mathbf{f}'_1(\mathbf{q}) - g_1(\mathbf{q})$, $Q_y = \mathbf{f}'_2(\mathbf{q}) - g_2(\mathbf{q})$

$$Q_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ e_z gh - u^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) & 2u & 0 & \frac{2}{3} \alpha_1 & 0 & \cdots & \frac{2}{2N+1} \alpha_N & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) & v & u & \frac{1}{3} \beta_1 & \frac{1}{3} \alpha_1 & \cdots & \frac{1}{2N+1} \beta_N & \frac{1}{2N+1} \alpha_N \\ -2u\alpha_1 & 2\alpha_1 & 0 & u & & & & \\ -u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & & u & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ -2u\alpha_N & 2\alpha_N & 0 & & & & u & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & & u \end{pmatrix}$$

$$Q_y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) & v & u & \frac{1}{3} \beta_1 & \frac{1}{3} \alpha_1 & \cdots & \frac{1}{2N+1} \beta_N & \frac{1}{2N+1} \alpha_N \\ e_z gh - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) & 0 & 2v & 0 & \frac{2}{3} \beta_1 & \cdots & 0 & \frac{2}{2N+1} \beta_N \\ -u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & & & & \\ -2v\beta_1 & 0 & 2\beta_1 & & v & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ -u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & \\ -2u\beta_N & 0 & 2\beta_N & & & & & v \end{pmatrix}$$

Quasilinear Eigenvalues The wavespeeds of this system in direction $\mathbf{n} = [n_1, n_2]$ are given by the eigenvalues of the matrix

$$n_1 Q_x + n_2 Q_y.$$

If all of the eigenvalues are real with a full set of eigenvectors, this this system is hyperbolic.

Convenient constants

$$\begin{aligned} d_0^1 &= n_1 \left(e_z g h - u^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) + n_2 \left(-uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \\ d_0^2 &= n_1 \left(-uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2 \left(e_z g h - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \\ d_i^1 &= n_1 (-2u\alpha_i) + n_2 (-u\beta_i - \alpha_i v) \\ d_i^2 &= n_1 (-u\beta_i - v\alpha_i) + n_2 (-2v\beta_i) \\ b_i^1 &= n_1 \frac{2}{2i+1} \alpha_i + n_2 \frac{1}{2i+1} \beta_i \\ b_i^2 &= n_2 \frac{1}{2i+1} \alpha_i \\ b_i^3 &= n_1 \frac{1}{2i+1} \beta_i \\ b_i^4 &= n_1 \frac{1}{2i+1} \alpha_i + n_2 \frac{2}{2i+1} \beta_i \\ c_i^1 &= n_1 2\alpha_i + n_2 \beta_i \\ c_i^2 &= n_1 \beta_i \\ c_i^3 &= n_2 \alpha_i \\ c_i^4 &= n_1 \alpha_i + n_2 2\beta_i \end{aligned}$$

Determinant of Block Matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| |A - BD^{-1}C|$$

$$\begin{aligned} \det(n_1 Q_x + n_2 Q_y - \lambda I) &= \begin{vmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\ d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ d_N^1 & c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\ d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde{\lambda} \end{vmatrix} \\ &= |D| |A - BD^{-1}C| \end{aligned}$$

$$A = \begin{pmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} \end{pmatrix} \quad (3)$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \end{pmatrix} \quad (4)$$

$$C = \begin{pmatrix} d_1^1 & c_1^1 & c_1^3 \\ d_1^2 & c_1^2 & c_1^4 \\ \vdots & \vdots & \vdots \\ d_N^1 & c_N^1 & c_N^3 \\ d_N^2 & c_N^2 & c_N^4 \end{pmatrix} \quad (5)$$

$$D = \begin{pmatrix} -\tilde{\lambda} & & & & \\ & -\tilde{\lambda} & & & \\ & & \ddots & & \\ & & & -\tilde{\lambda} & \\ & & & & -\tilde{\lambda} \end{pmatrix} \quad (6)$$

$$|D| = \tilde{\lambda}^{2N}$$

$$BD^{-1}C = \frac{1}{-\tilde{\lambda}} \begin{pmatrix} 0 & 0 & 0 \\ s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{pmatrix}$$

$$s_1 = \sum_{i=1}^N (b_i^1 d_i^1 + b_i^2 d_i^2) = \sum_{i=1}^N \left(-\frac{1}{2i+1} (4\alpha_i u n_1 + \beta_i u n_2 + 3\alpha_i v n_2) (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_2 = \sum_{i=1}^N (b_i^1 c_i^1 + b_i^2 c_i^2) = \sum_{i=1}^N \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_3 = \sum_{i=1}^N (b_i^1 c_i^3 + b_i^2 c_i^4) = \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_4 = \sum_{i=1}^N (b_i^3 d_i^1 + b_i^4 d_i^2) = \sum_{i=1}^N \left(-\frac{1}{2i+1} (3\beta_i u n_1 + \alpha_i v n_1 + 4\beta_i v n_2) (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_5 = \sum_{i=1}^N (b_i^3 c_i^1 + b_i^4 c_i^2) = \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_6 = \sum_{i=1}^N (b_i^3 c_i^3 + b_i^4 c_i^4) = \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right)$$

$$\begin{aligned}
A - BD^{-1}C &= \frac{1}{\tilde{\lambda}} \begin{pmatrix} -\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2 & \tilde{\lambda}n_1 & \tilde{\lambda}n_2 \\ \tilde{\lambda}d_0^1 + s_1 & \tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2 & \tilde{\lambda}un_2 + s_3 \\ \tilde{\lambda}d_0^2 + s_4 & \tilde{\lambda}vn_1 + s_5 & \tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6 \end{pmatrix} \\
\det(\tilde{\lambda}(A - BD^{-1}C)) &= (-\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2)((\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2)(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6) - (\tilde{\lambda}vn_1 + s_5)(\tilde{\lambda}un_2 + s_3)) \\
&\quad - \tilde{\lambda}n_1((\tilde{\lambda}d_0^1 + s_1)(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6) - (\tilde{\lambda}d_0^2 + s_4)(\tilde{\lambda}un_2 + s_3)) \\
&\quad + \tilde{\lambda}n_2((\tilde{\lambda}d_0^1 + s_1)(\tilde{\lambda}vn_1 + s_5) - (\tilde{\lambda}d_0^2 + s_4)(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2)) \\
&= (-\tilde{\lambda}^2 - (un_1 + vn_2)\tilde{\lambda})(\tilde{\lambda}^4 + (-un_1 - vn_2)\tilde{\lambda}^3 + (-s_2 - s_6)\tilde{\lambda}^2 + (un_1s_6 + vn_2s_2 - vn_1s_3 - un_2s_5)\tilde{\lambda} + s_2s_6 - s_3s_5) \\
&\quad - \tilde{\lambda}n_1(-d_0^1\tilde{\lambda}^3 + (d_0^1vn_2 - s_1 - d_0^2un_2)\tilde{\lambda}^2 + (d_0^1s_6 + vn_2s_1 - d_0^2s_3 - un_2s_4)\tilde{\lambda} + s_1s_6 - s_3s_4) \\
&\quad + \tilde{\lambda}n_2(d_0^2\tilde{\lambda}^3 + (d_0^1vn_1 - d_0^2un_1 + s_4)\tilde{\lambda}^2 + (d_0^1s_5 + vn_1s_1 - d_0^2s_2 - un_1s_4)\tilde{\lambda} + s_1s_5 - s_2s_4) \\
&\quad = -\tilde{\lambda}^6 + (s_2 + s_6 + (un_1 + vn_2)^2)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6)\tilde{\lambda}^3 \\
&\quad + (s_3s_5 - s_2s_6 + (u^2n_1n_2 + uvn_2^2)s_5 + (uvn_1^2 + v^2n_1n_2)s_3 - (u^2n_1^2 + uvn_1n_2)s_6 - (uvn_1n_2 + v^2n_2^2)s_2)\tilde{\lambda}^2 + (s_3s_5 - s_2s_6)(un_1 + \\
&\quad + d_0^1n_1\tilde{\lambda}^4 + (-d_0^1vn_1n_2 + s_1n_1 + d_0^2un_1n_2)\tilde{\lambda}^3 + (-d_0^1n_1s_6 - vn_1n_2s_1 + d_0^2n_1s_3 + un_1n_2s_4)\tilde{\lambda}^2 + (s_3s_4 - s_1s_6)n_1\tilde{\lambda} \\
&\quad + d_0^2n_2\tilde{\lambda}^4 + (d_0^1vn_1n_2 - d_0^2un_1n_2 + n_2s_4)\tilde{\lambda}^3 + (d_0^1n_2s_5 + vn_1n_2s_1 - d_0^2n_2s_2 - un_1n_2s_4)\tilde{\lambda}^2 + (s_1s_5 - s_2s_4)n_2\tilde{\lambda} \\
&\quad = -\tilde{\lambda}^6 + (s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4)\tilde{\lambda}^3 \\
&\quad + (s_3s_5 - s_2s_6 + (u^2n_1n_2 + uvn_2^2 + d_0^1n_2)s_5 + (uvn_1^2 + v^2n_1n_2 + d_0^2n_1)s_3 - (u^2n_1^2 + uvn_1n_2 + d_0^1n_1)s_6 - (uvn_1n_2 + v^2n_2^2 + d_0^2n_2) \\
&\quad + ((s_3s_5 - s_2s_6)(un_1 + vn_2) + (s_3s_4 - s_1s_6)n_1 + (s_1s_5 - s_2s_4)n_2)\tilde{\lambda}
\end{aligned}$$

Look at coefficients of polynomial in $\tilde{\lambda}$.

$$\begin{aligned}
&s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2 \\
&= \sum_{i=1}^N \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right) + \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right) + (un_1 + vn_2)^2 \\
&\quad + n_1^2 \left(e_z gh - u^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) + n_1 n_2 \left(-uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \\
&\quad + n_1 n_2 \left(-uv - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2^2 \left(e_z gh - v^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \\
&= \sum_{i=1}^N \left(\frac{5}{2i+1} (\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right) + e_z gh(n_1^2 + n_2^2) - \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i^2 n_1^2 + 2\alpha_i \beta_i n_1 n_2 + \beta_i^2 n_2^2) \right) \\
&\quad = \sum_{i=1}^N \left(\frac{4}{2i+1} (\alpha_i n_1 + \beta_i n_2)^2 \right) + e_z gh(n_1^2 + n_2^2)
\end{aligned}$$

$$\begin{aligned}
& vn_1 s_3 + un_2 s_5 + un_1 s_2 + vn_2 s_6 + s_1 n_1 + n_2 s_4 \\
&= \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i v n_1 n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&\quad + \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i u n_1 n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&\quad + \sum_{i=1}^N \left(\frac{1}{2i+1} (4\alpha_i u n_1^2 + \beta_i u n_1 n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&\quad + \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i v n_1 n_2 + 4\beta_i v n_2^2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&\quad + \sum_{i=1}^N \left(-\frac{1}{2i+1} (4\alpha_i u n_1^2 + \beta_i u n_1 n_2 + 3\alpha_i v n_1 n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&\quad + \sum_{i=1}^N \left(-\frac{1}{2i+1} (3\beta_i u n_1 n_2 + \alpha_i v n_1 n_2 + 4\beta_i v n_2^2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= \sum_{i=1}^N \left(\frac{1}{2i+1} (3\alpha_i v n_1 n_2 + 3\beta_i u n_1 n_2 - 3\alpha_i v n_1 n_2 - 3\beta_i u n_1 n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
& a_i = \alpha_i n_1 + \beta_i n_2 \\
& s_3 s_5 - s_2 s_6 \\
&= \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i n_2 a_i \right) \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i n_1 a_i \right) - \sum_{i=1}^N \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2) a_i \right) \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2) a_i \right) \\
&= \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{3}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right) - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} (4\alpha_i n_1 + \beta_i n_2) (\alpha_j n_1 + 4\beta_j n_2) a_i a_j \right) \right) \\
&= \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} (9\alpha_i \beta_j n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2) (\alpha_j n_1 + 4\beta_j n_2)) a_i a_j \right) \right) \\
&= \sum_{i=1}^N \left(\frac{1}{(2i+1)^2} (9\alpha_i \beta_i n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2) (\alpha_i n_1 + 4\beta_i n_2)) a_i a_i \right) \\
&+ \sum_{i=1}^N \left(\sum_{j=1}^{i-1} \left(\frac{1}{2i+1} \frac{1}{2j+1} (9\alpha_i \beta_j n_1 n_2 - (4\alpha_i n_1 + \beta_i n_2) (\alpha_j n_1 + 4\beta_j n_2) + 9\alpha_j \beta_i n_1 n_2 - (4\alpha_j n_1 + \beta_j n_2) (\alpha_i n_1 + 4\beta_i n_2)) a_i a_j \right) \right) \\
&= \sum_{i=1}^N \left(-\left(\frac{2}{2i+1} \right)^2 (\alpha_i n_1 + \beta_i n_2)^2 a_i a_i \right) + \sum_{i=1}^N \left(\sum_{j=1}^{i-1} \left(-8 \frac{1}{2i+1} \frac{1}{2j+1} (\alpha_i n_1 + \beta_i n_2) (\alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right) \\
&= -\sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{2}{2i+1} \frac{2}{2j+1} a_i^2 a_j^2 \right) \right) \\
&= -\left(\sum_{i=1}^N \left(\frac{2}{2i+1} a_i^2 \right) \right)^2
\end{aligned}$$

$$\begin{aligned}
& (u^2 n_1 n_2 + u v n_2^2 + d_0^1 n_2) s_5 \\
&= \left(n_2 n_1 \left(e_z g h - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_2^2 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \\
&= \left(e_z g h n_1 n_2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_1 n_2 s_5 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right) \\
& (u v n_1^2 + v^2 n_1 n_2 + d_0^2 n_1) s_3 \\
&= \left(-n_1^2 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_1 n_2 \left(e_z g h - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_1 n_2 s_3 - \left(\sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_1 n_2 s_3 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_j \beta_i n_1 n_2 a_i a_j \right) \right) \\
& (u^2 n_1^2 + u v n_1 n_2 + d_0^1 n_1) s_6 \\
&= \left(n_1^2 \left(e_z g h - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_1 n_2 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4 \beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_1^2 s_6 - \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4 \beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_1^2 s_6 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} \alpha_i n_1 (\alpha_j n_1 + 4 \beta_j n_2) a_i a_j \right) \right) \\
& (u v n_1 n_2 + v^2 n_2^2 + d_0^2 n_2) s_2 \\
&= \left(n_2 n_1 \left(- \sum_{i=1}^N \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2^2 \left(e_z g h - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^N \left(\frac{1}{2i+1} (4 \alpha_i n_1 + \beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_2^2 s_2 - \sum_{i=1}^N \left(\frac{1}{2i+1} \beta_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^N \left(\frac{1}{2i+1} (4 \alpha_i n_1 + \beta_i n_2) (\alpha_i n_1 + \beta_i n_2) \right) \\
&= e_z g h n_2^2 s_2 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} \beta_i n_2 (4 \alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right)
\end{aligned}$$

$$\begin{aligned}
& s_3 s_5 - s_2 s_6 + (u^2 n_1 n_2 + u v n_2^2 + d_0^1 n_2) s_5 + (u v n_1^2 + v^2 n_1 n_2 + d_0^2 n_1) s_3 \\
& - (u^2 n_1^2 + u v n_1 n_2 + d_0^1 n_1) s_6 - (u v n_1 n_2 + v^2 n_2^2 + d_0^2 n_2) s_2 \\
= & -4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h n_1 n_2 s_5 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right) \\
& + e_z g h n_1 n_2 s_3 - \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_j \beta_i n_1 n_2 a_i a_j \right) \right) \\
& - e_z g h n_1^2 s_6 + \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} \alpha_i n_1 (\alpha_j n_1 + 4 \beta_j n_2) a_i a_j \right) \right) \\
& - e_z g h n_2^2 s_2 + \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} \beta_i n_2 (4 \alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right) \\
= & -4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h (n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2) \\
& - 6 \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right) \\
& + \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} (\alpha_i \alpha_j n_1^2 + 8 \alpha_i \beta_j n_1 n_2 + \beta_i \beta_j n_2^2) a_i a_j \right) \right) \\
= & -4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h (n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2) \\
& + \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} (\alpha_i \alpha_j n_1^2 + 2 \alpha_i \beta_j n_1 n_2 + \beta_i \beta_j n_2^2) a_i a_j \right) \right) \\
= & -4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h (n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2) \\
& + \sum_{i=1}^N \left(\sum_{j=1}^N \left(\frac{1}{2i+1} \frac{1}{2j+1} a_i^2 a_j^2 \right) \right) \\
= & -3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h (n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2)
\end{aligned}$$

$$\begin{aligned}
& n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2 \\
&= n_1 n_2 \sum_{i=1}^N \left(\frac{3}{2i+1} \alpha_i n_2 a_i \right) \\
&\quad + n_1 n_2 \sum_{i=1}^N \left(\frac{3}{2i+1} \beta_i n_1 a_i \right) \\
&\quad - n_1^2 \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2) a_i \right) \\
&\quad - n_2^2 \sum_{i=1}^N \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2) a_i \right) \\
&= \sum_{i=1}^N \left(\frac{1}{2i+1} (3\alpha_i n_1 n_2^2 + 3\beta_i n_1^2 n_2 - (\alpha_i n_1^3 + 4\beta_i n_1^2 n_2) - (4\alpha_i n_1 n_2^2 + \beta_i n_2^3)) a_i \right) \\
&\quad = - \sum_{i=1}^N \left(\frac{1}{2i+1} (\alpha_i n_1^3 + \beta_i n_1^2 n_2 + \alpha_i n_1 n_2^2 + \beta_i n_2^3) a_i \right) \\
&\quad = - \sum_{i=1}^N \left(\frac{1}{2i+1} (n_1^2 a_i + n_2^2 a_i) a_i \right) \\
&\quad = -(n_1^2 + n_2^2) \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z g h (n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2) \\
&= -3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 - e_z g h (n_1^2 + n_2^2) \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right)
\end{aligned}$$

$$\begin{aligned}
& (s_3 s_5 - s_2 s_6)(u n_1 + v n_2) + (s_3 s_4 - s_1 s_6) n_1 + (s_1 s_5 - s_2 s_4) n_2 \\
&= -4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 (u n_1 + v n_2) + 4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 u n_1 + 4 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 v n_2 \\
&\quad = 0
\end{aligned}$$

$$\begin{aligned}
\det(\tilde{\lambda}(A - BD^{-1}C)) &= -\tilde{\lambda}^6 + \left(4 \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right) + e_z gh(n_1^2 + n_2^2)\right) \tilde{\lambda}^4 \\
&\quad - \left(3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right)^2 + e_z gh(n_1^2 + n_2^2) \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right) \tilde{\lambda}^2 \\
\det(A - BD^{-1}C) &= -\frac{1}{\tilde{\lambda}} \left(\tilde{\lambda}^2 - \left(3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right) + e_z gh(n_1^2 + n_2^2)\right)\right) \left(\tilde{\lambda}^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right) \\
\det(n_1 Q_x + n_2 Q_y + \lambda I) &= \det(D) \det(A - BD^{-1}C) \\
&= -\tilde{\lambda}^{2N-1} \left(\tilde{\lambda}^2 - \left(3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right) + e_z gh(n_1^2 + n_2^2)\right)\right) \left(\tilde{\lambda}^2 - \sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right) \\
\lambda &= un_1 + vn_2 \pm \sqrt{e_z gh(n_1^2 + n_2^2) + 3 \left(\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)\right)} \\
\lambda &= un_1 + vn_2 \pm \sqrt{\sum_{i=1}^N \left(\frac{1}{2i+1} a_i^2\right)} \\
\lambda &= un_1 + vn_2
\end{aligned}$$

Quasilinear Eigenvectors

$$\begin{aligned}
&(n_1 Q_x + n_2 Q_y - \lambda I) \mathbf{q} = 0 \\
&\begin{pmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\ d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ d_N^1 & c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\ d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde{\lambda} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \vdots \\ q_{2N+2} \\ q_{2N+3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Let $\lambda = un_1 + vn_2$

$$\begin{pmatrix} -n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\ d_0^1 & n_1 u & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_1 v & n_2 v & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^1 & c_1^3 & & & & & \\ d_1^2 & c_1^2 & c_1^4 & & & & & \\ \vdots & \vdots & \vdots & & & & & \\ d_N^1 & c_N^1 & c_N^3 & & & & & \\ d_N^2 & c_N^2 & c_N^4 & & & & & \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ \vdots \\ q_{2N+2} \\ q_{2N+3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Eigenvectors Zero Moments

$$\begin{aligned}
\lambda &= un_1 + vn_2 - \sqrt{e_z gh(n_1^2 + n_2^2)} \\
&\quad \begin{pmatrix} 1 \\ u - \frac{n_1}{n_1^2 + n_2^2} \sqrt{e_z gh(n_1^2 + n_2^2)} \\ v - \frac{n_2}{n_1^2 + n_2^2} \sqrt{e_z gh(n_1^2 + n_2^2)} \end{pmatrix} \\
\lambda &= un_1 + vn_2 + \sqrt{e_z gh(n_1^2 + n_2^2)} \\
&\quad \begin{pmatrix} 1 \\ u + \frac{n_1}{n_1^2 + n_2^2} \sqrt{e_z gh(n_1^2 + n_2^2)} \\ v + \frac{n_2}{n_1^2 + n_2^2} \sqrt{e_z gh(n_1^2 + n_2^2)} \end{pmatrix} \\
\lambda &= un_1 + vn_2 \\
&\quad \begin{pmatrix} 0 \\ 1 \\ -\frac{n_1}{n_2} \end{pmatrix}
\end{aligned}$$

One Moment

$$\begin{aligned}
\lambda &= un_1 + vn_2 - \sqrt{e_z gh(n_1^2 + n_2^2) + (\alpha_i n_1 + \beta_i n_2)^2} \\
&\quad (1) \\
\lambda &= un_1 + vn_2 + \sqrt{e_z gh(n_1^2 + n_2^2) + (\alpha_i n_1 + \beta_i n_2)^2} \\
&\quad (1) \\
\lambda &= un_1 + vn_2 \\
&\quad \begin{pmatrix} 1 \\ u \\ v \\ -\frac{1}{2} \frac{e_z gh(3\alpha_1 n_1^2 + 6\beta_1 n_1 n_2 - 3\alpha_1 n_2^2) - \alpha_1 (\alpha_1 n_1 + \beta_1 n_2)^2}{(\alpha_1 n_1 + \beta_1 n_2)^2} \\ \frac{1}{2} \frac{e_z gh(3\beta_1 n_1^2 - 6\alpha_1 n_1 n_2 - 3\beta_1 n_2^2) + \beta_1 (\alpha_1 n_1 + \beta_1 n_2)^2}{(\alpha_1 n_1 + \beta_1 n_2)^2} \end{pmatrix} \\
\lambda &= un_1 + vn_2 - \sqrt{\frac{1}{3}(\alpha_i n_1 + \beta_i n_2)^2} \\
&\quad \begin{pmatrix} 0 \\ 1 \\ -\frac{n_1}{n_2} \\ -\sqrt{3} \\ \sqrt{3} \frac{n_1}{n_2} \end{pmatrix} \\
\lambda &= un_1 + vn_2 + \sqrt{\frac{1}{3}(\alpha_i n_1 + \beta_i n_2)^2} \\
&\quad \begin{pmatrix} 0 \\ 1 \\ -\frac{n_1}{n_2} \\ \sqrt{3} \\ -\sqrt{3} \frac{n_1}{n_2} \end{pmatrix}
\end{aligned}$$