Discontinuous Galerkin Method for solving a Thin-Film Equation

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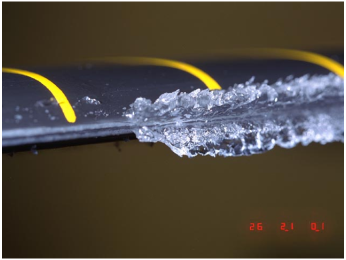
Overview

Motivation

- Aircraft Icing
- Runback



- Industrial Coating
- Paint Drying

Model Equations

- Navier-Stokes Equation

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + \left(\rho u^2 + p\right)_x &= \\
E_t + (u(E + p))_x &=
\end{aligned}
$$

- Asymptotic Limit, $\rho << L$
- Thin-Film Equation - 1D with u as fluid height.

$$
u_t + \left(f(x,t)u^2 - g(x,t)u^3\right)_x = \left(h(x,t)u^3 u_{xxx}\right)_x
$$

Current Model

- Simplified Expression

$$u_t + \left(u^2 - u^3\right)_x = \left(u^3 u_{xxx}\right)_x$$

- Operator Splitting

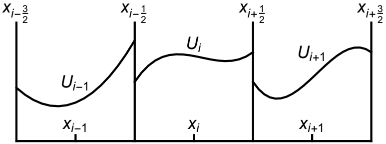
$$\begin{aligned} u_t + \left(u^2 - u^3\right)_x &= 0 \\ u_t - \left(u^3 u_{xxx}\right)_x &= 0 \end{aligned}$$

Introduction to Discontinuous Galerkin

- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $V_i = \left[x_{i-1/2}, x_{i+1/2}\right]$
- $\Delta x = x_{i+1/2} - x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$.



- Solution of order M on each cell

$$u\big|_{x \in V_i} \approx U_i = \sum_{k=1}^{M} U_i^k \phi^k(\xi)$$

- Legendre Basis - ϕ^k

Numerical Solutions

- Use canonical variable $\xi \in [-1, 1]$
- Let $\left\{\phi^k(\xi)\right\}$ be the Legendre polynomials.
- Solution of order M on each cell

$$u|_{x \in V_i} \approx U_i = \sum_{k=1}^{M} U_i^k \phi^k(\xi)$$

Convection

- Convection Equation

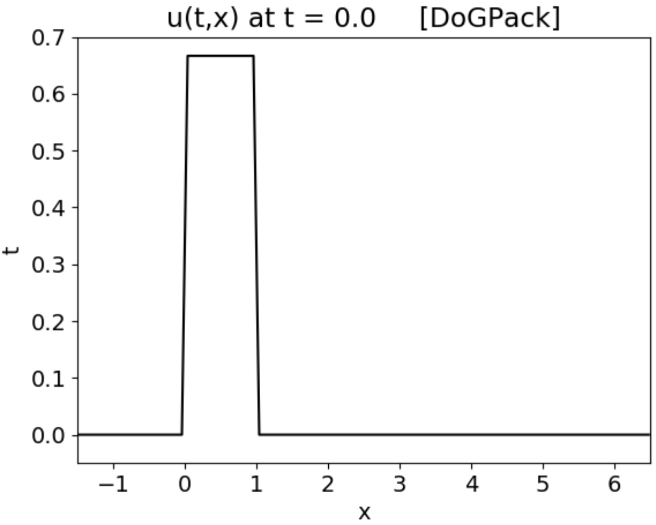
$$u_t + \left(u^2 - u^3\right)_x = 0$$

- Weak Form

$$\int_{-1}^{1} u_t \phi(\xi) + \frac{2}{\Delta x}\left(u^2 - u^3\right)_\xi \phi \, \mathrm{d}\xi$$

- Runge-Kutta Discontinuous Galerkin

Numerical Example - Square Wave



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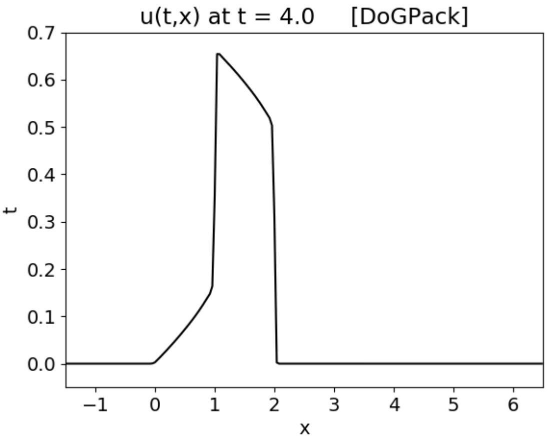

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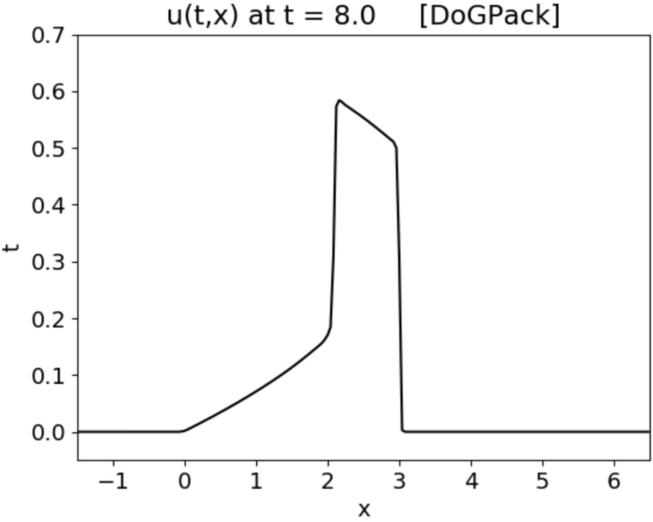

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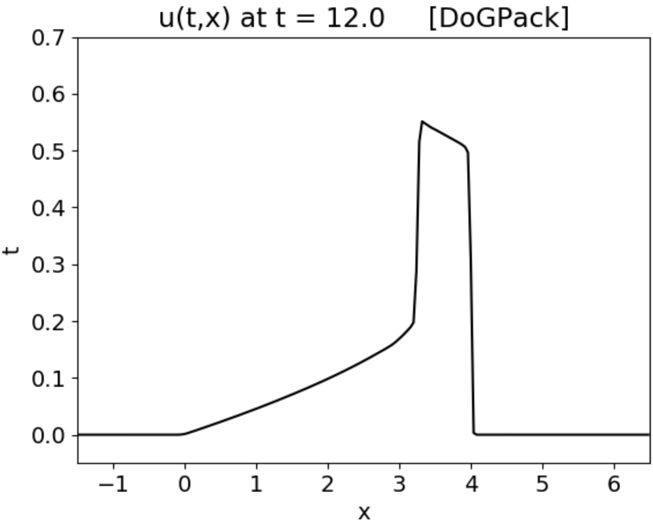

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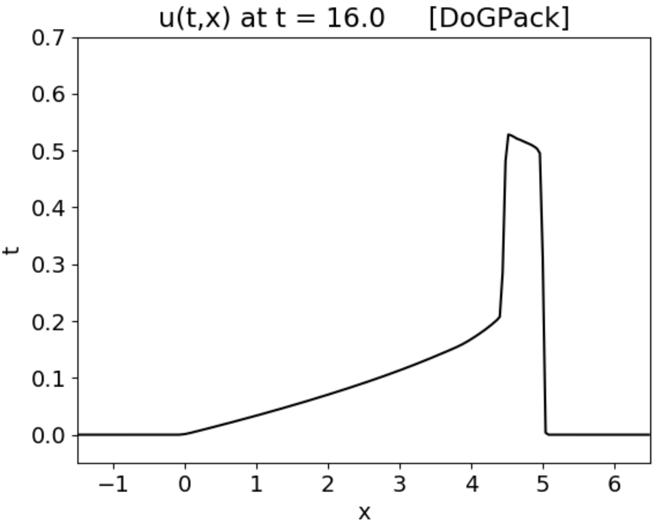

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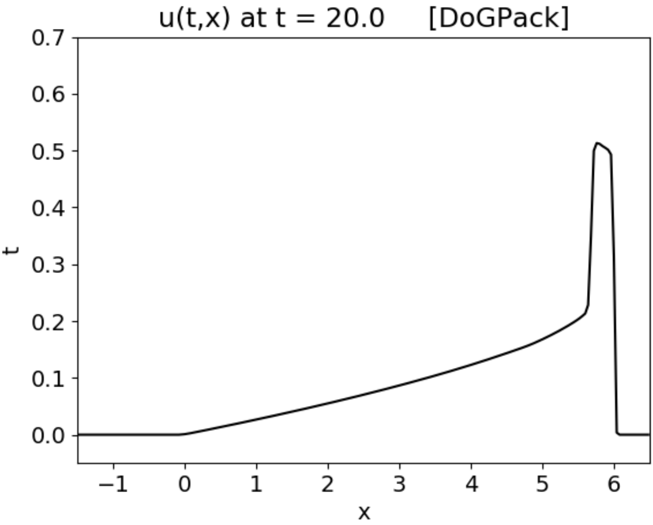

Diffusion

- Diffusion Equation

$$u_t - \left(u^3 u_{xxx}\right)_x = 0$$

- Local Discontinuous Galerkin

Multigrid Solver

Numerical Results

Future Work

- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models

Conclusion

- Questions?

Bibliography

[1] Y. Ha, Y.-J. Kim, and T.G. Myers. "On the numerical solution of a driven thin film equation". In: *J. Comp. Phys.* 227.15 (2008), pp. 7246–7263.

[2] J.A. Rossmanith. DoGPack. Available from http://www.dogpack-code.org/.