Derivation of Shallow Water Linearized Moment Equations

Shallow Water Moment Equations

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1} \right) + \frac{1}{2}ge_z h^2 \right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1} \right) \right)_y$$

$$= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j \right) + hg(e_x - e_z(h_b)_x)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1} \right) + \frac{1}{2}ge_z h^2 \right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1} \right) \right)_x$$

$$= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j \right) + hg\left(e_y - e_z(h_b)_y \right)$$

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k \right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k \right)_y$$

$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk}D_j\alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij} \right)\alpha_j \right)$$

$$(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k \right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k \right)_y$$

$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk}D_j\beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h}C_{ij} \right)\beta_j \right)$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \, d\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \, d\hat\zeta \right)\phi_k \, d\zeta$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \, d\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

To get to the Shallow Water Linearized Moment Equations, we assume that $\alpha_i = O(\varepsilon)$ and $\beta_i = O(\varepsilon)$ are drop all terms of $O(\varepsilon^2)$ in the moment equations. The momentum equations remain the same even though they contain

some of these terms.

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^N \frac{\alpha_j^2}{2j+1} \right) + \frac{1}{2}ge_zh^2 \right)_x + \left(h\left(u_mv_m + \sum_{j=1}^N \frac{\alpha_j\beta_j}{2j+1} \right) \right)_y$$

$$= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^N \alpha_j \right) + hg(e_x - e_z(h_b)_x)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^N \frac{\alpha_j\beta_j}{2j+1} \right) + \frac{1}{2}ge_zh^2 \right)_y + \left(h\left(u_mv_m + \sum_{j=1}^N \frac{\alpha_j\beta_j}{2j+1} \right) \right)_x$$

$$= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^N \beta_j \right) + hg\left(e_y - e_z(h_b)_y \right)$$

$$(h\alpha_i)_t + (2hu_m\alpha_i)_x + (hu_m\beta_i + hv_m\alpha_i)_y = u_mD_i - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j \right)$$

$$(h\beta_i)_t + (hu_m\beta_i + hv_m\alpha_i)_x + (2hv_m\beta_i)_y = v_mD_i - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^N \left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j \right)$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \, d\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

We can write down the shallow water linearized moments equations in the form

$$q_t + f_1(q)_x + f_2(q)_y = g_1(q)q_x + g_2(q)q_y + p. \tag{1}$$

In this case the unknown q will have the form

$$q = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{2}$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $n = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$n_1(f_1'(q) - g_1(q)) + n_2(f_2'(q) - g_2(q)).$$

If this matrix is diagonalizable with real eigenvalues for all directions n, then this system is considered hyperbolic.

Flux Functions

$$f_1(q) = \begin{pmatrix} hu \\ \frac{1}{2}e_zgh^2 + hu^2 + \sum_{j=1}^N \left(\frac{1}{2j+1}h\alpha_j^2 \right) \\ huv + \sum_{j=1}^N \left(\frac{1}{2j+1}h\alpha_j\beta_j \right) \\ 2hu\alpha_1 \\ \beta_1hu + \alpha_1hv \\ \vdots \\ 2hu\alpha_N \\ hu\beta_N + hv\alpha_N \end{pmatrix}, \quad f_2(q) = \begin{pmatrix} hv \\ huv + \sum_{j=1}^N \left(\frac{1}{2j+1}h\alpha_j\beta_j \right) \\ \frac{1}{2}e_zgh^2 + hv^2 + \sum_{j=1}^N \left(\frac{1}{2j+1}h\beta_j^2 \right) \\ hu\beta_1 + hv\alpha_1 \\ 2hv\beta_1 \\ \vdots \\ hu\beta_N + hv\alpha_N \\ 2hv\beta_N \end{pmatrix}$$

Nonconservative Matrices

$$
g_1(\boldsymbol{q}) =
\begin{pmatrix}
0 & & & & & & & & \\
0 & 0 & & & & & & & \\
 & 0 & 0 & & & & & & \\
 & & 0 & u & & & & & \\
 & & & v & 0 & & & & \\
 & & & & 0 & u & & & \\
 & & & & v & 0 & & & \\
 & & & & & & 0 & \ddots & \\
 & & & & & & & \ddots & \\
\end{pmatrix}
, \quad
g_2(\boldsymbol{q}) =
\begin{pmatrix}
0 & 0 & & & & & & & \\
 & 0 & 0 & & & & & & \\
 & & 0 & 0 & & & & & \\
 & & & 0 & u & & & & \\
 & & & v & 0 & & & & \\
 & & & & 0 & u & & & \\
 & & & & v & 0 & & & \\
 & & & & & & 0 & \ddots & \\
 & & & & & & & \ddots & \\
\end{pmatrix}
$$

Flux Jacobians

$$
\boldsymbol{f}_1'(\boldsymbol{q}) =
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2*i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2N+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & 2u & & & & \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & v & u & & & \\
\vdots & \vdots & \vdots & & \ddots & \ddots & & \\
-2u\alpha_N & 2\alpha_N & 0 & & & 0 & 2u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u \\
\end{pmatrix}
$$

$$
\boldsymbol{f}_2'(\boldsymbol{q}) =
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & u & & & \\
-2v\beta_1 & 0 & 2\beta_1 & & 2v & 0 & & \\
\vdots & \vdots & \vdots & & & \ddots & \ddots & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & u \\
-2u\beta_N & 0 & 2\beta_N & & & & & 2v \\
\end{pmatrix}
$$

Quasilinear Matrices, $Q_x = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), Q_y = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$
Q_x =
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & u & & & & \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & & u & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
-2u\alpha_N & 2\alpha_N & 0 & & & & u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & & u \\
\end{pmatrix}
$$

$$
Q_y =
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & & & & \\
-2v\beta_1 & 0 & 2\beta_1 & & v & & & \\
\vdots & \vdots & \vdots & & & \ddots & & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N & & & & v & \\
-2u\beta_N & 0 & 2\beta_N & & & & & v \\
\end{pmatrix}
$$

Quasilinear Eigenvalues The wavespeeds of this system in direction $\boldsymbol{n} = [n_1, n_2]$ are given by the eigenvalues of the matrix

$$n_1 Q_x + n_2 Q_y.$$

If all of the eigenvalues are real with a full set of eigenvectors, this this system is hyperbolic.

Convenient constants

$$d_0^1 = n_1 \left(e_z gh - u^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) + n_2 \left(-uv - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right)$$

$$d_0^2 = n_1 \left(-uv - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2 \left(e_z gh - v^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i^2 \right) \right)$$

$$d_i^1 = n_1(-2u\alpha_i) + n_2(-u\beta_i - \alpha_i v)$$

$$d_i^2 = n_1(-u\beta_i - v\alpha_i) + n_2(-2v\beta_i)$$

$$b_i^1 = n_1 \frac{2}{2i+1} \alpha_i + n_2 \frac{1}{2i+1} \beta_i$$

$$b_i^2 = n_2 \frac{1}{2i+1} \alpha_i$$

$$b_i^3 = n_1 \frac{1}{2i+1} \beta_i$$

$$b_i^4 = n_1 \frac{1}{2i+1} \alpha_i + n_2 \frac{2}{2i+1} \beta_i$$

$$c_i^1 = n_1 2\alpha_i + n_2 \beta_i$$

$$c_i^2 = n_1 \beta_i$$

$$c_i^3 = n_2 \alpha_i$$

$$c_i^4 = n_1 \alpha_i + n_2 2\beta_i$$

Determinant of Block Matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C|$$

$$\det(n_1 Q_x + n_2 Q_y - \lambda I) = \begin{vmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\ d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ d_N^1 & c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\ d_N^2 & c_N^2 & c_N^4 & & & & & -\tilde{\lambda} \end{vmatrix}$$

$$= |D||A - BD^{-1}C|$$

$$A = \begin{pmatrix} -\tilde{\lambda} - n_1 u - n_2 v & n_1 & n_2 \\ d_0^1 & n_1 u - \tilde{\lambda} & n_2 u \\ d_0^2 & n_1 v & n_2 v - \tilde{\lambda} \end{pmatrix} \tag{3}$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \end{pmatrix} \tag{4}$$

$$C = \begin{pmatrix} d_1^1 & c_1^1 & c_1^3 \\ d_1^2 & c_1^2 & c_1^4 \\ \vdots & \vdots & \vdots \\ d_N^1 & c_N^1 & c_N^3 \\ d_N^2 & c_N^2 & c_N^4 \end{pmatrix} \tag{5}$$

$$D = \begin{pmatrix} -\tilde{\lambda} & & & & \\ & -\tilde{\lambda} & & & \\ & & \ddots & & \\ & & & -\tilde{\lambda} & \\ & & & & -\tilde{\lambda} \end{pmatrix} \tag{6}$$

$$|D| = \tilde{\lambda}^{2N}$$

$$BD^{-1}C = \frac{1}{-\tilde{\lambda}} \begin{pmatrix} 0 & 0 & 0 \\ s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{pmatrix}$$

$$s_1 = \sum_{i=1}^{N} \left(b_i^1 d_i^1 + b_i^2 d_i^2 \right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1}(4\alpha_i u n_1 + \beta_i u n_2 + 3\alpha_i v n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_2 = \sum_{i=1}^{N} \left(b_i^1 c_i^1 + b_i^2 c_i^2 \right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1}(4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_3 = \sum_{i=1}^{N} \left(b_i^1 c_i^3 + b_i^2 c_i^4 \right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1}\alpha_i n_2(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_4 = \sum_{i=1}^{N} \left(b_i^3 d_i^1 + b_i^4 d_i^2 \right) = \sum_{i=1}^{N} \left(-\frac{1}{2i+1}(3\beta_i u n_1 + \alpha_i v n_1 + 4\beta_i v n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_5 = \sum_{i=1}^{N} \left(b_i^3 c_i^1 + b_i^4 c_i^2 \right) = \sum_{i=1}^{N} \left(\frac{3}{2i+1}\beta_i n_1(\alpha_i n_1 + \beta_i n_2) \right)$$

$$s_6 = \sum_{i=1}^{N} \left(b_i^3 c_i^3 + b_i^4 c_i^4 \right) = \sum_{i=1}^{N} \left(\frac{1}{2i+1}(\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$A - BD^{-1}C = \frac{1}{\tilde{\lambda}}\begin{pmatrix} -\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2 & \tilde{\lambda}n_1 & \tilde{\lambda}n_2 \\ \tilde{\lambda}d_0^1 + s_1 & \tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2 & \tilde{\lambda}un_2 + s_3 \\ \tilde{\lambda}d_0^2 + s_4 & \tilde{\lambda}vn_1 + s_5 & \tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6 \end{pmatrix}$$

$$\det\left(\tilde{\lambda}(A - BD^{-1}C)\right) = \left(-\tilde{\lambda}^2 - \tilde{\lambda}un_1 - \tilde{\lambda}vn_2\right)\left(\left(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\right)\left(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\right) - \left(\tilde{\lambda}vn_1 + s_5\right)\left(\tilde{\lambda}un_2 + s_3\right)\right)$$

$$-\tilde{\lambda}n_1\left(\left(\tilde{\lambda}d_0^1 + s_1\right)\left(\tilde{\lambda}vn_2 - \tilde{\lambda}^2 + s_6\right) - \left(\tilde{\lambda}d_0^2 + s_4\right)\left(\tilde{\lambda}un_2 + s_3\right)\right)$$

$$+\tilde{\lambda}n_2\left(\left(\tilde{\lambda}d_0^1 + s_1\right)\left(\tilde{\lambda}vn_1 + s_5\right) - \left(\tilde{\lambda}d_0^2 + s_4\right)\left(\tilde{\lambda}un_1 - \tilde{\lambda}^2 + s_2\right)\right)$$

$$= \left(-\tilde{\lambda}^2 - (un_1 + vn_2)\tilde{\lambda}\right)\left(\tilde{\lambda}^4 + (-un_1 - vn_2)\tilde{\lambda}^3 + (-s_2 - s_6)\tilde{\lambda}^2 + (un_1s_6 + vn_2s_2 - vn_1s_3 - un_2s_5)\tilde{\lambda} + s_2s_6 - s_3s_5\right)$$

$$-\tilde{\lambda}n_1\left(-d_0^1\tilde{\lambda}^3 + \left(d_0^1vn_2 - s_1 - d_0^2un_2\right)\tilde{\lambda}^2 + \left(d_0^1s_6 + vn_2s_1 - d_0^2s_3 - un_2s_4\right)\tilde{\lambda} + s_1s_6 - s_3s_4\right)$$

$$+\tilde{\lambda}n_2\left(d_0^2\tilde{\lambda}^3 + \left(d_0^1vn_1 - d_0^2un_1 + s_4\right)\tilde{\lambda}^2 + \left(d_0^1s_5 + vn_1s_1 - d_0^2s_2 - un_1s_4\right)\tilde{\lambda} + s_1s_5 - s_2s_4\right)$$

$$= -\tilde{\lambda}^6 + \left(s_2 + s_6 + (un_1 + vn_2)^2\right)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6)\tilde{\lambda}^3$$

$$+\left(s_3s_5 - s_2s_6 + \left(u^2n_1n_2 + uvn_2^2\right)s_5 + \left(uvn_1^2 + v^2n_1n_2\right)s_3 - \left(u^2n_1^2 + uvn_1n_2\right)s_6 - \left(uvn_1n_2 + v^2n_2^2\right)s_2\right)\tilde{\lambda}^2 + (s_3s_5 - s_2s_6)(un_1 +$$

$$+d_0^1n_1\tilde{\lambda}^4 + \left(-d_0^1vn_1n_2 + s_1n_1 + d_0^2un_1n_2\right)\tilde{\lambda}^3 + \left(-d_0^1n_1s_6 - vn_1n_2s_1 + d_0^2n_1s_3 + un_1n_2s_4\right)\tilde{\lambda}^2 + (s_3s_4 - s_1s_6)n_1\tilde{\lambda}$$

$$+d_0^2n_2\tilde{\lambda}^4 + \left(d_0^1vn_1n_2 - d_0^2un_1n_2 + n_2s_4\right)\tilde{\lambda}^3 + \left(d_0^1n_2s_5 + vn_1n_2s_1 - d_0^2n_2s_2 - un_1n_2s_4\right)\tilde{\lambda}^2 + (s_1s_5 - s_2s_4)n_2\tilde{\lambda}$$

$$= -\tilde{\lambda}^6 + \left(s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2\right)\tilde{\lambda}^4 + (vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4)\tilde{\lambda}^3$$

$$+\left(s_3s_5 - s_2s_6 + \left(u^2n_1n_2 + uvn_2^2 + d_0^1n_2\right)s_5 + \left(uvn_1^2 + v^2n_1n_2 + d_0^1n_1\right)s_3 - \left(u^2n_1^2 + uvn_1n_2 + d_0^1n_1\right)s_6 - \left(uvn_1n_2 + v^2n_2^2 + d_0^2n_2\right)\right)$$

$$+\left((s_3s_5 - s_2s_6)(un_1 + vn_2) + (s_3s_4 - s_1s_6)n_1 + (s_1s_5 - s_2s_4)n_2\right)\tilde{\lambda}$$

Look at coefficients of polynomial in $\tilde{\lambda}$.

$$s_2 + s_6 + (un_1 + vn_2)^2 + d_0^1n_1 + d_0^2n_2$$

$$= \sum_{i=1}^N\left(\frac{1}{2i+1}(4\alpha_in_1 + \beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + \sum_{i=1}^N\left(\frac{1}{2i+1}(\alpha_in_1 + 4\beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + (un_1 + vn_2)^2$$

$$+n_1^2\left(e_zgh - u^2 - \sum_{i=1}^N\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_1n_2\left(-uv - \sum_{i=1}^N\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right)$$

$$+n_1n_2\left(-uv - \sum_{i=1}^N\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2^2\left(e_zgh - v^2 - \sum_{i=1}^N\left(\frac{1}{2i+1}\beta_i^2\right)\right)$$

$$= \sum_{i=1}^N\left(\frac{5}{2i+1}(\alpha_in_1 + \beta_in_2)(\alpha_in_1 + \beta_in_2)\right) + e_zgh\left(n_1^2 + n_2^2\right) - \sum_{i=1}^N\left(\frac{1}{2i+1}\left(\alpha_i^2n_1^2 + 2\alpha_i\beta_in_1n_2 + \beta_i^2n_2^2\right)\right)$$

$$= \sum_{i=1}^N\left(\frac{4}{2i+1}(\alpha_in_1 + \beta_in_2)^2\right) + e_zgh\left(n_1^2 + n_2^2\right)$$

$$vn_1s_3 + un_2s_5 + un_1s_2 + vn_2s_6 + s_1n_1 + n_2s_4$$

$$= \sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_i vn_1n_2(\alpha_in_1 + \beta_in_2)\right)$$

$$+ \sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_i un_1n_2(\alpha_in_1 + \beta_in_2)\right)$$

$$+ \sum_{i=1}^{N}\left(\frac{1}{2i+1}\left(4\alpha_i un_1^2 + \beta_i un_1n_2\right)(\alpha_in_1 + \beta_in_2)\right)$$

$$+ \sum_{i=1}^{N}\left(\frac{1}{2i+1}\left(\alpha_i vn_1n_2 + 4\beta_i vn_2^2\right)(\alpha_in_1 + \beta_in_2)\right)$$

$$+ \sum_{i=1}^{N}\left(-\frac{1}{2i+1}\left(4\alpha_i un_1^2 + \beta_i un_1n_2 + 3\alpha_i vn_1n_2\right)(\alpha_in_1 + \beta_in_2)\right)$$

$$+ \sum_{i=1}^{N}\left(-\frac{1}{2i+1}\left(3\beta_i un_1n_2 + \alpha_i vn_1n_2 + 4\beta_i vn_2^2\right)(\alpha_in_1 + \beta_in_2)\right)$$

$$= \sum_{i=1}^{N}\left(\frac{1}{2i+1}(3\alpha_i vn_1n_2 + 3\beta_i un_1n_2 - 3\alpha_i vn_1n_2 - 3\beta_i un_1n_2)(\alpha_in_1 + \beta_in_2)\right)$$

$$= 0$$

$$a_i = \alpha_in_1 + \beta_in_2$$

$$s_3s_5 - s_2s_6$$

$$= \sum_{i=1}^{N}\left(\frac{3}{2i+1}\alpha_i n_2 a_i\right)\sum_{i=1}^{N}\left(\frac{3}{2i+1}\beta_i n_1 a_i\right) - \sum_{i=1}^{N}\left(\frac{1}{2i+1}(4\alpha_i n_1 + \beta_i n_2)a_i\right)\sum_{i=1}^{N}\left(\frac{1}{2i+1}(\alpha_i n_1 + 4\beta_i n_2)a_i\right)$$

$$= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{3}{2i+1}\frac{3}{2j+1}\alpha_i\beta_j n_1n_2 a_i a_j\right)\right) - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}(4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2)a_i a_j\right)\right)$$

$$= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}(9\alpha_i\beta_j n_1n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2))a_i a_j\right)\right)$$

$$= \sum_{i=1}^{N}\left(\frac{1}{(2i+1)^2}(9\alpha_i\beta_i n_1n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + 4\beta_i n_2))a_i a_i\right)$$

$$+ \sum_{i=1}^{N}\left(\sum_{j=1}^{i-1}\left(\frac{1}{2i+1}\frac{1}{2j+1}(9\alpha_i\beta_j n_1n_2 - (4\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + 4\beta_j n_2) + 9\alpha_j\beta_i n_1n_2 - (4\alpha_j n_1 + \beta_j n_2)(\alpha_i n_1 + 4\beta_i n_2))a_i a_j\right)\right)$$

$$= \sum_{i=1}^{N}\left(-\left(\frac{2}{2i+1}\right)^2(\alpha_i n_1 + \beta_i n_2)^2 a_i a_i\right) + \sum_{i=1}^{N}\left(\sum_{j=1}^{i-1}\left(-8\frac{1}{2i+1}\frac{1}{2j+1}(\alpha_i n_1 + \beta_i n_2)(\alpha_j n_1 + \beta_j n_2)a_i a_j\right)\right)$$

$$= -\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{2}{2i+1}\frac{2}{2j+1}a_i^2 a_j^2\right)\right)$$

$$= -\left(\sum_{i=1}^{N}\left(\frac{2}{2i+1}a_i^2\right)\right)^2$$

$$(u^2 n_1 n_2 + uv n_2^2 + d_0^1 n_2) s_5$$

$$= \left(n_2 n_1 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_2^2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= \left(e_z g h n_1 n_2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_5 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_i \beta_j n_1 n_2 a_i a_j \right) \right)$$

$$(uv n_1^2 + v^2 n_1 n_2 + d_0^2 n_1) s_3$$

$$= \left(-n_1^2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_1 n_2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_3 - \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \right) \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 (\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1 n_2 s_3 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{3}{2j+1} \alpha_j \beta_i n_1 n_2 a_i a_j \right) \right)$$

$$(u^2 n_1^2 + uv n_1 n_2 + d_0^1 n_1) s_6$$

$$= \left(n_1^2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i^2 \right) \right) - n_1 n_2 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1^2 s_6 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i n_1 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_1^2 s_6 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} \alpha_i n_1 (\alpha_j n_1 + 4\beta_j n_2) a_i a_j \right) \right)$$

$$(uv n_1 n_2 + v^2 n_2^2 + d_0^2 n_2) s_2$$

$$= \left(n_2 n_1 \left(- \sum_{i=1}^{N} \left(\frac{1}{2i+1} \alpha_i \beta_i \right) \right) + n_2^2 \left(e_z g h - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i^2 \right) \right) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_2^2 s_2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} \beta_i n_2 (\alpha_i n_1 + \beta_i n_2) \right) \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2)(\alpha_i n_1 + \beta_i n_2) \right)$$

$$= e_z g h n_2^2 s_2 - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left(\frac{1}{2i+1} \frac{1}{2j+1} \beta_i n_2 (4\alpha_j n_1 + \beta_j n_2) a_i a_j \right) \right)$$

$$s_3 s_5 - s_2 s_6 + \left(u^2 n_1 n_2 + u v n_2^2 + d_0^1 n_2\right) s_5 + \left(u v n_1^2 + v^2 n_1 n_2 + d_0^2 n_1\right) s_3$$
$$- \left(u^2 n_1^2 + u v n_1 n_2 + d_0^1 n_1\right) s_6 - \left(u v n_1 n_2 + v^2 n_2^2 + d_0^2 n_2\right) s_2$$

$$= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z g h n_1 n_2 s_5 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_i\beta_j n_1 n_2 a_i a_j\right)\right)$$

$$+ e_z g h n_1 n_2 s_3 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{3}{2j+1}\alpha_j\beta_i n_1 n_2 a_i a_j\right)\right)$$

$$- e_z g h n_1^2 s_6 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_i n_1(\alpha_j n_1 + 4\beta_j n_2)a_i a_j\right)\right)$$

$$- e_z g h n_2^2 s_2 + \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\beta_i n_2(4\alpha_j n_1 + \beta_j n_2)a_i a_j\right)\right)$$

$$= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z g h\left(n_1 n_2(s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2\right)$$

$$-6\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\alpha_i\beta_j n_1 n_2 a_i a_j\right)\right)$$

$$+\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\left(\alpha_i\alpha_j n_1^2 + 8\alpha_i\beta_j n_1 n_2 + \beta_i\beta_j n_2^2\right)a_i a_j\right)\right)$$

$$= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z g h\left(n_1 n_2(s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2\right)$$

$$+\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}\left(\alpha_i\alpha_j n_1^2 + 2\alpha_i\beta_j n_1 n_2 + \beta_i\beta_j n_2^2\right)a_i a_j\right)\right)$$

$$= -4\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z g h\left(n_1 n_2(s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2\right)$$

$$+\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\left(\frac{1}{2i+1}\frac{1}{2j+1}a_i^2 a_j^2\right)\right)$$

$$= -3\left(\sum_{i=1}^{N}\left(\frac{1}{2i+1}a_i^2\right)\right)^2 + e_z g h\left(n_1 n_2(s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2\right)$$

$$n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2$$

$$= n_1 n_2 \sum_{i=1}^{N} \left(\frac{3}{2i+1} \alpha_i n_2 a_i \right)$$

$$+ n_1 n_2 \sum_{i=1}^{N} \left(\frac{3}{2i+1} \beta_i n_1 a_i \right)$$

$$- n_1^2 \sum_{i=1}^{N} \left(\frac{1}{2i+1} (\alpha_i n_1 + 4\beta_i n_2) a_i \right)$$

$$- n_2^2 \sum_{i=1}^{N} \left(\frac{1}{2i+1} (4\alpha_i n_1 + \beta_i n_2) a_i \right)$$

$$= \sum_{i=1}^{N} \left(\frac{1}{2i+1} \big(3\alpha_i n_1 n_2^2 + 3\beta_i n_1^2 n_2 - (\alpha_i n_1^3 + 4\beta_i n_1^2 n_2) - (4\alpha_i n_1 n_2^2 + \beta_i n_2^3)\big) a_i \right)$$

$$= -\sum_{i=1}^{N} \left(\frac{1}{2i+1} \big(\alpha_i n_1^3 + \beta_i n_1^2 n_2 + \alpha_i n_1 n_2^2 + \beta_i n_2^3\big) a_i \right)$$

$$= -\sum_{i=1}^{N} \left(\frac{1}{2i+1} \big(n_1^2 a_i + n_2^2 a_i\big) a_i \right)$$

$$= -(n_1^2 + n_2^2) \sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right)$$

$$-3 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z gh \big(n_1 n_2 (s_3 + s_5) - n_1^2 s_6 - n_2^2 s_2\big)$$

$$= -3 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 - e_z gh \big(n_1^2 + n_2^2\big) \sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right)$$

$$(s_3 s_5 - s_2 s_6)(un_1 + vn_2) + (s_3 s_4 - s_1 s_6) n_1 + (s_1 s_5 - s_2 s_4) n_2$$

$$= -4 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 (un_1 + vn_2) + 4 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 un_1 + 4 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 vn_2$$

$$= 0$$

$$\det\big(\tilde{\lambda}(A - BD^{-1}C)\big) = -\tilde{\lambda}^6 + \left(4 \sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) + e_z gh \big(n_1^2 + n_2^2\big) \right) \tilde{\lambda}^4$$

$$- \left(3 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)^2 + e_z gh \big(n_1^2 + n_2^2\big) \sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right) \tilde{\lambda}^2$$

$$\det(A - BD^{-1}C) = -\frac{1}{\tilde{\lambda}} \left(\tilde{\lambda}^2 - \left(3 \left(\sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right) + e_z gh \big(n_1^2 + n_2^2\big) \right) \right) \left(\tilde{\lambda}^2 - \sum_{i=1}^{N} \left(\frac{1}{2i+1} a_i^2 \right) \right)$$

$$|C_N| = \begin{vmatrix} n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\ c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & \vdots & & & \ddots & & \\ c_N^1 & c_N^3 & & & & -\tilde{\lambda} & \\ c_N^2 & c_N^4 & & & & & -\tilde{\lambda} \end{vmatrix}$$

$$= -c_N^2 |C_N^1| + c_N^4 |C_N^2| - \tilde{\lambda} |C_N^3|$$

$$|C_N^1| = \begin{vmatrix} n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^3 & -\tilde{\lambda} & & & & \\ c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & & & \ddots & & \\ c_N^3 & & & & -\tilde{\lambda} \end{vmatrix}$$

$$= c_N^3 |B_N^1| + \tilde{\lambda} |C_N^4|$$

$$|B_N^1| = \begin{vmatrix} b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ -\tilde{\lambda} & & & & \\ & \ddots & & & \\ & & -\tilde{\lambda} \end{vmatrix}$$

$$= \left(b_N^1 b_N^4 - b_N^2 b_N^3 \right) \left(-\tilde{\lambda} \right)^{2N-2}$$

$$|C_N^4| = \begin{vmatrix} n_2 u & b_1^1 & b_1^2 & \cdots & b_{N-1}^2 & b_N^2 \\ n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_{N-1}^4 & b_N^4 \\ c_1^3 & -\tilde{\lambda} & & & & \\ c_1^4 & & -\tilde{\lambda} & & & \\ \vdots & & & \ddots & & \\ c_{N-1}^4 & & & & -\tilde{\lambda} \end{vmatrix}$$

$$\begin{vmatrix} n_1u-\tilde\lambda & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_1v & n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^1 & c_1^3 & -\tilde\lambda & & & & \\ c_1^2 & c_1^4 & & -\tilde\lambda & & & \\ \vdots & \vdots & & & \ddots & & \\ c_N^1 & c_N^3 & & & & -\tilde\lambda & \\ c_N^2 & c_N^4 & & & & & -\tilde\lambda \end{vmatrix} - n_1 \begin{vmatrix} d_0^1 & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^3 & -\tilde\lambda & & & & \\ d_1^2 & c_1^4 & & -\tilde\lambda & & & \\ \vdots & \vdots & & & \ddots & & \\ d_N^1 & c_N^3 & & & & -\tilde\lambda & \\ d_N^2 & c_N^4 & & & & & -\tilde\lambda \end{vmatrix}$$

$$+n_2 \begin{vmatrix} d_0^1 & n_1u-\tilde\lambda & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ d_0^2 & n_1v & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ d_1^1 & c_1^1 & -\tilde\lambda & & & & \\ d_1^2 & c_1^2 & & -\tilde\lambda & & & \\ \vdots & \vdots & & & \ddots & & \\ d_N^1 & c_N^1 & & & & -\tilde\lambda & \\ d_N^2 & c_N^2 & & & & & -\tilde\lambda \end{vmatrix}$$

$$|C_{N+1}| = \begin{vmatrix} n_1u-\tilde\lambda & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_1v & n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^1 & c_1^3 & -\tilde\lambda & & & & \\ c_1^2 & c_1^4 & & -\tilde\lambda & & & \\ \vdots & \vdots & & & \ddots & & \\ c_N^1 & c_N^3 & & & & -\tilde\lambda & \\ c_N^2 & c_N^4 & & & & & -\tilde\lambda \end{vmatrix}$$

$$= (-1)^{2N+3}c_N^2 \begin{vmatrix} n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^3 & -\tilde\lambda & & & & \\ c_1^4 & & -\tilde\lambda & & & \\ \vdots & & & \ddots & & \\ c_N^3 & & & & -\tilde\lambda \end{vmatrix} + (-1)^{2N+4}c_N^4 \begin{vmatrix} n_1u-\tilde\lambda & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_1v & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^1 & -\tilde\lambda & & & & \\ c_1^2 & & -\tilde\lambda & & & \\ \vdots & & & \ddots & & \\ c_N^1 & & & & -\tilde\lambda \end{vmatrix}$$

$$+(-1)^{4N+4}\tilde\lambda \begin{vmatrix} n_1u-\tilde\lambda & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 \\ n_1v & n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 \\ c_1^1 & c_1^3 & -\tilde\lambda & & & \\ c_1^2 & c_1^4 & & -\tilde\lambda & & \\ \vdots & \vdots & & & \ddots & \\ c_N^1 & c_N^3 & & & & -\tilde\lambda \end{vmatrix}$$

$$= (-1)^{2N+3}c_N^2(-1)^{2N+2}c_N^3 \begin{vmatrix} b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ -\tilde\lambda & & & & \\ & -\ddots & & & \\ & & \tilde\lambda & & \end{vmatrix} + (-1)^{2N+4}c_N^4 \begin{vmatrix} n_1u-\tilde\lambda & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\ n_1v & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\ c_1^1 & -\tilde\lambda & & & & \\ c_1^2 & & -\tilde\lambda & & & \\ \vdots & & & \ddots & & \\ c_N^1 & & & & -\tilde\lambda \end{vmatrix}$$

$$+(-1)^{4N+4}\tilde\lambda \begin{vmatrix} n_1u-\tilde\lambda & n_2u & b_1^1 & b_1^2 & \cdots & b_N^1 \\ n_1v & n_2v-\tilde\lambda & b_1^3 & b_1^4 & \cdots & b_N^3 \\ c_1^1 & c_1^3 & -\tilde\lambda & & & \\ c_1^2 & c_1^4 & & -\tilde\lambda & & \\ \vdots & \vdots & & & \ddots & \\ c_N^1 & c_N^3 & & & & -\tilde\lambda \end{vmatrix}$$

Quasilinear Eigenvectors