Nonconservative Products

1 Definition

Consider the nonconservative product

$$g(\boldsymbol{q})\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x},$$

where $g(\boldsymbol{q}) : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$ is continuous, but $\boldsymbol{q}$ is possibly discontinuous. In this case, the product is traditionally not well-defined at the discontinuities of $\boldsymbol{q}$. In order to define this product for discontinuous functions, $\boldsymbol{q}$, it is possible to regularize $\boldsymbol{q}$ with a path ϕ at discontinuities according to the theory laid out by Dal Maso, Le Floch, and Murat. To this end consider Lipschitz continuous paths, $\phi : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, that satisfy the following properties.

1. $\forall \boldsymbol{q}_L, \boldsymbol{q}_R \in \mathbb{R}^p$, $\boldsymbol{\phi}(0, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{q}_L$ and $\boldsymbol{\phi}(1, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{q}_R$
2. $\exists k > 0$, $\forall \boldsymbol{q}_L, \boldsymbol{q}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, $\left|\frac{\partial \boldsymbol{\phi}}{\partial s}(s, \boldsymbol{q}_L, \boldsymbol{q}_R)\right| \leq k|\boldsymbol{q}_L - \boldsymbol{q}_R|$ elementwise
3. $\exists k > 0$, $\forall \boldsymbol{q}_L, \boldsymbol{q}_R, \boldsymbol{u}_L, \boldsymbol{u}_R \in \mathbb{R}^p$, $\forall s \in [0,1]$, elementwise
$$\left|\frac{\partial \boldsymbol{\phi}}{\partial s}(s, \boldsymbol{q}_L, \boldsymbol{q}_R) - \frac{\partial \boldsymbol{\phi}}{\partial s}(s, \boldsymbol{u}_L, \boldsymbol{u}_R)\right| \leq k(|\boldsymbol{q}_L - \boldsymbol{u}_L| + |\boldsymbol{q}_R - \boldsymbol{u}_R|)$$

Once we have these paths, $\boldsymbol{\phi}$, we can define the nonconservative product.

Let $q : [a,b] \to \mathbb{R}^p$ be a function of bounded variation,let $g : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p$be a continuous function, and let ϕ satisfy the properties given above. Then there exists a unique real-valued bounded Borel measure μ on $[a,b]$ characterized by the two following properties.

1. if q is continuous on a Borel set $B \subset [a,b]$, then
$$\mu(B) = \int_B g(q)\frac{\mathrm{d}q}{\mathrm{d}x}\,\mathrm{d}x$$
2. if q is discontinuous at a point $x_0 \in [a,b]$, then
$$\mu(x_0) = \int_0^1 g(\phi(s; q(x_0^-), q(x_0^+)))\frac{\partial \phi}{\partial s}(s; q(x_0^-), q(x_0^+))\,\mathrm{d}s$$

By definition, this measure μ is the nonconservative product $g(q)\frac{\mathrm{d}q}{\mathrm{d}x}$ and will be denoted by

$$\mu = \left[g(q)\frac{\mathrm{d}q}{\mathrm{d}x}\right]_\phi$$

In higher dimensions the paths, $\boldsymbol{\phi}$ must also have the property that

4. $\boldsymbol{\phi}(s, \boldsymbol{q}_L, \boldsymbol{q}_R) = \boldsymbol{\phi}(1 - s, \boldsymbol{q}_L, \boldsymbol{q}_R)$

2 Weak Solutions

A function $\boldsymbol{q}$ of bounded variation is a weak solution to

$$\boldsymbol{q}_t + g(\boldsymbol{q})\boldsymbol{q}_x = 0$$

if

$$\boldsymbol{q}_t + [g(\boldsymbol{q})\boldsymbol{q}_x]_\phi = 0$$

as a bounded Borel measure on $\mathbb{R} \times \mathbb{R}_+$.

This is equivalent to finding $\boldsymbol{q}$ that satisfies,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \psi_t(t,x)\boldsymbol{q}(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}_+}\int_{\mathbb{R}} \psi(t,\cdot)[g(\boldsymbol{q}(t,\cdot))\boldsymbol{q}(t,\cdot)_x]_\phi\,\mathrm{d}t = \mathbf{0}$$

for all functions $\psi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R})$.

3 DG Weak Formulation