Local Discontinuous Galerkin Method for the Diffusion Equation

We would like to solve the 1D diffusion equation with a Discontinuous Galerkin Method. The 1D diffusion equation is given as

$$q_t = q_{xx}.$$

If we were to naively apply DG methods, we could discretize the domain and consider piecewise polynomial approximation. We would then multiply by a test function and integrate by parts.

$$\int_{I_j} u_t v \, \mathrm{d}x = \int_{I_j} u_{xx} v \, \mathrm{d}x$$

$$\int_{I_j} u_t v \, \mathrm{d}x = \left(\left(\hat{u}_x v^-\right)_{j+1/2} - \left(\hat{u}_x v^+\right)_{j-1/2} \right) - \int_{I_j} u_x v_x \, \mathrm{d}x$$

and we use the average numerical flux

$$(\hat{u}_x)_{j+1/2} = \frac{(u_x^-)_{j+1/2} + (u_x^+)_{j+1/2}}{2}$$

This method is convergent and stable but it converges to the wrong solution.

Local Discontinuous Galerkin Method The Local Discontinuous Galerkin method proposes a different approach. First rewrite the diffusion equation as a system of first order equations.

$$r = q_x$$
$$q_t = r_x$$

The LDG method becomes the process of finding $q_h, r_h \in V_h$ in the DG solution space, such that for all test functions $v_h, w_h \in V_h$ and for all j the following equations are satisfied

$$\int_{I_j} r_h w_h \, \mathrm{d}x = \int_{I_j} (q_h)_x w_h \, \mathrm{d}x$$
$$\int_{I_j} (q_h)_t v_h \, \mathrm{d}x = \int_{I_j} (r_h)_x v_h \, \mathrm{d}x$$

After integrating by parts, these equations are

$$\int_{I_j} r_h w_h \, \mathrm{d}x = \left(\left(\hat{q}_h w_h^-\right)_{j+1/2} - \left(\hat{q}_j w_h^+\right)_{j-1/2} \right) - \int_{I_j} q_h (w_h)_x \, \mathrm{d}x$$
$$\int_{I_j} (q_h)_t v_h \, \mathrm{d}x = \left(\left(\hat{r}_h v_h^-\right)_{j+1/2} - \left(\hat{r}_h v_h^+\right)_{j-1/2} \right) - \int_{I_j} r_h (v_h)_x \, \mathrm{d}x$$

A common choice of numerical fluxes are the so-called alternating fluxes.

$$\hat{r}_h = r_h^-$$
$$\hat{q}_h = q_h^+$$

Implementation If we consider a single cell I_j, do a linear transformation from $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$, and consider specifically the Legendre polynomial basis $\left\{\phi^k(\xi)\right\}$ with the following orthogonality property

$$\frac{1}{2} \int_{-1}^{1} \phi^j(\xi) \phi^k(\xi) \, \mathrm{d}\xi = \delta_{jk}$$

we can form a more concrete LDG method for implementing. The linear transformation can be expressed as

$$x = \frac{\Delta x}{2} \xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

or

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

After this tranformation the diffusion equation become

$$q_t = \frac{4}{\Delta x^2} q_{\xi\xi}$$

on the cell I_j. We can then write this as the following system of first order equations.

$$r = \frac{2}{\Delta x} q_\xi q_t \qquad = \frac{2}{\Delta x} r_\xi$$

With the Legendre basis, the numerical solution on I_j can be written as

$$\begin{aligned}
q|_{I_i} \approx q_h|_{I_i} &= \sum_{l=1}^{M} \left(Q_i^l \phi^l(\xi)\right) \\
r|_{I_i} \approx r_h|_{I_i} &= \sum_{l=1}^{M} \left(R_i^l \phi^l(\xi)\right)
\end{aligned}$$

Now plugging these into the system and multiplying by a Legendre basis and integrating gives. I will use the following shorthand for numerical fluxes using one of the alternating flux options.

$$\begin{aligned}
\hat{Q}_{i+1/2} &= \sum_{l=1}^{M} \left(Q_{i+1}^l \phi^l(-1)\right) \\
\hat{R}_{i+1/2} &= \sum_{l=1}^{M} \left(R_i^l \phi^l(1)\right)
\end{aligned}$$

$$\begin{aligned}
r &= \frac{2}{\Delta x} q_\xi \\
\sum_{l=1}^{M} \left(R_i^l \phi^l(\xi)\right) &= \frac{2}{\Delta x} \sum_{l=1}^{M} \left(Q_i^l \phi_\xi^l(\xi)\right) \\
\frac{1}{2}\int_{-1}^{1} \sum_{l=1}^{M} \left(R_i^l \phi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi &= \frac{1}{\Delta x}\int_{-1}^{1} \sum_{l=1}^{M} \left(Q_i^l \phi_\xi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi \\
R_i^k &= \frac{1}{\Delta x}\int_{-1}^{1} \sum_{l=1}^{M} \left(Q_i^l \phi_\xi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi \\
R_i^k &= -\frac{1}{\Delta x}\int_{-1}^{1} \sum_{l=1}^{M} \left(Q_i^l \phi^l(\xi)\right) \phi_\xi^k(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x}\left(\phi^k(1)\hat{Q}_{i+1/2} - \phi^k(-1)\hat{Q}_{i-1/2}\right)
\end{aligned}$$

$$
\begin{aligned}
q_t &= \frac{2}{\Delta x} r_\xi \\
\sum_{l=1}^{M} \left(\dot{Q}_i^l \phi^l(\xi)\right) &= \frac{2}{\Delta x} \sum_{l=1}^{M} \left(R_i^l \phi_\xi^l(\xi)\right) \\
\frac{1}{2} \int_{-1}^{1} \sum_{l=1}^{M} \left(\dot{Q}_i^l \phi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi &= \frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(R_i^l \phi_\xi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi \\
\dot{Q}_i^k &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(U_i^l \phi_\xi^l(\xi)\right) \phi^k(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^k(1) \hat{R}_{i+1/2} - \phi^k(-1) \hat{R}_{i-1/2}\right)
\end{aligned}
$$

Now this is a system of ODEs, there are $M \times N$ ODEs if M is the spacial order and N is the number of cells.

Matrix Representation Some common matrices and vectors that appear in these equations are

$$
\begin{aligned}
\boldsymbol{Q}_i &= \left[Q_i^l\right]_{l=1}^{M} \\
\boldsymbol{\phi}(\xi) &= \left[\phi^k(\xi)\right]_{k=1}^{M} \\
\Phi(\xi_1, \xi_2) &= \boldsymbol{\phi}(\xi_1) \boldsymbol{\phi}^T(\xi_2) \\
A &= [a_{kl}]_{k,l=1}^{M} \\
a_{kl} &= \int_{-1}^{1} \phi_\xi^k(\xi) \phi^l(\xi) \, \mathrm{d}\xi
\end{aligned}
$$

Also the numerical fluxes can be written as the following dot products

$$
\begin{aligned}
\hat{Q}_{i+1/2} &= \sum_{l=1}^{M} \left(Q_{i+1}^l \phi^l(-1)\right) \\
&= \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_{i+1} \\
\hat{R}_{i+1/2} &= \sum_{l=1}^{M} \left(R_i^l \phi^l(1)\right) \\
&= \boldsymbol{\phi}^T(1) \boldsymbol{R}_i
\end{aligned}
$$

$$
\begin{aligned}
R_i^k &= -\frac{1}{\Delta x} \int_{-1}^{1} \sum_{l=1}^{M} \left(Q_i^l \phi^l(\xi)\right) \phi_\xi^k(\xi) \, \mathrm{d}\xi + \frac{1}{\Delta x} \left(\phi^k(1) \hat{Q}_{i+1/2} - \phi^k(-1) \hat{Q}_{i-1/2}\right) \\
R_i^k &= -\frac{1}{\Delta x} \sum_{l=1}^{M} \left(Q_i^l \int_{-1}^{1} \phi^l(\xi) \phi_\xi^k(\xi) \, \mathrm{d}\xi\right) + \frac{1}{\Delta x} \left(\phi^k(1) \hat{Q}_{i+1/2} - \phi^k(-1) \hat{Q}_{i-1/2}\right) \\
R_i^k &= -\frac{1}{\Delta x} (A \boldsymbol{Q}_i)_k + \frac{1}{\Delta x} \left(\phi^k(1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_{i+1} - \phi^k(-1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_i\right) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} A \boldsymbol{Q}_i + \frac{1}{\Delta x} \left(\boldsymbol{\phi}(1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_{i+1} - \boldsymbol{\phi}(-1) \boldsymbol{\phi}^T(-1) \boldsymbol{Q}_i\right) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} A \boldsymbol{Q}_i + \frac{1}{\Delta x} (\Phi(1, -1) \boldsymbol{Q}_{i+1} - \Phi(-1, -1) \boldsymbol{Q}_i) \\
\boldsymbol{R}_i &= -\frac{1}{\Delta x} (A + \Phi(-1, -1)) \boldsymbol{Q}_i + \frac{1}{\Delta x} \Phi(1, -1) \boldsymbol{Q}_{i+1}
\end{aligned}
$$

$$
\begin{aligned}
Q_i^k &= -\frac{1}{\Delta x}\int_{-1}^{1}\sum_{l=1}^{M}\left(R_i^l\phi^l(\xi)\right)\phi_\xi^k(\xi)\,\mathrm{d}\xi + \frac{1}{\Delta x}\left(\phi^k(1)\hat{R}_{i+1/2} - \phi^k(-1)\hat{R}_{i-1/2}\right)\\
Q_i^k &= -\frac{1}{\Delta x}(A\boldsymbol{R}_i)_k + \frac{1}{\Delta x}\left(\phi^k(1)\boldsymbol{\phi}^T(1)\boldsymbol{R}_i - \phi^k(-1)\boldsymbol{\phi}^T(1)\boldsymbol{R}_{i-1}\right)\\
\boldsymbol{Q}_i &= -\frac{1}{\Delta x}(A - \Phi(1,1))\boldsymbol{R}_i - \frac{1}{\Delta x}\Phi(-1,1)\boldsymbol{R}_{i-1}
\end{aligned}
$$

Proving Stability In order to prove that this method is L^2 stable consider we sum both of the integral equations from before.

$$
\begin{aligned}
\int_{I_j}(u_h)_t v_h\,\mathrm{d}x + \int_{I_j} q_h w_h\,\mathrm{d}x = \left(\left(q_h^+ v_h^-\right)_{j+1/2} - \left(q_h^+ v_h^+\right)_{j-1/2}\right)\\
+\left(\left(u_h^- w_h^-\right)_{j+1/2} - \left(u_j^- w_h^+\right)_{j-1/2}\right) - \int_{I_j} q_h(v_h)_x\,\mathrm{d}x - \int_{I_j} u_h(w_h)_x\,\mathrm{d}x
\end{aligned}
$$

Consider using $v_h = u_h$ and $w_h = q_h$.

$$
\begin{aligned}
\int_{I_j}(u_h)_t u_h\,\mathrm{d}x + \int_{I_j} q_h q_h\,\mathrm{d}x = \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(q_h^+ u_h^+\right)_{j-1/2}\right)\\
+\left(\left(u_h^- q_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) - \int_{I_j} q_h(u_h)_x\,\mathrm{d}x - \int_{I_j} u_h(q_h)_x\,\mathrm{d}x
\end{aligned}
$$

Consider the following shorthand notation

$$
B_j = \int_{I_j}(u_h)_t u_h\,\mathrm{d}x + \int_{I_j} q_h q_h\,\mathrm{d}x
$$

$$
B_j = \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(q_h^+ u_h^+\right)_{j-1/2}\right) + \left(\left(u_h^- q_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) - \int_{I_j} q_h(u_h)_x\,\mathrm{d}x - \int_{I_j} u_h(q_h)_x\,\mathrm{d}x
$$

This can be simplified in several ways. First simplify the left hand side.

$$
\begin{aligned}
B_j &= \int_{I_j}(u_h)_t u_h\,\mathrm{d}x + \int_{I_j} q_h q_h\,\mathrm{d}x\\
B_j &= \frac{1}{2}\int_{I_j}\frac{\mathrm{d}}{\mathrm{d}t}\left(u_h^2\right)\mathrm{d}x + \int_{I_j} q_h^2\,\mathrm{d}x\\
B_j &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_j} u_h^2\,\mathrm{d}x + \int_{I_j} q_h^2\,\mathrm{d}x\\
B_j &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_{L^2(I_j)}^2 + \|q_h\|_{L^2(I_j)}^2
\end{aligned}
$$

Second the right hand side can be simplified.

$$
\begin{aligned}
\int_{I_j} q_h(u_h)_x\,\mathrm{d}x + \int_{I_j} u_h(q_h)_x\,\mathrm{d}x &= \int_{I_j} q_h(u_h)_x + u_h(q_h)_x\,\mathrm{d}x\\
&= \int_{I_j}(q_h u_h)_x\,\mathrm{d}x\\
&= (q_h^- u_h^-)_{j+1/2} - (q_h^+ u_h^+)_{j-1/2}
\end{aligned}
$$

Now

$$B_j = \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(q_h^+ u_h^+\right)_{j-1/2}\right) + \left(\left(u_h^- q_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) - \left((q_h^- u_h^-)_{j+1/2} - (q_h^+ u_h^+)_{j-1/2}\right)$$

$$B_j = \left(q_h^+ u_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}$$

Assuming periodic boundary conditions, and summing B_j over all cells

$$\begin{aligned}
\sum_{j=1}^{N} (B_j) &= \sum_{j=1}^{N} \left(\left(q_h^+ u_h^-\right)_{j+1/2} - \left(u_h^- q_h^+\right)_{j-1/2}\right) \\
&= -\left(u_h^- q_h^+\right)_{1/2} + \sum_{k=1}^{N} \left(\left(q_h^+ u_h^-\right)_{k+1/2} - \left(u_h^- q_h^+\right)_{k+1/2}\right) + \left(q_h^+ u_h^-\right)_{N+1/2} \\
&= 0
\end{aligned}$$

This shows that

$$\sum_{j=1}^{N} (B_j) = \sum_{j=1}^{N} \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_{L^2(I_j)}^2 + \|q_h\|_{L^2(I_j)}^2\right)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_{L^2}^2 + \|q_h\|_{L^2}^2 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_{L^2}^2 \leq 0$$