Local Discontinuous Galerkin Method for the Thin Film Diffusion Equation

We would like to solve the 1D thin film diffusion equation with a Discontinuous Galerkin Method. The 1D diffusion equation is given as

$$q_t = -\left(q^3 q_{xxx}\right)_x.$$

If we consider a single cell I_j, do a linear transformation from $x \in \left[x_{j-1/2}, x_{j+1/2}\right]$ to $\xi \in [-1, 1]$, and consider specifically the Legendre polynomial basis $\left\{\phi^k(\xi)\right\}$ with the following orthogonality property

$$\frac{1}{2}\int_{-1}^{1} \phi^j(\xi)\phi^k(\xi)\,\mathrm{d}\xi = \delta_{jk}$$

we can form a more concrete LDG method for implementing. The linear transformation can be expressed as

$$x = \frac{\Delta x}{2}\xi + \frac{x_{j-1/2} + x_{j+1/2}}{2}$$

or

$$\xi = \frac{2}{\Delta x}\left(x - \frac{x_{j-1/2} + x_{j+1/2}}{2}\right)$$

After this tranformation the diffusion equation become

$$q_t = -\frac{16}{\Delta x^4}\left(q^3 q_{\xi\xi\xi}\right)_\xi$$

on the cell I_j.

We can then write this as the following system of first order equations.

$$\begin{aligned}
r &= \frac{2}{\Delta x} q_\xi \\
s &= \frac{2}{\Delta x} r_\xi \\
u &= \frac{2}{\Delta x} q^3 s_\xi \\
q_t &= -\frac{2}{\Delta x} u_{xi}
\end{aligned}$$

Now approximate r, s, u, and q on I_j as

$$\begin{aligned}
r \approx r_h &= \sum_{k=1}^{M}\left(R_k \phi^k(\xi)\right) \\
s \approx s_h &= \sum_{k=1}^{M}\left(S_k \phi^k(\xi)\right) \\
u \approx u_h &= \sum_{k=1}^{M}\left(U_k \phi^k(\xi)\right) \\
q \approx q_h &= \sum_{k=1}^{M}\left(Q_k \phi^k(\xi)\right)
\end{aligned}$$

Plug into the equations and multiply by Legendre polynomials and integrate.