Derivation of Thin Film Equations

In this model we consider a thin film of liquid on a flat surface with a free interface. This liquid is driven by gravity, shear and normal forces on the surface, and surface tension (see Figure 1) We begin by considering the two dimensional incompressible Navier-Stokes equations, which have the form,

$$u_x + w_z = 0 \tag{1}$$
$$\rho(u_t + uu_x + wu_z) = -p_x + \mu\Delta u - \phi_x \tag{2}$$
$$\rho(w_t + uw_x + ww_z) = -p_z + \mu\Delta w - \phi_z \tag{3}$$

where ρ is the density, u is the horizontal velocity, w is the vertical velocity, p is the pressure, and ϕ is the force of gravity. Equation (1) is the incompressibility condition and also represents conservation of mass. Equations (2) and (3) represent the conservation of momementum in the x and z directions respectively. We take a no penetration and no slip boundary condition at the lower boundary and the kinimatic boundary condition at the upper boundary. These boundary conditions can be expressed as follows,

$$w = 0, u = 0 \qquad \text{at } z = 0 \tag{4}$$
$$w = h_t + uh_x \qquad \text{at } z = h \tag{5}$$

where h is the height of the liquid. We can also describe the stress tensor, $\boldsymbol{T}$, at the free surface, $z = h$, as

$$\boldsymbol{T} \cdot \boldsymbol{n} = (-\kappa\sigma + \Pi_0)\boldsymbol{n} + \left(\frac{\partial \sigma}{\partial s} + \tau_0\right)\boldsymbol{t} \qquad \text{at } z = h$$

where κ is the mean curvature, σ is the surface tension, and Π_0 and τ_0 are the normal and tangential components of the forcing respectively.

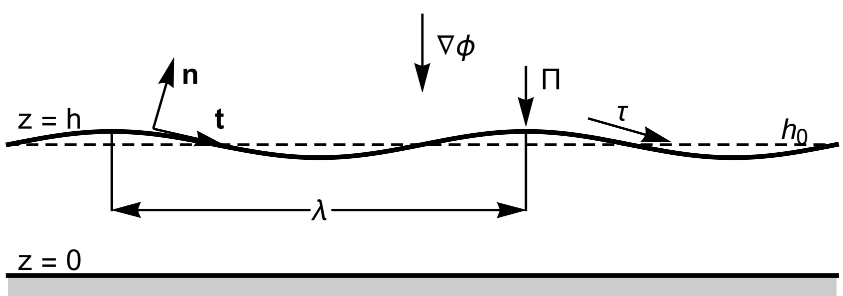

Figure 1: A diagram of the situation in question.

These equations completely describe the fluid, but we are now going to make a lubrication approximation, through a scaling argument. For the lubrication approximation we are going to assume that the average height of the liquid, h_0, is much smaller then the characterstic wavelength of the liquid, λ. We will denote the ratio of these two lengths as ε, that is

$$\varepsilon = \frac{h_0}{\lambda} \ll 1. \tag{6}$$

Now we nondimensionalize the rest of the variables with respect to this ratio. We denote the nondimensional variables as the uppercase variables. As we stated earlier the characteristic height is h_0 and the characteristic length is $\lambda = h_0/\varepsilon$, so the nondimensional length variables are

$$Z = \frac{z}{h_0}, \quad X = \frac{\varepsilon x}{h_0}, \quad H = \frac{h}{h_0}. \tag{7}$$

Let U_0 be the characteristic horizontal velocity, then

$$U = \frac{u}{U_0}, \quad W = \frac{w}{\varepsilon U_0} \tag{8}$$

where the nondimensional vertical velocity, W, follows from the continuity equation, equation (1). It follows that time will be scaled by λ/U_0, so nondimensional time is

$$T = \frac{\varepsilon U_0 t}{h_0}. \tag{9}$$

Finally we assume that the flow is locally parallel or equivalently $p_x \sim \mu u_{zz}$. This gives us the proper scaling for the pressure, gravity, and surface stresses,

$$P = \frac{\varepsilon h_0}{\mu U_0} p, \quad \Phi = \frac{\varepsilon h_0}{\mu U_0} \phi, \quad \Pi = \frac{\varepsilon h_0}{\mu U_0} \Pi_0, \quad \tau = \frac{h_0}{\mu U_0} \tau_0. \tag{10}$$

Lastly we can nondimensionalize the surface tension as

$$\Sigma = \frac{\varepsilon \sigma}{\mu U_0} \tag{11}$$

The following give some intermediate steps in how to nondimensionalize variables that include derivatives.

$$\begin{aligned}
u_x &= U_0 U_X \frac{\partial X}{\partial x} = \frac{\varepsilon U_0}{h_0} U_X \\
w_z &= \varepsilon U_0 W_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon U_0}{h_0} W_Z \\
u_t &= U_0 U_T \frac{\partial T}{\partial t} = \frac{\varepsilon U_0^2}{h_0} U_T \\
u u_x &= U_0 U U_0 U_X \frac{\partial X}{\partial x} = \frac{\varepsilon U_0^2}{h_0} U U_X \\
w u_z &= \varepsilon U_0 W U_0 U_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon U_0^2}{h_0} W U_Z \\
w_t &= \varepsilon U_0 W_T \frac{\partial T}{\partial t} = \frac{\varepsilon^2 U_0^2}{h_0} W_T \\
u w_x &= U_0 U \varepsilon U_0 W_X \frac{\partial X}{\partial x} = \frac{\varepsilon^2 U_0^2}{h_0} U W_X \\
w w_z &= \varepsilon U_0 W \varepsilon U_0 W_Z \frac{\partial Z}{\partial z} = \frac{\varepsilon^2 U_0^2}{h_0} W W_Z \\
p_x &= \frac{\mu U_0}{\varepsilon h_0} P_X \frac{\partial X}{\partial x} = \frac{\mu U_0}{h_0^2} P_X \\
p_z &= \frac{\mu U_0}{\varepsilon h_0} P_Z \frac{\partial Z}{\partial z} = \frac{\mu U_0}{\varepsilon h_0^2} P_Z \\
u_{xx} &= U_0 U_{XX} \left(\frac{\partial X}{\partial x} \right)^2 = \frac{\varepsilon^2 U_0}{h_0^2} U_{XX} \\
u_{zz} &= U_0 U_{ZZ} \left(\frac{\partial Z}{\partial z} \right)^2 = \frac{U_0}{h_0^2} U_{ZZ} \\
w_{xx} &= \varepsilon U_0 W_{XX} \left(\frac{\partial X}{\partial x} \right)^2 = \frac{\varepsilon^3 U_0}{h_0^2} W_{XX} \\
w_{zz} &= \varepsilon U_0 U_{ZZ} \left(\frac{\partial Z}{\partial z} \right)^2 = \frac{\varepsilon U_0}{h_0^2} W_{ZZ} \\
\phi_x &= \frac{\mu U_0}{\varepsilon h_0} \Phi_X \frac{\partial X}{\partial x} = \frac{\mu U_0}{h_0^2} \Phi_X \\
\phi_z &= \frac{\mu U_0}{\varepsilon h_0} \Phi_Z \frac{\partial Z}{\partial z} = \frac{\mu U_0}{\varepsilon h_0^2} \Phi_Z \\
h_t &= h_0 H_T \frac{\partial T}{\partial t} = \varepsilon U_0 H_T \\
u h_x &= U_0 U h_0 H_X \frac{\partial X}{\partial x} = \varepsilon U_0 U H_X
\end{aligned}$$

Substituting these nondimensional variables into the continuity equation (1), gives

$$\begin{aligned}
u_x + w_z &= 0 \\
\frac{\varepsilon U_0}{h_0} U_X + \frac{\varepsilon U_0}{h_0} W_Z &= 0 \\
U_X + W_Z &= 0
\end{aligned}$$

This computation also justifies the scaling of w as the nondimensional variables should also conserve mass.

We can nondimensionalize the conservation of momementum equations as follows,

$$\begin{aligned}
\rho(u_t + uu_x + wu_z) &= -p_x + \mu\Delta u - \phi_x \\
\rho\left(\frac{\varepsilon U_0^2}{h_0}U_T + \frac{\varepsilon U_0^2}{h_0}UU_X + \frac{\varepsilon U_0^2}{h_0}WU_Z\right) &= -\frac{\mu U_0}{h_0^2}P_X + \mu\left(\frac{\varepsilon^2 U_0}{h_0^2}U_{XX} + \frac{U_0}{h_0^2}U_{ZZ}\right) - \frac{\mu U_0}{h_0^2}\Phi_X \\
\frac{\varepsilon U_0^2 \rho}{h_0}(U_T + UU_X + WU_Z) &= \frac{\mu U_0}{h_0^2}\left(-P_X + \left(\varepsilon^2 U_{XX} + U_{ZZ}\right) - \Phi_X\right) \\
\frac{\varepsilon U_0 \rho h_0}{\mu}(U_T + UU_X + WU_Z) &= \left(-P_X + \left(\varepsilon^2 U_{XX} + U_{ZZ}\right) - \Phi_X\right)
\end{aligned}$$

and

$$\begin{aligned}
\rho(w_t + uw_x + ww_z) &= -p_z + \mu\Delta w - \phi_z \\
\rho\left(\frac{\varepsilon^2 U_0^2}{h_0}W_T + \frac{\varepsilon^2 U_0^2}{h_0}UW_X + \frac{\varepsilon^2 U_0^2}{h_0}WW_Z\right) &= -\frac{\mu U_0}{\varepsilon h_0^2}P_Z + \mu\left(\frac{\varepsilon^3 U_0}{h_0^2}W_{XX} + \frac{\varepsilon U_0}{h_0^2}W_{ZZ}\right) - \frac{\mu U_0}{\varepsilon h_0^2}\Phi_Z \\
\frac{\varepsilon^2 \rho U_0^2}{h_0}(W_T + UW_X + WW_Z) &= \frac{\mu U_0}{\varepsilon h_0^2}\left(-P_Z + \left(\varepsilon^4 W_{XX} + \varepsilon^2 W_{ZZ}\right) - \Phi_Z\right) \\
\varepsilon^3 \frac{\rho U_0 h_0}{\mu}(W_T + UW_X + WW_Z) &= \left(-P_Z + \varepsilon^2\left(\varepsilon^2 W_{XX} + W_{ZZ}\right) - \Phi_Z\right)
\end{aligned}$$

The boundary conditions are nondimensionalized as below,

$$\begin{aligned}
w = 0, u &= 0 && \text{at } z = 0 \\
\varepsilon U_0 W = 0, U_0 U &= 0 && \text{at } Z = 0 \\
W = 0, U &= 0 && \text{at } Z = 0
\end{aligned}$$

with

$$\begin{aligned}
w &= h_t + uh_x && \text{at } z = h \\
\varepsilon U_0 W &= \varepsilon U_0 H_T + \varepsilon U_0 U H_x && \text{at } Z = H \\
W &= H_T + UH_x && \text{at } Z = H
\end{aligned}$$

In order to nondimensionalize the stress tensor at the free surface, we consider the normal and tangential components of $\boldsymbol{T} \cdot \boldsymbol{n}$ separately. The normal and tangential vector at the surface can be expressed in terms of the free surface as

$$\boldsymbol{n} = \frac{\langle -h_x, 1\rangle}{(1 + h_x^2)^{1/2}} \quad \boldsymbol{t} = \frac{\langle 1, h_x\rangle}{(1 + h_x^2)^{1/2}} \tag{12}$$

and the mean curvature, κ, can be expressed in terms of h as

$$\kappa = -\frac{h_{xx}}{(1 + h_x^2)^{3/2}}. \tag{13}$$

We would now like to write the following vector equation in terms of its normal and tangential components.

$$\boldsymbol{T} \cdot \boldsymbol{n} = (-\kappa\sigma + \Pi_0)\boldsymbol{n} + \left(\frac{\partial \sigma}{\partial s} + \tau_0\right)\boldsymbol{t} \qquad \text{at } z = h$$

Note that we are using a Newtonian stress tensor which is given by

$$\boldsymbol{T} = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + \mu \begin{bmatrix} 2u_x & u_z + w_x \\ u_z + w_x & 2w_z \end{bmatrix} \tag{14}$$

The normal component of this vector equation is given by

$$\langle \boldsymbol{T} \cdot \boldsymbol{n}, \boldsymbol{n}\rangle = (-\kappa\sigma + \Pi_0) \tag{15}$$

First we will simplify the left hand side.

$$\begin{aligned}
a &= \left(1+h_x^2\right)^{1/2} \\
\langle \boldsymbol{T}\cdot\boldsymbol{n},\boldsymbol{n}\rangle &= \frac{1}{a^2}\begin{bmatrix}-h_x & 1\end{bmatrix}\begin{bmatrix}-p+2\mu u_x & \mu(u_z+w_x)\\ \mu(u_z+w_x) & -p+2\mu w_z\end{bmatrix}\begin{bmatrix}-h_x\\ 1\end{bmatrix} \\
&= \frac{1}{a^2}\begin{bmatrix}-h_x & 1\end{bmatrix}\begin{bmatrix}-h_x(-p+2\mu u_x)+\mu(u_z+w_x)\\ -h_x\mu(u_z+w_x)+-p+2\mu w_z\end{bmatrix} \\
&= \frac{1}{a^2}\left(h_x^2(-p+2\mu u_x)-\mu h_x(u_z+w_x)-\mu h_x(u_z+w_x)+-p+2\mu w_z\right) \\
&= \frac{1}{a^2}\left(h_x^2(-p+2\mu u_x)-2\mu h_x(u_z+w_x)-p+2\mu w_z\right) \\
&= \frac{1}{a^2}\left(\left(1+h_x^2\right)(-p)+2\mu\left(h_x^2 u_x+w_z\right)-2\mu h_x(u_z+w_x)\right) \\
&= -p+\frac{2\mu}{a^2}\left(\left(h_x^2 u_x+w_z\right)-h_x(u_z+w_x)\right)
\end{aligned}$$

Next simplify the right hand side.

$$(-\kappa\sigma+\Pi_0)=\frac{h_{xx}}{\left(1+h_x^2\right)^{3/2}}\sigma+\Pi_0$$

This gives the following scalar equation,

$$-p-\Pi_0+\frac{2\mu}{1+h_x^2}\left(\left(h_x^2 u_x+w_z\right)-h_x(u_z+w_x)\right)=\frac{h_{xx}}{\left(1+h_x^2\right)^{3/2}}\sigma. \tag{16}$$

The tangential component is given by

$$\langle \boldsymbol{T}\cdot\boldsymbol{n},\boldsymbol{t}\rangle=\left(\frac{\partial\sigma}{\partial s}+\tau_0\right) \tag{17}$$

First simplify the left hand side,

$$\begin{aligned}
\langle \boldsymbol{T}\cdot\boldsymbol{n},\boldsymbol{n}\rangle &= \frac{1}{a^2}\begin{bmatrix}1 & h_x\end{bmatrix}\begin{bmatrix}-p+2\mu u_x & \mu(u_z+w_x)\\ \mu(u_z+w_x) & -p+2\mu w_z\end{bmatrix}\begin{bmatrix}-h_x\\ 1\end{bmatrix} \\
&= \frac{1}{a^2}\begin{bmatrix}1 & h_x\end{bmatrix}\begin{bmatrix}-h_x(-p+2\mu u_x)+\mu(u_z+w_x)\\ -h_x\mu(u_z+w_x)+-p+2\mu w_z\end{bmatrix} \\
&= \frac{1}{a^2}\left(-h_x(-p+2\mu u_x)+\mu(u_z+w_x)+-h_x^2\mu(u_z+w_x)+h_x(-p+2\mu w_z)\right) \\
&= \frac{1}{a^2}\left(-h_x(2\mu u_x)+\mu(u_z+w_x)+-h_x^2\mu(u_z+w_x)+h_x(2\mu w_z)\right) \\
&= \frac{\mu}{a^2}\left(2h_x(w_z-u_x)+\left(1-h_x^2\right)(u_z+w_x)\right)
\end{aligned}$$

Next simplify the right hand side

$$\begin{aligned}
\left(\frac{\partial\sigma}{\partial s}+\tau_0\right) &= \frac{\partial\sigma}{\partial x}\frac{\partial x}{\partial s}+\tau_0 \\
&= \frac{\partial\sigma}{\partial x}\frac{1}{\left(1+h_x^2\right)^{1/2}}+\tau_0
\end{aligned}$$

This gives the following scalar equation.

$$\mu\left(2h_x(w_z-u_x)+\left(1-h_x^2\right)(u_z+w_x)\right)=\frac{\partial\sigma}{\partial x}\left(1+h_x^2\right)^{1/2}+\tau_0\left(1+h_x^2\right) \tag{18}$$

Lastly we will nondimensionalize these two equations.

$$-p-\pi+\frac{2\mu}{1+h_x^2}\left(\left(h_x^2u_x+w_z\right)-h_x(u_z+w_x)\right)=\frac{h_{xx}}{\left(1+h_x^2\right)^{3/2}}\sigma$$

$$-\frac{\mu U_0}{\varepsilon h_0}P-\frac{\mu U_0}{\varepsilon h_0}\Pi+\frac{2\mu}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2\frac{\varepsilon U_0}{h_0}U_X+\frac{\varepsilon U_0}{h_0}W_Z\right)-\varepsilon H_X\left(\frac{U_0}{h_0}U_Z+\frac{\varepsilon^2U_0}{h_0}W_X\right)\right)$$
$$=\frac{\varepsilon^2}{h_0}\frac{H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\frac{\mu U_0}{\varepsilon}\Sigma$$

$$\left(-\frac{\mu U_0}{\varepsilon h_0}P-\frac{\mu U_0}{\varepsilon h_0}\Pi+\frac{\varepsilon\mu U_0}{h_0}\frac{2}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2U_X+W_Z\right)-H_X\left(U_Z+\varepsilon^2W_X\right)\right)\right)$$
$$=\frac{\varepsilon\mu U_0}{h_0}\frac{H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\Sigma$$

$$\left(-\frac{\mu U_0}{\varepsilon h_0}P-\frac{\mu U_0}{\varepsilon h_0}\Pi+\frac{\varepsilon\mu U_0}{h_0}\frac{2}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2U_X+W_Z\right)-H_X\left(U_Z+\varepsilon^2W_X\right)\right)\right)$$
$$=\frac{\varepsilon\mu U_0}{h_0}\frac{H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\Sigma$$

$$\frac{\mu U_0}{\varepsilon h_0}\left(-P-\Pi+\frac{2\varepsilon^2}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2U_X+W_Z\right)-H_X\left(U_Z+\varepsilon^2W_X\right)\right)\right)=\frac{\mu U_0}{\varepsilon h_0}\frac{\varepsilon^2H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\Sigma$$

$$\left(-P-\Pi+\frac{2\varepsilon^2}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2U_X+W_Z\right)-H_X\left(U_Z+\varepsilon^2W_X\right)\right)\right)=\frac{\varepsilon^2H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\Sigma$$

and

$$\mu\left(2h_x(w_z-u_x)+\left(1-h_x^2\right)(u_z+w_x)\right)=\frac{\partial\sigma}{\partial x}\left(1+h_x^2\right)^{1/2}+\tau_0\left(1+h_x^2\right)$$

$$\mu\left(2\varepsilon H_X\left(\frac{\varepsilon U_0}{h_0}W_Z-\frac{\varepsilon U_0}{h_0}U_X\right)+\left(1-\varepsilon^2H_X^2\right)\left(\frac{U_0}{h_0}U_Z+\frac{\varepsilon^2U_0}{h_0}W_X\right)\right)$$
$$=\frac{\mu U_0}{h_0}\Sigma_X\left(1+\varepsilon^2H_X^2\right)^{1/2}+\frac{\mu U_0}{h_0}\tau\left(1+\varepsilon^2H_X^2\right)$$

$$\frac{\mu U_0}{h_0}\left(2\varepsilon^2H_X(W_Z-U_X)+\left(1-\varepsilon^2H_X^2\right)\left(U_Z+\varepsilon^2W_X\right)\right)$$
$$=\frac{\mu U_0}{h_0}\left(\Sigma_X\left(1+\varepsilon^2H_X^2\right)^{1/2}+\tau\left(1+\varepsilon^2H_X^2\right)\right)$$

$$2\varepsilon^2H_X(W_Z-U_X)+\left(1-\varepsilon^2H_X^2\right)\left(U_Z+\varepsilon^2W_X\right)$$
$$=\Sigma_X\left(1+\varepsilon^2H_X^2\right)^{1/2}+\tau\left(1+\varepsilon^2H_X^2\right)$$

The full nondimensional equations are thus

$$U_X+W_Z=0 \tag{19}$$

$$\frac{\varepsilon U_0\rho h_0}{\mu}(U_T+UU_X+WU_Z)=-P_X+\left(\varepsilon^2U_{XX}+U_{ZZ}\right)-\Phi_X \tag{20}$$

$$\varepsilon^3\frac{\rho U_0h_0}{\mu}(W_T+UW_X+WW_Z)=-P_Z+\varepsilon^2\left(\varepsilon^2W_{XX}+W_{ZZ}\right)-\Phi_Z \tag{21}$$

at $Z=0$

$$W=0,\quad U=0 \tag{22}$$

and at $Z=H$

$$W=H_T+UH_x \tag{23}$$

$$-P-\Pi+\frac{2\varepsilon^2}{1+\varepsilon^2H_X^2}\left(\left(\varepsilon^2H_X^2U_X+W_Z\right)-H_X\left(U_Z+\varepsilon^2W_X\right)\right)=\frac{\varepsilon^2H_{XX}}{\left(1+\varepsilon^2H_X^2\right)^{3/2}}\Sigma \tag{24}$$

$$2\varepsilon^2H_X(W_Z-U_X)+\left(1-\varepsilon^2H_X^2\right)\left(U_Z+\varepsilon^2W_X\right)=\Sigma_X\left(1+\varepsilon^2H_X^2\right)^{1/2}+\tau\left(1+\varepsilon^2H_X^2\right) \tag{25}$$

We can now let $\varepsilon \to 0$ which results in

$$U_X + W_Z = 0 \tag{26}$$
$$P_X + \Phi_X = U_{ZZ} \tag{27}$$
$$P_Z + \Phi_Z = 0 \tag{28}$$

at $Z = 0$,

$$W = 0, \quad U = 0 \tag{29}$$
$$\tag{30}$$

and at $Z = H$

$$W = H_T + UH_x \tag{31}$$
$$-P - \Pi = \bar{\Sigma} H_{XX} \tag{32}$$
$$U_Z = \Sigma_X + \tau \tag{33}$$

Note that we assume that the surface tension is large, so that $\bar{\Sigma} = \varepsilon^2 \Sigma = O(1)$. This is important in order to keep surface tension effects in the final equation.

Next we integrate the continuity equation over Z.

$$\int_0^H U_X + W_Z \, \mathrm{d}Z = 0$$
$$\int_0^H U_X \, \mathrm{d}Z + W|_{Z=0}^H = 0$$
$$\int_0^H U_X \, \mathrm{d}Z + H_T + UH_X = 0$$
$$H_T + \int_0^H U_X \, \mathrm{d}Z + UH_X = 0$$
$$H_T + \frac{\partial}{\partial X}\left(\int_0^H U \, \mathrm{d}Z\right) = 0$$

Using the boundary conditions we can solve for an expression of U as follows

$$\int_Z^H U_{ZZ} \, \mathrm{d}Z = \int_Z^H P_X + \Phi_X \, \mathrm{d}Z$$
$$U_Z|_Z^H = (P_X + \Phi_X)(H - Z)$$
$$(\tau + \Sigma_X) - U_Z = (P_X + \Phi_X)(H - Z)$$
$$\int_0^Z (\tau + \Sigma_X) - U_Z \, \mathrm{d}Z = \int_0^Z (P_X + \Phi_X)(H - Z) \, \mathrm{d}Z$$
$$(\tau + \Sigma_X)Z - U|_{Z=0}^Z = (P_X + \Phi_X)\left(HZ - \frac{1}{2}Z^2\right)$$
$$(\tau + \Sigma_X)Z - U = (P_X + \Phi_X)\left(HZ - \frac{1}{2}Z^2\right)$$
$$U = (\tau + \Sigma_X)Z + (P_X + \Phi_X)\left(\frac{1}{2}Z^2 - HZ\right)$$

The boundary conditions also give an expression for $P + \Phi$,

$$P_Z + \Phi_Z = 0$$
$$\int_Z^H P_Z + \Phi_Z \, \mathrm{d}Z = 0$$
$$P|_{Z=H} - P + \Phi|_{Z=H} - \Phi = 0$$
$$-\Pi - \bar{\Sigma} H_{XX} - P + \Phi|_{Z=H} - \Phi = 0$$
$$P + \Phi = \Phi|_{Z=H} - \Pi - \bar{\Sigma} H_{XX}$$

Plugging both of these into the integrated continuity equation gives,

$$
\begin{aligned}
H_T + \frac{\partial}{\partial X}\left(\int_0^H U \,\mathrm{d}Z\right) &= 0 \\
H_T + \frac{\partial}{\partial X}\left(\int_0^H (\tau + \Sigma_X)Z + (P_X + \Phi_X)\left(\frac{1}{2}Z^2 - HZ\right)\mathrm{d}Z\right) &= 0 \\
H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 + (P_X + \Phi_X)\left(\frac{1}{6}H^3 - \frac{1}{2}H^3\right)\right)_X &= 0 \\
H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}(P + \Phi)_X H^3\right)_X &= 0 \\
H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}\big(\Phi|_{Z=H} - \Pi - \bar{\Sigma}H_{XX}\big)_X H^3\right)_X &= 0 \\
H_T + \left(\frac{1}{2}(\tau + \Sigma_X)H^2 - \frac{1}{3}\big(\Phi|_{Z=H} - \Pi\big)_X H^3\right)_X &= -\big(\bar{\Sigma}H^3 H_{XXX}\big)_X
\end{aligned}
$$

This is our final Thin Film equation and taking all of the constants to be one gives

$$
H_T + \big(H^2 - H^3\big)_X = -\big(H^3 H_{XXX}\big)_X \tag{34}
$$