SHOCK WAVES FOR NONLINEAR HYPERBOLIC SYSTEMS IN NONCONSERVATIVE FORM

By

Philippe Le Floch

IMA Preprint Series # 593

October 1989

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PHILIPPE LE FLOCH*

Abstract. Based on a recent work by Dal Maso–Le Floch–Murat, we investigate a notion of weak solutions (with shock waves) to nonlinear hyperbolic systems in nonconservative form, in the functional framework of **functions of bounded variation**. This work is motivated by nonlinear hyperbolic systems in nonconservative form used in the modeling of great deformations of elastoplastic materials and the modeling of two-phase flows.

1. Introduction. This paper is based on a work in collaboration with G. Dal Maso and F. Murat. In [4], a definition was proposed for **a product in nonconservative form** $g(u)\frac{du}{dx}$ where $g : \mathbb{R}^p \to \mathbb{R}^p$ is a smooth function but $u :]a,b[\to \mathbb{R}^p$ may admit discontinuities. In this work the function u is assumed to be a function of **bounded variation** (B.V.) and the product $g(u)\frac{du}{dx}$ is defined as a Borel measure on $]a,b[$. Such a definition is needed when g is not the differential of a smooth function, that is when $g(u)\frac{du}{dx}$ does not admit a conservative form. In fact, the definition given in [4] is a generalization of the Volpert's product [37] which is useful even in the context of systems of conservation laws (see for instance DiPerna-Majda [6]). Furthermore, we recall that results of strong and weak stability of this product were obtained in [4].

The purpose of this paper is to apply the theory of [4] to **nonlinear hyperbolic systems in nonconservative form**, i.e. systems

$$\partial_t u + A(u)\partial_x u = 0, \quad u(x,t) \in \mathbb{R}^p, x \in \mathbb{R}, t > 0, \tag{1.1}$$

where the matrix A is **not** assumed to be a Jacobian matrix. It is well known that, generally speaking, nonlinear hyperbolic systems do not admit smooth solutions globally defined in time, but discontinuities occur in finite time (Lax [17]). Thus, weak solutions with shock waves must be considered. Due to the nonconservative form of the equations, the standard notion of weak solution in the sense of distributions does not apply to (1.1). In the framework of functions of **bounded variation**, Dal Maso-Le Floch-Murat proposed a notion of weak solutions to system (1.1). Roughly speaking, to define the nonconservative product $A(u)\partial_x u$ as a Borel measure, it is essential to fix a **Lipschitz family of paths** in $\mathbb{R}^p$ denoted by $\phi : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$. For any u_0, u_1 in $\mathbb{R}^p$, the application $\phi(;u_0,u_1) : [0,1] \to \mathbb{R}^p$ is a path connecting u_0 to u_1, i.e.

$$\phi(0;u_0,u_1) = u_0, \quad \phi(1;u_0,u_1) = u_1.$$

*This research was done while the author visited the Institute for Mathematics and its Applications at the University of Minnesota, Minneapolis, MN 55455 (USA).
Current address: Centre de Mathématiques Appliquées, Ecole Polytechnique, Unité Associée au C.N.R.S., (URAD 756) 91128 Palaiseau Cedex (FRANCE) E-Mail: UMAP067@FRORS12.BITNET

The product depends on this given family of paths and is denoted below by $[A(u)\partial_x u]_\phi$. Moreover, the family of paths ϕ depends on the physical problem under consideration. (See Section 2 for the precise definition of the product).

We recall that the Riemann problem (i.e. a Cauchy problem with piecewise constant initial data) for the nonconservative system (1.1) is solved by Dal Maso-Le Floch-Murat under standard assumptions. This result generalizes the Lax's theorem [16,17]. Furthermore, the Glimm's method is considered by Le Floch [18,19,20] (in the case of the Volpert's product).

In this paper, after a recall of these general results on nonlinear hyperbolic systems in nonconservative form, we are concerned with the question of the derivation of the family of path ϕ. We show in Section 3 that the family ϕ can be formally derived from a parabolic regularization of system (1.1), more precisely by means of smooth viscous profiles. Of course, the family of paths depend on the given regularization. This non-uniqueness is a specific feature of systems in nonconservative form.

This work is motivated by the nonlinear hyperbolic systems in nonconservative form which are often used in the modeling of the great deformations of elastoplastic materials ([1,2,8,15,25,36]) or the modeling of two-phase flows (i.e. mixtures of water and vapor, see [11,26,31]). Examples are given in Section 4 and we expect that the theory presented in this paper will be helpful to solve them.

In Section 5, we get systems in nonconservative form which – for weak solutions with bounded variation– are equivalent to the usual conservation laws of continuum mechanics. This result of equivalence for weak solutions is based on some properties of linearity of these equations (see Lemma 5.4 below). We mention also a system of two conservation laws which is non strictly hyperbolic. For this system, Korchinski ([14]) has proved that a generalized notion of solution must be introduced. Here, we give an interpretation of the results of [14] by using the theory of [4].

2. A nonconservative product for functions of bounded variation. In this section, we define the nonconservative product $g(u)\frac{du}{dx}$ and describe some elementary properties. Complements (in particular concerning the weak stability of this product) may be found in Dal Maso–LeFloch-Murat [4].

Let us motivate our definition by restricting ourselves to functions composed of two constant vectors u_0 and u_1 in $\mathbb{R}^p$ $(u_0 \neq u_1)$:

$$u(x) = u_0 + H(x - x_0)(u_1 - u_0), \qquad x \in]a, b[\tag{2.1}$$

where x_0 belongs to a given interval $]a, b[$ and $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

For each function $g : \mathbb{R}^p \to \mathbb{R}^p$, an idea to define the product $g(u)\frac{du}{dx}$ is to introduce a smooth regularization u^ϵ of the discontinuous function u. Then, in this particular case, if the total variation (see below) of u^ϵ remains uniformly bounded with respect to ϵ, then the formula

$$g(u)\frac{du}{dx} \equiv \lim_{\epsilon \to 0} g(u^\epsilon)\frac{du^\epsilon}{dx} \tag{2.2}$$

gives a sense to the nonconservative product as a bounded measure. Of course, (2.2) may depend on the sequence u^ϵ. Let us see how the limit depends on the regularization.

We introduce **a path** connecting u_0 and u_1 in $\mathbb{R}^p$, that is a Lipschitz continuous application $\phi : [0,1] \to \mathbb{R}^p$, satisfying $\phi(0) = u_0$, $\phi(1) = u_1$. Then, for $\epsilon > 0$, u^ϵ is defined by

$$u^\epsilon(x) = \begin{cases} u_0, & \text{if } x \in]a, x_0 - \epsilon[, \\ \phi(\frac{x - x_0 + \epsilon}{2\epsilon}), & \text{if } x \in]x_0 - \epsilon, x_0 + \epsilon[, \\ u_1, & \text{if } x \in]x_0 + \epsilon, b[. \end{cases} \tag{2.3}$$

LEMMA 2.1. *When ϵ tends to zero, we get*

$$g(u^\epsilon)\frac{du^\epsilon}{dx} \rightharpoonup C\delta_{x_0}$$

vaguely in the sense of measures on $]a, b[$, where δ_{x_0} is the Dirac measure at x_0 and the scalar C is given by

$$C = \int_0^1 g(\phi(s))\phi'(s)ds. \tag{2.4}$$

Hence, the limit of $g(u^\epsilon)\frac{du^\epsilon}{dx}$ depends on ϕ **except in** the particular case that there exists $f : \mathbb{R}^p \to \mathbb{R}$ satisfying $g = Df$ and in that case

$$C = f(u_1) - f(u_0).$$

We conclude that a definition to the nonconservative product $g(u)\frac{du}{dx}$ **must depend** on the path ϕ choosen in the regularization.

Thus, for the general definition below, we assume that any two points of $\mathbb{R}^p$ are connected by a **given path**, and we define the product with respect to this fixed family of paths. This definition gives a **tool** to handle with nonconservative products.

Let $BV(]a, b[, \mathbb{R}^p)$ be the space of functions of bounded variation, i.e. *functions* of $L^1(]a, b[, \mathbb{R}^p)$ whose first order derivative is a bounded Borel measure on the interval $]a, b[$

(Volpert [37], Volpert-Hudjaev [38] and Rudin [28]). The total variation of a BV function $u :]a,b[\to \mathbb{R}^p$ is by definition the mass of the measure of total variation $|\frac{du}{dx}|$

$$TV(u) = \int_{]a,b[} |\frac{du}{dx}|.$$

Recall that a BV function u admits a countable set of discontinuity points and, at each point of discontinuity x_0, a left-limit $u(x_0-)$ and a right-limit $u(x_0+)$. At a point of continuity x_0, we set $u(x_0-) = u(x_0+) = u(x_0)$.

From now on, we assume that **a fixed family of paths in** $\mathbb{R}^p$ is given, i.e. a Lipschitz continuous map $\phi : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ which satisfies the following properties:

$$\text{(Hyp.1)} \qquad \forall u_L, u_R \in \mathbb{R}^p, \forall s \in [0,1], \phi(0; u_L, u_R) = u_L \text{ and } \phi(1; u_L, u_R) = u_R,$$

$$\text{(Hyp.2)} \qquad \exists k > 0, \forall u_L, u_R \in \mathbb{R}^p, \forall s \in [0,1], \ |\frac{\partial \phi}{\partial s}(s; u_L, u_R)| \leq k|u_L - u_R|,$$

and

$$\text{(Hyp.3)} \qquad \begin{cases} \exists k > 0, \forall u_L, u_R, v_2, v_R \in \mathbb{R}^p, \forall s \in [0,1], \\ |\frac{\partial \phi}{\partial s}(s; u_L, u_R) - \frac{\partial \phi}{\partial s}(s; v_L, v_R)| \leq k \cdot (|u_L - v_L| + |u_R - v_R|), \end{cases}$$

(indeed, weaker assumptions would be sufficient, see [4]).

Then, the following definition of nonconservative product is based on this family ϕ.

THEOREM AND DEFINITION 2.1. *(Dal Maso, LeFloch, Murat [4])*

Let $u :]a,b[\to \mathbb{R}^p$ be a function of bounded variation and $g : \mathbb{R}^p \to \mathbb{R}^p$ be a continuous function. Then, there exists a unique real-valued bounded Borel measure μ on $]a,b[$ characterized by the two following properties:

1) if u is continuous on a Borel set $B \subset]a,b[$, then

$$\mu(B) = \int_B g(u) \frac{du}{dx}. \tag{2.5}$$

2) if u is discontinuous at a point x_0 of $]a,b[$, then

$$\mu(\{x_0\}) = \int_0^1 g(\phi(s; u(x_0-), u(x_0+))) \frac{\partial \phi}{\partial s}(s; u(x_0-), u(x_0+)) ds. \tag{2.6}$$

By definition, this measure μ is **the nonconservative product** *of $g(u)$ by $\frac{du}{dx}$ and is denoted by*

$$\mu = [g(u) \frac{du}{dx}]_\phi.$$

The essential (and difficult) question of the derivation of the family ϕ will be considered in Section 3 (in the case of nonlinear hyperbolic systems).

We could verify that the nonconservative product does not depend on the parametrization of the paths. Furthermore, Definition 2.1 is coherent with the usual distributional definition in the case of conservative products: if $f : \mathbb{R}^p \to \mathbb{R}$ is a continuously differentiable function, then we have

$$(2.7) \qquad [(Df)(u)\frac{du}{dx}]_\phi = \frac{d}{dx}(f(u)).$$

The left hand side in (2.7) is understood in the sense of the above definition, while the right hand side is understood in the sense of distributions.

Example 1. A simple example of paths ϕ is given by the (canonical) **family of straight lines** defined by

$$(2.8) \qquad \phi(s; u_L, u_R) = u_L + s \cdot (u_R - u_L), \quad \forall u_L, u_R \in \mathbb{R}^p, \forall s \in [0,1].$$

We denote by S this application.

If u is the step function considered in (2.1), i.e. then Definition 2.1 yields

$$(2.9) \qquad [g(u)\frac{du}{dx}]_\phi = \int_0^1 g(\phi(s; u_0, u_1))\frac{\partial \phi}{\partial s}(s; u_0, u_1)ds.\delta_{x_0},$$

which is exactly the limit found in Lemma 2.1. In particular, if u^ϵ is the regularization (2.3) of the function u, then we have

$$(2.10) \qquad g(u^\epsilon)\frac{du^\epsilon}{dx} \rightharpoonup [g(u)\frac{du}{dx}]_\phi.$$

The convergence holds vaguely in the sense of measures, that is: for every smooth function $\theta :]a,b[\to \mathbb{R}^p$

$$\lim_{\epsilon \to 0} \int_{]a,b[} \theta g(u^\epsilon)\frac{du^\epsilon}{dx} = \int_{]a,b[} \theta \cdot [g(u)\frac{du}{dx}]_\phi.$$

More general results of stability are proved in [4].

Result (2.9) becomes simpler when one uses the particular family of straight lines (2.8):

$$(2.11) \qquad [g(u)\frac{du}{dx}]_S = \int_0^1 g(u_0 + s(u_1 - u_0))ds(u_1 - u_0)\delta_{x_0}.$$

We may show that, in the particular case of straight lines, the nonconservative product given by Definition 2.1 coincides with a product introduced by Volpert [37] in 1967. Let us recall that the **averaged superposition** of a BV function $u :]a,b[\to \mathbb{R}^p$ by a continuous function $g : \mathbb{R}^p \to \mathbb{R}^p$ is the function $\hat{g}(u) :]a,b[\to \mathbb{R}^p$ defined by

$$\hat{g}(u)(x) = \int_0^1 g(u(x_0-) + s(u(x_0+) - u(x_0-))ds, \tag{2.12}$$

for **all** x in $]a,b[$. Of course, we have

$$\hat{g}(u)(x) = g(u(x)) \quad \text{for Lebesque - almost every } x \text{ in }]a,b[.$$

Then, it is proved in [37] that $\hat{g}(u)$ is a measurable function with respect to the Borel measure $\frac{du}{dx}$, so that the product $\hat{g}(u)\frac{du}{dx}$ makes sense as a bounded Borel measure. In fact, the results of Volpert hold in several space dimensions.

THEOREM 2.2. *([4])*
Let us consider the family S of straight lines in $\mathbb{R}^p$ defined by (2.8). Let u be in $BV(]a,b[,\mathbb{R}^p)$ and $g : \mathbb{R}^p \to \mathbb{R}^p$ be a continuous function. Then, we have

$$[g(u)\frac{du}{dx}]_S = \hat{g}(u)\frac{du}{dx}, \tag{2.13}$$

as Borel measures on $]a,b[$.

We return now to the general definition of nonconservative product and give a second example of paths.

Example 2. (Transformation of the canonical family of paths). Let $\psi : \mathbb{R}^p \to \mathbb{R}^p$ be a $\mathcal{C}^1$-diffeomorphism. From the canonical family S, we define a family of paths denoted by S_ψ, by setting

$$\phi(s; u_0, u_1) = \psi^{-1}(\psi(u_0) + s \cdot (\psi(u_1) - \psi(u_0))), \forall u_0, u_1 \in \mathbb{R}^P, \forall s \in [0,1]. \tag{2.14}$$

Let us notice the following property on S_ψ. Take u in $BV(]a,b[,\mathbb{R}^p)$ and g in $\mathcal{C}^1(\mathbb{R}^p,\mathbb{R}^p)$. If $v \in BV(]a,b[,\mathbb{R}^p)$ and $h \in \mathcal{C}^1(\mathbb{R}^p,\mathbb{R}^p)$ are defined by

$$v = \psi \circ u, h = g \circ \psi^{-1},$$

then we could easily show that

$$[g(u)\frac{du}{dx}]_{S_\psi} = [h(v)\frac{dv}{dx}]_S = \hat{h}(v)\frac{dv}{dx}. \tag{2.15}$$

Formula (2.15) shows that a nonconservative product $[g(u)\frac{du}{dx}]_\phi$ with the choice of family of paths $\phi = S_\psi$ may be transformed into a Volpert's product (i.e. with the family of straight lines). In particular, we emphasize that the **Volpert product is not invariant by change of variables.** Section 5 contains remarks on the problem of the choice of variables for a hyperbolic system.

We end with an example of family of paths which is useful in the theory of nonlinear hyperbolic systems (Section 3).

Example 3. Let u_* be in $\mathbb{R}^p$ and Ω be a neighborhood of $(0; u_*)$ in $\mathbb{R}^{2p}$. Consider a smooth application $(\epsilon_1, \dots, \epsilon_p, u) \in \Omega \to (\varphi_1(\epsilon_1, u), \dots, \varphi_p(\epsilon_p, u)) \in \mathbb{R}^p$ which satisfies

$$\varphi_1(0, u) = \dots = \varphi_p(0, u) = u, \quad \text{if } (0; u) \in \Omega, \tag{2.16}$$

and

$$\text{the vectors } \frac{\partial \varphi_i}{\partial \epsilon_i}(0; u_*), 1 \le i \le p, \text{ are linearly independent in } \mathbb{R}^p. \tag{2.17}$$

Then, by virtue of the theorem of implicit functions if Ω is small enough, for each u_0 in a neighborhood of u_*, the application

$$(\epsilon_1, \dots, \epsilon_p) \to \varphi_p(\epsilon_p, \varphi_{p-1}(\epsilon_{p-1}, \dots \varphi_1(\epsilon_1, u_0)) \dots)$$

is a smooth diffeomorphism onto a neighborhood of u_0. Thus, we can define a family of paths ϕ by setting for u_0, u_1 in a neighborhood of u_*

$$\phi(s; u_0, u_1) = \varphi_i((ps - i + 1)\epsilon_i, v_{i-1}) \text{ for } 1 \le i \le p \text{ and } \frac{i-1}{p} \le s \le \frac{i}{p}, \tag{2.18}$$

where the numbers $\epsilon_i (1 \le i \le p)$ and the vectors v_i $(0 \le i \le p)$ are characterized by

$$\begin{cases} v_0 = u_0, \\ v_i = \varphi_i(\epsilon_i, v_{i-1}), \quad 1 \le i \le p, \\ v_p = u_1. \end{cases} \tag{2.19}$$

3. Shock waves for systems in nonconservative form. We now apply the definition of Section 2 to the nonlinear hyperbolic systems in nonconservative form

$$\partial_t u + A(u) \partial_x u = 0, \qquad u(x, t) \in \mathbb{R}^p, x \in \mathbb{R}, t \ge 0, \tag{3.1}$$

where A is a smooth matrix-valued function. It is well-known that breakdown of solutions to (3.1) may occur even if the initial data $u(\cdot, 0)$ is smooth (see Lax [17]). But since (3.1) is in general not a system of conservation laws (A is not necessarily a Jacobian matrix), we can not use the standard notion of weak solutions to (3.1) in the sense of distributions.

However, following Le Floch [19] and Dal Maso–Le Floch–Murat [4], we can define weak solutions to these systems in the framework of functions of bounded variation.

Following the study of nonconservative products made in Section 2, we first assume below that **a family of paths** in $\mathbb{R}^p$ **is given**. Then, some results well-known for systems of conservation laws may be generalized to systems in nonconservative form (Riemann problem, Glimm method). Second, we focus on the (difficult) question of the derivation of the family of paths. This derivation necessarily must be made by using an "information" which is **not contained** in the hyperbolic system. This **non uniqueness** is indeed a fundamental feature of nonconservative products, but of course non-uniqueness is not surprising for nonlinear hyperbolic problems.

Let $\phi : [0,1] \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ be a **fixed** Lipschitz continuous family of paths satisfying (Hyp. 1)-(Hyp. 3) of Section 2.

DEFINITION 3.1. *A function u in $L^\infty(\mathbb{R}_+, BV(\mathbb{R}, \mathbb{R}^p))$ is a weak solution to the nonconservative system (3.1) if we have*

$$\partial_t u + [A(u)\partial_x u]_\phi = 0, \tag{3.2}$$

as bounded Borel measure on $\mathbb{R} \times \mathbb{R}_+$.

In (3.2), as in Proposition 2.1, the product $[A(u(\cdot,t))\partial_x u(\cdot,t))]_\phi$ is defined for almost every $t > 0$ as a Borel measure on $\mathbb{R}$ and is a bounded Lebesque-measurable function with respect to t. Equation (3.2) is equivalent to:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \partial_t\theta(x,t)u(x,t)dxdt + \int_{\mathbb{R}_+}(\int_{\mathbb{R}} \theta(\cdot,t)[A(u(\cdot,t))\partial_x u(\cdot,t)]_\phi)dt = 0, \tag{3.3}$$

for each function $\theta : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of class $\mathcal{C}^1$ with compact support.

Consider for instance the discontinuous function w given by

$$w(x,t) = u_0 + H(x - V.t)(u_1 - u_0), x \in \mathbb{R}, t \geq 0, \tag{3.4}$$

where u_0, u_1 are two constant vectors in $\mathbb{R}^p$ and V belongs to $\mathbb{R}$. Then, we could easily show that the function w is a weak solution to system (3.1) if and only if the following **generalized Rankine-Hugoniot jump relation** is satisfied:

$$-V \cdot (u_1 - u_0) + \int_0^1 A(\phi(s;u_0,u_1))\frac{\partial\phi}{\partial s}(s;u_0,u_1)ds = 0. \tag{3.5}$$

If ϕ is the family of straight lines, that is if the Volpert's product is used in (3.2) (see (2.8), (2.11)), this relation becomes simpler:

$$(-V.Id + \int_0^1 A(u_0 + s(u_1 - u_0))ds)(u_1 - u_0) = 0. \tag{3.6}$$

In the case that system (3.1) admits a conservative form, i.e. there exists a smooth function $f : \mathbb{R}^p \to \mathbb{R}$, such that

$$A(u) = Df(u), \quad \forall u \in \mathbb{R}^p, \tag{3.7}$$

then Definition 3.1 reduces to the usual definition of weak solutions in the sense of distributions (on the subject of systems of conservation laws, see Lax [16,17] or Smoller [30]). In particular, under assumption (3.7), relation (3.5) is exactly the **usual Rankine Hugoniot relation** (see [17])

$$-V \cdot (u_1 - u_0) + f(u_1) - f(u_0) = 0. \tag{3.8}$$

And, in this particular case, (3.5) does not depend on the family of paths ϕ.

Assume from now that system (3.1) is strictly hyperbolic with genuinely nonlinear characteristic fields or linearly degenerate one (see [17]). Then, we could study the jump relation (3.5) when $|u_0 - u_1| << 1$ exactly as Lax studied (3.8) for systems of conservation laws. Using the Lax entropy criterium, which may be immediately generalized to nonconservative systems, we define **shock curves** (and contact discontinuity curves) locally in the neighborhood of a state in $\mathbb{R}^p$. It is also easy to generalize the notions of rarefaction waves and **rarefaction curves** to systems in nonconservation form. We omit the details (see [4] and also [18]).

We now give the **existence result of solution to the Riemann problem**, i.e. to a Cauchy problem with a piecewise constant initial data of the form

$$u(x,0) = \begin{cases} u_L, & \text{if } x < 0, \\ u_R, & \text{if } x > 0, \end{cases} \tag{3.9}$$

where u_L and u_R are constant states in $\mathbb{R}^p$.

THEOREM 3.1. *Let ϕ be a family of paths satisfying (Hyp.1)-(Hyp.3) of Section 2. Assume that system (3.1) is strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields, and the family ϕ satisfies*

$$\frac{\partial \phi}{\partial u_1}(1; u_0, u_0) - \frac{\partial \phi}{\partial u_1}(0; u_0, u_0) = Id, \quad \forall u_0 \in \mathbb{R}^p. \tag{Hyp.4}$$

Then, for $|u_L - u_R|$ small enough, the Riemann problem (3.1)-(3.9) has a **solution with bounded variation** *u which depends only on $\frac{x}{t}$ and has the well-known Lax's structure. That is, u consists of $(p+1)$ constant states separated by shock waves, rarefaction waves or contact discontinuities.*

Thus, the solution to the Riemann problem associated with a system in nonconservative form looks like the standard solution to conservation laws. The only difference lies in the shock waves for which one has to write (3.5) instead of (3.8). Comparing (3.5) and (3.8), we note that the knowledge of the path ϕ connecting u_0 and u_1 is necessary to write (3.5) while (3.8) needs only the states u_0 and u_1.

We refer to Le Floch [18] for some results about the Glimm's method applied to systems in nonconservative form. The above results show that, for each family of paths ϕ, a mathematical theory of weak solutions to nonlinear hyperbolic systems in nonconservative form may be developed in the very usual framework of functions of bounded variation. We now turn to the fundamental difficulty which is to find the suitable family of paths for a given system in nonconservative form. To get a well-posed problem for system (3.1), we need – roughly speaking – an "information" which indeed is not contained into the hyperbolic system. We sketch now a method to be followed to derive the family of paths ϕ from a **parabolic regularization of the hyperbolic system** (3.1).

Such a difficulty with **non-uniqueness** of solutions to hyperbolic problems is well-known in the context of conservation laws. Namely, the system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \quad u(x,t) \in \mathbb{R}^p, \tag{3.10}$$

has in general many solutions (in the sense of distributions) corresponding to the same initial data $u(\cdot, 0)$. Then, given a smooth matrix-valued function $D \geq 0$, it is usual to consider the parabolic system ($\epsilon > 0$)

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x (D(u^\epsilon)\partial_x u^\epsilon), \quad u(x,t) \in \mathbb{R}^p, \tag{3.11}$$

with a smooth initial data. In certain cases, this problem admits a smooth solution u^ϵ and one expects that when ϵ tends to zero, u^ϵ tend to the solution u to the hyperbolic system (3.10). By definition, this limit-function u is the **physical** or **entropy** solution to the hyperbolic problem (Lax [17]).

Lax has shown that - at least formally - one can characterize this limit in the case that system (3.10) admits (for smooth solutions) an additional conservation law

$$\partial_t \eta(u) + \partial_x q(u) = 0, \quad u(x,t) \in \mathbb{R}^p, \tag{3.12}$$

where η is a convex smooth function and q is smooth. Lax [17] proves that if the solution u^ϵ of (3.11) satisfies certain a priori estimates and converges almost everywhere to a limit u, then the function u must satisfy the following **entropy inequality**

$$\partial_t \eta(u) + \partial_x q(u) \leq 0, \quad u(x,t) \in \mathbb{R}^p, \tag{3.13}$$

When (3.10) is a system of two strictly hyperbolic conservation laws, the rigourous proof of this derivation has been given by Di Perna [5] (see also Tartar [34], Rascle [27]).

We now want to generalize the previous approach to systems in nonconservative form

$$\partial_t u + A(u)\partial_x u = 0, \quad u(x,t) \in \mathbb{R}^p. \tag{3.14}$$

To this purpose, we consider a parabolic regularization of this system ($\epsilon > 0$)

$$\partial_t u^\epsilon + A(u^\epsilon)\partial_x u^\epsilon = \epsilon\partial_x(D(u^\epsilon)\partial_x u^\epsilon), \tag{3.15}$$

where $D \geq 0$ is a smooth matrix-valued function. As for conservation laws, we expect that, at least formally, a smooth solution u^ϵ to (3.15) tends to a weak solution to (3.14), provided that the family of paths ϕ is suitably determined. Thus, we may try to identify the family of paths ϕ needed in Definition 3.1. by the way of this regularization. These paths must indeed depend on the matrix D. We emphasize that all what follows is given only at a *formal* level.

To fix the ideas, assume below that u^ϵ is a solution of (3.15) which is bounded in BV norm uniformly with respect to ϵ, and the right hand-side of (3.15) satisfies

$$\epsilon\partial_x(D(u^\epsilon)\partial_x u^\epsilon) \to 0, \quad \epsilon \to 0, \tag{3.16}$$

(which can be ensured by suitable estimates on derivatives of u^ϵ). Then, the function u^ϵ tends almost everywhere to a BV function u. Which is the hyperbolic equation satisfied by this limit u?

Since we work with BV functions which are, roughly speaking, piecewise smooth (see Volpert [37]), we may try to distinguish between two cases: u is smooth in a domain of (x,t)-plan or u is continuous on the both sides of a smooth curve in (x,t)-plan. If u is continuous, then $A(u)$ is continuous and thus measurable with respect to the Borel measure $\frac{\partial u}{\partial x}$. In that case, we conjecture that:

$$A(u^\epsilon)\partial_x u^\epsilon \rightharpoonup A(u)\partial_x u$$

vaguely in the sense of measures, provided that some estimates on $\partial_x u^\epsilon$ are satisfied, so that we could get

$$\partial_t u + A(u)\partial_x u = 0.$$

Concerning now the case where u is piecewise smooth, let us restrict ourselves to the case of a **traveling wave** to system (3.15), that is a solution of the form

$$\begin{cases} u^\epsilon = w^\epsilon(x - s.t) = w^\epsilon(y), \\ \lim_{y\to-\infty} w^\epsilon(y) = w_L, \lim_{y\to-\infty} \frac{d}{dy} w^\epsilon(y) = 0, \\ \lim_{y\to+\infty} w^\epsilon(y) = w_R, \lim_{y\to+\infty} \frac{d}{dy} w^\epsilon(y) = 0, \end{cases} \tag{3.17}$$

where s is the speed of the wave and w_L, w_R ar two vectors in $\mathbb{R}^p$. A solution to (3.15) of the form (3.17) satisfies the **ordinary differential equation**:

$$(A(w_\epsilon) - s)w'_\epsilon = \epsilon(D(w_\epsilon)w'_\epsilon)'$$

Then, if we define $w : \mathbb{R} \to \mathbb{R}^p$ by

$$w(\frac{y}{\epsilon}) = w_\epsilon(y), \tag{3.18}$$

the function w satisfies an O.D.E. which **does not depend** on the parameter of regularization ϵ:

$$(A(w) - s)w' = (D(w)w')'. \tag{3.19}$$

Recall that in the conservative case, this equation (3.19) is usually integrated once, but generally speaking this is not possible in the nonconservative case.

We assume below that system (3.1) is strictly hyperbolic with genuinely nonlinear characteristic fields and the matrix-valued function A is smooth. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the p distinct eigenvalues of A and $r_1, r_2, \ldots, r_p$ be corresponding eigenvectors. Then, let us show how a family of paths in $\mathbb{R}^p$ can be constructed from solutions of O.D.E. (3.19). Given a vector u_* in $\mathbb{R}^p$, we search the vectors u_R in $\mathbb{R}^p$ satisfying the following property:

$$\begin{cases} \text{There exists } s \in \mathbb{R} \text{ and a solution } w : \mathbb{R} \to \mathbb{R}^p \text{ of (3.17)-} \\ \text{(3.19) where } w_L = u_* \text{ and } w_R = u_R. \end{cases} \tag{3.20}$$

By analogy with the the conservative systems, we may expect that, in a neighborhood of u_*, solutions of (3.20) consist of **p half-curves** φ_i, $1 \le i \le p$. Then changing the role of u_* leads to search the vectors u_L in $\mathbb{R}^p$ satisfying:

$$\begin{cases} \text{There exists } s \in \mathbb{R} \text{ and a solution } w : \mathbb{R} \to \mathbb{R}^p \text{ of (3.17)} \\ -\text{(3.19) where } w_L = u_L \text{ and } w_R = u_*. \end{cases} \tag{3.21}$$

That yields again **p half-curves** defined in a neighborhood of u_*. (See also Harabetian [10] and Stoufflet [32]).

More precisely, always following results on systems of conservation laws, the previous derivation gives p (smooth) applications ($i = 1, 2, \ldots, p$)

$$\varphi_i : \mathcal{O} \times \mathcal{U} \to \mathcal{U},$$

where $\mathcal{O}$ in a neighborhood of 0 in $\mathbb{R}^p$ and $\mathcal{U}$ is a subset of $\mathbb{R}^p$. Moreover, we expect that the applications would satisfy

$$\begin{cases} \varphi_i(0, u_*) = 0, \quad \forall u_* \in \Omega, i = 1, 2, \ldots, p, \\ \dfrac{\partial}{\partial \epsilon_i}\varphi_i(0, u_*) = r_i(u_*), \quad \forall u_* \in \Omega, i = 1, 2, \ldots, p. \end{cases} \tag{3.22}$$

(See for instance Majda-Pego [24] concerning the systems of conservation laws). Then, a family of paths ϕ in $\mathbb{R}^p$ can be constructed from the φ_i's as made in Example 3 of Section 2 ((2.17) is here a consequence of the strict hyperbolicity of system (3.1)). We denote below by ϕ_D this family of paths.

We are now concerned with the question of the convergence of the solutions u^ϵ to the regularized problem (3.15) towards a solution to (3.14) in the case that u^ϵ is a smooth travelling wave of (3.15). If the solution to (3.17)-(3.19), $u^\epsilon = w(\frac{x-st}{\epsilon})$, tends to the Heaviside-type function

$$u(x,t) = w_L + H(x - V.t)(w_R - w_L),$$

the, from the O.D.E. satisfied by w^ϵ, we deduce by integration

$$-V.(w_R - w_L) + \int_{-\infty}^{+\infty} A(w_\epsilon(y))w'_\epsilon(y)dy = \epsilon \int_{-\infty}^{+\infty} (D(w_\epsilon(y))w'_\epsilon(y))'dy. \tag{3.23}$$

But

$$\int_{-\infty}^{+\infty} A(w_\epsilon(y))w'_\epsilon(y)dy = \int_{-\infty}^{+\infty} A(w(y))w'(y)dy$$

and

$$\int_{-\infty}^{+\infty} (D(w_\epsilon(y))w'_\epsilon(y))'dy = 0,$$

so that the vectors w_L and w_R must satisfy the following **relation**

$$-V(w_R - w_L) + \int_{-\infty}^{+\infty} A(w(y))w'(y)dy = 0. \tag{3.24}$$

On the other hand, let us consider the limit-function $u = \lim_{\epsilon\to 0} u^\epsilon$ and the particular family of paths ϕ_D constructed previously (from the matrix D). Then, by Definition 3.1, the discontinuous function u is a weak solution to hyperbolic system (3.1) iff w_L and w_R satisfy the **generalized Rankine-Hugoniot relation** (3.5), i.e.

$$-V \cdot (w_R - w_L) + \int_0^1 A(\phi_D(s; w_L, w_R))\frac{\partial \phi_D}{\partial s}(s; w_L, w_R)ds = 0. \tag{3.25}$$

But, since w_L and w_R are by assumption connected by a solution $w : \mathbb{R} \to \mathbb{R}^p$ to (3.19), the path in (3.25) has been constructed to be of the form

$$\phi_D(s; w_L, w_R) = w(\psi(s)), s \in [0,1], \tag{3.26}$$

where ψ is an increasing smooth bijection from $]0,1[$ onto $\mathbb{R}$. (Recall here the nonconservative product is invariant by change of paramaterization of the paths). By (3.26), the jump relation (3.25) is equivalent to

$$-V \cdot (w_R - w_L) + \int_0^1 A(w(\psi(s)))\frac{\partial}{\partial s}(w(\psi(s)))ds = 0 \tag{3.27}$$

which is exactly (3.24).

Thus, we have proved:

THEOREM 3.2. *Let the family of paths in Definition 3.1 be the family ϕ_D given by the previous construction. If u^ϵ is a smooth travelling wave to the parabolic system (3.15) and u^ϵ tends to a Heaviside-type function u when ϵ tends to zero, then the function u is a weak solution to the hyperbolic system in nonconservative form.*

More generally, we may conjecture that the smooth solutions u^ϵ to a Cauchy problem for the system

$$\partial_t u^\epsilon + A(u^\epsilon)\partial_x u^\epsilon = \epsilon \partial_x (D(u^\epsilon)\partial_x u^\epsilon)$$

tend to a discontinous solution to the nonlinear hyperbolic system

$$\partial_t u + [A(u)\partial_x u]_{\phi_D} = 0,$$

where ϕ_D is the family of paths previously constructed from the matrix D. Indeed, such a result of convergence may be reasonably expected for instance in the particular case of the hyperbolic whose shock curves and rarefaction curves coincide (see Serre [29] in the case of systems of conservation laws).

4. Applications. This section is devoted to a brief presentation of some nonlinear hyperbolic systems which come from the modeling of two-phase flows and the modeling of the great deformations of elastic materials. Let us first discuss some two-phase flow models derived for instance by Stewart-Wendroff [31] and Ransom-Hicks [26]. The flow typically consists of a mixture of liquid and vapor, as those occuring in nuclear reactors cooled by water under pressure. For the modeling of such a flow, one write the conservation laws of Physics (conservation of mass, momentum and energy) for each phase (liquid and vapor). Interface conditions (transfers of mass, momentum and energy between the two phases) are needed to formally close the equations. Then, it turns out that due to the extremely complicated geometry of the domains occupied by the liquid and the vapor, it is necessary to use an **averaging procedure** to derive practical models (see [31], [26] and also Ishii [11]). We emphasize that this averaging procedure leads, from the conservation laws for the physical quantities describing each phase, to **equations in nonconservative form** for the averaged quantities.

A great variety of averaging procedures (in space, in time or statistical) exist in the literature of multiphase flows. We give here two models derived by Stewart-Wendroff [31] and Ransom-Hicks [26] for stratified flows. The fundamental assumption for the first following system is that the pressure is the same in the two phases. For simplicity, we assume here that the flow is isentropic. Let ρ_1, u_1 and α_1 (respectively ρ_2, u_2 and α_2) be the mass density, the velocity and the fraction of the liquid (resp. of the vapor). Then, if the effects of viscosity and transfer between phases are assumed to be absent (they appear as lower order source terms of the system below), the conservation laws of mass and momentum yield after averaging the following equations:

$$(4.1) \qquad \begin{cases} \partial_t(\alpha_i\rho_i) + \partial_x(\alpha_i\rho_i u_i) = 0, & i = 1,2, \\ \partial_t(\alpha_i\rho_i u_i) + \partial_x(\alpha_i\rho_i u_i^2) + \alpha_i \partial_x p = 0, & i = 1,2, \end{cases}$$

with

$$\alpha_1 + \alpha_2 = 1.$$

If the mass densities ρ_1 and ρ_2 are given as functions of the pressure

$$\rho_i = \rho_i(p), \quad i = 1, 2, \tag{4.2}$$

then (4.1) is a 4×4 system with the unknowns u_1, u_2, p and α_1. This system is in nonconservative form because of the terms $\alpha_1 \partial_x p$ and $(1 - \alpha_1)\partial_x p$. One may prove that it may be hyperbolic (four real characteristic speeds) or **of mixed type** (only two real characteristic speeds). A mathematical understanding of this simple model is important for the applications since models of this form are indeed used in numerical computations of multiphase flows to compute shock waves.

Let us mention a recent work by Toumi [35] which is a first step in the understanding of the nonconservative model (4.1). He remarks that adding the two last equations of (4.1) gives the conservation of total momentum (since $\alpha_1 \partial_x p + (1 - \alpha_1)\partial_x p = \partial_x p$) and assume an empiric law for the difference of the velocities between the two phases

$$u_2 - u_1 = g(\alpha_1), \tag{4.3}$$

where g is a given function of the fraction α_1 satisfying certain properties of convexity and smallness. In that case we get the following system of **three** equations

$$\begin{cases} \partial_t(\alpha_i \rho_i) + \partial_x(\alpha_i \rho_i u_i) = 0, \qquad i = 1, 2, \\ \partial_t(\alpha_1 \rho_1 u_1 + (1 - \alpha_1)\rho_2 u_2) + \partial_x(\alpha_1 \rho_1 u_1^2 + (1 - \alpha_1)\rho_2 u_2^2) + \partial_x p = 0, \end{cases} \tag{4.4}$$

Equations (4.2)–(4.4) yield a nonlinear hyperbolic system of three conservation laws with the unknowns u_1, p and α_1. The Riemann problem for this system is considered in [35].

A second kind of model assumes **two different pressions** p_1 and p_2 for the two phases of the flow. An equation for the evolution of the liquid fraction is then added. With the same notations as previously, we have the following system of 5 equations (see [31] and [26])

$$\begin{cases} \partial_t(\alpha_i \rho_i) + \partial_x(\alpha_i \rho_i u_i) = 0, \quad i = 1, 2 \\ \partial_t(\alpha_i \rho_i u_i) + \partial_x(\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i + (p_i - \overline{p})\partial_x \alpha_i = 0, \quad i = 1, 2, \\ \partial_t \alpha_1 + \overline{w} \partial_x \alpha_1 = 0. \end{cases} \tag{4.5}$$

In (4.5), $\overline{p}$ and $\overline{w}$ are given function of u_1, u_2, ρ_1 and ρ_2, i.e.

$$\overline{p} = \overline{p}(u_1, u_2, \rho_1, \rho_2), \quad \overline{w} = \overline{w}(u_1, u_2, \rho_1, \rho_2), \tag{4.6}$$

and an equation of state is assumed for the pressures p_i

$$p_i = p_i(\rho_i), i = 1, 2. \tag{4.7}$$

Then, (4.5)–(4.7) is a nonlinear hyperbolic system in nonconservative form which has always real characteristic speeds. Notice that these values may coincide at some point, so that the system is **not strictly hyperbolic**. This system has in fact been derived in order to eliminate the (mathematical and numerical) difficulties of the complex eigenvalues of model (4.1).

Finally, we mention a model, studied by Le Floch-Toumi [22], which may be viewed as a very simplified version of (4.5). The isentropic Euler equations for a single fluid of density ρ and velocity u in a nozzle of section α are usually written in the following nonconservative form

$$\begin{cases} \partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0, \\ \partial_t(\alpha\rho u) + \partial_x(\alpha\rho u^2) + \alpha\partial_x p(\rho) = 0, \\ \partial_t\alpha = 0, \end{cases} \tag{4.8}$$

where the third equation of (4.8) is just a convenient (for our purpose here) way to express that the section is independent of t. More generally, the results of [22] holds if α satisfies

$$\partial_t\alpha + w\partial_x\alpha = 0, \tag{4.9}$$

where w is some constant (velocity).

The understanding of this 3×3 system is a first step to study (4.5). If α is discontinuous in (4.8), that is if **the section** of the nozzle is **discontinuous** then the theory of Section 2 is needed to give a sense to the nonconservative product

$$\alpha\partial_x p(\rho).$$

For the sake of simplicity, we assume in [22] that the fluid obey the equation of state of a polytopic perfect gas

$$p = kp^\gamma, \qquad k > 0, \gamma > 1.$$

Let us define the sound speed by

$$c(\rho) = (\frac{\partial}{\partial\rho}p(\rho))^{\frac{1}{2}} = (k\gamma)^{\frac{1}{2}}\rho^{\frac{\gamma-1}{2}},$$

and set

$$\mathcal{S}_\pm = \{(\alpha, \rho, u) | \alpha > 0, \rho > 0, u \pm c(\rho) = 0\}, \tag{4.10}$$

and

$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-.$$

Outside the surface $\mathcal{S}$, system (4.8) admits three distinct and real characteristic eigenvalues

$$u - c(\rho), \quad u + c(\rho), \quad 0,$$

with a corresponding basis of eigenvectors. But, if (α_0, p_0, u_0) belongs to $\mathcal{S}_+$ (respectively $\mathcal{S}_-$), then

$$u_0 + c(\rho_0) = 0, \quad (\text{resp. } u_0 - c(\rho_0) = 0)$$

and the system admits only two eigenvectors at this point. One says that the system has a **parabolic degenerescency** on the surface $\mathcal{S}$ (see the work of Keyfitz-Kranzer [13] for the resolution of a 2×2 system possessing such a parabolic degenerescency). The Riemann problem (see Section 3) for the nonconservative system (4.8) in a zone of **strict hyperbolicity** is solved in Le Floch-Toumi [22] by using the general theory of Section 3. In particular, the family of paths needed to the definition of the nonconservative product $\alpha\partial_x p(\rho)$ is derived by regularization of the section of the nozzle.

The nonlinear deformations of an elastoplastic body may be described by an hyperbolic system of equations in nonconservative form. Namely, expressed in Eulerian coordinates, the conservation laws of mass, momentum and energy for a three-dimensional elastic body are ([15,25]):

$$(4.11) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0, \\ \partial_t(\rho \vec{v}) + \vec{\operatorname{div}}(\rho \vec{v} \otimes \vec{v}) = \vec{\operatorname{div}} \underline{\underline{\sigma}}, \\ \partial_t(\rho(\frac{1}{2}|\vec{v}|^2 + \mathcal{E})) + \operatorname{div}(\rho(\frac{1}{2}|\vec{v}|^2 + \mathcal{E})\vec{v}) = \operatorname{div}(\vec{v}\underline{\underline{\sigma}}) \end{cases}$$

where the density ρ, the velocity $\vec{v} = (v_i)_{1 \le i \le 3}$ and the internal energy $\mathcal{E}$ (for instance) are independent variables. But, as it is well known, a **constitutive relation** is needed for the stress tensor $\underline{\underline{\sigma}} = (\sigma^{ij})_{1 \le i,j \le 3}$. In the theory of elastoplastic materials, it is very usual to write an **equation in nonconservative form** (see Trangenstein-Colella [36] for a review)

$$(4.12) \qquad \partial_t \underline{\underline{\sigma}} + \vec{v}.\vec{\operatorname{grad}} \underline{\underline{\sigma}} + \underline{\underline{\sigma}}\,{}^t\underline{\underline{\Omega}} - \Omega \underline{\underline{\sigma}} = B \cdot D,$$

where the Eulerian deformation rate tensor D and the vorticity tensor Ω are defined by

$$(4.13) \qquad \begin{cases} D = (d_{ij})_{1 \le i,j \le 3} = (\frac{1}{2}(\frac{\partial}{\partial x_j} v_i + \frac{\partial}{\partial x_i} v_j))_{1 \le i,j \le 3}, \\ \Omega = (\omega_{ij})_{1 \le i,j \le 3} = (\frac{1}{2}(\frac{\partial}{\partial x_j} v_i - \frac{\partial}{\partial x_j} v_i))_{1 \le i,j \le 3}. \end{cases}$$

Moreover in (4.12), the product $B \cdot D$ of $B = (b_{ijkl})_{1\le i,j,k,l\le 3}$ by the tensor D is the tensor given by

$$B \cdot D = (\sum_{k,\ell} b_{ijk\ell} d_{k\ell})_{1\le i,j\le 3}. \tag{4.14}$$

For elastic materials, the coefficients $b_{ijk\ell}$ in (3.14) are assumed to be functions of the independent variables.

The left hand side of (4.12) is the so-called **Jaumann derivative** of the tensor σ which have a physical interpretation. This equation may be considered as a generalization, to evolution problems for great deformations, of the standard constitutive relations of quasistatic elasticity theory (see [36]).

We notice that a simple model is obtained from (4.11) by assuming that the unknowns of the system depend only on one direction (say x_1, denoted below by x) **and** that the stress in the material are only in this direction ($\sigma_{ij} = 0$ if $(i,j) \neq (1,1)$. In that case the system (4.11)-(4.14) becomes

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial(\rho v) + \partial_x(\rho v^2 - \sigma) = 0 \\ \partial_t(\rho(\frac{v^2}{2} + \mathcal{E})) + \partial_x(\rho(\frac{v^2}{2} + \mathcal{E})v - \sigma v) = 0 \\ \partial_t \sigma + v\partial_x \sigma - b\partial_x v = 0, \end{cases} \tag{4.15}$$

Here, for the sake of simplicity we have set $v = v_1$, $\sigma = \sigma_{11}$, $b = b_{1111}$. In general, b is a function of the variables $\rho, v, \mathcal{E}, \sigma$.

These hyperbolic systems in nonconservative form are indeed used in practice by numericians and physicists to compute shock waves in solids (Leroux-Colombeau [1,2]). This nonconservative formulation of the equation of state is also used for elastoplastic materials for which the coefficients $b_{ijk\ell}$ in (4.14) depend also on the derivatives of the unknowns ([36], Germain-Lee [8]). A mathematical theory of shock waves to these systems could be based on the ideas of Sections 2 and 3.

Remark that Plohr-Sharp [25] have recently proposed a conservative version of system (4.11) - (4.14) but only in the particular case of hyperelastic materials (the coefficients $b_{ijk\ell}$ in (4.12) are obtained from derivatives of an internal energy function). In Section 5, we are interested in the question of the equivalence between the nonconservative formulation and the conservative one. However, even in the case that a conservative formulation exists, the nonconservative formulation could be, from a numerical point of view, more adapted to the equations of state encountered in solid mechanics (where small variations of the density of solid correspond to large variations of the stress tensor, see [2]). Moreover, it seems that a conservative formulation is not available for elastoplastic materials.

5. Conservative and non-conservative forms of some systems of conservation laws. In this section, we are interested in nonconservative forms of systems of conservation laws. Of course, we want to find nonconservative equations which are, for weak BV solutions, equivalent to the original conservation laws.

We first consider below the elasticity system in Lagrangian coordinates and derive nonconservative form of this system (see also [1]). Our results give that, in the case of the one-dimensional deformations of an elastic body, the conservative and nonconservative forms of the system of elasticity - as considered by Plohr-Sharp [25]-are equivalent for weak solutions. For the case of three-dimensional deformations, the difficulty probably lies in the choice of the independent variables.

Second, we discuss an example, studied by Korchinski [14], of two nonstrictly hyperbolic conservation laws. Our idea is to get a nonconservative equation by integration of one equation of the system. Then, from the solution of the corresponding system in nonconservative form, we deduce the solution of the system of two conservation laws.

Example 1. The 2×2 system of elasticity (or p-system). We begin with the following nonlinear hyperbolic system of two conservation laws

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u - \partial_x \sigma = 0, \end{cases} \tag{5.1}$$

where v and u denote the specific volume and the velocity of an elastic material expressed in Lagrangian coordinates. The stress σ is usually an increasing function of v

$$\sigma = \Sigma(v). \tag{5.2}$$

The system (5.1)(5.2) is of course in conservation form. However, in the context of elasticity (see Section 4), we may find instead of the algebraic equation of state (5.2) **an evolution equation of state** in nonconservative form

$$\partial_t \sigma - k(v)\partial_x u = 0, \tag{5.3}$$

where k is a given function. If this function k is choosen to be

$$k(v) = \Sigma'(v),$$

the equation (5.3) results from the first equation of (5.1) by multiplication by $\Sigma'(v)$. Then, it is immediate that for smooth solutions the **two** equations (5.1) with the equation of state (5.2) are equivalent to the **three** equations (5.1)(5.3). In fact, this result is valid even for discontinuous solutions:

PROPOSITION 5.1. *Let Σ be an arbitrary (smooth) function of v. The system of two conservation laws*

$$\partial_t v - \partial_x u = 0, \quad \partial_t u - \partial_x \sigma = 0, \ \textit{with } \sigma = \Sigma(v) \tag{5.4}$$

and the 3×3 *system in non conservative form*

$$\partial_t v - \partial_x u = 0, \quad \partial_t u - \partial_x \sigma = 0, \quad \partial_t \sigma - [\Sigma'(v)\partial_x u]_s = 0, \tag{5.5}$$

are equivalent for weak solution in $L^\infty(\mathbb{R}_+, BV(\mathbb{R}))$. *Recall that, in (5.5),* S *denotes the family of straight lines defined by (2.8).*

Proof. If (v, u) is a smooth solution to the system of two equations (5.4), then (v, u, σ) with $\sigma = \Sigma(v)$ is a solution to (5.5). On the other hand, if (v, u, σ) is a smooth solution to (5.5) satisfying the property $\sigma = \Sigma(v)$, then (v, u) is a solution to (5.4). Thus, as in [20], it is sufficient to prove that the Rankine Hugoniot jump relations are the same for (5.4) and (5.5). On one hand, given two states (v_L, u_L) and (v_R, u_R), the jump relations for the system (5.1) are

$$\begin{cases} s \cdot (v_R - v_L) + u_R - u_L = 0 \\ s \cdot (u_R - u_L) + \Sigma(v_R) - \Sigma(v_L) = 0 \end{cases} \tag{5.6}$$

where s denotes the speed of the shock between (v_L, u_L) and (v_R, u_R). On the other hand, given two states (v_L, u_L, σ_L) and (v_R, u_R, σ_R), the jump relations for the system (5.5) (in the sense of Volpert) are

$$\begin{cases} s(v_R - v_L) + u_R - u_L = 0, \\ s(u_R - u_L) + \sigma_R - \sigma_L = 0, \\ s(\sigma_R - \sigma_L) + \displaystyle\int_0^1 \Sigma'(v_L + \alpha(v_R - v_L))d\alpha(u_R - u_L) = 0. \end{cases} \tag{5.7}$$

In the third equation of (5.7), we have

$$\int_0^1 \Sigma'(v_L + \alpha(v_R - v_L))d\alpha = \frac{\Sigma(v_R) - \Sigma(v_L)}{v_R - v_L} \qquad (\text{if } v_R - v_L \neq 0).$$

If, using the equation of state (5.2), we now **assume** that

$$\Sigma(v_L) = \sigma_L, \quad \Sigma(v_R) = \sigma_R,$$

that (5.7) becomes

$$\begin{cases} s(v_R - v_L) + u_R - u_L = 0, \\ s(u_R - u_L) + \sigma_R - \sigma_L = 0, \\ (\sigma_R - \sigma_L)\{s \cdot (v_R - v_L) + u_R - u_L\} = 0. \end{cases} \tag{5.8}$$

Thus the third equation of (5.8) is just a consequence of the first one and (5.6) and (5.7) are indeed equivalent jump relations. □

Example 2: The 3×3 system of gas dynamics. We turn now to the system of the three conservation laws (mass, momentum, energy) written again in mass Lagrangian coordinates

$$\partial_t v - \partial_x u = 0, \tag{5.9a}$$

$$\partial_t u + \partial_x p = 0, \tag{5.9b}$$

$$\partial_t (e + \frac{u^2}{2}) + \partial_x (pu) = 0. \tag{5.9c}$$

The independent variables are the specific volume v, the velocity u and the internal energy e. The pressure is a given function of the variables e and v, i.e.

$$p = P(e, v), \tag{5.10}$$

where P satisfies some thermodynamical constraints.

We claim that **the third equation** (5.9c) may be replaced by an equation on the internal energy or the pressure. Namely, for smooth solutions, (5.9c) gives

$$\partial_t e + u\partial_t u + u\partial_x p + p\partial_x u = 0.$$

But, by multiplication of (5.9a) and (5.9b) by p and u respectively, we have

$$u\partial_t u + u\partial_x p = 0, \quad p\partial_x u = p\partial_t v.$$

Thus (5.9c) becomes

$$\partial_t e + p\partial_t v = 0. \tag{5.11}$$

Let us now assume that the equation of state (5.10) may be written as a function of (p, v) for the variable e

$$e = \mathcal{E}(p, v). \tag{5.10bis}$$

PROPOSITION 5.2. *Assume that $\mathcal{E}$ is a (smooth) function of (p, v) Then, the 3×3 system of conservation laws*

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t (\mathcal{E}(p, v) + \frac{u^2}{2}) + \partial_x (pu) = 0, \end{cases} \tag{5.12}$$

and the 3×3 system in nonconservative form

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t \mathcal{E}(p, v) + [p\partial_t v]_s = 0, \end{cases} \tag{5.13}$$

are equivalent for weak solutions in $L^\infty(\mathbb{R}_+, BV(\mathbb{R}))$.

A variant of this result is

PROPOSITION 5.3. *Assume that $\mathcal{E}$ is a (smooth) function of (p, v) and set*

$$a(p,v) = \frac{\partial \mathcal{E}}{\partial p}(p,v), \quad b(p,v) = \frac{\partial \mathcal{E}}{\partial v}.$$

Then the systems (5.12), (5.13) and

$$\left\{\begin{array}{l} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ [a(p,v)\partial_t p]_S + [(b(p,v)+p)\partial_x u]_S = 0, \end{array}\right. \tag{5.14}$$

are equivalent for weak solutions. (S is the family of straight lines).

We notice that the third equation in (5.14) is an evolution equation for the pressure. As in the first example, this equation "contains" the equation of state (5.10).

It is easy to prove directly these results as made for Proposition 5.1. But it seems to us more interesting to understand "why it works" and to fit these results in a bit more general context. For this purpose, we notice that, to derive (5.13) and (5.14) from (5.9) (or (5.12)), we have just multiplied (5.9a) and (5.9b) but we have **never multiplied** (5.9c). Moreover, we could verify that the multiplication of (5.9a) and (5.9b) by any (non zero) functions of (v, u, p) does not change the jump relations associated with these two equations. We can explain this by **some properties of linearity** of the gas dynamics equations. The following Lemma we state now is concerned with the linearity properties of a nonlinear hyperbolic system.

LEMMA 5.4. *Let us consider a nonlinear hyperbolic system in nonconservative form*

$$[A_0(u)\partial_t u]_\phi + [A(u)\partial_x u]_\phi = 0, \quad u(x,t) \in \mathbb{R}^P, \tag{5.15}$$

where A_0 and A are given $p \times p$ matrix depending smoothly on the variable u. Set

$$A_0(u) = (a^o_{ij}(u))_{1 \le i,j \le p}, \quad A(u) = (a_{ij}(u))_{1 \le i,j \le p},$$

and assume that, for a fixed integer q, the functions a^0_{ij} and a_{ij} are constants independent of u for $i \le q$ and for any j. Let $B(u) = (b_{ij}(u))$ be a matrix satisfying

$$\left\{\begin{array}{l} (i) \displaystyle\int_0^1 B(u_0 + \alpha(u_1 - u_0))dx \text{ is invertible, for each } u_1, u_0 \text{ in } \mathbb{R}^p, \\ (ii)\ b_{ij}(u) \text{ is constant for any } i \text{ and for } j \ge q+1. \end{array}\right. \tag{5.16}$$

Then, system (5.15) and the system

$$[B(u)A_0(u)\partial_t u]_\phi + [B(u)A(u)\partial_x u]_\phi = 0, \tag{5.17}$$

are equivalent for weak solutions in $L^\infty \cap BV$.

To apply Lemma 1 to system (5.14) for instance, it suffices to rewrite it in the form (5.15)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \partial_t \begin{pmatrix} v \\ u \\ p \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & b+p & 1 \end{pmatrix} \partial_x \begin{pmatrix} v \\ u \\ p \end{pmatrix} = 0, \tag{5.18}$$

and to notice that the first and second lines of the matrix in (5.18) are constant.

Example 3. The nonconservative fomulation of elasticity equations. We discuss again the system of three conservation laws of Physics in Lagrangian coordinates but from a different point of view. Using standard notations (see for instance the paper of Plohr-Sharp [25]), the system under consideration is

$$\begin{cases} \partial_t F - \partial_x V = 0, \\ \rho_0 \cdot \partial_t V - \partial_x \sigma = 0, \\ \rho_0 \partial_t (\mathcal{E} + \dfrac{V}{2}) - \partial_x(\sigma V) = 0. \end{cases} \tag{5.19}$$

The internal energy $\mathcal{E}$ is assumed to be a given function of the specific entropy $\mathcal{S}$ and the gradient of deformation F:

$$\mathcal{E} = \mathcal{E}(\mathcal{S}, F). \tag{5.20}$$

The independent variables are choosen to be $F, \mathcal{S}$ and the velocity V, and, due to the Clausius-Duhem inequality, the stress σ expresses as a partial derivative of the energy function

$$\sigma = \rho_0 \frac{\partial \mathcal{E}}{\partial F}(\mathcal{S}, F). \tag{5.21}$$

Here ρ_0 is the (constant to simplify) density of the undeformed body. The equations (5.20) and (5.21) together with the system (5.19) yield a complete system of evolution equations for the deformation F, the velocity V and the entropy $\mathcal{S}$.

However, as indicated for the example 1, a nonconservative evolution equation on the stress σ is used in numerical computations instead of the algebraic **constitutive relation** (5.21). In a similar way as made in example 1, we want to replace (5.21) by an evolution equation. But, by contrast with the example 2, we obtain here a 4×4 system of equations. Let us make the assumption that the function $\mathcal{E}(\mathcal{S}, F)$ may be written as a function of the stress σ and the gradient of deformation F

$$\mathcal{E} = \tilde{\mathcal{E}}(\sigma, F).$$

If there exists, this function $\tilde{\mathcal{E}}$ is defined by the relation

$$\tilde{\mathcal{E}}(\rho_0 \frac{\partial \mathcal{E}}{\partial F}(\mathcal{S} F), F) = \tilde{\mathcal{E}}(\mathcal{S}, F).$$

PROPOSITION 5.5. *Let us consider the system (5.19) with the internal energy function $\mathcal{E}$ and the stress σ defined by the algebraic equations (5.20) and (5.21). Assume that the function $\mathcal{E}$ is also a function of σ and F*

$$\mathcal{E} = \mathcal{E}(\mathcal{S}, F) = \tilde{\mathcal{E}}(\sigma, F).$$

Then, let a and b be the following partial derivatives of the internal energy

$$a = \frac{\partial \tilde{\mathcal{E}}}{\partial \sigma}, \quad b = \frac{\partial \tilde{\mathcal{E}}}{\partial F}. \tag{5.22}$$

Then, the 3×3 conservative system (5.19)-(5.21) is equivalent for weak solutions in BV to the following nonconservative system of four equations

$$\begin{cases} \partial_t F - \partial_x V = 0, \\ \rho_0 \partial_t V - \partial_x \sigma = 0, \\ \rho_0 \partial_t (\mathcal{E}(\mathcal{S}, F) + \dfrac{V^2}{2}) - \partial_x (\sigma V) = 0, \\ [\rho_0 a(\sigma, F) \partial_t \sigma]_S + [(\rho_0 b(\sigma, F) - \sigma) \partial_x V]_S = 0. \end{cases} \tag{5.23}$$

Proof. If $(F, V, \mathcal{S})$ is a smooth solution to (5.19), then $(F, V, \mathcal{S}, \sigma)$ with $\sigma \stackrel{\text{def}}{=} \rho_0 \frac{\partial \mathcal{E}}{\partial F}(\mathcal{S}, F)$ is a smooth solution to (5.23). Namely, using $\rho_0 \partial_t V = \partial_x \sigma$, we have

$$\rho_0 \partial_t (\mathcal{E} + \frac{V^2}{2}) - \partial_x (\sigma V) = \rho_0 \partial_t \mathcal{E} + \rho_0 V \partial_t V - \sigma \partial_x V - V \partial_x \sigma = \rho_0 \partial_t \tilde{\mathcal{E}}(\sigma, F) - \sigma \partial_x V$$

thus with (5.22):

$$\rho_0 (a(\sigma, F) \partial_t \sigma + b(\sigma, F) \partial_t F) - \sigma \partial_x V = 0.$$

Since $\partial_x F = \partial_x V$, we find exactly

$$\rho_0 a(\sigma, F) \partial_t \sigma + (\rho_0 b(\sigma, F) - \sigma) \partial_x V = 0.$$

On the other hand, if $(F, V, \mathcal{S}, \sigma)$ is a smooth solution to (5.23) which satisfies

$$\sigma = \rho \frac{\partial \mathcal{E}}{\partial F}(\mathcal{S}, F),$$

then it is immediate that $(F, V, \mathcal{S})$ is a solution to the system of 3 equations (5.19). On the other hand with the same notations as previously and using the family of straight lines S, the jump relations for system (5.23) are

$$s(F_R - F_L) + (V_R - V_L) = 0, \tag{5.24a}$$

$$s\rho_0 (V_R - V_L) + (\sigma_R - \sigma_L) = 0, \tag{5.24b}$$

$$s\rho_0 (\mathcal{E}(V_R, \mathcal{S}_R) - \mathcal{E}(V_L, \mathcal{S}_2) + \frac{V_R^2}{2} - \frac{V_L^2}{2}) + (\sigma_R V_R - \sigma_L V_L) = 0, \tag{5.24c}$$

$$\text{(5.24d)} \qquad s\int_0^1 \rho_0 a(\sigma_\alpha, F_\alpha)d\alpha(\sigma_R-\sigma_L)+\int_0^1(-\rho_0 b(\sigma_\alpha, F_\alpha)d\alpha+\sigma_\alpha)d\alpha(V_R-V_L)=0.$$

In (5.24d), we have set

$$\text{(5.25)} \qquad \sigma_\alpha=\sigma_L+\alpha(\sigma_R-\sigma_L), \quad F_\alpha=F_L+\alpha(F_R-F_L), \quad \alpha\in[0,1].$$

Then, by (5.24a), we have

$$\text{(5.26)} \qquad \int_0^1 b(\sigma_\alpha, F_\alpha)d\alpha\cdot(V_R-V_L)=-s\int_0^1 b(\sigma_\alpha, F_\alpha)d\alpha(F_R-F_L),$$

so that by (5.22)

$$\begin{aligned}
&s\int_0^1 \rho_0 a(\sigma_\alpha, F_\alpha)d\alpha)(F_R-F_L)-\int_0^1 \rho_0 b(\sigma_\alpha, F_\alpha)d\alpha(F_R-F_L)\\
&=s\rho_0\int_0^1\{\frac{\partial\tilde{\mathcal{E}}}{\partial\sigma}(\sigma_\alpha, F_\alpha)(\sigma_R-\sigma_L)+\frac{\partial\tilde{\mathcal{E}}}{\partial F}(\sigma_a, F_\alpha)(F_R-F_L)\}d\alpha\\
&=s\rho_0(\mathcal{E}(\sigma_R, F_R)-\mathcal{E}(\sigma_L, F_L)).
\end{aligned}$$

But, using (5.24b), we may also compute

$$\begin{aligned}
s\rho_0(\frac{V_R^2}{2}-\frac{V_L^2}{2})+\sigma_R V_R-\sigma_L V_L &= -\frac{V_R+V_L}{2}(\sigma_R-\sigma_L)+\sigma_R V_R-\sigma_L V_L\\
&=\frac{1}{2}(\sigma_L+\sigma_R)(V_R-V_L).
\end{aligned}$$

We deduce that

$$\begin{aligned}
&s\int_0^1 \rho_0 a(\sigma_\alpha, F_\alpha)d\alpha\cdot(\sigma_R-\sigma_L)+\int_0^1(-\rho_0 b(\sigma_\alpha, F_\alpha)+\sigma_\alpha)d\alpha(V_R-V_L)\\
&=s\rho_0(\tilde{\mathcal{E}}(\sigma_R, F_R)-\tilde{\mathcal{E}}(\sigma_L, F_2))+\frac{1}{2}(\sigma_L+\sigma_R)(V_R-V_L)\\
&=s\rho_0(\tilde{\mathcal{E}}(\sigma_R, F_R)-\tilde{\mathcal{E}}(\sigma_L, F_L))+s\rho_0(\frac{V_R^2}{2}-\frac{V_L^2}{2})+\sigma_R V_R-\sigma_L V_L,
\end{aligned}$$

so that the equations (5.24) are equivalent to (5.24a)-(5.24c). But the jump relations for (5.19) are precisely these equations (5.24a)-(5.24c). □

Finally, we discuss the following **nonstrictly hyperbolic system** of two conservation laws studied by Korchinski [14]:

$$\text{(5.27)} \qquad \begin{cases}\partial_t u+\partial_x(u^2)=0,\\ \partial_t v+\partial_x(uv)=0.\end{cases}$$

Set

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A(U) = \frac{D}{DU}\begin{pmatrix} u^2 \\ uv \end{pmatrix} = \begin{pmatrix} 2u & 0 \\ v & u \end{pmatrix}.$$

The matrix $A(U)$ possesss two distinct eigenvalues

$$\lambda_1(U) = u, \quad \lambda(U) = 2u,$$

corresponding to a linearly degenerate field and a genuinely nonlinear one, **except** at $u = 0$ where the system changes its type. Namely, at a point $U = (u, v) = (0, v) \neq (0, 0)$, the eigenvalues coincide and $A(U)$ has only one eigenvector.

As it is well known, piecewise smooth solutions to Riemann problem with small initial data (Lax [17]) are obtained in the context of strictly hyperbolic systems of conservation laws. By contrast, Korchinski [14] solves the Riemann problem associated with this non-strictly hyperbolic system (5.27) and find solutions which are composed of Heaviside-type functions H and Dirac measures S. For instance, in the case that the initial data of the Riemann problem, (u_L, v_L) and (u_R, v_R), satisfy

$$u_L > 0, \quad u_R < 0, \tag{5.28}$$

[14] yields the following solution to (5.27)

$$\begin{cases} u(x,t) = u_L + H(x - s.t)(u_R - u_L), \\ v(x,t) = v_L + H(x - st)(v_R - v_L) + (u_L v_R - u_R v_L).t. \int(x - st), \\ \text{with } s = u_L + u_R, \end{cases} \tag{5.29}$$

where s is the speed of the wave. Of course, Korchinski needs to give a precise definition for the product uv. We refer to [14] for the details and we give here an interpretation of the results of [14] by use of the theory of nonconservative products of this paper.

The main question we are interested in is to show that (5.29) is indeed a "solution" to (5.27). To this purpose, we give below a **definition of measure-solution** to (5.27) which follows from the ideas of Section 3. We notice that if (u, v) is a (say smooth) solution to the conservation laws (5.27), such that v is integrable with respect to x in $\mathbb{R}$ and $u(x,t)$, $v(x,t)$ tend to zero when x, tends to infinity x then the function

$$w(x,t) = \int_{-\infty}^{x} v(y,t)dy \tag{5.30}$$

satisfies the following **hyperbolic equation in nonconservation form**:

$$\partial_t w + u\partial_x w = 0. \tag{5.31}$$

This remark motivates our definition below. To apply the theory of Section 3, we consider a given family of paths ϕ in $\mathbb{R}^2$ satisfying (Hyp. 1)-(Hyp.3). Then, we set:

DEFINITION 5.1. *Let u be in $L^\infty(\mathbb{R}_+, BV(\mathbb{R}))$ and v be in $L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}))$ where $\mathcal{M}(\mathbb{R})$ denote the space of locally bounded Borel measures. Then, one says that (u, v) is a solution to the system of conservation laws (5.27) if one has*

$$\begin{cases} \partial_t u + \partial_x(u^2) = 0, \\ \partial_t v + \partial_x([u.v]_\phi) = 0, \end{cases} \tag{5.32}$$

in the sense of distributions. In (5.32), the product of the function u by the measure v is given by Definition 2.1.

Then, restricting ourselves for simplicity to the case (5.28) (this is indeed the more interesting case), we prove now:

PROPOSITION 5.6. *Assume here that ϕ is the family of paths given by:*

$$(u(\alpha), v(\alpha)) = \begin{cases} (u_L + 3\alpha(u_* - u_L), v_L), & \text{for } \alpha \in [0, \frac{1}{3}], \\ (u_*, v_L + (3\alpha - 1)(v_R - v_L)), & \text{for } \alpha \in [\frac{1}{3}, \frac{2}{3}], \\ (u_* + (3\alpha - 2)(u_R - u_*), v_R), & \text{for } \alpha \in [\frac{2}{3}, 1], \end{cases} \tag{5.33}$$

for all (u_L, v_L) and (u_R, u_R) satisfying $u_L > 0$ and $u_R < 0$, and u_* being given by:

$$u_* = u_R + u_L. \tag{5.34}$$

Then, the pair (u, v) defined by (5.29) is a solution to system (5.27) in the sense of Definition 5.1.

Proof. The function u in (5.29) is clearly a weak solution to the first equation in (5.32) since $s = u_L + u_R$. On the other hand, from (5.29)-(5.33) and Definition 2.1 of nonconservative product, we have

$$[u.v]_\phi = u_L v_L + (u_R v_R - u_L v_L) H(x - s.t) + C.t.\delta(x - st),$$

where C is given by

$$C = \int_0^1 u(\alpha) v'(\alpha) d\alpha. \frac{u_L v_R - u_R v_L}{v_R - v_L}.$$

The path $(u(\alpha), v(\alpha))$ being defined by (5.33), we find

$$\int_0^1 u(\alpha) v'(\alpha) = \int_{1/3}^{2/3} u_*.3d\alpha.(v_R - v_L) = u_*.(v_R - v_L),$$

thus

$$C = u_*.(u_L v_R - u_R v_L),$$

i.e. with (5.34):

$$(5.35)\quad [uv]_\phi = u_L v_L + (u_R v_R - u_L v_L)H(x-st) + (u_R + u_L)(u_L v_R - u_R v_L) + \delta(x - st).$$

Then, from (5.29) and (5.35), we deduce

$$\begin{aligned}
\partial_t v + \partial_x [u.v]_\phi &= -s.(v_R - v_L).\delta + (u_L v_R - u_R v_L)(\delta - s.t.\delta') \\
&\quad + (u_R v_R - u_L v_L)\delta + (u_R + u_L)(u_L v_R - u_R v_L).t\delta' \\
&= \{-(u_R + u_L)(v_R - v_L) + (u_L v_R - u_R v_L) + (u_R v_R - u_L v_L)\}\delta \\
&\quad + \{-(u_R + u_L)(u_L v_R - u_R v_L) + (u_R + u_L)(u_L v_R - u_R v_L)\}.t.\delta' = 0
\end{aligned}$$

That ends the proof. □

Other cases than (5.28) can be treated similarly.

Definition 5.1 and Proposition 5.6 could be generalized to 2×2 systems of the form

$$\begin{cases} \partial_t u + \partial_x (a(u)u) = 0, \\ \partial_t v + a(u)\partial_x v = 0, \end{cases}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a given smooth function. See also Le Floch [21] and Forestier-Le Floch [7].

Acknowledgements. During my stay at the Institute for Mathematics and its Applications at the University of Minnesota, I had the advantage of friendly and valuable discussions with many participants of the different workshops. I wish to thank in particular Barbara Keyfitz, Tai Ping Liu, Brad Plohr and David Wagner. Many thanks also to Herbert Kranzer which has communicated to me the thesis of his student Korchinski. Of course, this work arises also from many interesting discussions with Gianni Dal Maso and Francois Murat.

REFERENCES

[1] Colombeau, J.F., Le Roux, A.Y., *Numerical methods for hyperbolic systems in nonconservative form using product of distributions*, Advances in comp. methods for p.d.e. VI– R. Vichnevetsky and R.S. Stepleman ed, Publi IMACS (1987).

[2] Colombeau, J.F., Le Roux, A.Y., *Numerical techniques in elastoplasticity*, in *the Proceedings on Nonlinear Hyperbolic Problems*, St Etienne 1986 eds. C. Carasso, Raviart, P.A., Serre, D., Lect. Notes in Maths. 1270, Springer-Verlag.

[3] Courant, R. Friedrichs, K.O., *Supersonic flow and shock waves*, Intersciences Publishers, Vol. 1, New York, 1968.

[4] Dal Maso, G., Le Floch, Ph., Murat, F., *Definition and weak stability of a non-conservative product*, Internal Report, Centre de Mathématiques Appliquées de l'Ecole Polytechnique (FRANCE), (February 1989) (to appear); and Note aux C.R. Acad. Sc. Paris (to appear).

[5] Di Perna, R.J., *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys, 91 (1983), pp. 1–30.

[6] DIPERNA, R.J., AND MAJDA, A., *The validity of nonlinear geometric optics for weak solutions of conservation laws*, Comm. Math. Phys., 98 (1985), pp. 313–347.

[7] FORESTIER, A, LE FLOCH, PH., *Multivalued solutions to some nonlinear and nonstrictly hyperbolic systems*, to appear.

[8] GERMAIN, P., LEE, E.H., *On shock waves in elastic-plastic solids*, J. Mech. Phys. Solids, 21, Pergamon Press (Great Britain) (1973), pp. 359–382.

[9] GLIMM, J., *Solutions in the large for nonlinear systems of equations*, Comm. Pure Appl. Math, 18 (1965), pp. 697–715.

[10] HARABETIAN, ED., *A numerical method for computing viscous shock layer*, in *Notes on Num. Fluid Mech. Vol 24,* J. Ballmann and R. Jeltsch (Ed.),, Nonlinear Hyp. Eq., Theory, Comp. Meth. and Appl., Vieweg, 1989.

[11] ISHII, N., *Thermo-Fluid Dynamic Theory of Two-Phase Flow*, Eyrolles (Paris) (1975).

[12] KEYFITZ, B.,, *A viscosity approximation to a system of conservation laws with no classical Riemann solution*, Proc. of Inter. Conf. on Hyp. Problems, Bordeaux 1988 (to appear).

[13] KEYFITZ, B., KRANZER, H., *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory*, Arch. Ration. Mech. Anal, 72 (1980), pp. 219–241.

[14] KORCHINSKI,, *Solution of a Riemann problem for a 2×2 system of conservation laws possessing no classical weak solution*, Ph.D Thesis Adelphi Univ. (1977).

[15] LANDAU, L., LIFSHITZ, *Elasticity*, Addison-Wesley, Reading, MA, 1959.

[16] LAX, P.D., *Hyperbolic systems of conservation laws II*, Comm. Pure and Appl. Math., 10 (1957), pp. 537–566.

[17] LAX, P.D., *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, CBMS Monograph, N^o SIAM (1973).

[18] LE FLOCH, PH., *Entropy weak solutions to nonlinear hyperbolic systems in nonconservative form*, Comm. in Partial Differential Equations, 13(6) (1988), pp. 669–727.

[19] LE FLOCH, PH., *Solutions faibles entropiques des systémes hyperboliques nonlinéaires sous forme nonconservative*, Note aux C.R. Acad. Sc. Paris, t. 306, Série 1 (1988), pp. 181–186.

[20] LE FLOCH, PH., *Entropy weak solutions to nonlinear hyperbolic systems in nonconservative form*, in *Proceedings of the Second International Conference on Nonlinear Hyperbolic Problems*, J. Ballmann, R. Jeltsch (Ed.), Vieweg (1989), pp. 362–373; (Aachen, FRG), (March 16th to 18th, 1988).

[21] LE FLOCH, PH., *An existence and uniqueness result for two nonstrictly hyperbolic systems*, in *the Proc. of the IMA Workshop on Nonlinear Evolution Equations that Change Type*, March 1989 (to appear).

[22] LE FLOCH, PH., TOUMI, I., to appear.

[23] LIU, T.P., *Admissible solutions of hyperbolic conservation laws*, Memoirs of the Amer. Math. Soc., n^o, Vol 30 (March 1981).

[24] MAJDA, A., PEGO, R., *Stable viscosity matrices for systems of conservation laws*, (to appear).

[25] PLOHR, B.J. SHARP, D.H., *A conservative Eulerian formulation of the equations for elastic flow*, Preprint n^o 192, (Los Alamos) (1988).

[26] RANSOM, V.H., HICKS, D.L., *Hyperbolic two-pressure models for two-phase Flow*, J. of Comp. Phys., 53 (1984), pp. 124–151.

[27] RASCLE, M., *Un résultat de compacité par compensation à coefficients variables*, Note aux CRAS Paris, 302 (1986), pp. 311–314.

[28] RUDIN, W., *Real and Complex Analysis*, Mc Graw-Hill editor, Inc., 1966.

[29] SERRE, D., *Solutions à variation bornée pour certains systèmes hyperboliques de lois de conservation*, J. of Diff. Equ., 68 (1987), pp. 137–168.

[30] SMOLLER, J., *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1983.

[31] STEWART, H.B., WENDROFF, B., *Two-phase flow: models and methods*, J. of Comp. Phys., 56 (1984), pp. 363–409.

[32] STOUFFLET, B., to appear.

[33] TANG, Z., TING, T.C.T., *Wave curves for the Riemann problem of plane waves in simple isotropic elastic solids*, Preprint, University of Illinois (Chicago) (to appear).

[34] TARTAR, L., *The compensated compactness method applied to systems of conservation laws*, J.M. Ball ed., Systems of Nonlinear P.D.E., pp. 263–285.

[35] Toumi, I., to appear.
[36] Trangenstein, J., Colella, Ph., *A high-order Godunov method for modeling finite deformations in Elastic-plastic solids*, (to appear).
[37] Volpert, A.I., *The space BV and quasilinear equations*, Math. Sbornik, 73(115) n°2 (1967), pp. 225–267.
[38] Volpert, A.I., Hudjaev, S.I., *Analysis in classes of discontinuous and equations of mathematical physics*, Nijhoff, Dordrecht (1985).

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