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Discontinuous Galerkin Method for solving a Thin-Film Equation

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September 30, 2017

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Motivation

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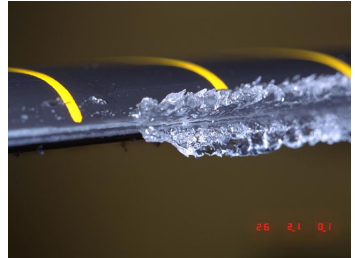
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- Aircraft Icing
- Runback



- Industrial Coating
- Paint Drying

Model Equations

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■ Navier-Stokes Equation

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = \frac{4}{3Re} u_{xx}$$

$$E_t + (u(E + p))_x = \frac{1}{Re} \left(\frac{2}{3} (u^2)_{xx} + \frac{\gamma}{(\gamma - 1)Pr} \left(\frac{p}{\rho} \right)_{xx} \right)$$

■ Asymptotic Limit, $\rho \ll L$

■ Thin-Film Equation - 1D with u as fluid height.

$$u_t + (f(x, t)u^2 - g(x, t)u^3)_x = -(h(x, t)u^3 u_{xxx})_x$$

Current Model

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■ Simplified Expression

$$u_t + (u^2 - u^3)_x = -(u^3 u_{xxx})_x$$

■ Operator Splitting

$$u_t + (u^2 - u^3)_x = 0$$

$$u_t + (u^3 u_{xxx})_x = 0$$

Introduction to Discontinuous Galerkin

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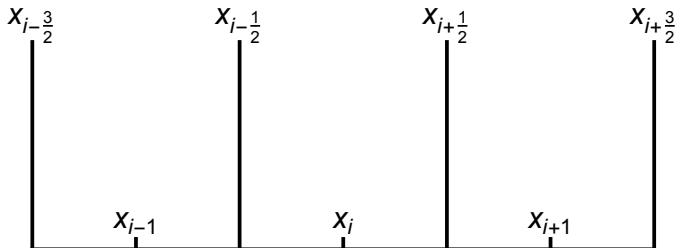
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- Partition the domain, $[a, b]$ as

$$a = x_{1/2} < \cdots < x_{i-1/2} < x_{i+1/2} < \cdots < x_{N+1/2} = b$$

- $V_i = [x_{i-1/2}, x_{i+1/2}]$
- $\Delta x = x_{i+1/2} - x_{i-1/2}$
- $x_i = \frac{x_{i+1/2} + x_{i-1/2}}{2}$.



Numerical Solutions

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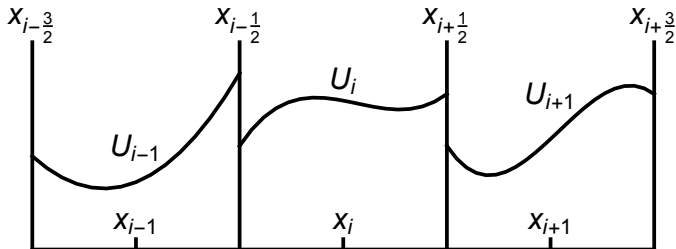
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- Use canonical variable $\xi \in [-1, 1]$
- Let $\{\phi^k(\xi)\}$ be the Legendre polynomials.
- Solution of order M on each cell

$$u|_{x \in V_i} \approx U_i = \sum_{k=1}^M U_i^k \phi^k(\xi)$$



Convection

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■ Convection Equation

$$u_t + \frac{2}{\Delta x} f(u)_\xi = 0$$
$$f(u) = u^2 - u^3$$

■ Weak Form

$$\int_{-1}^1 \left(u_t \phi(\xi) + \frac{2}{\Delta x} f(u)_\xi \phi(\xi) \right) d\xi = 0$$

■ Runge-Kutta Discontinuous Galerkin

$$\dot{U}_i^\ell = \frac{1}{\Delta x} \int_{-1}^1 f(U_i) \phi_\xi^\ell d\xi - \frac{1}{\Delta x} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2})$$

■ Rusanov Numerical Flux

$$\mathcal{F}_{j+1/2} = \frac{f(U_{i+1}(-1)) + f(U_i(1))}{2} \phi^\ell(1)$$

- Solve this system of ODEs with any Explicit Strong Stability Preserving (SSP) Runge-Kutta Method.

Numerical Example - Square Wave

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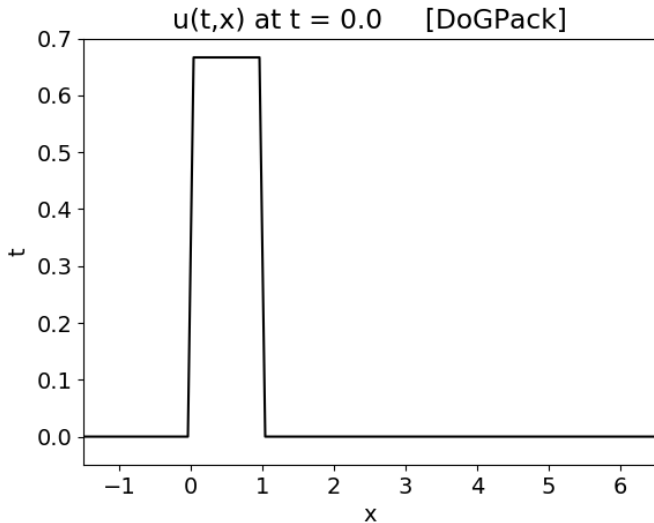
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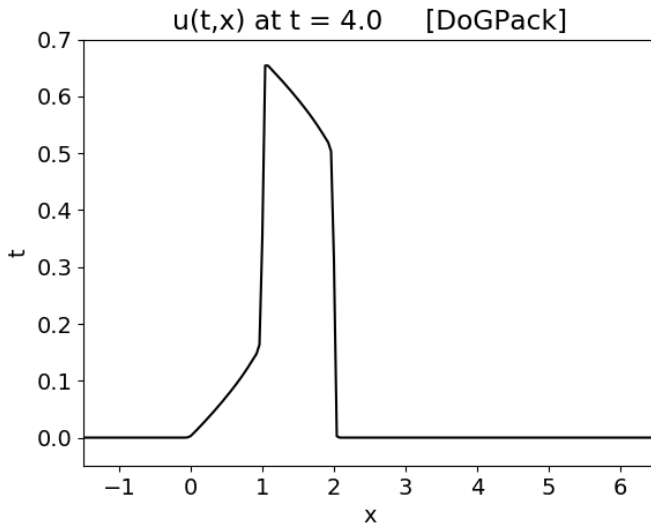
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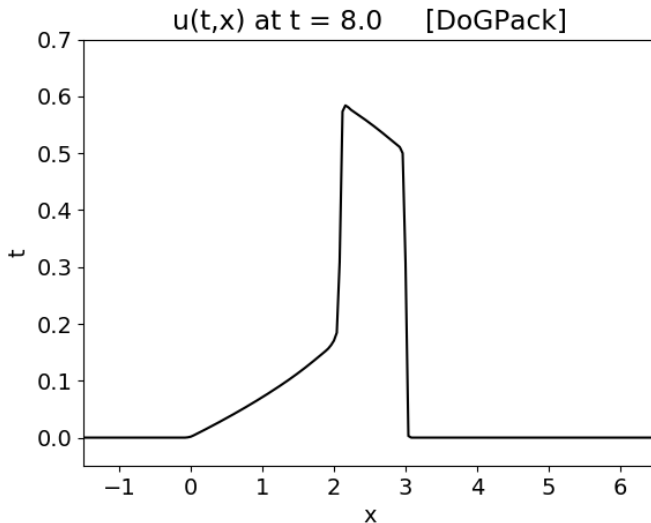
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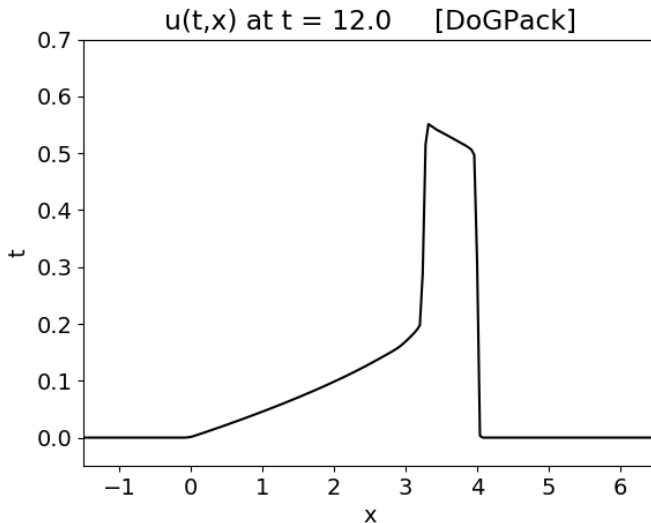
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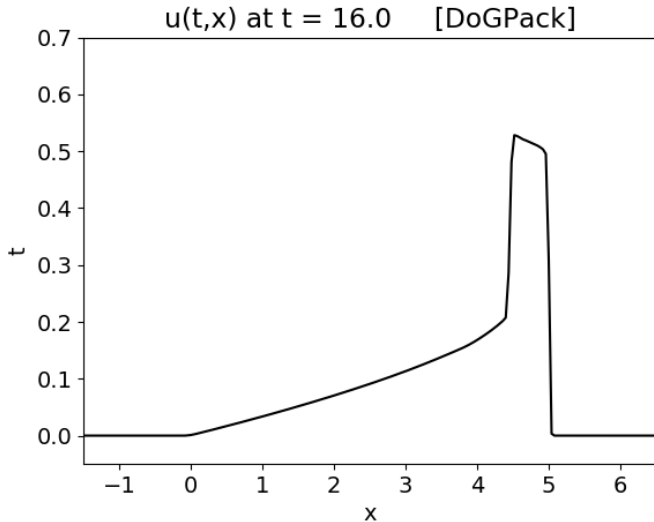
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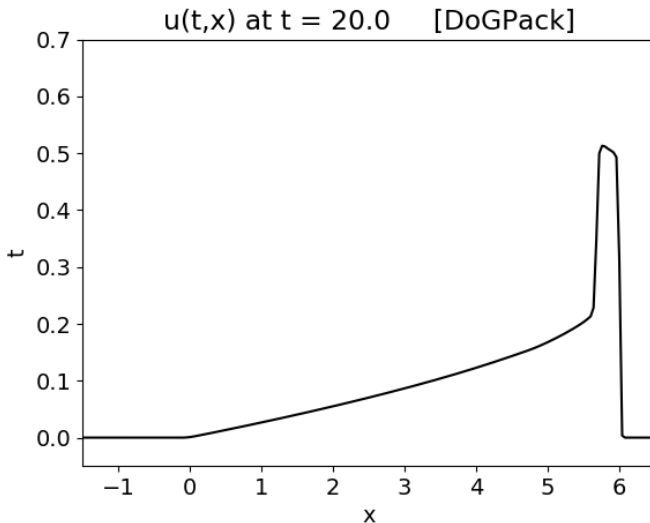
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Hyper-Diffusion

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■ Hyper-Diffusion Equation

$$u_t + \frac{16}{\Delta x^4} (u^3 u_{\xi\xi\xi})_{\xi} = 0$$

■ Local Discontinuous Galerkin (LDG)

$$q = \frac{2}{\Delta x} u_{\xi}$$

$$r = \frac{2}{\Delta x} q_{\xi}$$

$$s = \frac{2}{\Delta x} u^3 r_{\xi}$$

$$u_t = -\frac{2}{\Delta x} s_{\xi}$$

Local Discontinuous Galerkin

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$$\eta(\xi) = (U_i^n)^3$$

$$Q_i^\ell = -\frac{1}{\Delta x} \left(\int_{-1}^1 U_i \phi_\xi^\ell d\xi - \mathcal{F}(U)_{i+1/2}^\ell + \mathcal{F}(U)_{i-1/2}^\ell \right)$$

$$R_i^\ell = -\frac{1}{\Delta x} \left(\int_{-1}^1 Q_i \phi_\xi^\ell d\xi - \mathcal{F}(Q)_{i+1/2}^\ell + \mathcal{F}(Q)_{i-1/2}^\ell \right)$$

$$S_i^\ell = \frac{1}{\Delta x} \left(\int_{-1}^1 (R_i)_\xi \eta(\xi) \phi^\ell d\xi \right) \\ + \frac{1}{\Delta x} \left(\mathcal{F}(\eta)_{i+1/2} \mathcal{F}(R)_{i+1/2}^\ell - \mathcal{F}(\eta)_{i-1/2} \mathcal{F}(R)_{i-1/2}^\ell \right)$$

$$\dot{U}_i^\ell = \frac{1}{\Delta x} \left(\int_{-1}^1 S_i \phi_\xi^\ell d\xi - \mathcal{F}(S)_{i+1/2}^\ell + \mathcal{F}(S)_{i-1/2}^\ell \right)$$

Local Discontinuous Galerkin

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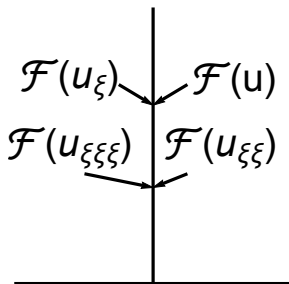
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$$\mathcal{F}(\eta)_{i+1/2} = \frac{1}{2}(\eta_{i+1}(-1) - \eta_i(1))$$

$$\mathcal{F}(\eta)_{i-1/2} = \frac{1}{2}(\eta_{i-1}(1) - \eta_i(-1))$$

$$\mathcal{F}(\ast)_{i+1/2}^\ell = \phi^\ell(1) \ast_{i+1/2}$$



Local Discontinuous Galerkin

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- Explicit SSP Runge Kutta
 - Severe time step restriction
 - $\Delta t \sim \Delta x^4$
 - $\Delta x = .1 \rightarrow \Delta t \approx 10^{-4}$
 - $\Delta x = .01 \rightarrow \Delta t \approx 10^{-8}$
- Implicit SSP Runge Kutta
 - Linear System Solver
 - Stabilized BiConjugate Gradient
 - MultiGrid Solver

Multigrid Solver

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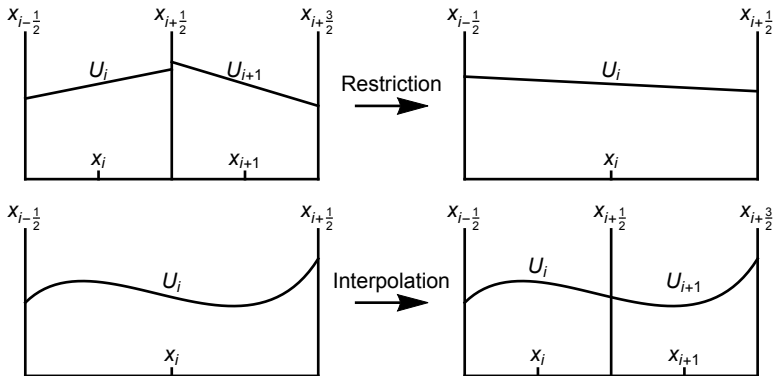
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■ Relaxation e.g. Jacobi Relaxation



Multigrid Solver

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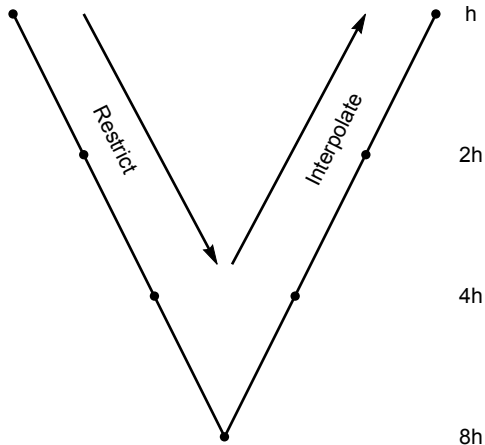
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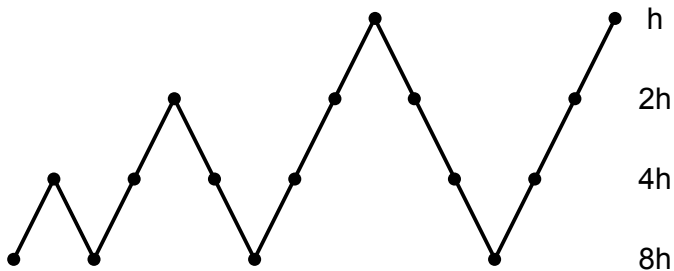
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Operator Splitting

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- Strang Splitting
 - 1 time step
 - 1/2 time step for convection
 - 1 time step for hyper-diffusion
 - 1/2 time step for convection
 - Second order splitting

Numerical Results - Riemann Problem

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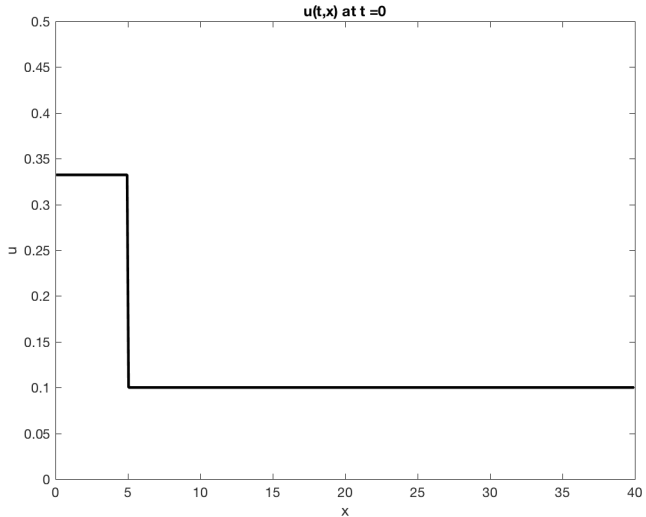
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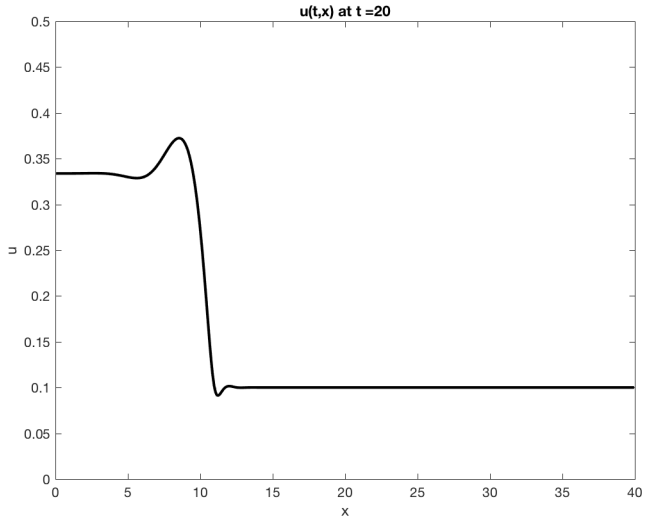
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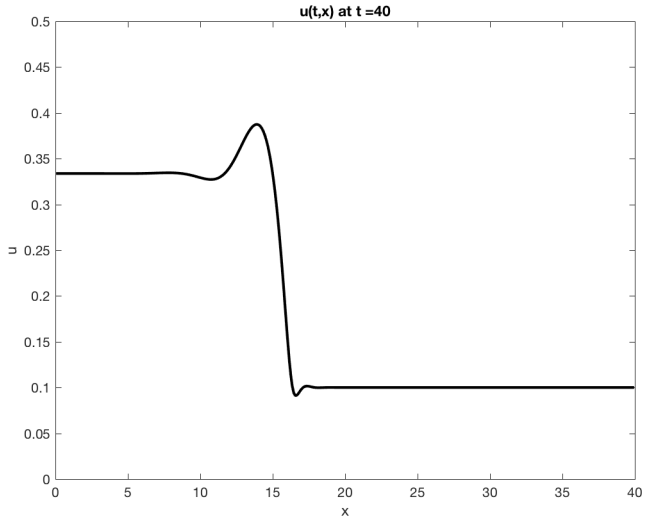
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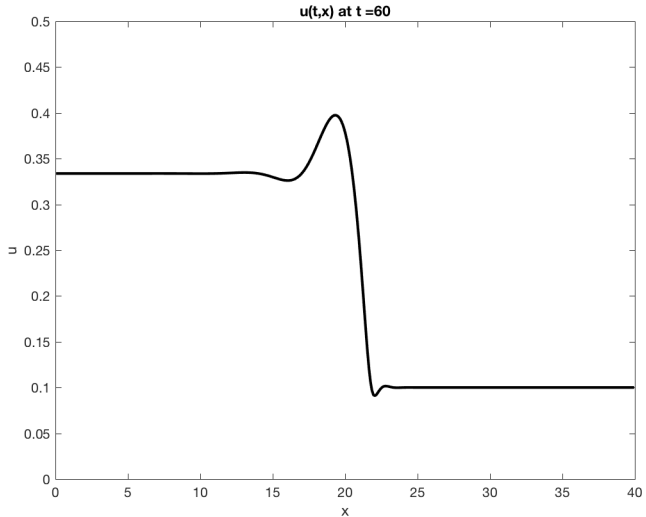
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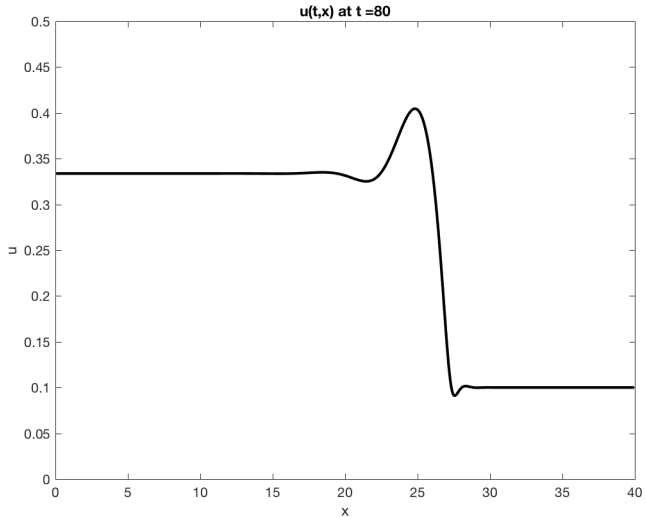
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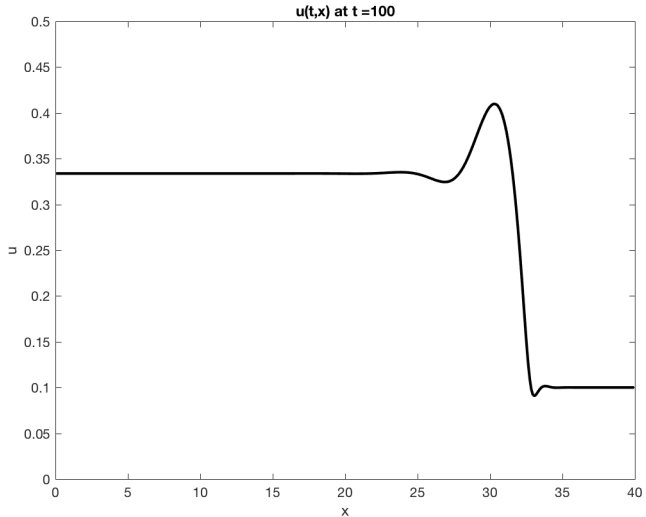
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Future Work

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- Higher dimensions
- Curved surfaces
- Space and time dependent coefficients
- Incorporation with air flow models
- Runge Kutta IMEX

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- Thanks
 - James Rossmannith
 - Alric Rothmayer
- Questions?

Bibliography

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