Derivation of Shallow Water Linearized Moment Equations

Shallow Water Moment Equations

$$h_t + (hu_m)_x + (hv_m)_y = 0$$

$$(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y$$

$$= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x)$$

$$(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x$$

$$= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big)$$

$$(h\alpha_i)_t + \left(2hu_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\alpha_k\right)_x + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_y$$

$$= u_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \alpha_k - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j\right)$$

$$(h\beta_i)_t + \left(hu_m\beta_i + hv_m\alpha_i + h\sum_{j,k=1}^{N} A_{ijk}\alpha_j\beta_k\right)_x + \left(2hv_m\beta_i + h\sum_{j,k=1}^{N} A_{ijk}\beta_j\beta_k\right)_y$$

$$= v_m D_i - \sum_{j,k=1}^{N} B_{ijk} D_j \beta_k - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right)$$

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \,\mathrm{d}\zeta$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right)\phi_k \,\mathrm{d}\zeta$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \,\mathrm{d}\zeta$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y$$

To get to the Shallow Water Linearized Moment Equations, we assume that $\alpha_i = O(\varepsilon)$ and $\beta_i = O(\varepsilon)$ are drop all terms of $O(\varepsilon^2)$ in the moment equations. The momentum equations remain the same even though they contain some of these terms.

$$
\begin{aligned}
h_t + (hu_m)_x + (hv_m)_y &= 0 \\
(hu_m)_t + \left(h\left(u_m^2 + \sum_{j=1}^{N} \frac{\alpha_j^2}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_x + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_y \\
= -\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N} \alpha_j\right) + hg(e_x - e_z (h_b)_x) \\
(hv_m)_t + \left(h\left(v_m^2 + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right) + \frac{1}{2} g e_z h^2\right)_y + \left(h\left(u_m v_m + \sum_{j=1}^{N} \frac{\alpha_j \beta_j}{2j+1}\right)\right)_x \\
= -\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N} \beta_j\right) + hg\Big(e_y - e_z (h_b)_y\Big) \\
(h\alpha_i)_t + (2hu_m\alpha_i)_x + (hu_m\beta_i + hv_m\alpha_i)_y = u_m D_i - (2i+1)\frac{\nu}{\lambda}\left(u_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\alpha_j\right) \\
(h\beta_i)_t + (hu_m\beta_i + hv_m\alpha_i)_x + (2hv_m\beta_i)_y = v_m D_i - (2i+1)\frac{\nu}{\lambda}\left(v_m + \sum_{j=1}^{N}\left(1 + \frac{\lambda}{h} C_{ij}\right)\beta_j\right) \\
C_{ij} &= \int_0^1 \phi_i' \phi_j' \,\mathrm{d}\zeta \\
D_i &= (h\alpha_i)_x + (h\beta_i)_y
\end{aligned}
$$

We can write down the shallow water linearized moments equations in the form

$$
\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{1}
$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$
\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{2}
$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $\boldsymbol{n} = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$
n_1(\boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q})) + n_2(\boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})).
$$

If this matrix is diaganalizable with real eigenvalues for all directions $\boldsymbol{n}$, then this system is considered hyperbolic.

Flux Functions

$$
\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix}
hu \\
\frac{1}{2} e_z g h^2 + hu^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j^2\right) \\
huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\
2hu\alpha_1 \\
\beta_1 hu + \alpha_1 hv \\
\vdots \\
2hu\alpha_N \\
hu\beta_N + hv\alpha_N
\end{pmatrix}, \quad
\boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix}
hv \\
huv + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\alpha_j\beta_j\right) \\
\frac{1}{2} e_z g h^2 + hv^2 + \sum_{j=1}^{N}\left(\frac{1}{2j+1} h\beta_j^2\right) \\
hu\beta_1 + hv\alpha_1 \\
2hv\beta_1 \\
\vdots \\
hu\beta_N + hv\alpha_N \\
2hv\beta_N
\end{pmatrix}
$$

Nonconservative Matrices

$$
g_1(\boldsymbol{q}) = \begin{pmatrix}
0 &&&&&&&& \\
0 & 0 &&&&&&& \\
& 0 & 0 &&&&&& \\
&& 0 & u &&&&& \\
&&& v & 0 &&&& \\
&&&& 0 & u &&& \\
&&&&& v & 0 && \\
&&&&&& 0 & \ddots & \\
&&&&&&& \ddots &
\end{pmatrix}, \quad
g_2(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 &&&&&&& \\
& 0 & 0 &&&&&& \\
&& 0 & 0 &&&&& \\
&&& 0 & u &&&& \\
&&&& v & 0 &&& \\
&&&&& 0 & u && \\
&&&&&& v & 0 & \\
&&&&&&& 0 & \ddots \\
&&&&&&&& \ddots
\end{pmatrix}
$$

Flux Jacobians

$$
\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2*i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2N+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & 2u &&&& \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 & v & u &&& \\
\vdots & \vdots & \vdots && \ddots & \ddots && \\
-2u\alpha_N & 2\alpha_N & 0 &&& 0 & 2u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & u
\end{pmatrix}
$$

$$
\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v & u &&& \\
-2v\beta_1 & 0 & 2\beta_1 && 2v & 0 && \\
\vdots & \vdots & \vdots &&& \ddots & \ddots & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & u \\
-2u\beta_N & 0 & 2\beta_N &&&&& 2v
\end{pmatrix}
$$

Quasilinear Matrices, $A = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), B = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right) & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \cdots & \frac{2}{2N+1}\alpha_N & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
-2u\alpha_1 & 2\alpha_1 & 0 & u &&&& \\
-u\beta_1 - v\alpha_1 & \beta_1 & \alpha_1 && u &&& \\
\vdots & \vdots & \vdots &&& \ddots && \\
-2u\alpha_N & 2\alpha_N & 0 &&&& u & \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&&& u
\end{pmatrix}
$$

$$
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right) & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \cdots & \frac{1}{2N+1}\beta_N & \frac{1}{2N+1}\alpha_N \\
e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right) & 0 & 2v & 0 & \frac{2}{3}\beta_1 & \cdots & 0 & \frac{2}{2N+1}\beta_N \\
-u\beta_1 - \alpha_1 v & \beta_1 & \alpha_1 & v &&&& \\
-2v\beta_1 & 0 & 2\beta_1 && v &&& \\
\vdots & \vdots & \vdots &&& \ddots && \\
-u\beta_N - v\alpha_N & \beta_N & \alpha_N &&&& v & \\
-2u\beta_N & 0 & 2\beta_N &&&&& v
\end{pmatrix}
$$

Quasilinear Eigenvalues The wavespeeds of this system in direction $\boldsymbol{n} = [n_1, n_2]$ are given by the eigenvalues of the matrix

$$n_1 A + n_2 B.$$

If all of the eigenvalues are real with a full set of eigenvectors, this this system is hyperbolic.

Convenient constants

$$
\begin{aligned}
d_0^1 &= n_1\left(e_z gh - u^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i^2\right)\right) + n_2\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) \\
d_0^2 &= n_1\left(-uv - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\alpha_i\beta_i\right)\right) + n_2\left(e_z gh - v^2 - \sum_{i=1}^{N}\left(\frac{1}{2i+1}\beta_i^2\right)\right) \\
d_i^1 &= n_1(-2u\alpha_i) + n_2(-u\beta_i - \alpha_i v) \\
d_i^2 &= n_1(-u\beta_i - v\alpha_i) + n_2(-2v\beta_i) \\
b_i^1 &= n_1\frac{2}{2i+1}\alpha_i + n_2\frac{1}{2i+1}\beta_i \\
b_i^2 &= n_2\frac{1}{2i+1}\alpha_i \\
c_i^1 &= n_1 2\alpha_i + n_2\beta_i \\
c_i^2 &= n_1\beta_i \\
c_i^3 &= n_2\alpha_i \\
c_i^4 &= n_1\alpha_i + n_2 2\beta_i
\end{aligned}
$$

$$
\det(n_1 A + n_2 B) = \begin{vmatrix}
0 & n_1 & n_2 & 0 & 0 & \cdots & 0 & 0 \\
d_0^1 & n_1 u - \tilde{\lambda} & n_2 u & b_1^1 & b_1^2 & \cdots & b_N^1 & b_N^2 \\
d_0^2 & n_1 v & n_2 v - \tilde{\lambda} & b_1^3 & b_1^4 & \cdots & b_N^3 & b_N^4 \\
d_1^1 & c_1^1 & c_1^3 & -\tilde{\lambda} & & & & \\
d_1^2 & c_1^2 & c_1^4 & & -\tilde{\lambda} & & & \\
\vdots & & & & & \ddots & & \\
d_N^1 & c_N^1 & & & & & -\tilde{\lambda} & \\
d_N^2 & c_N^2 & & & & & & -\tilde{\lambda}
\end{vmatrix}
$$

Quasilinear Eigenvectors