Generalized Shallow Water Equations

1 Generalized Shallow Water Equations

The generalized shallow water equations describe the movement of fluid when driven by gravity. The height of the fluid is given by h, the velocity profiles u^* and v^* in the x and y directions are approximated with the following Ansatz,

$$u^* = u + \sum_{j=1}^{N} (\alpha_j \phi_j)$$
$$v^* = v + \sum_{j=1}^{N} (\beta_j \phi_j),$$

where ϕ_j are the Legendre polynomials orthogonal on the domain $[0, 1]$, such that $\phi_j(0) = 1$. The first few of these Legendre polynomials are

$$\phi_0(\zeta) = 1, \qquad \phi_1(\zeta) = -2\zeta + 1, \qquad \phi_2(\zeta) = 6\zeta^2 - 6\zeta + 1, \qquad \phi_3(\zeta) = -20\zeta^3 + 30\zeta^2 - 12\zeta + 1.$$

Note that the mean velocities u and v can be expressed as coefficients of the constant moment, ϕ_0. They could be written as α_0 and β_0 respectively, but are given as u and v to match the standard shallow water equations.

The bottom topography is given by h_b, the kinematic viscosity ν, the slip length λ, the gravitational constant g, and the gravity direction $\boldsymbol{e} = [e_x, e_y, e_z]^T$.

The generalized shallow water equations are then given as follows.

$$h_t + (hu)_x + (hv)_x = 0 \tag{1}$$

$$\begin{aligned}(hu)_t &+ \left(hu^2 + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j^2\right) + \frac{1}{2}ge_zh^2\right)_x + \left(huv + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j\beta_j\right)\right)_y \\ &= -\frac{\nu}{\lambda}\left(u + \sum_{j=1}^{N}(\alpha_j)\right) + hge_x - hge_z(h_b)_x\end{aligned} \tag{2}$$

$$\begin{aligned}(hv)_t &+ \left(huv + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\alpha_j\beta_j\right)\right)_x + \left(hv^2 + h\sum_{j=1}^{N}\left(\frac{1}{2j+1}\beta_j^2\right) + \frac{1}{2}ge_zh^2\right)_y \\ &= -\frac{\nu}{\lambda}\left(v + \sum_{j=1}^{N}(\beta_j)\right) + hge_y - hge_z(h_b)_y\end{aligned} \tag{3}$$

$$\begin{aligned}(h\alpha_i)_t &+ \left(2hu\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\alpha_k)\right)\right)_x + \left(hu\beta_i + hv\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\beta_k)\right)\right)_y \\ &= u_mD_i - \sum_{j=1}^{N}\left(D_j\sum_{k=1}^{N}(B_{ijk}\alpha_k)\right) - (2i+1)\frac{\nu}{\lambda}\left(u + \sum_{j=1}^{N}\left(\left(1 + \frac{\lambda}{h}C_{ij}\right)\alpha_j\right)\right)\end{aligned} \tag{4}$$

$$\begin{aligned}(h\beta_i)_t &+ \left(hu\beta_i + hv\alpha_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\alpha_j\beta_k)\right)\right)_x + \left(2hv\beta_i + h\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(A_{ijk}\beta_j\beta_k)\right)\right)_y \\ &= v_mD_i - \sum_{j=1}^{N}\left(D_j\sum_{k=1}^{N}(B_{ijk}\beta_k)\right) - (2i+1)\frac{\nu}{\lambda}\left(v + \sum_{j=1}^{N}\left(\left(1 + \frac{\lambda}{h}C_{ij}\right)\beta_j\right)\right)\end{aligned} \tag{5}$$

where

$$A_{ijk} = (2i+1)\int_0^1 \phi_i\phi_j\phi_k \,\mathrm{d}\zeta \tag{6}$$

$$B_{ijk} = (2i+1)\int_0^1 \phi_i'\left(\int_0^\zeta \phi_j \,\mathrm{d}\hat{\zeta}\right)\phi_k \,\mathrm{d}\zeta \tag{7}$$

$$C_{ij} = \int_0^1 \phi_i'\phi_j' \,\mathrm{d}\zeta \tag{8}$$

$$D_i = (h\alpha_i)_x + (h\beta_i)_y \tag{9}$$

2 1D Equations

In one dimension the generalized shallow water equations will have the following form,

$$\boldsymbol{q}_t + \boldsymbol{f}(\boldsymbol{q})_x = g(\boldsymbol{q})\boldsymbol{q}_x + \boldsymbol{p}. \tag{10}$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$\boldsymbol{q} = [h, hu, h\alpha_1, h\alpha_2, \ldots]^T, \tag{11}$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeed of this system is given by the the eigenvalues of the matrix when the system is in quasilinear form. For this system we need to look at the eigenvalues of the matrix $f'(\boldsymbol{q}) - g(\boldsymbol{q})$. If all of the eigenvalues are real, then this system is considered hyperbolic. Also if the gradient of the eigenvalue with respect to the conserved variables dotted with the corresponding eigenvector is always positive or always negative, then we say that the system is convex. That is if

$$\nabla\lambda_i.\boldsymbol{v}_i < 0 \quad \text{or} \quad \nabla\lambda_i.\boldsymbol{v}_i > 0,$$

then the system is convex. If the dot product is zero, then we have a degenerate wave.

2.1 Zeroth Order

Flux Function

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z g h^2 + hu^2 \end{pmatrix} \tag{12}$$

Source Function

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z\frac{\partial}{\partial x}h_b - e_x)gh - \frac{\nu u}{\lambda} \end{pmatrix} \tag{13}$$

Flux Jacobian

$$\boldsymbol{f}'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 \\ e_z g h - u^2 & 2u \end{pmatrix} \tag{14}$$

Quasilinear Matrix

$$A = \begin{pmatrix} 0 & 1 \\ e_z g h - u^2 & 2u \end{pmatrix} \tag{15}$$

Quasilinear Matrix Eigenvalues

$$\lambda = u \pm \sqrt{gh} \tag{16}$$

Convexity

$$\nabla\lambda_i \cdot \boldsymbol{v}_i = \pm\frac{3}{2}\frac{g}{\sqrt{gh}} \tag{17}$$

2.2 First Order

Flux Function

$$\boldsymbol{f}(\boldsymbol{q}) \begin{pmatrix} hu \\ \frac{1}{2}e_z g h^2 + \frac{1}{3}\alpha_1^2 h + hu^2 \\ 2\alpha_1 hu \end{pmatrix} \tag{18}$$

Nonconservative Matrix

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{pmatrix} \tag{19}$$

Source Function

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1 + u)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h}+1)\alpha_1 + u)\nu}{\lambda} \end{pmatrix} \tag{20}$$

Flux Jacobian

$$\boldsymbol{f}'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - u^2 & 2u & \frac{2}{3}\alpha_1 \\ -2\alpha_1 u & 2\alpha_1 & 2u \end{pmatrix} \tag{21}$$

Quasilinear Matrix, $A = \boldsymbol{f}'(\boldsymbol{q}) - g(q)$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - u^2 & 2u & \frac{2}{3}\alpha_1 \\ -2\alpha_1 u & 2\alpha_1 & u \end{pmatrix} \tag{22}$$

Quasillinear Matrix Eigenvalues

$$\lambda = u \pm \sqrt{gh + \alpha_1^2}, u \tag{23}$$

Convexity

$$\nabla\lambda_1 \cdot \boldsymbol{v}_1 = -\frac{1}{2}\big(3gh + 4\alpha_1^2\big)\frac{\sqrt{gh + \alpha_1^2}}{gh^2 + h\alpha_1^2} \tag{24}$$

$$\nabla\lambda_2 \cdot \boldsymbol{v}_2 = \frac{\sqrt{gh + \alpha_1^2}}{h} \tag{25}$$

$$\nabla\lambda_3 \cdot \boldsymbol{v}_3 = -\frac{1}{2}\big(2gh + \alpha_1^2\big)\frac{\sqrt{gh + \alpha_1^2}}{gh^2 + h\alpha_1^2} \tag{26}$$

2.3 Second Order

Flux Function

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z g h^2 + hu^2 + \frac{1}{15}(5\alpha_1^2 + 3\alpha_2^2)h \\ \frac{4}{5}\alpha_1\alpha_2 h + 2\alpha_1 hu \\ 2\alpha_2 hu + \frac{2}{21}(7\alpha_1^2 + 3\alpha_2^2)h \end{pmatrix} \tag{27}$$

Nonconservative Matrix

$$g(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5}\alpha_2 + u & \frac{1}{5}\alpha_1 \\ 0 & 0 & \alpha_1 & \frac{1}{7}\alpha_2 + u \end{pmatrix} \tag{28}$$

Source Function

$$
\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1+\alpha_2+u)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h}+1)\alpha_1+\alpha_2+u)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h}+1)\alpha_2+\alpha_1+u)\nu}{\lambda} \end{pmatrix} \tag{29}
$$

Flux Jacobian

$$
\boldsymbol{f}'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - u^2 & 2u & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 \\ -\frac{4}{5}\alpha_1\alpha_2 - 2\alpha_1 u & 2\alpha_1 & \frac{4}{5}\alpha_2 + 2u & \frac{4}{5}\alpha_1 \\ -\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - 2\alpha_2 u & 2\alpha_2 & \frac{4}{3}\alpha_1 & \frac{4}{7}\alpha_2 + 2u \end{pmatrix} \tag{30}
$$

Quasilinear Matrix, $A = \boldsymbol{f}'(\boldsymbol{q}) - g(\boldsymbol{q})$

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - u^2 & 2u & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 \\ -\frac{4}{5}\alpha_1\alpha_2 - 2\alpha_1 u & 2\alpha_1 & \alpha_2 + u & \frac{3}{5}\alpha_1 \\ -\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - 2\alpha_2 u & 2\alpha_2 & \frac{1}{3}\alpha_1 & \frac{3}{7}\alpha_2 + u \end{pmatrix} \tag{31}
$$

Quasilinear Matrix Eigenvalues, need to be computed numerically

$$
\tag{32}
$$

2.4 Third Order

Flux Function

$$
\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2} e_z g h^2 + h u^2 + \frac{1}{105}(35\alpha_1^2 + 21\alpha_2^2 + 15\alpha_3^2) h \\ 2\alpha_1 h u + \frac{2}{35}(14\alpha_1\alpha_2 + 9\alpha_2\alpha_3) h \\ 2\alpha_2 h u + \frac{2}{21}(7\alpha_1^2 + 3\alpha_2^2 + 9\alpha_1\alpha_3 + 2\alpha_3^2) h \\ 2\alpha_3 h u + \frac{2}{15}(9\alpha_1\alpha_2 + 4\alpha_2\alpha_3) h \end{pmatrix} \tag{33}
$$

Nonconservative Matrix

$$
g(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5}\alpha_2 + u & \frac{1}{5}\alpha_1 - \frac{3}{35}\alpha_3 & \frac{3}{35}\alpha_2 \\ 0 & 0 & \alpha_1 - \frac{3}{7}\alpha_3 & \frac{1}{7}\alpha_2 + u & \frac{2}{7}\alpha_1 + \frac{1}{21}\alpha_3 \\ 0 & 0 & \frac{6}{5}\alpha_2 & \frac{4}{5}\alpha_1 + \frac{2}{15}\alpha_3 & \frac{1}{5}\alpha_2 + u \end{pmatrix} \tag{34}
$$

Source Function

$$
\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1+\alpha_2+\alpha_3+u)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h}+1)\alpha_1+(\frac{4\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h}+1)\alpha_2+\alpha_1+\alpha_3+u)\nu}{\lambda} \\ -\frac{7((\frac{4\lambda}{h}+1)\alpha_1+(\frac{24\lambda}{h}+1)\alpha_3+\alpha_2+u)\nu}{\lambda} \end{pmatrix} \tag{35}
$$

Flux Jacobian

$$
\boldsymbol{f}'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - \frac{1}{7}\alpha_3^2 - u^2 & 2u & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 & \frac{2}{7}\alpha_3 \\ -\frac{4}{5}\alpha_1\alpha_2 - \frac{18}{35}\alpha_2\alpha_3 - 2\alpha_1 u & 2\alpha_1 & \frac{4}{5}\alpha_2 + 2u & \frac{4}{5}\alpha_1 + \frac{18}{35}\alpha_3 & \frac{18}{35}\alpha_2 \\ -\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - \frac{6}{7}\alpha_1\alpha_3 - \frac{4}{21}\alpha_3^2 - 2\alpha_2 u & 2\alpha_2 & \frac{4}{3}\alpha_1 + \frac{6}{7}\alpha_3 & \frac{4}{7}\alpha_2 + 2u & \frac{6}{7}\alpha_1 + \frac{8}{21}\alpha_3 \\ -\frac{6}{5}\alpha_1\alpha_2 - \frac{8}{15}\alpha_2\alpha_3 - 2\alpha_3 u & 2\alpha_3 & \frac{6}{5}\alpha_2 & \frac{6}{5}\alpha_1 + \frac{8}{15}\alpha_3 & \frac{8}{15}\alpha_2 + 2u \end{pmatrix} \tag{36}
$$

Quasilinear Matrix, $A = \boldsymbol{f}'(\boldsymbol{q}) - g(\boldsymbol{q})$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ e_z gh - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - \frac{1}{7}\alpha_3^2 - u^2 & 2u & \frac{2}{3}\alpha_1 & \frac{2}{5}\alpha_2 & \frac{2}{7}\alpha_3 \\ -\frac{4}{5}\alpha_1\alpha_2 - \frac{18}{35}\alpha_2\alpha_3 - 2\alpha_1 u & 2\alpha_1 & \alpha_2 + u & \frac{3}{5}\alpha_1 + \frac{3}{5}\alpha_3 & \frac{3}{7}\alpha_2 \\ -\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - \frac{6}{7}\alpha_1\alpha_3 - \frac{4}{21}\alpha_3^2 - 2\alpha_2 u & 2\alpha_2 & \frac{1}{3}\alpha_1 + \frac{9}{7}\alpha_3 & \frac{3}{7}\alpha_2 + u & \frac{4}{7}\alpha_1 + \frac{1}{3}\alpha_3 \\ -\frac{6}{5}\alpha_1\alpha_2 - \frac{8}{15}\alpha_2\alpha_3 - 2\alpha_3 u & 2\alpha_3 & 0 & \frac{2}{5}\alpha_1 + \frac{2}{5}\alpha_3 & \frac{1}{3}\alpha_2 + u \end{pmatrix} \tag{37}$$

Eigenvalues, need to be computed numerically

$$\tag{38}$$

3 2D Equations

In two dimensions the generalized shallow water equations will have the following form,

$$\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = g_1(\boldsymbol{q})\boldsymbol{q}_x + g_2(\boldsymbol{q})\boldsymbol{q}_y + \boldsymbol{p}. \tag{39}$$

In this case the unknown $\boldsymbol{q}$ will have the form

$$\boldsymbol{q} = [h, hu, hv, h\alpha_1, h\beta_1, h\alpha_2, h\beta_2, \ldots]^T, \tag{40}$$

where the number of components depends on the number of moments in the velocity profiles.

The wavespeeds of the two dimensional system in the direction $\boldsymbol{n} = [n_1, n_2]$, are given by the eigenvalues of the matrix

$$n_1(\boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q})) + n_2(\boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})).$$

If this matrix is diaganalizable with real eigenvalues for all directions $\boldsymbol{n}$, then this system is considered hyperbolic.

3.1 Zeroth Order

The zeroth order system is exactly the standard shallow water equations, where only the average velocity is considered. This velocity profiles in this system only consider the constant moment. In this case the nonconservative product disappears and the equation has the following form.

$$\boldsymbol{q}_t + \boldsymbol{f}_1(\boldsymbol{q})_x + \boldsymbol{f}_2(\boldsymbol{q})_y = \boldsymbol{p}. \tag{41}$$

where

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \tag{42}$$

Flux Functions

$$\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z gh^2 + hu^2 \\ huv \end{pmatrix}, \qquad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv \\ \frac{1}{2}e_z gh^2 + hv^2 \end{pmatrix} \tag{43}$$

Source Function

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ -\left(e_z \dfrac{\partial}{\partial x}(h_b) - e_x\right)gh - \frac{\nu}{\lambda}u \\ -\left(e_z \dfrac{\partial}{\partial y}(h_b) - e_y\right)gh - \frac{\nu}{\lambda}v \end{pmatrix} \tag{44}$$

Flux Jacobians

$$\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 \\ e_z gh - u^2 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \qquad \boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ e_z gh - v^2 & 0 & 2v \end{pmatrix} \tag{45}$$

Quasilinear Matrices, $A = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), B = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ e_z g h - u^2 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ e_z g h - v^2 & 0 & 2v \end{pmatrix} \tag{46}$$

Wavespeed Eigenvalues

$$\lambda_{1,2} = n_1 u + n_2 v \pm \sqrt{e_z g h (n_0^2 + n_1^2)} \tag{47}$$

$$\lambda_3 = n_1 u + n_2 v \tag{48}$$

3.2 First Order

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \end{pmatrix} \tag{49}$$

Flux Functions

$$\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z g h^2 + \frac{1}{3}\alpha_1^2 h + hu^2 \\ \frac{1}{3}\alpha_1\beta_1 h + huv \\ 2\alpha_1 hu \\ \beta_1 hu + \alpha_1 hv \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ \frac{1}{3}\alpha_1\beta_1 h + huv \\ \frac{1}{2}e_z g h^2 + \frac{1}{3}\beta_1^2 h + hv^2 \\ \beta_1 hu + \alpha_1 hv \\ 2\beta_1 hv \end{pmatrix} \tag{50}$$

Nonconservative Matrices

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & v & 0 \end{pmatrix}, \quad g_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & v \end{pmatrix} \tag{51}$$

Source Function

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1 + u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y) g h - \frac{\nu(\beta_1 + v)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h}+1)\alpha_1 + u)\nu}{\lambda} \\ -\frac{3((\frac{4\lambda}{h}+1)\beta_1 + v)\nu}{\lambda} \end{pmatrix} \tag{52}$$

Flux Jacobians

$$\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ e_z g h - \frac{1}{3}\alpha_1^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 \\ -\frac{1}{3}\alpha_1\beta_1 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 \\ -2\alpha_1 u & 2\alpha_1 & 0 & 2u & 0 \\ -\beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & v & u \end{pmatrix} \tag{53}$$

$$\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{3}\alpha_1\beta_1 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 \\ e_z g h - \frac{1}{3}\beta_1^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 \\ -\beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & v & u \\ -2\beta_1 v & 0 & 2\beta_1 & 0 & 2v \end{pmatrix} \tag{54}$$

Quasilinear Matrices, $A = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), B = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ e_z gh - \frac{1}{3}\alpha_1^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 \\ -\frac{1}{3}\alpha_1\beta_1 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 \\ -2\alpha_1 u & 2\alpha_1 & 0 & u & 0 \\ -\beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & 0 & u \end{pmatrix} \tag{55}$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{3}\alpha_1\beta_1 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 \\ e_z gh - \frac{1}{3}\beta_1^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 \\ -\beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & v & 0 \\ -2\beta_1 v & 0 & 2\beta_1 & 0 & v \end{pmatrix} \tag{56}$$

Wavespeed Eigenvalues

$$\lambda_{1,2} = n_0 u + n_1 v \pm \sqrt{e_z gh(n_0^2 + n_1^2) + \left(\alpha_1 n_0 + \beta_1 n_1\right)^2} \tag{57}$$

$$\lambda_3 = n_0 u + n_1 v \tag{58}$$

$$\lambda_{4,5} = n_0 u + n_1 v \pm \frac{\sqrt{3}}{3}(\alpha_1 n_0 + \beta_1 n_1) \tag{59}$$

3.3 Second Order

$$\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \\ h\alpha_2 \\ h\beta_2 \end{pmatrix} \tag{60}$$

Flux Functions

$$\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z gh^2 + hu^2 + \frac{1}{15}(5\alpha_1^2 + 3\alpha_2^2)h \\ huv + \frac{1}{15}(5\alpha_1\beta_1 + 3\alpha_2\beta_2)h \\ \frac{4}{5}\alpha_1\alpha_2 h + 2\alpha_1 hu \\ \beta_1 hu + \alpha_1 hv + \frac{2}{5}(\alpha_2\beta_1 + \alpha_1\beta_2)h \\ 2\alpha_2 hu + \frac{2}{21}(7\alpha_1^2 + 3\alpha_2^2)h \\ \beta_2 hu + \alpha_2 hv + \frac{2}{21}(7\alpha_1\beta_1 + 3\alpha_2\beta_2)h \end{pmatrix}, \quad \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \frac{1}{15}(5\alpha_1\beta_1 + 3\alpha_2\beta_2)h \\ \frac{1}{2}e_z gh^2 + hv^2 + \frac{1}{15}(5\beta_1^2 + 3\beta_2^2)h \\ \beta_1 hu + \alpha_1 hv + \frac{2}{5}(\alpha_1\beta_1 + \alpha_2\beta_2)h \\ \frac{4}{5}\beta_1\beta_2 h + 2\beta_1 hv \\ \beta_2 hu + \alpha_2 hv + \frac{2}{21}(7\alpha_1\beta_1 + 3\alpha_2\beta_2)h \\ 2\beta_2 hv + \frac{2}{21}(7\beta_1^2 + 3\beta_2^2)h \end{pmatrix} \tag{61}$$

Nonconservative Matrices

$$g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\alpha_2 + u & 0 & \frac{1}{5}\alpha_1 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\beta_2 + v & 0 & \frac{1}{5}\beta_1 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 & \frac{1}{7}\alpha_2 + u & 0 \\ 0 & 0 & 0 & \beta_1 & 0 & \frac{1}{7}\beta_2 + v & 0 \end{pmatrix}, \quad g_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\alpha_2 + u & 0 & \frac{1}{5}\alpha_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\beta_2 + v & 0 & \frac{1}{5}\beta_1 \\ 0 & 0 & 0 & 0 & \alpha_1 & 0 & \frac{1}{7}\alpha_2 + u \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & \frac{1}{7}\beta_2 + v \end{pmatrix} \tag{62}$$

Source Function

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) gh - \frac{\nu(\alpha_1 + \alpha_2 + u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y) gh - \frac{\nu(\beta_1 + \beta_2 + v)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h} + 1)\alpha_1 + \alpha_2 + u)\nu}{\lambda} \\ -\frac{3((\frac{4\lambda}{h} + 1)\beta_1 + \beta_2 + v)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h} + 1)\alpha_2 + \alpha_1 + u)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h} + 1)\beta_2 + \beta_1 + v)\nu}{\lambda} \end{pmatrix} \tag{63}$$

Flux Jacobians

$$
\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_z g h - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \frac{2}{5}\alpha_2 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 \\
-\frac{4}{5}\alpha_1\alpha_2 - 2\alpha_1 u & 2\alpha_1 & 0 & \frac{4}{5}\alpha_2 + 2u & 0 & \frac{4}{5}\alpha_1 & 0 \\
-\frac{2}{5}\alpha_2\beta_1 - \frac{2}{5}\alpha_1\beta_2 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_2 + v & \frac{2}{5}\alpha_2 + u & \frac{2}{5}\beta_1 & \frac{2}{5}\alpha_1 \\
-\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - 2\alpha_2 u & 2\alpha_2 & 0 & \frac{4}{3}\alpha_1 & 0 & \frac{4}{7}\alpha_2 + 2u & 0 \\
-\frac{2}{3}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{2}{3}\beta_1 & \frac{2}{3}\alpha_1 & \frac{2}{7}\beta_2 + v & \frac{2}{7}\alpha_2 + u
\end{pmatrix} \tag{64}
$$

$$
\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 \\
e_z g h - \frac{1}{3}\beta_1^2 - \frac{1}{5}\beta_2^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 & 0 & \frac{2}{5}\beta_2 \\
-\frac{2}{5}\alpha_1\beta_1 - \frac{2}{5}\alpha_2\beta_2 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_1 + v & \frac{2}{5}\alpha_1 + u & \frac{2}{5}\beta_2 & \frac{2}{5}\alpha_2 \\
-\frac{4}{5}\beta_1\beta_2 - 2\beta_1 v & 0 & 2\beta_1 & 0 & \frac{4}{5}\beta_2 + 2v & 0 & \frac{4}{5}\beta_1 \\
-\frac{2}{3}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{2}{3}\beta_1 & \frac{2}{3}\alpha_1 & \frac{2}{7}\beta_2 + v & \frac{2}{7}\alpha_2 + u \\
-\frac{2}{3}\beta_1^2 - \frac{2}{7}\beta_2^2 - 2\beta_2 v & 0 & 2\beta_2 & 0 & \frac{4}{3}\beta_1 & 0 & \frac{4}{7}\beta_2 + 2v
\end{pmatrix} \tag{65}
$$

Quasilinear Matrices, $A = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), B = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_z g h - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \frac{2}{5}\alpha_2 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 \\
-\frac{4}{5}\alpha_1\alpha_2 - 2\alpha_1 u & 2\alpha_1 & 0 & \alpha_2 + u & 0 & \frac{3}{5}\alpha_1 & 0 \\
-\frac{2}{5}\alpha_2\beta_1 - \frac{2}{5}\alpha_1\beta_2 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{3}{5}\beta_2 & \frac{2}{5}\alpha_2 + u & \frac{1}{5}\beta_1 & \frac{2}{5}\alpha_1 \\
-\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - 2\alpha_2 u & 2\alpha_2 & 0 & \frac{1}{3}\alpha_1 & 0 & \frac{3}{7}\alpha_2 + u & 0 \\
-\frac{2}{3}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & -\frac{1}{3}\beta_1 & \frac{2}{3}\alpha_1 & \frac{1}{7}\beta_2 & \frac{2}{7}\alpha_2 + u
\end{pmatrix} \tag{66}
$$

$$
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 \\
e_z g h - \frac{1}{3}\beta_1^2 - \frac{1}{5}\beta_2^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 & 0 & \frac{2}{5}\beta_2 \\
-\frac{2}{5}\alpha_1\beta_1 - \frac{2}{5}\alpha_2\beta_2 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_1 + v & \frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2 & \frac{2}{5}\beta_2 & -\frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 \\
-\frac{4}{5}\beta_1\beta_2 - 2\beta_1 v & 0 & 2\beta_1 & 0 & \beta_2 + v & 0 & \frac{3}{5}\beta_1 \\
-\frac{2}{3}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{2}{3}\beta_1 & -\frac{1}{3}\alpha_1 & \frac{2}{7}\beta_2 + v & \frac{1}{7}\alpha_2 \\
-\frac{2}{3}\beta_1^2 - \frac{2}{7}\beta_2^2 - 2\beta_2 v & 0 & 2\beta_2 & 0 & \frac{1}{3}\beta_1 & 0 & \frac{3}{7}\beta_2 + v
\end{pmatrix} \tag{67}
$$

Wavespeed Eigenvalues need to be computed numerically

$$
\tag{68}
$$

3.4 Third Order

$$
\boldsymbol{q} = \begin{pmatrix} h \\ hu \\ hv \\ h\alpha_1 \\ h\beta_1 \\ h\alpha_2 \\ h\beta_2 \\ h\alpha_3 \\ h\beta_3 \end{pmatrix} \tag{69}
$$

Flux Functions

$$
\boldsymbol{f}_1(\boldsymbol{q}) = \begin{pmatrix} hu \\ \frac{1}{2}e_z g h^2 + hu^2 + \frac{1}{105}(35\alpha_1^2 + 21\alpha_2^2 + 15\alpha_3^2)h \\ huv + \frac{1}{105}(35\alpha_1\beta_1 + 21\alpha_2\beta_2 + 15\alpha_3\beta_3)h \\ 2\alpha_1 hu + \frac{2}{35}(14\alpha_1\alpha_2 + 9\alpha_2\alpha_3)h \\ \beta_1 hu + \alpha_1 hv + \frac{1}{35}(14\alpha_2\beta_1 + 14\alpha_1\beta_2 + 9\alpha_3\beta_2 + 9\alpha_2\beta_3)h \\ 2\alpha_2 hu + \frac{2}{21}(7\alpha_1^2 + 3\alpha_2^2 + 9\alpha_1\alpha_3 + 2\alpha_3^2)h \\ \beta_2 hu + \alpha_2 hv + \frac{1}{21}(14\alpha_1\beta_1 + 9\alpha_3\beta_1 + 6\alpha_2\beta_2 + 9\alpha_1\beta_3 + 4\alpha_3\beta_3)h \\ 2\alpha_3 hu + \frac{2}{15}(9\alpha_1\alpha_2 + 4\alpha_2\alpha_3)h \\ \beta_3 hu + \alpha_3 hv + \frac{1}{15}(9\alpha_2\beta_1 + 9\alpha_1\beta_2 + 4\alpha_3\beta_2 + 4\alpha_2\beta_3)h \end{pmatrix} \tag{70}
$$

$$
\boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} hv \\ huv + \frac{1}{105}(35\alpha_1\beta_1 + 21\alpha_2\beta_2 + 15\alpha_3\beta_3)h \\ \frac{1}{2}e_z g h^2 + hv^2 + \frac{1}{105}(35\beta_1^2 + 21\beta_2^2 + 15\beta_3^2)h \\ \beta_1 hu + \alpha_1 hv + \frac{1}{35}(14\alpha_1\beta_1 + 23\alpha_2\beta_2 + 9\alpha_3\beta_3)h \\ 2\beta_1 hv + \frac{2}{35}(14\beta_1\beta_2 + 9\beta_2\beta_3)h \\ \beta_2 hu + \alpha_2 hv + \frac{1}{21}(23\alpha_1\beta_1 + 6\alpha_2\beta_2 + 13\alpha_3\beta_3)h \\ 2\beta_2 hv + \frac{2}{21}(7\beta_1^2 + 3\beta_2^2 + 9\beta_1\beta_3 + 2\beta_3^2)h \\ \beta_3 hu + \alpha_3 hv + \frac{1}{15}(9\alpha_1\beta_1 + 13\alpha_2\beta_2 + 4\alpha_3\beta_3)h \\ 2\beta_3 hv + \frac{2}{15}(9\beta_1\beta_2 + 4\beta_2\beta_3)h \end{pmatrix} \tag{71}
$$

Nonconservative Matrices

$$
g_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\alpha_2 + u & 0 & \frac{1}{5}\alpha_1 - \frac{3}{35}\alpha_3 & 0 & \frac{3}{35}\alpha_2 & 0 \\ 0 & 0 & 0 & -\frac{1}{5}\beta_2 + v & 0 & \frac{1}{5}\beta_1 - \frac{3}{35}\beta_3 & 0 & \frac{3}{35}\beta_2 & 0 \\ 0 & 0 & 0 & \alpha_1 - \frac{3}{7}\alpha_3 & 0 & \frac{1}{7}\alpha_2 + u & 0 & \frac{2}{7}\alpha_1 + \frac{1}{21}\alpha_3 & 0 \\ 0 & 0 & 0 & \beta_1 - \frac{3}{7}\beta_3 & 0 & \frac{1}{7}\beta_2 + v & 0 & \frac{2}{7}\beta_1 + \frac{1}{21}\beta_3 & 0 \\ 0 & 0 & 0 & \frac{6}{5}\alpha_2 & 0 & \frac{4}{5}\alpha_1 + \frac{2}{15}\alpha_3 & 0 & \frac{1}{5}\alpha_2 + u & 0 \\ 0 & 0 & 0 & \frac{6}{5}\beta_2 & 0 & \frac{4}{5}\beta_1 + \frac{2}{15}\beta_3 & 0 & \frac{1}{5}\beta_2 + v & 0 \end{pmatrix} \tag{72}
$$

$$
g_2(\boldsymbol{q}) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\alpha_2 + u & 0 & \frac{1}{5}\alpha_1 - \frac{3}{35}\alpha_3 & 0 & \frac{3}{35}\alpha_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{5}\beta_2 + v & 0 & \frac{1}{5}\beta_1 - \frac{3}{35}\beta_3 & 0 & \frac{3}{35}\beta_2 \\ 0 & 0 & 0 & 0 & \alpha_1 - \frac{3}{7}\alpha_3 & 0 & \frac{1}{7}\alpha_2 + u & 0 & \frac{2}{7}\alpha_1 + \frac{1}{21}\alpha_3 \\ 0 & 0 & 0 & 0 & \beta_1 - \frac{3}{7}\beta_3 & 0 & \frac{1}{7}\beta_2 + v & 0 & \frac{2}{7}\beta_1 + \frac{1}{21}\beta_3 \\ 0 & 0 & 0 & 0 & \frac{6}{5}\alpha_2 & 0 & \frac{4}{5}\alpha_1 + \frac{2}{15}\alpha_3 & 0 & \frac{1}{5}\alpha_2 + u \\ 0 & 0 & 0 & 0 & \frac{6}{5}\beta_2 & 0 & \frac{4}{5}\beta_1 + \frac{2}{15}\beta_3 & 0 & \frac{1}{5}\beta_2 + v \end{pmatrix} \tag{73}
$$

Source Function

$$
\boldsymbol{p} = \begin{pmatrix} 0 \\ -(e_z \frac{\partial}{\partial x} h_b - e_x) g h - \frac{\nu(\alpha_1 + \alpha_2 + \alpha_3 + u)}{\lambda} \\ -(e_z \frac{\partial}{\partial y} h_b - e_y) g h - \frac{\nu(\beta_1 + \beta_2 + \beta_3 + v)}{\lambda} \\ -\frac{3((\frac{4\lambda}{h} + 1)\alpha_1 + (\frac{4\lambda}{h} + 1)\alpha_3 + \alpha_2 + u)\nu}{\lambda} \\ -\frac{3((\frac{4\lambda}{h} + 1)\beta_1 + (\frac{4\lambda}{h} + 1)\beta_3 + \beta_2 + v)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h} + 1)\alpha_2 + \alpha_1 + \alpha_3 + u)\nu}{\lambda} \\ -\frac{5((\frac{12\lambda}{h} + 1)\beta_2 + \beta_1 + \beta_3 + v)\nu}{\lambda} \\ -\frac{7((\frac{4\lambda}{h} + 1)\alpha_1 + (\frac{24\lambda}{h} + 1)\alpha_3 + \alpha_2 + u)\nu}{\lambda} \\ -\frac{7((\frac{4\lambda}{h} + 1)\beta_1 + (\frac{24\lambda}{h} + 1)\beta_3 + \beta_2 + v)\nu}{\lambda} \end{pmatrix} \tag{74}
$$

Flux Jacobians

$$
\boldsymbol{f}_1'(\boldsymbol{q}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
e_z gh - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - \frac{1}{7}\alpha_3^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \frac{2}{5}\alpha_2 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - \frac{1}{7}\alpha_3\beta_3 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 \\
-\frac{4}{5}\alpha_1\alpha_2 - \frac{18}{35}\alpha_2\alpha_3 - 2\alpha_1 u & 2\alpha_1 & 0 & \frac{4}{5}\alpha_2 + 2u & 0 & \frac{4}{5}\alpha_1 + \frac{18}{35}\alpha_3 \\
-\frac{2}{5}\alpha_2\beta_1 - \frac{1}{35}(14\alpha_1 + 9\alpha_3)\beta_2 - \frac{9}{35}\alpha_2\beta_3 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_2 + v & \frac{2}{5}\alpha_2 + u & \frac{2}{5}\beta_1 + \frac{9}{35}\beta_3 \\
-\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - \frac{6}{7}\alpha_1\alpha_3 - \frac{4}{21}\alpha_3^2 - 2\alpha_2 u & 2\alpha_2 & 0 & \frac{4}{3}\alpha_1 + \frac{6}{7}\alpha_3 & 0 & \frac{4}{7}\alpha_2 + 2u \\
-\frac{1}{21}(14\alpha_1 + 9\alpha_3)\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \frac{1}{21}(9\alpha_1 + 4\alpha_3)\beta_3 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{2}{3}\beta_1 + \frac{3}{7}\beta_3 & \frac{2}{3}\alpha_1 + \frac{3}{7}\alpha_3 & \frac{2}{7}\beta_2 + v \\
-\frac{6}{5}\alpha_1\alpha_2 - \frac{8}{15}\alpha_2\alpha_3 - 2\alpha_3 u & 2\alpha_3 & 0 & \frac{6}{5}\alpha_2 & 0 & \frac{6}{5}\alpha_1 + \frac{8}{15}\alpha_3 \\
-\frac{3}{5}\alpha_2\beta_1 - \frac{1}{15}(9\alpha_1 + 4\alpha_3)\beta_2 - \frac{4}{15}\alpha_2\beta_3 - \beta_3 u - \alpha_3 v & \beta_3 & \alpha_3 & \frac{3}{5}\beta_2 & \frac{3}{5}\alpha_2 & \frac{3}{5}\beta_1 + \frac{4}{15}\beta_3
\end{pmatrix} \tag{75}
$$

$$
\boldsymbol{f}_2'(\boldsymbol{q}) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - \frac{1}{7}\alpha_3\beta_3 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 & \frac{1}{7}\beta_3 \\
e_z gh - \frac{1}{3}\beta_1^2 - \frac{1}{5}\beta_2^2 - \frac{1}{7}\beta_3^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 & 0 & \frac{2}{5}\beta_2 & 0 \\
-\frac{2}{5}\alpha_1\beta_1 - \frac{23}{35}\alpha_2\beta_2 - \frac{9}{35}\alpha_3\beta_3 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_1 + v & \frac{2}{5}\alpha_1 + u & \frac{23}{35}\beta_2 & \frac{23}{35}\alpha_2 & \frac{9}{35}\beta_3 \\
-\frac{4}{5}\beta_1\beta_2 - \frac{18}{35}\beta_2\beta_3 - 2\beta_1 v & 0 & 2\beta_1 & 0 & \frac{4}{5}\beta_2 + 2v & 0 & \frac{4}{5}\beta_1 + \frac{18}{35}\beta_3 & 0 \\
-\frac{23}{21}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \frac{13}{21}\alpha_3\beta_3 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{23}{21}\beta_1 & \frac{23}{21}\alpha_1 & \frac{2}{7}\beta_2 + v & \frac{2}{7}\alpha_2 + u & \frac{13}{21}\beta_3 \\
-\frac{2}{3}\beta_1^2 - \frac{2}{7}\beta_2^2 - \frac{6}{7}\beta_1\beta_3 - \frac{4}{21}\beta_3^2 - 2\beta_2 v & 0 & 2\beta_2 & 0 & \frac{4}{3}\beta_1 + \frac{6}{7}\beta_3 & 0 & \frac{4}{7}\beta_2 + 2v & 0 \\
-\frac{3}{5}\alpha_1\beta_1 - \frac{13}{15}\alpha_2\beta_2 - \frac{4}{15}\alpha_3\beta_3 - \beta_3 u - \alpha_3 v & \beta_3 & \alpha_3 & \frac{3}{5}\beta_1 & \frac{3}{5}\alpha_1 & \frac{13}{15}\beta_2 & \frac{13}{15}\alpha_2 & \frac{4}{15}\beta_3 + v \\
-\frac{6}{5}\beta_1\beta_2 - \frac{8}{15}\beta_2\beta_3 - 2\beta_3 v & 0 & 2\beta_3 & 0 & \frac{6}{5}\beta_2 & 0 & \frac{6}{5}\beta_1 + \frac{8}{15}\beta_3 & 0
\end{pmatrix} \tag{76}
$$

Quasilinear Matrices, $A = \boldsymbol{f}_1'(\boldsymbol{q}) - g_1(\boldsymbol{q}), B = \boldsymbol{f}_2'(\boldsymbol{q}) - g_2(\boldsymbol{q})$

$$
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
e_z gh - \frac{1}{3}\alpha_1^2 - \frac{1}{5}\alpha_2^2 - \frac{1}{7}\alpha_3^2 - u^2 & 2u & 0 & \frac{2}{3}\alpha_1 & 0 & \frac{2}{5}\alpha_2 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - \frac{1}{7}\alpha_3\beta_3 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 \\
-\frac{4}{5}\alpha_1\alpha_2 - \frac{18}{35}\alpha_2\alpha_3 - 2\alpha_1 u & 2\alpha_1 & 0 & \alpha_2 + u & 0 & \frac{3}{5}\alpha_1 + \frac{3}{5}\alpha_3 \\
-\frac{2}{5}\alpha_2\beta_1 - \frac{1}{35}(14\alpha_1 + 9\alpha_3)\beta_2 - \frac{9}{35}\alpha_2\beta_3 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{3}{5}\beta_2 & \frac{2}{5}\alpha_2 + u & \frac{1}{5}\beta_1 + \frac{12}{35}\beta_3 \\
-\frac{2}{3}\alpha_1^2 - \frac{2}{7}\alpha_2^2 - \frac{6}{7}\alpha_1\alpha_3 - \frac{4}{21}\alpha_3^2 - 2\alpha_2 u & 2\alpha_2 & 0 & \frac{1}{3}\alpha_1 + \frac{9}{7}\alpha_3 & 0 & \frac{3}{7}\alpha_2 + u \\
-\frac{1}{21}(14\alpha_1 + 9\alpha_3)\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \frac{1}{21}(9\alpha_1 + 4\alpha_3)\beta_3 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & -\frac{1}{3}\beta_1 + \frac{6}{7}\beta_3 & \frac{2}{3}\alpha_1 + \frac{3}{7}\alpha_3 & \frac{1}{7}\beta_2 \\
-\frac{6}{5}\alpha_1\alpha_2 - \frac{8}{15}\alpha_2\alpha_3 - 2\alpha_3 u & 2\alpha_3 & 0 & 0 & 0 & \frac{2}{5}\alpha_1 + \frac{2}{5}\alpha_3 \\
-\frac{3}{5}\alpha_2\beta_1 - \frac{1}{15}(9\alpha_1 + 4\alpha_3)\beta_2 - \frac{4}{15}\alpha_2\beta_3 - \beta_3 u - \alpha_3 v & \beta_3 & \alpha_3 & -\frac{3}{5}\beta_2 & \frac{3}{5}\alpha_2 & -\frac{1}{5}\beta_1 + \frac{2}{15}\beta_3
\end{pmatrix} \tag{77}
$$

$$
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3}\alpha_1\beta_1 - \frac{1}{5}\alpha_2\beta_2 - \frac{1}{7}\alpha_3\beta_3 - uv & v & u & \frac{1}{3}\beta_1 & \frac{1}{3}\alpha_1 & \frac{1}{5}\beta_2 & \frac{1}{5}\alpha_2 & \frac{1}{7}\beta_3 \\
e_z gh - \frac{1}{3}\beta_1^2 - \frac{1}{5}\beta_2^2 - \frac{1}{7}\beta_3^2 - v^2 & 0 & 2v & 0 & \frac{2}{3}\beta_1 & 0 & \frac{2}{5}\beta_2 & 0 \\
-\frac{2}{5}\alpha_1\beta_1 - \frac{23}{35}\alpha_2\beta_2 - \frac{9}{35}\alpha_3\beta_3 - \beta_1 u - \alpha_1 v & \beta_1 & \alpha_1 & \frac{2}{5}\beta_1 + v & \frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2 & \frac{23}{35}\beta_2 & -\frac{1}{5}\alpha_1 + \frac{23}{35}\alpha_2 + \frac{3}{35}\alpha_3 & \frac{9}{35}\beta_3 \\
-\frac{4}{5}\beta_1\beta_2 - \frac{18}{35}\beta_2\beta_3 - 2\beta_1 v & 0 & 2\beta_1 & 0 & \beta_2 + v & 0 & \frac{3}{5}\beta_1 + \frac{3}{5}\beta_3 & 0 \\
-\frac{23}{21}\alpha_1\beta_1 - \frac{2}{7}\alpha_2\beta_2 - \frac{13}{21}\alpha_3\beta_3 - \beta_2 u - \alpha_2 v & \beta_2 & \alpha_2 & \frac{23}{21}\beta_1 & \frac{2}{21}\alpha_1 + \frac{3}{7}\alpha_3 & \frac{2}{7}\beta_2 + v & \frac{1}{7}\alpha_2 & \frac{13}{21}\beta_3 \\
-\frac{2}{3}\beta_1^2 - \frac{2}{7}\beta_2^2 - \frac{6}{7}\beta_1\beta_3 - \frac{4}{21}\beta_3^2 - 2\beta_2 v & 0 & 2\beta_2 & 0 & \frac{1}{3}\beta_1 + \frac{9}{7}\beta_3 & 0 & \frac{3}{7}\beta_2 + v & 0 \\
-\frac{3}{5}\alpha_1\beta_1 - \frac{13}{15}\alpha_2\beta_2 - \frac{4}{15}\alpha_3\beta_3 - \beta_3 u - \alpha_3 v & \beta_3 & \alpha_3 & \frac{3}{5}\beta_1 & \frac{3}{5}\alpha_1 - \frac{6}{5}\alpha_2 & \frac{13}{15}\beta_2 & -\frac{4}{5}\alpha_1 + \frac{13}{15}\alpha_2 - \frac{2}{15}\alpha_3 & \frac{4}{15}\beta_3 + v \\
-\frac{6}{5}\beta_1\beta_2 - \frac{8}{15}\beta_2\beta_3 - 2\beta_3 v & 0 & 2\beta_3 & 0 & 0 & 0 & \frac{2}{5}\beta_1 + \frac{2}{5}\beta_3 & 0
\end{pmatrix} \tag{78}
$$

Wavespeed Eigenvalues need to be computed numerically

$$
\tag{79}
$$