A formal treatment of a weighted mean on $\mathbb{R}^d$

Definition

Let $d \in \mathbb{N}$, $d \geq 1$, and let $v, w \in \mathbb{R}^d$. For each index $i \in \{1, \ldots, d\}$ define the scalar

$$a_i \;:=\; 2 - \frac{1}{i}. \tag{1}$$

Let

$$S_v \;:=\; \sum_{k=1}^{d} a_k v_k, \qquad S_w \;:=\; \sum_{k=1}^{d} a_k w_k, \qquad B \;:=\; 2d - 1. \tag{2}$$

Define the mean $m(v, w) \in \mathbb{R}^d$ componentwise by the given formula

$$m_i \;:=\; \frac{1}{2}\Big(v_i a_i + \frac{S_v - v_i a_i + S_w}{B}\Big), \tag{3}$$

for each $i = 1, \ldots, d$. The displayed formula in (3) is taken as the definition.
It is convenient to rewrite (3) in equivalent algebraic forms. A short computation yields

$$m_i \;=\; v_i a_i \frac{B-1}{2B} + \frac{S_v + S_w}{2B}. \tag{4}$$

Writing $a = (a_1, \ldots, a_d)^\mathsf{T}$ and 1 for the d-vector of ones, one obtains the matrix representation

$$m \;=\; M_v v + M_w w, \tag{5}$$

where the $d \times d$ matrices M_v and M_w are

$$M_v \;=\; \tfrac{1}{2}\,\mathrm{diag}(a) + \tfrac{1}{2B}\,1\,a^\mathsf{T}, \qquad M_w \;=\; \tfrac{1}{2B}\,1\,a^\mathsf{T}. \tag{6}$$

The remainder of this document records and proves consequences of the definition given in (3).

Consequences and proofs

Consequence 1 (Linearity and homogeneity)**.** *The mapping $(v, w) \mapsto m(v, w)$ is linear in v and in w separately and homogeneous of degree one in each argument.*

Proof. From equation $m_i \;=\; v_i a_i \frac{B-1}{2B} + \frac{S_v+S_w}{2B}$ the dependence of m_i on components of v and w is an affine-linear combination with coefficients that do not depend on v or w (they depend only on a and B). Explicitly, for scalars $\alpha, \beta \in \mathbb{R}$ and vectors $v, v', w, w' \in \mathbb{R}^d$ we have

$$m(\alpha v, \beta w) = \alpha m(v, 0) + \beta m(0, w),$$

since each m_i is a linear combination of the coordinates of v and of w. Thus $(v, w) \mapsto m(v, w)$ is linear in each argument and homogeneous of degree one. $\square$

Consequence 2 (Non-local dependence)**.** *Each coordinate m_i depends on every coordinate of v and w through the scalars S_v and S_w.*

Proof. Equation $m_i = \frac{1}{2}\big(v_i a_i + \dfrac{S_v - v_i a_i + S_w}{B}\big)$ exhibits the dependence: the terms S_v and S_w are sums over all indices $k = 1, \ldots, d$, so changing any coordinate v_k or w_k alters S_v or S_w and hence alters m_i. Therefore m_i is not determined solely by v_i and w_i. □

Consequence 3 (Asymmetry in the two arguments)**.** *The operation $m(v, w)$ is not symmetric in general; the direct coefficient multiplying v_i differs from that multiplying w_i.*

Proof. Compare equation $m_i = v_i a_i \frac{B-1}{2B} + \frac{S_v+S_w}{2B}$ with the analogous formula for $m_i(w, v)$ which is

$$m_i(w, v) = w_i a_i \tfrac{B-1}{2B} + \frac{S_w + S_v}{2B} = w_i a_i \tfrac{B-1}{2B} + \frac{S_v + S_w}{2B}.$$

The direct multiplicative coefficient of v_i in $m_i(v, w)$ is $a_i \frac{B-1}{2B}$ while the direct multiplicative coefficient of w_i in $m_i(v, w)$ (i.e. the coefficient multiplying w_i when expressing m_i in terms of the entries of w) is $\frac{a_i}{2B}$ when one expands equation $m_i = \frac{1}{2}\big(v_i a_i + \dfrac{S_v - v_i a_i + S_w}{B}\big)$ and isolates the contribution of the single coordinate w_i. Because $a_i \frac{B-1}{2B} \neq \frac{a_i}{2B}$ for $d \geq 1$ (indeed $B - 1 \neq 1$ unless $d = 1$), the two arguments play different roles; therefore $m(v, w) \neq m(w, v)$ in general. □

Consequence 4 (Special case $d = 1$)**.** *If $d = 1$ then $m(v, w)$ reduces to the ordinary midpoint of scalars.*

Proof. For $d = 1$ we have $a_1 = 2 - 1/1 = 1$ and $B = 2 \cdot 1 - 1 = 1$. Substituting into equation $m_i = \frac{1}{2}\big(v_i a_i + \dfrac{S_v - v_i a_i + S_w}{B}\big)$ yields

$$m_1 = \tfrac{1}{2}\big(v_1 + (v_1 - v_1 + w_1)\big) = \tfrac{v_1+w_1}{2},$$

the usual arithmetic mean. □

Consequence 5 (Zero-input behaviour)**.** *If $v = 0$ then $m(v, w)$ is the constant vector whose every component equals $S_w/(2B)$. If $w = 0$ then every component equals $S_v/(2B)$ plus the diagonal correction coming from v.*

Proof. Substitute $v = 0$ into equation $m_i = v_i a_i \frac{B-1}{2B} + \frac{S_v+S_w}{2B}$. Since $v_i = 0$ for all i and $S_v = 0$, one obtains $m_i = S_w/(2B)$ for every i, hence the constant vector. The other claim follows by symmetry of the formula in S_v and S_w while noting the diagonal term when $w = 0$. □

Consequence 6 (Equal inputs and idempotence)**.** *If $w = v$ then $m(v, v) \neq v$ in general; the mapping is not idempotent. The condition for idempotence is a linear equation on v.*

Proof. Set $w = v$ in equation $m_i = v_i a_i \frac{B-1}{2B} + \frac{S_v+S_w}{2B}$ to obtain

$$m_i(v, v) = v_i a_i \tfrac{B-1}{2B} + \frac{2S_v}{2B} = v_i a_i \tfrac{B-1}{2B} + \frac{S_v}{B}.$$

Requiring $m(v, v) = v$ leads to the linear system

$$v_i = v_i a_i \tfrac{B-1}{2B} + \frac{S_v}{B}, \qquad i = 1, \ldots, d,$$

or equivalently

$$\big(1 - a_i \tfrac{B-1}{2B}\big) v_i = \frac{S_v}{B}.$$

This is a system of d linear equations with unknowns v_i and generically has at most a one-parameter family of solutions (since the right-hand side is the same across indices). Thus idempotence fails for a generic v. □

Consequence 7 (Continuity and monotonicity)**.** *The mapping m is continuous and componentwise nondecreasing in each coordinate of v and w. Moreover each partial derivative exists and is nonnegative.*

Proof. From the linear algebraic representations in equations $m_i = v_i a_i \frac{B-1}{2B} + \frac{S_v+S_w}{2B}$ the partial derivatives exist and are constant:

$$\frac{\partial m_i}{\partial v_j} = \begin{cases} \dfrac{a_i(B-1)}{2B} + \dfrac{a_j}{2B}, & j = i, \\ \dfrac{a_j}{2B}, & j \neq i, \end{cases} \qquad \frac{\partial m_i}{\partial w_j} = \frac{a_j}{2B}.$$

Since each $a_j > 0$ and $B > 0$, these partial derivatives are nonnegative (indeed positive for every j), hence m is componentwise nondecreasing and continuous. □

Consequence 8 (Rank and operator structure)**.** *The matrix M_w is rank one, and M_v equals a diagonal matrix plus the same rank-one matrix; therefore M_w has one nonzero singular value and M_v is a rank-one perturbation of a diagonal matrix.*

Proof. Equation $M_w = \frac{1}{2B} 1\, a^\mathsf{T}$ expresses M_w as an outer product of the vectors 1 and a scaled by $\frac{1}{2B}$, hence it is of rank at most one; because $a \neq 0$ and $1 \neq 0$ it is precisely rank one. Equation $M_v = \frac{1}{2}\operatorname{diag}(a) + \frac{1}{2B} 1\, a^\mathsf{T}$ expresses M_v as a diagonal matrix plus the rank-one matrix M_w, completing the proof. □

Consequence 9 (Sum of components)**.** *The sum $\sum_{i=1}^d m_i$ equals $\dfrac{(3d-2)S_v + dS_w}{2B}$.*

Proof. Sum equation $m_i = v_i a_i \frac{B-1}{2B} + \dfrac{S_v + S_w}{2B}$ over $i = 1, \dots, d$ to obtain

$$\sum_{i=1}^{d} m_i = \frac{B-1}{2B} \sum_{i=1}^{d} a_i v_i + d\frac{S_v + S_w}{2B} = \frac{B-1}{2B} S_v + \frac{d(S_v + S_w)}{2B}.$$

Combine terms using $B = 2d - 1$ to compute $\dfrac{B-1}{2B} S_v + \dfrac{dS_v}{2B} = \dfrac{(3d-2)S_v}{2B}$, hence

$$\sum_{i=1}^{d} m_i = \frac{(3d-2)S_v + dS_w}{2B},$$

as claimed. □

Consequence 10 (Failure of translation invariance)**.** *Adding the same constant c to every coordinate of both v and w does not, in general, add c to every coordinate of $m(v, w)$.*

Proof. Let $v' = v + c\mathbb{1}$ and $w' = w + c\mathbb{1}$. Then $S_{v'} = S_v + c\sum_{k=1}^d a_k$ and $S_{w'} = S_w + c\sum_{k=1}^d a_k$. Using equation $m_i = v_i a_i \frac{B-1}{2B} + \dfrac{S_v + S_w}{2B}$ we obtain

$$m_i(v', w') = (v_i + c)a_i \tfrac{B-1}{2B} + \frac{S_v + S_w + 2c\sum_{k=1}^d a_k}{2B}.$$

Subtracting $m_i(v, w)$ gives

$$m_i(v', w') - m_i(v, w) = c\, a_i \tfrac{B-1}{2B} + c\frac{\sum_{k=1}^d a_k}{B},$$

which depends on i through a_i. Thus the increment is not equal to c for all i unless special and generally non-generic relations hold among the a_i; therefore translation invariance fails. □

Consequence 11 (Fixed points)**.** *The vectors $x \in \mathbb{R}^d$ satisfying $m(x, x) = x$ are solutions of the linear system*

$$\big(I - \tfrac{1}{2}\operatorname{diag}(a)\big)x = \frac{S_x}{B}\,\mathbb{1},$$

and therefore (if any nontrivial solutions exist) they lie in a one-parameter affine family determined by this system.

Proof. Using equation $m_i(v, v) = v_i a_i \frac{B-1}{2B} + \dfrac{S_v}{B}$ and setting $v = x$, the fixed-point equation $m(x, x) = x$ becomes

$$x_i = x_i a_i \tfrac{B-1}{2B} + \frac{S_x}{B}, \qquad i = 1, \ldots, d.$$

Rewriting yields

$$\big(1 - a_i \tfrac{B-1}{2B}\big)\, x_i = \frac{S_x}{B}.$$

In vector form this is $(I - \frac{1}{2}\operatorname{diag}(a))x = (S_x/B)\mathbb{1}$, as stated. The right-hand side is a scalar multiple of $\mathbb{1}$, so the system imposes that the entries of the left-hand side are equal; generically the solution space is at most one-dimensional (an affine line) or empty depending on compatibility. □

Consequence 12 (Large-d asymptotics)**.** *As $d \to \infty$ with the entries of v and w uniformly bounded, the contribution from the global sums S_v and S_w to each m_i is $O(1/d)$ while the direct term involving v_i is $\Theta(1)$; hence the direct v_i-term dominates.*

Proof. Since $B = 2d - 1 \sim 2d$ one has $\frac{1}{2B} = O(1/d)$. From equation $m_i = v_i a_i \frac{B-1}{2B} + \dfrac{S_v + S_w}{2B}$ the global term $\dfrac{S_v + S_w}{2B}$ is $O(1/d)$ provided $S_v, S_w = O(d)$ (which holds if v, w have bounded entries because $S_v = \sum_k a_k v_k$ is then $O(d)$). The coefficient of v_i equals $a_i \frac{B-1}{2B} \to a_i/2$ as $d \to \infty$, which is bounded away from zero for each fixed i. This establishes the claim. □

Consequence 13 (Geometric interpretation: not the Euclidean midpoint)**.** *In general the vector $m(v, w)$ does not lie on the line segment joining v and w in $\mathbb{R}^d$; therefore it is not the Euclidean midpoint.*

Proof. If $m(v, w)$ lay on the line segment joining v and w then there would exist a scalar $t \in [0, 1]$ such that $m(v, w) = tv + (1 - t)w$ for all v, w. But equation $m = M_v v + M_w w$ shows that the coefficients in front of v and w are matrices, not scalar multiples of the identity. If $m(v, w) =$

$tv + (1-t)w$ for all v, w then one would have $M_v = tI$ and $M_w = (1-t)I$, which contradicts the rank-one structure of M_w unless a is a scalar multiple of 1 and the diagonal part matches — which does not occur for the given $a_i = 2 - 1/i$ (they are nonconstant). Hence in general no such scalar t exists and $m(v, w)$ need not lie on the segment. Concretely, take $v = e_1$ and $w = e_2$ (standard basis vectors); a direct computation using equation $m_i = v_i a_i \frac{B-1}{2B} + \dfrac{S_v + S_w}{2B}$ yields coordinates that cannot be expressed as a convex combination of e_1 and e_2 with the same scalar across components. $\square$

Consequence 14 (Symmetry condition and characterization)**.** *The equality $m(v, w) = m(w, v)$ holds if and only if*

$$(M_v - M_w)(v - w) = 0,$$

where M_v and M_w are the matrices in equation $M_v = \frac{1}{2}\operatorname{diag}(a) + \frac{1}{2B} 1\, a^{\mathsf{T}}, \quad M_w = \frac{1}{2B} 1\, a^{\mathsf{T}}$. Moreover, m is symmetric for all v, w if and only if $M_v = M_w$, which does not occur for the present choice of a_i unless $d = 1$.

Proof. From equation $m = M_v v + M_w w$ we have

$$m(v, w) - m(w, v) = (M_v - M_w)(v - w).$$

Hence $m(v, w) = m(w, v)$ precisely when $(M_v - M_w)(v - w) = 0$. For symmetry for all pairs v, w we would require $M_v = M_w$, but comparing equation $M_v = \frac{1}{2}\operatorname{diag}(a) + \frac{1}{2B} 1\, a^{\mathsf{T}}$ with $M_w = \frac{1}{2B} 1\, a^{\mathsf{T}}$ shows $M_v = M_w$ iff $\frac{1}{2}\operatorname{diag}(a) = 0$, which is impossible since $a_i > 0$; the exceptional case $d = 1$ yields equality because the diagonal contribution collapses to a scalar that matches. Thus the mapping is not symmetric in general, but symmetry may hold for special pairs (v, w) satisfying the linear relation $(M_v - M_w)(v - w) = 0$. $\square$

Consequence 15 (Effect of the index i)**.** *The index i influences the weights $a_i = 2 - \frac{1}{i}$ appearing in every formula: the direct coefficient multiplying v_i equals $a_i \frac{B-1}{2B}$ while the influence of the j-th coordinate of v or w on m_i occurs through the factor $a_j/(2B)$. In particular a_i is strictly increasing in i and $a_i \to 2^-$ as $i \to \infty$, so higher-index coordinates receive larger direct weight from the v-diagonal term.*

Proof. Differentiate a_i with respect to i in the discrete sense: for integers $i \geq 1$ one computes $a_{i+1} - a_i = \left(2 - \frac{1}{i+1}\right) - \left(2 - \frac{1}{i}\right) = \frac{1}{i} - \frac{1}{i+1} > 0$, so a_i is strictly increasing in i. Moreover $\lim_{i\to\infty} a_i = 2$. From equation $m_i = v_i a_i \frac{B-1}{2B} + \dfrac{S_v + S_w}{2B}$ the direct multiplicative weight on v_i increases with i, while the contribution of a fixed coordinate j to every m_i scales with $a_j/(2B)$ and thus increases with the index j. $\square$