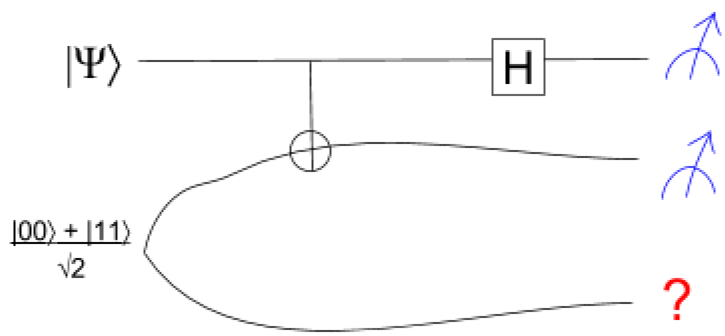
Lecture 10: Thurs Feb 16

Quantum Teleportation (Continued)

So let's say Alice wants to get a qubit over to Bob, but they do not share a quantum communication channel. They do, however, have a classical channel and preshared entanglement.

How should Alice go about this?

You can play around with testing different combinations of operations, and you'd eventually discover that what works is:



The qubit Alice wants to transmit is

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The entangle qubits form a Bell Pair.

The state starts at:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Then Alice applies cNOT ($|\Psi\rangle$ controls her entangled qubit):

$$\frac{1}{\sqrt{2}} [\ \alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|111\rangle\ ]$$

Alice Hadamard's her first qubit:

$$\frac{1}{2} [\ \alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\ ]$$

At which point Alice measures both her qubits in the $|0\rangle,|1\rangle$ basis.
This leads to four possible outcomes:

If Alice Sees	00	01	10	11
Then Bob's qubit is	$\alpha|0\rangle + \beta|1\rangle$	$\alpha|1\rangle + \beta|0\rangle$	$\alpha|0\rangle - \beta|1\rangle$	$\alpha|0\rangle - \beta|1\rangle$

We're deducing information about by Bob's state using the partial measurement rule. If Alice sees 00, then we narrow down the state of the entire system to the possibilities that fit, i.e. $|000\rangle$ and $|001\rangle$.

What is Bob's state, if he knows that Alice measured, but not knowing the measurement? It's an even mixture of all four possibilities, which is the Maximally Mixed State. This makes sense given the No Communication Theorem. Until Alice sends information over, Bob's qubit doesn't depend on $|\Psi\rangle$.

Now, Alice sends Bob her measurements via a classical channel.

If the first bit is 1, he applies $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

If the second bit is 1, he applies $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

These transformations will bring Bob's qubit to the state $\alpha|0\rangle + \beta|1\rangle = |\Psi\rangle$.
That means they've successfully sent over a qubit without a quantum channel!

This protocol works *even if* Alice doesn't know what $|\Psi\rangle$ is.

For this protocol to work, Alice had to measure her **syndrome** bits. These measurements were destructive (since we can't ensure that they'll be made in a basis orthonormal to $|\Psi\rangle$, and thus Alice doesn't have $|\Psi\rangle$ at the end.

Something to think about: Where is $|\Psi\rangle$ after Alice's measurement, but before Bob does his operations?

<u>How do people come up with this stuff? I can't picture how anyone trying to solve this problem would even begin their search…</u>

Well it's worth pointing out that quantum mechanics was discovered in 1926 and that quantum teleportation was only discovered in the 90's. These sorts of properties *can* be hard to find. Oftentimes someone tries to prove that something is impossible, and in doing so eventually figures out a way to get it done.

<u>Aren't we fundamentally sending infinitely more information than two classical bits if we've sent over enough information to perfectly describe an arbitrary qubit, since the qubit's amplitudes can be encoded in an arbitrarily complex way?</u>

I suppose, but you only really obtain the information that you can measure, which is significantly less. Amplitudes may exist physically, but they're different from other physical properties like length, in that they seem to act a lot more like probabilities.

For some $\alpha|0\rangle + \beta|1\rangle$ you could say that β is a binary expansion that encodes the complete works of Shakespeare—the rules of quantum mechanics don't put a limit on the amount information that it takes to encode a qubit. With that said, you could also encode the probability of a classical coin to do that.

If we can teleport one qubit, the next question we may want to ask is:

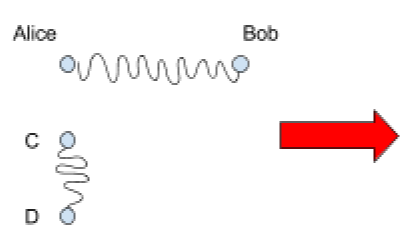
<u>Can we go further? What would it take to teleport an arbitrary quantum state?</u>

There's two things we need to note about the protocol first.

- It destroys Alice's version of $|\Psi\rangle$ (as expected from the No Cloning Theorem)
- It destroys the entanglement (this can be phrased as "Alice and Bob *used* a unit of entanglement")

First, we can notice that the qubit that's transmitted doesn't have to be unentangled.

You could run the protocol and have $|\Psi\rangle$ be half of another Bell Pair. That

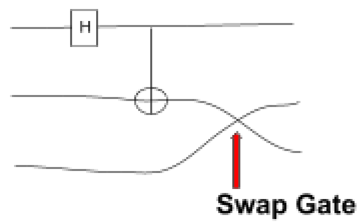


would entangle the fourth qubit to Bob's qubit (You can check this via calculation). That's not a particularly interesting operation, since it lands you where you started, with one qubit of entanglement between Alice and Bob, but it does have an interesting implication.

It suggests that it should be possible to transmit an n-qubit entangled state, by sending each over at a time, thus using n ebits of preshared entanglement.

One further crazy consequence of this is that two qubits don't need to interact directly to become entangled.

A simple example would be this:



In this circuit the 1st and 3rd qubit become entangled without direct contact.

<u>What does it take for Alice and Bob to get entangled anyways?</u>
The obvious way is for Alice to create a Bell Pair and send one of the qubits to Bob. In most practical experiments, the entangled qubits are created somewhere between Alice and Bob and are then sent off to them.

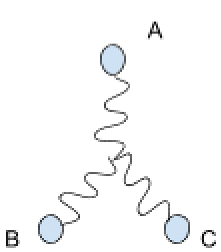
An even more surprising consequence is…
Entanglement Swapping

If Alice has a qubit $|\Psi\rangle$ that's entangled with Bob, she can send it over by using an ebit of entanglement.

This only requires measuring Alice's qubits and applying local transformations to Bob's qubits. This process is often used in practical experiments.

By the way, quantum teleportation has been demonstrated experimentally plenty of times.

A few more comments on the nature of entanglement:



We've seen the Bell Pair, and what it's good for. There's an analogue of it to three-party entanglement called **The GHZ State**: $\frac{|000\rangle + |111\rangle}{\sqrt{2}}$. We'll see applications of it later in the course, but for now we'll use it to show an interesting conceptual point.

Let's say that Alice, Bob, and Charlie share 3 classically correlated states. If all three of them get together, they can see that their qubits are classically correlated, and the same can be said if only two of them are together.

But now suppose that Charlie is gone. Can Alice and Bob use the entanglement between them to do quantum teleportation?

No. The trick here is that Charlie can measure without Alice and Bob knowing, which would remove their qubits from superposition, and thus would make the quantum teleportation protocol fail.

A different way to see this is to look at the density matrix of shared by Alice and Bob

$$\rho_{AB} = \begin{pmatrix} \frac{1}{2} & & & \\ & & & \\ & & & \\ & & & \frac{1}{2} \end{pmatrix}$$

And notice that it's different than the density matrix of a Bell Pair shared by Alice and Bob

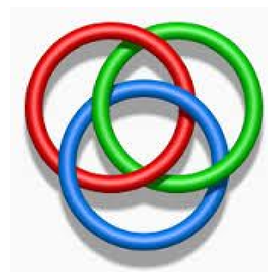
$$\rho_{AB} = \begin{pmatrix} \frac{1}{2} & & & \frac{1}{2} \\ & & & \\ & & & \\ \frac{1}{2} & & & \frac{1}{2} \end{pmatrix}$$

Remember: This gets derived by $|\Psi\rangle\langle\Psi|$

This is actually represents a generalization of…

The Monogamy of Entanglement

Simply put, this means that if Alice has a qubit that is maximally entangled with Bob, then it can't also be maximally entangled with Charlie.



With GHZ, you can only see the entanglement if you have all three together. This is often analogized to the Borromean Rings (right), a grouping of three rings in a way that all three are linked together, without any two being linked together.

There are other 3-qubit states which aren't like that…

In the W State, $\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, there's ***some*** entanglement between Alice and Bob, and there's ***some*** entanglement between Alice and Charlie, but neither pair is ***maximally entangled***.

So how do you quantify how much entanglement exists between two states?

It's worth noting that we sort of get to decide what we think a measure of entanglement *ought* to mean. We've seen how it can be useful to think of quantities of entanglement as a resource, so we can phrase the question as "How many 'Bell Pairs of entanglement' is this?"

It's not immediately obvious whether different kinds of entanglement would be good for different things. That's actually the case for large multi-party states, but with just Alice and Bob, it turns out that you can just measure in 'number of Bell Pairs of entanglement'.

Given $\sum_{i,j} \alpha_{ij}|i\rangle_A|j\rangle_B$, how many Bell Pairs is this worth?

Our first observation here should be that given any bipartite state, you can always find a representation of it with Alice and Bob representing their qubits in bases that are orthonormal. So we can write the state as $\Sigma\, \lambda_i|v_i\rangle|w_i\rangle$

such that all $|v_i\rangle$'s are orthonormal,

and all $|w_i\rangle$'s are orthonormal.

We get vectors in this form through…

Schmidt Decomposition

Given a the matrix $A = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ & \ddots & \\ \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix}$ representing the entire quantum state.

We can multiply by two unitary matrices to get a diagonal matrix:

$UAV = \Lambda$ U and V can be found efficiently using linear algebra

Essentially this means that we're rotating Alice's and Bob's states into an orthogonal basis.

We then have $\begin{pmatrix} |\lambda_i|^2 \\ \vdots \\ |\lambda_n|^2 \end{pmatrix}$ and we can just ask for the Shannon entropy of this to figure out how many Bell Pairs that's equal to.