Lecture 3: Tues Jan 24

Tensor Products are a way of building bigger vectors out of smaller ones.

Let's apply a NOT operation to the first bit, and do nothing to the second bit. That's really the same as defining function f as f(00) = 10, f(01) = 11, f(10) = 00, f(11) = 01. So we can fill in the tensor product as follows:

$$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{matrix}00\\01\\10\\11\end{matrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{matrix}00\\01\\10\\11\end{matrix}$$

$$\begin{matrix}00 & 01 & 10 & 11\end{matrix}$$

A **Quantum State** is a unit vector in $\mathbb{C}^N$ referring to the state of a quantum system.

Formally a quantum state could exist in any dimension. Physics courses cover infinitely dimensional vectors, but we'll stick to discrete systems (which is to say that when we make a measurement, there's a discrete number of variables to be read (with continuous outcomes).

- What does quantum mechanics say about the universe being discrete or continuous at the base level? It suggests a strange, hybrid picture. There's an infinite number of possibilities, but a discrete outcome. Formalisms of quantum mechanics technically contain infinite possibilities, like a system with two variables $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ has uncountably infinite possible amplitudes (given the only restriction is that $|\alpha|^2 + |\beta|^2 = 1$), but you could do that in classical mechanics as well by just making a complex formation about the probabilities of flipping coins.

The **Qubit** is the simplest quantum system.

It's a two-state system (we label these '0' and '1') whose amplitudes sum to 1.

A one-state quantum system would just be (1). Not very interesting!

As an alternative to vector notation, we have **Ket Notation**.

$$\begin{matrix}\alpha\\\beta\end{matrix}\begin{matrix}0\\1\end{matrix} = \alpha|0\rangle + \beta|1\rangle$$

Note that $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

and that $|\Psi\rangle$ is the variable we'll usually use for kets.

Why do we use ket notation?

One main advantage is that practically speaking, we'll usually care mostly about really sparse vectors (where most values are 0), so it's easier to represent only the values we are talking about.

It's really just a formalism to make life easier, we can put anything in ket notation. Look, this is Shrodinger's Cat in ket notation: $|$  $\rangle + |$  $\rangle$.

Often you'll need to take the transpose of a vector $\begin{pmatrix}\alpha\\\beta\end{pmatrix} \rightarrow (\alpha\ \beta)$ or for complex values $\begin{pmatrix}\alpha\\\beta\end{pmatrix} \rightarrow (\alpha^*\ \beta^*)$

Using the complex conjugate allows you to define a norm
$||v||^2 = v^T v$

Then we get $\begin{pmatrix} \alpha \end{pmatrix}$

$$v^T v = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \beta \end{pmatrix}$$
$$= \alpha^*\alpha + \beta^*\beta$$
$$= |\alpha|^2 + |\beta|^2$$

What does this look like in ket notation?

$\begin{pmatrix} \alpha \end{pmatrix}$

Just like we have the **ket** $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for $\begin{pmatrix} \beta \end{pmatrix}$

We define the **bra** $\langle\Psi| = \alpha^*\langle 0| + \beta^*\langle 1|$ for $\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$

And we define $\langle x|y\rangle$ as the inner product of ket $|x\rangle$ with ket $|y\rangle$

Therefore $\langle\Psi|\Psi\rangle = 1$.
So $\langle v|w\rangle = \langle w|v\rangle^*$.

$\begin{pmatrix} \alpha \end{pmatrix}$

Remember: the way we change quantum states is by applying linear transformations. $(U)\begin{pmatrix} \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$

A linear transformation is **Unitary** if $|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2$

Unitary Matrices correspond to unitary transformations.

We've got the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and permutation matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which are the only stochastic UMs.

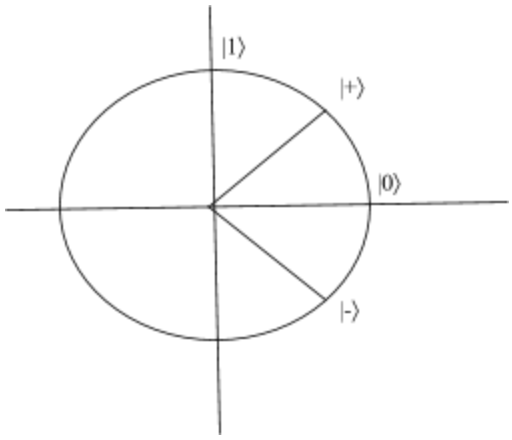
Others include

$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ <- which maps $|0\rangle$ -> $|1\rangle$ $\begin{pmatrix} 0 & -i \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ <- Note: Euler's Equation says $e^{i\theta} = \cos\theta + i*\sin\theta$

All real possible states of a qubit define a circle and all complex possibile states define a sphere. That's because these states are all the quantum vectors of length 1.

We define:

$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$
$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$



Unitary Transformations are norm-preserving linear transformations.

For any angle θ you could have $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ which grabs a vector and rotates it θ degrees.

For example $R_{\pi/4} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

To get the transform of $(ABv)^\dagger$ $v^\dagger B^\dagger A^\dagger$

<u>What does it mean that a unitary matrix preserves the 2-norm?</u>

It means applying a unitary transformation retains $\langle\Psi|\Psi\rangle$

$\langle\Psi|\Psi\rangle = (U|\Psi\rangle)^\dagger\, U|\Psi\rangle = \langle\Psi|U^\dagger U|\Psi\rangle$

So for this to always hold, $U^\dagger U$ has to be I. Which means $U^{-1} = U^\dagger$

That in turn implies that the rows of U must be an orthogonal unit basis .

So you can check if the rows or columns form an orthogonal unit basis (this isn't part of the definition of unitary matrices or anything, but because unitary matrices will always preserve the inner products).

An **Orthogonal Matrix** is both unitary and real.

They are the product of rotations and reflections.

Some examples:

$R_{\pi/4}|0\rangle = |+\rangle$

$R_{\pi/4}|+\rangle = |1\rangle$ You'll get a full revolution after applying $R_{\pi/4}$ eight times.

$R_{\pi/4}|1\rangle = -|-\rangle$

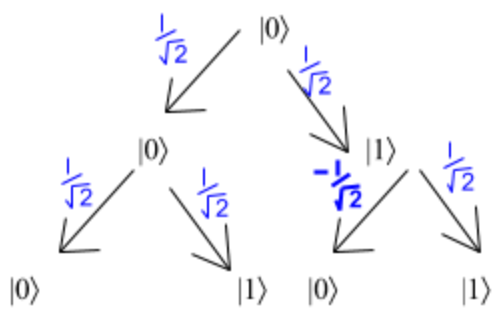
In the classical world

½ probability of a random event + ½ probability of a random event = just random

But in the quantum world

You can apply a transformation to a superpositioned state and get a specific answer

Anything interesting in quantum mechanics can be explained in terms of **interference**.



The $|0\rangle$ amplitude can go to states 0 and 1 equally.

There were two different amplitudes on the 0 state but they cancel each other out.

$|0\rangle$ states interfere destructively

$|1\rangle$ states interfere constructively

No matter what unitary transformation you apply: If $|0\rangle$ goes to $U|0\rangle$, then $-|0\rangle$ goes to $-U|0\rangle$.

The zero state and the minus zero state are indistinguishable mathematically, which is to say:

Global phase is unobservable.

Multiplying your entire quantum state by a scalar is like if last night someone moved the entire universe twenty feet to the left. We can only really measure things relative to other things:

Relative phase *is* observable.

To distinguish between the states $|+\rangle$ and $|-\rangle$ we can rotate and then measure them.

There are no second chances. Once you measure, the outcome is set.

So you can distinguish some states via repeated measurement.