Lecture 6: Thurs Feb 2

Last time we discussed the **Bell Pair**, and how if Alice measures her qubit in any basis, the state of Bob's qubit collapses to whichever state she got for hers. That being said, there's a formalism that tells us that Bob can't do anything to distinguish which basis Alice makes her measurement in, and thus no information travels instantaneously. This brings us to…

Mixed States

Which are probability distributions over quantum superposition.

We define a mixed state as a distribution over quantum states, so $\{p_i, |\Psi_i\rangle\} = p_1, |\Psi_1\rangle, \ldots, p_n, |\Psi_n\rangle$

Thus, we can think of a pure state as a degenerate state of a mixed state where all probabilities are 1.

Note that these don't have to be orthogonal

The tricky thing about mixed states is that they have to preserve the property we discussed above (that the basis Alice measures in doesn't affect Bob's state), which is to say that if we used the $\{p_i, |\Psi_i\rangle\}$ notation, we'd be allowing multiple instances of the notation to represent the same state. For example $\frac{|00\rangle + |01\rangle}{2}$ could be represented in the $|0\rangle,|1\rangle$ basis or the $|+\rangle,|-\rangle$ basis. To avoid this, we'll use…

Density Matrices

represented as $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

$|\Psi_i\rangle\langle\Psi_i|$ is the **outer product** of Ψ with itself.

It's the matrix you get by multiplying

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \begin{pmatrix} \alpha_1^* & \ldots & \alpha_N^* \end{pmatrix} = \begin{pmatrix} |\alpha_1|^2 & & \alpha_i\alpha_j^* \\ & \ldots & \\ \alpha_i^*\alpha_j & & |\alpha_N|^2 \end{pmatrix}$$

Note that $\alpha_i\alpha_j^* = \alpha_i^*\alpha_j$ which means that the matrix is it's own conjugate transpose

$A = A^\dagger$ That makes it a **Hermitian Matrix**.

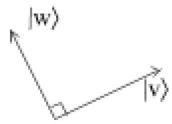
Some examples: $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore an even mixture of them would be $\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \begin{pmatrix} ½ & 0 \\ 0 & ½ \end{pmatrix} = \frac{I}{2}$.

Similarly: $|+\rangle\langle +| = \begin{pmatrix} ½ & ½ \\ ½ & ½ \end{pmatrix}$ $|-\rangle\langle -| = \begin{pmatrix} ½ & -½ \\ -½ & ½ \end{pmatrix}$

And $\frac{|+\rangle\langle +| + |-\rangle\langle -|}{2} = \begin{pmatrix} ½ & 0 \\ 0 & ½ \end{pmatrix} = \frac{I}{2}$.

Note that a mixture of $|0\rangle$ and $|1\rangle$ is different from a superposition of $|0\rangle$ and $|1\rangle$ (aka $|+\rangle$), and so they have different density matrices. However, the mixture of $|0\rangle$ and $|1\rangle$ and the mixture of $|+\rangle$ and $|-\rangle$ have the same density matrix: which makes sense because Alice converting between the two bases in our example above should maintain Bob's density matrix representation of the state.



In fact, this is true of whichever basis Alice chooses, and so for orthogonal vectors $|v\rangle$ and $|w\rangle$ we have that $\dfrac{|v\rangle\langle v| + |w\rangle\langle w|}{2} = \dfrac{I}{2}$.

Measuring ρ in the basis $|1\rangle, \dots, |N\rangle$ gives us the probability of $|i\rangle$ to be:

$$\Pr[|i\rangle] = \rho_{ii} = \langle i|\, \rho\, |i\rangle$$

Which is represented by the diagonal entries of the density matrix.

You don't need to square the value or anything because the Born Rule is already encoded in the density matrix (i.e. $(\alpha_1)(\alpha_1^*) =)|\alpha_1|^2$

That means that a density state which is a diagonal matrix is just a fancy way of writing a classical probability distribution. $\begin{pmatrix} p_1 & & \\ & \dots & \\ & & p_N \end{pmatrix}$

While a pure state would look like $|\Psi\rangle\langle\Psi| = \begin{pmatrix} ½ & ½ \\ ½ & ½ \end{pmatrix}$

What if we want to measure a density matrix in a different basis?

Measuring ρ in $\{|v\rangle, |w\rangle\}$ will give $\Pr[|v\rangle] = \rho_{ii} = \langle v|\, \rho\, |v\rangle$

You can think of density matrices as encoding not one but infinite probability distributions because you can measure it in any basis.

The matrix I/2 we've encountered above, as the even mixture of $|0\rangle$ and $|1\rangle$ (and also that of $|+\rangle$ and $|-\rangle$) is called the **Maximally Mixed State**. This state is basically just the outcome of a classical coin flip, which gives it a special property:

Regardless of the basis we measure it in, both outcomes will be equally likely.

So for some basis $|v\rangle, |w\rangle$ you get the probabilities $\langle v|\, I/2\, |v\rangle = ½ \langle v \mid v\rangle = ½$

$$\langle w|\, I/2\, |w\rangle = ½ \langle w \mid w\rangle = ½$$

This explains why Alice is unsuccessful in sending a message to Bob: the maximally mixed state in any other basis is *still the maximally mixed state*.

So how do we handle unitary transformations with density matrices?

Since $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, Applying U to ρ means:

$$\sum_i p_i (U|\Psi_i\rangle)(U|\Psi_i\rangle)^\dagger = \sum_i p_i U|\Psi_i\rangle\langle\Psi_i|U^\dagger = U\rho U^\dagger$$

You can pull out the U's since it's the same one applied to each mixture.

It's worth noting that getting n^2 values in the density matrix isn't some abstraction, you really need all those extra parameters. What do the off-diagonal entries represent?

$|+\rangle\langle+| = (\ \tfrac{1}{2}\ \tfrac{1}{2}\ )$

$(\ \tfrac{1}{2}\ \tfrac{1}{2}\ )$ These are where all the 'quantumness' resides.

It's where the interference between qubits is represented.

They can be different depending on relative phase:

$|+\rangle$ has positive off-diagonal entries

$|-\rangle$ has negative off-diagonal entries

$|i\rangle\langle i| = (\ \tfrac{1}{2}\ \ -i/2)$

$(\ i/2\ \ \tfrac{1}{2}\ )$

Later we'll see that as a quantum system interacts with the environment, the off-diagonal states get pushed down.

$(\ \tfrac{1}{2}\ \varepsilon\ )$

The density matrices in experimental quantum papers look like $(\ \varepsilon\ \tfrac{1}{2}\ )$.

The bigger the off-diagonal values, the better the experiment: because it represents them seeing more of the quantum effect.

Which matrices can arise as density matrices?

We're effective asking: What constraints does the form $\sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ put on the matrix?

It must be:

- Square
- Hermitian
- $\sum_i p_{ii} = 1$ (which is to say: the **trace**, $\mathrm{Tr}(\rho) = 1$)

Could $M = (\tfrac{1}{2}\ -10)$ be a density matrix?

$(-10\ \ \tfrac{1}{2})$

No. Measuring this in $|+\rangle,|-\rangle$ would give $(\ \tfrac{1}{\sqrt{2}}\ \ \tfrac{1}{\sqrt{2}}\ )\ M\ \begin{pmatrix} \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} \end{pmatrix} = 19/2$. Bad!

Remember that you can view each ρ as $U\rho U^\dagger$, whose diagonal has to be a probability distribution for all U. If we want that condition to hold, then in linear algebra terms, we need to add the restriction:

- All eigenvalues are non-negative (aka being **PSD: Positive Semidefinite**)

As a refresher: For the matrix ρ, the eigenvectors $|\Psi\rangle$ hold the equation:

$\rho|\Psi\rangle = \lambda|\Psi\rangle$ for some eigenvalue λ

If we had a negative eigenvector

$\langle\Psi|\rho|\Psi\rangle = \lambda$ would be < 0, which is nonsense.

Could we have missed a condition? Let's check.

<u>We claim: any square, Hermitian, PSD matrix arises as a density matrix of a quantum state.</u>

For such a ρ, find a representation of it in the form $\sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

Then there exist eigenvectors $p|\Psi_i\rangle = \lambda_i|\Psi_i\rangle$ for each row $\lambda_i \geq 0$

$\langle\Psi_i|\rho|\Psi_i\rangle = \lambda_i$ so $\Sigma\lambda_i = \Sigma\langle i|\rho|i\rangle = \rho_{ii} = 1$

So you can say that for $\Sigma \lambda |\Psi_i\rangle\langle\Psi_i|$:

λ are the eigenvalues and $|\Psi_i\rangle$ are the eigenvalues.

This process of obtaining eigenvalues and eigenvectors is called **eigendecomposition**.

We know eigenvalues will be real because the matrix is Hermitian,
They're non-negative because the matrix is PSD.

One quantity you can always compute for density matrices is:

Rank

rank(ρ) = the number of non-zero λ_i's
(the number of rows with no eigenvectors)

A density matrix of rank(n) must look like $\begin{pmatrix} p_1 & & 0 \\ & \ldots & \\ 0 & & p_n \end{pmatrix}$

And a density matrix of rank(1) represents a pure state.

Rank being at most n means that every mixed state
can be written as a mixture of at most n pure states.

In general, rank tells you the number of pure states that you have to mix to reach this mixed state.

Now, consider the pure 2 qubit state $\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}$.

We'll give the first qubit to Alice and the second to Bob.

How does Bob calculate his density matrix?

By picking some orthogonal basis for Alice's side.

You can rewrite Alice's part as $\sqrt{2/3}\,|0\rangle|+\rangle + \sqrt{1/3}\,|1\rangle|0\rangle$, which lets you calculate Bob's d.m.:

$$\tfrac{2}{3}|+\rangle\langle+| + \tfrac{1}{3}|0\rangle\langle 0|$$
$$= \tfrac{2}{3}\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix} + \tfrac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac23 & \frac13 \\ \frac13 & \frac13 \end{pmatrix}$$

In general, if you have a bipartite pure state, it'll look like $\sum_{i,j=1}^{N} \alpha_{ij}|i\rangle|j\rangle = |\Psi\rangle$

And you can get Bob's local density matrix

$$(\rho_{Bob})_{j,j'} = \sum_i \alpha_{ij}\,\alpha_{ij'}{}^*$$

This process of going from part of a mixed state to a whole pure state is called **Tracing Out**.

The Key Points:

1) A density matrix encodes all and only what is physically observable

- 2 quantum states will lead to different probabilities *iff* they have different d.m.'s

2) No-Communication Theorem
    - If Alice and Bob share an entangled state, nothing Alice chooses to do will have any effect on Bob's density matrix.
        - In other words, there's no observable effect on Bob's end. Which is the fundamental reason that quantum mechanics *is* compatible with the physical limitations of reality.