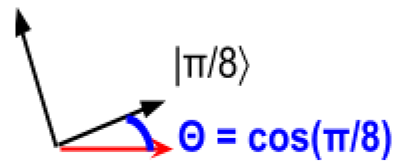
Lecture 14: Thurs March 2

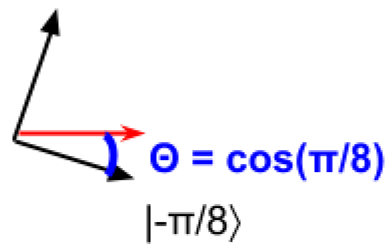
The Bell/CHSH Game (Continued)

Last time we talked about the CHSH Game and how we can use entanglement to create a better strategy than the classical one.



So why does this strategy work 85% of the time?

Lets consider the case where Alice gets $x = 0$ and measures $|0\rangle$.
She'll set $a = 0$, and they'll win the game if Bob sets $b = 0$.



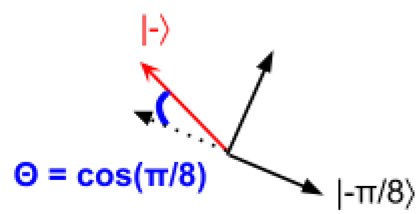
So what are the odds that Bob outputs 0?

Given that Alice measured her qubit already, Bob's qubit collapsed to the $|0\rangle$ state.

So if $y = 0$, Bob measures the $|0\rangle$ state in a basis rotated by $\pi/8$ clockwise. He outputs 0 if he measures $|\pi/8\rangle$. We know the probability of measuring a quantum state in a different basis is the cosine of the angle between the two vectors. Thus, the odds that Bob outputs 0 is $\cos^2(\pi/8) \approx 85\%$.

The same calculation is done for the case where $y = 1$. The angle between vectors is still $\pi/8$. In fact, you can extrapolate this result for all the cases where either x or y is 0.

Note that we can assume Alice measured first because of the No Communication Theorem.



The interesting case is where a and b are both set to 1.

This case requires that Alice measured in the $|+\rangle,|-\rangle$ basis and got $|-\rangle$. So what is Bob's probability of getting $|1\rangle$?

Still $\cos^2(\pi/8)$, because the angle between $|-\rangle$ and $-|\pi/8\rangle$ is $\pi/8$, and global phase doesn't matter.

The reason this game relates to hidden variable theories is that if all correlation between particles could be explained as "if anyone asks, we're both 0," you'd predict that Alice and Bob would win only ¾'s of the time (because that's how good they can do by pre-sharing arbitrary amounts of classical information). So you could refute local realism by running this experiment repeatedly—without having to presuppose that quantum mechanics is true.

Does Alice and Bob's ability to succeed more than ¾ of the time mean that they are communicating?

No, we know that's not possible (No Communication Theorem). We can more explicitly work out what Alice and Bob's density matrixes look like over time to check this.

Bob's initial density matrix is $\begin{pmatrix} ½ & 0 \\ 0 & ½ \end{pmatrix}$ and after Alice measures it's still $\begin{pmatrix} ½ & 0 \\ 0 & ½ \end{pmatrix}$.

So in that sense, no signal has been communicated from Alice to Bob. Nevertheless, if you know Alice's measurement and outcome you can predict Bob's measurement to update his density matrix. That shouldn't worry us though, since even classically if you condition on what Alice sees you can change your predictions.

Imagine a hierarchy of possibilities within physics of what the universe allows. You'd have Classical Local Realism at the bottom, where you can determine all outcomes of all measurements you make, and you only need to use probability when you have incomplete information about local objects.

At the top of the hierarchy is a Faster-Than-Light Science-Fiction Utopia where Alice and Bob can communicate instantaneously, you can travel faster than light, and so forth.

A priori people tend to believe that reality must be one or the other, and so reading pop-science articles that negate classical local realism, they think, "Okay, then we must live in a FTL sci-fi utopia."

Instead, the truth is a subtle midterm, which is perhaps so subtle that no science fiction writer would have the imagination to create, where there are no hidden variables, but there's no faster-than-light communication either.

Maybe no science fiction writer ever nailed how our universe works because it's hard to come up with a plot that requires Alice and Bob to win the CHSH game 85% of the time instead of 75%.

If we ran the experiment and Alice and Bob were winning CHSH more than 75% of the time, *and* we kept the assumption that the world is classical, then we would have to suppose that faster-than-light communication is occurring. Instead we suppose the likelier alternative: quantum mechanics is at play.

So where is that $\cos(\pi/8)$ coming from anyways? That seems so arbitrary…

It may seem like that value is simply coming from our particular approach to the problem. Maybe if we came at it another way we could improve on the $\cos^2(\pi/8)$ probability.

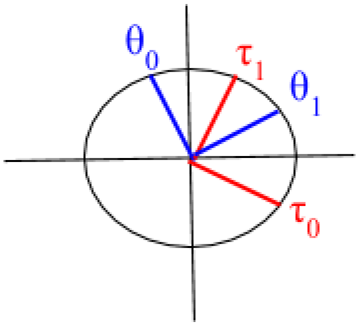
This was answered by...

Tsirelson's Inequality

A cousin of the Bell Inequality from the late 1980s.

Says that even if Alice and Bob share arbitrary amounts of entanglement, quantum mechanics can truly only help you win CHSH up to $\cos^2(\pi/8) \approx 85\%$.

It requires a bit too much machinery to prove here.
What we can do is show that out of strategies similar to the one we used, ours is optimal.



Let's say that Alice has two angles:

θ_0, the angle she outputs if she receives a 0 and

θ_1, the one she outputs if she receives a 1.

Similarly, Bob has τ_0 and τ_1.

The same rules apply from the solution we constructed earlier for the CHSH game. All we're doing here is changing the chosen vectors into variables to try and show that there's no better vectors to chose than the ones we did.

We can then say that the probability of success for Alice and Bob is:

$$P[\text{success}] = \tfrac{1}{4}\,[\, \cos^2(\theta_0 - \tau_0) + \cos^2(\theta_0 - \tau_1) + \cos^2(\theta_1 - \tau_0) + \sin^2(\theta_1 - \tau_1)]$$

^ [1] ^ [2] ^ [3]

Why?

1. We assume each outcome has an equal chance of occurring.
2. Alice and Bob win (in most cases) if they output the same bit, so we measure the cosine between their output angles.
3. Unless, both receive a 1. In this case we measure the chance of their angles being different, which is their sine.

Now we use some high-school trigonometry to get

$$= \tfrac{1}{2} + \tfrac{1}{8}\,[\cos(2(\theta_0 - \tau_0)) + \cos(2(\theta_0 - \tau_1)) + \cos(2(\theta_1 - \tau_0)) - \cos(2(\theta_1 - \tau_1))]$$

And we can abstract out the 2's on the cosines by understanding that we could adjust our original vectors to account for them.

We can also think of these cosines as the inner product of two vectors.

$$= \tfrac{1}{2} + \tfrac{1}{8}\,[U_0 \cdot V_0 + U_0 \cdot V_1 + U_1 \cdot V_0 - U_1 \cdot V_1]$$

$$= \tfrac{1}{2} + \tfrac{1}{8}\,[U_0\,(V_0 + V_1) + U_1\,(V_0 - V_1)]$$

Since these are all unit vectors, they're bounded by the norms

$$\leq \tfrac{1}{2} + \tfrac{1}{8}\,[\|V_0 + V_1\| + \|V_0 - V_1\|]11$$

And from here, we can use the parallelogram inequality to bound it further

$$\leq \tfrac{1}{2} + \tfrac{1}{8}\sqrt{2(\|V_0 + V_1\|^2 + \|V_0 - V_1\|^2}$$

Which equals

$$= 1/2 + (\sqrt{2}/8)\sqrt{4}$$

$$= \tfrac{1}{4}\,(2 + \sqrt{2})$$

Which wouldn't you know it, brings us to

$$\mathbf{= \cos^2(\pi/8)} \qquad \approx 85\%$$

So $\cos^2(\pi/8)$ really is the maximum winning percentage for the CHSH game.

There's been a trend in the last 10-15 years to study theories that would go past quantum mechanics (past Tsirelson's Inequality), but that would still avoid faster-than-light travel. In such a reality, it's been proven that if Alice and Bob want to schedule something on a calendar, they could agree on a date over only one bit on communication. That's better than can be done under the rules of quantum mechanics!

Testing the Bell Inequality

When Bell proposed his inequality, it was meant only as a conceptual point about quantum mechanics, but by the 1980s it was on it's way to becoming a feasible experiment. Alan Aspect (and others) ran the experiment, and his results were consistent with quantum mechanics.

He didn't quite get to 85% given the usual difficulties that affect quantum experiments, but he was able to reach a high statistical confidence that he was producing wins greater than 80% of the time.

This showed that you can use entanglement to win the CHSH game. Perhaps more impressive is that winning the CHSH game at > ¾ probability provides evidence that entanglement is there.

Most physicists shrugged, already sold on quantum mechanics (and the existence of entanglement), but others looked for holes in the experiment, because it refutes the classical view of the world.

They pointed out two loopholes in the existing experiment, essentially saying "if you squint enough, classical local realism might still be possible":

1. Detector Inefficiency

Sometimes detectors fail to detect a photon or they detect non-existent photons. Enough noise in the experiments could skew the data.

2. The Locality Issue

Taking the measurement and storing it on a computer takes microseconds, which by physics standards isn't negligible. Unless Alice and Bob and the referee are *very* far away from each other, there could be a sort of "local hidden variable conspiracy" going on, where as soon as Alice measures, some particle (unknown to physicists) flies over to Bob and says "hey, Alice's qubit measured to 0. You should measure to 0 too."

Aspect was able to close [2], but only in experiments still subject to [1].
By the 2000s, others were able to close [1], but only in experiments still subject to [2].
In 2016, a bunch of teams were finally able to close both loopholes simultaneously.

There are still people who deny the existence of entanglement, but through increasingly solipsistic arguments. For example…

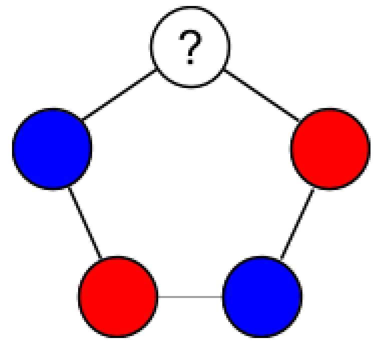
Superdeterminism

is a theory that says classical local realism is still the law of the land.

Explains the results of CHSH experiments by saying "We only *think* Alice and Bob can choose bases randomly," and that there's a grand cosmic conspiracy involving all of our minds, our computers, and our random number generators with the purpose of ensuring that Alice and Bob win the CHSH game at > ¾ probability by rigging the measurement bases. That's all it does.

Nobel Laureate Gerard 't Hooft advocates superdeterminism, so it's not like the idea lacks serious supporters, but Professor Aaronson is on board with entanglement.

Now we'll look at other non-local games to see what other tasks the Bell Inequality can help with. First, we have…



The Odd Cycle Game

There's a cycle with an odd number of vertices.

Alice and Bob claim that they have a two-coloring of the cycle, but basic graph theory tells us that this isn't possible.

Alice and Bob will agree on a strategy in advance (pre-sharing an arbitrary number of bits/ebits) to try to convince the referee that they've found one anyways.

The referee asks two obvious consistency checks:

- He can ask them both the color of vertex v (in the two-coloring they've found).
  - They pass if $v_A = v_B$
- He can ask Alice the color of vertex v and Bob the color of adjacent vertex w.
  - They pass if $v \neq w$

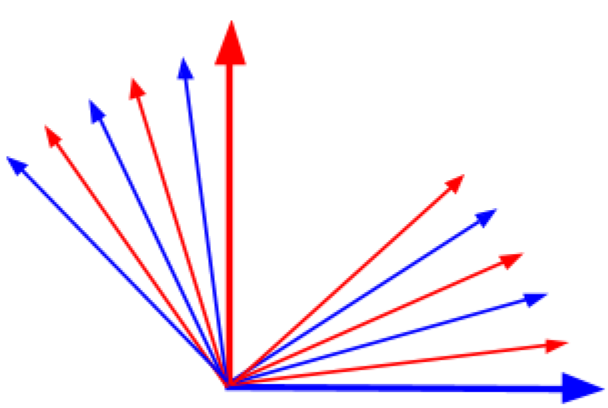
We take one run of the game to mean the referee asking a question once, and getting a response. Without loss of generality, answers are always RED or BLUE, and the cycle has size n.
What strategy provides the best probability that Alice and Bob will pass the test and win the game?

We know that the <u>classical strategy</u> has Pr[win] < 1, because for Alice and Bob to agree on a perfect solution ahead of time, they'd have to find a two-coloring (impossible). The best they can do is agree on a coloring for all but one of the vertices, which gets them Pr[win] $\leq 1 - \frac{1}{2n}$.

We claim that with the <u>quantum strategy</u> has Pr[win] $\approx 1 - \frac{1}{n^2}$.
First, Alice and Bob share a bell pair, $\frac{|00\rangle + |11\rangle}{2}$.
Alice and Bob each measure their qubit on a basis depending on the vertex they're asked about.

The measurement bases each differ by 2π/n, so they're evenly spaced between $|0\rangle$ and $|1\rangle$.

The first basis has 0 map to answering BLUE and 1 to answering RED. The second has 0 mapped to RED, and 1 to BLUE. They continue alternating.



So when Alice and Bob are asked about the same vertex, they both measure in the same basis, and thus both answer the same color.

When Alice and Bob are asked about adjacent vertices, we get a similar situation to the CHSH game, where the probability of Bob measuring his qubit to the same value as Alice's is the distance between the two vectors. So they answer incorrectly with probability $\sin^2\theta = \sin^2(1/n) \approx \frac{1}{n^2}$.

Another such game is…

The Magic Square Game

Alice and Bob claim that they can fill a 3x3 grid with 0's and 1's such that:

- Every row has an even sum
- Every column has an odd sum

The referee asks Alice to provide a random row of the grid, and Bob to provide a random column.

		0
0	1	1
		0

You can see that this grid can't actually be created by examining the total sum of the grid. The first rule requires it to be even, the second requires it to be odd. That means there's no classical strategy where Alice and Bob always win.

Mermin (the author of our textbook) discovered a quantum strategy where Alice and Bob can always win with only 2 ebits.

We wont write out this strategy.