Lecture 11: Tues Feb 21

For a classical probability distribution $D = (P_1, ..., P_n)$, we say its **Shannon Entropy** is

$$H(D) = \sum_{i=1}^{n} P_i \log_2 \frac{1}{P_i}$$

Von Neumann Entropy is generalization of Shannon Entropy from distributions to mixed states.
We say that the Von Neumann Entropy of a mixed state ρ is

$$S(\rho) = \sum_{i=1}^{n} \lambda_i \log_2 1/\lambda_i$$

You could say that Von Neumann Entropy *is* the Shannon Entropy of the vector of eigenvalues of the density matrix of ρ. If you diagonalize the density matrix, it represents a probability distribution over n orthogonal outcomes, and taking the Shannon Entropy of *that* gives you the Von Neumann Entropy of your quantum state.

Another way to think about it:

Say you took all the possible changes in bases of some quantum state. The Von Neumann Entropy of the quantum state would be the minimum of their Shannon Entropies.

$$S(\rho) = \min \begin{cases} H(U\rho U^\dagger) \\ H(U\rho U^\dagger) \\ H(\quad \vdots \quad) \end{cases} \qquad \text{each } U\rho U^\dagger \text{ looks like } \begin{pmatrix} x_1 & 0 & \dots \\ 0 & x_2 & 0 \\ 0 & \vdots & \end{pmatrix}$$

Why? Because any measurement basis results in some amount of uncertainty in the result. Most bases will have a degree of probabilistic uncertainty in the measurement, but the basis with the minimum Shannon Entropy can be said to be measurement basis that will provide the maximum amount of information about the quantum state.

So the Von Neumann Entropy of any pure state is 0, because there's always some measurement basis with a certain outcome.

You could choose to measure $|+\rangle$ in the $|0\rangle,|1\rangle$ basis and you'll have complete uncertainty, and an entropy of 1. But if you measure $|+\rangle$ in the $|+\rangle,|-\rangle$ basis, you have an entropy of 0, because you'll always measure it at $|+\rangle$.

So $S(|+\rangle) = 0$.

The Von Neumann Entropy of I/2 is 1.

Similarly, the maximum Von Neumann Entropy of an n-qubit state is N.

We can now talk about how much entropy is in a bipartite pure state.

Entanglement Entropy

Given Alice and Bob share a bipartite, pure state $|\Psi\rangle = \sum_{i,j} \alpha_{ij}|i\rangle_A|j\rangle_B$

To quantify the entanglement entropy, we'll trace out Bob's part, and look at the Von Neumann Entropy of Alice's side, $S(\rho_A)$, by asking: If Alice made an optimal measurement, how much could she learn about Bob's state?

$$S(\rho_A) = S(\rho_B) = H\{\lambda_i\}$$

^ This is the shannon entropy of these λ's, which you can get by diagonalizing Alice's (or Bob's) matrix, or by putting $|\Psi\rangle$ in schmidt form.

The Entanglement Entropy of $|\Psi\rangle \otimes |\Psi\rangle$ is 0.
The Entanglement Entropy of $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is 1.

You can think of entanglement entropy as either:

- The number of Bell Pairs it would take to create the state
- The number of Bell Pairs that you can extract from the state

It's not immediately obvious that these two values would be the same.

A sample calculation...
$|\Psi\rangle = \frac{3}{5} |0\rangle_A|+\rangle_B + \frac{4}{5} |1\rangle_A|-\rangle_B$ This is in Schmidt Form: Alice is in the X basis, Bob is in Y.
$E = (\frac{3}{5})^2 \log_2 ((5/3)^2) + (\frac{4}{5})^2 \log_2 ((5/4)^2)$
$= \sim .942$
That means that if Alice and Bob share 1000 instances of $|\Psi\rangle$, they'd be able to teleport about 942 qubits.

So for any bipartite, pure state we may want to know how many ebits of entanglement it corresponds to. There are two values to consider:

<u>The Entanglement of Formation</u> $E_F(\rho_{AB})$
Which is the number of ebits Alice and Bob need to create one copy of the state
<u>The Distillable Entanglement</u> $E_D(\rho_{AB})$
Which is the number of ebits Alice and Bob could extract from one copy of the state

It turns out that $E_F >> E_D$, which is to say that there exist bipartite, pure states which take a lot of entanglement to make and but that you can only extract a fraction of the entanglement you put in.

We say that a mixed state ρ_{AB} is *separable* if it can be written as a mixture of product states.

i.e. $\rho_{AB} = \sum_i p_i |v_i\rangle\langle v_i| + |w_i\rangle\langle w_i|$

The paper (Gurvits, 2003) proves a pretty crazy fact:
If you're given a density matrix, deciding whether ρ_{AB} is separable or entangled is NP-Complete. As a result, there's no nice characterization for telling apart mixed and unmixed states (Since that would prove P = NP).

There are endless paper writing opportunities in trying to classify types of entanglement, since looking at $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is much different from looking at $\frac{|000\rangle + |111\rangle}{\sqrt{2}}$

Interpretations of Quantum Mechanics

Now we're in a position to step back and ask,"What is quantum mechanics telling us about the nature of reality?" It should be no surprise that there isn't a consensus, but it's still worth looking at the philosophical debate, as its positions have often corresponded to breakthroughs in quantum mechanics (we'll see an example of this with the Bell Inequality).

Most discussions about the implications of quantum mechanics to our understanding of reality center around The Measurement Problem.

In most physics texts (and in this class), measurement is introduced as an unanalysed primitive that we don't question. There's a fundamental weirdness about it that stems from the fact that quantum mechanics seems to follow both:

1. Unitary Evolution
   when no one is watching $|\Psi\rangle \rightarrow U|\Psi\rangle$
2. Measurement
   Which collapses states to a single possibility $|\Psi\rangle \rightarrow i$ with probability $|\langle\Psi|i\rangle|^2 = |\alpha|^2$

In other words, quantum mechanics generally seems to work in a way that's gradual, continuous, and reversible most of the time (1), except for during (2), which is the only time we see it work in a way that's probabilistic, irreversible, and sudden. So we can alternatively phrase the question as:

"How does the universe know when to apply unitary evolution and when to apply measurement?"

People have argued about this for about 100 years, and the discussion is perhaps best compared to the discussion surrounding the nature of consciousness (which has gone on for millennia) in that they both devolve into people talking in circles about each other.

It's worth discussing the three main schools of thought, starting with…

The Copenhagen Interpretation

The prefered interpretation of most of the founders of quantum mechanics and was proposed by Bohr (hence the name) and Heisenberg.

It basically says that there are two different worlds: the quantum world and the physical world. We live in the physical world, which only has classical information, but in doing experiments we've discovered that there also exists the quantum world "beneath" it, which has quantum information.

Measurement, in this view, is the operation that bridges the two worlds.

It lets us "peek under the hood" into the quantum world and see what's going on.

Bohr wrote long tracks saying that just to make statements about the quantum world in the classical world is to suppose that there exists a boundary between them, and that we should never make an error in trying to conflate the two. His point of view essentially says "if you don't understand this, then you're just stuck in the old way of thinking, and you need to change".

The next interpretation, which is closely related is…

S.U.A.C. : "Shut Up And Calculate!"

The prefered interpretation of most current current researchers, academics in the field.

It says that at the end of the day, quantum mechanics works (it correctly predicts the results of experiments). If something seems confusing about it, then that's because there's something wrong with our current understanding of it.

You could say that the Copenhagen interpretation is basically just S.U.A.C. without the S.U. part. After seeing something weird, instead of shutting up, they'll write volumes and volumes about how we can't find a deeper truth.

The popularity of this point of view corresponds to most researchers thinking, "yes, this is how we do things in practice". It seems likely that the popularity of this view isn't going to last forever, because at the end of the day, people will want to understand more about what physical states are truly made of.

Schrödinger's Cat

There were physicists in the 30s and 40s who never accepted the Copenhagen interpretation, namely Einstein and Schrödinger, and they came up with plenty of examples to show just how untenable it is to have a rigid boundary between worlds if you think hard about it.

The most famous of these is Schrödinger's Cat, which first appears with Einstein saying that if you think of a pile of gunpowder as being inherently unstable, you could model it as a quantum state which looks like |⟩ + |⟩

Then Schrödinger comes along and adds some flair by asking, "What happens if we create a quantum state that corresponds to a superposition of a state in which a cat is alive and one where the cat is dead?" He allows for the assumption that the cat is isolated by putting it in a box. |⟩ + |⟩

The point of the thought experiment is that the formal rules of quantum mechanics should apply whenever you have distinguishable states, and thus you should also be able to have linear combinations of such states. It seems patently obvious that at some point we're implicitly crossing the boundary between the worlds, and thus we should have to say something about the nature of what's going on before measurement. Otherwise we'd devolve into extreme solipsism in saying that the cat only exists once we've opened the box to observe it.

Wigner's Friend

Is similar thought experiment. It says that Wigner could be put in a superposition of thinking one thought or another, modeled as $\frac{1}{\sqrt{2}}\left(|\text{Wigner}_0\rangle + |\text{Wigner}_1\rangle \right)$.

We can look at the state of him and a friend that's not aware of his state.

$$|\text{Friend}\rangle \otimes \frac{1}{\sqrt{2}}\left(|\text{Wigner}_0\rangle + |\text{Wigner}_1\rangle \right)$$

Whichever branch Wigner is in is what he believes (either one thought or the other) after the experiment has been performed. But from his friend's point of view, the experiment hasn't been performed. Then the two can talk, making the state

$$\frac{1}{\sqrt{2}}\left(|\text{Friend}_0\rangle|\text{Wigner}_0\rangle + |\text{Friend}_1\rangle|\text{Wigner}_1\rangle \right)$$

But then what happens if another friend comes along, and then another?

The point is to highlight the incompatibility of the perspectives of two observers: one ascribes a pure state other mixed state. We need some way of regarding measurement as fictitious *or* believing in only local truth.