Steerability of the sum of two Gaussian functions. *

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The sum of two Gaussians is defined as follow:

$$g(x,y) = e^{-\left(x^2+y^2\right)/2\sigma_1} + e^{-\left(x^2+y^2\right)/2\sigma_2}. \tag{1}$$

Next, differentiate with respect to the x direction:

$$\begin{aligned} g_x(x,y) &= \frac{\partial g(x,y)}{\partial x} \\ &= -x[e^{-\left(x^2+y^2\right)/2\sigma_1} + e^{-\left(x^2+y^2\right)/2\sigma_2}]. \end{aligned} \tag{2}$$

The question of Steerability involves representing the derivative of a function in any direction. A compact way of working with a directional derivative can be obtained by working with the function in polar form. Represent the derivative in polar form:

$$g_x(r,\theta) = -r\cos(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]. \tag{3}$$

The first condition of Steerability states that the directional derivative at any orientation can be obtained by a rotation of the function. We show this by showing that the derivative in the y direction can be obtained by suitably rotating the derivative in the x direction. The x and y orientations are orthogonal, so we rotate by $\pi/2$:

$$\begin{aligned} g_x(r,\theta-\pi/2) &= -r\cos(\theta-\pi/2)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \\ &= -r(\cos(\theta)\cos(\pi/2) + \sin(\theta)\sin(\pi/2))[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \text{ (by angular difference identity)} \\ &= -r\sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]. \end{aligned} \tag{4}$$

Similar to the derivative in the x direction, the derivative in y is the following (with a sin because of the polar representation):

$$\begin{aligned} g_y(x,y) &= \frac{\partial g(x,y)}{\partial y} \\ &= -y[e^{-\left(x^2+y^2\right)/2} + e^{-\left(x^2+y^2\right)/2\sigma_2}],. \\ g_y(r,\theta) &= -r\sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \end{aligned} \tag{5}$$

So, $g_x(r,\theta-\pi/2) = g_y(r,\theta-\pi/2)$.

Next we show that any arbitrary directional derivative can be obtained from the derivatives in the x and y directions, another essential property of Steerability:

$$\begin{aligned} g_a(r,\theta) &= -r\cos(\theta-\alpha)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}], \text{ offset by } \alpha \text{ for arbitrary orientation} \\ &= -r(\cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha))[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \\ &= \cos(\alpha)\left(-r\cos(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]\right) + \sin(\alpha)\left(-r\sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]\right) \\ &= \cos(\alpha)g_x + \sin(\alpha)g_y \end{aligned} \tag{6}$$

Therefore, the directional derivative at any orientation can be produced by a linear combination of x and y directional derivatives, a requirement of Steerability.

*Uses David Heeger's proof method found at https://www.cns.nyu.edu/ david/handouts/steerable.pdf