

# Steerability of the sum of two Gaussian functions. \*

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The sum of two Gaussians is defined as follow:

$$g(x, y) = e^{-(x^2+y^2)/2\sigma_1} + e^{-(x^2+y^2)/2\sigma_2}. \quad (1)$$

Next, differentiate with respect to the  $x$  direction:

$$\begin{aligned} g_x(x, y) &= \frac{\partial g(x, y)}{\partial x} \\ &= -x[e^{-(x^2+y^2)/2\sigma_1} + e^{-(x^2+y^2)/2\sigma_2}]. \end{aligned} \quad (2)$$

The question of Steerability involves representing the derivative of a function in any direction. A compact way of working with a directional derivative can be obtained by working with the function in polar form. Represent the derivative in polar form:

$$g_x(r, \theta) = -r \cos(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]. \quad (3)$$

The first condition of Steerability states that the directional derivative at any orientation can be obtained by a rotation of the function. We show this by showing that the derivative in the  $y$  direction can be obtained by suitably rotating the derivative in the  $x$  direction. The  $x$  and  $y$  orientations are orthogonal, so we rotate by  $\pi/2$ :

$$\begin{aligned} g_x(r, \theta - \pi/2) &= -r \cos(\theta - \pi/2)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \\ &= -r(\cos(\theta) \cos(\pi/2) + \sin(\theta) \sin(\pi/2))[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \text{ (by angular difference identity)} \\ &= -r \sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}]. \end{aligned} \quad (4)$$

Similar to the derivative in the  $x$  direction, the derivative in  $y$  is the following (with a *sin* because of the polar representation):

$$\begin{aligned} g_y(x, y) &= \frac{\partial g(x, y)}{\partial y} \\ &= -y[e^{-(x^2+y^2)/2\sigma_1} + e^{-(x^2+y^2)/2\sigma_2}], \\ g_y(r, \theta) &= -r \sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \end{aligned} \quad (5)$$

So,  $g_x(r, \theta - \pi/2) = g_y(r, \theta - \pi/2)$ .

Next we show that any arbitrary directional derivative can be obtained from the derivatives in the  $x$  and  $y$  directions, another essential property of Steerability:

$$\begin{aligned} g_a(r, \theta) &= -r \cos(\theta - \alpha)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}], \text{ offset by } \alpha \text{ for arbitrary orientation} \\ &= -r(\cos(\theta) \cos(\alpha) + \sin(\theta) \sin(\alpha))[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \\ &= \cos(\alpha) \left( -r \cos(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \right) + \sin(\alpha) \left( -r \sin(\theta)[e^{-r^2/2\sigma_1} + e^{-r^2/2\sigma_2}] \right) \\ &= \cos(\alpha)g_x + \sin(\alpha)g_y \end{aligned} \quad (6)$$

Therefore, the directional derivative at any orientation can be produced by a linear combination of  $x$  and  $y$  directional derivatives, a requirement of Steerability.

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\*Uses David Heeger's proof method found at <https://www.cns.nyu.edu/~david/handouts/steerable.pdf>